Fall Semester, 2025 **Instructor**: M.J. DeJong
CE 225 – Dynamics of Structures

Homework #3

Due: Monday, September 22

1) In a forced vibration test under harmonic excitation it was noted that the amplitude of motion at $\omega = \omega_n$ was exactly three times the amplitude at an excitation frequency 20% higher than ω_n. Determine the damping ratio of the system.

2) A machine is supported on four steel springs for which damping can be neglected. The natural frequency of vertical vibration of the machine–spring system is 200 cycles per minute. The machine generates a vertical force $p(t) = p_0 \sin \omega t$. The amplitude of the resulting steady state vertical displacement of the machine is $u_0 = 0.2$ inches when the machine is running at 20 revolutions per minute (rpm), 1.042 in. at 180 rpm, and 0.0248 in. at 600 rpm. Calculate the amplitude of vertical motion of the machine if the steel springs are replaced by four rubber isolators that provide the same stiffness but introduce damping equivalent to $\zeta = 30\%$ for the system. Comment on the effectiveness of the isolators at various machine speeds.

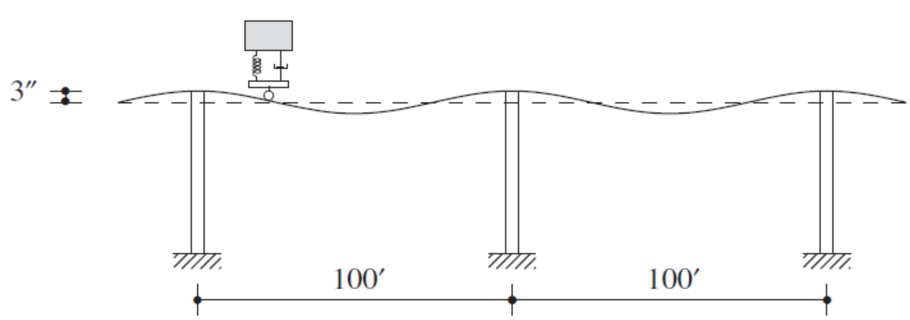
3) Consider an industrial machine of mass m supported on spring-type isolators of total stiffness k. The machine operates at a frequency of f Hertz with a force unbalance of p_o.

(a) Determine an expression giving the fraction of force transmitted to the foundation as a function of the forcing frequency f and the static deflection $\delta_{st} = mg/k$. Consider only the steady-state response. Note: your expressions should include δ_{st} and f.

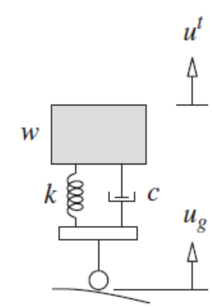
(b) Determine the static deflection δ_{st} for the force transmitted to be 20% of p_o if $f = 10$ Hz.

4) An automobile is traveling along a multispan elevated roadway supported every 100 ft. Long-term creep has resulted in a 4-in. deflection at the middle of each span (see Figure a). The roadway profile can be approximated as sinusoidal with an amplitude of 2 in. and a period of 100 ft. The SDF system shown if Figure (b) is a simple idealization of an automobile, appropriate for a "first approximation" study of the ride quality of the vehicle. When fully loaded, the weight of the automobile is 3600 lbs. The stiffness of the automobile suspension system is 700 lb/in., and its viscous damping coefficient is such that the damping ratio of the system is 50%. Assuming the automobile does not lift off the road surface, determine the maximum and minimum contact force between the road and the automobile when the automobile is traveling at 65 mph. Would the automobile actually lift off the road?

*Note: this problem uses the scenario of Example 3.4 in the textbook.



(a)



(b)

Problem # 1.

Data : @ $\omega_1 = \omega_n$ (resonance) $\rightarrow (u_0)_1 \rightarrow$ peak amplitude of motion at resonance.

