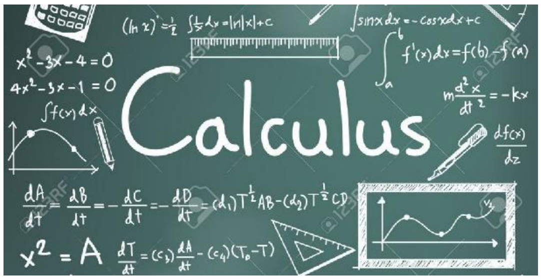
MODULE 2: **DERIVATIVES**

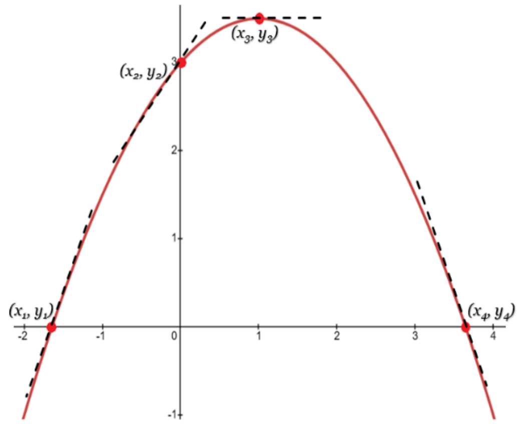


At the end of the module the students should be able to:

1. use the basic principles of algebra and limits to define the derivative;
2. apply derivatives to the rate of change of a function;
3. derive the differentiation rules;
4. compute the derivative of functions using differentiation rules; and
5. find the maximum and minimum values of a function and solve problem set using differentiation.

TANGENT LINE TO A CURVE

- ❑ The slope of a line can be used to describe the rate at which the line rises or falls. For a line, this rate (or slope) is the same at every point on the line. For graphs (or curves) other than lines, the rate at which the graph rises or falls, changes from point to point.
  - ❑ From the figure, the graph rises more quickly at the point (x_1, y_1) than it is at point (x_2, y_2). Then the graph levels off at the point (x_3, y_3) and is falling at point (x_4, y_4).
- ❑ To determine the rate at which a curve rises or falls at a point, we use the **tangent line** to the curve at the point.
  - ❑ The tangent line to the graph of the function (*f*) at a point is the line that best approximates the graph of *f* at that point.



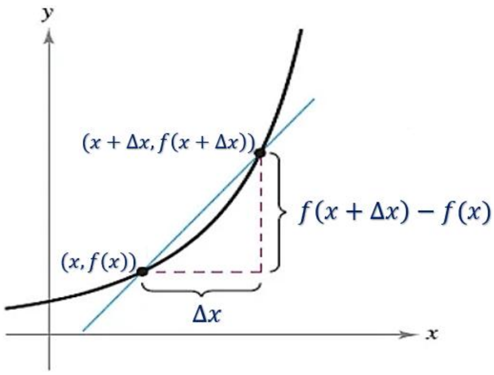
The slope of a curve changes from one point to another.

SLOPE OF A CURVE

- **Slope of the graph**
  - The slope *m* of the graph of *y = f(x)* at *(x, f(x))* is equal to the slope of its tangent line at *(x, f(x))*, and it is determined by the formula

$$m = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

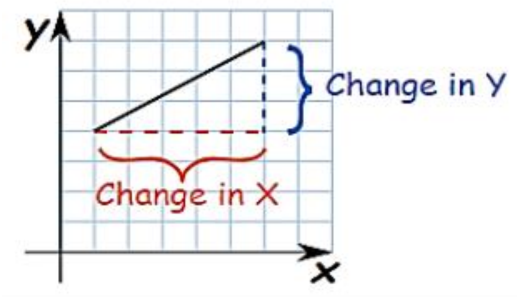
provided this limit exists.



- **Secant Line**
  - A line through two points on the curve.

INTRODUCTION TO DERIVATIVES

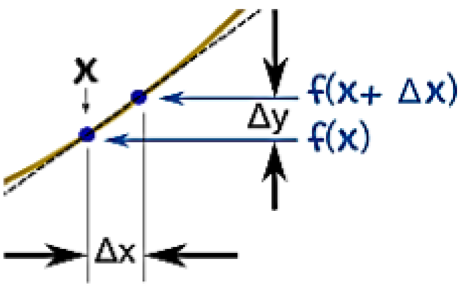
We know from basic algebra that a line has the form f(x) = mx + b, where m is the slope. We measure the slope as the distance traveled up (along the vertical axis) divided by the corresponding distance traveled across (along the horizontal axis): this is what we call "rise over run." We can also call the rise Δy, since it is the change in y; the run we can call Δx, since it is the change in x. Then,



$$\textit{slope} = \frac{\textit{change in x}}{\textit{change in y}} = \frac{\Delta y}{\Delta x}$$

From the diagram at the right, we see that:

- x changes from x to $x + \Delta x$
- y changes from $f(x)$ to $f(x + \Delta x)$



DERIVATIVE

A function is said to be **differentiable** at *x* if its derivative exists at *x*, and the process of finding the derivative is called **differentiation**.

