# 计算物理

# Lecture 11

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# Today's lecture

- Challenges in numerical integration
  - Using a change of variables
  - Dealing with singularities & improper integrals
- Multidimensional integration

# Change of variables: motivation

- Romberg integration works especially well when $f'(a)=f'(b)$ since the first error term must cancel

$$\int_a^b f(x)dx = \frac{h}{2}\left[f(a)+f(b)\right]+ h\sum_{i=1}^{n-1} f(a+ih)$$

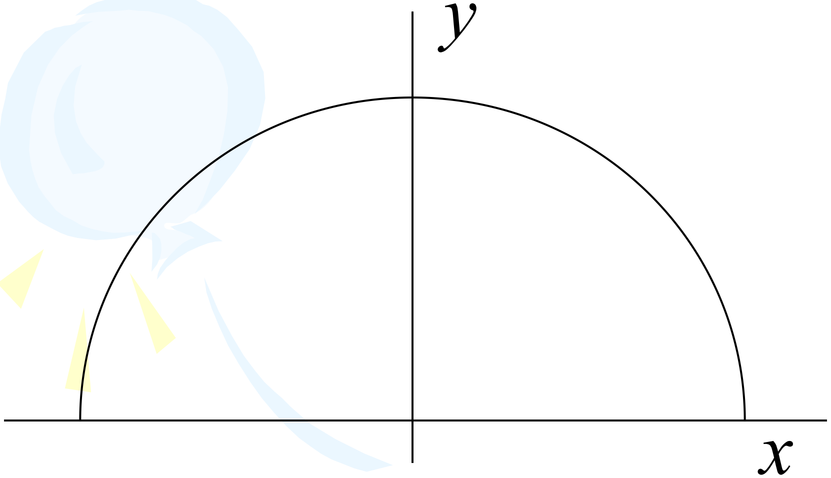
$$+\boxed{\frac{h^2}{12}\left[f'(a)-f'(b)\right]}$$

$$+\frac{h^4}{720}\left[f'''(a)-f'''(b)\right]+\ldots$$

**Euler-Maclaurin integration rule.**

# Change of variables: motivation

However, if derivatives become infinite at the end points there can be trouble converging




Consider $f(x) = \sqrt{1 - x^2}$,

Area of half circle ：

$$A = \int_{-1}^{1} \sqrt{1 - x^2}\, dx$$

But $f'(x) = -\dfrac{x}{\sqrt{1 - x^2}} \to \infty$

for $x = \pm 1$

In fact *all* derivatives are $\infty$ at $\pm 1$

- so error is not determined by the power of *h*!

# Apply a change of variables

$$\text{Let } x = \cos\theta, \Rightarrow \sqrt{1\text{-}x^2} = \sin\theta, \text{ and } dx = \text{-}\sin\theta\, d\theta$$
$$\text{At } x = 1, \text{ we have } \theta = 0$$
$$\text{At } x = \text{-}1, \text{ we have } \theta = \pi$$
$$\therefore A = \int_{-1}^{1}\sqrt{1-x^2}\,dx = -\int_{\pi}^{0}\sin^2\theta d\theta = \int_{0}^{\pi}\sin^2\theta d\theta \qquad (1)$$
Derivatives are now:
$$f'(\theta) = 2\sin\theta\cos\theta, \text{ and } f'(0) = f'(\pi) = 0$$
$$f''(\theta) = 2\cos 2\theta,$$
$$\text{so } f'''(\theta) = \text{-}4\sin 2\theta, \text{ and } f'''(0) = f'''(\pi) = 0$$

- All odd derivatives are zero – we can expect the trapezoid rule to be very accurate!
  - In fact it gives the exact answer with just 2 zones!

# Check for convergence: Romberg Ratio

- The Romberg integrator needs to know whether things are converging or not
- Define the Romberg ratio:

$$Rt_j = \frac{R_{j-1,1} - R_{j,1}}{R_{j,1} - R_{j+1,1}} = \frac{\text{error at step } j}{\text{error at step } j+1}$$

$$\approx \frac{h_{j-1}^2 - (h_{j-1}^2/4)}{(h_{j-1}^2/4) - (h_{j-1}^2/16)} \sim 4$$

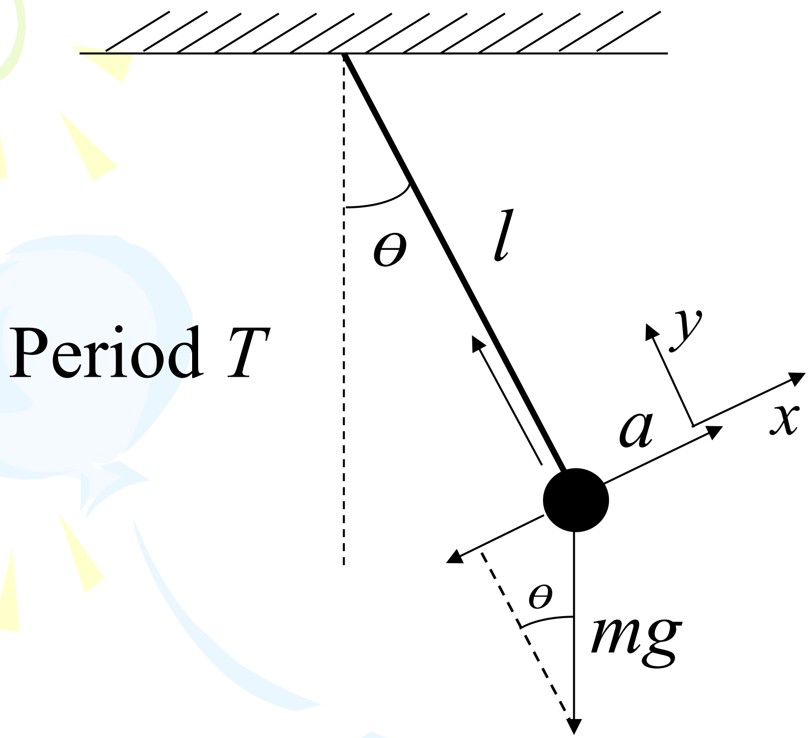
- So provided this ratio stays around 4 things are converging as they should
- If $Rt_j$<3.5 (say) then that is a signal things are not converging and we need to check some things…

