Solusi Ujian Akhir Semester

Metode Numerik dan Teknik Komputasi

1. **Bab Interpolasi Numerik**

   a) $x_s = 1.3233$

   Ada 2 pendekatan untuk soal (a), yaitu menggunakan metode 1) *Lagrange* atau 2) *Newton Gregory Forward* (NGF). *Lagrange* dipilih karena menghasilkan *error* yang paling kecil (mahasiswa sudah membuktikannya saat Tugas Besar) sedangkan *Newton Gregory Forward* dipilih karena titik yang dicari berada di antara x_0 dan x_1 serta *equispaced* (nilai bedanya sama/ $x_1 – x_0 = x_2 – x_1 = dst..$). Jika mahasiswa menggunakan *Lagrange*, nilai maksimal yang diperoleh adalah **20 point**. Sedangkan untuk NGF, **15 point**. Perincian untuk masing-masing metode dijelaskan sebagai berikut.

   1. Menggunakan *Lagrange*. Persamaan yang digunakan adalah:

$$
\begin{aligned}
f(x_s) = {} & \frac{(x_s - x_1)(x_s - x_2)(x_s - x_3)(x_s - x_4)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)(x_0 - x_4)} f_0 \\
& + \frac{(x_s - x_0)(x_s - x_2)(x_s - x_3)(x_s - x_4)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)(x_1 - x_4)} f_1 \\
& + \frac{(x_s - x_0)(x_s - x_1)(x_s - x_3)(x_s - x_4)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)(x_2 - x_4)} f_2 \\
& + \frac{(x_s - x_0)(x_s - x_1)(x_s - x_2)(x_s - x_4)}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)(x_3 - x_4)} f_3 \\
& + \frac{(x_s - x_0)(x_s - x_1)(x_s - x_2)(x_s - x_3)}{(x_4 - x_0)(x_4 - x_1)(x_4 - x_2)(x_4 - x_3)} f_4
\end{aligned}
$$

$$
\begin{aligned}
f(1.3233) = {} & \frac{(1.3233 - 1.1781)(1.3233 - 0.7854)(1.3233 - 0.3927)(1.3233 - 0)}{(1.5708 - 1.1781)(1.5708 - 0.7854)(1.5708 - 0.3927)(1.5708 - 0)} \times 1 \\
& + \frac{(1.3233 - 1.5708)(1.3233 - 0.7854)(1.3233 - 0.3927)(1.3233 - 0)}{(1.1781 - 1.5708)(1.1781 - 0.7854)(1.1781 - 0.3927)(1.1781 - 0)} \times 0.9239 \\
& + \frac{(1.3233 - 1.5708)(1.3233 - 1.1781)(1.3233 - 0.3927)(1.3233 - 0)}{(0.7854 - 1.5708)(0.7854 - 1.1781)(0.7854 - 0.3927)(0.7854 - 0)} \times 0.7071 \\
& + \frac{(1.3233 - 1.5708)(1.3233 - 1.1781)(1.3233 - 0.7854)(1.3233 - 0)}{(0.3927 - 1.5708)(0.3927 - 1.1781)(0.3927 - 0.7854)(0.3927 - 0)} \times 0.3827 \\
& + \frac{(1.3233 - 1.5708)(1.3233 - 1.1781)(1.3233 - 0.7854)(1.3233 - 0.3927)}{(0 - 1.5708)(0 - 1.1781)(0 - 0.7854)(0 - 0.3927)} \times 0
\end{aligned}
$$

$$
\begin{aligned}
f(1.3233) = {} & \frac{(0.1452)(0.5379)(0.9306)(1.3233)}{(0.3927)(0.7854)(1.1781)(1.5708)} \times 1 \\
& + \frac{(-0.2475)(0.5379)(0.9306)(1.3233)}{(-0.3927)(0.3927)(0.78540)(1.1781)} \times 0.9239 \\
& + \frac{(-0.2475)(0.1452)(0.9306)(1.3233)}{(-0.7854)(-0.3927)(0.3927)(0.7854)} \times 0.7071 \\
& + \frac{(-0.2475)(0.1452)(0.5379)(1.3233)}{(-1.1781)(-0.78540)(-0.39270)(0.3927)} \times 0.3827 \\
& + \frac{(-0.2475)(0.1452)(0.5379)(0.9306)}{(-1.5708)(-1.1781)(-0.7854)(-0.3927)} \times 0 \\
f(1.3233) = {} & \frac{0.0962}{0.5708} \times 1 \ + \ \frac{-0.16394}{-0.1427} \times 0.9239 \ + \ \frac{-0.0442}{0.0951} \times 0.7071 \\
& + \frac{-0.0256}{-0.1427} \times 0.3827 \ + \ \frac{-0.0180}{0.5708} \times 0 \\
f(1.3233) = {} & 0.1685 \ + \ 1.0614 \ + \ -0.32864 \ + \ 0.0687 \ + \ 0 \\
f(1.3233) = {} & 0.96996
\end{aligned}
$$