Notations:

$$\frac{\Delta \boldsymbol{y}}{\Delta \boldsymbol{x}} \qquad \frac{\boldsymbol{dy}}{\boldsymbol{dx}} \qquad \boldsymbol{y}' \qquad \boldsymbol{f}'(\boldsymbol{x}) \qquad \frac{\boldsymbol{dy}}{\boldsymbol{dx}}[\boldsymbol{f}(\boldsymbol{x})] \quad \boldsymbol{D}_{\boldsymbol{x}}(\boldsymbol{y})$$

Follow these steps:

- Fill in this slope formula:

$$\frac{\Delta x}{\Delta y} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

- Simplify it as best we can
- Then make Δx shrink toward zero

The notation $\frac{dy}{dx}$ is read as the "derivative of *y* with respect to *x*". Thus,

$$\frac{\boldsymbol{dy}}{\boldsymbol{dx}} = \lim_{\Delta \boldsymbol{x} \to \boldsymbol{0}} \frac{\Delta \boldsymbol{y}}{\Delta \boldsymbol{x}}$$

Thus, we have

$$\frac{\boldsymbol{dy}}{\boldsymbol{dx}} = \lim_{\Delta \boldsymbol{x} \to \boldsymbol{0}} \frac{\Delta \boldsymbol{y}}{\Delta \boldsymbol{x}} = \lim_{\Delta \boldsymbol{x} \to \boldsymbol{0}} \frac{\boldsymbol{f}(\boldsymbol{x} + \Delta \boldsymbol{x}) - \boldsymbol{f}(\boldsymbol{x})}{\Delta \boldsymbol{x}}$$

The **derivative of *f* at *x*** is given by

$$\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

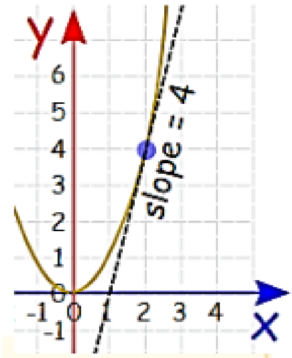
provided the limit exists.

Example 1: Find the Derivative of:

$$f(x) = x^2$$

Solution: Substitute $x + \Delta x$ to x:

$$\begin{aligned}
f(x+\Delta x) &= (x+\Delta x)^2 \\
&= x^2 + 2(x+\Delta x) + (\Delta x)^2 \\
\frac{dy}{dx} &= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{(x^2 + 2x\Delta x + (\Delta x)^2) - (x^2)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{x^2 + 2x\Delta x + (\Delta x)^2 - x^2}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{2x\Delta x + (\Delta x)^2}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{2x\Delta x}{\Delta x} + \lim_{\Delta x \to 0} \frac{(\Delta x)^2}{\Delta x} \\
&= \lim_{\Delta x \to 0} 2x + \lim_{\Delta x \to 0} \Delta x \\
&= 2x + 0 \\
\frac{dy}{dx} &= \mathbf{2x}
\end{aligned}$$



$\frac{dy}{dx} = 2x$ means that, for the function x^2, the slope or "rate of change" at any point is $\mathbf{2x}$.

When $x = 2$, the slope is $2x = \mathbf{4}$.

$x = 5$, the slope is $2x = \mathbf{10}$, and so on.

Example 2: Find the derivative of $f(x) = x^3$

Solution: If $f(x) = x^3$, then we can find $f(x + \Delta x)$

$$f(x+\Delta x) = (x+\Delta x)^3$$
$$= x^3 + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3$$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{x^3 + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 - x^3}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{\Delta x(3x^2 + 3x\Delta x + (\Delta x)^2)}{\Delta x}$$
$$= \lim_{\Delta x \to 0} (3x^2 + 3x\Delta x + (\Delta x)^2)$$
$$= \lim_{\Delta x \to 0} 3x^2 + \lim_{\Delta x \to 0} 3x\Delta x + \lim_{\Delta x \to 0} (\Delta x)^2$$
$$= 3x^2 + 0 + 0$$
$$f'(x) = \mathbf{3x^2}$$

Example 3: Find the derivative of $f(x) = 2x^2 + 2x - 3$

Solution:

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{2(x+\Delta x)^2 + 2(x+\Delta x) - 3 - (2x^2 + 2x - 3)}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{2(x^2 + 2x\Delta x + (\Delta x)^2) + 2(x+\Delta x) - 3 - (2x^2 + 2x - 3)}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{2x^2 + 4x\Delta x + 2(\Delta x)^2 + 2x + 2\Delta x - 3 - 2x^2 - 2x + 3}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{4x\Delta x + 2(\Delta x)^2 + 2\Delta x}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{\Delta x(4x + 2\Delta x + 2)}{\Delta x}$$
$$= \lim_{\Delta x \to 0} (4x + 2\Delta x + 2)$$
$$= \lim_{\Delta x \to 0} 4x + \lim_{\Delta x \to 0} 2\Delta x + \lim_{\Delta x \to 0} 2$$
$$= 4x + 0 + 2$$
$$f'(x) = \mathbf{4x + 2}$$

Name: ____ Score: ______________

Date: _______________________ Section: __________________

ACTIVITY

INTRODUCTION TO DERIVATIVE

Find the Derivative of the following functions. Show all necessary solutions.

