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THEORETICAL PEARLS

Self-interpretation in lambda calculus

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This editorial emphasizes results from the theory behind functional programming (including lambda calculus, type theory and term-rewriting systems) that are particularly beautiful, and which have short and elegant proofs. Readers are encouraged to send comments or contributions to:

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Programming languages which are capable of interpreting themselves have been fascinating computer scientists. Indeed, if this is possible then a 'strange loop' (in the sense of Hofstadter, 1979) is involved. Nevertheless, the phenomenon is a direct consequence of the existence of universal languages. Indeed, if all computable functions can be captured by a language, then so can the particular job of interpreting the code of a program of that language. Self-interpretation will be shown here to be possible in lambda calculus.

The set of λ-*terms*, notation Λ, is defined by the following abstract syntax

$$\Lambda = V|\Lambda\Lambda|\lambda V.\Lambda$$

where

$$V = v|V'$$

is the set $\{v, v', v'', v''', \ldots\}$ of *variables*. Arbitrary variables are usually denoted by x, y, z, ... and λ-terms by M, N, L, ... A *redex* is a λ-term of the form

$$(\lambda x.M)N$$

and has as *contractum*

$$M[x := N],$$

that is, the result of substituting N for (the free occurrences of) x in M. Stylistically, it can be said that λ-terms represent functional programs including their input. A *reduction machine* executes such terms by trying to reduce them to normal form; that is, redexes are continuously replaced by their contracta until hopefully no more redexes are present. If such a normal form can be reached, then this is the output of the functional program; otherwise, the program diverges.

From the point of view of a reduction machine, a λ-term M can be considered as an executable. It 'itches' at many places: all redexes want to be reduced.

1 Definition

(i) Each λ-term M has a unique natural number $\#M$ as code. One way of coding is

$$\#v^{(i)} = \langle 0, i \rangle,$$
$$\#(MN) = \langle 1, \langle \#M, \#N \rangle\rangle,$$
$$\#(\lambda x . M) = \langle 2, \langle \#x, \#M \rangle\rangle,$$

where $\langle -, - \rangle$ is some effective coding of pairs of numbers as a single number, for example,

$$\langle n, m \rangle = \tfrac{1}{2}(n+m)(n+m+1)+m.$$

(ii) Let $\ulcorner 0 \urcorner, \ulcorner 1 \urcorner, \ulcorner 2 \urcorner, \ldots$ be some set of numerals (λ-terms representing the natural numbers). We take the *Church numerals* $\ulcorner n \urcorner \equiv \lambda fx . f^n(x)$.

Write $\ulcorner M \urcorner \equiv \ulcorner \#M \urcorner$, the *internal* λ*-code* of M. Now $\ulcorner M \urcorner$ does not itch: being a Church numeral it is in normal form.

Write FV(M) for the set of free variables of M. A λ-term M is *closed* if $FV(M) = \varnothing$. the set of closed λ-terms is denoted by Λ^0.

2 Definition

(i) An *interpreter* (or *evaluator*) is an (external) function $E: \Lambda \to \Lambda$ such that

$$E(\ulcorner M \urcorner) \equiv M.$$

(ii) A *self-interpreter* is a λ-term E such that for $M \in \Lambda^0$ one has

$$\mathsf{E}\ulcorner M \urcorner =_\beta M. \tag{1}$$

Here $=_\beta$ (or simply $=$) denotes convertibility between elements of Λ.

3 Remarks

(i) Equation (1) cannot hold for open terms containing free variables. Indeed, E has at most a finite number of free variables and $\ulcorner M \urcorner$ being a numeral has none, but on the right-hand side M may have arbitrarily many free variables.

(ii) Define the *quote* to be the function $Q: \Lambda \to \Lambda$ such that

$$Q(M) \equiv \ulcorner M \urcorner.$$

A *self-quote* is a λ-term Q such that (say for closed terms M)

$$\mathsf{Q}M =_\beta \ulcorner M \urcorner.$$

Such a self-quote does not exist, however. Indeed, the existence of Q implies

$$\ulcorner \mathsf{II} \urcorner =_\beta \mathsf{Q}(\mathsf{II}) =_\beta \mathsf{QI} =_\beta \ulcorner \mathsf{I} \urcorner.$$

Since numerals are in β-normal form it follows that $\ulcorner \mathsf{II} \urcorner \equiv \ulcorner \mathsf{I} \urcorner$, so $\#(\mathsf{II}) = \#\mathsf{I}$. However, II and I are different terms, and so have different codes, which is a contradiction.