# Check derivatives

- First thing to check is the first derivatives at $x=a$ & $x=b$
  - If large or infinite then a change of variables is probably necessary

- It is possible that you may hit a point where $f'(a), f'(b), f'''(a), f'''(b)$ are still well behaved but convergence is slow
  - In this case you force you integrator to proceed but keep a check on the interim results
  - Eventually if there is convergence you will get back to a Romberg ratio of $Rt_j \sim 4$

# Example: Pendulum without small angle approximation




- Along $x$ direction:

$$-mg\sin\theta = ma$$

Since $dx = l d\theta$, we get

$$a = \frac{d^2x}{dt^2} = l\frac{d^2\theta}{dt^2}$$

$$\Rightarrow \frac{d^2\theta}{dt^2} = -\frac{g}{l}\sin\theta \quad (1)$$

Note under small angle approx. $\sin\theta \rightarrow \theta$, to give a solution:

$$\theta(t) = \theta_0 \cos\sqrt{\frac{g}{l}}t \equiv \theta_0 \cos\omega t \qquad \text{where } \omega = \sqrt{\frac{g}{l}}$$

# Rewriting as an integral

- Eqn (1) is a second order ODE which we could solve using a numerical ODE solver
- However we can translate this particular problem into an integral as follows:

Multiply (1) by $\dot{\theta}$

$$\Rightarrow \dot{\theta}\ddot{\theta} = -\frac{g}{l}\dot{\theta}\sin\theta$$

$$\Rightarrow \int \dot{\theta}\frac{d\dot{\theta}}{dt}dt = -\frac{g}{l}\int \sin\theta \frac{d\theta}{dt}dt$$

cancel $dt$'s & integrate wrt $\dot{\theta}, \theta$

$$\Rightarrow \frac{1}{2}\dot{\theta}^2 = \frac{1}{2}\left(\frac{d\theta}{dt}\right)^2 = \frac{g}{l}\cos\theta + c$$

# Further manipulation

We can establish the constant $c$ by imposing that at $t=0,\ \theta=\theta_0,\ \dot{\theta}=0$ hence $c=-\frac{g}{l}\cos\theta_0$

Substitute for $c$ and take square roots to get

$$\frac{d\theta}{dt}=\sqrt{\frac{2g}{l}\left(\cos\theta-\cos\theta_0\right)}$$

$$\Rightarrow dt=\sqrt{\frac{l}{2g}}\frac{d\theta}{\sqrt{\cos\theta-\cos\theta_0}}$$

Since we set $\theta=\theta_0$ at $t=0$, the period, $T$, must be 4 times the time taken to go from $\theta=0$ to $\theta=\theta_0$.

$$\therefore T=4\int_{t=0}^{t=\theta_0}dt=4\sqrt{\frac{l}{2g}}\int_0^{\theta_0}\frac{d\theta}{\sqrt{\cos\theta-\cos\theta_0}} \quad (2)$$

# Immediate problems

- This is an *elliptic integral of the first kind* and must be evaluated numerically – however the upper limit produces a singularity so change variables:

We first rearrange using the trig identity

$$\cos\theta = 1 - 2\sin^2(\theta/2)$$

$$\Rightarrow \cos\theta - \cos\theta_0 = 2[\sin^2(\theta_0/2) - \sin^2(\theta/2)],$$

so we now have

$$T = 4\sqrt{\frac{l}{g}}\int_0^{\theta_0} \frac{d\theta}{2\sqrt{\sin^2(\theta_0/2) - \sin^2(\theta/2)}} \qquad (2)$$

We next set $\sin\xi = \dfrac{\sin(\theta/2)}{\sin(\theta_0/2)}$

# Change of variables to $x$

- Here are all the steps:

$$\text{Setting } \sin\xi = \frac{\sin(\theta/2)}{\sin(\theta_0/2)} \text{ gives}$$

$$\text{at } \theta = 0,\ \sin\xi = 0 \Rightarrow \xi = 0$$

$$\text{at } \theta = \theta_0, \sin\xi = 1 \Rightarrow \xi = \frac{\pi}{2}$$

$$\text{and for the differential substitution} \quad \cos\xi \mathrm{d}\xi = \frac{\cos(\theta/2)}{\sin(\theta_0/2)}\frac{d\theta}{2}$$

$$\Rightarrow d\theta = \frac{2\sin(\theta_0/2)\sqrt{1-\sin^2\xi}}{\sqrt{1-\sin^2(\theta/2)}}d\xi = \frac{2\sqrt{\sin^2(\theta_0/2)-\sin^2(\theta/2)}}{\sqrt{1-k^2\sin^2\xi}}d\xi$$

$$\text{where } k = \sin(\theta_0/2) \Rightarrow k < 1$$

$$\therefore T = 4\sqrt{\frac{l}{g}}\int_0^{\pi/2}\frac{2\sqrt{\sin^2(\theta_0/2)-\sin^2(\theta/2)}}{2\sqrt{\sin^2(\theta_0/2)-\sin^2(\theta/2)}\sqrt{1-k^2\sin^2\xi}}d\xi \quad (2)$$