1. $f(x) = 3x + 5$

2. $f(x) = 6x - 7$

3. $f(x) = x^2 - 7$

4. $f(x) = 3x^2 + 2x - 2$

5. $f(x) = x^3 - 2x + 1$

DERIVATIVES OF ALGEBRAIC FUNCTIONS

An algebraic function is any function that cab ne built from the identity function $y = x$ by forming liner combinations, products, quotients, and fractional powers.

Differentiation Rules

1. **The derivative of a constant is equal to zero**

$$\boldsymbol{\frac{d}{dx}(k) = 0}$$

where $\boldsymbol{k}$ is any constant

Example: Differentiate $f(x) = 5$

Answer: $\boldsymbol{f'(x) = 0}$

Proof:

$$\begin{aligned}\frac{dy}{dx} &= \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}\\ &= \lim_{\Delta x \to 0} \frac{5-5}{\Delta x}\\ &= \lim_{\Delta x \to 0} \frac{0}{\Delta x}\\ &= \lim_{\Delta x \to 0} 0\\ &= \mathbf{0}\end{aligned}$$

2. **General Rule for differentiation**

$$\boldsymbol{\frac{d}{dx}x = 1}$$

Example: Differentiate $f(x) = x$

Answer: $\boldsymbol{f'(x) = 1}$

Proof:

$$\begin{aligned}\frac{dy}{dx} &= \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}\\ &= \lim_{\Delta x \to 0} \frac{x+\Delta x-x}{\Delta x}\\ &= \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x}\\ &= \lim_{\Delta x \to 0} 1\\ &= \mathbf{1}\end{aligned}$$

3. **Simple Power Rule**

$$\frac{d}{dx}x^n = nx^{n-1}$$

where $n \in R$ and $n \neq 0$

Example: Differentiate $f(x) = x^3$

Answer: $f'(x) = 3x^2$

Proof:

$$\begin{aligned}
\frac{dy}{dx} &= \lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}\\
&= \lim_{\Delta x\to 0}\frac{(x+\Delta x)^3 - x^3}{\Delta x}\\
&= \lim_{\Delta x\to 0}\frac{x^3 + 3x^2\Delta x + 3x\Delta x^2 + \Delta x^3 - x^3}{\Delta x}\\
&= \lim_{\Delta x\to 0}\frac{3x^2\Delta x + 3x\Delta x^2 + \Delta x^3}{\Delta x}\\
&= \lim_{\Delta x\to 0}\frac{\Delta x(3x^2 + 3x\Delta x + \Delta x^2)}{\Delta x}\\
&= \lim_{\Delta x\to 0}(3x^2 + 3x\Delta x + \Delta x^2)\\
&= \lim_{\Delta x\to 0} 3x^2 + \lim_{\Delta x\to 0} 3x\Delta x + \lim_{\Delta x\to 0} \Delta x^2\\
&= \mathbf{3x^2}
\end{aligned}$$

4. **The derivative of a constant multiplied by a function is equal to the constant multiplied by the derivative of the function**

$$\frac{d}{dx}[k \bullet f(x)] = k\frac{d}{dx}[f(x)]$$

Example: Differentiate $f(x) = 5x$

Answer: $\mathbf{f'(x) = 5}$

Proof:

$$\begin{aligned}
\frac{dy}{dx} &= \lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}\\
&= \lim_{\Delta x\to 0}\frac{5(x+\Delta x)-5x}{\Delta x}\\
&= \lim_{\Delta x\to 0}\frac{5x+5\Delta x-5x}{\Delta x}\\
&= \lim_{\Delta x\to 0}\frac{5\Delta x}{\Delta x}\\
&= \mathbf{5}
\end{aligned}$$

5. **The derivative of a sum is equal to the sum of the derivatives**

$$\frac{d}{dx}[f(x)+g(x)] = \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)]$$

Example: Differentiate

$$f(x) = x^3 + 5x$$

Answer:

$$f'(x) = 3x^2 + 5$$

Proof:

$$\begin{aligned}
\frac{dy}{dx} &= \lim_{\Delta x\to 0} \frac{[(x+\Delta x)^3+5(x+\Delta x)+2]-[x^3-5x+2]}{\Delta x} \\
&= \lim_{\Delta x\to 0} \frac{[x^3+3x^2\Delta x+3x\Delta x^2+\Delta x^3+5x+5\Delta x+2]-[x^3+5x+2]}{\Delta x} \\
&= \lim_{\Delta x\to 0} \frac{3x^2\Delta x+3x\Delta x^2+\Delta x^3+5\Delta x}{\Delta x} \\
&= \lim_{\Delta x\to 0} \frac{\Delta x(3x^2+3x\Delta x+\Delta x^2+5)}{\Delta x} \\
&= \lim_{\Delta x\to 0} (3x^2+5) \\
&= \mathbf{3x^2 + 5}
\end{aligned}$$

