Towards Strong Relative Pseudomonads

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Abstract

Synthetic measure theory uses commutative monads to develop an entirely categorical language of measures and integration. This language has proven practically useful in the development of higher-order statistical programming languages.

There is another purely categorical notion of integration: coends. Certain coends arise from the presheaf construction, a monad-like structure that fails to be a model of synthetic measure theory for several reasons. There are multiple ways in which one could attempt to fix these problems.

In this report, we make a start at defining a strong relative pseudomonad that would be suitable as the backbone of an extended synthetic measure theory. At each step, we show how the presheaf construction gives rise to the required structure and how this structure satisfies the necessary axioms.

Unfortunately, the strength of a monad does not lend itself to being generalised to relative pseudomonads. We are thus neither able to give a complete definition of a strong relative pseudomonad nor do we manage to extend synthetic measure theory to admit the presheaf construction as a model.

Research Ethics Approval

This project was planned in accordance with the Informatics Research Ethics policy. It did not involve any aspects that required approval from the Informatics Research Ethics committee.

Declaration

I declare that this thesis was composed by myself, that the work contained herein is my own except where explicitly stated otherwise in the text, and that this work has not been submitted for any other degree or professional qualification except as specified.

(*Franz Miltz*)

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Chapter 1

Introduction

1.1 Synthetic measure theory

Kock observed that it is possible to characterise measure theory abstractly by a commutative monad over a locally small cartesian closed category satisfying two further axioms [8].

For practical purposes, it makes sense to consider models that account for traditional Lebesgue integration. However, function spaces are never measurable [2] so the category of measurable spaces is not cartesian closed. This means that conventional measure theory is not a model of synthetic measure theory. It is therefore interesting to find models that are close to conventional measure theory.

Measurable function spaces are desirable for other reasons, e.g. to formalise higher-order probability theory, which is a problem that arises naturally when considering probabilisitic programming languages. It is possible to extend measurable spaces to quasi-Borel spaces in order to achieve cartesian closure [6]. While some care is required when comparing quasi-Borel spaces to measure theory and rephrasing probability theory in these new terms, they bring with them several convenient properties. In particular, quasi-Borel spaces form a model of synthetic measure theory and the associated integral is precisely the measure space integral [16].

Since synthetic measure theory turns out to be a practically useful tool, it is now also valuable to consider models that are far removed from conventional measure theory. One may think of this as testing the limits of the theory.

A classical example of a monad from functional programming is the list monad. This fails to be commutative. If we forget about the ordering of elements, we obtain the powerset monad taking sets to their powersets. This monad may be endowed with a commutative strong structure that, moreover, forms a model of synthetic measure theory. In this model, the measures over a set X are subsets $\mu \subseteq X$ and the integral of a function

$f : X \to \{0,1\}$ with respect to μ is

$$\int_X \mu(dx) f(x) = \begin{cases} 0 & \text{if } \forall x \in \mu . f(x) = 0 \\ 1 & \text{if } \exists x \in \mu . f(x) = 1 \end{cases}$$

In other words, if we consider f to be the characteristic function of a subset $B \subseteq X$ then the integral is zero if and only if $\mu \cap B = \emptyset$.

It is worth noting that Kock's synthetic measure theory is not the only attempt at formalising probability theory categorically. An alternative approach was taken by Fritz in [5] who developed Markov categories by generalising probability theory directly. This allows for more abstract reasoning, thereby improving clarity. In particular, Fritz' theory unifies different types of probability theory and does not rely on measure theoretic probability theory to the same degree.

1.2 Presheaves

Category theory has its own notion of integration: coends. A coend of a functor $\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{D}$ is a particular colimit in $\mathcal{D}$. We tend to think of coends as integrals because they are denoted by an integral symbol and because they behave like integrals.

However, as of right now there have been little to no formal developments relating coends to integration. The motivation for this project is to establish such a connection by considering a suitable model of synthetic measure theory. To construct this model, we require the following:

1. a cartesian closed category $\mathcal{C}$ and
2. a commutative monad T on $\mathcal{C}$ such that
3. the extension operation of T involves coends.

These conditions are contradictory. Coends are not unique, so defining an extension operation involves choosing particular coends. It is then not reasonable to demand that this choice is consistent in the sense that all the monad laws are satisfied up to equality. Thus we will not obtain a monad in the strict sense. Fortunately, monads have been generalised to pseudomonads to account for this problem [13].

Now the obvious candidate model is the presheaf construction, taking each small category $\mathbb{C}$ to its presheaf category $\widehat{\mathbb{C}} = [\mathbb{C}^{\mathrm{op}}, \mathbf{Set}]$. This is a great choice because a functor $\mathbb{C} \to \widehat{\mathbb{C}}$ may be thought of as a functor $\mathbb{C}^{\mathrm{op}} \times \mathbb{C} \to \mathbf{Set}$. Thus coends are closely related to presheaves. Secondly, it is known that the presheaf construction exhibits monad-like behaviour [4]. Unfortunately, the presheaf category of a small category is itself not necessarily small. Thus the presheaf construction fails to be a pseudomonad.

One way to turn the presheaf construction into a pseudomonad is by considering small presheaves [3]. This is still insufficient for our pruposes, as the functor category $[\mathbb{C}, \mathbb{D}]$ of small categories $\mathbb{C}$ and $\mathbb{D}$ is itself not small. This violates our first requirement of cartesian closure. We have tried and failed to find a cartesian closed 2-category on which the presheaf construction remains a pseudomonad.

We thus opt for relative pseudomonads [4]. This generalisation allows for the possibility that the pseudofunctor underlying a pseudomonad is itself not an endomorphism. We know that the presheaf construction is a relative pseudomonad. Before we can extend synthetic measure theory to include relative pseudomonads, we need to define the corresponding notion of strength. This turns out to be a difficult problem that we are not able to solve entirely.

1.3 Related work

There are four pieces of work that this project builds on:

- Kocks synthetic measure theory from [8] is our main motivation. It is therefore especially helpful in making sure what requirements our theory needs to satisfy and what simplifications we are able to make.
- Fiore et al. developed the theory theory of relative pseudomonads in [4]. This is our main reference point for the first part of our project.
- Paquet and Saville defined strong pseudomonads in [14]. This serves as a guide for the second part of our theory and will most likely be a valuable reference for future work.
- Uustalu outlined in [17] how to construct a strong relative monad. While this extended abstract does justify the correctness of the definition, it has allowed us to come up with a sensible structure of a strong relative pseudomonad nonetheless.

1.4 Contributions

The following developments in this report are worth noting:

- We partially define strong relative pseudomonads with a particular focus on generalising synthetic measure theory.
- We show that our definition coincides with relative pseudomonads in the sense of [4].
- We show how our definition yields the structure of a strong pseudomonad in the sense of [14].
- We demonstrate how to endow the presheaf construction with the necessary structure and verify that it satisfies the corresponding axioms.

1.5 Overview

We structure this report as follows:

- Chapter 2 revisits introductory category theory. We define commutative monads and relative monads in order to develop an intuition for later developments. We then focus our attention towards coends, getting used to the notation and proving results that will be useful later on.

- Chapter 3 rigorously introduces 2-categories, pseudofunctors, and pseudonatural transformations. We pay particular attention to the strictness requirements, making sure that our definitions are not more general than required.
- Chapter 4 consists entirely of novel developments: Step by step we work towards the definition of a strong relative pseudomonad. Along the way we show how the structures and axioms relate to previous work and to the presheaf construction.
- Chapter 5 contains a critical evaluation.
- Chapter 6 summarises the remaining work and lists possible extensions.

Chapter 2

1-categories

There are two purely categorical notions of integration that we are interested in: the integral arising from Kock's synthetic measure theory [8] and coends as an integral of certain functors.

We begin this chapter by revisiting some introductory category theory. We then proceed to define commutative monads, the backbone of synthetic measure theory. After that, we shift our attention to coends, outlining in what sense the coend of a functor may be thought of as an integral.

2.1 Categories, functors, and natural transformations

This section presents standard constructions from the literature such as [11].

Recall that a category consists of objects, morphisms between objects, and a composition operation that is associative and has identities.

Let $\mathcal{C}$ be a category. For objects $X, Y \in \mathcal{C}$ we denote by $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ the class of all morphisms $X \to Y$ and by id_X the identity $X \to X$. Moreover, we will drop the composition $\circ$ and subscripts whenever it is convenient to do so. Such notational conventions will prove particularly useful when considering larger structures later on.

One of the most important categories is **Set** whose objects are sets and whose morphisms are functions. Composition is just the usual composition of functions and the identities are the usual identity functions.

Each category $\mathcal{C}$ has a dual $\mathcal{C}^{\mathrm{op}}$ called the opposite category. The objects of $\mathcal{C}^{\mathrm{op}}$ are exactly the objects of $\mathcal{C}$ but the morphisms $Y \to X$ in $\mathcal{C}^{\mathrm{op}}$ are exactly the morphisms $X \to Y$ in $\mathcal{C}$. Composition in $\mathcal{C}^{\mathrm{op}}$ is the same as in $\mathcal{C}$.

Given two categories $\mathcal{C}$ and $\mathcal{D}$, we have the product category $\mathcal{C} \times \mathcal{D}$ which has as objects all pairs (X,Y) with $X \in \mathcal{C}$ and $Y \in \mathcal{D}$ and hom-sets $\mathrm{Hom}((X,Y),(X',Y')) = \mathrm{Hom}(X,Y) \times \mathrm{Hom}(X',Y')$. Composition is defined pointwise so the identites are $(\mathrm{id}, \mathrm{id})$.

Functors are morphisms between categories. They act on objects and morphisms in a way that preserves identities and distributes over composition. One particularly important functor is $\mathrm{Hom} : \mathbb{C}^{\mathrm{op}} \times \mathbb{C} \to \mathbf{Set}$, where $\mathbb{C}^{\mathrm{op}}$ is the opposite category. It takes a pairs of objects in a small category to its hom-set and the action on morphisms is given by composition: $\mathrm{Hom}(f,g)(h) = ghf$. We may also fix an object $X \in \mathbb{C}$ to obtain the functors

$$\mathrm{Hom}(X,-) : \mathbb{C} \to \mathbf{Set}, \qquad \mathrm{Hom}(-,X) : \mathbb{C}^{\mathrm{op}} \to \mathbf{Set}.$$

Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ be functors. The composite $GF : \mathcal{C} \to \mathcal{E}$ is defined as $GF = G(F-)$. We thus have the category **CAT** of all categories and all functors between them and its full subcategory **Cat** of small categories.

Let $F, G : \mathcal{C} \to \mathcal{D}$ be parallel functors. A natural transformation $\phi : F \Rightarrow G$ consists of morphisms $\phi_X : FX \to GX$ in $\mathcal{D}$, for all $X \in \mathcal{C}$, that satisfy the naturality condition: for all $f : X \to Y$ in $\mathcal{C}$, $\phi_Y \circ Ff = Gf \circ \phi_X$.

We can compose natural transformations $F \Rightarrow G$ and $G \Rightarrow H$ by composing the components. Thus, for all categories $\mathcal{C}$ and $\mathcal{D}$, we have the functor category $[\mathcal{C}, \mathcal{D}]$ with functors $\mathcal{C} \to \mathcal{D}$ as objects and natural transformations as morphisms.

Consider functors $F, G : \mathcal{C} \to [\mathcal{D}, \mathcal{E}]$. A natural transformation $\phi : F \Rightarrow G$ has components $\phi_C : FC \to GC$ in $[\mathcal{D}, \mathcal{E}]$. This means that each ϕ_C is itself a natural transformation with components $(\phi_C)_D : (FC)D \to (GC)D$ in $\mathcal{E}$. Such functors and natural transformations will play a major role in later chapters. We will make sure to keep our notation precise to assist the reader in peeling back the various layers of indirection.

