

# Introspective Kripke models and normalisation-by-evaluation for the $\lambda^\square$ -calculus

Miëtek Bak<sup>1</sup>

*Least Fixed Ltd, Cambridge, UK*

---

## Abstract

We consider the  $\lambda^\square$ -calculus, an extension of the simply typed  $\lambda$ -calculus with a *type of quoted programs* that corresponds to the  $\square$  connective of the modal logic S4. We present a novel class of *introspective Kripke models*, constructed in continuation-passing style, and prove the syntax of the  $\lambda^\square$ -calculus sound and complete with respect to these models. We fully formalise the arguments in AGDA, a dependently typed total functional programming language based on intensional type theory. The composition of our soundness and completeness proofs is an AGDA program that performs normalisation-by-evaluation for the  $\lambda^\square$ -calculus.

*Keywords:* constructive logic, continuation-passing style, completeness, Curry-Howard correspondence, intensionality, intuitionistic logic, Kripke semantics, meta-programming, modal logic S4, normalisation-by-evaluation, quotation, self-interpretation, typed  $\lambda$ -calculus

---

## 1 Introduction

I'd like to have a total functional programming language that works like Lisp; that is, a language that allows programs to be quoted, inspected, and evaluated, in which types are propositions and programs are proofs. Some people have been using the modal logic S4 for similar purposes. I quickly found a reasonable  $\lambda$ -calculus based on S4, that is, the  $\lambda^\square$ -calculus, and I set out to write an interpreter for it.

$$\frac{\Delta \ni A}{\Delta; \Gamma \vdash A} \quad \frac{\Delta; \emptyset \vdash A}{\Delta; \Gamma \vdash \square A} \quad \frac{\Delta; \Gamma \vdash \square A \quad \Delta, A; \Gamma \vdash C}{\Delta; \Gamma \vdash C}$$

It's well-known that if we prove a language sound and complete with respect to some class of models, then an interpreter falls out — as long as the proofs are constructive. Unfortunately, all I could find was classical proofs of completeness for S4, and so I decided to do the proofs myself.

---

<sup>1</sup> [mietek@bak.io](mailto:mietek@bak.io)

It's also well-known how to prove STLC sound and complete with respect to intuitionistic Kripke semantics. Since the  $\lambda^\square$ -calculus is an extension of STLC, I thought it should be simple to extend intuitionistic Kripke semantics and obtain the desired proofs. Famous last words! Eventually, I realised that the interpretation of  $\square A$  should be a syntactically-justified interpretation of  $A$ ; that is, a pair consisting of a derivation of  $A$  and an interpretation of  $A$ .

$$\begin{aligned}\Delta; \Gamma \Vdash \square A &= \forall \Delta' \supseteq \Delta, \Gamma' \supseteq \Gamma. \Delta'; \Gamma' \Vdash A \\ \Delta; \Gamma \Vdash A &= \Delta; \emptyset \vdash A \times \Delta; \Gamma \Vdash A\end{aligned}$$

This puts the syntax in the semantics — peeks beneath the veil sooner than expected. We can still keep the abstraction of a Kripke semantics, as long as we can discard it when we need to.

$$\mathfrak{M} = \langle \mathcal{W}, -\mathcal{V}-, -\geq-, \text{id}_a, -\circ_a-, \text{rel}_v, \lfloor - \rfloor, \lfloor - \rfloor_a \rangle$$

Now, we can try to write the interpretation of  $\square A$  using the model.

$$\begin{aligned}\mathfrak{M} \mid w \Vdash \square A &= \forall w' \geq w. \mathfrak{M} \mid w' \Vdash A \\ \mathfrak{M} \mid w \Vdash A &= \mathfrak{m}[w]; \emptyset \vdash A \times \mathfrak{M} \mid w \Vdash A\end{aligned}$$

However, adding this interpretation to intuitionistic Kripke semantics allows us to prove soundness, but not completeness. The problem with proving completeness with respect to the modified semantics is similar to the problem with proving completeness of full STLC, with disjunction and the empty type. Some people have already come up with a solution to that, which is a CPS transformation of the semantics.

$$\begin{aligned}\mathfrak{M} \mid w \Vdash \square A &= \forall w' \geq w. \mathfrak{M} \mid w' \text{mk} \Vdash A \\ \mathfrak{M} \mid w \text{k} \Vdash A &= \forall C, w' \geq w. \frac{\forall w'' \geq w'. \frac{\mathfrak{M} \mid w'' \Vdash A}{\lfloor w'' \rfloor \vdash_{\text{nm}} C}}{\lfloor w' \rfloor \vdash_{\text{nm}} C} \\ \mathfrak{M} \mid w \text{mk} \Vdash A &= \mathfrak{m}[w]; \emptyset \vdash A \times \mathfrak{M} \mid w \text{k} \Vdash A\end{aligned}$$

It turns out that CPS transforming the modified semantics allows us to prove soundness and completeness, and so, to write an interpreter for the  $\lambda^\square$ -calculus.

$$\Delta; \Gamma \vDash A = \forall \mathfrak{M}, w. \frac{\mathfrak{M} \mid w \text{mk} \Vdash \Delta \quad \mathfrak{M} \mid w \text{k} \Vdash \Gamma}{\mathfrak{M} \mid w \text{k} \Vdash A}$$

### Theorem 1.1 (Soundness)

$$\downarrow : \frac{\Delta; \Gamma \vdash A}{\Delta; \Gamma \vDash A}$$

**Theorem 1.2 (Completeness)**

$$\uparrow : \frac{\Delta; \Gamma \models A}{\Delta; \Gamma \vdash_{\text{nm}} A}$$

**Corollary 1.3 (Normalisation)** *Every program of the  $\lambda^\square$ -calculus has a normal form.*

$$\text{nm} : \frac{\Delta; \Gamma \vdash A}{\Delta; \Gamma \vdash_{\text{nm}} A}$$

**Proof.** By composition of soundness (Theorem ??) and completeness (Theorem ??).

$$\text{nm} = \uparrow \circ \downarrow$$

□

I've done that now, and I think this interpreter can be used to decide  $\beta\eta$ -equivalence according to the following convertibility relation, but I haven't managed to prove this yet.

To prove completeness, we need a canonical model.

**Theorem 1.4** *There exists a canonical model.*

**Proof.**

$$\mathfrak{M}_u = \langle Cx, -\mathcal{V}_u-, -\supseteq^2-, \text{id}_r^2, -\circ_r^2-, \text{ren}_{\text{nm}}^2, \text{id}, \text{id} \rangle$$

$$\Delta; \Gamma \mathcal{V}_u x = \Delta; \Gamma \vdash_{\text{nm}}^{\text{tv}} x$$

□

We first prove soundness and completeness with respect to this canonical model, and then use that to prove completeness proper.

$$\downarrow_u : \frac{\Delta; \Gamma \vdash_{\text{nt}} A}{\mathfrak{M}_u \mid \Delta; \Gamma \Vdash A} \quad \uparrow_u : \frac{\mathfrak{M}_u \mid \Delta; \Gamma \Vdash A}{\Delta; \Gamma \vdash_{\text{nm}} A}$$

**Contributions**

Lorem ipsum dolor sit amet, consectetur adipiscing elit. Ut purus elit, vestibulum ut, placerat ac, adipiscing vitae, felis. Curabitur dictum gravida mauris. Nam arcu libero, nonummy eget, consectetur id, vulputate a, magna. Donec vehicula augue eu neque. Pellentesque habitant morbi tristique senectus et netus et malesuada fames ac turpis egestas. Mauris ut leo. Cras viverra metus rhoncus sem. Nulla et lectus vestibulum urna fringilla ultrices. Phasellus eu tellus sit amet tortor gravida placerat. Integer sapien est, iaculis in, pretium quis, viverra ac, nunc. Praesent eget sem vel leo ultrices bibendum. Aenean faucibus. Morbi dolor nulla, malesuada eu, pulvinar at, mollis ac, nulla. Curabitur auctor semper nulla. Donec varius orci eget risus. Duis nibh mi, congue eu, accumsan eleifend, sagittis quis, diam. Duis eget orci sit amet orci dignissim rutrum.

