CE 225: Dynamic of Structures **Fall 2025**

Discussion 3: Arbitrary Forcing Functions

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Plan for today

The plan for today's discussion includes:

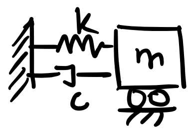
1. Conceptual review of transmissibility (study summary).
2. Example of solving the EOM for an arbitrary forcing function.

1 Transmissibility

$$TR = \frac{\sqrt{1 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2}}{\sqrt{\left(1 - \left(\frac{\omega}{\omega_n}\right)^2\right)^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2}}$$

This can mean two different things, depending on the type of EOM that we have.

(a) Harmonic Force Case



$\rightarrow p(t) = p_o \cdot \sin(\omega t)$

If our system is subjected to a harmonic force, then the EOM looks like:

$$\boxed{m\ddot{u} + c\dot{u} + ku = p_0 \sin\omega t} \tag{1.1}$$

$f_T = c\dot{u} + ku$

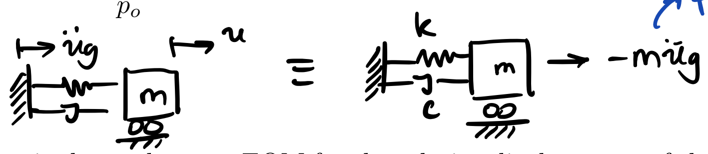
find max of this

In this case:

$$TR = \frac{f_{T,\max}}{p_o}$$

$\nearrow p(t).$

(b) Harmonic Ground Motion Case



If our system is subjected to a ground motion at its base, then our EOM for the relative displacement of the system w/r to the ground looks like:

$$\boxed{m\ddot{u} + c\dot{u} + ku = -m\ddot{u}_g} \tag{1.2}$$

$\rightarrow p(t) = -m\ddot{u}_g(t)$

$= -m\ddot{u}_{go} \sin(\omega t)$

$p_o.$

Where $\ddot{u}_g = \ddot{u}_{go} \sin(\omega t)$

In this case, we can use the Transmissibility equation for two purposes: (1) to find the ratio of the motion transmitted from the ground to the system:

$$TR = \frac{u_o^t}{u_{go}} = \frac{\ddot{u}_o^t}{\ddot{u}_{go}}$$

→ Peak total motion on our system.

→ peak motion of the ground.

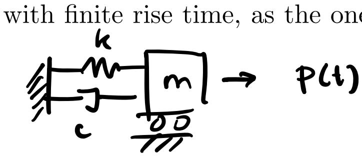
Or, if we compare our equations (1.1) and (1.2), we can define $p_o = -m\ddot{u}_{go}$ and use the transmissibility equation to find a relation between the force transmitted to our system, and the inertial force generated by the ground motion:

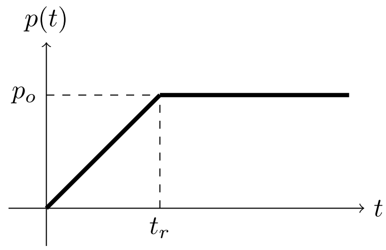
$$TR = \frac{f_{T,\max}}{p_o} = \frac{f_{T,\max}}{m\ddot{u}_{go}}$$

2 Arbitrary Forcing Functions

Example

An undamped SDF system, starting from rest, is subjected to a step force with finite rise time, as the one shown in the Figure below.





$$p(t) = \begin{cases} \dfrac{p_o t}{t_r} & t \le t_r \\ p_o & t > t_r \end{cases}$$

The response of the system can be shown to be:

$$u(t) = (u_{st})_o \left(\frac{t}{t_r} - \frac{\sin \omega_n t}{\omega_n t_r} \right) \quad \text{for } t \le t_r \tag{2.3}$$

$$u(t) = (u_{st})_o \left\{ 1 + \frac{1}{\omega_n t_r} \left((1 - \cos(\omega_n t_r)) \sin(\omega_n (t - t_r)) - \sin(\omega_n t_r) \cos(\omega_n (t - t_r)) \right) \right\} \quad \text{for } t > t_r \tag{2.4}$$

Equation (2.4) can be simplified through trigonometric identities to:

$$u(t) = (u_{st})_o \left\{ 1 - \frac{1}{\omega_n t_r} \left[\sin(\omega_n t) - \sin(\omega_n (t - t_r)) \right] \right\}$$

Derive these results with:

(a) Using the convolution integral. (Duhamel's Integral).

(b) Solving the EOM with analytical procedure.

