MATRIKULASI

MATEMATIKA DASAR

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Pokok Bahasan

1. Pertidaksamaan Linier
   - Interval
   - Penyelesaian Pertidaksamaan
2. Fungsi dan Limit
   - Fungsi
   - Limit
3. Trigonometri
4. Turunan
5. Integral
   - Integral Tak Tentu
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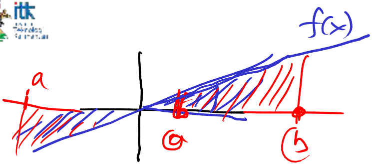
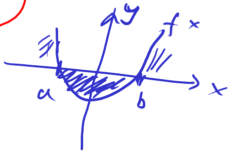
   - Integral Tentu

Integral Tak Tentu





Secara geometri integral merupakan suatu luasan daerah pada kurva tertentu. Jika diberikan suatu fungsi $f(x)$ yang kontinu tak negatif pada interval $[a, b]$, maka yang dimaksud dengan

$$\int_a^b f(x)dx \tag{13}$$

adalah luasan dibawah kurva $f(x)$.

$g(x) = x + 3$
$f(x) = x + 2$

$\frac{d}{dx} f(x) = 1$

$\frac{d}{dx} g(x) = 1$

$\int 1 \, dx \to x + c$

Integral Tak Tentu

Integral juga disebut sebagai **anti turunan**. Beberapa contoh integral tak tentu sebagai anti turunan dapat dilihat pada tabel berikut

Turunan	Anti Turunan
$\frac{d}{dx}[x] = 1$	$\int 1 dx = x + C$
$\frac{d}{dx}[x^{n+1}] = x^n$	$\int x^n dx = \frac{x^{n+1}}{n+1} + C$
$\frac{d}{dx}[\sin x] = \cos x$	$\int \cos x \, dx = \sin x + C$
$\frac{d}{dx}[-\cos x] = \sin x$	$\int \sin x \, dx = -\cos x + C$
$\frac{d}{dx}[\tan x] = \sec^2 x$	$\int \sec^2 x \, dx = \tan x + C$
$\frac{d}{dx}[-\cot x] = \operatorname{cosec} x$	$\int \operatorname{cosec} x \, dx = -\cot x + C$
$\frac{d}{dx}[\sec x] = \sec x \tan x$	$\int \sec x \tan x \, dx = \sec x + C$
$\frac{d}{dx}[-\operatorname{cosec} x] = \operatorname{cosec} x \cot x$	$\int \operatorname{cosec} x \cot x \, dx = -\operatorname{cosec} x + C$

$\int x^n dx = \frac{1}{n+1} x^{(n+1)} + c$

Integral Tak Tentu

Secara umum integral tak tentu dapat ditulis

$$\int ax^n dx = \frac{a}{n+1}x^{n+1} + C \tag{14}$$

Integral Tak Tentu

Integral tak tentu mempunyai sifat – sifat sebagai berikut:

1. Pengali konstan dapat dikeluarkan dari operasi integral

$$\int c f(x)dx = c \int f(x)dx \tag{15}$$

2. Integral dari penjumlahan dan pengurangan fungsi integran dapat dinyatakan sebagai jumlahan atau pengurangan dari masing-masing integral fungsi yang berkaitan

$$\int [f(x) \pm g(x)] = \int f(x)dx \pm \int g(x)dx \tag{16}$$

Integral dengan Substitusi

Selain mengintegralkan secara langsung, terdapat beberapa teknik pengintegralan, salah satunya integral dengan substitusi. Berikut langkah – langkah pengerjaan integral dengan teknik substitusi:

1. Tentukan suatu fungsi tertentu sebagai u, yaitu $u = g(x)$.
2. Hitung $\frac{du}{dx} = g'(x)$.
3. Substitusi $u = g(x)$ dan $du = g'(x)dx$. Perhatikan bahwa, pada step ini integrase harus dalam suku-suku u, sehingga tidak ada suku-suku variabel x.
4. Selesaikan integral tersebut (masih dalam suku-suku u).
5. Ganti kembali u dengan $g(x)$, sehingga diperoleh hasil dengan variabel x