Nam dui ligula, fringilla a, euismod sodales, sollicitudin vel, wisi. Morbi auctor lorem non justo. Nam lacus libero, pretium at, lobortis vitae, ultricies et, tellus.

Donec aliquet, tortor sed accumsan bibendum, erat ligula aliquet magna, vitae ornare odio metus a mi. Morbi ac orci et nisl hendrerit mollis. Suspendisse ut massa. Cras nec ante. Pellentesque a nulla. Cum sociis natoque penatibus et magnis dis parturient montes, nascetur ridiculus mus. Aliquam tincidunt urna. Nulla ullamcorper vestibulum turpis. Pellentesque cursus luctus mauris.

## 2 Syntax

**Definition 2.1** Syntactic entailment.

$$\begin{array}{c}
\text{mv} : \frac{\Delta \ni A}{\Delta; \Gamma \vdash A} \quad \text{v} : \frac{\Gamma \ni A}{\Delta; \Gamma \vdash A} \\
\lambda : \frac{\Delta; \Gamma, A \vdash B}{\Delta; \Gamma \vdash A \supset B} \quad -\$- : \frac{\Delta; \Gamma \vdash A \supset B \quad \Delta; \Gamma \vdash A}{\Delta; \Gamma \vdash B} \\
\ulcorner - \urcorner : \frac{\Delta; \emptyset \vdash A}{\Delta; \Gamma \vdash \Box A} \quad \llcorner - \lrcorner : \frac{\Delta; \Gamma \vdash \Box A \quad \Delta, A; \Gamma \vdash C}{\Delta; \Gamma \vdash C}
\end{array}$$

**Example 2.2** Axioms of S4.

$$\begin{array}{l}
D : \Delta; \Gamma \vdash \Box(A \supset B) \supset \Box A \supset \Box B \\
D = \lambda (\lambda (\llcorner \text{v} 1 \lrcorner \urcorner \text{mv} 1 \$ \text{mv} 0 \urcorner)) \\
T : \Delta; \Gamma \vdash \Box A \supset A \\
T = \lambda (\llcorner \text{v} 0 \lrcorner \urcorner \text{mv} 0) \\
4 : \Delta; \Gamma \vdash \Box A \supset \Box \Box A \\
4 = \lambda (\llcorner \text{v} 0 \lrcorner \urcorner \urcorner \text{mv} 0 \urcorner \urcorner)
\end{array}$$

**Definition 2.3** Normal forms and neutral forms.

$$\begin{array}{c}
\lambda : \frac{\Delta; \Gamma, A \vdash_{\text{nm}} B}{\Delta; \Gamma \vdash_{\text{nm}} A \supset B} \\
\ulcorner - \urcorner : \frac{\Delta; \emptyset \vdash A}{\Delta; \Gamma \vdash_{\text{nm}} \Box A} \quad \llcorner - \lrcorner : \frac{\Delta; \Gamma \vdash_{\text{nt}} \Box A \quad \Delta, A; \Gamma \vdash_{\text{nm}} C}{\Delta; \Gamma \vdash_{\text{nm}} C} \\
\text{nt} : \frac{\Delta; \Gamma \vdash_{\text{nt}}^{\text{tv}} x}{\Delta; \Gamma \vdash_{\text{nm}}^{\text{tv}} x} \\
\text{mv} : \frac{\Delta \ni A}{\Delta; \Gamma \vdash_{\text{nt}} A} \quad \text{v} : \frac{\Gamma \ni A}{\Delta; \Gamma \vdash_{\text{nt}} A} \\
-\$- : \frac{\Delta; \Gamma \vdash_{\text{nt}} A \supset B \quad \Delta; \Gamma \vdash_{\text{nm}} A}{\Delta; \Gamma \vdash_{\text{nt}} B}
\end{array}$$

Lorem ipsum dolor sit amet, consectetur adipiscing elit. Ut purus elit, vestibulum ut, placerat ac, adipiscing vitae, felis. Curabitur dictum gravida mauris. Nam arcu libero, nonummy eget, consectetur id, vulputate a, magna. Donec vehicula augue eu neque. Pellentesque habitant morbi tristique senectus et netus et malesuada

fames ac turpis egestas. Mauris ut leo. Cras viverra metus rhoncus sem. Nulla et lectus vestibulum urna fringilla ultrices. Phasellus eu tellus sit amet tortor gravida placerat. Integer sapien est, iaculis in, pretium quis, viverra ac, nunc. Praesent eget sem vel leo ultrices bibendum. Aenean faucibus. Morbi dolor nulla, malesuada eu, pulvinar at, mollis ac, nulla. Curabitur auctor semper nulla. Donec varius orci eget risus. Duis nibh mi, congue eu, accumsan eleifend, sagittis quis, diam. Duis eget orci sit amet orci dignissim rutrum.

Nam dui ligula, fringilla a, euismod sodales, sollicitudin vel, wisi. Morbi auctor lorem non justo. Nam lacus libero, pretium at, lobortis vitae, ultricies et, tellus. Donec aliquet, tortor sed accumsan bibendum, erat ligula aliquet magna, vitae ornare odio metus a mi. Morbi ac orci et nisl hendrerit mollis. Suspendisse ut massa. Cras nec ante. Pellentesque a nulla. Cum sociis natoque penatibus et magnis dis parturient montes, nascetur ridiculus mus. Aliquam tincidunt urna. Nulla ullamcorper vestibulum turpis. Pellentesque cursus luctus mauris.

Nulla malesuada porttitor diam. Donec felis erat, congue non, volutpat at, tincidunt tristique, libero. Vivamus viverra fermentum felis. Donec nonummy pellentesque ante. Phasellus adipiscing semper elit. Proin fermentum massa ac quam. Sed diam turpis, molestie vitae, placerat a, molestie nec, leo. Maecenas lacinia. Nam ipsum ligula, eleifend at, accumsan nec, suscipit a, ipsum. Morbi blandit ligula feugiat magna. Nunc eleifend consequat lorem. Sed lacinia nulla vitae enim. Pellentesque tincidunt purus vel magna. Integer non enim. Praesent euismod nunc eu purus. Donec bibendum quam in tellus. Nullam cursus pulvinar lectus. Donec et mi. Nam vulputate metus eu enim. Vestibulum pellentesque felis eu massa.

Quisque ullamcorper placerat ipsum. Cras nibh. Morbi vel justo vitae lacus tincidunt ultrices. Lorem ipsum dolor sit amet, consectetur adipiscing elit. In hac habitasse platea dictumst. Integer tempus convallis augue. Etiam facilisis. Nunc elementum fermentum wisi. Aenean placerat. Ut imperdiet, enim sed gravida sollicitudin, felis odio placerat quam, ac pulvinar elit purus eget enim. Nunc vitae tortor. Proin tempus nibh sit amet nisl. Vivamus quis tortor vitae risus porta vehicula.

Fusce mauris. Vestibulum luctus nibh at lectus. Sed bibendum, nulla a faucibus semper, leo velit ultricies tellus, ac venenatis arcu wisi vel nisl. Vestibulum diam. Aliquam pellentesque, augue quis sagittis posuere, turpis lacus congue quam, in hendrerit risus eros eget felis. Maecenas eget erat in sapien mattis porttitor. Vestibulum porttitor. Nulla facilisi. Sed a turpis eu lacus commodo facilisis. Morbi fringilla, wisi in dignissim interdum, justo lectus sagittis dui, et vehicula libero dui cursus dui. Mauris tempor ligula sed lacus. Duis cursus enim ut augue. Cras ac magna. Cras nulla. Nulla egestas. Curabitur a leo. Quisque egestas wisi eget nunc. Nam feugiat lacus vel est. Curabitur consectetur.