6. **The derivative of a difference is equal to the difference of the derivatives**

$$\frac{d}{dx}[f(x)-g(x)] = \frac{d}{dx}[f(x)] - \frac{d}{dx}[g(x)]$$

Example: Differentiate $f(x) = x^3 - 5x$

Answer: $f'(x) = 3x^2 - 5$

Proof:

$$\begin{aligned}
\frac{dy}{dx} &= \lim_{\Delta x\to 0} \frac{[(x+\Delta x)^3-5(x+\Delta x)]-[x^3-5x]}{\Delta x} \\
&= \lim_{\Delta x\to 0} \frac{[x^3+3x^2\Delta x+3x\Delta x^2+\Delta x^3-5x-5\Delta x]-[x^3-5x]}{\Delta x} \\
&= \lim_{\Delta x\to 0} \frac{3x^2\Delta x+3x\Delta x^2+\Delta x^3-5\Delta x}{\Delta x} \\
&= \lim_{\Delta x\to 0} \frac{\Delta x(3x^2+3x\Delta x+\Delta x^2-5)}{\Delta x} \\
&= \lim_{\Delta x\to 0} (3x^2+5) \\
&= \mathbf{3x^2 - 5}
\end{aligned}$$

Name: ______________________________ Score: ____________

Date: ________________ Section: ______________

ACTIVITY

DERIVATIVE OF ALGEBRAIC FUNCTIONS

Find the Derivative of the following functions. Show all necessary solutions.

1. $f(x) = 5$
2. $f(x) = -2x$
3. $f(x) = 6x^3 - 9x + 4$
4. $f(x) = -2x + 2$
5. $y = 2t^4 - 10t^2 + 13t$
6. $f(x) = -2x^2 - 5$
7. $f(x) = \frac{5}{x^5} + \frac{3}{x^2}$
8. $f(x) = \frac{5}{x^5}$
9. $f(x) = 2x^4 + x^3 - x^2 + 4$
10. $f(x) = \frac{1}{3x^2}$
11. $f(x) = x^3 + 2$
12. $f(x) = \sqrt{x}$
13. $g(z) = 4z^7 - 3z^{-7} + 9z$
14. $h(y) = y^{-4} - 9y^{-3} + 8y^{-2} + 12$
15. $f(x) = 10\sqrt[5]{x^3} - \sqrt{x^7} + 6\sqrt[3]{x^8} - 3$
16. $f(t) = \frac{4}{t} - \frac{1}{6t^3} + \frac{8}{t^5}$
17. $h(x) = \frac{4x^3 - 7x + 8}{x}$
18. $f(y) = \frac{y^5 - 5y^3 + 2y}{y^3}$
19. $s(t) = 3t^4 - 40t^3 + 126t^2 - 9$
20. $y = \sqrt{x} + 8\sqrt[3]{x} - 2\sqrt[4]{x}$

THE PRODUCT RULE

The derivative of a product of two functions is the first times the derivative of the second, plus the second times the derivative of the first. If u and v are two functions of x, then the derivative of the product uv is given by

$$\boldsymbol{\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}}$$

Example 1: Determine the derivative of

$$y = (2x^2 + 6x)(2x^3 + 5x^2)$$

Solution: Using the substitution:

$u = 2x^2 + 6x$ $\quad du = 4x + 6$

$v = 2x^3 + 5x^2$ $\quad dv = 6x^2 + 10x$

$$\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$= (2x^2 + 6x)(6x^2 + 10x) + (2x^3 + 5x^2)(4x + 6)$$

$$= (12x^4 + 20x^3 + 36x^3 + 60x^2) + (8x^4 + 12x^3 + 20x^3 + 30x^2)$$

$$= \mathbf{20x^4 + 88x^3 + 90x^2}$$

Example 2: Determine the derivative of

$$y = (x^3 - 6x)(2 - 4x^3)$$

Solution: Using the substitution:

$u = x^3 - 6x$ $\quad du = 3x^2 - 6$

$v = 2 - 4x^3$ $\quad dv = -12x^2$

$$\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$= (x^3 - 6x)(-12x^2) + (2 - 4x^3)(3x^2 - 6)$$

$$= -12x^5 + 72x^3 + 6x^2 - 12x^5 - 12 + 24x^3$$

$$= \mathbf{-24x^5 + 96x^3 + 6x^2 - 12}$$

Name: ____ Score: ____________

Date: ___________________ Section: _______________

ACTIVITY

PRODUCT RULE

Find the Derivative of the following functions. Show all necessary solutions.

