LECTURE 18 - EQUATIONS OF MOTION - MDOF SYSTEMS - PART II

CE 225

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EXAMPLE 1: LUMPED MASS WITH ROTATIONAL INERTIA

EXAMPLE 1: LUMPED MASS WITH ROTATIONAL INERTIA

(CONTINUED)

$$\boxed{\underline{\underline{k}} = \frac{EI}{h^3}\begin{bmatrix} 12 & 6h \\ 6h & 4h^2 \end{bmatrix} \qquad \underline{\underline{m}} = \begin{bmatrix} m & 0 \\ 0 & J \end{bmatrix}}$$

EXAMPLE 2: BEAM WITH POINT MASSES

Use 'Stiffness Method' to derive EOM.

Deflection at tip:

$$u_1 = \frac{f_{11}L^3}{3EI} + \left[\frac{f_{21}(L/2)^3}{3EI} + \frac{f_{21}(L/2)^2}{2EI}\frac{L}{2}\right] = 1 \tag{1}$$

Deflection at midspan:

$$u_2 = \frac{f_{21}(L/2)^3}{3EI} + \frac{f_{11}(L/2)^3}{3EI} + \frac{\left(f_{11}\frac{L}{2}\right)(L/2)^2}{2EI} = 0 \tag{2}$$

Example 2: Beam with point masses

Solve (1) and (2) simultaneously $\begin{cases} f_{11} = k_{11} = \dfrac{48}{7}\dfrac{2EI}{L^3} \\ f_{21} = k_{21} = \dfrac{48}{7}\left(\dfrac{-5EI}{L^3}\right) \end{cases}$

Deflection at tip:

$$u_1 = \frac{f_{12}L^3}{3EI} + \left[\frac{f_{22}(L/2)^3}{3EI} + \frac{f_{12}(L/2)^2}{2EI}\frac{L}{2}\right] = 0$$

$$\longrightarrow f_{22} = k_{22} = \frac{48}{7}\frac{16EI}{L^3}$$

Therefore, EOM:

$$\boxed{\begin{bmatrix} mL/4 & 0 \\ 0 & mL/2 \end{bmatrix} \underline{\ddot{u}} + \frac{48}{7}\frac{EI}{L^3}\begin{bmatrix} 2 & -5 \\ -5 & 16 \end{bmatrix} \underline{u} = \begin{bmatrix} p_1(t) \\ p_2(t) \end{bmatrix}}$$

FLEXIBILITY METHOD

Theory:

$$\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{bmatrix} = \begin{bmatrix} \hat{f}_{11} & \hat{f}_{12} & \cdots & \hat{f}_{1N} \\ \hat{f}_{21} & \hat{f}_{22} & \cdots & \hat{f}_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{f}_{N1} & \hat{f}_{N2} & \cdots & \hat{f}_{NN} \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_N \end{bmatrix} \longrightarrow \quad \underline{u} = \underline{\underline{\hat{f}}}\,\underline{f} \quad \longrightarrow \quad \underline{\underline{\hat{f}}} = \underline{\underline{k}}^{-1}$$

If $f_1 = 1$ and all other forces are zero $\longrightarrow$ $\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{bmatrix} = \begin{bmatrix} \hat{f}_{11} \\ \hat{f}_{21} \\ \vdots \\ \hat{f}_{N1} \end{bmatrix}$

**Therefore, the displacements that result form the forces $f_1 = 1$ and $f_2 = f_3 = ... = f_N = 0f$ actually provide the first column of the flexibility matrix $\underline{\underline{\hat{f}}}$

Then repeat for other DOFs.