@ $\omega_2 = 1.2\,\omega_n \rightarrow (u_0)_2 \rightarrow$ peak amplitude of motion @ ω_2.

given : $(u_0)_1 = 3 \cdot (u_0)_2$

System is damped, and : $\dfrac{u_0}{(u_{st})_0} = R_d = \dfrac{1}{\sqrt{\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2}}$

@ $\omega = \omega_n \rightarrow \dfrac{\omega}{\omega_n} = 1 \Rightarrow R_d = \dfrac{1}{2\zeta} \Rightarrow \dfrac{(u_0)_1}{(u_{st})_0} = \dfrac{1}{2\zeta}$

@ $\omega = 1.2\omega_n \rightarrow \dfrac{\omega}{\omega_n} = 1.2 \Rightarrow$ $R_d = \dfrac{1}{\sqrt{\left(1-\underbrace{(\omega/\omega_n)^2}_{1.2}\right)^2 + \left(2\zeta\underbrace{\omega/\omega_n}_{1.2}\right)^2}} = \dfrac{(u_0)_2}{(u_{st})_0}$

And : $(u_0)_1 = 3(u_0)_2$

$\Rightarrow \cancel{(u_{st})_0} \cdot \dfrac{1}{2\zeta} = 3 \cdot \cancel{(u_{st})_0} \cdot \dfrac{1}{\sqrt{\left(1-(1.2)^2\right)^2 + (2\zeta\,1.2)^2}} \rightarrow$ solve for ζ.

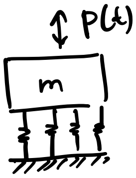
$\left(\dfrac{1}{2\zeta}\right)^2 = \dfrac{9}{(1-1.2^2)^2 + (2\zeta \cdot 1.2)^2} \Rightarrow 9(2\zeta)^2 = (1-1.44)^2 + (2.4\zeta)^2$

$\Rightarrow 36\zeta^2 = 0.44^2 + 2.4^2 \cdot \zeta^2$

$\Rightarrow \zeta^2 (36 - 2.4^2) = 0.44^2 \Rightarrow \zeta = \sqrt{\dfrac{0.44^2}{36 - 2.4^2}} = 0.08$

$\Rightarrow \boxed{\zeta = 8.0\%}$

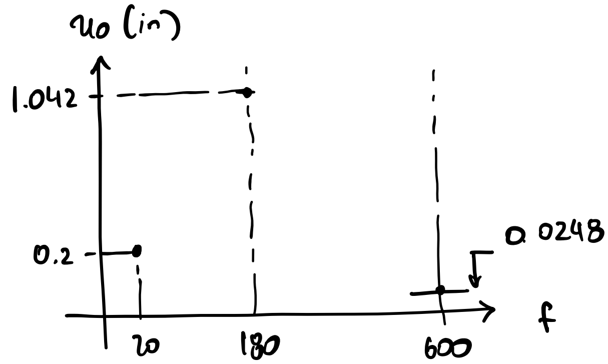
Problem # 2.



$f_n = 200\ (\text{cycles}/\text{minute}) \times \frac{1\ \text{min}}{60\ \text{sec}} = \frac{10}{3}\ (Hz).$

$\omega_n = 2\pi f_n = 20.94\ (\text{rad}/\text{sec}).$

$P(t) = p_0 \cdot \sin(\omega t) \rightarrow$ Harmonic forcing.



@ $f = 20$ rpm $\rightarrow u_0 = 0.2$ (in)

@ $f = 180$ rpm $\rightarrow u_0 = 1.042$ (in)

@ $f = 600$ rpm $\rightarrow u_0 = 0.0248$ (in)

if no damping (neglected) $\rightarrow R_d = \left| \frac{1}{1-(\omega/\omega_n)^2} \right|$

@ $f = 20$ rpm $\rightarrow \frac{\omega}{\omega_n} = \frac{20(2\pi)}{200(2\pi)} = 0.1 \rightarrow R_d = 1.01$;

@ $f = 180$ rpm $\rightarrow \frac{\omega}{\omega_n} = \frac{180(2\pi)}{200(2\pi)} = 0.9 \rightarrow R_d = 5.263$;

@ $f = 600$ rpm $\rightarrow \frac{\omega}{\omega_n} = \frac{600(2\pi)}{200(2\pi)} = 3.0 \rightarrow R_d = 0.125$;

From test 1 $\rightarrow (u_{st})_0 \cdot R_d = 0.2$ (in) $\Rightarrow (u_{st})_0 \approx 0.198''$

test 2 $\rightarrow (u_{st})_0\ R_d = 1.042'' \Rightarrow (u_{st})_0 \approx 0.198''$.

test 3 $\rightarrow (u_{st})_0 \cdot R_d = 0.0248'' \Rightarrow (u_{st})_0 \approx 0.198''$.