2. Menggunakan *Newton Gregory Forward* (**NGF**) karena nilai beda variabelnya sama (*equispaced* / $x_1 – x_0 = x_2 – x_1 = dst..$) dan titik x_s yang dicari di antara x_0 dan x_1.

Langkah Pertama. Membuat tabel beda. Jika tabel bedanya benar seperti yang di bawah ini, beri nilai **5 point**.

n	x_n	$f_n(x)$	$\Delta f_n(x)$	$\Delta^2 f_n(x)$	$\Delta^3 f_n(x)$	$\Delta^4 f_n(x)$
0	1.5708	1.0000				
			-0.0761			
1	1.1781	0.9239		-0.1406		
			-0.2168		0.0330	
2	0.7854	0.7071		-0.1076		0.0164
			-0.3244		0.0494	
3	0.3927	0.3827		-0.0583		
			-0.3827			
4	0.0000	0.0000				

Langkah Kedua. Penggunaan persamaan NGF. Jika benar semua, beri nilai **10 point**. Jika persamaan benar namun proses perhitungannya salah, beri nilai **8 point.**

$$
\begin{aligned}
s &= \frac{x_s - x_0}{x_1 - x_0} \\
&= \frac{1.3233 - 1.5708}{1.1781 - 1.5708} \\
&= \frac{-0.2475}{-0.3927} \\
&= 0.6302
\end{aligned}
$$

$$
\begin{aligned}
f(x_s) &= f_0 + s\Delta f_0 + \frac{s(s-1)}{2!}\Delta^2 f_0 + \frac{s(s-1)(s-2)}{3!}\Delta^3 f_0 + \frac{s(s-1)(s-2)(s-3)}{4!}\Delta^4 f_0 \\
&= 1.0000 \ + \ (0.6302)(-0.0761) \ + \ \frac{(0.6302)(0.6302-1)}{2}(-0.1407) \\
&\quad + \frac{(0.6302)(0.6302-1)(0.6302-2)}{6}(0.0330) \\
&\quad + \frac{(0.6302)(0.6302-1)(0.6302-2)(0.6302-3)}{24}(0.0164) \\
&= 1.0000 - 0.0480 + 0.0164 + 0.0018 - 0.0005 \\
&= 0.9697
\end{aligned}
$$

b) $x_s = 0.9748$

Ada 2 pendekatan dalam menyelesaikan permasalahan interpolasi pada titik ini (x_s) yaitu dengan menggunakan metode *Lagrange* (karena variabelnya sedikit dan metode paling akurat dari 5 metode yang diajarkan) atau dengan menggunakan *Stirling* (karena titik x_s pada **Tabel 1** berada di tengah)

1. Metode *Lagrange*

$$
\begin{aligned}
f(x_s) &= \frac{(x_s - x_1)(x_s - x_2)(x_s - x_3)(x_s - x_4)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)(x_0 - x_4)} f_0 \\
&\quad + \frac{(x_s - x_0)(x_s - x_2)(x_s - x_3)(x_s - x_4)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)(x_1 - x_4)} f_1 \\
&\quad + \frac{(x_s - x_0)(x_s - x_1)(x_s - x_3)(x_s - x_4)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)(x_2 - x_4)} f_2 \\
&\quad + \frac{(x_s - x_0)(x_s - x_1)(x_s - x_2)(x_s - x_4)}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)(x_3 - x_4)} f_3 \\
&\quad + \frac{(x_s - x_0)(x_s - x_1)(x_s - x_2)(x_s - x_3)}{(x_4 - x_0)(x_4 - x_1)(x_4 - x_2)(x_4 - x_3)} f_4
\end{aligned}
$$