1. $y = (2x + 2)(x - 7)$

2. $y = 4x^4(5x - 3)$

3. $y = (3x + 2)(4x^2 - 2x)$

4. $y = (2x^3 - 4x^2 + 2x)(2x^2 - 1)$

5. $y = (x^4 + 2)(x^3 - 3x)$

THE QUOTIENT RULE

The quotient rule is a formula for taking the derivative of a quotient of two functions. If you have function $f(x) = u$ in the numerator and the function $g(x) = v$ in the denominator, then the derivative can be found using the formula:

$$\boldsymbol{\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}}$$

Example 1: Find the derivative of

$$y = \frac{2x^3}{4-x}$$

Solution: Using the substitution:

$u = 2x^3$ $du = 6x^2$

$v = 4 - x$ $dv = -1$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$= \frac{(4-x)(6x^2) - (2x^3)(-1)}{(4-x)^2}$$

$$= \frac{24x^2 - 6x^3 + 2x^3}{(4-x)^2}$$

$$= \boldsymbol{\frac{24x^2 - 4x^3}{(4-x)^2}}$$

Example 2: Find $\frac{dy}{dx}$ if

$$y = \frac{4x^2}{x^3 + 3}$$

Solution: Using the substitution:

$u = 4x^2$ $du = 8x$

$v = x^3 + 3$ $dv = 3x^2$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$= \frac{(x^3 + 3)(8x) - (4x^2)(3x^2)}{(x^3 + 3)^2}$$

$$= \frac{8x^4 + 24x - 12x^4}{(x^3 + 3)^2}$$

$$= \boldsymbol{\frac{-4x^4 + 24x}{(x^3 + 3)^2}}$$

Example 3: Find the derivative of

$$f(x) = \frac{x^3 + 2x}{x - 1}$$

Solution: Using the substitution:

$u = x^3 + 2x$ $du = 3x^2 + 2$

$v = x - 1$ $dv = 1$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$= \frac{(x - 1)(3x^2 + 2) - (x^3 + 2x)(1)}{(x - 1)^2}$$

$$= \frac{3x^3 + 2x - 3x^2 - 2 - x^3 - 2x}{(x - 1)^2}$$

$$= \boldsymbol{\frac{2x^3 - 3x^2 - 2}{(x - 1)^2}}$$

Name: ____ Score: ____________

Date: ____________________ Section: ________________

ACTIVITY

QUOTIENT RULE

Find the Derivative of the following functions. Show all necessary solutions.

1. $f(x) = \frac{x^2-4}{2x+1}$

2. $f(x) = \frac{x^2-5x+4}{3x+4}$

3. $f(x) = \frac{5x+7}{2x^2-5x+1}$

4. $f(x) = \frac{x^2+3x+2}{3x^2-1}$

5. $f(x) = \frac{7x^3}{2x+1}$

THE CHAIN RULE

The Chain Rules states that the derivative of $f(g(x))$ is $f'(g(x))\bullet g'(x)$. In other words, it helps us differentiate composite functions.

Suppose that we have two functions $f(x)$ and $g(x)$ and they are both differentiable.

1. If we define $F(x) = (f \circ g)(x)$ then the derivative of $F(x)$ is,
$$F'(x) = f'\big(g(x)\big) \bullet g'(x)$$

2. If we have $y = f(u)$ and $u = g(x)$ then the derivative of y is,
$$\boldsymbol{\frac{dy}{dx} = \frac{dy}{du} \bullet \frac{du}{dx}}$$

Example 1: Use the chain rule to differentiate

$$R(z) = \sqrt{5z-8}$$

Solution:

$$f(z) = \sqrt{z} \qquad g(z) = 5z - 8$$

$$f'(z) = \frac{1}{2\sqrt{z}} \qquad g'(z) = 5$$

$$\begin{aligned} R'(z) &= f'\left((5z-8)^{\frac{1}{2}}\right) \bullet \mathrm{g}'(5z-8) \\ &= \left(\tfrac{1}{2}\right)(5z-8)^{-\frac{1}{2}} \bullet (5) \\ &= \frac{1}{2\sqrt{5z-8}}(5) \\ &= \boldsymbol{\frac{5}{2\sqrt{5z-8}}} \end{aligned}$$

Example 2: Differentiate

$$f(x) = (6x^2 + 7x)^4$$

Solution:

$$\begin{aligned} f'(x) &= 4(6x^2+7x)^3(12x+7) \\ &= \boldsymbol{4(12x+7)(6x^2+7x)^3} \\ &= \boldsymbol{(48x+28)(6x^2+7x)^3} \end{aligned}$$

Example 3: Differentiate

$$y = \sqrt[3]{1 - 8z}$$

Solution: $y' = (1 - 8z)^{\frac{1}{3}}$

$$= \left(\frac{1}{3}\right)(1 - 8z)^{-\frac{2}{3}}(-8)$$

$$= -\frac{8}{3}(1 - 8z)^{-\frac{2}{3}} \text{ or } -\frac{8}{3\sqrt[3]{(1 - 8z)^2}}$$

Example 4: Differentiate

$$y = (2x^3 - 2x^2 - 5)^{12}$$

Solution: $f'(x) = 12(2x^3 - 2x^2 - 5)^{11}(6x^2 - 4x)$

$$= 12(6x^2 - 4x)(2x^3 - 2x^2 - 5)^{11}$$

$$= (72x^2 - 48x)(2x^3 - 2x^2 - 5)^{11}$$

Example 5: Differentiate

$$y = \sqrt{7x - 5}$$

Solution: $y = (7x - 5)^{\frac{1}{2}}$

$$y' = (7x - 5)^{\frac{1}{2}}$$

$$= \left(\frac{1}{2}\right)(7x - 5)^{-\frac{1}{2}} \bullet (7)$$

$$= \frac{7}{2\sqrt{7x - 5}}$$

Name: ____ Score: ____________

Date: ____________________ Section: ________________

ACTIVITY

CHAIN RULE

Find the Derivative of the following functions. Show all necessary solutions.

