LECTURE 5 - HARMONIC FORCING (PART 3)

CE 225

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Harmonic Forcing

Force Transmission

Recall Solution: $u(t) = (u_{st})_0 R_d \sin(\omega t - \phi)$

$$\therefore \quad \dot{u}(t) = (u_{st})_0 R_d \omega \cos(\omega t - \phi)$$

$$f_T = ku(t) + c\dot{u}(t)$$

$$\rightarrow$$

$$\therefore \quad f_{T,\text{max}} = (u_{st})_0 R_d \sqrt{k^2 + (c\omega)^2}$$

Transmissibility (TR) $= \dfrac{f_{T,max}}{p_0} = R_d \sqrt{1 + \left(\dfrac{c}{k}\omega\right)^2}$

$$\longrightarrow \boxed{TR = R_d \sqrt{1 + \left(2\zeta \frac{\omega}{\omega_n}\right)^2}}$$

HARMONIC FORCING

TRANSMISSIBILITY FUNCTION

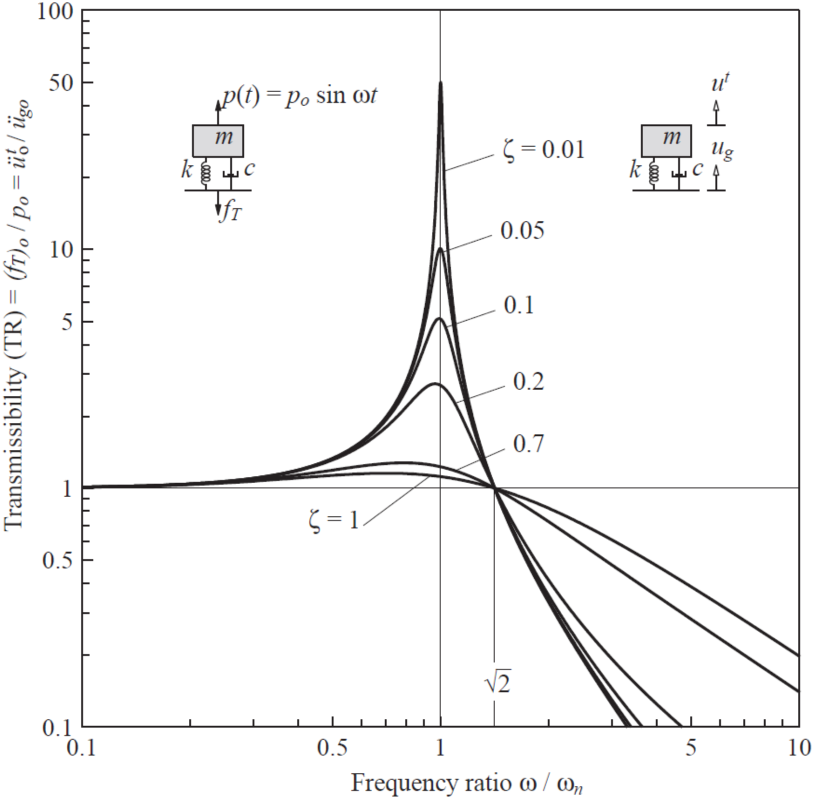

Figure 3.5.1 Transmissibility for harmonic excitation. Force transmissibility and ground motion transmissibility are identical.

HARMONIC FORCING

FIDGET SPINNER EXAMPLE

Calculate force transmitted at $f = 10$ Hz

Recall: $f_n = 8.6$ Hz and $\zeta \approx 2\%$

HARMONIC GROUND MOTION

EOM:

Recall $\longrightarrow$ $$\begin{cases} u(t) = \dfrac{-m\ddot{u}_{g0}}{k} R_d \sin(\omega t - \phi) \\ \ddot{u}(t) = \dfrac{m\ddot{u}_{g0}}{k} R_d \omega^2 \sin(\omega t - \phi) \end{cases}$$

Total response $\longrightarrow$ $\ddot{u}^t(t) = \ddot{u}_g(t) + \ddot{u}(t)$

HARMONIC GROUND MOTION

TRANSMISSIBILITY FUNCTION

Max Response: $$\frac{(\ddot{u}^t)_{\max}}{(\ddot{u}_g)_{\max}} = \frac{\ddot{u}_0^t}{\ddot{u}_{g0}} = \frac{\max\left(\ddot{u}_{g0}\sin(\omega t) + \frac{m\ddot{u}_{g0}}{k}R_d\omega^2\sin(\omega t - \phi)\right)}{\ddot{u}_{g0}}$$

$$\frac{\ddot{u}_0^t}{\ddot{u}_{g0}} = \text{TR} = R_d\sqrt{1 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2}$$

Also: $$\frac{(u^t)_{\max}}{(u_g)_{\max}} = \frac{u_0^t}{u_{g0}} = \text{TR}$$

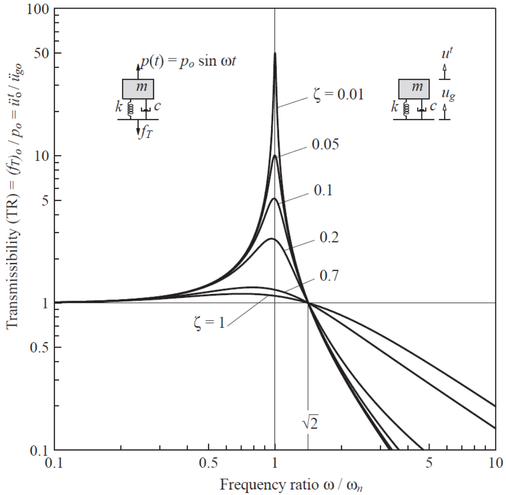

Figure 3.5.1 Transmissibility for harmonic excitation. Force transmissibility and ground motion transmissibility are identical.

