Introspective Kripke models and normalisation-by-evaluation for the $\lambda^{\Box}$-calculus

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Abstract

We consider the $\lambda^{\Box}$-calculus, an extension of the simply typed λ-calculus with a *type of quoted programs* that corresponds to the $\Box$ connective of the modal logic S4. We present a novel class of *introspective Kripke models,* constructed in continuation-passing style, and prove the syntax of the $\lambda^{\Box}$-calculus sound and complete with respect to these models. We fully formalise the arguments in Agda, a dependently typed total functional programming language based on intensional type theory. The composition of our soundness and completeness proofs is an Agda program that performs normalisation-by-evaluation for the $\lambda^{\Box}$-calculus.

Keywords: constructive logic, continuation-passing style, completeness, Curry-Howard correspondence, intensionality, intuitionistic logic, Kripke semantics, meta-programming, modal logic S4, normalisation-by-evaluation, quotation, self-interpretation, typed λ-calculus

1 Introduction

I'd like to have a total functional programming language that works like Lisp; that is, a language that allows programs to be quoted, inspected, and evaluated, in which types are propositions and programs are proofs. Some people have been using the modal logic S4 for similar purposes. I quickly found a reasonable λ-calculus based on S4, that is, the $\lambda^{\Box}$-calculus, and I set out to write an interpreter for it.

$$\mathsf{mv}\, i : \frac{\Delta \ni A}{\Delta; \Gamma \vdash A} \qquad \ulcorner M \urcorner : \frac{\Delta; \varnothing \vdash A}{\Delta; \Gamma \vdash \Box A} \qquad \llcorner M \lrcorner\, N : \frac{\Delta; \Gamma \vdash \Box A \quad \Delta, A; \Gamma \vdash C}{\Delta; \Gamma \vdash C}$$

It's well-known that if we prove a language sound and complete with respect to some class of models, then an interpreter falls out — as long as the proofs are constructive. Unfortunately, all I could find was classical proofs of completeness for S4, and so I decided to do the proofs myself.

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It's also well-known how to prove STLC sound and complete with respect to intuitionistic Kripke semantics. Since the $\lambda^{\Box}$-calculus is an extension of STLC, I thought it should be simple to extend intuitionistic Kripke semantics and obtain the desired proofs. Famous last words! Eventually, I realised that the interpretation of $\Box A$ should be a syntactically-justified interpretation of A; that is, a pair consisting of a derivation of A and an interpretation of A.

$$\Delta;\Gamma \;{}^{\mathsf{m}}\!\Vdash A \;=\; \Delta;\varnothing \vdash A \;\times\; \Delta;\Gamma \Vdash A$$

$$\Delta;\Gamma \Vdash \Box A \;=\; \forall\, \Delta' \supseteq \Delta, \Gamma' \supseteq \Gamma.\; \Delta';\Gamma' \;{}^{\mathsf{m}}\!\Vdash A$$

This puts the syntax in the semantics — peeks beneath the veil sooner than expected. We can still keep the abstraction of a Kripke semantics, as long as we can discard it when we need to.

$$\mathfrak{M} \;=\; \langle\, \mathcal{W},\, \mathcal{V},\, -{\geq}-,\, \mathsf{id_a},\, -{\circ_\mathsf{a}}-,\, \mathsf{acc_v},\, \lfloor - \rfloor,\, \lfloor - \rfloor_\mathsf{a} \,\rangle$$

$$\mathfrak{M} \,|\, w \;{}^{\mathsf{m}}\!\Vdash A \;=\; {}^{\mathsf{m}}\lfloor w \rfloor;\varnothing \vdash A \;\times\; \mathfrak{M} \,|\, w \Vdash A$$

$$\mathfrak{M} \,|\, w \Vdash \Box A \;=\; \forall\, w' \geq w.\; \mathfrak{M} \,|\, w' \;{}^{\mathsf{m}}\!\Vdash A$$

However, adding this interpretation to intuitionistic Kripke semantics allows us to prove soundness, but not completeness. The problem with proving completeness with respect to the modified semantics is similar to the problem with proving completeness of full STLC, with disjunction and the empty type. Some people have already come up with a solution to that, which is a CPS transformation of the semantics.

$$\mathfrak{M} \,|\, w \;{}^{\partial}\!\Vdash A \;=\; \forall\, C, w' \geq w.\; \frac{\forall\, w'' \geq w'.\; \dfrac{\mathfrak{M} \,|\, w'' \Vdash A}{\lfloor w'' \rfloor \vdash_{\mathsf{nf}} C}}{\lfloor w' \rfloor \vdash_{\mathsf{nf}} C}$$

$$\mathfrak{M} \,|\, w \;{}^{\mathsf{m}\partial}\!\Vdash A \;=\; {}^{\mathsf{m}}\lfloor w \rfloor;\varnothing \vdash A \;\times\; \mathfrak{M} \,|\, w \;{}^{\partial}\!\Vdash A$$

$$\mathfrak{M} \,|\, w \Vdash \Box A \;=\; \forall\, w' \geq w.\; \mathfrak{M} \,|\, w' \;{}^{\mathsf{m}\partial}\!\Vdash A$$

It turns out that CPS transforming the modified semantics allows us to prove soundness and completeness, and so, to write an interpreter for the $\lambda^{\Box}$-calculus. I've done that now, and I think this interpreter can be used to decide $\beta\eta$-equivalence according to the following convertibility relation, but I haven't managed to prove this yet.

$$\Delta;\Gamma \vDash A \;=\; \forall\, \mathfrak{M}, w.\; \frac{\mathfrak{M} \,|\, w \;{}^{\mathsf{m}\partial}\!\Vdash^{\star} \Delta \quad \mathfrak{M} \,|\, w \;{}^{\partial}\!\Vdash^{\star} \Gamma}{\mathfrak{M} \,|\, w \;{}^{\partial}\!\Vdash A}$$

$$\uparrow\!- : \frac{\Delta;\Gamma \vdash A}{\Delta;\Gamma \vDash A} \qquad \downarrow\!- : \frac{\Delta;\Gamma \vDash A}{\Delta;\Gamma \vdash_{\mathsf{nf}} A}$$

To prove completeness, we need a canonical model. We first prove soundness and completeness with respect to this canonical model, and then use that to prove completeness proper.

$$\mathfrak{M}_{\mathsf{c}} \;=\; \langle\, \mathcal{C}^2,\ -\vdash_{\mathsf{nf}} *,\ -\supseteq^2-,\ \mathsf{id}^2_{\mathsf{r}},\ -\circ^2_{\mathsf{r}}-,\ \mathsf{ren}^2_{\mathsf{nf}},\ \mathsf{id},\ \mathsf{id}\,\rangle$$

$$\uparrow_{\mathsf{c}}- : \frac{\Delta;\Gamma \vdash_{\mathsf{ne}} A}{\Delta;\Gamma \;{}^{\partial}\!\Vdash A} \qquad \downarrow_{\mathsf{c}}- : \frac{\Delta;\Gamma \;{}^{\partial}\!\Vdash A}{\Delta;\Gamma \vdash_{\mathsf{nf}} A}$$

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