### 3 Semantics

**Definition 3.1 (Introspective Kripke models)** An *introspective Kripke model* is a tuple that consists of a set of *worlds*  $\mathcal{W}$ , a *valuation relation* on worlds and type variables  $-\mathcal{V}-$ , an *accessibility relation* on worlds  $-\geq-$ , a *world introspection function*, and an *accessibility introspection function*. Accessibility is reflexive and

transitive, as witnessed by  $\text{id}_a$  and  $-\circ_a-$ , and valuation is monotonic with respect to accessibility, as witnessed by  $\text{rel}_v$ . World introspection  $\lfloor - \rfloor$  maps worlds to contexts, and accessibility introspection  $\lfloor - \rfloor_a$  maps elements of the accessibility relation to order-preserving embeddings on contexts.

$$\mathfrak{M} = \langle \mathcal{W}, -\mathcal{V}-, -\geq-, \text{id}_a, -\circ_a-, \text{rel}_v, \lfloor - \rfloor, \lfloor - \rfloor_a \rangle$$

We write  $w \mathcal{V} x$  to mean the valuation for  $x$  at  $w$ , and  $w' \geq w$  to say that  $w'$  is accessible from  $w$ . We use the word *relocating* to mean applying the monotonicity property for valuations. For example, if  $\eta$  witnesses that  $w'$  is accessible from  $w$ , we read  $\text{rel}_v \eta v$  as relocating  $v$  via  $\eta$ .

As shorthand, we use *modal introspection*  $\mathfrak{m}[-]$  to map directly to the modal projection of a context, and  $\mathfrak{m}[-]_a$  to map directly to the modal projection of an order-preserving embedding on contexts.

**Definition 3.2 (Values and value continuations)** A *value* for  $A$  at  $w$  of  $\mathfrak{M}$ , written  $\mathfrak{M} | w \Vdash A$ , is defined by induction on the type, mutually with value continuations and justified value continuations.

For type variables  ${}^{\text{tv}}x$  (3.2.1), the value at  $w$  is given by the valuation for  $x$  at  $w$ . For function types  $A \supset B$  (3.2.2), the value at  $w$  is a function that, for every  $w'$  accessible from  $w$ , takes a value continuation for  $A$  at  $w'$  and returns a value continuation for  $B$  at  $w'$ . For quotation types  $\Box A$  (3.2.3), the value at  $w$  is a justified value continuation for  $A$  at  $w$ .

A *value continuation* for  $A$  at  $w$  of  $\mathfrak{M}$ , written  $\mathfrak{M} | w^k \Vdash A$  (3.2.4), is a function that, for every type  $C$  and  $w'$  accessible from  $w$ , takes a continuation that, for every  $w''$  accessible from  $w'$ , consumes a value for  $A$  at  $w''$  and produces a normal derivation for  $C$  under  $\lfloor w'' \rfloor$ . The result is a normal derivation for  $C$  under  $\lfloor w' \rfloor$ .

A *justified value continuation* for  $A$  at  $w$  of  $\mathfrak{M}$ , written  $\mathfrak{M} | w^{\text{mk}} \Vdash A$  (3.2.5), is a simple product of a derivation for  $A$  under  $\mathfrak{m}[w]; \emptyset$  and a value continuation for  $A$  at  $w$ .

$$\mathfrak{M} | w \Vdash {}^{\text{tv}}x = w \mathcal{V} x \quad (3.2.1)$$

$$\mathfrak{M} | w \Vdash A \supset B = \forall w' \geq w. \frac{\mathfrak{M} | w'^k \Vdash A}{\mathfrak{M} | w'^k \Vdash B} \quad (3.2.2)$$

$$\mathfrak{M} | w \Vdash \Box A = \mathfrak{M} | w^{\text{mk}} \Vdash A \quad (3.2.3)$$

$$\mathfrak{M} | w^k \Vdash A = \forall C, w' \geq w. \frac{\forall w'' \geq w'. \frac{\mathfrak{M} | w'' \Vdash A}{\lfloor w'' \rfloor \vdash_{\text{nm}} C}}{\lfloor w' \rfloor \vdash_{\text{nm}} C} \quad (3.2.4)$$

$$\mathfrak{M} | w^{\text{mk}} \Vdash A = \mathfrak{m}[w]; \emptyset \vdash A \times \mathfrak{M} | w^k \Vdash A \quad (3.2.5)$$

**Lemma 3.3 (Projections of justified value continuations)** Given a justified value continuation for  $A$  at  $w$  of  $\mathfrak{M}$ , there exists a derivation for  $A$  under  $\mathfrak{m}[w]; \emptyset$ , being the *syntactic projection*, and a value continuation for  $A$  at  $w$ , being the

*semantic projection.*

$$\text{syn} : \frac{\mathfrak{M} | w^{\text{mk}} \Vdash A}{\mathfrak{m}[w]; \emptyset \vdash A} \quad \text{sem} : \frac{\mathfrak{M} | w^{\text{mk}} \Vdash A}{\mathfrak{M} | w^{\text{k}} \Vdash A}$$

Proved by simple product elimination.

**Lemma 3.4 (Monotonicity of values and value continuations with respect to accessibility)** For every  $w'$  accessible from  $w$  of  $\mathfrak{M}$ , given a value (value continuation; justified value continuation) for  $A$  at  $w$ , there exists a value (value continuation; justified value continuation) for  $A$  at  $w'$ .

$$\begin{aligned} \text{rel} : \frac{w' \geq w \quad \mathfrak{M} | w \Vdash A}{w' \Vdash A} \\ \text{krel} : \frac{w' \geq w \quad \mathfrak{M} | w^{\text{k}} \Vdash A}{w' \text{k} \Vdash A} \quad \text{mkrel} : \frac{w' \geq w \quad \mathfrak{M} | w^{\text{mk}} \Vdash A}{w' \text{mk} \Vdash A} \end{aligned}$$

**Proof.** By mutual induction on the type, and lemmas ?? and 3.3.

$$\begin{aligned} \text{rel} \{^{\text{tv}}x\} \quad \eta v &= \text{rel}_v \eta v \\ \text{rel} \{A \supset B\} \quad \eta f &= \eta' \mapsto f(\eta \circ_a \eta') \\ \text{rel} \{\Box A\} \quad \eta p &= \text{mkrel} \eta p \\ \text{krel} \eta k &= \eta' f \mapsto k(\eta \circ_a \eta') f \\ \text{mkrel} \eta p &= \text{mren} \mathfrak{m}[\eta]_a (\text{syn } p), \text{krel} \eta (\text{sem } p) \end{aligned}$$

□

We extend the use of the word *relocating* to mean applying the monotonicity property for values (value continuations; justified value continuations).