(c) Using superposition

(a) Convolution Integral

For undamped systems, starting from rest subjected to a force defined by $p(t)$, we can find the response as:

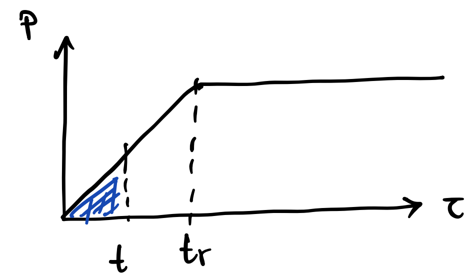
$$\boxed{u(t) = \frac{1}{m\omega_n}\int_0^t p(\tau)\sin\left(\omega_n(t-\tau)\right)d\tau} \rightarrow \text{undamped case.}$$

We also have the damped version of the convolution integral as:

$$u(t) = \frac{1}{m\omega_D}\int_0^t p(\tau)e^{-\zeta\omega_n(t-\tau)}\sin\left(\omega_D(t-\tau)\right)d\tau$$

$$p(\tau) = \begin{cases} \frac{p_0 \tau}{t_r} & \tau \le t_r \\ p_0 & \tau > t_r \end{cases}$$

(i) $t \le t_r$



$$u(t) = \frac{1}{m\omega_n}\int_0^t \frac{p_0\tau}{t_r} \cdot \sin\left(\omega_n(t-\tau)\right)d\tau \leftarrow \text{integration by parts}$$

$$= \frac{1}{m\cdot\omega_n} \cdot \frac{p_0}{t_r}\int_0^t \tau\cdot\sin\left(\omega_n(t-\tau)\right)d\tau$$

$$u = \tau \rightarrow du = d\tau$$

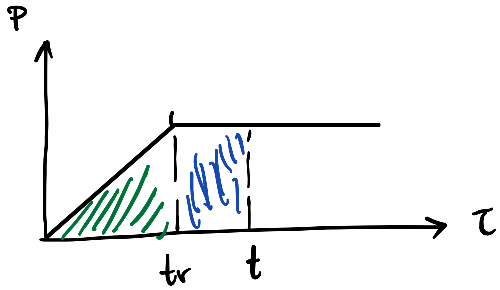
$$dv = \sin\left(\omega_n(t-\tau)\right)d\tau \rightarrow v = \frac{\cos\left(\omega_n(t-\tau)\right)}{\omega_n}$$

$$= \frac{1}{m\omega_n}\frac{p_0}{t_r}\left[\tau\cdot\frac{\cos\left(\omega_n(t-\tau)\right)}{\omega_n}\Bigg|_0^t - \int_0^t \frac{\cos\left(\omega_n(t-\tau)\right)}{\omega_n}\cdot d\tau\right]$$

$$= \frac{p_0}{m\omega_n^2 t_r}\left[t - \frac{\sin\left(\omega_n(t-\tau)\right)}{\omega_n}\Bigg|_0^t\right]$$

$$= \frac{p_0}{k\, t_r}\left[t - \left(-\frac{\sin(\omega_n t)}{\omega_n}\right)\right] = (u_{st})_0\left(\frac{t}{t_r} - \frac{\sin(\omega_n t)}{\omega_n t_r}\right).$$

(ii) $t > t_d$



$$\Rightarrow u(t) = \frac{1}{m\omega_n}\int_0^{t_r} \frac{P_0 \tau}{t_r} \cdot \sin\left(\omega_n (t-\tau)\right) d\tau + \frac{1}{m\omega_n}\int_{t_r}^{t} P_0 \cdot \sin\left(\omega_n (t-\tau)\right) d\tau$$

$$= \frac{1}{m\omega_n} \cdot \frac{P_0}{t_r}\left[\tau \cdot \frac{\cos\left(\omega_n (t-\tau)\right)}{\omega_n}\Bigg|_0^{t_r} - \frac{\sin\left(\omega_n(t-\tau)\right)}{\omega_n^2}\Bigg|_0^{t_r} \right]$$

$$+ \frac{1}{m\cdot\omega_n} \cdot P_0 \cdot \int_{t_r}^{t} \sin\left(\omega_n (t-\tau)\right) d\tau$$

$$= \frac{P_0}{m\cdot\omega_n^2\cdot t_r} \cdot \left[t_r \cdot \cos\left(\omega_n(t-t_r)\right) - (0) - \frac{1}{\omega_n} \cdot \left(\sin\left(\omega_n(t-t_r)\right) - \sin(\omega_n t) \right) \right]$$

$$+ \frac{P_0}{m\cdot\omega_n} \cdot - \frac{\cos\left(\omega_n(t-\tau)\right)}{-\omega_n}\Bigg|_{t_r}^{t}$$

$$= \frac{P_0}{m\cdot\omega_n^2}\left[\cancel{\cos\left(\omega_n(t-t_r)\right)} - \frac{1}{\omega_n} \cdot \sin\left(\omega_n(t-t_r)\right) + \frac{1}{\omega_n} \cdot \sin(\omega_n t) \right]$$

$$+ \frac{P_0}{m\cdot\omega_n^2}\left[\underbrace{\cos\left(\omega_n(t-t)\right)}_{1} - \cancel{\cos\left(\omega_n(t-t_r)\right)} \right]$$

$$= \frac{P_0}{k}\left[1 - \frac{1}{\omega_n}\left(\sin(\omega_n t) - \sin\left(\omega_n(t-t_r)\right) \right) \right]$$

$\square$.