$\overline{(u_{st})_0} = 0.198''$.

Adding damping but not k means ω_n and $(u_{st})_0$ remain unchanged. The only thing that changes is R_d.

$$R_d = \frac{1}{\sqrt{\left(1-(\omega/\omega_n)^2\right)^2 + \left(2\zeta\, \omega/\omega_n\right)^2}}$$ with $\zeta = 0.3$.

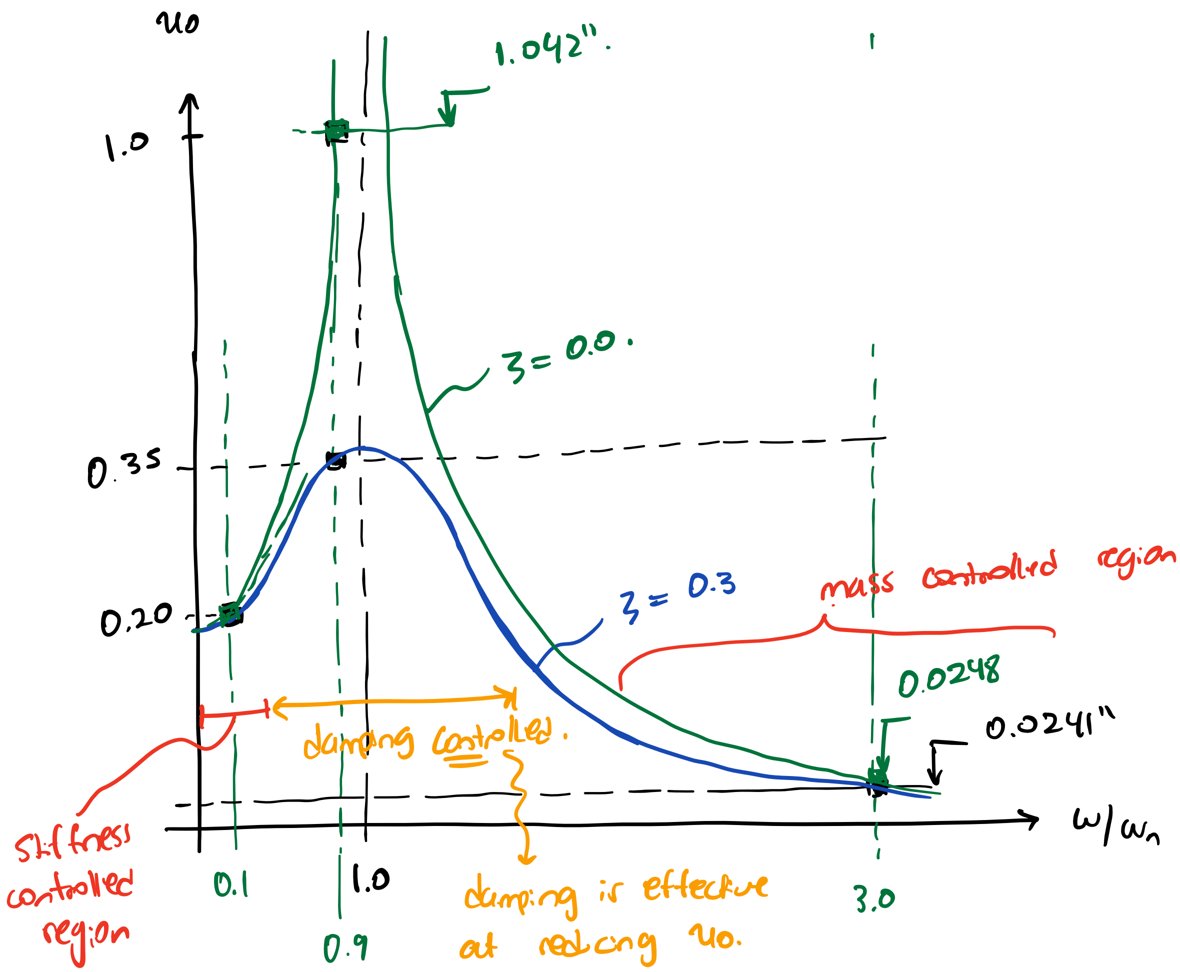
$u_0 = R_d\,(u_{st})_0$

with $\zeta = 0.3$:

@ $f = 20$ rpm $\rightarrow \frac{\omega}{\omega_n} = 0.1 \Rightarrow R_d = 1.0083 \rightarrow u_0 = 1.0083 \cdot 0.198'' = 0.2''$

@ $f = 180$ rpm $\rightarrow \frac{\omega}{\omega_n} = 0.9 \Rightarrow R_d = 1.7469 \rightarrow u_0 = 1.7469 \cdot 0.198'' = 0.35''$

@ $f = 600$ rpm $\rightarrow \frac{\omega}{\omega_n} = 3.0 \Rightarrow R_d = 0.1220 \rightarrow u_0 = 0.0241''$.



Additional damping provided by isolators is only effective near resonance.

Problem #3.

a) E.O.M. $m\ddot{u} + ku = P_0 \cdot \sin(\omega t)$.

u: dynamic displacement.

$u = \underline{\underline{u^t}} - \delta_{st}$ $\qquad \delta_{st} = \frac{mg}{k}$

$\downarrow$ total deformation on the spring.

For steady state:

$u(t) = (u_{st})_0 \; R_d \; \sin(\omega t - \phi)$; $\quad R_d = \left| \frac{1}{1-(\omega/\omega_n)^2} \right|$ for undamped system.

