CE 225: Dynamic of Structures **Fall 2025**

Discussion 1: Equations of Motion and Free Vibration

Instructor: Matthew DeJong *GSI: Miguel A. Gomez*

1 Logistics

- GSI: Miguel A. Gomez. (508 Davis).
- Email: miguel.gomez.f@berkeley.edu
- Office hours: Wed 3-4pm & Fri 11am-12pm, 504 Davis
- Homework: **bCourses** $\rightarrow$ **Files**
- Submission: **bCourses** $\rightarrow$ **Gradescope**
- Class recording: **bCourses** $\rightarrow$ **Media Gallery**

2 Objective

The purpose of the discussion section is to go over selected topics that accompany but may not have been covered in details during lecture. We may go over problems that are tangentially related to, but not direct copies of, homework assignment problems.

3 Summary of basic concepts

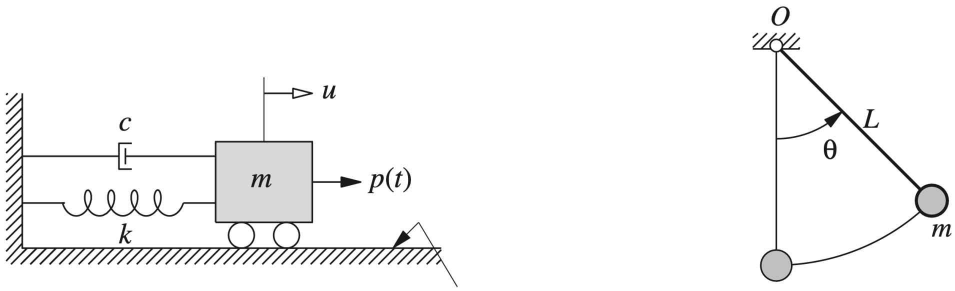


Figure 3.1: Left: Mass-spring-damper system. Right: Simple pendulum.

(a) Inertial forces

Translational Motion

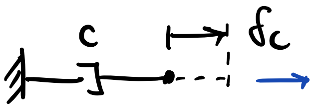
$$f_I = m\ddot{u}_t$$

↳ total acceleration measured from an inertial reference frame.

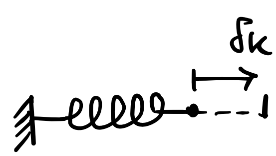
Rotational Motion

$$M_I = J\cdot\ddot{\theta}$$

(b) Damping and spring forces

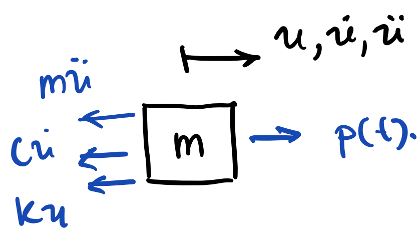


$\rightarrow F_c = c\cdot\dot{\delta}_c = c\cdot\dot{u}$



$\rightarrow F_k = k\cdot\delta_k = k\cdot u.$

(c) Free body diagram



(d) Equation of motion

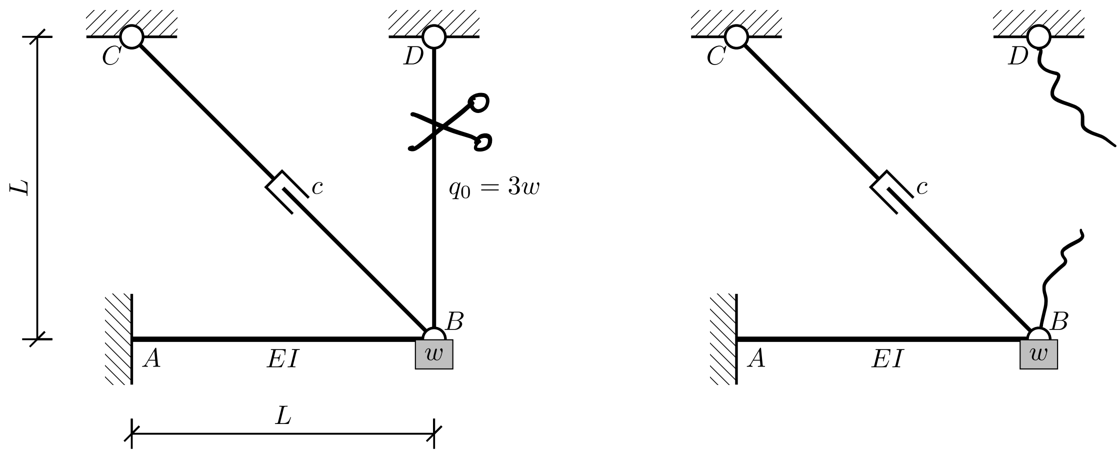
$$\boxed{m\ddot{u} + c\dot{u} + ku = p(t)}$$