Contoh 1

$$\int ax^n\,dx = \frac{a}{n+1}x^{n+1} + C$$

Tentukan

$$\int 5x^{\frac{3}{5}}\,dx = \frac{5}{\frac{3}{5}+1}x^{\frac{3}{5}+1} + C$$

$\frac{5}{5}$ $\quad \frac{3}{5}+\frac{5}{5} = \frac{3+5}{5}$

$$= \frac{5}{\frac{3+5}{5}}x^{\frac{3+5}{5}} + C$$

$$= \frac{5}{\frac{8}{5}}x^{\frac{8}{5}} + C$$

$$= 5 \cdot \frac{5}{8}x^{\frac{8}{5}} + C$$

$$\frac{25}{8}x^{\frac{8}{5}} + C \quad //$$

Contoh 2

$\int f(x)\,dx$

Tentukan

$$\int \underbrace{(2x^2+3)}_{u}{}^{25} 4x\,dx$$

$2x^2+3 = u$

$4x = \frac{du}{dx}$

$\frac{du}{4x} = dx$

$= \int u^{25} \cdot \cancel{4x} \cdot \frac{du}{\cancel{4x}}$

$= \int u^{25}\,du$

$= \frac{1}{25+1} u^{25+1} + C$

$= \frac{1}{26} u^{26} + C$

$= \frac{1}{26}(2x^2+3)^{26} + C$

Integral Tentu

$\int f(x)\,dx = F(x)$

- Integral tertentu merupakan suatu integral yang memiliki batas integrasi, yaitu batas bawah, yang disimbolkan dengan a dan batas atas, yang disimbolkan dengan b. Jika suatu integral memiliki batas, maka hasil integral tersebut adalah tunggal.
- Misalkan $F(x)$ merupakan suatu fungsi anti turunan dari fungsi $f(x)$ pada interval $[a, b]$, maka

$$\int_a^b f(x)dx = F(b) - F(a) \tag{17}$$

Integral Tentu

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Berikut sifat – sifat dari integral tertentu: $F(a) - F(a) = 0$

1. Jika $a = b$ maka

$$\int_a^b f(x)dx = \int_a^a f(x)dx = 0 \tag{18}$$

2. Jika f adalah fungsi yang terintegral pada interval $[a, b]$, maka

$$\int_a^b f(x)dx = -\int_b^a f(x)dx \tag{19}$$

$$= -\left(F(a) - F(b)\right)$$

$$= F(b) - F(a).$$

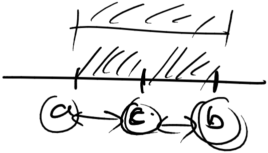
Integral Tentu

3. Jika $f(x)$ dan $g(x)$ merupakan fungsi yang dapat diintegralkan pada interval $[a, b]$ dan k adalah suatu konstanta, maka

$$\int_a^b k f(x) dx = k \int_a^b f(x) dx \quad (20)$$

$$\int_a^b \left[f(x) \pm g(x) \right] dx = \int_a^b f(x) dx \pm \int_a^b g(x) dx \quad (21)$$

Integral Tentu



4. Jika $f(x)$ terintegralkan pada interval $[a, b]$, dimana c adalah suatu titik diantara interval $[a, b]$, maka

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx \tag{22}$$

5. Integral tertentu tidak bergantung pada variabel yang digunakan

$$\int_a^b f(x)dx = \int_a^b f(t)dt = \int_a^b f(u)du \tag{23}$$

$$= \int_a^b f(k)\,dk$$

Integral Tentu

$[b, c]$

$f(x) > g(x)$

$\int_b^c f(x)\,dx > \int_b^c f(x)\,dx$

6. Jika $f(x)$ dan $g(x)$ terintegralkan pada interval $[a, b]$ dan $f(x) \leq g(x)$ pada $x \in [a, b]$, maka

$$\int_a^b f(x)dx \leq \int_a^b g(x)dx \tag{24}$$

Contoh 3

$\pi \quad \frac{\pi}{2} \quad 0$

s c
c -s

$$\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

Tentukan

$$\int_0^{\frac{\pi}{2}} (\sin x + \cos x)dx$$

$$= \int_0^{\pi/2} \sin x\,dx + \int_0^{\pi/2} \cos x\,dx$$

$$= -\cos x \Big|_0^{\pi/2} + \sin x \Big|_0^{\pi/2}$$

$$\left[-\cos\left(\tfrac{\pi}{2}\right) - \left(-\cos(0)\right)\right] + \left[\sin\left(\tfrac{\pi}{2}\right) - \sin(0)\right]$$