The transmitted force is: $f_T(t) = k \cdot u$

So, the peak transmitted force is:

$$(f_T)_0 = k \cdot u_0 = k \cdot R_d \cdot (u_{st})_0 = \cancel{k} \left| \frac{1}{1-(\omega/\omega_n)^2} \right| \cdot \frac{P_0}{\cancel{k}}.$$

we are asked for:

$$\frac{(f_T)_0}{P_0} = \left| \frac{1}{1-(\omega/\omega_n)^2} \right|$$

we are given $\delta_{st} = \frac{mg}{k} \Rightarrow \frac{k}{m} = \frac{g}{\delta_{st}} \Rightarrow \omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{g}{\delta_{st}}}$

and $\omega = 2\pi f$:

$$\frac{(f_T)_0}{P_0} = \left| \frac{1}{1 - \left(\frac{2\pi f}{g/\delta_{st}} \right)^2} \right| = \left| \frac{1}{1 - (2\pi f)^2 \cdot \frac{\delta_{st}}{g}} \right|$$

Concept: the larger δ_{st}, the more flexible the system is. $\rightarrow \omega_n \downarrow$

b) want $\frac{(f_T)_0}{P_0} = 0.2$ with $f = 10$ Hz.

Solve: $$0.2 = \left| \frac{1}{1 - (2\pi \cdot f)^2 \cdot \frac{\delta_{st}}{g}} \right|$$

$$\rightarrow 5 = \left| 1 - (2\pi f)^2 \cdot \frac{\delta_{st}}{g} \right|$$

$$\rightarrow 5 = 1 - (2\pi f)^2 \cdot \frac{\delta_{st}}{g} \quad \text{or} \quad 5 = -1 + (2\pi f)^2 \cdot \frac{\delta_{st}}{g}$$