Divide by m $\quad \ddot{u} + \frac{c}{m}\cdot\dot{u} + \frac{k}{m}\cdot u = p(t)/m$

Define $\frac{c}{m} = 2\zeta\omega_n$ and $\frac{k}{m} = \omega_n^2$

$$\boxed{\ddot{u} + 2\zeta\omega_n\dot{u} + \omega_n^2 u = p(t)/m}$$

Example 1: The structure shown below consists of a cantilever beam, with a point mass of weight w, attached to its free end. The mass is also attached to a viscous damper of constant c, which is placed in the configuration indicated in the Figure. The structure is initially at rest, connected to a vertical cable, which



is pre-stressed with a force equal to 3 times the weight of the point mass $q_0 = 3w$.

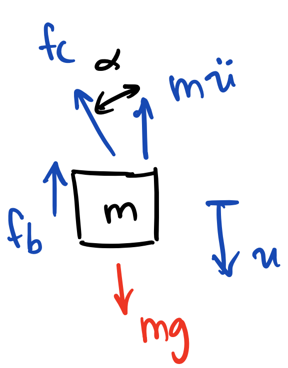
Starting from this position, in an instant, the cable is cut, and the system starts vibrating.

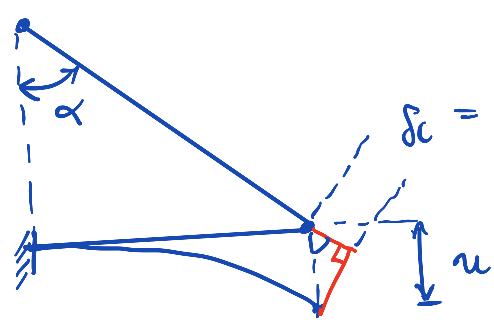
a) Determine the equation of motion for the free vibration of the system.

b) ~~Determine the equation of motion for the free vibration of the system~~. Solve EOM.

c) If after 3 full cycles the amplitude of the motion is 0.5 times what it was initially, determine an expression for c.

(a) F.B.D of the mass (after cable is cut).





$\delta_c = u \cdot \cos(\alpha)$

$\dot{\delta}_c = \dot{u} \cdot \cos(\alpha)$.

$\Rightarrow \quad m\ddot{u} + f_c \cdot \cos(\alpha) + f_b = mg$

$\Rightarrow \quad m\ddot{u} + c \cdot \cos^2(\alpha) \cdot \dot{u} + \underbrace{\frac{3EI}{L}}_{k.} \cdot u = \underline{mg}$

u: when beam is horizontal.

Example 1, continued...

What does constant R.H.S. tell me? $\rightarrow$ oscillations will happen around $u_{st} \neq 0$.