$$= [0+1] + [1-0]$$

$$= 1+1 = 2$$

Contoh 4

$$f(x) = x + 1 \leftarrow x \geq 0$$

$$f(x) = 2x \leftarrow \; < 0$$

Diberikan

$$f(x) = \begin{cases} x + 1, & x \geq 0 \\ 2x, & x < 0 \end{cases}$$

Hitunglah $\int_{-1}^{1} f(x)\,dx$

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

$$\int_{-1}^{1} f(x)\,dx = \underbrace{\int_{-1}^{0} f(x)\,dx}_{f(x) = 2x} + \underbrace{\int_{0}^{1} f(x)\,dx}_{f(x) = x+1}$$

$$\boxed{\int_{-1}^{1} f(x)\,dx} = \underbrace{\int_{-1}^{0} f(x)\,dx}_{f(x)=2x} + \underbrace{\int_{0}^{1} f(x)\,dx}_{f(x)=x+1}$$

$$= \int_{-1}^{0} 2x\,dx + \int_{0}^{1} x + 1\,dx \qquad x^0$$

$$= \frac{2}{1+1} x^{1+1} \Big|_{-1}^{0} + \frac{1}{1+1} x^{1+1} + \frac{1}{0+1} x^{0+1} \Big|_{0}^{1}$$

$$= x^2 \Big|_{-1}^{0} + \left(\frac{1}{2}x^2 + x\right)\Big|_{0}^{1}$$

$$= \left(0^2 - (-1)^2\right) + \left(\underline{\frac{1}{2}(1)^2} + \underline{1} \underbrace{- \left(\frac{1}{2}(0)^2 + 0\right)}_{0}\right)$$

$$= -1 + \frac{1}{2} + 1 = \frac{1}{2}$$

//

Contoh 5

Selesaikan integral berikut

$$\int_0^1 2x(x^2+3)^5 \, dx$$

$u = x^2 + 3$

$x = 1 \rightarrow u = 1^2 + 3 = 4$

$x = 0 \rightarrow u = 0^2 + 3 = 3$

$u = x^2 + 3$

$\frac{du}{dx} = 2x$

$dx = \frac{du}{2x}$

$$\int_0^1 2x(x+3)^5 \, dx = \int_0^1 \cancel{2x} \cdot u^5 \cdot \frac{du}{\cancel{2x}}$$

$$= \int_0^1 u^5 \, du$$

$$= \frac{1}{5+1} u^{5+1} \Big|_{0=x}^{1=x}$$

$$= \frac{1}{6} u^6 \Big|_3^4$$

Cara 1

kembalikan u ke x

$$= \frac{1}{6} u^6 \Big|_0^1 = \frac{1}{6}(x^2+3)^6 \Big|_0^1$$

$$= \left[\frac{1}{6}(1^2+3)^6\right] - \left[\frac{1}{6}(0^2+3)^6\right]$$

$$= \left[\frac{4^6}{6}\right] - \left[\frac{3^6}{6}\right]$$

$$= \frac{1}{6}(4^6 - 3^6)$$

$$= \frac{3367}{6}$$

Cara 2

ubah batas x menjadi u

$$\text{batas}\left|\begin{matrix} 1 \leftarrow x \\ 0 \leftarrow x \end{matrix}\right. \Rightarrow u = x^2 + 3 \Rightarrow \begin{matrix} x = 1 \rightarrow u = 1^2 + 3 \\ u = 4 \\ x = 0 \rightarrow u = 0^2 + 3 \\ u = 3 \end{matrix}$$