$$\frac{\delta_{st}}{g} = \frac{-4}{(2\pi f)^2} < 0$$

$\downarrow$

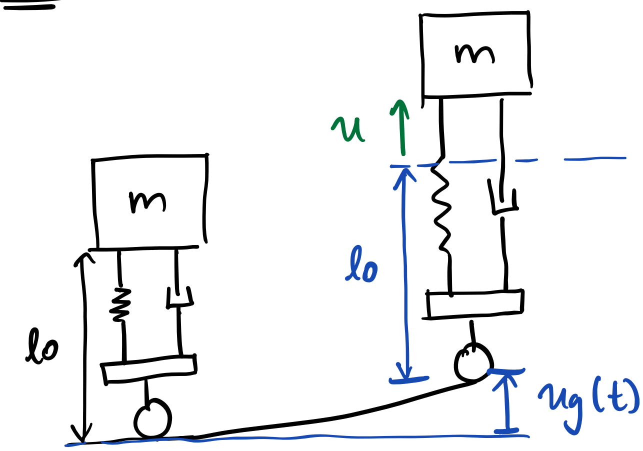
(not possible ! $\delta_{st} < 0$)

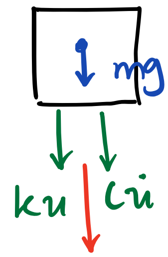
$$\frac{\delta_{st}}{g} = \frac{6}{(2\pi f)^2}$$

$$\Rightarrow \delta_{st} = \frac{6g}{(2\pi f)^2}$$

$$\Rightarrow \delta_{st} = \frac{6 \cdot 386\ (in/s^2)}{(2\pi \cdot 10\ (1/s))^2} = 0.587''.$$

Problem # 4.





$f_I = m\,\ddot{u}^t = m(\ddot{u} + \ddot{u}_g)$

$\Rightarrow$ EOM: $m(\ddot{u} + \ddot{u}_g) + c\dot{u} + ku = -mg$

$\Rightarrow \quad m\ddot{u} + c\dot{u} + ku = -mg - m\ddot{u}_g$

If we measure u from the position of static equilibrium $\delta_{st} = \frac{mg}{k}$ then our E.O.M. becomes:

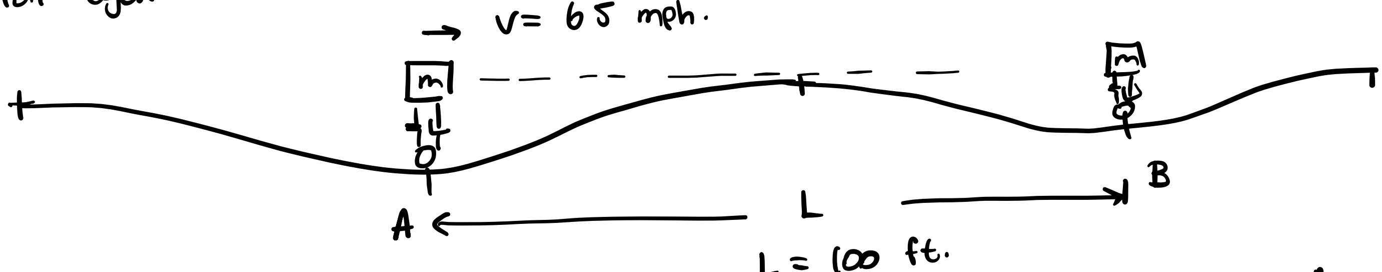
$$\boxed{m\ddot{u} + c\dot{u} + ku = -m\ddot{u}_g}$$

where u is measured from δ_{st}.

Now, what is $\ddot{u}_g(t)$?

$u_g(t) = u_o \cdot \sin(\omega t)$, where $\omega = \frac{2\pi}{T}$ and T is the time that takes for the car to go from one midspan to the other (one full cycle of the sinusoid :

distance travelled $\underline{\underline{L = v \times \text{time}}}$.



if L= 100 ft, the time that takes for the car to go from A $\rightarrow$ B is $T = L/v$ where T is the period of the "ground motion".