$$f(0.9748) = \frac{(0.9748-1.1781)(0.9748-0.7854)(0.9748-0.3927)(0.9748-0)}{(1.5708-1.1781)(1.5708-0.7854)(1.5708-0.3927)(1.5708-0)} \times 1$$
$$+ \frac{(0.9748-1.5708)(0.9748-0.7854)(0.9748-0.3927)(0.9748-0)}{(1.1781-1.5708)(1.1781-0.7854)(1.1781-0.3927)(1.1781-0)} \times 0.9239$$
$$+ \frac{(0.9748-1.5708)(0.9748-1.1781)(0.9748-0.3927)(0.9748-0)}{(0.7854-1.5708)(0.7854-1.1781)(0.7854-0.3927)(0.7854-0)} \times 0.7071$$
$$+ \frac{(0.9748-1.5708)(0.9748-1.1781)(0.9748-0.7854)(0.9748-0)}{(0.3927-1.5708)(0.3927-1.1781)(0.3927-0.7854)(0.3927-0)} \times 0.3827$$
$$+ \frac{(0.9748-1.5708)(0.9748-1.1781)(0.9748-0.7854)(0.9748-0.3927)}{(0-1.5708)(0-1.1781)(0-0.7854)(0-0.3927)} \times 0$$

$$f(0.9748) = \frac{(-0.2033)(0.1894)(0.5821)(0.9748)}{(0.3927)(0.7854)(1.1781)(1.5708)} \times 1$$
$$+ \frac{(-0.5960)(0.1894)(0.5821)(0.9748)}{(-0.3927)(0.3927)(0.7854)(1.1781)} \times 0.9239$$
$$+ \frac{(-0.5960)(-0.2033)(0.5821)(0.9748)}{(-0.7854)(-0.3927)(0.3927)(0.7854)} \times 0.7071$$
$$+ \frac{(-0.5960)(-0.2033)(0.1894)(0.9748)}{(-1.1781)(-0.7854)(-0.3927)(0.3927)} \times 0.3827$$
$$+ \frac{(-0.5960)(-0.2033)(0.1894)(0.5821)}{(-1.5708)(-1.1781)(-0.7854)(-0.3927)} \times 0$$

$$f(0.9748) = \frac{-0.0218}{0.57076} \times 1 \ + \ \frac{-0.064053}{-0.14269} \times 0.9239$$
$$+ \frac{0.068754}{0.095127} \times 0.7071 \ + \ \frac{0.022371}{-0.14269} \times 0.3827$$
$$+ \frac{0.013359}{0.57076} \times 0$$

$$f(0.9748) = -0.038195 \ + \ 0.41474 \ + \ 0.51106 \ + \ -0.060000 \ + \ 0$$
$$f(0.9748) = 0.82760$$

2. Metode *Stirling*, karena titik x_s berada diantara x_{-1} dan x_0 dan nilai beda variabelnya adalah sama ($x_{-1} - x_{-2} \ = \ x_0 - x_{-1} \cdots \mathrm{dst.}$)

n	x_n	$f_n(x)$	$\Delta f_n(x)$	$\Delta^2 f_n(x)$	$\Delta^3 f_n(x)$	$\Delta^4 f_n(x)$
-2	1.5708	1.0000				
			-0.0761			
-1	1.1781	0.9239		-0.1406		
			-0.2168		0.0330	
0	0.7854	0.7071		-0.1076		0.0164
			-0.3244		0.0494	
1	0.3927	0.3827		-0.0583		

			-0.3827			
2	0.0000	0.0000				

$$
\begin{aligned}
s &= \frac{x_s - x_0}{h} \\
&= \frac{0.9748 - 0.7854}{1.1781 - 1.5708} \\
&= \frac{0.18940}{-0.39270} \\
&= -0.4823
\end{aligned}
$$

$$
\begin{aligned}
f(x_s) = f_0 &+ \left| \begin{matrix} s \\ 1 \end{matrix} \right| \frac{\Delta f_{-1} + \Delta f_0}{2} + \frac{\left| \begin{matrix} s+1 \\ 2 \end{matrix} \right| + \left| \begin{matrix} s \\ 2 \end{matrix} \right|}{2} \Delta^2 f_{-1} \\
&+ \left| \begin{matrix} s+1 \\ 3 \end{matrix} \right| \frac{\Delta^3 f_{-2} + \Delta^3 f_{-1}}{2} + \frac{\left| \begin{matrix} s+2 \\ 4 \end{matrix} \right| + \left| \begin{matrix} s+1 \\ 4 \end{matrix} \right|}{2} \Delta^4 f_{-2}
\end{aligned}
$$

dimana,

$$
\begin{aligned}
\left| \begin{matrix} s \\ 1 \end{matrix} \right| &= s \\
&= -0.4823
\end{aligned}
$$