How to find u_{st}? $\left.\begin{array}{l}\ddot{u} = 0 \\ \dot{u} = 0 \\ u = u_{st}\end{array}\right\}$ in eom.

$$k \cdot u_{st} = mg \Rightarrow \boxed{u_{st} = \frac{mg}{k}}$$

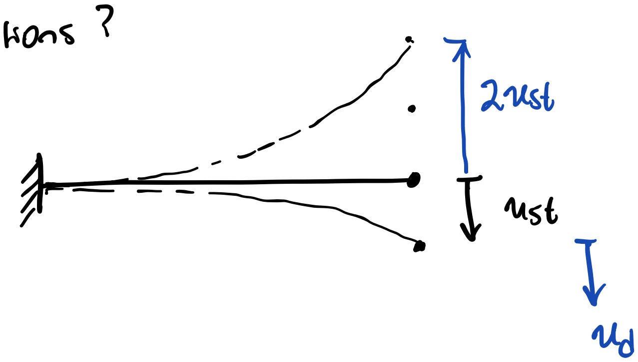
So, if we define $u_d = u - u_{st}$,

$$m\ddot{u}_d + c \cdot \cos^2(\alpha) \cdot \dot{u}_d + \frac{3EI}{L} \cdot u_d = 0 \quad \nearrow \text{free vibration.}$$

(b) What are the initial conditions?

$\dot{u}(0) = 0$

$u(0) = -3u_{st}$.



In free vibration we have:

$$u(t) = e^{-\zeta\omega_n t}\left[A \cdot \cos(\omega_d t) + B \cdot \sin(\omega_d t)\right].$$

$$u(0) = -3u_{st} \Rightarrow (1)\left[A \cdot (1) + B(0)\right] = -3u_{st} \Rightarrow \boxed{A = -3u_{st}}$$

$$\dot{u}(t) = -\zeta\omega_n \cdot e^{-\zeta\omega_n t}\left[A \cdot \cos(\omega_d t) + B \cdot \sin(\omega_d t)\right]$$
$$+ e^{-\zeta\omega_n t}\left[-A \cdot \omega_d \cdot \sin(\omega_d t) + B \cdot \omega_d \cdot \cos(\omega_d t)\right]$$

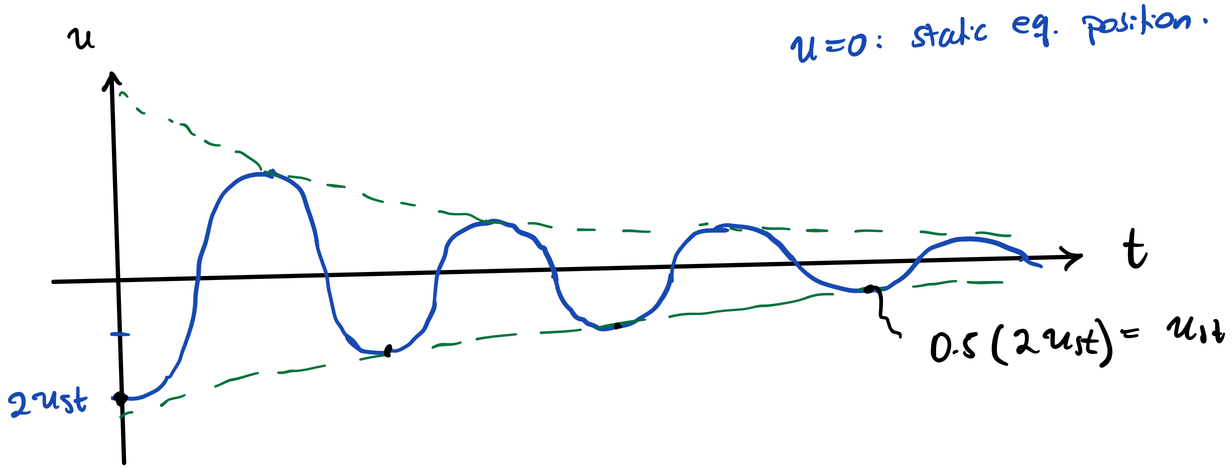
$$\dot{u}(0) = 0 \Rightarrow -\zeta\omega_n(1)\left[A \cdot (1) + B \cdot (0)\right]$$
$$+ (1) \cdot \left[-A \cdot \omega_d \cdot (0) + B \cdot \omega_d \cdot (1)\right] = 0$$

$$\Rightarrow -\zeta\omega_n \cdot A + B \cdot \omega_d = 0 \Rightarrow B = \frac{\zeta\omega_n \cdot A}{\omega_d} = -\frac{3u_{st} \cdot \zeta\omega_n}{\omega_d}.$$

