

BUILD YOUR OWN PROOF ASSISTANT

Miëtek Bak

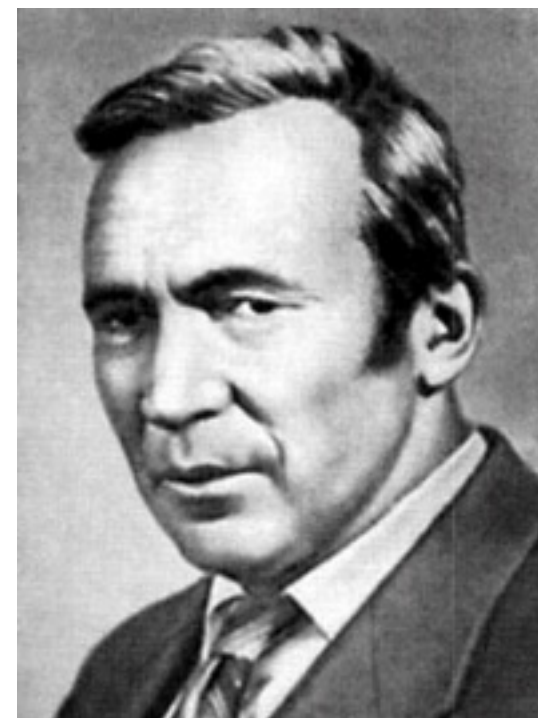
Least Fixed, Cambridge



L.E.J. Brouwer



A. Heyting



A. Kolmogorov

$$a \supset b$$

$$\begin{array}{c} a \\ \vdots \\ b \end{array}$$



H.B. Curry



W.A. Howard

$$a \supset b$$

$$\lambda x : a \rightarrow y : b$$



G. Gentzen

$$\begin{array}{c}
 \text{---} \text{ var } x \\
 a \\
 \vdots \\
 b \\
 \hline
 a \supset b \quad \text{lam } x : a
 \end{array}$$

$$\begin{array}{c}
 a \supset b \quad a \\
 \hline
 b \quad \text{app}
 \end{array}$$

$$\begin{array}{c}
 \frac{}{\text{var } x} \\
 a \\
 \vdots \\
 b \\
 \hline
 a \supset b \quad \text{lam } x : a \\
 \\
 \frac{a \supset b \quad a}{b} \text{ app}
 \end{array}$$

$$(a \supset b \supset c) \supset (a \supset b) \supset a \supset c$$

$$\frac{\frac{\text{---} \text{ var } x}{a} \vdots b}{a \supset b} \text{ lam } x : a$$

$$\frac{a \supset b \quad a}{b} \text{ app}$$

$$\frac{(a \supset b) \supset a \supset c}{(a \supset b \supset c) \supset (a \supset b) \supset a \supset c} \text{ lam } f : a \supset b \supset c$$

$$\frac{\frac{\text{---} \text{ var } x}{a} \vdots b}{a \supset b} \text{ lam } x : a$$

$$\frac{a \supset b \quad a}{b} \text{ app}$$

$$\frac{\frac{a \supset c}{(a \supset b) \supset a \supset c} \text{ lam } g : a \supset b}{(a \supset b \supset c) \supset (a \supset b) \supset a \supset c} \text{ lam } f : a \supset b \supset c$$

$$\begin{array}{c}
 \frac{}{\text{var } x} \\
 \frac{a}{\vdots} \\
 \frac{b}{\text{lam } x : a} \\
 \hline
 a \supset b
 \end{array}
 \quad
 \frac{a \supset b \quad a}{b} \text{ app}$$

$$\begin{array}{c}
 \frac{c}{\text{lam } x : a} \\
 \frac{a \supset c}{\text{lam } g : a \supset b} \\
 \hline
 (a \supset b) \supset a \supset c
 \end{array}
 \quad
 \frac{}{\text{lam } f : a \supset b \supset c}$$

$$(a \supset b \supset c) \supset (a \supset b) \supset a \supset c$$

$$\begin{array}{c}
 \frac{}{a} \text{ var } x \\
 \vdots \\
 b \\
 \hline
 a \supset b \text{ lam } x : a \\
 \\
 \frac{a \supset b \quad a}{b} \text{ app}
 \end{array}$$

$$\begin{array}{c}
 \frac{b \supset c \quad b}{\quad} \text{ app} \\
 \\
 \frac{c}{\quad} \text{ lam } x : a \\
 a \supset c \\
 \hline
 (a \supset b) \supset a \supset c \text{ lam } g : a \supset b \\
 \\
 \hline
 (a \supset b \supset c) \supset (a \supset b) \supset a \supset c \text{ lam } f : a \supset b \supset c
 \end{array}$$

$$\frac{\frac{\frac{}{\text{var } x} \quad a}{\vdots} \quad b}{a \supset b} \text{ lam } x : a$$

