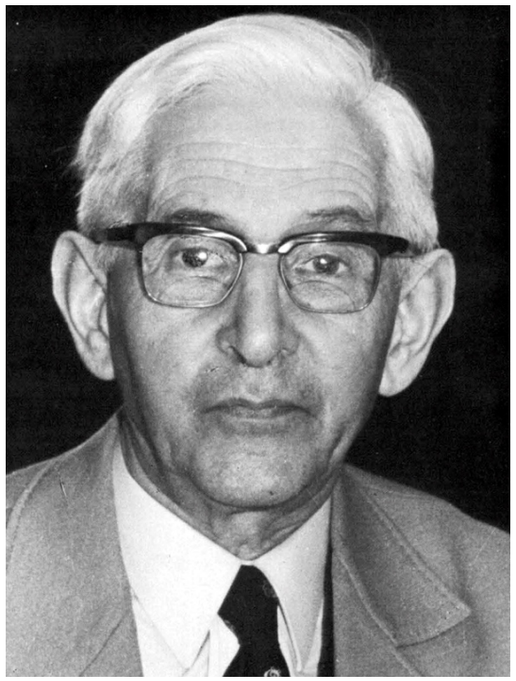
Build your own proof assistant

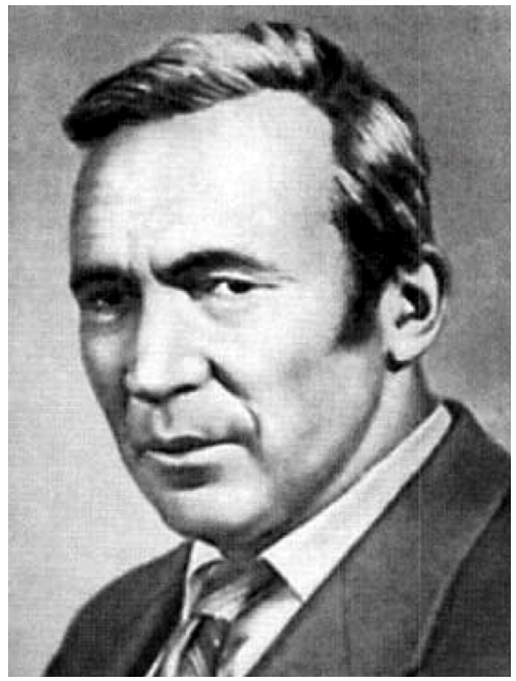
Miëtek Bak

Least Fixed, Cambridge

L.E.J. Brouwer



A. Heyting



A. Kolmogorov

$$a \supset b \qquad \begin{matrix} a \\ \vdots \\ b \end{matrix}$$

H.B. Curry

W.A. Howard

$$a \supset b \qquad\qquad \lambda\, x : a \to y : b$$

G. Gentzen

$$\frac{}{a}\ \text{var}\ x$$

$$\vdots$$

$$\frac{b}{a \supset b}\ \text{lam}\ x : a \qquad \frac{a \supset b \qquad a}{b}\ \text{app}$$

$$\dfrac{\dfrac{}{a}\,\text{var}\,x \\ \vdots \\ b}{a \supset b}\,\text{lam}\,x : a$$

$$\dfrac{a \supset b \qquad a}{b}\,\text{app}$$

$$(a \supset b \supset c) \supset (a \supset b) \supset a \supset c$$

$$\dfrac{\begin{array}{c}\\ a\\ \vdots\\ b\end{array}}{a \supset b}\ \text{lam}\ x : a \qquad \dfrac{\text{—}}{a}\ \text{var}\ x$$

$$\dfrac{a \supset b \qquad a}{b}\ \text{app}$$

$$\dfrac{(a \supset b) \supset a \supset c}{(a \supset b \supset c) \supset (a \supset b) \supset a \supset c}\ \text{lam}\, f : a \supset b \supset c$$

$$\dfrac{}{a}\ \text{var}\ x$$
$$\vdots$$
$$\dfrac{b}{a \supset b}\ \text{lam}\ x : a$$