2.2 Cartesian structure

Now recall that the product of objects $X, Y \in \mathcal{C}$ is an object $X \times Y$ together with morphisms $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$. It is universal in the sense that, for all other pairs of morphisms $f : W \to X$ and $g : W \to Y$, there is a unique morphism $\langle f, g \rangle : W \to X \times Y$ such that the following commutes:

$$\begin{tikzcd} & W \arrow[dl, "f"'] \arrow[d, dashed, "\langle f,g \rangle"] \arrow[dr, "g"] & \\ X & X \times Y \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & Y \end{tikzcd} \tag{2.1}$$

We say that a diagram commutes if every two paths from the same source to the same sink are equal. We will make heavy use of commutative diagrams, both as axioms that we postulate and as statements that we prove. In the latter case, the commutativity may be left implicit. Unfortunately, diagrams tend to get much more complex in later chapters. Therefore it will not be possible to write down every proof in full detail. We will, however, occasionally include links to quiver [18], a graphical editor for commutative diagrams. For example, the diagram 2.1 corresponds to this quiver link. The purpose of these links is twofold: On the one hand they should serve as a useful guide through the less obvious proofs and on the other we hope that they may serve as resource for future reference. However, we believe that this report is entirely self-contained and therefore the reader is not required to engage with the quiver links whatsoever.

We now take the product of morphisms $f : X \to X'$ and $g : Y \to Y'$ to be the morphism $f \times g : X \times X' \to Y \times Y'$ given by $f \times g = \langle f\pi_1, g\pi_2 \rangle$. Thus we have a functor $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$. It is important to realises that, in general, products are not unique. Fortunately, they are unique *up to canonical isomorphism*. Therefore specifying such a product functor amounts to choosing a product for each pair of objects.

As the product functor depends on the choice of products, we do not expect it to be associative or commutative in the strict sense. However, there are natural isomorphisms with components

$$\alpha_{X,Y,Z} : (X \times Y) \times Z \cong X \times (Y \times Z), \qquad \gamma_{X,Y} : X \times Y \cong Y \times X. \tag{2.2}$$

It is worth pointing out that γ is its own inverse. That is, $\gamma^{-1}_{X,Y} = \gamma_{Y,X}$.

An object $1 \in \mathcal{C}$ is terminal if, for all $X \in \mathcal{C}$, there is a unique morphism $X \to 1$. It turns out that terminal objects are units for multiplication. That is, there is a natural isomorphism with components

$$\lambda_X : 1 \times X \cong X.$$

Combining this with γ from (2.2) yields the right unitor $\rho = \lambda\gamma : X \times 1 \cong X$.

To tie all of this together, a cartesian category is a category with a choice of all (binary) products and a distinguished terminal object.

Example 2.1. Two particular cartesian structures are going to be of interest to us: on **Set** and on **CAT** (and thereby **Cat**). In **Set**, we take the product of two sets to be the cartesian product with the usual projections $(x, y) \mapsto x$ and $(x, y) \mapsto y$. The terminal object $1 \in$ **Set** is a distinguished singleton $1 = \{*\}$. In **CAT**, the product of categories $\mathcal{C}$ and $\mathcal{D}$ is the product category $\mathcal{C} \times \mathcal{D}$ with the obvious projection functors and the terminal category $\mathbb{1} \in$ **CAT** is a distinguished category with a single object and a single (identity) morphism. Note that, for small categories $\mathbb{C}$ and $\mathbb{D}$, $\mathbb{C} \times \mathbb{D}$ is small so **Cat** inherits the cartesian structure.

2.3 Monads

Monads are central to the study of category theory. They arise naturally in practical settings, e.g. to model computation with side effects, as well as for theoretical purposes, e.g. monad algebras as a generalisation of algebraic theories.

We adapt the no-iteration definition of a monad [12] because it more easily generalises to relative monads [1].

Definition 2.2. Let $\mathcal{C}$ be a category. A *monad T on $\mathcal{C}$* consists of

1. for all $X \in \mathcal{C}$, an object $TX \in \mathcal{C}$;
2. for all $X \in \mathcal{C}$, a morphism $\eta_X : X \to TX$ in $\mathcal{C}$;
3. for all $X, Y \in \mathcal{C}$, a map

$$(-)^*_{X,Y} : \mathrm{Hom}\,(X, TY) \to \mathrm{Hom}\,(TX, TY)$$

such that, for all $X, Y, Z \in \mathcal{C}$, $f : X \to TY$, and $g : Y \to TZ$, the following commute:

$$\begin{tikzcd} TX \arrow[r, bend left, equal] \arrow[r, bend right, "\eta^*"'] & TX \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "\eta"] \arrow[dr, "f"'] & TY \arrow[d, "f^*"] \\ & TY \end{tikzcd} \qquad \begin{tikzcd} TX \arrow[r, "f^*"] \arrow[dr, "(g^* f)^*"'] & TY \arrow[d, "g^*"] \\ & TZ \end{tikzcd} \tag{2.3}$$

The double line in the first diagram refers to the identity $TX \to TX$.

Let us now investigate the powerset monad which we touched on in chapter 1.

Example 2.3. The powerset construction $\mathcal{P}$ takes each set X to its powerset $\mathcal{P}X$. We add a monad structure like so: the unit map takes elements to singletons, i.e. $\eta\,(x) = \{x\}$. The extension of a function $f : X \to \mathcal{P}Y$ takes $A \in \mathcal{P}X$ to the union $f^T\,(A) = \bigcup \{f\,(x) : x \in A\}$. This is known as the powerset monad.

Every monad T on $\mathcal{C}$ gives rise to an endofunctor $\mathcal{C} \to \mathcal{C}$ with object map $X \mapsto TX$ and morphism map $f \mapsto (\eta f)^T$. We abuse notation and denote this endofunctor simply by T and write $Tf : TX \to TY$, as usual. In the case of the powerset monad, the functorial action takes a function $f : X \to Y$ to the direct image map $A \mapsto \{f\,(x) : x \in A\}$.

2.4 Commutative monads

Given that a monad is just a monoid in the category of endofunctors [11], one might expect that the commutativity of a monad is related to the commutativity of the corresponding monoid. This is not the case. The commutativity required for synthetic measure theory refers to the strength of a monad. For our purposes, a monad is strong if it interacts well with the cartesian structure of the corresponding category.

Fix a cartesian category $\mathcal{C}$. A strength σ for a monad T on $\mathcal{C}$ is a natural transformation with components

$$\sigma_{X,Y} : X \times TY \to T\,(X \times Y)$$

following certain conditions. A strong monad is a monad T equipped with a strength σ.

Such a strong monad is then called commutative [7], if the costrength

$$\tau_{X,Y} : TX \times Y \to T(X \times Y)$$

given by the composite

$$\begin{tikzcd} TX \times Y \arrow[r, "\tau"] \arrow[d, "\gamma"'] & T(X \times Y) \\ Y \times TX \arrow[r, "\sigma"'] & T(Y \times X) \arrow[u, "T\gamma"'] \end{tikzcd} \tag{2.4}$$

makes the diagram below commute:

$$\begin{tikzcd} TX \times TY \arrow[r, "\sigma"] \arrow[d, "\tau"'] & T(TX \times Y) \arrow[r, "T\tau"] & T^2(X \times Y) \arrow[d, "\mathrm{id}^*"] \\ T(X \times TY) \arrow[r, "T\sigma"'] & T^2(X \times Y) \arrow[r, "\mathrm{id}^*"'] & T(X \times Y) \end{tikzcd} \tag{2.5}$$

Example 2.4. In the case of **Set**, the product is the usual cartesian product of sets. The powerset monad is a commutative monad with strength and costrength given by $\sigma(x,B) = \{x\} \times B$ and $\tau(A,y) = A \times \{y\}$, respectively.

2.5 Relative monads

This section closely follows [1].

In section 2.3, we chose to present the no-iteration definition of a monad because it avoids repeated applications of the object map. This now leads to an obvious generalisation of monads that are functors $\mathcal{J} \to \mathcal{C}$ rather than endofunctors. We require another functor $\mathcal{J} \to \mathcal{C}$ to relate objects in $\mathcal{J}$ to those in the image of the relative monad. The resulting definition is remarkably similar to 2.2.

Definition 2.5. Let $J : \mathcal{J} \to \mathcal{C}$ be a functor. A *relative monad T over J* consists of

1. for all $X \in \mathcal{J}$, an object $TX \in \mathcal{C}$;
2. for all $X \in \mathcal{J}$, a morphism $\eta_X : JX \to TX$ in $\mathcal{C}$;
3. for all $X, Y \in \mathcal{J}$, a map

$$(-)^*_{X,Y} : \mathrm{Hom}(JX, TY) \to \mathrm{Hom}(TX, TY)$$

such that, for all $X, Y, Z \in \mathcal{J}$, $f : JX \to TY$, and $g : JY \to TZ$, the following commute:

$$\begin{tikzcd} TX \arrow[r, bend left, equal] \arrow[r, bend right, "\eta^*"'] & TX \end{tikzcd} \qquad \begin{tikzcd} JX \arrow[r, "\eta"] \arrow[dr, "f"'] & TY \arrow[d, "f^*"] \\ & TY \end{tikzcd} \qquad \begin{tikzcd} TX \arrow[r, "f^*"] \arrow[dr, "(g^* f)^*"'] & TY \arrow[d, "g^*"] \\ & TZ \end{tikzcd} \tag{2.6}$$

Example 2.6. Let T be a monad on $\mathcal{C}$. Then T is a relative monad over the identity $\mathcal{C} \to \mathcal{C}$. Moreover, for any functor $J : \mathcal{J} \to \mathcal{C}$, we have a relative monad T' over J with object map $T'X = TJX$.

Our motivation for presenting this definition is the presheaf construction. It behaves like a relative monad **Cat** $\to$ **CAT** but fails to satisfy the axioms in the strict sense because coends, which we shall introduce next, are unique only up to isomorphism.

While our definition of a monad is easy to generalise to the relative case, the same cannot be said for commutative monads. The axioms of strong and commutative monads involve repeated applications of the object map (see 2.5). We therefore have to find alternative conditions to impose. Fortunately, [17] already contains the definition of a strong relative moand that we will make use of. However, we will encounter similar problems when identifying suitable coherence conditions in chapter 4.

2.6 Coends

Let us now turn our attention to the categorical integral that we are hoping to capture in a model of synthetic measure theory. There are two dual notions of an integral of a functor $\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{D}$: ends and coends. We will focus on the latter because coends arise naturally when considering the Yoneda embedding, one of the most fundamental constructions in category theory.

Rather than defining coends directly, we begin by defining a more general structure:

Definition 2.7. Let $F : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{D}$ be a functor. A *cowedge of F* consists of

1. an object $W \in \mathcal{D}$;
2. for all $X \in \mathcal{C}$, a morphism $w_X : F(X,X) \to W$

such that, for all $f : X \to Y$ in $\mathcal{C}$,

$$\begin{tikzcd} F(Y,X) \arrow[r, "F(Y,f)"] \arrow[d, "F(f,X)"'] & F(Y,Y) \arrow[d, "w_Y"] \\ F(X,X) \arrow[r, "w_X"'] & W \end{tikzcd} \tag{2.7}$$

Let (W, w_X) and (W', w'_X) be cowedges of F. A morphism $h : W \to W'$ is a morphism of cowedges $(W, w_X) \to (W', w'_X)$ if, for all $X \in \mathcal{C}$, $hw_X = w'_X$.

Examples of cowedges that are both interesting and specific are hard to come by. What the following lacks in specificity it makes up for in importance.

Example 2.8. Let $P : \mathcal{C}^{\mathrm{op}} \to$ **Set** be a functor and $W \in \mathcal{C}$ an object. Consider the functor $\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to$ **Set** given by

$$Y, X \mapsto PY \times \mathrm{Hom}(W, X).$$

and, for each $X \in \mathcal{C}$, the function

$$w_X : PX \times \mathrm{Hom}(W,X) \to PW$$

given by $x, f \mapsto (Pf)(x)$. Now fix $f : W \to X$, $g : X \to Y$, and $y \in PY$. Then

$$w(y, gf) = P(gf)(y) = PfPg(y) = w(Pg(y), f).$$

I.e. the diagram

$$\begin{tikzcd}
PY \times \mathrm{Hom}(W,X) \arrow[r, "Pg\times\mathrm{Hom}(W,X)"] \arrow[d, "PY\times\mathrm{Hom}(W,g)"'] & PX \times \mathrm{Hom}(W,X) \arrow[d, "w_Z"] \\
PY \times \mathrm{Hom}(W,Y) \arrow[r, "w_Y"'] & PW
\end{tikzcd}$$

commutes. Thus (PW, w_X) is a cowedge.