Kleene already in (1936) showed that there is a self-interpreter E for the lambda calculus. One would think that E is defined by recursion on the structure of its argument. There is, however, a difficulty: closed terms are not built up inductively, but formed as a subset of the wider class of open terms. Kleene avoided this problem by building up closed terms from combinators S, K and I (actually, he worked with the λI-calculus and used combinators like S, B, C and I). The construction was as follows

$$\ulcorner M \urcorner \xrightarrow{\text{CL}} \ulcorner M_{\text{CL}} \urcorner \xrightarrow{\mathsf{E}_{\text{CL}}} M_{\text{CL}} =_\beta M$$

where CL is a compiler from λ-terms to combinatory terms, and E_{CL} is an interpreter for combinatory terms. The translation CL gives, for example,

$$(\lambda z.zz)_{\text{CL}} \equiv \mathsf{SII}\ (=_\beta \lambda z.\mathsf{I}z(\mathsf{I}z) =_\beta \lambda z.zz).$$

P. de Bruin (my former student) gave an essentially simpler construction of a self-interpreter for the λ-calculus. He used an idea from denotational semantics. In the following construction, F plays the role of an environment in the sense that it determines the values of the free variables.

4 Theorem (Kleene, 1936)
There exists a self-interpreter E for the lambda calculus.

Proof (de Bruin)
By the representability of computable functions there is a term E_0 such that

$$\begin{aligned} \mathsf{E}_0 \ulcorner x \urcorner F &=_\beta F \ulcorner x \urcorner, \\ \mathsf{E}_0 \ulcorner MN \urcorner F &=_\beta F(\mathsf{E}_0 \ulcorner M \urcorner F)(\mathsf{E}_0 \ulcorner N \urcorner F), \\ \mathsf{E}_0 \ulcorner \lambda x.M \urcorner F &=_\beta \lambda x.(\mathsf{E}_0 \ulcorner M \urcorner F_{[\ulcorner x \urcorner \mapsto x]}), \end{aligned}$$

where $F_{[\ulcorner x \urcorner \mapsto x]} = F'x$, with

$$\begin{aligned} F'x\ulcorner x \urcorner &=_\beta x, \\ F'x\ulcorner y \urcorner &=_\beta F\ulcorner y \urcorner, \quad \text{if } y \not\equiv x. \end{aligned}$$

By induction on the structure of $M \in \Lambda$ it can be shown that

$$\mathsf{E}_0 \ulcorner M \urcorner F =_\beta M[x_1 := F\ulcorner x_1 \urcorner, \ldots, x_n := F\ulcorner x_n \urcorner] \tag{2}$$

(simultaneous substitution), where $\{x_1, \ldots, x_n\} = \text{FV}(M)$. Now we can take

$$\mathsf{E} \equiv \lambda m.\mathsf{E}_0 m\mathsf{I}.$$

Indeed, for closed M it follows by equation (2) that

$$\mathsf{E}\ulcorner M \urcorner =_\beta \mathsf{E}_0 \ulcorner M \urcorner \mathsf{I} =_\beta M. \qquad \square$$

Using the self-interpreter E it can be shown that certain λ-terms exist without giving details. We first introduce some λ-terms inspired by the language LISP.

5 Definition

$$\begin{aligned} \mathsf{cons} &\equiv \lambda xyz.zxy; \\ \mathsf{nil} &\equiv \lambda xyz.y; \\ \mathsf{null} &\equiv \lambda x.x(\lambda abcd.d). \end{aligned}$$

6 Proposition

(i) null nil $= \lambda xy.x \equiv$ true;
(ii) null (cons a b) $= \lambda xy.y \equiv$ false.
(iii) Moreover, there exist terms car and cdr such that

$$\text{car (cons a b)} = a;$$
$$\text{cdr (cons a b)} = b.$$

Proof

(i), (ii). Easy. (iii) Take car $\equiv \lambda x.x(\lambda ab.a)$ and cdr $\equiv \lambda x.x(\lambda ab.b)$. □

7 Notation

Write

$$a:b \equiv \text{cons a b};$$
$$\langle\,\rangle \equiv \text{nil};$$
$$\langle x_1, \ldots, x_{n+1}\rangle \equiv x_1 : \langle x_2, \ldots, x_{n+1}\rangle.$$

For example, $\langle a, b\rangle \equiv a:b:\text{nil} \equiv$ (cons a (cons b nil)).

The following problem was raised by Dr Wim Vree of the University of Amsterdam.