$$\dfrac{a \supset b \qquad a}{b}\ \text{app}$$

$$\dfrac{\dfrac{\dfrac{a \supset c}{a \supset b \supset c}\ \text{lam}\ g : a \supset b}{(a \supset b) \supset a \supset c}\ \text{lam}\ f : a \supset b \supset c}{(a \supset b \supset c) \supset (a \supset b) \supset a \supset c}$$

$$\dfrac{\dfrac{}{a}\ \text{var}\ x}{\vdots}\ \text{lam}\ x : a$$
$$a \supset b$$

$$\dfrac{a \supset b \qquad a}{b}\ \text{app}$$

$$\dfrac{\dfrac{\dfrac{}{c}\ \text{lam}\ x : a}{a \supset c}\ \text{lam}\ g : a \supset b}{(a \supset b) \supset a \supset c}\ \text{lam}\ f : a \supset b \supset c$$
$$(a \supset b \supset c) \supset (a \supset b) \supset a \supset c$$

$$\dfrac{\dfrac{}{a}\,\text{var}\,x}{\vdots}\;\dfrac{}{b}\;\text{lam}\,x : a$$

$$a \supset b$$

$$\dfrac{a \supset b \qquad a}{b}\;\text{app}$$

$$\dfrac{\dfrac{b \supset c \qquad\qquad b}{c}\;\text{app}}{a \supset c}\;\text{lam}\,x : a$$

$$\dfrac{}{(a \supset b) \supset a \supset c}\;\text{lam}\,g : a \supset b$$

$$\dfrac{}{(a \supset b \supset c) \supset (a \supset b) \supset a \supset c}\;\text{lam}\,f : a \supset b \supset c$$

$$\dfrac{\dfrac{\dfrac{}{a}\,\text{var}\,x}{\vdots}\,b}{a \supset b}\,\text{lam}\,x : a$$

$$\dfrac{\dfrac{}{a \supset b}\,\text{var}\,f \quad \dfrac{}{a}\,\text{var}\,x}{b}\,\text{app}$$

$$\dfrac{\dfrac{\dfrac{}{a \supset b \supset c}\,\text{var}\,f \quad \dfrac{}{a}\,\text{var}\,x}{b \supset c}\,\text{app} \qquad \dfrac{}{b}}{c}\,\text{app}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{a \supset b \supset c}\,\text{var}\,f \quad \dfrac{}{a}\,\text{var}\,x}{b \supset c}\,\text{app} \qquad \dfrac{}{b}}{c}\,\text{app}}{a \supset c}\,\text{lam}\,x : a$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{a \supset b \supset c}\,\text{var}\,f \quad \dfrac{}{a}\,\text{var}\,x}{b \supset c}\,\text{app} \qquad \dfrac{}{b}}{c}\,\text{app}}{a \supset c}\,\text{lam}\,x : a}{(a \supset b) \supset a \supset c}\,\text{lam}\,g : a \supset b$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{a \supset b \supset c}\,\text{var}\,f \quad \dfrac{}{a}\,\text{var}\,x}{b \supset c}\,\text{app} \qquad \dfrac{}{b}}{c}\,\text{app}}{a \supset c}\,\text{lam}\,x : a}{(a \supset b) \supset a \supset c}\,\text{lam}\,g : a \supset b}{(a \supset b \supset c) \supset (a \supset b) \supset a \supset c}\,\text{lam}\,f : a \supset b \supset c$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{a}\,\text{var}\,x}{\vdots}}{b}}{a \supset b}\,\text{lam}\,x : a$$