We notice that, for any cowedge (W, w_X), postcomposition with a morphism $f : W \to W'$ yields another cowedge (W', fw_X), making f into a cowedge homomorphism. Now a coend is an initial cowedge:

Definition 2.9. Let $F : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{D}$ be a functor. A *coend of F* is a cowedge (E, e_X) of F such that, for all cowedges (W, w_X) of F, there exists a unique cowedge homomorphism $(E, e_X) \to (W, w_X)$.

Coends are special colimits (see [10, Remark 1.2.3]). So, while there is no guarantee that a particular functor has a coend, any two coends of the same functor are canonically isomorphic. This allows us to fix a choice of coends without loss of generality, just as we discussed for products.

Observe a useful property of coends: Consider functors $F, G : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{D}$, a natural transformation $\phi : F \Rightarrow G$ between them, and assume that we have chosen coends of F and G. By precomposing the coend diagram of G with ϕ, we obtain a cowedge of F. Thus, for each choice of coends, there exists a unique cowedge homomorphism induced by ϕ.

Thus a choice of coends assigns objects to functors $\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{D}$ and morphisms to natural transformations between such functors. This observations suggests that one might want to think of coends as a functor $[\mathcal{C}^{\mathrm{op}} \times \mathcal{C}, \mathcal{D}] \to \mathcal{D}$. There is one problem, however: just like measure theoretic integrals, statements about coends rely on their existence in the first place. Fortunately, all coends in this thesis will be of functors of the form $\mathbb{C}^{\mathrm{op}} \times \mathbb{C} \to \mathbf{Set}$ where $\mathbb{C}$ is small. As **Set** is cocomplete, all such coends exist and are indeed functorial. It is possible to construct coends in **Set** explicitly by taking disjoint unions and quotienting them by a particular equivalence relation. Unfortunately, the resulting construction is far from being intuitive so we will not investigate it further.

Notation 2.10. The notation that arises from the formal definition of coends does not scale well to more complicated calculations. We improve it as follows:

- For a functor $F : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{D}$, we will denote the object and morphism of a chosen coend by
$$\int^C F(C,C) \qquad \text{and} \qquad q_X : F(X,X) \to \int^C F(C,C). \tag{2.8}$$
Here the variable C is bound by the integral. That is, the choice of symbol is arbitrary.
- For functors $F, G : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{D}$, a natural transformation $\phi : F \Rightarrow G$, and choices of coends, we denote the morphism of cowedges induced by ϕ by
$$\int^C \phi_{C,C} : \int^C F(C,C) \to \int^C G(C,C). \tag{2.9}$$

We already saw an example of a coend in 2.8. We establish the universality by constructing a cowedge homomorphism to the chosen coend.

Example 2.11. The cowedge in 2.8 is a coend. To see this fix a choice of coend and consider the morphism h given by the composite

$$\begin{tikzcd} PW \arrow[r, "\langle PW, \Delta\mathrm{id} \rangle"] \arrow[dr, "h"'] & PW \times \mathrm{Hom}(W,W) \arrow[d, "q"] \\ & \int^C PC \times \mathrm{Hom}(W,C) \end{tikzcd} \tag{2.10}$$

Here $\Delta\mathrm{id}$ is the constant function $x \mapsto \mathrm{id}$. Now let $x \in PX$ and $f : W \to X$. Then

$$(hw)(x,f) = h(Pf(x))a = q(Pf(x), \mathrm{id}) = q(x,f).$$

where the last equality holds due to 2.7. Thus the diagram

$$\begin{tikzcd} PX \times \mathrm{Hom}(W,X) \arrow[r, "w"] \arrow[dr, "q"'] & PW \arrow[d, "h"] \\ & \int^C PC \times \mathrm{Hom}(W,C) \end{tikzcd}$$

commutes, showing that h is a cowedge homomorphism. By universality of the coend we conclude that we have an isomorphism

$$h : PW \cong \int^C PC \times \mathrm{Hom}(W,C).$$

This is sometimes referred to as the co-Yoneda lemma.

2.7 Properties of coends

We are now in a position to make precise the functoriality of the coend:

Lemma 2.12. Let $\mathbb{C}$ be a small category. Every choice of coends for all functors $\mathbb{C}^{\mathrm{op}} \times \mathbb{C} \to \mathbf{Set}$ induces a functor $[\mathbb{C}^{\mathrm{op}} \times \mathbb{C}, \mathbf{Set}] \to \mathbf{Set}$ given by

$$F \mapsto \int^{C} F(C,C), \qquad \phi \mapsto \int^{C} \phi_{C,C}.$$

Proof. Fix $F, G, H : \mathbb{C}^{\mathrm{op}} \times \mathbb{C} \to \mathbf{Set}$. Consider natural transformations $\phi : F \Rightarrow G$ and $\psi : G \Rightarrow H$. Now

$$\left(\int^{C} \psi_{C,C}\right)\left(\int^{C} \phi_{C,C}\right) = \int^{C} (\psi\phi)_{C,C}$$

because both sides are cowedge homomorphisms

$$\int^{C} F(C,C) \to \int^{C} H(C,C).$$

By universality of the coend on the left, this is unique.

Preservation of identities follows by a similar argument. □

While it is not at all obvious in what sense coends describe integration, we observe that they tend to behave like integrals in the analytic sense. For example, we have the following result about scalar multiplication:

Lemma 2.13. Let $F : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{D}$ be a functor where $\mathcal{D}$ is cartesian, and let $D \in \mathcal{D}$. For any choice of coend of F, there is a canonical isomorphism

$$D \times \int^{C} F(C,C) \cong \int^{C} D \times F(C,C).$$

Proof. We have a functor $D \times - : \mathcal{D} \to \mathcal{D}$ with inverse $\pi_2 \circ -$. In particular, $D \times -$ is cocontinuous. By [10, Theorem 1.2.7], we have

$$(D \times -)\left(\int^{C} F(C,C)\right) \cong \int^{C} (D \times -)F(C,C),$$

as required. □

Due to the canonical isomorphism of products $A \times B \cong B \times A$, we immediately have a similar statement for scalar multiplication on the right.

The final property that relates coends to conventional integrals is the Fubini rule:

Theorem 2.14 ([10, Theorem 1.3.1])**.** Let $F : \mathcal{B}^{\mathrm{op}} \times \mathcal{B} \times \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{D}$ be a functor. Then there are canonical isomorphisms

$$\int^{B}\int^{C} F(B,B,C,C) \cong \int^{B,C} F(B,B,C,C) \cong \int^{C}\int^{B} F(B,B,C,C).$$

That is, if one of the coends above exists, so do the other and there are unique isomorphisms of cowedges between them.

Chapter 3

2-categories

To account for the possibility of axioms holding only up to a specified isomorphism, we need to consider 2-categories. We avoid confusion by referring to the usual categories as 1-categories. Since 2-categorical notions tend to be straightforward generalisations of their 1-categorical counterparts, many of the developments in this chapter are going to seem unremarkable. The difficulty lies in choosing which axioms we allow to hold only up to a coherent isomorphism. This type of problem rarely arises in 1-category theory but is a central to the study of 2-categories.

Notation 3.1. It will be convenient to fix some notational conventions for the remainder of this report. Unless otherwise indicated,

- curly upper-case letters denote 1-categories, e.g. $\mathcal{C}$;
- blackboard bold upper-case letters denote small 1-categories, e.g. $\mathbb{C}$;
- plain upper-case letters denote objects, e.g. X.

3.1 Definition

The idea of a 2-category is straightforward. In mathematics, it has proven useful to replace equality by different notions of isomorphisms. Commutative diagrams in 1-categories are nothing but equalities between morphisms, so we would like to study isomorphisms between morphisms instead. Thus 2-categories have morphisms between morphisms. To disambiguate we call morphisms between objects 1-cells and morphisms between morphisms 2-cells.

The concept of morphisms between morphisms should be familiar already. The 1-category **CAT** has functors as morphisms and we know that natural transformations are just morphisms between functors. We will make use of this and extend **CAT** to a 2-category **2CAT**.

While the additional structure brings with it a lot of freedom, it comes with a price. Firstly, 2-categorical concepts contain more data that needs to be specified and more axioms that need to be verified. This makes 2-category theory more complex. Secondly, due to this additional complexity, it is not always desirable to state axioms up to isomorphism. Thus, one has to make an informed choice about the degree of strictness required at each step along the way.

Even when defining a 2-category itself, strictness plays a role: It is possible to demand that the axioms hold only up to isomorphism. This consideration leads to weak 2-categories or bicategories. (see [9]) We note that the weakness of the former has nothing to do with the strength of a monad as in 2.4. We do not require the full generality of bicategories and are thus going to restrict our attention to (strict) 2-categories:

Definition 3.2. A *2-category* $\mathbf{C}$ consists of

1. a class of objects $\mathrm{Obj}_{\mathbf{C}}$;
2. for all objects X, Y, a 1-category $\mathrm{Hom}_{\mathbf{C}}[X,Y]$ with composition $\bullet$ whose objects $f : X \to Y$ are called 1-cells and whose morphisms $\mathbf{u} : f \Rightarrow g$ are called 2-cells;
3. for all $X \in \mathbf{C}$, an identiy 1-cell $\mathrm{id}_X : X \to X$;
4. for all $X, Y, Z \in \mathbf{C}$, a composition functor

$$\circ_{X,Y,Z} : \mathrm{Hom}_{\mathbf{C}}[Y,Z] \times \mathrm{Hom}_{\mathbf{C}}[X,Y] \to \mathrm{Hom}_{\mathbf{C}}[X,Z]; \tag{3.1}$$

such that the following hold:

1. for all composable 1-cells f, g, h, $h \circ (g \circ f) = (h \circ g) \circ f$;
2. for all $f : X \to Y$, $\mathrm{id}_Y \circ f = f = f \circ \mathrm{id}_X$.

We note that there are two ways to compose 2-cells:

1. For 1-cells $f, g, h : X \to Y$ and 2-cells $\mathbf{u} : f \Rightarrow g$, $\mathbf{v} : g \Rightarrow h$ we have the vertical composite $\mathbf{v} \bullet \mathbf{u} : f \Rightarrow h$ given by the composition in $\mathrm{Hom}[X,Y]$. In a diagram:

$$\begin{tikzcd}[column sep=large] X \arrow[r, bend left=70, "f"] \arrow[r, "g" description] \arrow[r, bend right=70, "h"'] \arrow[r, phantom, shift left=3ex, "\Downarrow \mathbf{u}"] \arrow[r, phantom, shift right=3ex, "\Downarrow \mathbf{v}"] & Y \end{tikzcd}$$

2. For 1-cells $f, g : X \to Y$, $h, k : Y \to Z$ and 2-cells $\mathbf{u} : f \Rightarrow g$, $\mathbf{v} : h \Rightarrow k$ we have the horizontal composite $\mathbf{v} \circ \mathbf{u} : h \circ f \Rightarrow k \circ g$ given by the composition functor $\circ_{X,Y,Z}$. In a diagram:

$$\begin{tikzcd}[column sep=large] X \arrow[r, bend left=50, "f"] \arrow[r, bend right=50, "g"'] \arrow[r, phantom, "\Downarrow \mathbf{u}"] & Y \arrow[r, bend left=50, "h"] \arrow[r, bend right=50, "k"'] \arrow[r, phantom, "\Downarrow \mathbf{v}"] & Z \end{tikzcd}$$

Notation 3.3. We adopt similar conventions as for 1-categories:

- $X \in \mathbf{C}$ means $X \in \mathrm{Obj}_{\mathbf{C}}$;
- We drop the 1-cell composition and therefore the horizontal composition of 2-cells. Thus $\mathbf{vu} : hf \Rightarrow kg$ means $\mathbf{v} \circ \mathbf{u} : h \circ f \Rightarrow k \circ g$.
- To avoid unnecessary subscripts we identify objects and 1-cells with their respective identities. Thus we might write $h\mathbf{u}X : hfX \Rightarrow hgX$ to mean

  $$\mathbf{id}_h \circ \mathbf{u} \circ \mathbf{id}_{\mathrm{id}_X} : h \circ f \circ \mathrm{id}_X \Rightarrow h \circ g \circ \mathrm{id}_X \,.$$

  Note how we identify $\mathbf{id}_{\mathrm{id}_X}$ with id_X and thus with X.