**Lemma 3.5 (Kripke continuation monad)** Value continuations form a monad.

$$\text{unit} : \frac{\mathfrak{M} | w \Vdash A}{\mathfrak{M} | w^{\text{k}} \Vdash A} \quad \text{bind} : \frac{\mathfrak{M} | w^{\text{k}} \Vdash A \quad \forall w' \geq w. \frac{\mathfrak{M} | w' \Vdash A}{\mathfrak{M} | w' \text{k} \Vdash C}}{\mathfrak{M} | w^{\text{k}} \Vdash C}$$

**Proof.** By lemma 3.4.

$$\begin{aligned} \text{unit } a &= \eta f \mapsto f \text{id}_a (\text{rel } \eta a) \\ \text{bind } k f &= \eta f' \mapsto k \eta (\eta' a \mapsto \\ &\quad f(\eta \circ_a \eta') a \text{id}_a (\eta'' b \mapsto \\ &\quad f'(\eta' \circ_a \eta'') b)) \end{aligned}$$

□

**Definition 3.6 (Environments)** An (*ordinary*) *environment* for  $\Xi$  at  $w$  of  $\mathfrak{M}$ , written  $\mathfrak{M} \mid w \Vdash \star$ , is a mapping that assigns to every type in  $\Xi$  a value continuation at  $w$  of  $\mathfrak{M}$ . A *justified environment* for  $\Xi$  at  $w$  of  $\mathfrak{M}$ , written  $\mathfrak{M} \mid w \text{mk} \Vdash \star$ , is a mapping that assigns to every type in  $\Xi$  a justified value continuation at  $w$  of  $\mathfrak{M}$ .

$$\mathfrak{M} \mid w \Vdash \star \Xi = \text{All} (\mathfrak{M} \mid w \Vdash -) \Xi$$

$$\mathfrak{M} \mid w \text{mk} \Vdash \star \Xi = \text{All} (\mathfrak{M} \mid w \text{mk} \Vdash -) \Xi$$

We skip the word *ordinary* when possible. We also abuse the word *environment* to mean a justified environment together with an ordinary environment. For example, given a context  $\Delta; \Gamma$  and a world  $w$  of  $\mathfrak{M}$ , the *current environment* refers to a justified environment  $\delta$  together with an ordinary environment  $\gamma$ , where  $\delta : \mathfrak{M} \mid w \text{mk} \Vdash \star \Delta$  and  $\gamma : \mathfrak{M} \mid w \Vdash \star \Gamma$ .

**Lemma 3.7 (Projections of justified environments)** Given a justified environment for  $\Xi$  at  $w$  of  $\mathfrak{M}$ , there exists a simultaneous substitution for  $\Xi$  under  $\mathfrak{m}[w]; \emptyset$  and an environment for  $\Xi$  at  $w$  of  $\mathfrak{M}$ .

$$\text{syn} \star : \frac{\mathfrak{M} \mid w \text{mk} \Vdash \star \Xi}{\mathfrak{m}[w]; \emptyset \vdash \star \Xi} \quad \text{sem} \star : \frac{\mathfrak{M} \mid w \text{mk} \Vdash \star \Xi}{\mathfrak{M} \mid w \Vdash \star \Xi}$$

Proved by induction on the justified environment and lemma 3.3.

**Lemma 3.8 (Monotonicity of environments with respect to accessibility)** For every  $w'$  accessible from  $w$ , given an environment (justified environment) for  $\Xi$  at  $w$  of  $\mathfrak{M}$ , there exists an environment (justified environment) for  $\Xi$  at  $w'$  of  $\mathfrak{M}$ .

$$\text{krel} \star : \frac{w' \geq w \quad \mathfrak{M} \mid w \Vdash \star \Xi}{w' \Vdash \star \Xi} \quad \text{mkrel} \star : \frac{w' \geq w \quad \mathfrak{M} \mid w \text{mk} \Vdash \star \Xi}{w' \text{mk} \Vdash \star \Xi}$$

Proved by induction on the environment (justified environment) and lemma 3.4.

**Lemma 3.9 (Environment lookup)** Given an environment for  $\Xi$  at  $w$  of  $\mathfrak{M}$ , for every type  $A$  in  $\Xi$ , there exists a value continuation for  $A$  at  $w$  of  $\mathfrak{M}$ .

$$\text{lookup} : \frac{\mathfrak{M} \mid w \Vdash \star \Xi \quad \Xi \ni A}{\mathfrak{M} \mid w \Vdash A}$$

Proved by induction on the environment.

**Definition 3.10 (Semantic entailment)** We say that  $\Delta; \Gamma$  *semantically entails*  $A$  when, for every world  $w$  of every model  $\mathfrak{M}$ , given a justified environment for  $\Delta$  at  $w$  of  $\mathfrak{M}$  and an environment for  $\Gamma$  at  $w$  of  $\mathfrak{M}$ , there exists a value continuation for  $A$  at  $w$  of  $\mathfrak{M}$ .

$$\Delta; \Gamma \models A = \forall \mathfrak{M}, w. \frac{\mathfrak{M} \mid w \text{mk} \Vdash \star \Delta \quad \mathfrak{M} \mid w \Vdash \star \Gamma}{\mathfrak{M} \mid w \Vdash A}$$



**Theorem 3.11 (Soundness)** If  $\Delta; \Gamma$  syntactically entails  $A$ , then  $\Delta; \Gamma$  semantically entails  $A$ .

$$\Downarrow : \frac{\Delta; \Gamma \vdash A}{\Delta; \Gamma \models A}$$

**Proof.** By induction on the derivation and lemmas 3.9, 3.7, 3.5, and 3.8.

$$\Downarrow (\text{mv } i) = \delta \gamma \mapsto \text{lookup} (\text{sem} \star \delta) i \quad (3.11.1)$$

$$\Downarrow (\text{v } i) = \delta \gamma \mapsto \text{lookup } \gamma i \quad (3.11.2)$$

$$\Downarrow (\lambda \mathcal{D}) = \delta \gamma \mapsto \text{unit} (\eta k \mapsto \quad (3.11.3)$$

$$\Downarrow \mathcal{D} (\text{mkrel} \star \eta \delta) (\text{krel} \star \eta \gamma, k))$$

$$\Downarrow (\mathcal{D} \$ \mathcal{E}) = \delta \gamma \mapsto \text{bind} (\Downarrow \mathcal{D} \delta \gamma) (\eta f \mapsto \quad (3.11.4)$$

$$f \text{id}_a (\Downarrow \mathcal{E} (\text{mkrel} \star \eta \delta) (\text{krel} \star \eta \gamma)))$$

$$\Downarrow (\ulcorner \mathcal{D} \urcorner) = \delta \gamma \mapsto \text{unit} (\text{m}_{\text{sub}} (\text{syn} \star \delta) \mathcal{D}, \Downarrow \mathcal{D} \delta \emptyset) \quad (3.11.5)$$

$$\Downarrow (\lfloor \mathcal{D} \rfloor \mathcal{E}) = \delta \gamma \mapsto \text{bind} (\Downarrow \mathcal{D} \delta \gamma) (\eta p \mapsto \quad (3.11.6)$$

$$\Downarrow \mathcal{E} (\text{mkrel} \star \eta \delta, p) (\text{krel} \star \eta \gamma))$$

□

$\Downarrow$ , pronounced ‘reflect’, is The proof of soundness is a monadic evaluator for  $\lambda^\square$ -derivations: a function that *reflects* syntactic objects as semantic objects, abstracting over the specifics of any particular model. Given a justified environment  $\delta$  and an environment  $\gamma$ , both at some world of some model, evaluation consumes a derivation for  $A$  under  $\Delta; \Gamma$  and produces a value continuation for  $A$ , still at the same world of the same model.

Modal variables (3.11.1) and variables (3.11.2) are looked up in the semantic projection of  $\delta$  and in  $\gamma$ , respectively.

In the case of function abstraction (3.11.3), we first suppose that we can access via  $\eta$  some world in which the value continuation  $k$  is the function argument. Then, we evaluate the function body  $\mathcal{D}$  in an environment obtained by relocating the current environment via  $\eta$  and extending it with  $k$ .