# Arrive at numerically tractable integral

- After cancelling like terms in the numerator & denominator we get

$$T = 4\sqrt{\frac{l}{g}}\int_0^{\pi/2}\frac{1}{\sqrt{1-k^2\sin^2\xi}}d\xi \qquad (3)$$

- (3) is a perfectly well behaved function: there are no issues at the end points or the interior ($k$=1 is impossible since $k=\sin(\theta_0/2)$ and $\theta_0<\pi/2$)
  - Thus amenable to numerical (e.g. Romberg) integration
- Note that the integral of the form

$$\boxed{K(k) = \int_0^{\pi/2}\frac{1}{\sqrt{1-k^2\sin^2\xi}}d\xi}$$

is called a *complete elliptic integral of the first kind* and can be found tabulated for various $k$

# Special values

$$K(0)=\frac{\pi}{2}$$

$$K\left(\frac{\sqrt{2}}{2}\right)=\frac{1}{4\sqrt{\pi}}\Gamma\left(\frac{1}{4}\right)^2$$

$$K\left(\frac{1}{4}\left(\sqrt{6}-\sqrt{2}\right)\right)=\frac{3^{\frac{1}{4}}}{2^{\frac{7}{3}}\pi}\Gamma\left(\frac{1}{3}\right)^3$$

$$K\left(\frac{1}{4}\left(\sqrt{6}+\sqrt{2}\right)\right)=\frac{3^{\frac{3}{4}}}{2^{\frac{7}{3}}\pi}\Gamma\left(\frac{1}{3}\right)^3$$

$$K\left(2\sqrt{-4-3\sqrt{2}}\right)=\frac{\left(2-\sqrt{2}\right)\pi^{\frac{3}{2}}}{4\,\Gamma\left(\frac{3}{4}\right)^2}$$