Example 3.4. The most important 2-category that we will consider is **2CAT** which has as objects all categories, as 1-cells all functors, and as 2-cells all natural transformations. That is, for all $\mathcal{C}, \mathcal{D} \in \mathbf{2CAT}$, $\mathrm{Hom}\,[\mathcal{C}, \mathcal{D}]$ is the functor category $[\mathcal{C}, \mathcal{D}]$. Composition of 1-cells is the usual composition of functors.

Given two 2-categories, we obtain the product 2-category which is entirely analogous to the product 1-category:

Definition 3.5. Let $\mathbf{C}, \mathbf{D}$ be 2-categories. The *product 2-category* $\mathbf{C} \times \mathbf{D}$ has

1. as objects all pairs (X, X') with $X \in \mathbf{C}$ and $X' \in \mathbf{D}$;
2. for all objects (X, X') and (Y, Y'), the hom-category

   $$\mathrm{Hom}\left[(X,Y),(X',Y')\right] = \mathrm{Hom}\,[X,Y] \times \mathrm{Hom}\left[X',Y'\right];$$

3. for all objects (X, X'), the identity 1-cell $(\mathrm{id}_X, \mathrm{id}_{X'})$;
4. the composition functor given by pointwise composition in the respective 2-categories, i.e. $(g, g')(f, f') = (gf, g'f')$.

3.2 Cartesian structure

Kock's commutative monads are strong with respect to the cartesian structure of a 1-category. In order to formalise commutative relative pseudomonads, we require a similar structure on 2-categories. We are fortunate in that our main 2-category of interest, **2CAT**, admits strict products. That is, the required 1-cell equations hold up to equality and not merely up to isomorphism.

While it is straightforward to define larger product structures (e.g. [15]), the notion of strength of a monad only requires binary products and a terminal object which serves as the identity.

The standard definition of a 1-categorical product that we outlined in 2.2 requires an alternative formulation before we may generalise it elegantly. We invite the reader to verify that adapting the following definition to 1-categories in the obvious way does indeed yields a product in the usual sense.

Definition 3.6. A *cartesian structure for a 2-category* $\mathbf{C}$ consists of

1. an object $1 \in \mathbf{C}$;
2. for all $W \in \mathbf{C}$, an isomorphism of categories $\mathrm{Hom}[W,1] \cong \mathbb{1}$;
3. for all $X,Y \in \mathbf{C}$:
   (a) an object $X \times Y \in \mathbf{C}$;
   (b) 1-cells $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ called projections;
   (c) for all $W \in \mathbf{C}$, an isomorphism of categories

$$\begin{tikzcd} \mathrm{Hom}[W,X \times Y] \arrow[rr, bend left, "(\pi_1\circ-{,}\pi_2\circ-)"] \arrow[rr, phantom, "\cong"] & & \mathrm{Hom}[W,X] \times \mathrm{Hom}[W,Y] \arrow[ll, bend left, "\langle - \rangle"] \end{tikzcd} \tag{3.2}$$

   The functor $\langle - \rangle$ is called tupling.

Example 3.7. Let $\mathcal{C},\mathcal{D}$ be 1-categories. It is known that the product of categories $\mathcal{C} \times \mathcal{D}$ is a product in **CAT**. Let π_1,π_2 denote the corresponding projections in **CAT**. These are 1-cells in **2CAT**. Define the tupling functor by

$$\langle F,G \rangle = (F-,G-)$$
$$\langle \phi,\psi \rangle_X = (\phi_X,\psi_X)$$

By using the universal properties of the 1-categorical product it is straightforward to verify that this is indeed an inverse to $(\pi_1 \circ -, \pi_2, \circ -)$. Thus the 1-categorical cartesian structure of **CAT** extends to a 2-categorical cartesian structure of **2CAT**.

3.3 Pseudofunctors

Category theory is in many ways about studying morphisms rather than objects. We are therefore interested in defining the notion of a morphism between 2-categories which we call pseudofunctors. Just as a functor specifies where objects and morphisms are mapped to, a pseudofunctor is a map on objects, 1-cells, and 2-cells.

There is an important difference, however: pseudofunctors are not the most strict morphism between 2-categories. In particular, we allow for the possibility that the functor axioms hold only up to a canonical isomorphism. This is achieved by specifying these isomorphisms as part of the structure: preservation of identities is witnessed by **i** and distributivity is witnessed by **d**.

Now that the functoriality axioms are part of the structure, the axioms of a pseudofunctor serve a different purpose entirely. While the structural 2-cells ensure that certain 1-cells are isomorphic, we want to avoid the possibility of deriving multiple different isomorphisms between the same 1-cells from this structure. This is referred to as coherence. While proving coherence directly is hard in general, many structures come with a sufficient set of conditions that lead to coherence.

Definition 3.8. Let $\mathbf{C},\mathbf{D}$ be 2-categories. Then a *pseudofunctor* $F : \mathbf{C} \to \mathbf{D}$ consists of

1. for all $X \in \mathbf{C}$, an object $FX \in \mathbf{D}$;
2. for all $X, Y \in \mathbf{C}$, a functor $F_{X,Y} : \mathrm{Hom}[X,Y] \to \mathrm{Hom}[FX,FY]$;
3. for all $X \in \mathbf{C}$, an invertible 2-cell

$$\begin{tikzcd} FX \arrow[d, bend right=60, "F_{X,X}(\mathrm{id}_X)"'] \arrow[d, bend left=60, equal] \arrow[d, phantom, "\xRightarrow{\mathbf{i}_X}"] \\ FX \end{tikzcd} \tag{3.3}$$

4. for all $f : X \to Y$ and $g : Y \to Z$ in $\mathbf{C}$, an invertible 2-cell

$$\begin{tikzcd} FX \arrow[rr, "Ff"] \arrow[dr, "F(gf)"'] \arrow[rr, phantom, shift right=4, "\xRightarrow{\mathbf{d}_{f,g}}"] & & FY \arrow[dl, "Fg"] \\ & FZ & \end{tikzcd} \tag{3.4}$$

such that

1. for all composable 1-cells f, g, h,

$$\begin{tikzcd} F(hgf) \arrow[r, "\mathbf{d}"] \arrow[d, "\mathbf{d}"'] & FhF(gf) \arrow[d, "Fh\mathbf{d}"] \\ F(hg)Ff \arrow[r, "\mathbf{d}Ff"'] & FhFgFf \end{tikzcd} \tag{3.5}$$

2. for all 1-cells f,

$$\begin{tikzcd} FfF(\mathrm{id}) \arrow[r, "Ff\mathbf{i}"] \arrow[d, "\mathbf{d}^{-1}"'] & Ff\mathrm{id} \arrow[d, equal] \\ F(f\mathrm{id}) \arrow[r, equal] & Ff \end{tikzcd} \qquad \begin{tikzcd} F(\mathrm{id})Ff \arrow[r, "\mathbf{i}Ff"] \arrow[d, "\mathbf{d}^{-1}"'] & \mathrm{id}Ff \arrow[d, equal] \\ F(\mathrm{id}f) \arrow[r, equal] & Ff \end{tikzcd} \tag{3.6}$$

A 2-functor is a pseudofunctor whose 2-cells $\mathbf{i}$ and $\mathbf{d}$ are identities.

As we will be dealing with at most one pseudofunctor with non-identity 2-cells, our notation is unambiguous. One 2-functor that will be of interest is the product 2-functor that allows us to take products not just of objects but also of 1-cells and 2-cells:

Example 3.9. Analogously to the 1-categorical case (see 2.2), we observe that taking binary products in a cartesian 2-category yields a 2-functor $\mathbf{C} \times \mathbf{C} \to \mathbf{C}$, given by the following structure:

1. for all $(X, Y) \in \mathbf{C} \times \mathbf{C}$, we have the object $X \times Y \in \mathbf{C}$;

2. for all $X, X', Y, Y' \in \mathbf{C}$, the functor on hom-categories is given by the composite

$$\begin{tikzcd}
\mathrm{Hom}[X,Y] \times \mathrm{Hom}[X',Y'] \arrow[r, "(-\circ\pi_1, -\circ\pi_2)"] \arrow[dr, "(\times)_{(X,X'),(Y,Y')}"'] & \mathrm{Hom}[X \times X', Y] \times \mathrm{Hom}[X \times X', Y'] \arrow[d, "\langle - _k \rangle"] \\
& \mathrm{Hom}[X \times X', Y \times Y']
\end{tikzcd} \tag{3.7}$$

We need to verify that 3.3 and 3.4 are identities. We verify, for all $X, X' \in \mathbf{C}$,

$$\mathrm{id}_X \times \mathrm{id}_{X'} = \langle \pi_1, \pi_2 \rangle = \mathrm{id}_{X \times X'}$$

where the last equality follows by the universal property of the product category. Similarly, for 1-cells $(f, f') : (X, X') \to (Y, Y')$ and $(g, g') : (Y, Y') \to (Z, Z')$ in $\mathbf{C} \times \mathbf{C}$, we notice $gf \times g'f' = \langle gf\pi_1, g'f'\pi_2 \rangle$. In the diagram

$$\begin{tikzcd}
X & X \times X' \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[d, "\langle f\pi_1, f'\pi_2 \rangle"'] & X' \arrow[d, "f'"] \\
Y & Y \times Y' \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[d, "\langle g\pi_1, g'\pi_2 \rangle"'] & Y' \arrow[d, "g'"] \\
Z & Z \times Z' \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & Z'
\arrow[from=1-1, to=2-1, "f"'] \arrow[from=2-1, to=3-1, "g"']
\end{tikzcd} \tag{3.8}$$

each tile commutes by definition of the product. Thus the whole square commutes, proving the claim.

There are two functors corresponding to a particular relative monad: on the one hand there is the functor that the monad is relative to and on the other we have the functor induced by the monad [1]. Similarly, a relative pseudomonad has two corresponding pseudofunctors. It is for this reason that we require two degrees of strictness: while the inclusion **2Cat** $\to$ **2CAT** is a 2-functor, the pseudofunctor induced by the presheaf construction is not. To limit complexity, we aim to make use of the strictness of the former while allowing for the non-strictness of the latter.

The presheaf construction gives rise to a pseudofunctor. We will show this in the next chapter (see 4.11). For now, let us briefly investigate what this structure looks like without spelling out the details:

Example 3.10. Consider the pseudofunctor structure $\widehat{-} : \mathbf{2Cat} \to \mathbf{2CAT}$:

1. for all $\mathbb{X} \in \mathbf{2Cat}$, $\widehat{\mathbb{X}} = [\mathbb{X}^{\mathrm{op}}, \mathbf{Set}]$;
2. for all $F : \mathbb{X} \to \mathbb{Y}$ and $P \in \widehat{\mathbb{X}}$,

$$\widehat{F}P = \int^X PX \times \mathrm{Hom}(-, FX);$$

3. the natural isomorphism $\mathbf{i}$ has as components natural isomorphisms

$$\int^X PX \times \mathrm{Hom}(-, X) \cong P;$$

4. for all composable functors F and G, the natural isomorphism $\mathbf{d}_{F,G}$ has as components isomorphisms
$$\int^X PX \times \mathrm{Hom}(-,(GF)X) \cong \int^Y \left(\int^X PX \times \mathrm{Hom}(Y,FX)\right) \times \mathrm{Hom}(-,GY).$$

3.4 Inclusions of cartesian 2-categories

Before we can endow the presheaf construction with a strength, we need to capture some of the additional properties of the inclusion. It only makes sense to define a strong pseudomonad relative to a 2-functor $\mathbf{J} \to \mathbf{C}$ if both its domain and codomain are cartesian and the cartesian structures agree. We could treat this in full generality by considering monoidal pseudofunctors, similar to [17]. However, the inclusion $\mathbf{2Cat} \to \mathbf{2CAT}$ preserves products and terminal objects we are able to impose and satisfy much stricter conditions.

An inclusion functor is injective on objects and morphisms. Similarly, we demand an inclusion pseudofunctor be injective on objects, 1-cells, and 2-cells. Moreover, we require inclusions to preserve the cartesian structure.

