1 Derivation of Point-to-Plane Minimization

Consider the Chen-Medioni (point-to-plane) framework for ICP. Assume we have a collection of points (p_i, q_i) with normals n_i. We want to determine the optimal rotation and translation to be applied to the first collection of points (i.e., the p_i) to bring them into alignment with the second (i.e., the q_i). Thus, we want to minimize the alignment error

$$\mathcal{E} = \sum_i \left[(Rp_i + t - q_i) \cdot n_i \right]^2 \tag{1}$$

with respect to the rotation R and translation t.

The rotation is a nonlinear function, incorporating sines and cosines of the rotation angles. If, however, we assume that incremental rotations will be small, it is possible to linearize the rotations, approximating $\cos\theta$ by 1 and $\sin\theta$ by θ. For example, in the case of rotation in x,

$$\begin{aligned} R_{x,\alpha} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix} \\ &\approx \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -\alpha \\ 0 & \alpha & 1 \end{pmatrix}. \end{aligned}$$

Thus, the full rotation may be approximated as

$$R \approx \begin{pmatrix} 1 & -\gamma & \beta \\ \gamma & 1 & -\alpha \\ -\beta & \alpha & 1 \end{pmatrix}, \tag{2}$$

for rotations α, β, and γ around the x, y, and z axes, respectively.

Substituting Equation (2) into (1) we obtain

$$\begin{aligned} \mathcal{E} = \sum_i \big[& (p_{i,x} - \gamma p_{i,y} + \beta p_{i,z} + t_x - q_{i,x}) n_{i,x} + \\ & (\gamma p_{i,x} + p_{i,y} - \alpha p_{i,z} + t_y - q_{i,y}) n_{i,y} + \\ & (-\beta p_{i,x} + \alpha p_{i,y} + p_{i,z} + t_z - q_{i,z}) n_{i,z} \ \big]^2, \end{aligned}$$

which may be rewritten as

$$\begin{aligned} \mathcal{E} = \sum_i \big[& (p_i - q_i) \cdot n_i + t \cdot n_i + \\ & \alpha (p_{i,y} n_{i,z} - p_{i,z} n_{i,y}) + \\ & \beta (p_{i,z} n_{i,x} - p_{i,x} n_{i,z}) + \\ & \gamma (p_{i,x} n_{i,y} - p_{i,y} n_{i,x}) \ \big]^2. \end{aligned}$$

Defining

$$c = p \times n$$

and

$$r = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix},$$

the alignment error may be written as

$$\mathcal{E} = \sum_i \left[(p_i - q_i) \cdot n_i + t \cdot n_i + r \cdot c_i\right]^2.$$

We now minimize $\mathcal{E}$ with respect to α, β, γ, t_x, t_y, and t_z by setting the partial derivatives to zero:

$$\begin{aligned}
\frac{\partial \mathcal{E}}{\partial \alpha} &= \sum_i 2\, c_{i,x} \left[(p_i - q_i) \cdot n_i + t \cdot n_i + r \cdot c_i\right] &= 0 \\
\frac{\partial \mathcal{E}}{\partial \beta} &= \sum_i 2\, c_{i,y} \left[(p_i - q_i) \cdot n_i + t \cdot n_i + r \cdot c_i\right] &= 0 \\
\frac{\partial \mathcal{E}}{\partial \gamma} &= \sum_i 2\, c_{i,z} \left[(p_i - q_i) \cdot n_i + t \cdot n_i + r \cdot c_i\right] &= 0 \\
\frac{\partial \mathcal{E}}{\partial t_x} &= \sum_i 2\, n_{i,x} \left[(p_i - q_i) \cdot n_i + t \cdot n_i + r \cdot c_i\right] &= 0 \\
\frac{\partial \mathcal{E}}{\partial t_y} &= \sum_i 2\, n_{i,y} \left[(p_i - q_i) \cdot n_i + t \cdot n_i + r \cdot c_i\right] &= 0 \\
\frac{\partial \mathcal{E}}{\partial t_z} &= \sum_i 2\, n_{i,z} \left[(p_i - q_i) \cdot n_i + t \cdot n_i + r \cdot c_i\right] &= 0
\end{aligned}$$

These equations may be collected and written in matrix form:

$$\sum_i \begin{pmatrix}
c_{i,x}c_{i,x} & c_{i,x}c_{i,y} & c_{i,x}c_{i,z} & c_{i,x}n_{i,x} & c_{i,x}n_{i,y} & c_{i,x}n_{i,z} \\
c_{i,y}c_{i,x} & c_{i,y}c_{i,y} & c_{i,y}c_{i,z} & c_{i,y}n_{i,x} & c_{i,y}n_{i,y} & c_{i,y}n_{i,z} \\
c_{i,z}c_{i,x} & c_{i,z}c_{i,y} & c_{i,z}c_{i,z} & c_{i,z}n_{i,x} & c_{i,z}n_{i,y} & c_{i,z}n_{i,z} \\
n_{i,x}c_{i,x} & n_{i,x}c_{i,y} & n_{i,x}c_{i,z} & n_{i,x}n_{i,x} & n_{i,x}n_{i,y} & n_{i,x}n_{i,z} \\
n_{i,y}c_{i,x} & n_{i,y}c_{i,y} & n_{i,y}c_{i,z} & n_{i,y}n_{i,x} & n_{i,y}n_{i,y} & n_{i,y}n_{i,z} \\
n_{i,z}c_{i,x} & n_{i,z}c_{i,y} & n_{i,z}c_{i,z} & n_{i,z}n_{i,x} & n_{i,z}n_{i,y} & n_{i,z}n_{i,z}
\end{pmatrix}
\begin{pmatrix} \alpha \\ \beta \\ \gamma \\ t_x \\ t_y \\ t_z \end{pmatrix}
= -\sum_i \begin{pmatrix}
c_{i,x}\,(p_i - q_i) \cdot n_i \\
c_{i,y}\,(p_i - q_i) \cdot n_i \\
c_{i,z}\,(p_i - q_i) \cdot n_i \\
n_{i,x}\,(p_i - q_i) \cdot n_i \\
n_{i,y}\,(p_i - q_i) \cdot n_i \\
n_{i,z}\,(p_i - q_i) \cdot n_i
\end{pmatrix}.$$

This is a linear matrix equation of the form $Cx = b$, where C is the 6×6 "covariance matrix" accumulated from the c_i and n_i, x is a 6×1 vector of unknowns, and b is a 6×1 vector that also depends on the data points. The equation may be solved using standard methods (A is symmetric, so Cholesky decomposition is the preferred algorithm), yielding the optimal incremental rotation and translation.