$$\frac{a \supset b \quad a}{b} \text{ app}$$

$$\frac{a \supset b \supset c \quad a}{b \supset c} \text{ app}$$

$$\frac{b \supset c \quad b}{c} \text{ app}$$

$$\frac{c}{a \supset c} \text{ lam } x : a$$

$$\frac{}{(a \supset b) \supset a \supset c} \text{ lam } g : a \supset b$$

$$\frac{(a \supset b) \supset a \supset c}{(a \supset b \supset c) \supset (a \supset b) \supset a \supset c} \text{ lam } f : a \supset b \supset c$$

$$\frac{\frac{\frac{\text{---}}{a} \text{ var } x}{\vdots} b}{a \supset b} \text{ lam } x : a$$

$$\frac{a \supset b \quad a}{b} \text{ app}$$

$$\frac{\text{---}}{a \supset b \supset c} \text{ var } f$$

$$\frac{a \supset b \supset c \quad a}{b \supset c} \text{ app}$$

$$\frac{b \supset c \quad b}{c} \text{ app}$$

$$\frac{c}{a \supset c} \text{ lam } x : a$$

$$\frac{a \supset c}{(a \supset b) \supset a \supset c} \text{ lam } g : a \supset b$$

$$\frac{(a \supset b) \supset a \supset c}{(a \supset b \supset c) \supset (a \supset b) \supset a \supset c} \text{ lam } f : a \supset b \supset c$$

$$\frac{\frac{\frac{\text{---}}{a} \text{ var } x}{\vdots} b}{a \supset b} \text{ lam } x : a$$

$$\frac{a \supset b \quad a}{b} \text{ app}$$

$$\frac{\frac{\text{---}}{a \supset b \supset c} \text{ var } f \quad \frac{\text{---}}{a} \text{ var } x}{\text{---}} \text{ app}$$

$$\frac{b \supset c \quad b}{c} \text{ app}$$

$$\frac{c}{a \supset c} \text{ lam } x : a$$

$$\frac{\text{---}}{(a \supset b) \supset a \supset c} \text{ lam } g : a \supset b$$

$$\frac{\text{---}}{(a \supset b \supset c) \supset (a \supset b) \supset a \supset c} \text{ lam } f : a \supset b \supset c$$

$$\begin{array}{c}
\frac{}{\text{var } x} \\
a \\
\vdots \\
b \\
\hline
a \supset b \quad \text{lam } x : a
\end{array}$$

$$\frac{a \supset b \quad a}{b} \text{ app}$$

$$\begin{array}{c}
\frac{}{\text{var } f} \quad \frac{}{\text{var } x} \\
a \supset b \supset c \quad a \quad a \supset b \quad a \\
\hline
b \supset c \quad b \\
\hline
c \quad \text{app} \\
\hline
c \\
\hline
a \supset c \quad \text{lam } x : a \\
\hline
(a \supset b) \supset a \supset c \quad \text{lam } g : a \supset b \\
\hline
(a \supset b \supset c) \supset (a \supset b) \supset a \supset c \quad \text{lam } f : a \supset b \supset c
\end{array}$$

$$\begin{array}{c}
\frac{}{\text{var } x} \\
a \\
\vdots \\
b \\
\hline
a \supset b \quad \text{lam } x : a \\
\\
\frac{a \supset b \quad a}{b} \text{ app}
\end{array}$$

$$\begin{array}{c}
\frac{}{\text{var } f} \quad \frac{}{\text{var } x} \quad \frac{}{\text{var } g} \\
a \supset b \supset c \quad a \quad a \supset b \quad a \\
\hline
b \supset c \quad b \\
\hline
c \quad \text{app} \\
\\
\frac{}{\text{lam } x : a} \\
a \supset c \\
\\
\frac{}{\text{lam } g : a \supset b} \\
(a \supset b) \supset a \supset c \\
\\
\frac{}{\text{lam } f : a \supset b \supset c} \\
(a \supset b \supset c) \supset (a \supset b) \supset a \supset c
\end{array}$$

$$\frac{\frac{\frac{\text{---}}{a} \text{ var } x}{\vdots} b}{a \supset b} \text{ lam } x : a$$

$$\frac{a \supset b \quad a}{b} \text{ app}$$

$$\frac{\frac{\frac{\text{---}}{a \supset b \supset c} \text{ var } f}{\text{---}} \text{ app} \quad \frac{\frac{\text{---}}{a} \text{ var } x}{\text{---}} \text{ app} \quad \frac{\frac{\frac{\text{---}}{a \supset b} \text{ var } g}{\text{---}} \text{ app} \quad \frac{\text{---}}{a} \text{ var } x}{\text{---}} \text{ app}}{b \supset c} \text{ app}$$

$$\frac{c}{a \supset c} \text{ lam } x : a$$

$$\frac{\text{---}}{(a \supset b) \supset a \supset c} \text{ lam } g : a \supset b$$