Definition 3.11. An *inclusion of cartesian 2-categories* consists of

1. cartesian 2-categories $\mathbf{J}$ and $\mathbf{C}$;
2. a 2-functor $J : \mathbf{J} \to \mathbf{C}$;

such that

1. J is injective on objects, 1-cells, and 2-cells;
2. for all $X, Y \in \mathbf{J}$,
$$J1 = 1, \qquad J(X \times Y) = JX \times JY, \qquad J(\pi_1) = \pi_1, \qquad J(\pi_2) = \pi_2.$$

Example 3.12. Define $\mathbf{2Cat}$ as the 2-category with objects all small categories and, for all $\mathbb{C}, \mathbb{D} \in \mathbf{2Cat}$, the hom-category $\mathrm{Hom}[\mathbb{C}, \mathbb{D}] = [\mathbb{C}, \mathbb{D}]$. We now have the obvious inclusion $J : \mathbf{2Cat} \to \mathbf{2CAT}$ that is the identity on objects and hom-categories. Define the cartesian structure on $\mathbf{2Cat}$ in the same way that we defined it for $\mathbf{2CAT}$. This is justified as the product of small categories is itself small. Thus J forms an inclusion of cartesian 2-categories.

3.5 Pseudonatural transformations

Naturally, we have morphisms between pseudofunctors. The naturality of a pseudonatural transformation is witnessed by a structural 2-cell which obeys a coherence axiom. Once again, we have two levels of strictness. It is worth noting, however, that these are not related to the strictness of the underlying pseudofunctors.

Definition 3.13. Let $F, G : \mathbf{C} \to \mathbf{D}$ be pseudofunctors. A *pseudonatural transformation* $\phi : F \Rightarrow G$ consists of

1. for all $X \in \mathbf{C}$, a 1-cell $\phi_X : FX \to GX$;
2. for all $f : X \to Y$ in $\mathbf{C}$, a naturality 2-cell

$$
\begin{tikzcd}
FX \arrow[r, "Ff"] \arrow[d, "\phi"'] \arrow[dr, phantom, "\Longrightarrow \mathbf{n}_f"] & FY \arrow[d, "\phi"] \\
GX \arrow[r, "Gf"] & GY
\end{tikzcd}
\tag{3.9}
$$

such that, for all composable 1-cells $f, g \in \mathbf{C}$,

$$
\begin{tikzcd}
GgGf\phi \arrow[r, "Gg\mathbf{n}"] \arrow[d, "\mathbf{d}^{-1}\phi"'] & Gg\phi Ff \arrow[r, "\mathbf{n}Ff"] & \phi FgFf \arrow[d, "\phi\mathbf{d}^{-1}"] \\
G(gf)\phi \arrow[rr, "\mathbf{n}"'] & & \phi F(gf)
\end{tikzcd}
\tag{3.10}
$$

A 2-natural transformation is a pseudonatural transformation whose naturality 2-cell $\mathbf{n}$ is the identity.

Example 3.14. Let $\mathbf{C}$ be a cartesian 2-category. We saw that the product 2-functor is merely an extension of the usual product functor. Because $\mathbf{n}$ is the identity, each natural transformation between product functors yields a 2-natural transformation between product 2-functors. Thus the usual isomorphisms λ, α, γ are in fact 2-natural transformations.

Chapter 4

Strong inclusion pseudomonads

We have now developed all the theory required to present our approach to defining a generalisation of strong monads that admits the presheaf construction as a model. We do so in three major steps, each of which we will justify by applying it to the presheaf construction and comparing it to related work

Firstly, we present a structure that is similar to that of a relative pseudomonad. We justify the differences and show how the presheaf construction gives rise to such a structure. Secondly, we state the axioms that this structure is required to satisfy in order to induce a relative pseudomonad. Finally, we extend the structure in a way that induces a strong pseudomonad structure in the case where the inclusion is the identity. Unfortunately, we are not able to postulate axioms that are sufficient to show that this induced structure satsifies the axioms of a strong pseudomonad.

Fix an inclusion of cartesian 2-categories $J : \mathbf{J} \to \mathbf{C}$.

4.1 Prestrong structure

A strong monad is a monad equipped with a suitable natural transformation. Hence the obvious way to define a strong relative pseudomonad is by equipping a relative pseudomonad with a suitable pseudonatural transformation. This approach would have two notable benefits: Firstly, it would allow us to leverage already existing results about relative pseudomonads without any overhead. Secondly, it would provide us with an intuitive connection to strong pseudomonads that would, presumably, make some of the axioms easier to postulate.

We investigate another approach. Rather than adding the strength as additional structure, we choose to build it in to the extension operator. That is, we only allow 1-cells $f : W \times JX \to TY$ to be extended to $f^\dagger : W \times TX \to TY$. This relates the extension to the cartesian structure directly. Another reason why we believe that this idea is worth entertaining is because it becomes very pleasant to describe commutativity of such a structure. One can demand the existence of the following natural isomorphism:

$$\begin{tikzcd}
\mathrm{Hom}[JX \times JY, TZ] \arrow[r, "(-)^\dagger"] \arrow[d, "-\circ\gamma"'] & \mathrm{Hom}[JX \times TY, TZ] \arrow[r, "-\circ\gamma"] & \mathrm{Hom}[TY \times JX, TZ] \arrow[d, "(-)^\dagger"] \\
\mathrm{Hom}[JY \times JX, TZ] \arrow[d, "(-)^\dagger"'] & \cong & \mathrm{Hom}[TY \times TX, TZ] \arrow[d, "-\circ\gamma"] \\
\mathrm{Hom}[JY \times TX, TZ] \arrow[r, "-\circ\gamma"'] & \mathrm{Hom}[TX \times JY, TZ] \arrow[r, "(-)^\dagger"'] & \mathrm{Hom}[TX \times TY, TZ]
\end{tikzcd}$$

Given that synthetic measure theory requires a commutative monad, it is possible that this alternative structure is more convenient in the context of a generalised synthetic measure theory.

Definition 4.1. A *strong J-pseudomonad structure* consists of

1. for all $X \in \mathbf{J}$, an object $TX \in \mathbf{C}$;
2. for all $X \in \mathbf{J}$, a 1-cell $\eta_X : JX \to TX$ in $\mathbf{C}$;
3. for all $X, Y \in \mathbf{J}$ and $W \in \mathbf{C}$, a functor

$$(-)^\dagger_{X,Y,W} : \mathrm{Hom}\,[W \times JX, TY] \to \mathrm{Hom}\,[W \times TX, TY]; \tag{4.1}$$

4. for all $f : JW \times JX \to TY$ in $\mathbf{C}$, an invertible 2-cell

$$\begin{tikzcd}
W \times JX \arrow[rr, "W\times\eta"] \arrow[dr, "f"'] \arrow[rr, phantom, shift right=4ex, "\xRightarrow{\mathbf{r}_f}"] & & W \times TX \arrow[dl, "f^\dagger"] \\
& TY &
\end{tikzcd} \tag{4.2}$$

5. for all $X \in \mathbf{J}$, an invertible 2-cell

$$\begin{tikzcd}
1 \times TX \arrow[d, bend right=60, "\lambda"'] \arrow[d, bend left=60, "(\eta\lambda)^\dagger"] \arrow[d, phantom, "\xRightarrow{\mathbf{l}_X}"] \\
TX
\end{tikzcd} \tag{4.3}$$

6. for all $i : V \to W$, $f : JX \to TY$, and $g : W \times JY \to TZ$ in $\mathbf{C}$, an invertible 2-cell

$$\begin{tikzcd}
V \times TX \arrow[rr, "i\times(f\lambda)^\dagger\lambda^{-1}"] \arrow[dr, "(g^T(i\times f))^T"'] \arrow[rr, phantom, shift right=4ex, "\xRightarrow{\mathbf{c}_{f,g,i}}"] & & W \times TY \arrow[dl, "g^T"] \\
& TZ &
\end{tikzcd} \tag{4.4}$$

We note that this extension operator is defined on a narrower selection of 1-cells than the extension operator of a relative pseudomonad. This is only a temporary limitation. Using the left unitor λ, we obtain the usual extension functor

$$(-)^*_{X,Y} = (-\lambda_{JX})^\dagger \lambda_{JX}^{-1}; \tag{4.5}$$

as in the diagram

$$\begin{tikzcd} \mathrm{Hom}[JX,TY] \arrow[r, "(-)^*"] \arrow[d, "-\circ\lambda"'] & \mathrm{Hom}[TX,TY] \\ \mathrm{Hom}[1\times JX,TY] \arrow[r, "(-)^\dagger"'] & \mathrm{Hom}[1\times TX,TY] \arrow[u, "-\circ\lambda^{-1}"'] \end{tikzcd} \tag{4.6}$$

This will help us state the axioms in the following section more cleanly.

The purpose of 4.1 is to define a theory that admits the presheaf construction as a model. While one might expect this result to be a straightforward statement, some work is required to define the structure itself. One must pay special attention when working with several layers of indirection. For example, we have a natural transformation whose components are functors $\lambda_{\mathbb{X}} : 1 \times \mathbb{X} \to \mathbb{X}$. This problem is amplified by the fact that our main point of study are objects $[\mathbb{X}^{\mathrm{op}}, \mathbf{Set}]$ which are themselves categories with functors as objects and natural transformations as morphisms.

Nonetheless, we define the entire structure in detail:

Example 4.2. The *prestrong presheaf construction* consists of:

1. for all $\mathbb{X} \in \mathbf{2Cat}$, the object $\widehat{\mathbb{X}} = [\mathbb{X}^{\mathrm{op}}, \mathbf{Set}]$;
2. for all $\mathbb{X} \in \mathbf{2Cat}$, the unit functor $\eta_{\mathbb{X}}$ given by $\eta_{\mathbb{X}}(X) = \mathrm{Hom}(-,X)$;
3. for all $\mathbb{X}, \mathbb{Y} \in \mathbf{2Cat}$ and $\mathcal{W} \in \mathbf{2CAT}$, the extension functor $(-)^\dagger_{\mathbb{X},\mathbb{Y},\mathcal{W}}$ given by

   (a) for all $F : \mathcal{W} \times \mathbb{X} \to \widehat{\mathbb{Y}}$, $W \in \mathcal{W}$, and $P \in \widehat{\mathbb{X}}$

   $$F^\dagger(W,P) = \int^X PX \times F(W,X)(-) \tag{4.7}$$

   and, for all morphisms f in $\mathcal{W}$ and ϕ in $\widehat{\mathbb{X}}$,

   $$F^\dagger(f,\phi) = \int^X \phi_X \times F(f,X)(-); \tag{4.8}$$

   (b) for all $F,G : \mathcal{W} \times \mathbb{X} \to \widehat{\mathbb{Y}}$, $\phi : F \Rightarrow G$, $W \in \mathcal{W}$, and $P \in \widehat{\mathbb{X}}$,

   $$\phi^\dagger_{W,P} = \int^X PX \times (\phi_{W,P})_{(-)}; \tag{4.9}$$

4. for all $F : \mathcal{W} \times \mathbb{X} \to \widehat{\mathbb{Y}}$, $W \in \mathcal{W}$, and $X \in \mathbb{X}$, the natural transformation $(\mathbf{r}_F)_{W,X}$ has the components

$$\begin{tikzcd} F(W,X)Y \arrow[r, "\langle \Delta\mathrm{id}, F(W,X)Y \rangle"] \arrow[dr, "((\mathbf{r}_F)_{W,X})_Y"'] & \mathrm{Hom}(X,X) \times F(W,X)Y \arrow[d, "q"] \\ & \int^{X'} \mathrm{Hom}(X',X) \times F(W,X')Y \end{tikzcd} \tag{4.10}$$

5. for all $P \in \widehat{\mathbb{X}}$, the natural transformation $(\mathbf{l}_{\mathbb{X}})_P$ has the components

$$\begin{tikzcd} PX \arrow[r, "\langle PX, \Delta\mathrm{id} \rangle"] \arrow[dr, "((\mathbf{l}_{\mathbb{X}})_P)_X"'] & PX \times \mathrm{Hom}(X,X) \arrow[d, "q"] \\ & \int^{X'} PX' \times \mathrm{Hom}(X,X') \end{tikzcd} \tag{4.11}$$

6. for all $F : \mathbb{X} \to \widehat{\mathbb{Y}}$, $G : \mathcal{W} \times \mathbb{Y} \to \widehat{\mathbb{Z}}$, $I : \mathcal{V} \to \mathbb{W}$, $P \in \widehat{\mathbb{X}}$, and $V \in \mathcal{V}$, the natural isomorphism $(\mathbf{c}_{F,G,I})_{V,P}$ has as components the canonical cowedge isomorphisms given by

$$\begin{tikzcd} PX \times ((FX)Y \times G(IV,Y)Z) \arrow[r, bend left, "\cong"] \arrow[d, "PX \times q"'] & (PX \times (FX)Y) \times G(IV,Y)Z \arrow[d, "q \times G(IV,Y)(-)"] \\ PX \times \int^Y (FX)Y \times G(IV,Y)Z \arrow[d, "q"'] & \left(\int^X PX \times (FX)Y\right) \times G(IV,Y)Z \arrow[d, "q"] \\ \int^X PX \times \int^Y (FX)Y \times G(IV,Y)Z \arrow[r, bend right, "((\mathbf{c}_{F,G,I})_{V,P})_Z"'] & \int^Y \left(\int^X PX \times (FX)Y\right) \times G(IV,Y)Z \end{tikzcd}$$

Even from this detailed description it is not clear that we have provided the necessary structure: we have to make sure that the structural 2-cells are in fact invertible. For $\mathbf{c}$, this is immediate and we already showed the result for $\mathbf{l}$ in 2.11. For $\mathbf{r}$, the proof is similar:

Lemma 4.3. The function (4.10) is an isomorphism.

