

LECTURE 6 - ARBITRARY STEP FORCING FUNCTIONS

CE 225

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UC Berkeley

September 16, 2025

STEP LOAD OF INFINITE DURATION

DERIVATION

$$u(t) = \frac{p_0}{k} \left(1 - e^{-\zeta \omega_n t} \left(\cos \omega_D t + \frac{\zeta}{\sqrt{1 - \zeta^2}} \sin \omega_D t \right) \right)$$

STEP LOAD OF INFINITE DURATION

UNIT STEP RESPONSE FUNCTION

Dyn. Amplification: $\zeta = 0 \longrightarrow DA = 2$

Rewrite: $u(t) = p_0 g(t)$

where: $g(t) = \frac{1}{k} \left[1 - e^{-\zeta \omega_n t} \left(\cos \omega_D t + \frac{\zeta}{\sqrt{1 - \zeta^2}} \sin \omega_D t \right) \right]$ = Unit Step Response Function

FINITE STEP LOAD

DERIVATION

RAMP FUNCTION (UNDAMPED)

DERIVATION

$$u(t) = u_h(t) + u_p(t) = C \cos \omega_n t + D \sin \omega_n t + \frac{p_0}{k t_d} t$$

RAMP FUNCTION (UNDAMPED)

DERIVATION

Initial Conditions: $u(0) = \dot{u}(0) = 0$

$$u(t) = \frac{p_0}{k} \left(\frac{t}{t_d} - \frac{\sin \omega_n t}{\omega_n t_d} \right) \quad \longrightarrow \quad u_{ramp}(t) = \frac{p_0}{k t_d} \left(t - \frac{\sin \omega_n t}{\omega_n} \right)$$

RAMP PULSE (UNDAMPED)

PLOT

$$u(t) = u_{ramp}(t) - u_{ramp}(t - t_d) - p_0 g(t - t_d)$$

IMPULSE LOADING

DERIVATION

$$\text{Impulse} = \int_{t_1}^{t_2} F dt = \int_{t_1}^{t_2} m \frac{d\dot{u}}{dt} dt = m(v_2 - v_1) = mv_2 - mv_1$$

Solution for impulse: $u(t) = p_0[g(t) - g(t - t_d)] =$

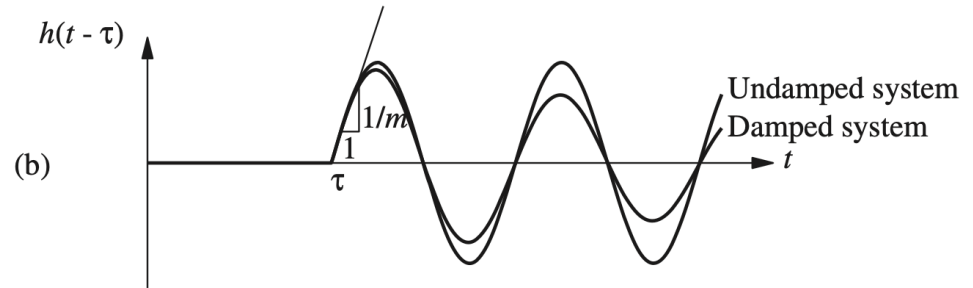
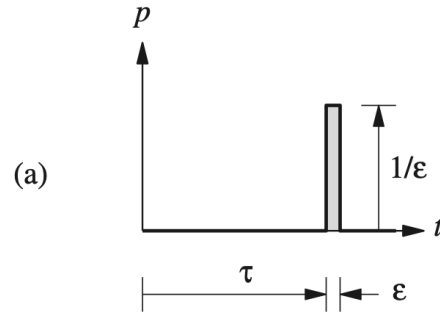
$$\longrightarrow \lim_{t_d \rightarrow 0} u(t) = \frac{dg}{dt} = \frac{d}{dt} \left[\frac{1}{k} \left(1 - e^{-\zeta \omega_n t} (\cos \omega_D t + \frac{\zeta}{\sqrt{1 - \zeta^2}} \sin \omega_D t) \right) \right]$$

Unit Impulse Response Function:

$$h(t) = \frac{1}{m\omega_D} e^{-\zeta \omega_n t} \sin \omega_D t$$

IMPULSE LOADING

PLOT



RESPONSE TO ARBITRARY LOADING

CONVOLUTION INTEGRAL

Impulse Magnitude:

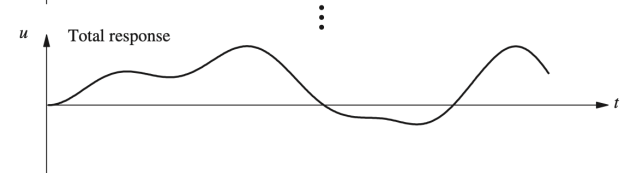
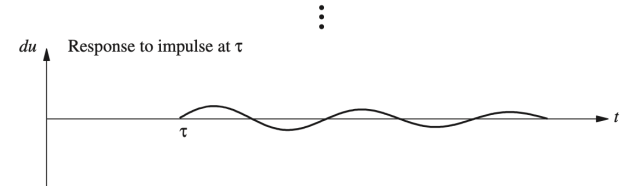
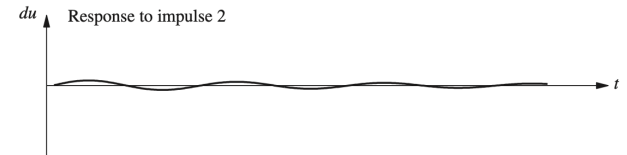
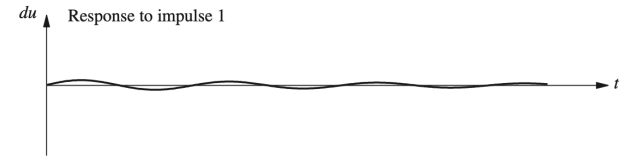
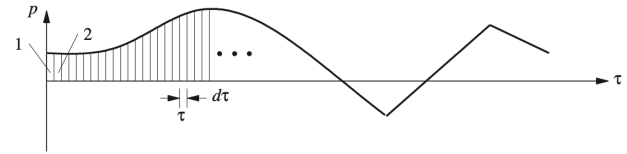
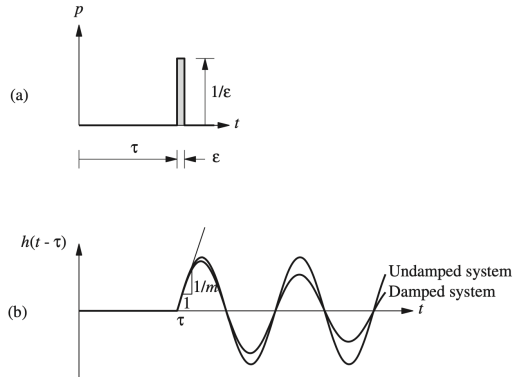
Response at time t :

Apply superposition: Find response at time t due to all impulses from time 0 to time t

$$u(t) = \frac{1}{m\omega_D} \int_0^t p(\tau) e^{-\zeta\omega_n(t-\tau)} \sin[\omega_D(t-\tau)] d\tau \rightarrow \text{Duhamel's Integral}$$

RESPONSE TO ARBITRARY LOADING

PLOT



RESPONSE TO ARBITRARY LOADING

GENERAL RESPONSE FOR FORCING + IC'S

Response to Arbitrary forcing: $u(t) = \int_0^t h(t - \tau)P(\tau)d\tau = h * P$

Response to Arbitrary forcing + IC's: $u(t) =$

$$\rightarrow u(t) = e^{-\zeta\omega_n t} \left[u(0) \cos(\omega_D t) + \frac{\dot{u}_0 + \zeta\omega_n u(0)}{\omega_d} \sin \omega_D t \right] + h * P$$

RAMP FUNCTION (AGAIN)

USING CONVOLUTION (IC'S = 0)

Convolution: $u(t) = \int_0^t p(\tau)h(t - \tau)d\tau = \int_0^t \left(p_0 \frac{\tau}{t_d}\right) \frac{1}{m\omega_n} \sin [\omega_n(t - \tau)] d\tau$

Define: $x = \omega_n(t - \tau) \longrightarrow dx = -\omega_n d\tau \rightarrow d\tau = -\frac{dx}{\omega_n} \longrightarrow \begin{cases} \tau = t \rightarrow x = 0 \\ \tau = 0 \rightarrow x = \omega_n t \end{cases}$

$$\boxed{\frac{p_0}{k} \left(\frac{t}{t_d} - \frac{\sin \omega_n t}{\omega_n t_d} \right)} \rightarrow \text{Same as before!}$$

RAMP FUNCTION (AGAIN)

PLOT