The **complete elliptic integral of the second kind** $E$ is defined as

$$E(k) = \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2 \xi}\, d\xi$$

$$E(0) = \frac{\pi}{2}$$

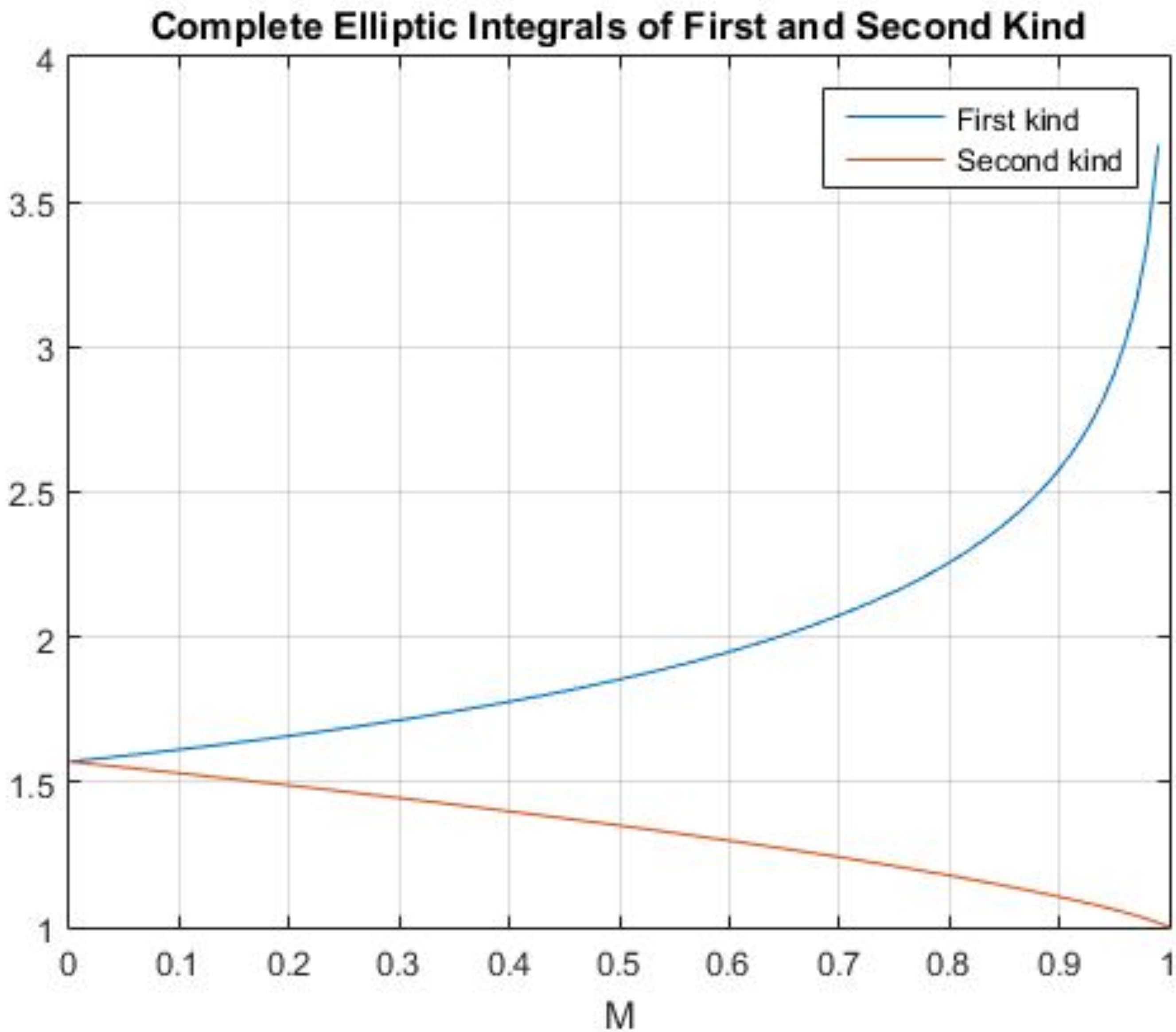
$$E(1) = 1$$

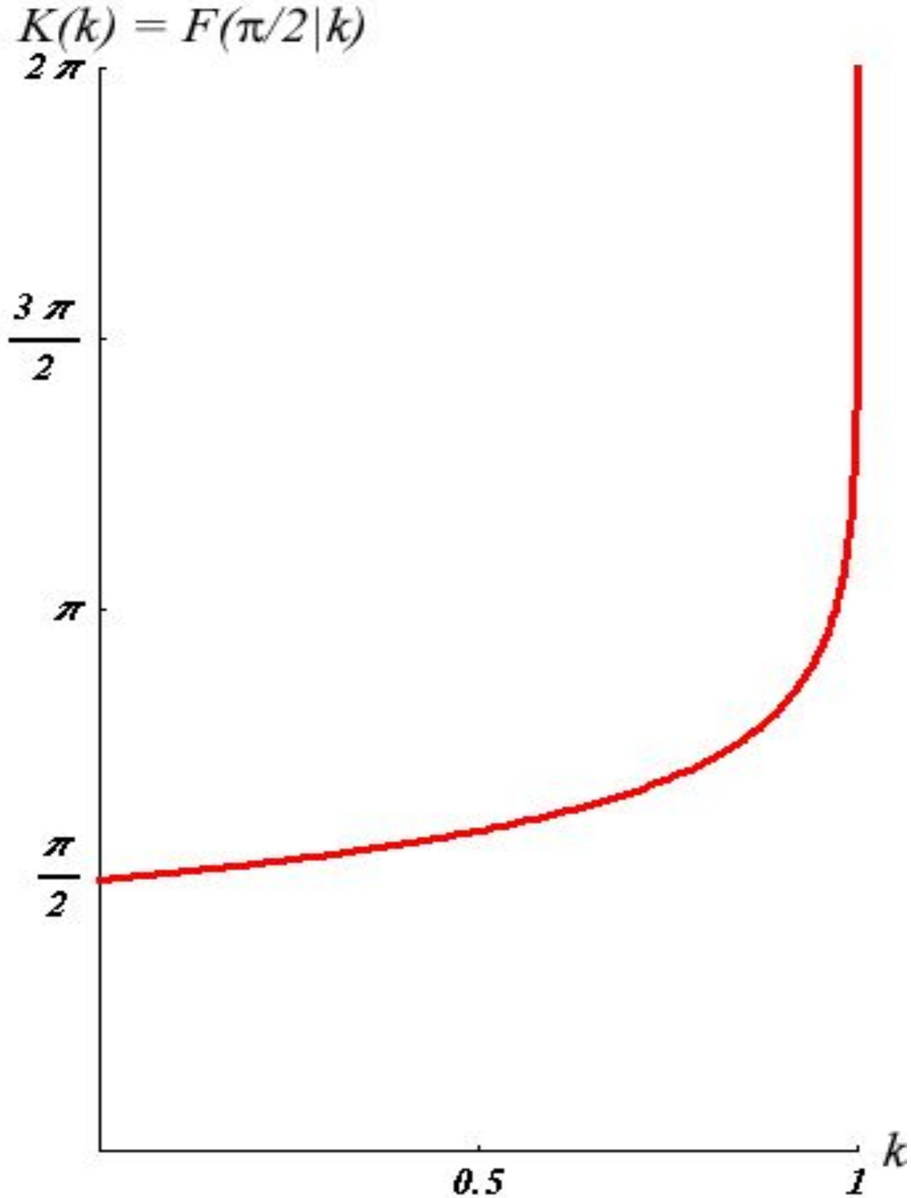
$$E\left(\frac{\sqrt{2}}{2}\right) = \pi^{\frac{3}{2}} \Gamma\left(\frac{1}{4}\right)^{-2} + \frac{1}{8\sqrt{\pi}} \Gamma\left(\frac{1}{4}\right)^{2}$$

$$E\left(\frac{1}{4}\left(\sqrt{6} - \sqrt{2}\right)\right) = 2^{\frac{1}{3}} 3^{-\frac{3}{4}} \pi^2 \Gamma\left(\frac{1}{3}\right)^{-3} + 2^{-\frac{10}{3}} 3^{-\frac{1}{4}} \pi^{-1} \left(\sqrt{3} + 1\right) \Gamma\left(\frac{1}{3}\right)^{3}$$

$$E\left(\frac{1}{4}\left(\sqrt{6} + \sqrt{2}\right)\right) = 2^{\frac{1}{3}} 3^{-\frac{1}{4}} \pi^2 \Gamma\left(\frac{1}{3}\right)^{-3} + 2^{-\frac{10}{3}} 3^{\frac{1}{4}} \pi^{-1} \left(\sqrt{3} - 1\right) \Gamma\left(\frac{1}{3}\right)^{3}$$

$$E\left(2\sqrt{-4 - 3\sqrt{2}}\right) = \frac{\left(2 + \sqrt{2}\right)\left(\pi^2 + 4\Gamma\left(\frac{3}{4}\right)^{4}\right)}{4\sqrt{\pi}\,\Gamma\left(\frac{3}{4}\right)^{2}}$$