So:

$$u(t) = e^{-\zeta \omega_n t}\left[-3u_{st}\cdot\cos(\omega_d t) - 3\cdot u_{st}\cdot\zeta\cdot\frac{\omega_n}{\omega_d}\cdot\sin(\omega_d t)\right].$$

$u=0$: static eq. position.



(c) Logarithmic decrement. $j = 3$

$u_i = 2u_{st}$ $\quad u_{i+3} = u_{st}$ $\Rightarrow$ $\ln\left(\frac{u_i}{u_{i+j}}\right) = 2\pi j\zeta$

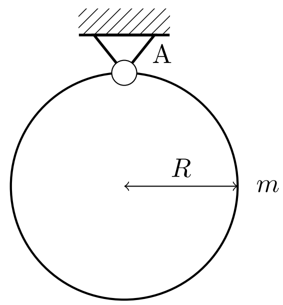
$$\Rightarrow \ln\left(\frac{2u_{st}}{u_{st}}\right) = 2\pi\cdot 3\cdot\zeta \Rightarrow \zeta = 0.037 = 3.7\%.$$

and $\zeta = \frac{c}{c_r} = \frac{c}{2m\cdot\omega_n} \Rightarrow \boxed{c = 2m\cdot\omega_n\cdot\zeta = 0.037\cdot 2\cdot m\cdot\omega_n}$

Example 2: A uniform ring with mass m and radius R is connected to the ceiling with a pin at point A. As shown in the lecture, the equation of motion of the system is:

$$(mR^2 + J_c)\ddot{\theta} + mgR\sin(\theta) = 0$$

$\sin(\theta) \sim \theta$



$J_c = mR^2$

$\Rightarrow \underbrace{2mR^2}_{m}\ddot{\theta} + \underbrace{mgR}_{k}\theta = 0$

Take $R = 10$ in, $g = 386$ in/s^2 and:

a) Compute the natural frequency ω_n of the ring.

b) A small rock with mass $2m$ is thrown towards the ring in horizontal direction with a velocity of $v_0 = 10$ in/s. The impact happens at the center of the ring, at a vertical distance R from point A. Right after the impact, the rock's horizontal component of the velocity is zero, and it just falls to the ground. Write the equation that describes the rotation of the ring at time t after the impact. Assume no damping.

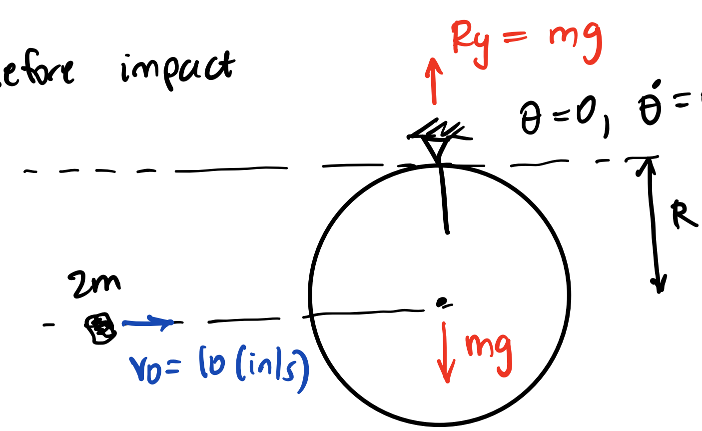
c) (Extra) Solve (2) again, but now considering that the ring has a damping ratio of 2% of the critical damping.