1. $y = (3x^3 + 2x^2 - x + 5)^{\frac{3}{4}}$

2. $y = (4x^4 + x - 5)^{12}$

3. $y = \sqrt[5]{x^2 - 5}$

4. $y = (x - 2)(4x - 1)^3$

5. $y = \frac{(x-1)^3}{x+2}$

DERIVATIVES OF TRIGONOMETRIC FUNCTIONS

The basic trigonometric functions include the following six trigonometric functions:

- Sine (sin x)
- Cosine (cos x)
- Tangent (tan x)
- Cotangent (cot x)
- Secant (sec x)
- Cosecant (csc x)

All these functions are continuous and differentiable in their domains.

Derivative of Trigonometric Functions

$$\frac{d}{dx}(\sin(x)) = \cos(x) \qquad \frac{d}{dx}(\sin(u)) = \cos(u)\frac{du}{dx}$$

$$\frac{d}{dx}(\cos(x)) = -\sin(x) \qquad \frac{d}{dx}(\cos(u)) = -\sin(u)\frac{du}{dx}$$

$$\frac{d}{dx}(\tan\ (x)) = \sec^2(x) \qquad \frac{d}{dx}(\tan\ (u)) = \sec^2(u)\frac{du}{dx}$$

$$\frac{d}{dx}(\csc(x)) = -\csc\ (x)\cot(x) \qquad \frac{d}{dx}(\csc(u)) = -\csc\ (u)\cot(u)\frac{du}{dx}$$

$$\frac{d}{dx}(\sec(x)) = \sec\ (x)\tan(x) \qquad \frac{d}{dx}(\sec(u)) = \sec\ (u)\tan(u)\frac{du}{dx}$$

$$\frac{d}{dx}(\cot(x)) = -\csc^2\ (x) \qquad \frac{d}{dx}(\cot(u)) = -\csc^2\ (u)\frac{du}{dx}$$

Some Trigonometric Identities

Reciprocal Identities:

$$\sin x = \frac{1}{\csc x} \qquad \csc x = \frac{1}{\sin x} \qquad \tan x = \frac{1}{\cot x}$$

$$\cos x = \frac{1}{\sec x} \qquad \sec x = \frac{1}{\csc x} \qquad \cot x = \frac{1}{\tan x}$$

Quotient Identities:

$$\tan x = \frac{\sin x}{\cos x} \qquad \cot x = \frac{\cos x}{\sin x}$$

Pythagorean identities:

$$\sin^2 x + \cos^2 x = 1$$
$$\sec^2 x = \tan^2 x + 1$$
$$\csc^2 x = 1 + \cot^2 x$$

Double Angle Formulas:

$$\sin 2x = 2\sin x\cos x$$
$$\begin{aligned}\cos 2x &= \cos^2 x - \sin^2 x\\ &= 2\cos^2 x - 1\\ &= 1 - 2\sin^2 x\end{aligned}$$
$$\tan 2x = \frac{2\tan x}{1-\tan^2 x}$$

Example 1: Differentiate the following trigonometric functions:

1. $y = \sin(5x)$

   Solution: $y' = \cos(5x)(5)$

   $= \mathbf{5\ cos\ (5x)}$

2. $y = \sin(5x^4)$

   Solution: $y' = \cos(5x^4)(20x^3)$

   $= \mathbf{20x^3\ cos\ (5x^4)}$

3. $y = \sin(6x - 5)$

   Solution: $y' = \cos(6x - 5)(6)$

   $= \mathbf{6\ cos\ (6x - 5)}$

4. $y = x\sin(7x)$

   Solution: $u = x \qquad v = \sin(7x)$

   $du = 1 \qquad dv = 7\cos(7x)$

   $y' = (x)(7\cos(7x)) + \sin(7x)(1)$

   $= \mathbf{7x\cos(7x) + \sin(7x)}$

5. $y = \cos(9x)$

   Solution: $y' = -\sin(9x)(9)$

   $= \mathbf{-9\ sin\ (9x)}$

Name: ____ Score: ____________

Date: ___________________ Section: _______________

ACTIVITY

DERIVATIVE OF TRIGONOMETRIC FUNCTIONS

Find the Derivative of the following trigonometric functions. Show all necessary solutions.