Complete Elliptic Integrals of First and Second Kind
First kind
Second kind
4
3.5
3
2.5
2
1.5
1
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
M



K(k) = F(π/2|k)
2 π
3 π / 2
π
π / 2
0.5
1
k

# Improper integrals

- Thus far we have considered only definite integrals with finite limits, what about

$$I = \int_0^\infty f(x)dx$$

- Firstly, if $I$ is divergent then a numerical integration is pointless
- If $I$ is convergent we must still handle the upper limit somehow
- There are two ways of handling this type of integral
  - Break up the integral into two parts
  - Use a single change of variables to make the limits tractable

# Split the integral

- Write $I = \int_0^\infty f(x)dx = \underbrace{\int_0^a f(x)dx}_{\text{Proper integral } I_1} + \underbrace{\int_a^\infty f(x)dx}_{\text{Improper integral } I_2}$
- The value of $a$ needs to be chosen with care
- $I_1$ we can evaluate numerically, with say a Romberg integrator (provided $f$ is finite everywhere!)
- For $I_2$ we apply a change of variables to turn the integral into one with finite limits

# Change of variables

Suppose we set $x = 1/y$, then
at $x = a, \quad y = 1/a$

$$x = \infty, \;\; y = 0 \qquad\qquad \text{and} \;\; dx = -\frac{dy}{y^2}$$

Then we write $I_2$ as follows :

$$I_2 = \int_a^{\infty} f(x)dx = \int_0^{1/a} \frac{f(1/y)}{y^2} dy$$

- Provided $f(1/y) \sim y^n$ & $n>1$ then there are no singularities in the integrand and we can apply Romberg integration
  - This is a reasonable expectation. We really asking for asymptotic behaviour of $f(x) \sim 1/x^2$ or lower. Without a fairly rapid approach to zero we wouldn't expect the integral to be bounded.

# Removable singularities

- It is still possible that after our substitution we wind up with removable singularities
- Consider

$$\frac{y \sin y}{y^2} \sim \frac{\sin y}{y} \to 1 \text{ as } y \to 0$$

- However, when you calculate sin 0/0 the computer will give a divide by zero error
- There is clearly no "magic algorithm" that works in all cases
  - Different substitutions may be required, series expansions may be necessary or even other methods be employed before invoking numerical integration

# Other useful substitutions

$$1)\ \text{Set } x = \frac{1+y}{1-y},\ \text{then } dx = \frac{2dy}{(1-y)^2}\ \ \&\ y = \frac{x-1}{x+1}$$

$$\text{at } x = 0,\quad y = -1$$

$$x = \infty,\ \ y = +1$$

$$2)\ \text{Set } x = \frac{y}{1-y},\ \text{then } dx = \frac{dy}{(1-y)^2}\ \ \&\ y = \frac{x}{x+1}$$

$$\text{at } x = 0,\quad y = 0$$

$$x = \infty,\ \ y = +1$$

- Notice that both these substitutions can be used on the entire range [0,∞) – there is no need to break up the integral
  – It may be the case though that it is better to do this for numerical convergence reasons

# Singularities in the integrand

Consider,

$$I = \int_0^\infty \frac{1}{(1+x)\sqrt{x}}dx, \text{ break this up as suggested before}$$

$$= \underbrace{\int_0^a \frac{1}{(1+x)\sqrt{x}}dx}_{I_1} + \underbrace{\int_a^\infty \frac{1}{(1+x)\sqrt{x}}dx}_{I_2}$$

- Our primary concern is $I_1$ due to the singularity at 0
- Firstly though, let's look at $I_2$ and how we would approach that

# Try substitutions in $I_2$

$x = 1/y$, gives

$$I_2 = \int_0^{\frac{1}{a}} \frac{\sqrt{y}}{y^2(1+1/y)} dy = \int_0^{\frac{1}{a}} \frac{1}{\sqrt{y}(1+y)} dy$$

Exactly the same form as $I_1$, which has singularity at $x = 0$!

$x = 1/y^2, \;\Rightarrow\; dx = -2dy/y^3$

$$I_2 = 2\int_0^{\frac{1}{\sqrt{a}}} \frac{y}{y^3(1+1/y^2)} dy = 2\int_0^{\frac{1}{\sqrt{a}}} \frac{1}{(1+y^2)} dy$$

- The latter example is perfectly well behaved (no singularities in $[0,1/\sqrt{a}]$ )
  - So we can deal with $I_2$ without a problem

# Approaches to help us integrate $I_1$

- Use another substitution
- Can do a power series expansion
  - Not covered here, but see some references?
- We may be able to isolate the singularity (say by using a partial fraction expansion)
- We'll look at both substitution (again) and isolation