(a) $\omega_n = \sqrt{\dfrac{k}{m}} = \sqrt{\dfrac{\cancel{m}g\cancel{R}}{2\cancel{m}R^{\cancel{2}}}} = \sqrt{\dfrac{g}{2R}} = \sqrt{\dfrac{386\ (in/s^2)}{2\cdot 10\ (in)}} = 4.39\ (rad/s)$

$f_n = \dfrac{\omega_n}{2\pi} = \dfrac{4.39}{2\pi} = 0.7\ (Hz) \rightarrow T_n = \dfrac{1}{f_n} = 1.43\ (sec).$

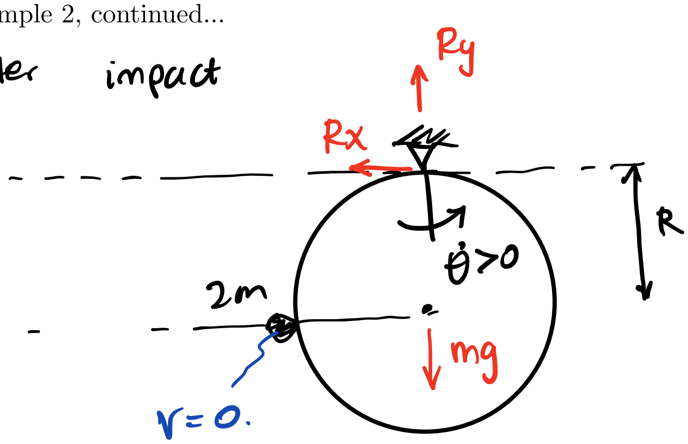
(b) Before impact

$\theta = 0,\ \dot{\theta} = 0,\ \ddot{\theta} = 0.$



Example 2, continued...

After impact



Recall:

$$\left(\sum m_i v_i\right)^- = \left(\sum m_i v_i\right)^+$$

Conservation of the angular momentum:

$$\left(\sum J_i \dot{\theta}_i\right)^- = \left(\sum J_i \dot{\theta}_i\right)^+$$

$$\Rightarrow \quad 2m \cdot v_0 \cdot R = J_A \cdot \dot{\theta}_0 \qquad J_A = 2mR^2$$

$$\Rightarrow \quad 2mv_0 R = 2mR^2 \cdot \dot{\theta}_0 \quad \Rightarrow \quad \dot{\theta}_0 = \frac{2mv_0 R}{2mR^2} = \frac{v_0}{R}.$$

Our E.O.M. is:

$$2mR^2 \ddot{\theta} + mgR\theta = 0$$

$$\Leftrightarrow \quad \ddot{\theta} + \frac{g}{2R} \cdot \theta = 0$$

So:

$$\theta(t) = A \cdot \cos(\omega_n t) + B \cdot \sin(\omega_n t) \qquad \omega_n = \sqrt{g/2R}.$$

$$\theta(0) = 0 \quad \Rightarrow \quad A \cdot (1) + B \cdot (0) = 0 \quad \Rightarrow \quad A = 0.$$

$$\dot{\theta}(t) = B \cdot \omega_n \cdot \cos(\omega_n t)$$

$$\dot{\theta}(0) = \dot{\theta}_0 \quad \Rightarrow \quad B = \dot{\theta}_0 / \omega_n \qquad \text{and} \quad \dot{\theta}_0 = \frac{v_0}{R} = \frac{10 \text{ in/s}}{10} = 1 \text{ (rad/sec)}.$$

$$\Rightarrow \quad B = \frac{1 \text{ (rad/sec)}}{4.39 \text{ (rad/sec)}} = 0.228.$$

$$0.228 \text{ rad} \times \frac{180^\circ}{\text{rad}} = 13.05^\circ$$

not too small but ok.

$$\Rightarrow \quad \theta(t) = 0.228 \cdot \sin(\omega_n t) \quad \text{(rad)}$$

(c) if we have damping $\zeta = 0.02$.

$$\theta(t) = e^{-\zeta\omega_n t}\left[A\cdot\cos(\omega_d t) + B\cdot\sin(\omega_d t)\right].$$

need:

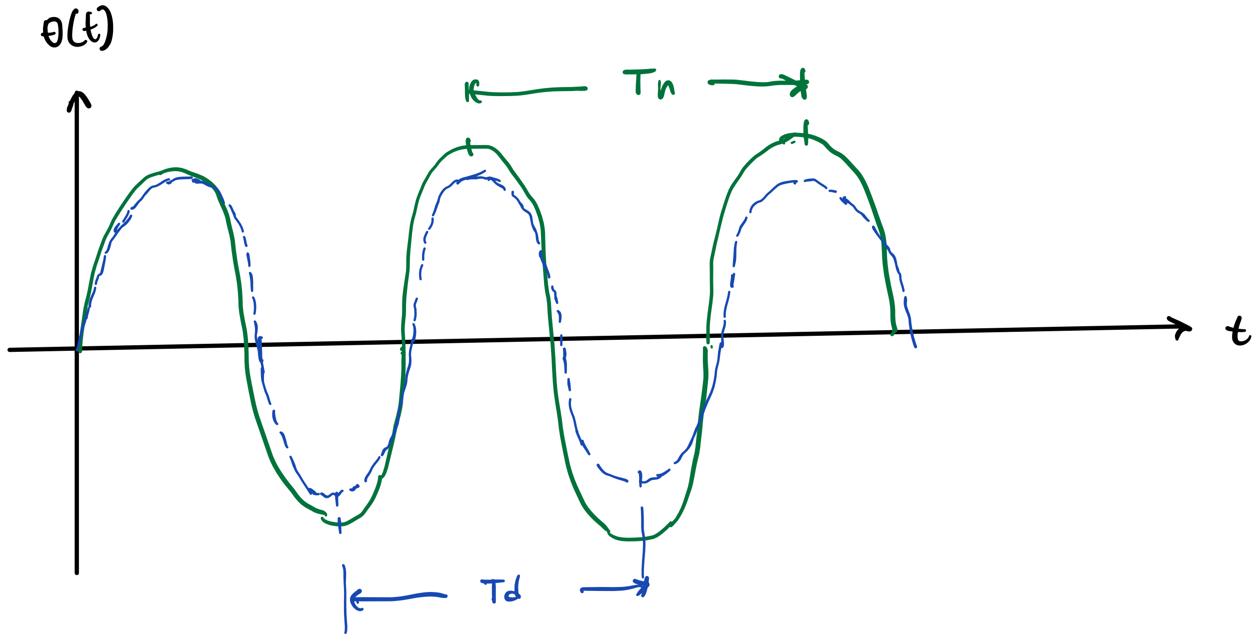
$$\omega_d = \omega_n\sqrt{1-\zeta^2} = 4.39\cdot\underbrace{\sqrt{1-0.02^2}}_{0.9998} \rightarrow \text{almost } 1.$$

$\omega_d \sim \omega_n$.

To find A and B we use the initial conditions.

$\theta(0) = 0 \Rightarrow (1)\left[A\cdot(1) + B\cdot(0)\right] = 0 \Rightarrow \boxed{A = 0.}$

Same for $\dot\theta(0) = \dot\theta(0) \rightarrow \boxed{B = \frac{\theta_0}{\omega_d}}$ → basically same as undamped



4 Free vibration

Free vibration (with $\zeta < 1$):

$$\boxed{u(t) = e^{-\zeta\omega_n t}\left[A\cos(\omega_D t) + B\sin(\omega_D t)\right]} \qquad \omega_D = \omega_n\sqrt{1-\zeta^2}$$