$$= \frac{1}{6} u^6 \Big|_3^4 = \frac{1}{6} 4^6 - \frac{1}{6} 3^6$$

①

$$= \frac{1}{6}\left(4^6 - 3^6\right)$$

$$= \frac{3367}{6}$$

Soal Latihan

1. Selesaikan:

a. $\int (x^2 + x^3)dx$

b. $\int \frac{\cos x}{\sin^2 x} dx$

c. $\int \sqrt[3]{t} dt$

d. $\int \frac{t^2 - 2t^4}{t^4} dt$

e. $\int \sin^2 x \, dx$

f. $\int_{-1}^{1} \sqrt{1 - x^2} dx$

g. $\int_{-1}^{1} \sqrt{1 - x^2} dx$

h. $\int_{1}^{4} \frac{s^4 - 8}{s^2} ds$

i. $\int_{0}^{1} x^3 \sqrt{x^2 + 3} dx$

j. $\int_{0}^{\frac{\pi}{4}} \tan^2 x \sec^2 x \, dx$

1.a

$$\int (x^2 + x^3)\,dx = \int x^2\,dx + \int x^3\,dx$$

$$= \frac{1}{2+1} x^{2+1} + \frac{1}{3+1} x^{3+1} + C$$

$$= \frac{1}{3} x^3 + \frac{1}{4} x^4 + C$$

1.b

$$\int \frac{\cos x}{\sin^2 x}\,dx = \int \frac{\cos x}{\sin x \cdot \sin x}\,dx$$

$$= \int \boxed{\frac{\cos x}{\sin x}} \boxed{\frac{1}{\sin x}}\,dx$$

$$= \int \boxed{\cot x} \cdot \boxed{\operatorname{cosec} x}\,dx$$

$$= -\operatorname{cosec} x + C$$

dari Tabel ☺

1.C $\int \sqrt[3]{t}\, dt = \int t^{\frac{1}{3}}\, dt = \frac{1}{\frac{1}{3}+1} t^{\frac{1}{3}+1} + C$

$$= \frac{1}{\frac{1}{3}+\frac{3}{3}} t^{\frac{1}{3}+\frac{3}{3}} + C$$

$$= \frac{1}{\frac{4}{3}} t^{\frac{4}{3}} + C$$

$$\frac{\frac{1}{1}}{\frac{4}{3}} = \frac{1}{1} \times \frac{3}{4}$$

$$= \frac{3}{4}$$

$$= \frac{3}{4} t^{\frac{4}{3}} + C \quad //$$

1.D. $\int \frac{t^2 - 2t^4}{t^4}\, dt = \int \frac{t^2}{t^4} - \frac{2t^4}{t^4}\, dt$

$= \int t^{2-4} - 2t^{4-4}\, dt$

$= \int t^{-2} - 2\, dt = \int t^{-2}\, dt - \int 2\, dt$

$= \frac{1}{-2+1} t^{-2+1} - \frac{2}{0+1} t^{0+1}$

$= \frac{1}{-1} t^{-1} - 2t$

$= -t^{-1} - 2t$

$= -\frac{1}{t} - 2t$

i.e. $\int \sin^2 x \, dx$

$$\boxed{\cos 2A = 1 - 2\sin^2 A}$$

$$\cos 2A - 1 = -2\sin^2 A$$

$$\frac{\cos 2A - 1}{-2} = \sin^2 A$$

$$\frac{1-\cos 2A}{2} = \sin^2 A$$

$$\int \frac{1-\cos(2x)}{2}\,dx = \frac{1}{2}\int 1-\cos(2x)\,dx$$

$$= \frac{1}{2}\left[\int 1\,dx - \int \cos(2x)\,dx\right]$$

$\int 1\,dx$

$= x$

$u = 2x$

$\frac{du}{dx} = 2$

$dx = \frac{du}{2}$

$\int \cos u \cdot \frac{du}{2} = \frac{\sin u}{2}$

$u = 2x \rightarrow \frac{\sin u}{2} = \frac{\sin 2x}{2}$

$$= \frac{1}{2}\left[x - \frac{\sin 2x}{2}\right] + C$$

1.f.

$$\int_{-1}^{1} \sqrt{1-x^2}\, dx$$

$$\sqrt{1-x^2} = k$$

$$1^2 - x^2 = k^2$$

$$1^2 = k^2 + x^2$$

misal.