$v = 65 \frac{\text{miles}}{\text{hour}} \times 5280 \frac{\text{ft}}{\text{mile}} \times \frac{1\text{ hr}}{3600\text{ sec}} = 95.33\ (\text{ft}/\text{s}).$

$T = \frac{100\text{ ft}}{95.33\ (\text{ft/sec})} \Rightarrow T = 1.05\text{ sec} \Rightarrow \omega = \frac{2\pi}{T} = 5.98\ \frac{\text{rad}}{\text{sec.}}$

And the ground motion:

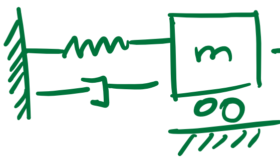
$$u_g(t) = u_{go} \cdot \sin(\omega t) \quad ; \quad u_{go} = 2'' \quad \omega = 5.98 \text{ rad/sec.}$$

$$\ddot{u}_g(t) = -\omega^2 \cdot u_{go} \cdot \underline{\underline{\sin(\omega t)}}.$$

So, as E.O.M:

$$m\ddot{u} + c\dot{u} + ku = -\omega^2 \cdot u_{go} \cdot \sin(\omega t)$$

This is equivalent to the following problem:

(*)  $\rightarrow p(t) = -\omega^2 \cdot u_{go} \cdot \sin(\omega t) = P_o \cdot \sin(\omega t)$

where $\underline{\underline{P_o = \omega^2 \cdot u_{go}}}.$

The contact force will be the sum of the weight of the system, plus the force transmitted through the spring and dashpot on the equivalent system (*)

$$C_F = m \cdot g + \underbrace{k \cdot u + c\dot{u}}_{\text{force transmitted through spring + damper on (*).}}$$

@ steady state:

$$u(t) = R_d \cdot (u_{st})_o \cdot \sin(\omega t - \phi) \qquad (u_{st})_o = \frac{P_o}{k} = \frac{m\omega^2 u_{go}}{k}$$

$$k = m \cdot \omega_n^2$$

$$\rightarrow (u_{st})_o = u_{go} \cdot \frac{\omega^2}{\omega_n^2} \quad ;$$

And:

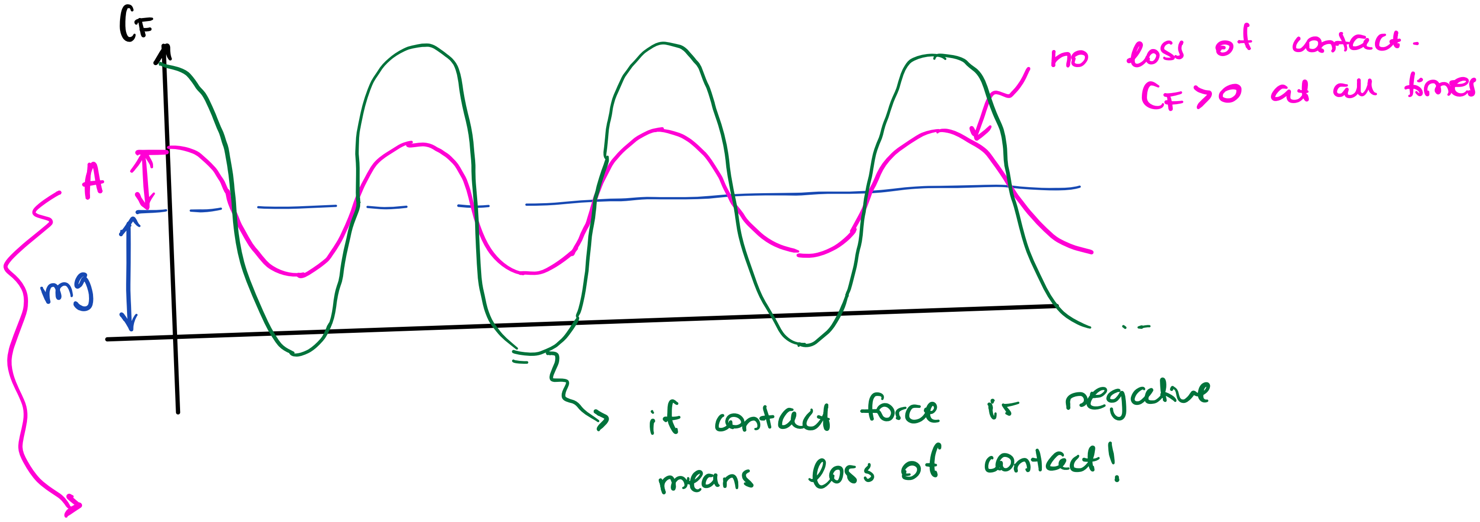
$$u(t) = \underbrace{R_d \cdot u_{go} \cdot \frac{\omega^2}{\omega_n^2}}_{u_o} \cdot \sin(\omega t - \phi)$$