Proof. For each $X' \in \mathbb{X}$, consider the function

$$w\left(X'\right) : \mathrm{Hom}(X',X) \times F(W,X')Y \to F(W,X)Y$$

given by $(f,u) \mapsto (F(W,f)Y)u$. We note that, for all 1-cells $f : X_1 \to X_2$ in $\mathbb{X}$, the following commutes:

$$\begin{tikzcd}
\mathrm{Hom}(X_2,X)\times F(W,X_1)Y \arrow[r, "\mathrm{Hom}(f,X)\times F(W,X_1)Y"] \arrow[d, "\mathrm{Hom}(X_2,X)\times F(W,f)Y"'] & \mathrm{Hom}(X_1,X)\times F(W,X_1)Y \arrow[d, "w(X_1)"] \\
\mathrm{Hom}(X_2,X)\times F(W,X_2)Y \arrow[r, "w(X_2)"'] & F(W,X)Y
\end{tikzcd}$$

Thus $F(W,X)Y$ and w define a cowedge. Consider the diagram

$$\begin{tikzcd}
\mathrm{Hom}(X',X)\times F(W,X')Y \arrow[r, "w(X')"] \arrow[ddr, "q"'] & F(W,X)Y \arrow[d, "\langle \Delta\mathrm{id}, F(W,X)Y\rangle"'] \arrow[dd, bend left, "((\mathbf{r}_F)_{W,X})_Y"] \\
& \mathrm{Hom}(X,X)\times F(W,X)Y \arrow[d, "q"'] \\
& \int^{X'} \mathrm{Hom}(X',X)\times F(W,X')Y
\end{tikzcd} \tag{4.12}$$

Let $g : X' \to X$ and let $u \in F(W,X')Y$. Chasing elements we find

$$\begin{aligned}
(q_X \circ \langle \Delta\mathrm{id}, F(W,X)Y\rangle \circ w(X'))(g,u) &= (q_X \circ \langle \Delta\mathrm{id}, F(W,X)Y\rangle)((F(W,g)Y)u) \\
&= q_X(\mathrm{id}, (F(W,g)Y)u) \\
&= q_{X'}(g,u)
\end{aligned}$$

where the last step follows from the cowedge property of the coend. Thus (4.12) commutes, making (4.10) into a cowedge morphism. By universality of the coend it follows that this must be an isomorphism. □

4.2 Induced relative pseudomonad structure

For our generalisation to make sense, we require a strong J-pseudomonad to induce a relative pseudomonad over J. The structure 4.1 already resembles that of a relative pseudomonad so it is straightforward to see how the former gives rise to the latter.

Definition 4.4. Let T be a strong J-pseudomonad. The *relative pseudomonad structure induced by* T consists of

- for all $X, Y \in \mathbf{J}$, the extension functor $(-)^*$ as in 4.5;
- for all 1-cells $f : JX \to TY$ and $g : JY \to TZ$, the 2-cell

$$\mathbf{m}_{f,g} = \mathbf{c}_{f,g\lambda}\lambda^{-1}_{TX}$$

as in the diagram

$$
\begin{tikzcd}
TX \arrow[r, "\lambda^{-1}"'] \arrow[rrrr, bend left, "f^*"] \arrow[drr, bend right, "(g^*f)^*"'] & 1 \times TX \arrow[rr, "1\times f^*"] \arrow[dr, "((g\lambda)^\dagger(1\times f))^\dagger"'] & \arrow[d, phantom, "\overset{\mathbf{c}}{\Longrightarrow}"] & 1 \times TY \arrow[dl, "(g\lambda)^\dagger"] & TY \arrow[l, "\lambda^{-1}"] \arrow[dll, bend left, "g^*"] \\
 & & TZ & &
\end{tikzcd}
\tag{4.13}
$$

- for all $f : JX \to TY$, the 2-cell

$$
\mathbf{e}_f = \mathbf{r}_{f\lambda}\lambda^{-1}_{JX}
$$

as in the diagram

$$
\begin{tikzcd}
JX \arrow[r, "\lambda^{-1}"'] \arrow[rrrr, bend left, "\eta"] \arrow[drr, bend right, "f"'] & 1 \times JX \arrow[rr, "1\times\eta"] \arrow[dr, "f\lambda"'] & \arrow[d, phantom, "\overset{\mathbf{r}}{\Longrightarrow}"] & 1 \times TX \arrow[dl, "(f\lambda)^\dagger"] & TX \arrow[l, "\lambda^{-1}"] \arrow[dll, bend left, "f^*"] \\
 & & TY & &
\end{tikzcd}
\tag{4.14}
$$

- for all $X \in \mathbf{J}$, the 2-cell

$$
\mathbf{t}_X = \mathbf{l}_{\eta_X}\lambda^{-1}_{TX}
$$

as in the diagram

$$
\begin{tikzcd}
TX \arrow[r, "\lambda^{-1}"] \arrow[dr, bend right, "\eta^{T\lambda}"'] & 1 \times TX \arrow[d, bend right, "(\eta\lambda)^T"'] \arrow[d, bend left, "\lambda"] \arrow[d, phantom, "\overset{\mathbf{l}}{\Longrightarrow}"] \\
 & TX
\end{tikzcd}
\tag{4.15}
$$

When considering the presheaf construction, the induced relative pseudomonad structure consists of several isomorphisms that may be familiar from coend calculus. While we are not concerned with the details, it is worth investigating the structure nonetheless:

Example 4.5. For the presheaf construction, we obtain the following relative pseudomonad structure:

1. for all $F, G : \mathbb{X} \to \widehat{\mathbb{Y}}$, $\phi : F \Rightarrow G$, and $P \in \widehat{\mathbb{X}}$,

$$
F^*(P) = \int^X PX \times FX(-), \qquad (\phi^*)_P = \int^X PX \times \phi_X;
$$

2. for all $X \in \mathbb{X}$, $\eta_{\mathbb{X}}(X) = \mathrm{Hom}(-, X)$;

3. for all $F : \mathbb{X} \to \widehat{\mathbb{Y}}$ and $X \in \mathbb{X}$, there is a natural isomorphism with components

$$(\mathbf{e}_F)_X : FX \to \int^{X'} \mathrm{Hom}(X', X) \times FX';$$

4. for all $F : \mathbb{X} \to \widehat{\mathbb{Y}}$, $G : \mathbb{Y} \to \widehat{\mathbb{Z}}$, and $P \in \widehat{\mathbb{X}}$, $(\mathbf{m}_{F,G})_P$ is a natural isomorphism with components

$$\int^X PX \times \int^Y (FX)Y \times (GY)Z \to \int^Y \left(\int^X PX \times (FX)Y \right) \times (GY)Z;$$

5. for all $P \in \widehat{\mathbb{X}}$, $\mathbf{t}$ has as components isomorphisms

$$((\mathbf{t}_{\mathbb{X}})_P)_X : \int^{X'} \mathrm{Hom}(X, X') \times PX' \to PX$$

4.3 Prestrong axioms

A prestrong J-pseudomonad should induce a relative pseudomonad over J. It is therefore not surprising that the conditions which we impose are entirely analogous to those stated in [4].

Definition 4.6. A *prestrong J-pseudomonad* is a prestrong J-pseudomonad structure such that

1. $\mathbf{r}_f$ is natural in f;
2. $\mathbf{c}_{f,g,i}$ is natural in f, g, and i;
3. for all $f : W \times JX \to TY$,

$$\begin{tikzcd} f^\dagger \arrow[r, "\mathbf{r}"] \arrow[d, equal] & (f^\dagger(1 \times \eta))^\dagger \arrow[d, "\mathbf{c}"] \\ f^\dagger & f^\dagger(1 \times \eta^*) \arrow[l, "f^\dagger(1 \times \mathbf{l}\lambda^{-1})"] \end{tikzcd} \tag{4.16}$$

4. for all suitable 1-cells f, g, h, i, j,

$$\begin{tikzcd} & ((h^\dagger(j \times g))^\dagger(i \times f))^\dagger \arrow[dl, "(\mathbf{c}(i \times f))^\dagger"'] \arrow[dr, "\mathbf{c}"] & \\ (h^\dagger(ji \times g^* f))^\dagger \arrow[d, "\mathbf{c}"] & & (h^\dagger(j \times g))^\dagger(i \times f^*) \arrow[d, "\mathbf{c}(i \times f^*)"] \\ h^\dagger(ji \times (g^* f)^\dagger) \arrow[rr, "h^\dagger(ji \times \mathbf{c}\lambda^{-1})"] & & h^\dagger(ji \times g^* f^*) \end{tikzcd} \tag{4.17}$$

Now that we have stated some axioms, we need to make sure that they are sensible. The first step is to verify that the presheaf construction remains a model.

Proposition 4.7. The prestrong presheaf construction in 4.2 is a prestrong J-pseudomonad.

Proof. We verify the axioms:

1. We note that

$$
\begin{tikzcd}
F(W,X)Y \arrow[rr, "(\phi_{W,X})_Y"] \arrow[d, "\langle \Delta\mathrm{id}, F(W,X)Y\rangle"] \arrow[dd, bend right=60, "\mathbf{r}"'] \arrow[drr, phantom, "\langle \Delta\mathrm{id}, -\rangle\text{-nat}"] & & G(W,X)Y \arrow[d, "\langle \Delta\mathrm{id}, G(W,X)Y\rangle"] \arrow[dd, bend left=60, "\mathbf{r}"] \\
\mathrm{Hom}(X,X) \times F(W,X)Y \arrow[rr, bend right=10, "\mathrm{Hom}(X,X)\times(\phi_{W,X})_Y"'] \arrow[d, "q"] & \mathbf{r}\text{-def} \quad \mathbf{r}\text{-def} & \mathrm{Hom}(X,X) \times G(W,X)Y \arrow[d, "q"] \\
\int^{X'} \mathrm{Hom}(X',X) \times F(W,X')Y \arrow[rr, bend right=10, "\int^{X'} \mathrm{Hom}(X',X)\times(\phi_{W,X})_Y"'] & & \int^{X'} \mathrm{Hom}(X',X) \times F(W,X')Y
\end{tikzcd}
$$

   commutes due to (2.9) on the right. This shows naturality of $\mathbf{r}$.

2. Naturality of $\mathbf{c}$ follows from naturality of the underlying isomorphism.[1]

3. We note that the following commutes:

$$
\begin{tikzcd}
PX \times F(W,X)Y \arrow[r, bend left=10, "PX\times\mathbf{r}"] \arrow[dr, "PX\times\langle\Delta\mathrm{id}, F(W,X)Y\rangle"'] \arrow[ddd, equal] & PX \times \int^{X'} \mathrm{Hom}(X',X) \times F(W,X')Y \\
\mathbf{r}\text{-def} & PX \times (\mathrm{Hom}(X,X) \times F(W,X)Y) \arrow[u, bend right, "PX\times q"'] \arrow[d, "\alpha"] \\
\mathbf{r}\text{-def} & (PX \times \mathrm{Hom}(X,X)) \times F(W,X)Y \arrow[d, bend left, "q\times F(W,X)Y"] \\
PX \times F(W,X)Y \arrow[ur, "\langle PX, \Delta\mathrm{id}\rangle\times F(W,X)Y"] \arrow[r, bend right=10, "\mathbf{l}\times F(W,X')Y"'] & \left(\int^{X} PX \times \mathrm{Hom}(X',X)\right) \times F(W,X)Y
\end{tikzcd}
$$

   Postcomposition with the appropriate canonical morphisms yields (4.16).[2]

4. The proof of (4.17) is similar to the above.[3]

[1] Full diagram of naturality of $\mathbf{c}_{F,G,J}$ in F and G in quiver.
[2] Full diagram for proof of (4.16) in quiver.
[3] Full diagram for proof of (4.17) in quiver.