$$\frac{\text{---}}{(a \supset b \supset c) \supset (a \supset b) \supset a \supset c} \text{ lam } f : a \supset b \supset c$$

$$\begin{array}{c}
\text{lam } f : a \supset b \supset c \quad \frac{(a \supset b \supset c) \supset (a \supset b) \supset a \supset c}{(a \supset b) \supset a \supset c} \\
\text{lam } g : a \supset b \quad \frac{(a \supset b) \supset a \supset c}{a \supset c} \\
\text{lam } x : a \quad \frac{a \supset c}{c} \\
\text{app} \quad \frac{\quad}{b \supset c} \quad \quad \quad b \\
\text{app} \quad \frac{\quad}{a \supset b \supset c} \quad \quad a \quad \quad \text{app} \quad \frac{\quad}{a \supset b} \quad \quad a \\
\text{var } f \quad \frac{\quad}{a \supset b \supset c} \quad \text{var } x \quad \frac{\quad}{a} \quad \text{var } g \quad \frac{\quad}{a \supset b} \quad \text{var } x \quad \frac{\quad}{a}
\end{array}$$

$$\begin{array}{c}
\text{lam } f : a \supset b \supset c \quad \frac{(a \supset b \supset c) \supset (a \supset b) \supset a \supset c}{(a \supset b) \supset a \supset c} \\
\text{lam } g : a \supset b \quad \frac{(a \supset b) \supset a \supset c}{a \supset c} \\
\text{lam } x : a \quad \frac{a \supset c}{c} \\
\text{app} \quad \frac{\quad}{b \supset c} \quad \frac{\quad}{b} \\
\text{app} \quad \frac{\quad}{\text{var } f \$ \text{ var } x} \quad \text{app} \quad \frac{\quad}{\text{var } g \$ \text{ var } x}
\end{array}$$

$$\begin{array}{c}
\text{lam } f : a \supset b \supset c \quad \frac{(a \supset b \supset c) \supset (a \supset b) \supset a \supset c}{(a \supset b) \supset a \supset c} \\
\text{lam } g : a \supset b \quad \frac{(a \supset b) \supset a \supset c}{a \supset c} \\
\text{lam } x : a \quad \frac{a \supset c}{c} \\
\text{app} \quad \frac{\quad}{\quad}
\end{array}$$

$(\text{var } f \$ \text{ var } x) \$ (\text{var } g \$ \text{ var } x)$

$$\begin{array}{c}
(a \supset b \supset c) \supset (a \supset b) \supset a \supset c \\
\text{lam } f : a \supset b \supset c \text{ } \frac{\quad}{\quad} \\
(a \supset b) \supset a \supset c \\
\text{lam } g : a \supset b \text{ } \frac{\quad}{\quad} \\
a \supset c \\
\text{lam } x : a \text{ } \frac{\quad}{\quad}
\end{array}$$

$(\text{var } f \$ \text{ var } x) \$ (\text{var } g \$ \text{ var } x)$

$$\text{lam } f : a \supset b \supset c \frac{(a \supset b \supset c) \supset (a \supset b) \supset a \supset c}{(a \supset b) \supset a \supset c}$$

$$\text{lam } g : a \supset b \frac{}{}$$

$\text{lam } x \rightarrow$
 $(\text{var } f \$ \text{ var } x) \$ (\text{var } g \$ \text{ var } x)$

$$\text{lam } f: a \supset b \supset c \frac{(a \supset b \supset c) \supset (a \supset b) \supset a \supset c}{}$$

lam $g \rightarrow$
 lam $x \rightarrow$
 (var f \$ var x) \$ (var g \$ var x)

$$S : (a \supset b \supset c) \supset (a \supset b) \supset a \supset c$$

$S = \text{lam } f \rightarrow$
 $\quad \text{lam } g \rightarrow$
 $\quad \quad \text{lam } x \rightarrow$
 $\quad \quad \quad (\text{var } f \$ \text{ var } x) \$ (\text{var } g \$ \text{ var } x)$

$$S : \forall \{a \ b \ c\} \rightarrow T \ ((a \Rightarrow b \Rightarrow c) \Rightarrow (a \Rightarrow b) \Rightarrow a \Rightarrow c)$$

$$\begin{aligned}
 S = & \text{lam } \backslash f \rightarrow \\
 & \text{lam } \backslash g \rightarrow \\
 & \text{lam } \backslash x \rightarrow \\
 & \quad (\text{var } f \$ \text{ var } x) \$ (\text{var } g \$ \text{ var } x)
 \end{aligned}$$

(code reading)

<https://github.com/mietek/haskell-exchange-2015>

P. Wadler

Propositions as types

J. Carette, O. Kiselyov, C. Shan

Finally tagless, partially evaluated

F. Pfenning

Constructive logic

A.S. Troelstra

History of constructivism in the twentieth century

Thanks for listening!