# Substitution

Let's try $x = y^2$

$\Rightarrow \; y = \sqrt{x}, \quad \text{and} \quad dx = 2ydy$

Limits $x = [0, a]$ become

$$y = [0, \sqrt{a}\,]$$

$$I_1 = \int_0^a \frac{1}{(1+x)\sqrt{x}} dx = 2\int_0^{\sqrt{a}} \frac{1}{(1+y^2)} dy$$

- We have removed the singularity and the result can easily be calculated via Romberg integration

# Isolation

We know that as $x \to 0$, so $f(x) = \dfrac{1}{(1+x)\sqrt{x}} \to \dfrac{1}{\sqrt{x}}$

Try adding and subtracting the singularity:

$$f(x) = \frac{1}{(1+x)\sqrt{x}} = \frac{1}{(1+x)\sqrt{x}} - \frac{1}{\sqrt{x}} + \frac{1}{\sqrt{x}} = \frac{-\sqrt{x}}{(1+x)} + \frac{1}{\sqrt{x}}$$

$$\Rightarrow I_1 = \int_0^a f(x)dx = -\int_0^a \frac{\sqrt{x}}{(1+x)}dx + \int_0^a \frac{1}{\sqrt{x}}dx$$

This has isolated the singularity. If $I_1$ is tractable, the singularity must be integrable. Clearly here

$$\int_0^a \frac{1}{\sqrt{x}}dx = 2\sqrt{x}\Big|_0^a = 2\sqrt{a} \Rightarrow I_1 = 2\sqrt{a} - \int_0^a \frac{\sqrt{x}}{(1+x)}dx$$

# What to take away from this…

- Every integral will be different
- You must be prepared to handle each one individually!

# Integrals over infinite ranges

$$I = \int_0^{\infty} f(x)dx$$

solution is to change variables: $y = \dfrac{x}{1+x} \Rightarrow x = \dfrac{y}{1-y}$ & $dx = \dfrac{dy}{(1-y)^2}$

$$\int_0^{\infty} f(x)dx = \int_0^1 \frac{1}{(1-y)^2} f\left(\frac{y}{1-y}\right)dy$$

We can make two changes of variables: $y = x - a$ & $z = y/(1+y)$ to calculate

$$\int_a^{\infty} f(x)dx = \int_0^1 \frac{1}{(1-z)^2} f\left(\frac{z}{1-z} + a\right)dy$$

# Multidimensional Numerical Integration

- At first it seems a straightforward extension of 1*d* ideas
  – However, if in 1*d* we needed say a 100 function evaluations 2*d* will require $100\times100$!
    - 3*d* $100\times100\times100$… etc
    - Quickly gets very numerically expensive – sometimes we have to resort to guestimating using Monte Carlo methods
  – Secondly, multidimensional integrals often have unusual boundaries
    - Defining the limits may be tricky

- As a first step lets consider integration over a rectangle

# Multidimensional Numerical Integration

**"Divide and Conquer"**

$$\int_{x_a}^{x_b}\int_{y_a}^{y_b} f(x,y)\,dy\,dx$$

$$\int_{x_a}^{x_b} f(x)\,dx$$

$$f(x) = \int_{y_a}^{y_b} f(x,y)\,dy$$

**Solve by Series Integration**

*Newton-Cotes*
*Trapezoidal*
*3-Point Simpson*

For N points in each integral calculation there are $N^2$ calculations

# Simple 2*d* integration

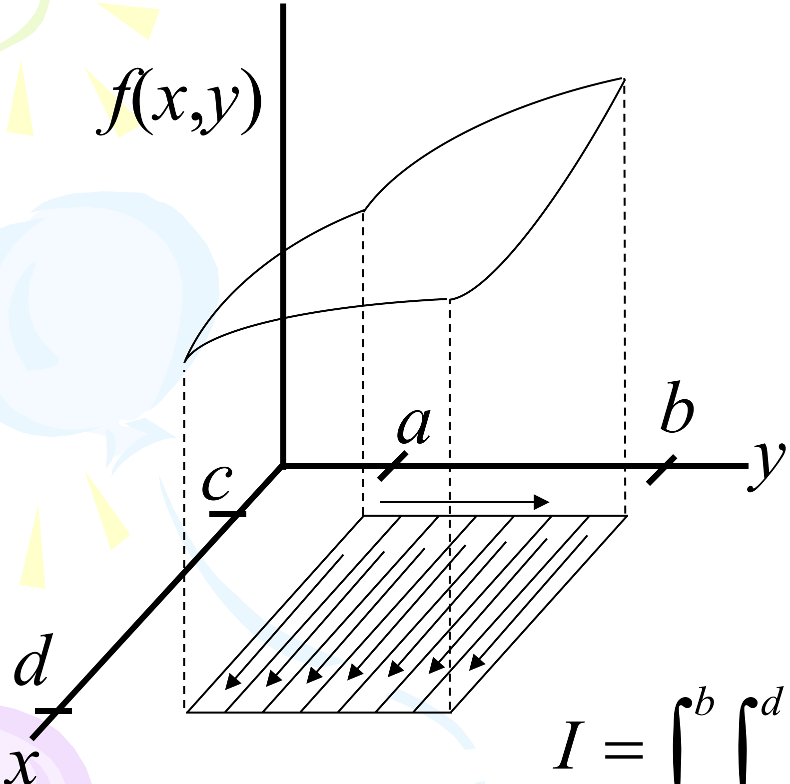
Consider,

$$I = \int_a^b \int_c^d f(x,y)dxdy$$

$$= \int_a^b F(y)dy \quad \text{where } F(y) = \int_c^d f(x,y)dx$$

- Thus there are two "sweeps"
  - 1d integrator evaluates $F(y)$ from $f(x,y)$ (clearly we must do this for all values of $y$)
  - 1d integrator evaluates $I$ from $F(y)$