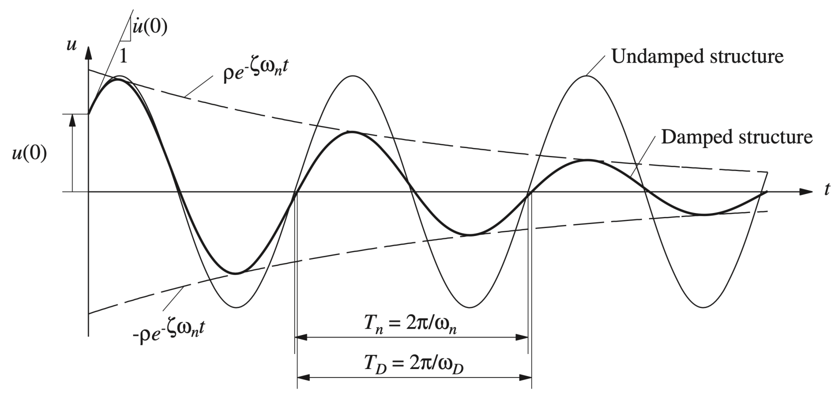


Figure 4.2: Effects of damping on free vibration

5 Rotational Inertia

Rotational Inertia is the counterpart of **mass** when an object is subject to rotational motion instead of translational motion. The rotational inertia is a scalar value that describes the resistance to change of rotational (angular) velocity of an object around a given axis.

Fig. 5.3 shows the rotational inertia for some simple shapes.

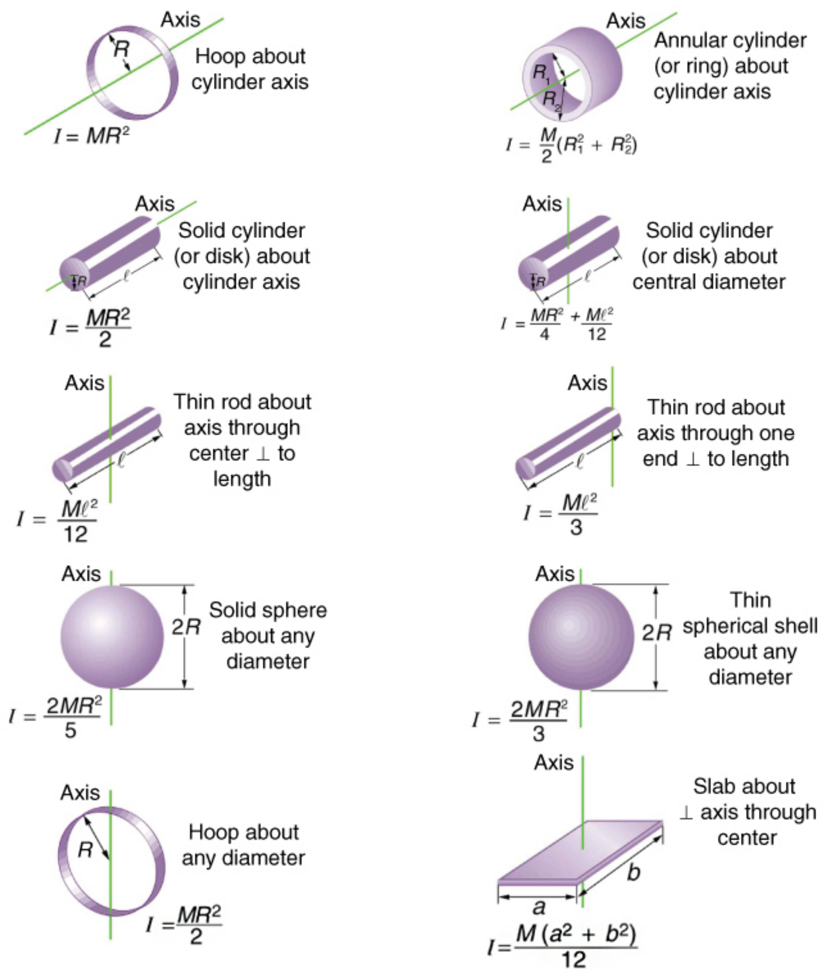


Figure 5.3: Rotational inertia of some simple shapes under rotation

The parallel axis theorem (Eq. 5.1) allows us to find the rotational inertia of an object about a point o as long as we know the rotational inertia of the shape around its centroid c, mass m and distance d between points o and c.

$$I_o = I_c + md^2 \tag{5.1}$$