$x = \sin u$ ..

$$\frac{dx}{du} = \frac{d}{du} \sin u$$

$$\frac{dx}{du} = \cos u$$

$dx = \cos u \, du$ ∴

$$\int_{-1}^{1} \sqrt{1-x^2}\, dx = \int_{-1}^{1} \sqrt{\boxed{1-\sin^2 u}} \cdot \cos u \, du$$

$$\boxed{\begin{aligned} &\cos^2 x + \sin^2 x = 1 \\ &\qquad \cos^2 x = 1 - \sin^2 x \end{aligned}}$$

$$= \int_{-1}^{1} \sqrt{\cos^2 u} \cdot \cos u \, du$$

$$= \int_{-1} \cos u \cdot \cos u \, du$$

$$= \int_{-1}^{1} \cos u \cdot \cos u \cdot du$$

$$= \int_{-1}^{1} \cos^2 u \, du$$

$$\cos 2A = 2\cos^2 A - 1$$

$$\cos 2A + 1 = 2\cos^2 A$$

$$\boxed{\frac{\cos 2A + 1}{2} = \cos^2 A}$$

$$= \int_{-1}^{1} \frac{\cos 2u + 1}{2} \, du = \frac{1}{2} \int_{-1}^{1} (\cos 2u + 1) \, du$$

$$= \frac{1}{2} \left[\boxed{\int_{-1}^{1} \cos 2u \, du} + \boxed{\int_{-1}^{1} 1 \, du} \right]$$

1 $= \int_{-1}^{1} \cos 2u \, du = \int_{-1}^{1} \cos k \, \frac{1}{2} \, dk$

$= \frac{1}{2} \int_{-1}^{1} \cos k \, dk$

$= \frac{1}{2} \left[\sin k \right] \Big|_{-1}^{1}$

$= \frac{1}{2} \left[\sin 2u \right] \Big|_{-1}^{1}$

$2u = k$

$\frac{d}{du} 2u = \frac{d}{du} k$

$2 = \frac{dk}{du}$

$du = \frac{1}{2} dk$

2 $\int_{-1}^{1} 1 \, du = u \Big|_{-1}^{1}$

$$= \frac{1}{2}\left[\frac{1}{2}\left[\sin 2u\right]\Big|_{-1}^{1} + u\Big|_{-1}^{1}\right]$$

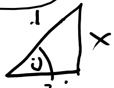
$$\boxed{\sin 2x = 2\sin x\cos x}$$

$$= \frac{1}{2}\left[\frac{1}{\cancel{2}}\left(\cancel{2}\sin u\cos u\right)\Big|_{-1}^{1} + u\Big|_{-1}^{1}\right]$$

$$= \frac{1}{2}\left[\sin u\cos u\Big|_{-1}^{1} + u\Big|_{-1}^{1}\right]$$

$$= \frac{1}{2}\left[x\sqrt{1-x^2}\Big|_{-1}^{1} + \sin^{-1}(x)\Big|_{-1}^{1}\right]$$

$$\sin u = x$$



$y = ?$

$$y^2 = 1^2 - x^2$$

$$y = \sqrt{1-x^2}$$

$$\cos u = \frac{\sqrt{1-x^2}}{1}$$

$$\cos u = \sqrt{1-x^2}$$

$$\sin u = x$$

$$u = \sin^{-1}(x)$$

$$= \frac{1}{2}\left[x\sqrt{1-x^2}\,\Big|_{-1}^{1} + \sin^{-1}(x)\,\Big|_{-1}^{1} \right]$$

$$= \frac{1}{2}\left[\left(1\cdot\sqrt{1-1^2} - (-1)\underbrace{\sqrt{1-(-1)^2}}_{(1-1)=0}\right) + \left(\sin^{-1}(1) - \sin^{-1}(-1)\right) \right]$$