(b) Solving the EOM

(i) $t \le t_r$ $\quad p(t) = \frac{p_0 t}{t_r}$

$m\ddot{u} + ku = \frac{p_0}{t_r} t \rightarrow u(t) = u_h(t) + u_p(t)$

$= A \cdot \cos(\omega_n t) + B \cdot \sin(\omega_n t) + u_p(t)$

1st degree polynomial

$u_p(t) = C + Dt$

$\dot{u}_p(t) = D$

$\ddot{u}_p(t) = 0$

into EOM $\rightarrow m \cdot (0) + k(C + Dt) = \frac{p_0}{t_r} \cdot t$

$\Rightarrow kC + kDt = \frac{p_0}{t_r} \cdot t + 0$

So: $kC = 0 \Rightarrow \boxed{C = 0}$ and $kD = \frac{p_0}{t_r} \Rightarrow \boxed{D = \frac{p_0}{k t_r} = \frac{(u_{st})_0}{t_r}}$

and $u_p(t) = (u_{st})_0 \cdot \frac{t}{t_r}$.

Now, the solution is:

$u(t) = A \cdot \cos(\omega_n t) + B \cdot \sin(\omega_n t) + (u_{st})_0 \cdot \frac{t}{t_r}$

IC's:
$u(0) = 0$
$\dot{u}(0) = 0$

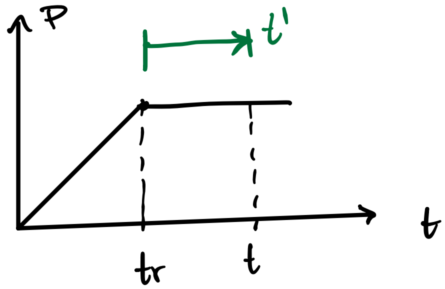
$u(0) = 0 \rightarrow A \cdot (1) + B \cdot (0) + (0) = 0 \Rightarrow \boxed{A = 0.}$

$\dot{u}(t) = \omega_n B \cdot \cos(\omega_n t) + \frac{(u_{st})_0}{t_r} \rightarrow @\ t = 0 \rightarrow \dot{u}(0) = \omega_n B + \frac{(u_{st})_0}{t_r} = 0$

$\rightarrow \boxed{B = -\frac{(u_{st})_0}{t_r \cdot \omega_n}}$

$u(t) = (u_{st})_0 \left[\frac{t}{t_r} - \frac{\sin(\omega_n t)}{\omega_n \cdot t_r} \right].$

(ii) $t > t_r$



EOM: $m\ddot{u} + ku = P_0$

$t'(0)$ @ $t = t_r$

$\Rightarrow \quad t' = t - t_r.$

$$u(t') = A\cdot\cos(\omega_n t') + B\cdot\sin(\omega_n t') + u_p.$$

$$u_p = C \rightarrow \begin{matrix}\dot{u}_p(0) = 0 \\ \ddot{u}_p(0) = 0\end{matrix} \rightarrow 0 + kC = P_0 \Rightarrow C = \frac{P_0}{k} = (u_{st})_0.$$

$$\Rightarrow \quad u(t') = A\cdot\cos(\omega_n t') + B\cdot\sin(\omega_n t') + (u_{st})_0$$

now, need initial conditions @ $t' = 0$ (or $t = t_r$) $\rightarrow$ $\begin{matrix}u_{ramp}(t_r) \\ \dot{u}_{ramp}(t_r)\end{matrix}$

$$u_{ramp}(t) = (u_{st})_0\left(\frac{t}{t_r} - \frac{\sin(\omega_n t)}{t_r\cdot\omega_n}\right) \Rightarrow u_{ramp}(t_r) = (u_{st})_0\left[1 - \frac{\sin(\omega_n t_r)}{t_r\cdot\omega_n}\right]$$

$$\dot{u}_{ramp}(t) = (u_{st})_0\left(\frac{1}{t_r} - \frac{\cos(\omega_n t)}{\omega_n}\right) \Rightarrow \dot{u}_{ramp}(t_r) = (u_{st})_0\left[\frac{1}{t_r} - \frac{\cos(\omega_n t_r)}{t_r}\right]$$