To perform function application (3.11.4), we start by evaluating the function  $\mathcal{D}$  in the current environment, obtaining a value  $f$  at some world accessible via  $\eta$ . By definition, the value of a function of type  $A \supset B$  is a meta-level function that, given a value continuation of  $A$  at some accessible world, returns a value continuation of  $B$  at the same world. Evaluating the function argument  $\mathcal{E}$  in the current environment relocated via  $\eta$  gives us a value continuation, which we use to call  $f$ .

Normalisation-by-evaluation is also known as *reduction-free normalisation* [4,20], because performing substitution at the meta level frees us from having to implement it at the object level. However, in the case of quotation (3.11.5), object-level modal substitution (??) is required.

XXX

Lorem ipsum dolor sit amet, consectetur adipiscing elit. Ut purus elit, vestibulum ut, placerat ac, adipiscing vitae, felis. Curabitur dictum gravida mauris. Nam arcu libero, nonummy eget, consectetur id, vulputate a, magna. Donec vehicula au-

gue eu neque. Pellentesque habitant morbi tristique senectus et netus et malesuada fames ac turpis egestas. Mauris ut leo. Cras viverra metus rhoncus sem. Nulla et lectus vestibulum urna fringilla ultrices. Phasellus eu tellus sit amet tortor gravida placerat. Integer sapien est, iaculis in, pretium quis, viverra ac, nunc. Praesent eget sem vel leo ultrices bibendum. Aenean faucibus. Morbi dolor nulla, malesuada eu, pulvinar at, mollis ac, nulla. Curabitur auctor semper nulla. Donec varius orci eget risus. Duis nibh mi, congue eu, accumsan eleifend, sagittis quis, diam. Duis eget orci sit amet orci dignissim rutrum.

## 4 Completeness

**Definition 4.1** *A universal model for the  $\lambda^\square$ -calculus.*

$$\mathfrak{M}_u = \langle Cx, -\mathcal{V}_u-, -\supseteq^2-, \text{id}_r^2, -\circ_r^2-, \text{ren}_{nm}^2, \text{id}, \text{id} \rangle$$

$$\Delta; \Gamma \mathcal{V}_u x = \Delta; \Gamma \vdash_{nm}^{\text{tv}} x$$

**Lemma 4.2** *Soundness and completeness of the  $\lambda^\square$ -calculus with respect to the universal model  $\mathfrak{M}_u$ .*

$$\downarrow_u : \frac{\Delta; \Gamma \vdash_{nt} A}{\mathfrak{M}_u \mid \Delta; \Gamma \Vdash A} \quad \uparrow_u : \frac{\mathfrak{M}_u \mid \Delta; \Gamma \Vdash A}{\Delta; \Gamma \vdash_{nm} A}$$

**Proof.** By mutual induction on the type.

$$\downarrow_u \{^{\text{tv}}x\} \quad \mathcal{D} = \text{unit} (^{\text{nt}}\mathcal{D}) \tag{4.2.1}$$

$$\downarrow_u \{A \supset B\} \mathcal{D} = \text{unit} (\eta k \mapsto \downarrow_u (\text{ren}_{nt}^2 \eta \mathcal{D} \$ \uparrow_u k)) \tag{4.2.2}$$

$$\downarrow_u \{\Box A\} \quad \mathcal{D} = \eta f \mapsto \perp \text{ren}_{nt}^2 \eta \mathcal{D} \sqcup (f (^{\text{mwk}^2} \text{id}_r^2) (^{\text{mv}0}, \downarrow_u^{\text{mv}0})) \tag{4.2.3}$$

$$\uparrow_u \{^{\text{tv}}x\} \quad k = k \text{id}_r^2 (\eta \mathcal{D} \mapsto \mathcal{D}) \tag{4.2.4}$$

$$\uparrow_u \{A \supset B\} k = k \text{id}_r^2 (\eta f \mapsto \lambda (\uparrow_u (f (^{\text{wk}^2} \text{id}_r^2) (\downarrow_u^{\text{v}0})))) \tag{4.2.5}$$

$$\uparrow_u \{\Box A\} \quad k = k \text{id}_r^2 (\eta p \mapsto \ulcorner \text{syn } p \urcorner) \tag{4.2.6}$$

□

**Lemma 4.3** *Identity environments.*

$$^{\text{m}}\text{id}_e : \Delta; \Gamma \text{mk} \Vdash_\star \Delta \quad \text{id}_e : \Delta; \Gamma \text{k} \Vdash_\star \Gamma$$

Each proved by induction on the environment.

**Theorem 4.4 (Completeness)**

$$\uparrow : \frac{\Delta; \Gamma \models A}{\Delta; \Gamma \vdash_{nm} A}$$

**Proof.** By lemma 4.2 and 4.3.

$$\uparrow f = \uparrow_u (f ^{\text{m}}\text{id}_e \text{id}_e)$$

□

**Corollary 4.5 (Normalisation)**

$$\text{nm} : \frac{\Delta; \Gamma \vdash A}{\Delta; \Gamma \vdash_{\text{nm}} A}$$

**Proof.** By theorem 3.11 and 4.4.

$$\text{nm} = \uparrow \circ \downarrow$$

□

Lorem ipsum dolor sit amet, consectetur adipiscing elit. Ut purus elit, vestibulum ut, placerat ac, adipiscing vitae, felis. Curabitur dictum gravida mauris. Nam arcu libero, nonummy eget, consectetur id, vulputate a, magna. Donec vehicula augue eu neque. Pellentesque habitant morbi tristique senectus et netus et malesuada fames ac turpis egestas. Mauris ut leo. Cras viverra metus rhoncus sem. Nulla et lectus vestibulum urna fringilla ultrices. Phasellus eu tellus sit amet tortor gravida placerat. Integer sapien est, iaculis in, pretium quis, viverra ac, nunc. Praesent eget sem vel leo ultrices bibendum. Aenean faucibus. Morbi dolor nulla, malesuada eu, pulvinar at, mollis ac, nulla. Curabitur auctor semper nulla. Donec varius orci eget risus. Duis nibh mi, congue eu, accumsan eleifend, sagittis quis, diam. Duis eget orci sit amet orci dignissim rutrum.

Nam dui ligula, fringilla a, euismod sodales, sollicitudin vel, wisi. Morbi auctor lorem non justo. Nam lacus libero, pretium at, lobortis vitae, ultricies et, tellus. Donec aliquet, tortor sed accumsan bibendum, erat ligula aliquet magna, vitae ornare odio metus a mi. Morbi ac orci et nisl hendrerit mollis. Suspendisse ut massa. Cras nec ante. Pellentesque a nulla. Cum sociis natoque penatibus et magnis dis parturient montes, nascetur ridiculus mus. Aliquam tincidunt urna. Nulla ullamcorper vestibulum turpis. Pellentesque cursus luctus mauris.

Nulla malesuada porttitor diam. Donec felis erat, congue non, volutpat at, tincidunt tristique, libero. Vivamus viverra fermentum felis. Donec nonummy pellentesque ante. Phasellus adipiscing semper elit. Proin fermentum massa ac quam. Sed diam turpis, molestie vitae, placerat a, molestie nec, leo. Maecenas lacinia. Nam ipsum ligula, eleifend at, accumsan nec, suscipit a, ipsum. Morbi blandit ligula feugiat magna. Nunc eleifend consequat lorem. Sed lacinia nulla vitae enim. Pellentesque tincidunt purus vel magna. Integer non enim. Praesent euismod nunc eu purus. Donec bibendum quam in tellus. Nullam cursus pulvinar lectus. Donec et mi. Nam vulputate metus eu enim. Vestibulum pellentesque felis eu massa.