8 Problem

Does there exist a λ-term F such that for all $n \in \mathbb{N}$ one has

$$F\ulcorner n\urcorner = \lambda x_1 \ldots x_n . \langle x_1, \ldots, x_n\rangle ? \tag{3}$$

Solution

Write $M_n \equiv \lambda x_1 \ldots x_n . \langle x_1, \ldots, x_n\rangle$. Clearly, $\# M_n$ is computable from n, say $\# M_n = g(n)$ with g recursive. Let g be λ-defined by G, say. Then

$$G\ulcorner n\urcorner = \ulcorner g(n)\urcorner = \ulcorner M_n\urcorner.$$

Then $F = \lambda n.\mathsf{E}(Gn)$ satisfies equation (3)

$$F\ulcorner n\urcorner = \mathsf{E}(G\ulcorner n\urcorner) = \mathsf{E}\ulcorner M_n\urcorner = M_n.$$ □

At first, Vree thought the answer to Problem 8 was negative. After seeing the positive answer, he came up with a more constructive solution.

Constructive solution

One can find a λ-term rev such that for all n

$$\text{rev}\,\langle x_1, \ldots, x_n\rangle = \langle x_n, \ldots, x_1\rangle.$$

(For example, $\qquad \text{rev} = \lambda L_1.\text{rev}' L_1 \langle\,\rangle,$

with $\qquad \text{rev}'(a:b)\,L_2 = \text{rev}'\,b\,(a:L_2)$
$\qquad\qquad \text{rev}'\,\text{nil}\,L_2 = L_2.$

So take $\text{rev}' \equiv Y(\lambda r L_1 L_2.\,\text{if}[\text{null}\,L_1]\,\text{then}\,[L_2]\,\text{else}\,[r\,(\text{cdr}\,L_1)\,((\text{car}\,L_1):L_2)])$, where Y is the fixed point combinator and if X then Y else Z is simply XYZ.)

Construct a λ-term V such that

$$\begin{aligned} V\ulcorner n+1\urcorner &= \lambda Lx.(V\ulcorner n\urcorner(x:L)), \\ V\ulcorner 0\urcorner &= \mathsf{rev}. \end{aligned}$$

(For example, $V = Y(\lambda vn.\,\mathsf{if}[\mathsf{zero?}\,n]\,\mathsf{then}\ \ \mathsf{rev}\ \ \mathsf{else}\ \ [\lambda Lx.\,v(\mathsf{pred}\,n)(x:L)])$, where pred represents the predecessor function.)

Then $F = \lambda n.\,V\,n\,\mathsf{nil}$ satisfies equation (3). Indeed

$$\begin{aligned} F\ulcorner n\urcorner x_1 \ldots x_n &= V\ulcorner n\urcorner\,\mathsf{nil}\,x_1 \ldots x_n \\ &= (\lambda Lx.\,V\ulcorner n-1\urcorner(x:L))\,\mathsf{nil}\,x_1 \ldots x_n \\ &= V\ulcorner n-1\urcorner(x_1:\mathsf{nil})\,x_2 \ldots x_n \\ &= (\lambda Lx.\,V\ulcorner n-2\urcorner(x:L))\,(x_1:\mathsf{nil})\,x_2 \ldots x_n \\ &= V\ulcorner n-2\urcorner(x_2:x_1:\mathsf{nil}))\,x_3 \ldots x_n \\ &\ldots \\ &= V\ulcorner 0\urcorner(x_n:\ldots:x_1:\mathsf{nil}) \\ &= \mathsf{rev}\,\langle x_n, \ldots, x_1\rangle \\ &= \langle x_1, \ldots, x_n\rangle. \end{aligned}$$

□

A concrete λ-term satisfying equation (3) is the following

$$\begin{aligned} F \equiv \lambda n.&(\lambda ab.\,b(aab))(\lambda ab.\,b(aab)) \\ &(\lambda vn.[n(\lambda xyz.\,y)(\lambda xy.\,x)] \\ &\quad[\lambda L.(\lambda ab.\,b(aab))(\lambda ab.\,b(aab)) \\ &\qquad(\lambda rL_1L_2.[L_1(\lambda abcd.\,d)][L_2][r(L_1(\lambda ab.\,b))(\lambda z.\,z(L_1(\lambda ab.\,a))L_2)]) \\ &\qquad L(\lambda xyz.\,y)] \\ &\quad[\lambda Lx.\,v(\lambda yz.\,n(\lambda pq.\,q(py))(\lambda w.\,z)(\lambda t.\,t))(\lambda z.\,zxL)]) \\ &n(\lambda xyz.\,y). \end{aligned}$$

9 Exercises

(i) Show that there is no λ-term G such that for all $n \in \mathbb{N}$ one has

$$Gx_1 \ldots x_n = \langle x_1, \ldots, x_n\rangle. \tag{4}$$

(ii) Construct a λ-term H such that for all $n \in \mathbb{N}$ one has

$$H\ulcorner n\urcorner x_1 \ldots x_n = \lambda z.\,zx_1 \ldots x_n. \tag{5}$$

References

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Kleene, S. C. 1936. λ-definability and recursiveness. *Duke Math. J.* 2, 340–353.