$$u(t') = u_{ramp}(t_r)$$

$$\Rightarrow \quad A\cdot(1) + B\cdot(0) + (u_{st})_0 = (u_{st})_0 - (u_{st})_0\cdot\frac{\sin(\omega_n t_r)}{t_r\cdot\omega_n}$$

$$\Rightarrow \quad \boxed{A = -(u_{st})_0\cdot\frac{\sin(\omega_n t_r)}{\omega_n t_r}}$$

$$\dot{u}(t'=0) = \dot{u}_{ramp}(t_r)$$

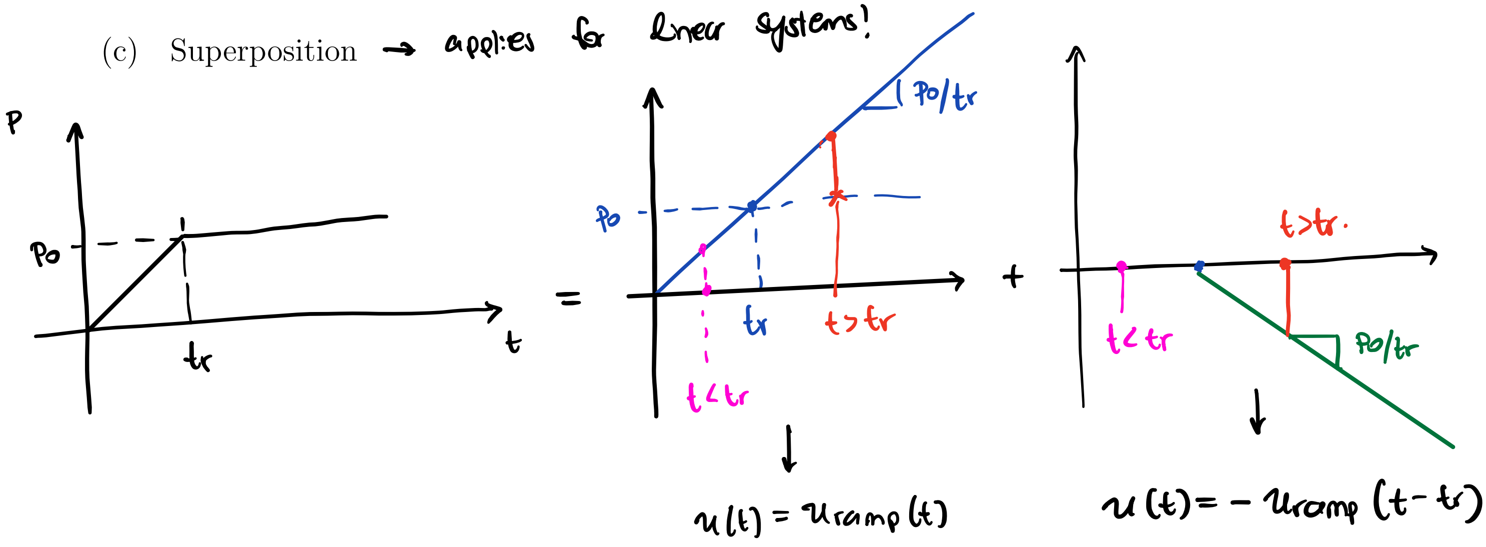
$$\Rightarrow \quad B = (u_{st})_0\left[\frac{1}{\omega_n t_r} - \frac{\cos(\omega_n t_r)}{\omega_n t_r}\right]$$

So: $$u(t') = (u_{st})_0\left[-\sin\frac{(\omega_n t_r)}{\omega_n t_r}\cos(\omega_n t') + \frac{1}{\omega_n t_r}\left(1 - \cos(\omega_n t_r)\right)\cdot\sin(\omega_n t') + 1\right]$$

with $t' = t - t_r$

$$u(t) = (u_{st})_0\left[-\frac{\sin(\omega_n t_r)}{\omega_n t_r}\cdot\cos(\omega_n(t - t_r)) + \frac{1}{\omega_n t_r}\left(1 - \cos(\omega_n t_r)\right)\sin(\omega_n(t - t_r)) + 1\right]$$

(c) Superposition → applies for linear systems!



for $t > t_r$

$$u(t) = u_{ramp}(t) - u_{ramp}(t - t_r)$$

$$= (u_{st})_0 \left[\frac{t}{t_r} - \frac{\sin(\omega_n t)}{t_r} - \frac{t - t_r}{t_r} + \frac{\sin(\omega_n (t - t_r))}{\omega_n t_r} \right]$$

$$= (u_{st})_0 \left[1 - \frac{1}{\omega_n t_r} \left(\frac{\sin(\omega_n t_r)}{\omega_n t_r} - \frac{\sin(\omega_n (t - t_r))}{\omega_n t_r} \right) \right].$$