$$\dot{u}(t) = \omega \cdot u_o \cdot \cos(\omega t - \phi)$$

The contact force will then be:

$$C_F = mg + \omega \cdot u_o \cdot \cos(\omega t - \phi) \cdot c + u_o \cdot \sin(\omega t - \phi) \cdot k$$

and we want to find max and min of C_F... let's see a plot:



A is the amplitude of:

$$\omega \cdot u_0 \cdot \cos(\omega t - \phi) \cdot c + u_0 \cdot \sin(\omega t - \phi) \cdot k$$

$$A = \sqrt{(\omega \cdot u_0 \cdot c)^2 + u_0^2 k^2} = u_0 \sqrt{(\omega \cdot c)^2 + k}$$

The max. and min contact forces are given by:

$$C_{F,max} = mg + A$$

$$C_{F,min} = mg - A$$

Need A... we know $u_0 = R_d \cdot u_{g0} \left(\frac{\omega}{\omega_n}\right)^2$, the term under $\sqrt{\ }$:

$$\sqrt{(c\omega)^2 + k^2} = k\sqrt{\frac{c^2\omega^2}{k^2} + 1} \quad ; \quad c = 2m\omega_n\zeta, \quad k = m \cdot \omega_n^2$$

$$\rightarrow \quad k\sqrt{\frac{(2\cancel{m}\omega_n\zeta \cdot \omega)^2}{(\cancel{m} \cdot \omega_n^{\cancel{2}})^2} + 1} = k\sqrt{\left(2\zeta\frac{\omega}{\omega_n}\right)^2 + 1}$$

And

$$A = R_d \cdot u_{g0} \left(\frac{\omega}{\omega_n}\right)^2 \cdot k \cdot \sqrt{1 + (2\zeta\,\omega/\omega_n)^2}$$

with $k = m \cdot \omega_n^2$

$$A = R_d \cdot u_{g0} \cdot \omega^2 \cdot m \cdot \sqrt{1 + (2\zeta\,\omega/\omega_n)^2}$$

$$= \underbrace{u_{g0} \cdot \omega^2 \cdot m}_{P_0 \text{ (defined above)}} \boxed{\sqrt{1 + (2\zeta\,\omega/\omega_n)^2} \cdot R_d}$$

TR: transmissibility! (see lecture notes).

$\Rightarrow \quad A = P_0 \cdot TR.$

So, max and min contact forces are given by:

$$\boxed{\begin{aligned} C_{max} &= mg + P_0 \cdot TR \\ C_{min} &= mg - P_0 \cdot TR \end{aligned}}$$

If we know that we can apply transmissibility, we can skip the derivation of the TR...

Now, plug in values: $m = w/g.$

$$P_0 = (2\,in) \times (5.98\ \tfrac{rad}{s})^2 \cdot \left(\frac{3{,}600\ lb}{386\ (in/s^2)}\right) = 669\ (lbs).$$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{700\ lb/in \times 386\ in/s^2}{3{,}600\ lb}} = 8.66\ (rad/sec) \rightarrow \frac{\omega}{\omega_n} = 0.691$$

$$TR = \sqrt{\frac{1+(2\zeta(\omega/\omega_n))^2}{(1-(\omega/\omega_n)^2)^2 + (2\zeta\,\omega/\omega_n)^2}} = 1.403$$

$$C_{max} = 3{,}600 + 1.403 \cdot 669 = 4{,}536.6\ (lbs).$$

$$C_{min} = 3{,}600 - 1.403 \cdot 669 = 2{,}663.4\ (lbs)\ > 0\ !$$

So, the car does not lift off the ground.

question... is there a speed that makes the car lift off the ground?