□

Only a little more work is required to make sure that a prestrong J-pseudomonad does indeed correspond to a relative pseudomonad over J.

Theorem 4.8. Let T be a strong J-pseudomonad. Then the relative pseudomonad structure induced by T is a relative pseudomonad in the sense of [4, Definition 3.1].

Proof. We have the obvious correspondence between the structure in [4] and 4.4: $(-)^*_{X,Y}$ is identical, i_X is η_X, $\mu_{g,f}$ is $\mathbf{m}_{f,g}$, η_f is $\mathbf{e}_f$, θ_X is $\mathbf{t}_X$. We now verify the axioms:

- naturality of $\mathbf{e}$ is just naturality of $\mathbf{r}$;
- naturality of $\mathbf{m}$ is just naturality of $\mathbf{c}$;
- [4, (3.2)] holds by postcomposing (4.16) with λ^{-1} as in the diagram

$$
\begin{tikzcd}
f^* \arrow[rrr, "(\mathbf{e}_f)^*"] \arrow[ddd, equal] \arrow[dr, "-\circ\lambda"] & & & (f^*\eta)^* \arrow[dl, "-\circ\lambda"'] \arrow[ddd, "\mathbf{m}"] \\
& (f\lambda)^\dagger \arrow[r, "\mathbf{r}"] \arrow[d, equal] & ((f\lambda)^\dagger(1\times\eta))^\dagger \arrow[d, "\mathbf{c}"] & \\
& (f\lambda)^\dagger \arrow[dl, "-\circ\lambda^{-1}"'] & (f\lambda)^\dagger(1\times(\eta\lambda)^\dagger) \arrow[l, "(f\lambda)^\dagger(1\times\mathbf{l})"'] & \\
f^* & & & f^*\eta^* \arrow[lll, "f^*\mathbf{t}"] \arrow[ul, "-\circ\lambda"']
\end{tikzcd}
$$

- [4, (3.1)] holds by postcomposing (4.17) with λ^{-1} just like above.

□

We may thus conclude that the definitions that we have stated so far fit in with already established work.

4.4 Induced pseudofunctor

Before we move on to the strong structure of a J-pseudomonad, we take some time to appreciate what we have developed so far. Relative monads induce functors [1]. It is therefore desirable that relative pseudomonads induce pseudofunctors. This was not explicitly proven in [4]. We will show how to obtain the pseudofunctor induced by a prestrong J-pseudomonad.

Fix a prestrong J-pseudomonad T. Analogous to 3.8, we define the following:

Definition 4.9. The *pseudofunctor structure induced by T* consists of

1. for all $X \in \mathbf{J}$, the object $TX \in \mathbf{C}$;

2. for all $X, Y \in \mathbf{J}$, the functor $T_{X,Y}$ is the composite

$$\begin{tikzcd} \mathrm{Hom}[X,Y] \arrow[r, "T_{X,Y}"] \arrow[d, "J_{X,Y}"'] & \mathrm{Hom}[TX,TY] \\ \mathrm{Hom}[JX,JY] \arrow[r, "\eta\circ-"'] & \mathrm{Hom}[JX,TY] \arrow[u, "(-)^{T\lambda}"'] \end{tikzcd} \tag{4.18}$$

3. for all $X \in \mathbf{J}$, the 2-cell $\mathbf{i}_X = \mathbf{l}_X{}^{-1}\lambda^{-1}$;
4. for all $f : X \to Y$ and $g : Y \to Z$ in $\mathbf{J}$, the 2-cell

$$\mathbf{d}_{f,g} = \mathbf{m}_{\eta Jf,\eta Jg} \bullet (\mathbf{e}_{\eta Jg}Jf)^{T\lambda}$$

as in the diagram

$$\begin{tikzcd} TX \arrow[rr, bend left, "Tf"] \arrow[rr, "(\eta Jf)^*"] \arrow[d, "(\eta JgJf)^*"] \arrow[d, bend right=60, "T(gf)"'] \arrow[drr, "((\eta Jg)^*\eta Jf)^*"] & & TY \arrow[d, "(\eta Jg)^*"'] \arrow[d, bend left=60, "Tg"] \arrow[dl, phantom, "\Rightarrow \mathbf{m}"] \\ TZ \arrow[rr, equal] \arrow[r, phantom, "\Rightarrow (\mathbf{e}Jf)^*"] & & TZ \end{tikzcd} \tag{4.19}$$

Example 4.10. The prestrong presheaf construction induces a pseudofunctor structure $\widehat{-} : \mathbf{2Cat} \to \mathbf{2CAT}$ as described in 3.10.

Similar to the induced relative pseudomonad structure, we can prove a general statement and no additional work is required to show that the presheaf construction induces a pseudofunctor:

Proposition 4.11. The pseudofunctor structure induced by T is a pseudofunctor.

Proof. We verify the axioms:

1. We have the commuting diagram

$$
\begin{tikzcd}
T(hgf) \arrow[r, "(\mathbf{e}J(gf))^*"] \arrow[d, "(\mathbf{e}Jf)^*"'] \arrow[dr, phantom, "\mathbf{e}\text{-nat}"] & (Th\eta J(gf))^* \arrow[r, "\mathbf{m}"] \arrow[d, "(\mathbf{e}Jf)^*"'] \arrow[ddd, bend left, "((\eta Jh)^*\mathbf{e}Jf)^*"] \arrow[ddr, phantom, "\mathbf{m}\text{-nat}"] & ThT(gf) \arrow[ddddd, "Th(\mathbf{e}Jf)^*"] \\
(T(hg)\eta Jf)^{T\lambda} \arrow[r, "((\mathbf{e}Jg)^*\eta Jf)^*"'] \arrow[d, "\mathbf{m}"'] & ((Th\eta Jg)^*\eta Jf)^* \arrow[dddl, "\mathbf{m}"'] \arrow[dd, "(\mathbf{m}\eta Jf)^*"] & \\
T(hg)Tf \arrow[dd, "(\mathbf{e}Jg)^*Tf"'] \arrow[r, phantom, "\mathbf{n}\text{-nat}"] & & \\
 & ((\eta Jh)^*(\eta Jg)^*\eta Jf)^* \arrow[ddr, "\mathbf{m}"] & \\
(Th\eta Jg)^*Tf \arrow[d, "\mathbf{m}Tf"'] \arrow[r, phantom, "(4.17)"] & & \\
ThTgTf & & Th((\eta Jg)^*\eta Jf)^* \arrow[ll, "(\eta Jh)^*\mathbf{m}"]
\end{tikzcd}
$$

(4.20)

where the remaining face follows from [4, Lemma 3.2 (i)]. We have shown 3.5.

2. The conditions 3.6 are just [4, (3.1)].

□

4.5 Strong structure

We are now ready for our final step towards the definition of a strong relative pseudomonad. As we are not be able to prove all the results that we would have liked, we provide some additional insight into how one might come up with the definition of a strong J-pseudomonad structure.

Firstly, we notice that the prestrong structure already gives rise to a 1-cell $X \times TY \to T(X \times Y)$ by extending the unit $\eta_{X\times Y}$. We will therefore think of $\eta^\dagger$ as the strength. Secondly, we observe that of the four structural 2-cells in [14, Definitions 8 and 9] only one involves repeated applications of the object map. The others are thus easily translated. To avoid the repeated applications in the problematic case, we take inspiration from the definition of a strong relative monad in [17]. The result is the rather unintuitive family of invertible 2-cells $\mathbf{q}$ which will allow us to construct the usual pentagon in the case where J is the identity.

Definition 4.12. A *strong J-pseudomonad structure* consists of

1. a prestrong J-pseudomonad T;

2. for all $X,Y,Z \in \mathbf{J}$, an invertible 2-cell

$$\begin{tikzcd}
(JX \times JY) \times TZ \arrow[rr, "\eta^\dagger"] \arrow[d, "\alpha"'] \arrow[rr, phantom, shift right=4ex, "\Longrightarrow \mathbf{p}_{X,Y,Z}"] & & T((X \times Y) \times Z) \arrow[d, "T\alpha"] \\
JX \times (JY \times TZ) \arrow[r, "\eta^\dagger"'] & JX \times T(Y \times Z) \arrow[r, "\eta^\dagger"'] & T(X \times (Y \times Z))
\end{tikzcd} \tag{4.21}$$

3. for all invertible 2-cells

$$\begin{tikzcd}
JW \times JX \arrow[rr, "JW \times f"] \arrow[dr, "g"'] \arrow[rr, phantom, shift right=3ex, "\overset{\mathbf{u}}{\Longrightarrow}"] & & JW \times TY \arrow[dl, "\eta^\dagger"] \\
& T(W \times Y) &
\end{tikzcd}$$

an invertible 2-cell

$$\begin{tikzcd}
JW \times TX \arrow[r, "\eta^\dagger"] \arrow[d, "JW \times f^*"'] \arrow[dr, phantom, "\overset{\mathbf{q_u}}{\Longrightarrow}"] & T(W \times X) \arrow[d, "g^*"] \\
JW \times TY \arrow[r, "\eta^\dagger"'] & T(W \times Y)
\end{tikzcd} \tag{4.22}$$

4. for all $f : JW \times JX \to TY$ in $\mathbf{C}$, an invertible 2-cell

$$\begin{tikzcd}
JW \times TX \arrow[rr, "\eta^\dagger"] \arrow[dr, "f^\dagger"'] \arrow[rr, phantom, shift right=3ex, "\overset{\mathbf{s}_f}{\Longrightarrow}"] & & T(W \times X) \arrow[dl, "f^*"] \\
& TY &
\end{tikzcd} \tag{4.23}$$

Let us now verify that it is indeed possible to obtain such a structure for the presheaf construction. Of particular interest to us is the structural 2-cells $\mathbf{q_u}$ because they are different to all the others that we have seen so far.

Example 4.13. We extend the prestrong presheaf construction as follows:

1. for all $X \in \mathbb{X}$, $Y \in \mathbb{Y}$, and $P \in \widehat{\mathbb{Z}}$, $\left(\mathbf{p}_{\mathbb{X},\mathbb{Y},\mathbb{Z}}\right)_{X,Y,P}$ is the isomorphism of coends

$$\begin{aligned}
&\int^{X',Y',Z'} \left(\int^{Z} PZ \times \mathrm{Hom}(((X',Y'),Z'),((X,Y),Z)) \right) \times \mathrm{Hom}(-,(X',(Y',Z'))) \\
&\cong \int^{Y',Z'} \left(\int^{Z} PZ \times \mathrm{Hom}((Y',Z'),(Y,Z)) \right) \times \mathrm{Hom}(-,(X,(Y',Z')))
\end{aligned}$$

which may be obtained by composition of $\mathbf{l}$ and $\mathbf{c}$;[4]

[4]Construction of $\mathbf{p}$ in quiver

2. for all natural isomorphisms with components

$$\mathbf{u}_{W,X} : G(W,X) \cong \int^{Y} (FX)Y \times \mathrm{Hom}(-,(W,Y))$$

we have the natural isomorphism $\mathbf{q}_{\mathbf{u}}$ whose components are themselves natural isomorphisms

$$\begin{aligned}&\int^{Y} \left(\int^{X} PX \times (FX)Y\right) \times \mathrm{Hom}\,(-,(W,Y))\\ &\cong \int^{W',X'} \left(\int^{X} PX \times \mathrm{Hom}((W',X'),(W,X))\right) \times G(W',X')(-)\end{aligned}$$

which are obtained by composing $\mathbf{c}$, $\mathbf{l}$, and $\mathbf{u}$;[5]

3. for all $F : \mathbb{W} \times \mathbb{X} \to \widehat{\mathbb{Y}}$, the natural isomorphism $\mathbf{s}_F$ has as components further natural isomorphisms

$$\begin{aligned}&\int^{X} PX \times F(W,X)(-)\\ &\cong \int^{X} PX \times \int^{W',X'} F(W',X')(-) \times \mathrm{Hom}((W',X'),(W,X))\\ &\cong \int^{X} PX \times \int^{W',X'} \mathrm{Hom}((W',X'),(W,X)) \times F(W',X')(-)\\ &\cong \int^{W',X'} \left(\int^{X} PX \times \mathrm{Hom}((W',X'),(W,X))\right) \times F(W',X')(-).\end{aligned}$$

4.6 Towards an induced strong pseudomonad

The structure 4.12 requires some further axioms to be useful. Ideally, we would like for the structural 2-cells to be coherent. However, proving such a result may be very difficult. A more attainable goal is to show that, in the case where the inclusion is the identity, we obtain a strong pseudomonad.