# 2d integration illustrated

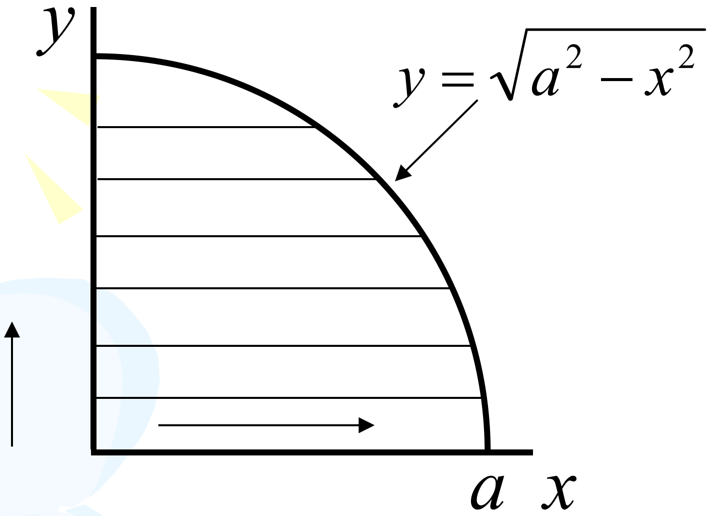



- This diagram shows precisely how the "sweeps" come about
- There are $N$ $x$-sweeps followed by a single $y$ sweep
- We of course could do things the other way round
  - i.e. $y$-sweeps before the $x$-sweeps:

$$I = \int_a^b \int_c^d f(x,y)\,dx\,dy$$

$$= \int_c^d G(x)\,dx \quad \text{where } G(x) = \int_a^b f(x,y)\,dy$$

# Other integration limits (in 2d)

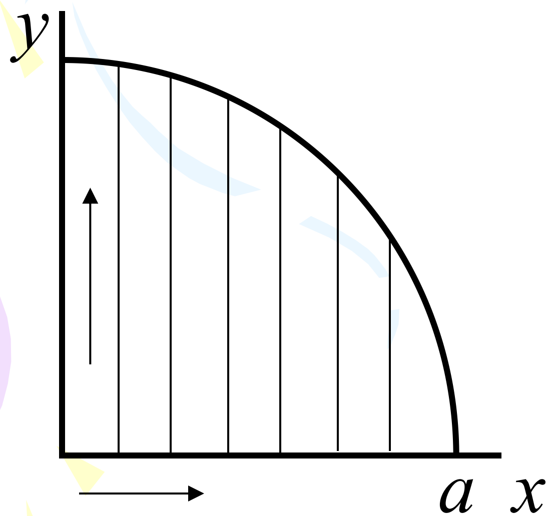



- You may frequently encounter integrals defined by a curve in the $x$-$y$ plane

Limits on $y$: $0 \rightarrow a$

Limits on $x$: $0 \rightarrow \sqrt{a^2\text{-}y^2}$

Thus $I = \int_0^a \int_0^{\sqrt{a^2-y^2}} f(x,y)\,dx\,dy$




However, we could do $y$-sweeps then $x$-sweeps

Then $I = \int_0^a \int_0^{\sqrt{a^2-x^2}} f(x,y)\,dy\,dx$

# In polar coordinates

- This particular problem strongly favours a change of variables:

$$r=\sqrt{x^2+y^2},\quad x=r\cos\theta$$
$$\theta=\tan^{-1}(y/x),\quad y=r\sin\theta$$

Which under this change of variables gives

$$dx\,dy=\frac{\partial(x,y)}{\partial(r,\theta)}dr\,d\theta$$

where the Jacobian $\frac{\partial(x,y)}{\partial(r,\theta)}$ is given by

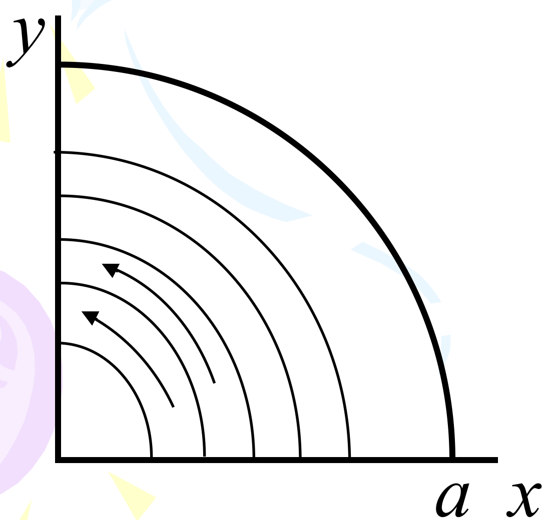
$$\frac{\partial(x,y)}{\partial(r,\theta)}=\begin{vmatrix}\partial x/\partial r & \partial x/\partial\theta\\ \partial y/\partial r & \partial y/\partial\theta\end{vmatrix}$$
$$=\begin{vmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{vmatrix}=r\cos^2\theta+r\sin^2\theta=r$$

# Change of limits

- The limits are obvious here: $r \in [0, a], \quad \theta \in [0, \pi/2]$
- Note that while the limits in $x$ & $y$ are from 0 to $a$ respectively so that

$$\text{min value for } y/x = 0;\ \tan^{-1} 0 = 0$$

$$\text{max value for } y/x = \infty;\ \tan^{-1} \infty = \pi/2$$




$$\text{Thus } I = \int_0^a \int_0^{\sqrt{a^2-y^2}} f(x,y)\,dx\,dy$$

$$= \int_0^a \int_0^{\pi/2} f(r\cos\theta, r\sin\theta)\, r\,d\theta\,dr$$

Corresponds to $n$ $\theta$-sweeps followed by 1 $r$-sweep.