$$
\begin{aligned}
\left| \begin{matrix} s+1 \\ 2 \end{matrix} \right| + \left| \begin{matrix} s \\ 2 \end{matrix} \right| &= \frac{(s+1)(s)}{2!} + \frac{(s)(s-1)}{2!} \\
&= \frac{(-0.4823+1)(-0.4823)}{2!} + \frac{(-0.4823)(-0.4823-1)}{2!} \\
&= -0.12484 + 0.35746 \\
&= 0.23262
\end{aligned}
$$

$$
\begin{aligned}
\left| \begin{matrix} s+1 \\ 3 \end{matrix} \right| &= \frac{(s+1)(s)(s-1)}{3!} \\
&= \frac{(-0.4823+1)(-0.4823)(-0.4823-1)}{6} \\
&= 0.061685
\end{aligned}
$$

$$\begin{vmatrix} s+2 \\ 4 \end{vmatrix} + \begin{vmatrix} s+1 \\ 4 \end{vmatrix} = \frac{(s+2)(s+1)(s)(s-1)}{4!} + \frac{(s+1)(s)(s-1)(s-2)}{4!}$$
$$= \frac{(-0.4823+2)(-0.4823+1)(-0.4823)(-0.4823-1)}{24}$$
$$+ \frac{(-0.4823+1)(-0.4823)(-0.4823-1)(-0.4823-2)}{24}$$
$$= 0.023405 + (-0.038280)$$
$$= -0.014875$$

sehingga,

$$f(x_s) = 0.7071 + (-0.4823)\frac{(-0.2168) + (-0.3244)}{2}$$
$$+ \frac{0.23262}{2}(-0.1076) + (0.061685)\frac{0.0330 + 0.0494}{2}$$
$$+ \frac{-0.014875}{2}0.0164$$
$$= 0.7071 + (0.13051) + (-0.012515) + (-0.00012198)$$
$$= 0.82497$$

2. **Bab Interpolasi Numerik.** Kelebihan dan kekurangan metode *Newton Gregory Forward*, *Stirling* dan *Lagrange*.

   a) *Newton Gregory Forward* :

   + Metode yang efektif untuk mencari nilai $f(x)$ di sekitar titik awal,

   - Hanya dapat digunakan menyelesaikan persoalan interpolasi *equispaced*,

   - Menyelesaikan permasalahan untuk nilai x_s terletak di antara x_0 dan x_1.

   b) *Stirling* :

   + Metode yang efektif untuk mencari nilai $f(x)$ di sekitar titik tengah,

   - Hanya dapat digunakan menyelesaikan persoalan interpolasi *equispaced*,

   - Menyelesaikan permasalahan untuk nilai x_s terletak di antara x_0 dan x_1 atau di antara x_{-1} dan x_0.

   c) *Lagrange* :

   + Dapat digunakan menyelesaikan persoalan interpolasi *equispaced* dan *non-equispaced*,

   + Dapat digunakan untuk mencari nilai $f(x)$ di sekitar titik awal, tengah dan akhir,

   + Tidak membutuhkan tabel beda dalam penyelesaian masalah,

   - Jika nilai variabel x dan nilai fungsinya $f(x)$ terlalu banyak, maka perhitungan menjadi kompleks.

3. Bab Diferensiasi Numerik

a) Titik $x_s = 1.4321$ berada di titik awal antara $x_0 = 0$ dan $x_1 = 0.3142$ sehingga menggunakan Metode *Newton Gregory Forward.*

n	x_n	$f_n(x)$	$\Delta f_n(x)$	$\Delta^2 f_n(x)$	$\Delta^3 f_n(x)$	$\Delta^4 f_n(x)$	$\Delta^5 f_n(x)$
0	0.0000	0.0000					
			0.7167				
1	0.3142	0.7167		0.1369			
			0.8536		-0.0546		
2	0.6283	1.5704		0.0823		0.0113	
			0.9359		-0.0433		0.0042
3	0.9425	2.5063		0.0390		0.0155	
			0.9750		-0.0278		
4	1.2566	3.4812		0.0112			
			0.9862				
5	1.5708	4.4674					

$$
\begin{aligned}
f'(x) = \frac{d(f(x))}{dx} = \quad & \frac{1}{h}\left[\Delta f_0 + \frac{2s+1}{2!}\Delta^2 f_0 + \frac{3s^2+6s+2}{3!}\Delta^3 f_0\right. \\
& + \frac{4s^3+18s^2+22s+6}{4!}\Delta^4 f_0 \\
& \left. + \frac{5s^4+40s^3+105s^2+100s+24}{5!}\Delta^5 f_0\right]
\end{aligned}
$$

b)

c)

4.