We are not able to state any further axioms or prove any of the results above. Instead we are going to outline how one can obtain the structure of a strong pseudomonad. We begin by constructing the strength and proceed by adding the structural 2-cells that promote it to a strength of the induced pseudofunctor and subsequently the induced pseudomonad.

Fix a strong J-pseudomonad structure T.

We have already discussed how to obtain the 1-cells $JW \times TX \to T(W \times X)$ that resemble the components of a pseudonatural transformation. We are now able to extend this structure to include the naturality 2-cell. Combining these, we obtain the structure of a pseudonatural transformation.

Definition 4.14. The *strength induced by T* consists of

[5] Construction of $\mathbf{q}$ in quiver.

1. for all $W, X \in \mathbf{J}$, the 1-cell
$$\sigma_{W,X} = (\eta_{W\times X})^{\dagger} : JW \times TX \to T(W \times X) \tag{4.24}$$

2. for all $f : W \to W'$ and $g : X \to X'$ in $\mathbf{J}$, the naturality 2-cell
$$\mathbf{n}_{f,g} = \mathbf{c} \bullet (\mathbf{r}J(f \times g))^{\dagger} \bullet \mathbf{s}^{-1}$$
as in the diagram
$$\begin{tikzcd}
JW \times TX \arrow[rr, "\eta^{\dagger}"] \arrow[d, equal] \arrow[ddrr, "(\eta(Jf\times Jg))^{\dagger}"] & {} \arrow[d, phantom, "\Downarrow \mathbf{s}^{-1}"] & T(W\times X) \arrow[dd, "T(f\times g)"] \\
JW \times TX \arrow[dd, "Jf\times Tg"'] \arrow[ddrr, "(\eta^{\dagger}(Jf\times\eta Jg))^{\dagger}"'] & {} \arrow[d, phantom, "\Downarrow (\mathbf{r}(Jf\times Jg))^{\dagger}"] & \\
 & {} \arrow[d, phantom, "\Downarrow \mathbf{c}"] & T(W'\times X') \arrow[d, equal] \\
JW'\times TX' \arrow[rr, "\eta^{\dagger}"'] & {} & T(W'\times X')
\end{tikzcd} \tag{4.25}$$

With the axioms we have included so far, we are not able to prove that this is indeed a pseudonatural transformation. Further, we have not been able to identify any intuitive or obvious axioms that would achieve this goal.[6].

Example 4.15. For the strong presheaf construction, the strength is given by

1. the functor $\sigma : \mathbb{W} \times \widehat{\mathbb{X}} \to \widehat{\mathbb{W} \times \mathbb{X}}$ acting on objects by
$$\sigma(W,P) = \int^{X} PX \times \mathrm{Hom}(-,(W,X));$$

2. the natural transformation $\mathbf{n}_{F,G} : JF \times \widehat{G} \to \widehat{F \times G}$ which has as components natural isomorphisms
$$\begin{aligned}&\int^{X',Y'} \left(\int^{Y} PY \times \mathrm{Hom}((X',Y'),(X,Y)) \right) \times \mathrm{Hom}(-,(FX',GY')) \\ &\cong \int^{Y'} \left(\int^{Y} PY \times \mathrm{Hom}(Y',GY) \right) \times \mathrm{Hom}(-,(FX,Y')).\end{aligned}$$

In order for a strong J-pseudomonad to induce a strong pseudomonad in the sense of [14], we require J to be the identity. We describe how one obtains the appropriate structure:

[6]For future reference, the diagram that we require to commute

Definition 4.16. Let $J : \mathbf{C} \to \mathbf{C}$ be the identity and T a strong J-pseudomonad structure. The *strong pseudomonad structure induced by* T consists of

1. the strength 4.14;
2. for all $X \in \mathbf{C}$, the invertible 2-cell $\mathbf{x}_X = \mathbf{s}_{\eta\lambda} \bullet \mathbf{l}_X$;
3. for all $X, Y, Z \in \mathbf{C}$, the invertible 2-cell $\mathbf{y}_{X,Y,Z} = \mathbf{p}_{X,Y,Z}$;
4. for all $X, Y \in \mathbf{C}$, the invertible 2-cell $\mathbf{w}_{X,Y}$ given by

$$\begin{tikzcd}
W \times T^2X \arrow[rr, "X\times \mathrm{id}^*"] \arrow[d, dashed, "\eta^\dagger"'] \arrow[drr, phantom, "\xRightarrow{\mathbf{q}^{-1}}"] & & W \times TX \arrow[d, dashed, "\eta^\dagger"] \\
T(W \times TX) \arrow[rr, "(\eta^\dagger)^*"] \arrow[d, "(\eta\eta^\dagger)^*"'] \arrow[drr, "(\lambda^\dagger(1\times\eta\eta^\dagger))^\dagger\lambda^{-1}" description] \arrow[dr, phantom, "\xRightarrow{\mathbf{c}^{-1}}"] & \arrow[dr, phantom, "\xRightarrow{(\mathbf{r}^{-1}(1\times\eta^\dagger))^\dagger\lambda^{-1}}"] & T(W \times X) \arrow[d, equal] \\
T^2(W \times X) \arrow[rr, "\mathrm{id}^*"] & & T(W \times X)
\end{tikzcd} \tag{4.26}$$

5. for all $X, Y \in \mathbf{C}$, the invertible 2-cell $\mathbf{z}_{X,Y} = \mathbf{r}_\eta^{-1}$.

Beyond the previously mentioned pseudonaturality, it is straightforward to postulate axioms that ensure the coherence axioms for strong pseudofunctors are satisfied. This is because none of these coherence axioms contain repeated applications of the pseudofunctor. Thus the notion generalises from endofunctors to arbitrary pseudofunctors.

Chapter 5

Evaluation

It is now time to reflect on what we have and have not achieved. We begin by highlighting two alternative approaches that we could have taken to solve the problem. We then assess the quality of our results. Finally, we focus on the presentation of these results and the compromises involved.

5.1 Small presheaves instead of relative pseudomonads

We worked towards extending synthetic measure theory to admit the presheaf construction as a model by generalising to relative pseudomonads. This has made it very difficult to develop the required theory but almost trivial to establish the model.

While we believe that this path can lead to success, it may not be the simplest. In the early stages of this project we were aiming to restrict **2CAT** to a suitable cartesian closed 2-category on which the presheaf construction is a pseudomonad. We tried to work with small presheaves on locally presentable categories but this failed. After spending a significant amount of time with this approach, we decided that it would be safer to follow the potentially longer route of relative pseudomonads as it would allow us to make some progress rightaway.

This does not mean, however, that it is impossible to restrict the presheaf construction and the underlying category appropriately so that generalising synthetic measure theory to pseudomonads is sufficient. In a sense, these considerations are likely to lead to an entirely new set of problems: formulating the theory would become signficantly easier, but constructing a non-trivial model would be difficult.

5.2 Strength as a natural transformation

The next thing that has to be criticised is our choice of changing the extension operator to incorporate strength, rather than adding the strength to a relative pseudomonad as a suitable pseudonatural transformation. Whether this approach is fruitful remains to be seen. However, there are several notable disadvantages.

Firstly, it requires a certain amount of reinventing the wheel. Rather than sticking with the already established theory of relative pseudomonads, we had to rewrite the definition entirely. While the result is similar, a lot of time and effort went into making everything consistent. Given that we have not been able to reap the benefits of the new structure, it is unclear whether this detour will eventually pay off.

Secondly, the new structure is in many ways less elegant than usual relative pseudomonads. For example, have a look a the 2-cell families **c** and **m** in the case of the presheaf construction. The former involves additional parts that are not required for the latter. This makes it more difficult to distinguish important details from the overall noise.

5.3 Quality of the definitions

Inspecting where the complexity is inspires confidence. This is because the largest diagrams arise whenever coends are involved. This is deceptive, however. Showing coend-related results has, for the most part, been a straightforward mechanical task. The vast majority of work went into the three core definitions 4.1, 4.6, and 4.12. The fact that all the developments related to our theory, in particular the proofs and the induced structures, are short and succinct suggests that we have indeed developed a suitable language to reason about strong relative pseudomonads.

5.4 Presentation

The complexity of the expressions has repeatedly led to diagrams whose width was several times what could fit the page. We have therefore not been able to include as much detail as we would have liked. This means that some proofs may be harder to follow than is appropriate. In any case, we do not expect anyone to be able to reconstruct every step of our developments without some pen and paper.

This problem is not new to category theorists. There have been several approaches to deal with large diagrams that usually require even more notational shortcuts. See [13] and [14] for some related examples. While this would have allowed us to condense more information onto the page, it would have also meant hiding a signficiant amount of complexity and thus required long explanations as to what is going on. We are doubtful whether this would have been possible within the scope of this report.

To aid the reader we have decided to include quiver links to some particularly large diagrams. There are several reasons why this is not a solution that can be relied upon, though. Firstly, there is the technical problem that the content of those links is not strictly part of the report. Secondly, the quiver server will, eventually, go offline. If we were to rely on the service, then a significant part of the content would be lost. Of course, the latter problem may be solved by installing quiver locally.

Machine verified proofs may be another way to maintain rigour while improving the presentation. This way we would be able to hide some technical details in the comforting knowledge that everything has been made to work as intended. Those who are interested would be welcome to read the corresponding source code. While this is in a sense the optimal solution it is also idealistic: formalising our work like this would vastly increase our time investment. This would be a particularly risky bet given that we have not been able to show that things are going to work out in the end.

5.5 Target audience

This report is supposed to be targeted towards undergraduate students. Meeting this requirement has been difficult. While there are a few courses that define categories, functors, and natural transformations, category theory is not taught at the university in its own right. This has led us to take some major shortcuts. For example, we would have liked to investigate our attempt at restricting our model as described in 5.1. Unfortunately, this would have required us to define locally presentable categories from the ground up which is impossible with the space that we are given.

While this report may not provide a comprehensive introduction to category theory, it still offers valuable insights and contributions that can be understood by readers with varying degrees of knowledge in the subject. To fully comprehend the technical details of the report, only a basic understanding of category theory is required. As a result, undergraduate students with some familiarity with natural transformations should be able to understand almost all the technical details of our work. The only exceptions are more technical arguments such as the existence of coends in **Set**, which have to be taken for granted. However, our explanations make it possible for readers without prior knowledge to follow the main ideas and results presented.

Chapter 6

Future work

Given that we have not been able to establish the presheaf construction as a model of a generalised synthetic measure theory, a lot of future work remains.

The first step should be to make sure that the induced strength is a pseudonatural transformation. It is hard to say how difficult a problem this would be, but more axioms related to the interplay of the strong structural 2-cells **u** and **s** with their prestrong counterparts are required. Such axioms will certainly prove useful for later developments.

Secondly, there is the problem of postulating axioms so that the induced strong pseudomonad satisfies the coherence conditions of a strong pseudomonad in the sense of [14]. This has the potential to be a lot of work: there are five axioms for which there are no obvious counterparts. The unusual 2-cell **q** in 4.12 is a good example of the ingenuity that might be required to make this work. It is safe to say that the notion of strength does not lend itself to being generalised to relative monads.

Thirdly, once strong relative pseudomonads have been defined it is time for the most interesting part. Generalising synthetic measure theory. Studying the measure theory arising from the presheaf construction should be satisfying in its own right. After all, this is the motivation for this whole project. How difficult of a task this will end up being depends on the quality of the definitions obtained in the previous steps.

Finally, formalising our theory in a theorem proving language seems like a logical extension. The axioms required to make a complete definition of a strong relative pseudomonad are bound to be complex and the addition of synthetic measure theory will not improve the situation. This means that it will become even harder to verify the correctness of the statements involved.

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