Quisque ullamcorper placerat ipsum. Cras nibh. Morbi vel justo vitae lacus tincidunt ultrices. Lorem ipsum dolor sit amet, consectetur adipiscing elit. In hac habitasse platea dictumst. Integer tempus convallis augue. Etiam facilisis. Nunc elementum fermentum wisi. Aenean placerat. Ut imperdiet, enim sed gravida sollicitudin, felis odio placerat quam, ac pulvinar elit purus eget enim. Nunc vitae tortor. Proin tempus nibh sit amet nisl. Vivamus quis tortor vitae risus porta vehicula.

Fusce mauris. Vestibulum luctus nibh at lectus. Sed bibendum, nulla a faucibus semper, leo velit ultricies tellus, ac venenatis arcu wisi vel nisl. Vestibulum diam. Aliquam pellentesque, augue quis sagittis posuere, turpis lacus congue quam, in hendrerit risus eros eget felis. Maecenas eget erat in sapien mattis porttitor. Vestibulum porttitor. Nulla facilisi. Sed a turpis eu lacus commodo facilisis. Morbi fringilla, wisi in dignissim interdum, justo lectus sagittis dui, et vehicula libero dui cursus dui. Mauris tempor ligula sed lacus. Duis cursus enim ut augue. Cras ac magna. Cras nulla. Nulla egestas. Curabitur a leo. Quisque egestas wisi eget nunc. Nam feugiat lacus vel est. Curabitur consectetur.

## 5 Conclusion

Lorem ipsum dolor sit amet, consectetur adipiscing elit. Ut purus elit, vestibulum ut, placerat ac, adipiscing vitae, felis. Curabitur dictum gravida mauris. Nam arcu libero, nonummy eget, consectetur id, vulputate a, magna. Donec vehicula augue eu neque. Pellentesque habitant morbi tristique senectus et netus et malesuada fames ac turpis egestas. Mauris ut leo. Cras viverra metus rhoncus sem. Nulla et lectus vestibulum urna fringilla ultrices. Phasellus eu tellus sit amet tortor gravida placerat. Integer sapien est, iaculis in, pretium quis, viverra ac, nunc. Praesent eget sem vel leo ultrices bibendum. Aenean faucibus. Morbi dolor nulla, malesuada eu, pulvinar at, mollis ac, nulla. Curabitur auctor semper nulla. Donec varius orci eget risus. Duis nibh mi, congue eu, accumsan eleifend, sagittis quis, diam. Duis eget orci sit amet orci dignissim rutrum.

Nam dui ligula, fringilla a, euismod sodales, sollicitudin vel, wisi. Morbi auctor lorem non justo. Nam lacus libero, pretium at, lobortis vitae, ultricies et, tellus. Donec aliquet, tortor sed accumsan bibendum, erat ligula aliquet magna, vitae ornare odio metus a mi. Morbi ac orci et nisl hendrerit mollis. Suspendisse ut massa. Cras nec ante. Pellentesque a nulla. Cum sociis natoque penatibus et magnis dis parturient montes, nascetur ridiculus mus. Aliquam tincidunt urna. Nulla ullamcorper vestibulum turpis. Pellentesque cursus luctus mauris.

Nulla malesuada porttitor diam. Donec felis erat, congue non, volutpat at, tincidunt tristique, libero. Vivamus viverra fermentum felis. Donec nonummy pellentesque ante. Phasellus adipiscing semper elit. Proin fermentum massa ac quam. Sed diam turpis, molestie vitae, placerat a, molestie nec, leo. Maecenas lacinia. Nam ipsum ligula, eleifend at, accumsan nec, suscipit a, ipsum. Morbi blandit ligula feugiat magna. Nunc eleifend consequat lorem. Sed lacinia nulla vitae enim. Pellentesque tincidunt purus vel magna. Integer non enim. Praesent euismod nunc eu purus. Donec bibendum quam in tellus. Nullam cursus pulvinar lectus. Donec et mi. Nam vulputate metus eu enim. Vestibulum pellentesque felis eu massa.

Quisque ullamcorper placerat ipsum. Cras nibh. Morbi vel justo vitae lacus tincidunt ultrices. Lorem ipsum dolor sit amet, consectetur adipiscing elit. In hac habitasse platea dictumst. Integer tempus convallis augue. Etiam facilisis. Nunc elementum fermentum wisi. Aenean placerat. Ut imperdiet, enim sed gravida sollicitudin, felis odio placerat quam, ac pulvinar elit purus eget enim. Nunc vitae tortor. Proin tempus nibh sit amet nisl. Vivamus quis tortor vitae risus porta vehicula.

Fusce mauris. Vestibulum luctus nibh at lectus. Sed bibendum, nulla a faucibus semper, leo velit ultricies tellus, ac venenatis arcu wisi vel nisl. Vestibulum diam. Aliquam pellentesque, augue quis sagittis posuere, turpis lacus congue quam, in hendrerit risus eros eget felis. Maecenas eget erat in sapien mattis porttitor. Vestibulum porttitor. Nulla facilisi. Sed a turpis eu lacus commodo facilisis. Morbi fringilla, wisi in dignissim interdum, justo lectus sagittis dui, et vehicula libero dui cursus dui. Mauris tempor ligula sed lacus. Duis cursus enim ut augue. Cras ac magna. Cras nulla. Nulla egestas. Curabitur a leo. Quisque egestas wisi eget nunc. Nam feugiat lacus vel est. Curabitur consectetur.

## Acknowledgements

The author is deeply grateful to Andreas Abel, Guillaume Allais, Ahmad Salim Al-Sibahi, Roy Dyckhoff, Michael Gabbay, Paolo Giarrusso, Tom Jack, Roman Kireev, Jerzy Marcinkowski, Darryl McAdams, Conor McBride, Dominic Orchard, Maciej Piróg, Ida Szubert, Tarmo Uustalu, Andrea Vezzosi, and Tomasz Wierzbicki, for many fruitful discussions over the years.

Furthermore, the author thanks Sergei Artemov, Andrej Bauer, Jacques Carette, Danko Ilik, Alex Kavvos, Jon Sterling, and the anonymous reviewers, for comments that helped improve this work.

## Citations

Abel (2013) [1]. Alechina *et al.* (2001) [2]. Altenkirch (1993) [3]. Altenkirch, Hofmann, and Streicher (1995) [4]. Altenkirch and Reus (1999) [5]. Artemov (2001) [6]. Artemov and Bonelli (2007) [7]. Bauer (2016) [8]. Berger and Schwichtenberg (1991) [9]. Bierman and de Paiva (2000) [10]. Boolos (1994) [11]. Božić and Došen (1984) [12]. Brown and Palsberg (2016) [13]. Chapman (2009) [15]. C. Coquand (1993) [16]. C. Coquand (2002) [17]. T. Coquand and Dybjer (1997) [18]. Danvy (1996) [19]. Danvy (2005) [20]. Danvy, Keller, and Puech (2014) [21]. Davies and Pfenning (2001) [22]. de Bruijn (1972) [14]. Dybjer and Filinski (2002) [23]. Dyckhoff (2016) [24]. Ewald (1986) [25]. Fischer Servi (1984) [26]. Fitting (2005) [27]. Gabbay and Nanevski (2013) [28]. Girard *et al.* (1989) [29]. Gödel (1933) [30]. Iemhoff (2001) [31]. Ilik (2010) [32]. Ilik (2013) [33]. Joachimski and Matthes (2003) [34]. Kovacs (2017) [35]. Kripke (1965) [36]. Lindley (2005) [37]. Martin-Löf (1975) [38]. McBride (2005) [39]. McCarthy *et al.* (1962) [40]. McKinsey and Tarski (1948) [41]. Meyer and Wand (1985) [42]. Mkrtchyan (1997) [43]. Nanevski (2002) [44]. Nanevski (2004) [45]. Nanevski, Pfenning, and Pientka (2008) [46]. Norell (2007) [47]. Ono (1977) [48]. Pfenning and Davies (2001) [49]. Pientka and Abel (2015) [50]. Plotkin and Stirling (1986) [51]. Sheard (2001) [52]. Simpson (1994) [53]. Stump (2016) [54]. Turner (2004) [55]. Wadler (2015) [56]. Wickline, Lee, and Pfenning (1998) [57]. Wijesekera (1990) [58].