$$= \frac{1}{2}\left[(0-0) + \left(\sin^{-1}(1) - \sin^{-1}(-1)\right) \right]$$

$$= \frac{1}{2}\left(\overbrace{\sin^{-1}(1)}^{\pi/2} - \underbrace{\sin^{-1}(-1)}_{\frac{\pi}{2}} \right)$$

$$= \frac{1}{2}\left(\boxed{2,37679} \right)$$

$$= 1,57 \approx \frac{\pi}{2}$$

//

1.h $\displaystyle\int_1^4 \frac{s^4-8}{s^2}\,ds = \int_1^4 \frac{s^4}{s^2}\,ds - \int_1^4 \frac{8}{s^2}\,ds$

$$= \int_1^4 s^2\,ds - \int_1^4 8s^{-2}\,ds$$

$$= \frac{1}{3}s^3\Big|_1^4 - \frac{8}{-1}s^{-1}\Big|_1^4$$

$$= \frac{1}{3}s^3\Big|_1^4 + 8s^{-1}\Big|_1^4$$

$$= \ldots$$

1.i $\int_0^1 x^3\sqrt{x^2+3}\,dx$

$u = x^2+3$

$\frac{du}{dx} = 2x$

$dx = \frac{1}{2x}\cdot du$

$u = x^2+3$

$x^2 = u-3$

$$= \int_0^1 x^3\sqrt{u}\cdot\frac{1}{2x}\cdot du$$

$$= \int_0^1 \frac{\cancel{2x}}{2}\cdot x^2\sqrt{u}\cdot\frac{1}{\cancel{2x}}\,du$$

$$= \int_0^1 \frac{1}{2}(u-3)\sqrt{u}\,du$$

$$= \frac{1}{2}\int_0^1 (u-3)u^{\frac{1}{2}}\,du$$

$$= \frac{1}{2}\int_0^1 u^{3/2} - 3u^{1/2}\,du$$

$$= \frac{1}{2}\int_0^1 u^{3/2} - 3u^{1/2}\, du$$ $\leftarrow$ dlm x , kita ubah ke u

$$= \frac{1}{2}\left[\frac{1}{1+3/2} u^{3/2+1}\Big|_0^1 - \frac{3}{\frac{1}{2}+1} u^{\frac{1}{2}+1}\Big|_0^1\right]$$

$u = x^2 + 3$

$x = 1 \rightarrow u = (1)^2 + 3$
$= 4$

$x = 0 \rightarrow u = (0)^2 + 3$
$= 3$

$$= \frac{1}{2}\left[\frac{1}{5/2} u^{5/2}\Big|_3^4 - \frac{3}{3/2} u^{3/2}\Big|_3^4\right]$$

$$= \frac{1}{2}\left[\frac{1}{5/2}(4)^{5/2} - \frac{1}{5/2}(3)^{5/2} - \left(\frac{3}{3/2}(4)^{3/2} - \frac{3}{3/2}(3)^{3/2}\right)\right]$$

$$= \frac{1}{2}\left[\underbrace{\frac{1}{5/2}(4)^{5/2} - \frac{1}{5/2}(3)^{5/2}} - \underbrace{\left(\frac{3}{3/2}(4)^{3/2} - \frac{3}{3/2}(3)^{3/2}\right)}\right]$$

$$= \frac{1}{2}\left[(-1{,}6) - (-2{,}08)\right]$$

Pakai kalkulator!

$$= 0{,}48$$

1.j $\displaystyle\int_0^{\pi/4} \tan^2 x \underline{\sec^2 x\, dx} = \int_0^{\pi/4} u^2 \cdot du$

mis:

$u = \tan x$

$\frac{du}{dx} = \sec^2 x$

$du = \underline{\sec^2 x\, dx}$

$$= \frac{1}{3}u^3 \Big|_0^{\pi/4} = \frac{1}{3}\tan^3 x \Big|_0^{\pi/4}$$

$$= \frac{1}{3}\tan^3(\pi/4) - \frac{1}{3}\tan^3(0) = \frac{1}{3}$$

Hore!