Flexibility Method

Example 2 (again)

$$f_1 = 1 \quad f_2 = 0$$

$$f_{11} = \frac{1L^3}{3EI}$$

$$f_{21} = \frac{(L/2)^3}{3EI} + \frac{(L/2)(L/2)^2}{2EI} = \frac{5}{48}\frac{L^3}{EI}$$

$$f_1 = 0 \quad f_2 = 1$$

$$f_{22} = \frac{(L/2)^3}{3EI} = \frac{2L^3}{48EI}$$

$$\underline{\underline{f}} = \frac{L^3}{48EI}\begin{bmatrix} 16 & 5 \\ 5 & 2 \end{bmatrix} \longrightarrow$$ Now, $\underline{\underline{k}} = \underline{\underline{f}}^{-1}$ yields same as before!

MORE GENERALIZED APPROACH (STATIC CONDENSATION)

EXAMPLE 2 (THIRD TIME)

The equation of motion is given by:

$$\begin{bmatrix} m\frac{L}{4} & 0 & 0 & 0 \\ 0 & m\frac{L}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \underline{\ddot{u}} + \frac{8EI}{L^3} \begin{bmatrix} 12 & -12 & -3L & -3L \\ -12 & 24 & 3L & 0 \\ -3L & 3L & L^2 & \frac{L^2}{2} \\ -3L & 0 & \frac{L^2}{2} & L^2 \end{bmatrix} \underline{u} = \begin{bmatrix} p_1 \\ p_2 \\ 0 \\ 0 \end{bmatrix}$$

However, we have more DOF's than are needed (only two masses!).

MORE GENERALIZED APPROACH (STATIC CONDENSATION)

EXAMPLE 2 (THIRD TIME)

Rewrite EOMs in the following form:

$$\begin{bmatrix} \underline{\underline{m}}_{tt} & \underline{\underline{0}} \\ \underline{\underline{0}} & \underline{\underline{0}} \end{bmatrix} \begin{bmatrix} \ddot{\underline{u}}_t \\ \ddot{\underline{u}}_0 \end{bmatrix} + \begin{bmatrix} \underline{\underline{k}}_{tt} & \underline{\underline{k}}_{t0} \\ \underline{\underline{k}}_{0t} & \underline{\underline{k}}_{00} \end{bmatrix} \begin{bmatrix} \underline{u}_t \\ \underline{u}_0 \end{bmatrix} = \begin{bmatrix} \underline{p}_t \\ \underline{p}_0 \end{bmatrix}$$

Write 2 separate equations in the following form:

$$\underline{\underline{m}}_{tt}\ddot{\underline{u}}_t + \underline{\underline{k}}_{tt}\underline{u}_t + \underline{\underline{k}}_{t0}\underline{u}_0 = \underline{p}_t$$

$$\underline{\underline{k}}_{0t}\underline{u}_t + \underline{\underline{k}}_{00}\underline{u}_0 = \underline{0} \qquad \longrightarrow \qquad \boxed{\underline{u}_0 = -\underline{\underline{k}}_{00}^{-1}\underline{\underline{k}}_{0t}\underline{u}_t}$$

Plug $\underline{u}_0$ into first equation: $\underline{\underline{m}}_{tt}\ddot{\underline{u}}_t + \underline{\underline{k}}_{tt}\underline{u}_t + \underline{\underline{k}}_{t0}[-\underline{\underline{k}}_{00}^{-1}\underline{\underline{k}}_{0t}\underline{u}_t] = \underline{p}_t$

Rearrange: $\underline{\underline{m}}_{tt}\ddot{\underline{u}}_t + [\underline{\underline{k}}_{tt} - \underline{\underline{k}}_{0t}^T\underline{\underline{k}}_{00}^{-1}\underline{\underline{k}}_{0t}]\underline{u}_t = \underline{p}_t \longrightarrow$ Note: $[\underline{\underline{k}}_{tt} - \underline{\underline{k}}_{0t}^T\underline{\underline{k}}_{00}^{-1}\underline{\underline{k}}_{0t}] = \frac{48}{7}\frac{EI}{L^3}\begin{bmatrix} 2 & -5 \\ -5 & 16 \end{bmatrix}$

REVIEW