# LECTURE 6 - ARBITRARY STEP FORCING FUNCTIONS

## CE 225

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# STEP LOAD OF INFINITE DURATION

## DERIVATION

$$\boxed{u(t) = \frac{p_0}{k}\left(1 - e^{-\zeta\omega_n t}(\cos\omega_D t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_D t)\right)}$$

# STEP LOAD OF INFINITE DURATION

## UNIT STEP RESPONSE FUNCTION

Dyn. Amplification: $\zeta = 0 \longrightarrow DA = 2$

Rewrite: $u(t) = p_0 g(t)$

where: $$\boxed{g(t) = \frac{1}{k}\left[1 - e^{-\zeta\omega_n t}(\cos\omega_D t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_D t)\right]} = \text{Unit Step Response Function}$$

# FINITE STEP LOAD

## DERIVATION

# RAMP FUNCTION (UNDAMPED)

## DERIVATION

$$u(t) = u_h(t) + u_p(t) = C \cos \omega_n t + D \sin \omega_n t + \frac{p_0}{k t_d} t$$

# Ramp Function (Undamped)

## Derivation

Initial Conditions: $u(0) = \dot{u}(0) = 0$

$$u(t) = \frac{p_0}{k}\left(\frac{t}{t_d} - \frac{\sin\omega_n t}{\omega_n t_d}\right) \quad \longrightarrow \quad \boxed{u_{ramp}(t) = \frac{p_0}{k t_d}\left(t - \frac{\sin\omega_n t}{\omega_n}\right)}$$

# RAMP PULSE (UNDAMPED)

## PLOT

$$\boxed{u(t) = u_{ramp}(t) - u_{ramp}(t - t_d) - p_0 g(t - t_d)}$$

# Impulse Loading

## Derivation

Impulse $= \int_{t_1}^{t_2} F dt = \int_{t_1}^{t_2} m \frac{d\dot{u}}{dt} dt = m(v_2 - v_1) = mv_2 - mv_1$

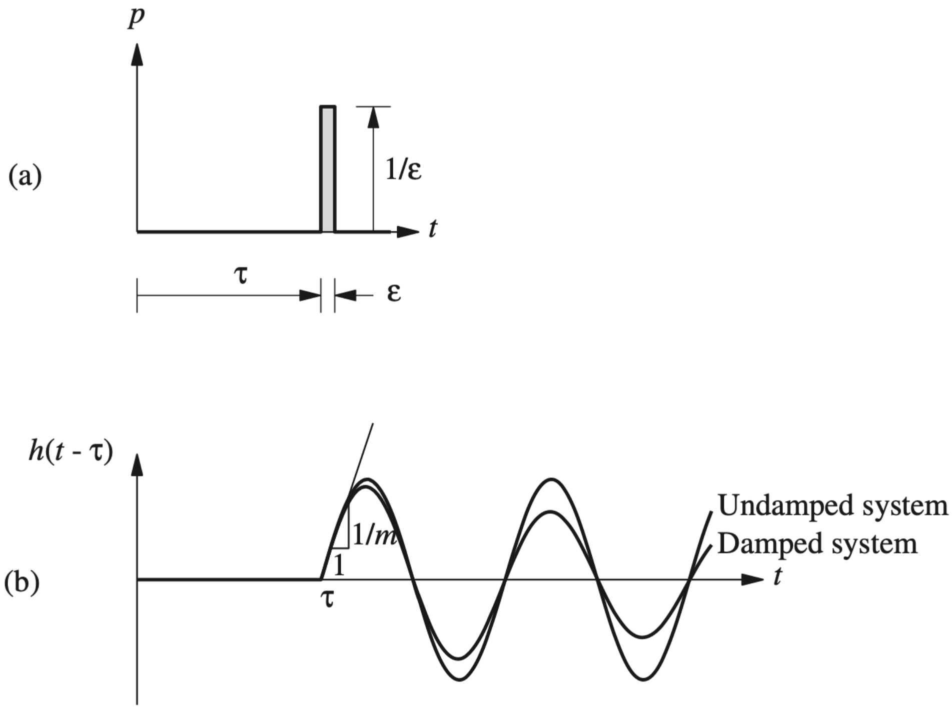
Solution for impulse: $u(t) = p_0[g(t) - g(t - t_d)] =$

$$\longrightarrow \lim_{t_d \to 0} u(t) = \frac{dg}{dt} = \frac{d}{dt}\left[\frac{1}{k}\left(1 - e^{-\zeta\omega_n t}(\cos\omega_D t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_D t)\right)\right]$$