1. $y = \sec(5x)$

2. $y = \tan(2x) - \cot(2x)$

3. $y = \cot(3x) + \csc\,(3x)$

4. $y = 4\sec(x) - 2\csc(x)$

5. $y = \sin\,(3x^3 - 2x^2 + 5x + 1)$

6. $y = 2\sec(x)\tan(x)$

7. $y = (2x^2 + 4)[\sin(2x)]$

8. $y = \frac{x+1}{\cos(x)}$

9. $y = \frac{\tan(3x)}{\cos(3x)-4}$

10. $y = 2\sin^5(3x) + \cos^3(3x)$

DERIVATIVE OF INVERSE TRIGONOMETRIC FUNCTIONS

The inverse trigonometric functions can be denoted as

$$\arcsin(x)\,, \arccos(x)\,, \arctan(x)\,, \operatorname{arccot}(x)\,, \operatorname{arcsec}(x)\,, \operatorname{arccsc}(x)$$

$\frac{d}{dx}(\arcsin(x)) = \frac{1}{\sqrt{1-x^2}}, x \neq \pm 1$

$\frac{d}{dx}(\arccos(x)) = -\frac{1}{\sqrt{1-x^2}}, x \neq \pm 1$

$\frac{d}{dx}(\arctan(x)) = \frac{1}{1+x^2}$

$\frac{d}{dx}(\operatorname{arccot}(x)) = -\frac{1}{1+x^2}$

$\frac{d}{dx}(\operatorname{arcsec}(x)) = \frac{1}{|x|\sqrt{x^2-1}}\,, x \neq \pm 1, 0$

$\frac{d}{dx}(\operatorname{arccsc}(x)) = -\frac{1}{|x|\sqrt{x^2-1}}\,, x \neq \pm 1, 0$

$\frac{d}{dx}(\arcsin(u) = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$

$\frac{d}{dx}(\arccos(u)) = -\frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$

$\frac{d}{dx}(\arctan(u)) = \frac{1}{1+u^2}\frac{du}{dx}$

$\frac{d}{dx}(\operatorname{arccot}(u)) = -\frac{1}{1+u^2}\frac{du}{dx}$

$\frac{d}{dx}(\operatorname{arcsec}(u)) = \frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx}$

$\frac{d}{dx}(\operatorname{arccsc}(x)) = -\frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx}$

Example: Differentiate the following functions:

1. $y = \arctan\frac{1}{x}$

   Solution: By chain rule

$$y' = \frac{1}{1+\left(\frac{1}{x}\right)^2}\frac{d}{dx}\left(\frac{1}{x}\right)$$
$$= \frac{1}{1+\left(\frac{1}{x}\right)^2}\left(-\frac{1}{x^2}\right)$$
$$= -\frac{\mathbf{1}}{\mathbf{1}+\boldsymbol{x}^{\mathbf{2}}}$$

2. $y = \arcsin(x-1)$

   Solution: By chain rule

$$y' = \frac{1}{\sqrt{1-(x-1)^2}}$$
$$= \frac{1}{\sqrt{1-(x^2-2x+1)}}$$
$$= \frac{1}{\sqrt{-x^2+2x}}$$
$$= \frac{\mathbf{1}}{\sqrt{\mathbf{2}\boldsymbol{x}-\boldsymbol{x}^{\mathbf{2}}}}$$

3. $y = \text{arccot}(x^2)$

Solution: By chain rule

$$y' = \frac{1}{1+(x^2)^2}\,(2x)$$

$$= \boldsymbol{\frac{2x}{1+x^4}}$$

4. $y = \text{arccot}\left(\frac{1}{x^2}\right)$

Solution: By chain rule

$$y' = \frac{1}{1+\left(\frac{1}{x^2}\right)^2}\left(-\frac{2}{x^3}\right)$$

$$= \frac{2x^4}{(x^4+1)x^3}$$

$$= \boldsymbol{\frac{2x}{1+x^4}}$$

Name: ____ Score: ______________

Date: _______________________ Section: __________________

ACTIVITY

DERIVATIVE OF INVERSE TRIGONOMETRIC FUNCTIONS

Find the Derivative of the following inverse trigonometric functions. Show all necessary solutions.

1. $f(x) = \arcsin(x^2 - 1)$
2. $f(x) = \arcsin\sqrt{x+1}$
3. $y = \arccos(5x^2)$
4. $f(x) = \arccos(3x^2 - 4)$
5. $f(x) = \operatorname{arccsc}(3 - x^2)$
6. $y = \operatorname{arcsec}(6x^2)$
7. $y = \operatorname{arccot}(7x^2 - 3)$
8. $y = \arctan(4x^2 - 2)$
9. $f(x) = (3x)\arccos(2x)$
10. $y = (4x)\arctan(2x)$

DERIVATIVE OF EXPONENTIAL FUNCTIONS

$$\frac{d}{dx}(e^x) = e^x \qquad\qquad \frac{d}{dx}(a^x) = a^x \ln a$$

$$\frac{d}{dx}(e^u) = e^u \frac{du}{dx} \qquad\qquad \frac{d}{dx}(a^u) = a^u \ln a \frac{du}{dx}$$