## References

- [1] Abel, A., “Normalization by evaluation: dependent types and impredicativity,” Habilitation thesis, Ludwig-Maximilians-Universität München (2013).  
<http://www.cse.chalmers.se/~abela/habil.pdf>

- [2] Alechina, N., M. Mendler, V. de Paiva and E. Ritter, *Categorical and Kripke semantics for constructive  $S_4$  modal logic*, in: *Computer Science Logic (CSL'01)*, LNCS **2142** (2001), pp. 292–307.  
[http://doi.org/10.1007/3-540-44802-0\\_21](http://doi.org/10.1007/3-540-44802-0_21)
- [3] Altenkirch, T., *Proving strong normalization of CC by modifying realizability semantics*, in: *Types for Proofs and Programs (TYPES'93)*, LNCS **806** (1993), pp. 3–18.  
[http://doi.org/10.1007/3-540-58085-9\\_70](http://doi.org/10.1007/3-540-58085-9_70)
- [4] Altenkirch, T., M. Hofmann and T. Streicher, *Categorical reconstruction of a reduction free normalization proof*, in: *Category Theory and Computer Science (CTCS'95)*, (LNCS) **953** (1995), pp. 182–199.  
[http://doi.org/10.1007/3-540-60164-3\\_27](http://doi.org/10.1007/3-540-60164-3_27)
- [5] Altenkirch, T. and B. Reus, *Monadic presentations of lambda terms using generalized inductive types*, in: *Computer Science Logic (CSL'99)*, LNCS **1683** (1999), pp. 453–468.  
[http://doi.org/10.1007/3-540-48168-0\\_32](http://doi.org/10.1007/3-540-48168-0_32)
- [6] Artemov, S. N., *Explicit provability and constructive semantics*, Bulletin of Symbolic Logic **7** (2001), pp. 1–36.  
<http://doi.org/10.2307/2687821>
- [7] Artemov, S. N. and E. Bonelli, *The intensional lambda calculus*, in: *Logical Foundations of Computer Science (LFCS'07)*, LNCS **4514** (2007), pp. 12–25.  
[http://doi.org/10.1007/978-3-540-72734-7\\_2](http://doi.org/10.1007/978-3-540-72734-7_2)
- [8] Bauer, A., *On self-interpreters for Gödel's System T*, in: *Types for Proofs and Programs (TYPES'16)*, 2016, accepted.  
<http://math.andrej.com/wp-content/uploads/2016/01/self-interpreter-for-T.pdf>
- [9] Berger, U. and H. Schwichtenberg, *An inverse of the evaluation functional for typed lambda-calculus*, in: *Logic in Computer Science (LICS'91)* (1991), pp. 203–211.  
<http://doi.org/10.1109/LICS.1991.151645>
- [10] Bierman, G. M. and V. de Paiva, *On an intuitionistic modal logic*, Studia Logica **65** (2000), pp. 383–416.  
<http://doi.org/10.1023/A:1005291931660>
- [11] Boolos, G., *An  $S_4$ -preserving proof-theoretical treatment of modality*, in: *The Logic of Provability*, Cambridge University Press, 1994 pp. 155–164.  
<http://doi.org/10.1017/CB09780511625183.014>
- [12] Božić, M. and K. Došen, *Models for normal intuitionistic modal logics*, Studia Logica **43** (1984), pp. 217–245.  
<http://doi.org/10.1007/BF02429840>
- [13] Brown, M. and J. Palsberg, *Breaking through the normalization barrier: a self-interpreter for F-omega*, in: *Principles of Programming Languages (POPL'16)* (2016), pp. 5–17.  
<http://doi.org/10.1145/2837614.2837623>
- [14] de Bruijn, N. G., *Lambda calculus notation with nameless dummies, a tool for automatic formula manipulation, with application to the Church-Rosser theorem*, Indagationes Mathematicae (Proceedings) **75** (1972), pp. 381–392.  
[http://doi.org/10.1016/1385-7258\(72\)90034-0](http://doi.org/10.1016/1385-7258(72)90034-0)
- [15] Chapman, J. M., “Type checking and normalisation,” Ph.D. thesis, University of Nottingham (2009).  
<http://jmchapman.github.io/papers/thesis.pdf>
- [16] Coquand, C., *From semantics to rules: a machine assisted analysis*, in: *Computer Science Logic (CSL'93)*, LNCS **832** (1993), pp. 91–105.  
<http://doi.org/10.1007/BFb0049326>
- [17] Coquand, C., *A formalised proof of the soundness and completeness of a simply typed lambda-calculus with explicit substitutions*, Higher-Order and Symbolic Computation **15** (2002), pp. 57–90.  
<http://doi.org/10.1023/A:1019964114625>
- [18] Coquand, T. and P. Dybjer, *Intuitionistic model constructions and normalization proofs*, Mathematical Structures in Computer Science **7** (1997), pp. 75–94.  
<http://doi.org/10.1017/S0960129596002150>
- [19] Danvy, O., *Type-directed partial evaluation*, in: *Principles of Programming Languages (POPL'96)* (1996), pp. 242–257.  
<http://doi.org/10.1145/237721.237784>
- [20] Danvy, O., *From reduction-based to reduction-free normalization*, Electronic Notes in Theoretical Computer Science **124** (2005), pp. 79–100.  
<http://doi.org/10.1016/j.entcs.2005.01.007>