$$\dfrac{\dfrac{a \supset b \quad a}{b}\,\text{app}}{a \supset b}\,\text{lam}\,x : a$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{a \supset b \supset c}\,\text{var}\,f \quad \dfrac{}{a}\,\text{var}\,x}{b \supset c}\,\text{app} \quad \dfrac{\dfrac{}{a \supset b}\,\text{var}\,g \quad \dfrac{}{a}\,\text{var}\,x}{b}\,\text{app}}{c}\,\text{app}}{a \supset c}\,\text{lam}\,x : a}{(a \supset b) \supset a \supset c}\,\text{lam}\,g : a \supset b}{(a \supset b \supset c) \supset (a \supset b) \supset a \supset c}\,\text{lam}\,f : a \supset b \supset c$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{(a \supset b \supset c) \supset (a \supset b) \supset a \supset c}\,\text{lam}\, f : a \supset b \supset c}{(a \supset b) \supset a \supset c}\,\text{lam}\, g : a \supset b}{a \supset c}\,\text{lam}\, x : a}{c}}{\dfrac{\dfrac{\dfrac{}{a \supset b \supset c}\,\text{var}\, f \quad \dfrac{}{a}\,\text{var}\, x}{b \supset c}\,\text{app} \quad \dfrac{\dfrac{}{a \supset b}\,\text{var}\, g \quad \dfrac{}{a}\,\text{var}\, x}{b}\,\text{app}}{b}\,\text{app}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{(a \supset b \supset c) \supset (a \supset b) \supset a \supset c}\,\mathrm{lam}\, f : a \supset b \supset c}{(a \supset b) \supset a \supset c}\,\mathrm{lam}\, g : a \supset b}{a \supset c}\,\mathrm{lam}\, x : a}{c}}{\dfrac{\dfrac{\mathrm{var}\, f \quad \mathrm{var}\, x}{b \supset c}\,\mathrm{app} \qquad \dfrac{\mathrm{var}\, g \quad \mathrm{var}\, x}{b}\,\mathrm{app}}{}\,\mathrm{app}}$$

$$\dfrac{\dfrac{\dfrac{(a \supset b \supset c) \supset (a \supset b) \supset a \supset c}{\text{lam}\, f : a \supset b \supset c}}{(a \supset b) \supset a \supset c}\,\text{lam}\, g : a \supset b}{a \supset c}\,\text{lam}\, x : a$$

$$\dfrac{c}{\text{app}}$$

$$\text{var}\, f\ \$\ \text{var}\, x \qquad \text{var}\, g\ \$\ \text{var}\, x$$

$$\dfrac{\dfrac{\dfrac{}{(a \supset b \supset c) \supset (a \supset b) \supset a \supset c}\,\mathrm{lam}\, f : a \supset b \supset c}{(a \supset b) \supset a \supset c}\,\mathrm{lam}\, g : a \supset b}{a \supset c}\,\mathrm{lam}\, x : a$$

$$(\mathrm{var}\, f\ \$\ \mathrm{var}\, x)\ \$\ (\mathrm{var}\, g\ \$\ \mathrm{var}\, x)$$

$$\mathrm{lam}\, f : a \supset b \supset c \;\frac{(a \supset b \supset c) \supset (a \supset b) \supset a \supset c}{\mathrm{lam}\, g : a \supset b \;\dfrac{(a \supset b) \supset a \supset c}{\quad}}$$

```
lam x →
    (var f $ var x) $ (var g $ var x)
```

$$\mathrm{lam}\, f : a \supset b \supset c \;\frac{}{(a \supset b \supset c) \supset (a \supset b) \supset a \supset c}$$

```
lam g →
  lam x →
    (var f $ var x) $ (var g $ var x)
```

$$\mathbf{S} : (a \supset b \supset c) \supset (a \supset b) \supset a \supset c$$

```
S = lam f →
      lam g →
        lam x →
          (var f $ var x) $ (var g $ var x)
```

$$\mathrm{S} : \forall\ \{\mathrm{a}\ \mathrm{b}\ \mathrm{c}\} \rightarrow \mathrm{T}\ ((a \Rightarrow b \Rightarrow c) \Rightarrow (a \Rightarrow b) \Rightarrow a \Rightarrow c)$$

```
S = lam \f →
      lam \g →
        lam \x →
          (var f $ var x) $ (var g $ var x)
```

Live demo

Agda, Idris, and Haskell implementations of minimal implicational logic, or simply-typed λ calculus

https://github.com/mietek/haskell-exchange-2015

J. Carette, O. Kiselyov, C. Shan
Finally tagless, partially evaluated:
Tagless staged interpreters for simpler typed languages

F. Pfenning
Constructive logic

A.S. Troelstra
History of constructivism in the twentieth century

P. Wadler
Propositions as types

Thanks for listening!

https://mietek.io/

hello@mietek.io