Unit Impulse Response Function: $\boxed{h(t) = \frac{1}{m\omega_D} e^{-\zeta\omega_n t}\sin\omega_D t}$

# IMPULSE LOADING

## PLOT



# RESPONSE TO ARBITRARY LOADING

## CONVOLUTION INTEGRAL

Impulse Magnitude:

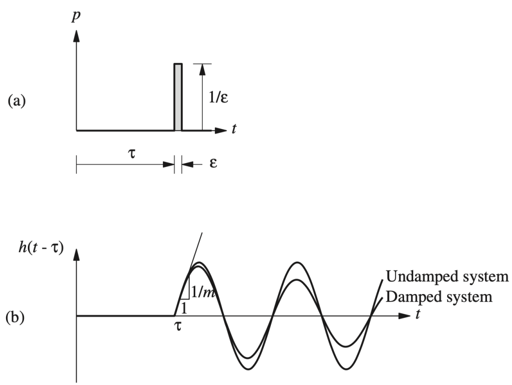
Response at time $t$:

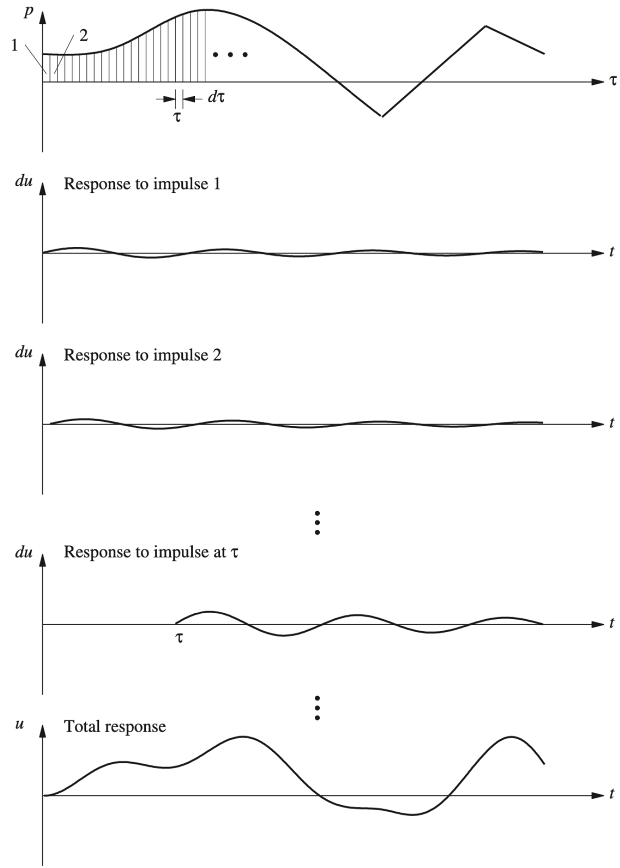
Apply superposition: Find response at time $t$ due to all impulses from time 0 to time $t$

$$\boxed{u(t) = \frac{1}{m\omega_D}\int_0^t p(\tau)e^{-\zeta\omega_n(t-\tau)}\sin[\omega_D(t-\tau)]d\tau} \rightarrow \quad \text{Duhamel's Integral}$$

# RESPONSE TO ARBITRARY LOADING

## PLOT



# RESPONSE TO ARBITRARY LOADING

## GENERAL RESPONSE FOR FORCING + IC'S

Response to Arbitrary forcing: $u(t) = \int_0^t h(t-\tau)P(\tau)d\tau = h * P$

Response to Arbitrary forcing + IC's: $u(t) =$

$$\longrightarrow u(t) = e^{-\zeta\omega_n t}\left[u(0)\cos(\omega_D t) + \frac{\dot{u}_0 + \zeta\omega_n u(0)}{\omega_d}\sin\omega_D t\right] + h * P$$

# Ramp Function (again)

## Using Convolution (IC's = 0)

Convolution: $u(t) = \int_0^t p(\tau)h(t-\tau)d\tau = \int_0^t \left(p_0 \frac{\tau}{t_d}\right) \frac{1}{m\omega_n} \sin\left[\omega_n(t-\tau)\right] d\tau$

Define: $x = \omega_n(t-\tau) \longrightarrow dx = -\omega_n d\tau \rightarrow d\tau = -\frac{dx}{\omega_n} \longrightarrow \begin{cases} \tau = t \rightarrow x = 0 \\ \tau = 0 \rightarrow x = \omega_n t \end{cases}$

$\boxed{\frac{p_0}{k}\left(\frac{t}{t_d} - \frac{\sin\omega_n t}{\omega_n t_d}\right)}$ $\rightarrow$ Same as before!

# RAMP FUNCTION (AGAIN)

## PLOT