Example 1: Differentiate the following trigonometric functions:

1. $y = e^{2x+5}$

   Solution: $y' = e^{2x+5}(2)$

   $= \boldsymbol{2e^{2x+5}}$

2. $y = e^{5x^2}$

   Solution: $y' = e^{5x^2}(10x)$

   $= \boldsymbol{10x\, e^{5x^2}}$

3. $y = 9^{5x}$

   Solution: $a = 9 \qquad u = 5x$

   $y' = 9^{5x} \ln 9\,(5)$

   $= \boldsymbol{5\, ln\, (9) 9^{5x}}$

4. $y = 5^{4x^2-3}$

   Solution: $a = 5 \qquad u = 4x^2 - 3$

   $y' = 5^{4x^2-3} \ln 5\,(8x)$

   $= \boldsymbol{8\, ln\, (5) 5^{4x^2-3}\, x}$

Name: ____ Score: ______________

Date: _______________________ Section: ___________________

ACTIVITY

DERIVATIVE OF EXPONENTIAL FUNCTIONS

Find the Derivative of the following exponential functions. Show all necessary solutions.

1. $f(x) = e^{5x^4}$
2. $f(x) = e^{5x^2}$
3. $f(x) = e^{3x^5}$
4. $f(x) = e^{4x^2}$
5. $f(x) = 2e^x - 8^x$
6. $f(x) = 7^{\sin(3x)}$
7. $f(x) = 7^{\cos(x)}(x^2 + 1)$
8. $f(x) = \frac{e^{5x^2}}{e^{2x^4-1}}$
9. $f(x) = e^{3x^2}(3x^5 - 4)$
10. $f(x) = \frac{x^4+3}{e^{5x^5}}$

DERIVATIVE OF LOGARITHMIC FUNCTIONS

The derivative of the logarithmic function $y = \ln x$ is given by:

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

It can also be written in a few other ways as well. The following are equivalent:

$$\frac{d}{dx}\log_e x = \frac{1}{x}$$

$$\frac{d}{dx}\ln(x) = \frac{1}{x}$$

Derivative of $y = \ln u$, where u is a function of x

$$\frac{dy}{dx} = \frac{1}{u}\frac{du}{dx}$$

If $u = f(x)$ is a function of x, and $y = \log_b u$is a logarithm with base b, then we can obtain the derivative of the logarithm function with base b using:

$$\frac{dy}{dx} = (\log_b e)\frac{u'}{u}$$

Properties of Logarithms:

$\log_a 1 = 0$	because	$a^0 = 1$
$\log_a x = 1$	because	$a^1 = a$
$\log_a a^x = x$	because	$a^{\log_a x} = x$

If $\log_a x = \log_a y$, then $x = y$

Product Property: $\log_a(uv) = \log_a u + \log_a v$

Quotient Property: $\log_a\left(\frac{u}{v}\right) = \log_a u - \log_a v$

Power Property: $\log_a u^n = n \log_a u$

Example: Find the derivatives of the following functions:

1. $y = \ln(2x)$

Solution:

$$y = \ln(2) + \ln(x)$$

$$y' = 0 + \frac{1}{x}$$

$$= \mathbf{\frac{1}{x}}$$

2. $y = \ln(x^2)$

Solution:

$$y = 2\ \ln(x)$$

$$y' = 2\left(\frac{1}{x}\right)$$

$$= \mathbf{\frac{2}{x}}$$

3. $y = 2\ \ln(3x^2 - 1)$

Solution:

$$u = 3x^2 - 1$$
$$du = 6x$$

$$y' = 2\left(\frac{6x}{3x^2-1}\right)$$

$$= \mathbf{\frac{12x}{3x^2 - 1}}$$

4. $y = \log_2 6x$

Solution:

$$y = \log_2 6 + \log_2 x$$

$$y' = 0 + \log_2 e\left(\frac{1}{x}\right)$$

$$= \mathbf{\frac{\log_2 e}{x}}$$

5. $y = 3\log_7 (x^2 + 1)$

Solution:

$$y' = 3\ (\log_7 e)\left(\frac{2x}{x^2+1}\right)$$

$$= \mathbf{3\log_7 e\left(\frac{2x}{x^2 + 1}\right)}$$

Name: ______________________________ Score: ____________

Date: ______________________ Section: ________________

ACTIVITY

DERIVATIVE OF LOGARITHMIC FUNCTIONS

Find the Derivative of the following logarithmic functions. Show all necessary solutions.

1. $y = \log_3 (4x^2)$
2. $y = \ln(x^2 - 2x)$
3. $f(x) = \ln(2x^3)$
4. $f(x) = \ln(11x^7)$
5. $y = x \ln x - x$
6. $y = \frac{\ln x}{x}$
7. $y = \frac{\ln x}{x}$
8. $f(x) = \log_e (7x^{-2})$
9. $y = \ln(\sin x)$
10. $y = \log_2(\cos x)$

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