- [21] Danvy, O., C. Keller and M. Puech, *Typeful normalization by evaluation*, in: *Types for Proofs and Programs (TYPES'14)* (2014), pp. 72–88.  
<http://doi.org/10.4230/LIPICs.TYPES.2014.72>
- [22] Davies, R. and F. Pfenning, *A modal analysis of staged computation*, Journal of the ACM **48** (2001), pp. 555–604.  
<http://doi.org/10.1145/382780.382785>
- [23] Dybjer, P. and A. Filinski, *Normalization and partial evaluation*, in: *Applied Semantics (APPSEM'00)*, LNCS **2395**, Springer, 2002 pp. 137–192.  
[http://doi.org/10.1007/3-540-45699-6\\_4](http://doi.org/10.1007/3-540-45699-6_4)
- [24] Dyckhoff, R., *Some remarks on proof-theoretic semantics*, in: *Advances in Proof-Theoretic Semantics*, Trends in Logic **43**, Springer, 2016 pp. 79–93.  
[http://doi.org/10.1007/978-3-319-22686-6\\_5](http://doi.org/10.1007/978-3-319-22686-6_5)
- [25] Ewald, W. B., *Intuitionistic tense and modal logic*, Journal of Symbolic Logic **51** (1986), pp. 166–179.  
<http://doi.org/10.2307/2273953>
- [26] Fischer Servi, G., *Axiomatizations for some intuitionistic modal logics*, Rendiconti del Seminario Matematico Università Politecnico di Torino **42** (1984), pp. 179–194, to appear online.  
<http://www.seminariomatematico.unito.it/rendiconti/cartaceo/42-3.html>
- [27] Fitting, M., *The logic of proofs, semantically*, Annals of Pure and Applied Logic **132** (2005), pp. 1–25.  
<http://doi.org/10.1016/j.apal.2004.04.009>
- [28] Gabbay, M. J. and A. Nanevski, *Denotation of contextual modal type theory: syntax and meta-programming*, Journal of Applied Logic **11** (2013), pp. 1–29.  
<http://doi.org/10.1016/j.jal.2012.07.002>
- [29] Girard, J.-Y., P. Taylor and Y. Lafont, “Proofs and Types,” Cambridge University Press, 1989.  
<http://paultaylor.eu/stable/prot.pdf>
- [30] Gödel, K., *Eine Interpretation des intuitionistischen Aussagenkalküls*, in: *Kurt Gödel: Collected Works, Vol. I: Publications 1929–1936*, Oxford University Press, 1933/1986 pp. 300–303.  
<http://global.oup.com/academic/product/collected-works-9780195039641>
- [31] Iemhoff, R., “Provability logic and admissible rules,” Ph.D. thesis, University of Amsterdam (2001).  
<http://phil.uu.nl/~iemhoff/Mijn/Papers/proeve.pdf>
- [32] Ilik, D., “Constructive completeness proofs and delimited control,” Ph.D. thesis, École Polytechnique X (2010).  
<http://pastel.archives-ouvertes.fr/tel-00529021>
- [33] Ilik, D., *Continuation-passing style models complete for intuitionistic logic*, Annals of Pure and Applied Logic **164** (2013), pp. 651–662.  
<http://doi.org/10.1016/j.apal.2012.05.003>
- [34] Joachimski, F. and R. Matthes, *Short proofs of normalization for the simply-typed  $\lambda$ -calculus, permutative conversions and Gödel’s T*, Archive for Mathematical Logic **42** (2003), pp. 59–87.  
<http://doi.org/10.1007/s00153-002-0156-9>
- [35] Kovács, A., “A machine-checked correctness proof of normalization by evaluation for simply typed lambda calculus,” Master’s thesis, Eötvös Loránd University (2017).  
<http://github.com/AndrasKovacs/stlc-nbe/blob/separate-PSh/thesis.pdf>
- [36] Kripke, S. A., *Semantical analysis of intuitionistic logic I*, Studies in Logic and the Foundations of Mathematics **40** (1965), pp. 92–130.  
[http://doi.org/10.1016/S0049-237X\(08\)71685-9](http://doi.org/10.1016/S0049-237X(08)71685-9)
- [37] Lindley, S., “Normalisation by evaluation in the compilation of typed functional programming languages,” Ph.D. thesis, University of Edinburgh (2005).  
<http://www.era.lib.ed.ac.uk/handle/1842/778>
- [38] Martin-Löf, P., *An intuitionistic theory of types: predicative part*, Studies in Logic and the Foundations of Mathematics **80** (1975), pp. 73–118.  
[http://doi.org/10.1016/S0049-237X\(08\)71945-1](http://doi.org/10.1016/S0049-237X(08)71945-1)
- [39] McBride, C., *Epigram: practical programming with dependent types*, in: *Advanced Functional Programming (AFP'05)*, LNCS **832** (2005), pp. 91–105.  
[http://doi.org/10.1007/11546382\\_3](http://doi.org/10.1007/11546382_3)
- [40] McCarthy, J., P. W. Abrahams, D. J. Edwards, T. P. Hart and M. I. Levin, “LISP 1.5 Programmer’s Manual,” MIT Press, 1962.  
<http://softwarepreservation.org/projects/LISP/book/LISP%201.5%20Programmers%20Manual.pdf>

- [41] McKinsey, J. C. C. and A. Tarski, *Some theorems about the sentential calculi of Lewis and Heyting*, Journal of Symbolic Logic **13** (1948), pp. 1–15.  
<http://doi.org/10.2307/2268135>
- [42] Meyer, A. R. and M. Wand, *Continuation semantics in typed lambda-calculi*, in: *Logics of Programs (LP’85)*, LNCS **193**, Springer, 1985 pp. 219–224.  
[http://doi.org/10.1007/3-540-15648-8\\_17](http://doi.org/10.1007/3-540-15648-8_17)
- [43] Mkrtychev, A., *Models for the logic of proofs*, in: *Logical Foundations of Computer Science (LFCS’97)*, LNCS **1234**, Springer, 1997 pp. 266–275.  
[http://doi.org/10.1007/3-540-63045-7\\_27](http://doi.org/10.1007/3-540-63045-7_27)
- [44] Nanevski, A., *Meta-programming with names and necessity*, in: *International Conference on Functional Programming (ICFP’02)* (2002), pp. 206–217.  
<http://doi.org/10.1145/581478.581498>
- [45] Nanevski, A., “Functional programming with names and necessity,” Ph.D. thesis, Carnegie Mellon University (2004).  
<https://software.imdea.org/~aleks/thesis/CMU-CS-04-151.pdf>
- [46] Nanevski, A., F. Pfenning and B. Pientka, *Contextual modal type theory*, ACM Transactions on Computational Logic **9** (2008), pp. 23:1–23:49.  
<http://doi.org/10.1145/1352582.1352591>
- [47] Norell, U., “Towards a practical programming language based on dependent type theory,” Ph.D. thesis, Chalmers University of Technology (2007).  
<http://www.cse.chalmers.se/~ulfn/papers/thesis.pdf>
- [48] Ono, H., *On some intuitionistic modal logics*, Publications of the Research Institute for Mathematical Sciences **13** (1977), pp. 687–722.  
<http://doi.org/10.2977/prims/1195189604>
- [49] Pfenning, F. and R. Davies, *A judgmental reconstruction of modal logic*, Mathematical Structures in Computer Science **11** (2001), pp. 511–540.  
<http://doi.org/10.1017/S0960129501003322>
- [50] Pientka, B. and A. Abel, *Well-founded recursion over contextual objects*, in: *Typed Lambda Calculi and Applications (TLCA’15)*, 2015, pp. 273–287.  
<http://doi.org/10.4230/LIPIcs.TLCA.2015.273>
- [51] Plotkin, G. D. and C. Stirling, *A framework for intuitionistic modal logics*, in: *Theoretical Aspects of Reasoning about Knowledge (TARK’86)* (1986), pp. 399–406.  
[http://tark.org/proceedings/tark\\_mar19\\_86/p399-plotkin.pdf](http://tark.org/proceedings/tark_mar19_86/p399-plotkin.pdf)
- [52] Sheard, T., *Accomplishments and research challenges in meta-programming*, in: *Semantics, Applications, and Implementation of Program Generation (SAIG’01)*, LNCS **2196**, Springer, 2001 pp. 2–44.  
[http://doi.org/10.1007/3-540-44806-3\\_2](http://doi.org/10.1007/3-540-44806-3_2)
- [53] Simpson, A., “The proof theory and semantics of intuitionistic modal logic,” Ph.D. thesis, University of Edinburgh (1994).  
<http://homepages.inf.ed.ac.uk/als/Research/thesis.pdf>
- [54] Stump, A., *Intuitionistic logic and Kripke semantics*, in: *Verified Functional Programming in Agda*, ACM and Morgan & Claypool, 2016 pp. 215–246.  
<http://doi.org/10.1145/2841316>
- [55] Turner, D. A., *Total functional programming*, Journal of Universal Computer Science **10** (2004), pp. 751–768.  
<http://doi.org/10.3217/jucs-010-07-0751>
- [56] Wadler, P., *Propositions as types*, Communications of the ACM **58** (2015), pp. 75–84.  
<http://doi.org/10.1145/2699407>
- [57] Wickline, P., P. Lee and F. Pfenning, *Run-time code generation and Modal-ML*, in: *Programming Language Design and Implementation (PLDI’98)* (1998), pp. 224–235.  
<http://doi.org/10.1145/277650.277727>
- [58] Wijesekera, D., *Constructive modal logics I*, Annals of Pure and Applied Logic **50** (1990), pp. 271–301.  
[http://doi.org/10.1016/0168-0072\(90\)90059-B](http://doi.org/10.1016/0168-0072(90)90059-B)