HARMONIC FORCING

ENERGY DISSIPATION: VISCOUS DAMPING AT STEADY STATE

In one cycle:

Energy dissipated: $E_D = \int_0^{\frac{2\pi}{\omega}} c\dot{u}^2 dt = \int_0^{\frac{2\pi}{\omega}} c[\omega u_0 \cos(\omega t - \phi)]^2 dt = c\omega^2 u_0^2 \left[\frac{\pi}{\omega}\right]$

$$\boxed{E_D = 2\pi\zeta k \frac{\omega}{\omega_n} u_0^2}$$

Energy input: $E_I = \int (\text{force})(\text{displacement}) \longrightarrow E_I = \int_0^{\frac{2\pi}{\omega}} p(t)du = \int_0^{\frac{2\pi}{\omega}} p(t)\dot{u}dt$

$$= \int_0^{\frac{2\pi}{\omega}} p_0 \sin(\omega t)[\omega u_0 \cos(\omega t - \phi)]dt$$

$$\boxed{E_I = p_0 u_0 \pi \sin\phi}$$

HARMONIC FORCING

ENERGY DISSIPATION: VISCOUS DAMPING AT STEADY STATE

At Steady State $\longrightarrow$ Energy in = Energy out

$$E_I = E_D \longrightarrow \quad p_0 u_0 \pi \sin\phi = 2\pi\zeta k \frac{\omega}{\omega_n} u_0^2 \longrightarrow \quad p_0 \sin\phi = 2\zeta k \frac{\omega}{\omega_n} u_0$$

At resonance $\longrightarrow \dfrac{\omega}{\omega_n} = 1,\ \phi = \dfrac{\pi}{2} \longrightarrow \boxed{\dfrac{1}{2\zeta} = \dfrac{u_0}{p_0/k}}$

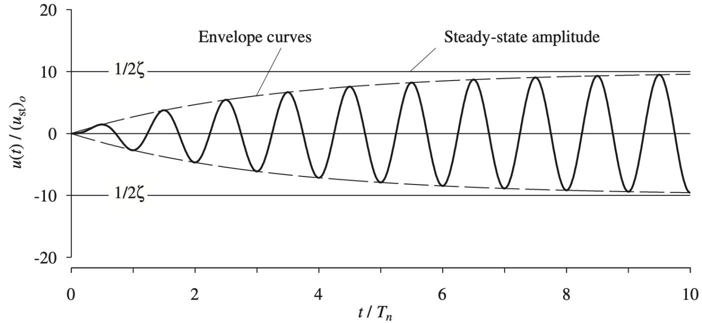

Figure 3.2.2 Response of damped system with $\zeta = 0.05$ to sinusoidal force of frequency $\omega = \omega_n$; $u(0) = \dot{u}(0) = 0$.

HARMONIC FORCING

PLOT DAMPING FORCE

$$f_D(t) = c\dot{u} = c\omega u_0 \cos(\omega t - \phi) \xrightarrow{\cos x = \sqrt{1-\sin^2 x}}$$

Rearrange $\rightarrow \quad \left(\frac{f_D(t)}{c\omega u_0}\right)^2 = 1 - \left(\frac{u(t)}{u_0}\right)^2 \quad \rightarrow \quad \therefore \left(\frac{f_D(t)}{c\omega u_0}\right)^2 + \left(\frac{u(t)}{u_0}\right)^2 = 1$

Instead, typically:

Harmonic Forcing
Equivalent Viscous Damping

In reality, damping is not purely viscous, but it can be useful to define:

$$c_{\text{equivalent}} = c_{eq} \text{ or } \zeta_{\text{equivalent}} = \zeta_{eq}$$

For example, define ζ_{eq} by measuring "stored energy" (E_S) and "dissipated energy" (E_D) per cycle:

We need relation between $E_D,\ E_s$ and ζ_{eq}

Recall: $\zeta = \dfrac{E_D}{2\pi k \frac{\omega}{\omega_n} u_0^2} \quad \longrightarrow \quad \zeta = \dfrac{E_D}{2\pi \frac{\omega}{\omega_n} 2\left(\frac{1}{2} k u_0^2\right)}$

Harmonic Forcing

Damping Notes

Ideally, "test" damping at resonance, where the response is most sensitive to damping (not always possible).

$$\longrightarrow \text{If: } \omega = \omega_n \longrightarrow \boxed{\zeta = \frac{1}{4\pi}\frac{E_D}{E_s}}$$

Notes:

- Testing at other frequencies would give different value of damping. (Damping is effectively frequency dependent)
- However, using the ζ found at $\omega = \omega_n$ often provides an acceptable approximation because:
  - Good approximation when it is most important, near resonance.
  - "Less good" at other frequencies, where damping is less important.

NONLINEAR SYSTEMS

Notes:

ζ_{eq} is a "bigger" assumption for nonlinear systems, especially with macroscale 'damage',
for which $k_{sec},\ E_s,\ E_D$ are dependent on $\dfrac{\omega}{\omega_n}$, u_0

For earthquakes: Large input range for $\dfrac{\omega}{\omega_n}$ and a range of u_0, therefore typically model inelastic responses directly, or use Inelastic Spectra.