# Quickly gets very numerically expensive

Example: Atomic Physics

$^4$Be

$$I=\int_0^1 dx_1 \int_0^1 dx_2 \ldots \int_0^1 dx_{12} f(x_1, x_2, \ldots x_{12})$$

3 Dimension/electron * 4 electrons = 12 Dimensions

For 100 steps in each integration there are $100^{12} = 10^{24}$ calculations

Assuming 1 Giga evaluations/sec

It would take over $10^7$ years!!!!

# Summary

- These examples have shown that numerical integration needs to be treated with care
  - Examine behaviour at end points, apply changes of variables if needs be
- Improper integrals can be handled by
  - Changes of variables
  - Breaking up the integral into separate parts
- Multidimensional integration is considerably more computational expensive

# Next Lecture

- ODE

# Homework 11： 05/22/2019

**Problem 1:** Use the method discussed in this lecture to write the codes to integrate

a) $I = \int_{-1}^{1} \sqrt{1-x^2}\, dx$ $\qquad I = \int_{0}^{\pi} \sin^2 \theta d\theta$

b) $T(\theta_0) = \int_0^{\theta_0} \frac{d\theta}{\sqrt{2\cos\theta - 2\cos\theta_0}}$ $\qquad K(k) = \int_0^{\pi/2} \frac{1}{\sqrt{1-k^2\sin^2\xi}} d\xi$

$k = \sin(\theta_0/2)$ or $\theta_0 = 2\arcsin(k)$

**Plot the results with different *k* (prefer to use xmgrace)**

c) $I = \int_0^{\infty} \frac{1}{(1+x)\sqrt{x}} dx$

## **Problem 2.** Gaussian integral via numerical integration

$$I = \int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$$

Write a program which uses your integration module to performs the Gaussian integral

$$f(x) = e^{-x^2}$$

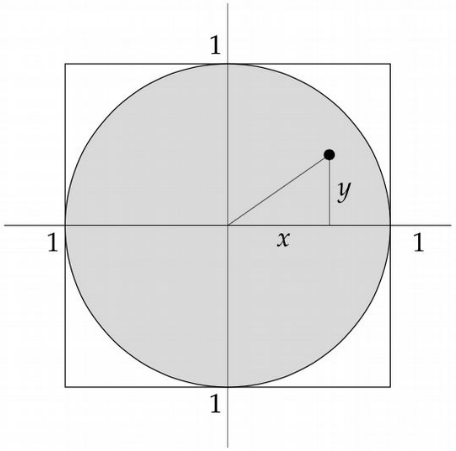
from $-\infty$ to $\infty$ via numerical integration. The value for integral should be $\sqrt{\pi}$.

Have your program compare your results of the numerical integration to the true numerical value.

# Problem 3. Volume of a n-dimensional hypersphere:

This exercise asks you to estimate the volume of a sphere of unit radius with any dimensions. Consider the equivalent problem in two dimensions, the area of a circle of unit radius:

$$I = \int\int_{-1}^{1} f(x, y)dxdy$$




The area of the circle, the shaded area shown, is given by the integral $I$ where $f(x, y) = 1$ everywhere inside the circle and zero everywhere outside, *i.e*,

$$f(x, y) = \begin{cases} 1 & \text{if } x^2 + y^2 \leq 1 \\ 0 & \text{Otherwise} \end{cases}$$

**a**) calculate hyper volume as a function of the dimensions from $n = 0$ to $n = 12$. (12,11,10…???)

**b**) Print results to the screen and generate a graph of the hyper volume vs dimension.

(**Hints:** You will get high credit with higher $n$ **project 3**?)

# Volume of a n-dimensional hypersphere:

$$V_n = \frac{\pi^{\frac{n}{2}} R^n}{\Gamma(\frac{n}{2}+1)} = C_n R^n$$

Where Γ is gamma function.

$$\Gamma\left(\frac{n}{2}+1\right) = \left(\frac{n}{2}\right)!$$ （**For Even *n*=2*k***），

$$\Gamma\left(\frac{n}{2}+1\right) = \sqrt{\pi}\frac{n!!}{2^{(n+1)/2}}$$ （**For odd *n*=2*k*+1**）

$$C_n = \frac{\pi^k}{k!}$$ （**For Even *n*=2*k***），

$$C_n = C_{2k+1} = \frac{2^{2k+1} k! \, \pi^k}{(2k+1)!}$$ （**For odd *n*=2*k*+1**）

# Error Estimation

- Trapezoidal error for *f* on [*a*, *b*]
  - Where M = max value of |f ''(x)| on [a, b]

$$|E_n| \le \frac{(b-a)^3}{12n^2} \cdot M$$

- Simpson's error for *f* on [*a*, *b*]
  - Where *K* = max value of $|f^{(4)}(x)|$ on [*a*, *b*]

$$|E_n| \le \frac{(b-a)^5}{180n^4} \cdot K$$