LECTURE 9 - NUMERICAL METHODS (PART 2)

CE 225

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METHOD 3: ASSUMED VARIATION OF ACCELERATION

AVERAGE ACCELERATION METHOD

Assume constant acceleration over time step Δt_i

METHOD 3: ASSUMED VARIATION OF ACCELERATION

AVERAGE ACCELERATION METHOD

Plug in $\tau = \Delta t_i \;\rightarrow\; \boxed{\dot{u}(\Delta t_i) = \frac{\Delta t_i}{2}(\ddot{u}_{i+1} + \ddot{u}_i) + \dot{u}_i}$ (1)

Integrate:

Plug in $\tau = \Delta t_i \;\rightarrow\; \boxed{u(\Delta t_i) = \frac{(\Delta t_i)^2}{4}(\ddot{u}_{i+1} + \ddot{u}_i) + \dot{u}_i(\Delta t_i) + u_i}$ (2)

Want to "eliminate" $\ddot{u}_{i+1}$ which we don't know!

Also want to solve for equilibrium at Δt_i

METHOD 3: ASSUMED VARIATION OF ACCELERATION

AVERAGE ACCELERATION METHOD

From Eq (1): $\Delta \dot{u}_i = \dot{u}_{i+1} - \dot{u}_i = \frac{\Delta t_i}{2}(\ddot{u}_{i+1} + \ddot{u}_i)$

(3)

From Eq (2): $\Delta u_i = u_{i+1} - u_i =$ (4)

Solve Eq. (4) for $\Delta \ddot{u}_i$ $\longrightarrow$ $$\boxed{\Delta \ddot{u}_i = \frac{4}{(\Delta t_i)^2}(\Delta u_i - \dot{u}_i \Delta t_i) - 2\ddot{u}_i} \quad (5)$$

Plug $\Delta \ddot{u}_i$ from Eq. (5) into Eq. (3) & solve for $\Delta \dot{u}_i$ $\longrightarrow$ $$\boxed{\Delta \dot{u}_i = \frac{2\Delta u_i}{\Delta t_i} - 2\dot{u}_i} \quad (6)$$

METHOD 3: ASSUMED VARIATION OF ACCELERATION

AVERAGE ACCELERATION METHOD

Plug into EOM at $t = \Delta t_i$:

$$m\left(\frac{4}{(\Delta t_i)^2}(\Delta u_i - \dot{u}_i \Delta t_i) - 2\ddot{u}_i\right) + c\left(\frac{2\Delta u_i}{\Delta t_i} - 2\dot{u}_i\right) + k\Delta u_i = \Delta p_i$$

Re-arrange to form:

$$\boxed{\hat{k}\Delta u_i = \Delta \hat{p}_i} \tag{7}$$

where:

$$\hat{k} = k + \frac{2c}{\Delta t} + \frac{4}{(\Delta t)^2}m$$

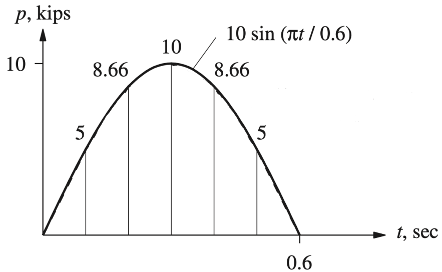
$$\Delta \hat{p}_i = \Delta p_i + \left(\frac{4m}{\Delta t} + 2c\right)\dot{u}_i + 2m\ddot{u}_i$$

$$\boxed{\Delta u_i = \frac{\Delta \hat{p}_i}{\hat{k}}} \quad \longrightarrow \quad \boxed{u_{i+1} = u_i + \Delta u_i}$$

METHOD 3: ASSUMED VARIATION OF ACCELERATION

AVERAGE ACCELERATION METHOD - SOLUTION PROCEDURE

Solution



1.0 *Initial calculations*

$$m = 0.2533 \qquad k = 10 \qquad c = 0.1592$$
$$u_0 = 0 \qquad \dot{u}_0 = 0 \qquad p_0 = 0$$

1.1 $\ddot{u}_0 = \dfrac{p_0 - c\dot{u}_0 - ku_0}{m} = 0.$

1.2 $\Delta t = 0.1.$

1.3 $a_1 = \dfrac{4}{(\Delta t)^2}m + \dfrac{2}{\Delta t}c = 104.5; \quad a_2 = \dfrac{4}{\Delta t}m + c = 10.29; \quad$ and

$a_3 = m = 0.2533.$

1.4 $\hat{k} = k + a_1 = 114.5.$

2.0 *Calculations for each time step,* $i = 0, 1, 2, \ldots$

2.1 $\hat{p}_{i+1} = p_{i+1} + a_1 u_i + a_2 \dot{u}_i + a_3 \ddot{u}_i = p_{i+1} + 104.5\, u_i + 10.29\, \dot{u}_i + 0.2533\, \ddot{u}_i .$

2.2 $u_{i+1} = \dfrac{\hat{p}_{i+1}}{\hat{k}} = \dfrac{\hat{p}_{i+1}}{114.5}.$

2.3 $\dot{u}_{i+1} = \dfrac{2}{\Delta t}(u_{i+1} - u_i) - \dot{u}_i .$

2.4 $\ddot{u}_{i+1} = \dfrac{4}{(\Delta t)^2}(u_{i+1} - u_i) - \dfrac{4}{\Delta t}\dot{u}_i - \ddot{u}_i .$

3.0 *Repetition for the next time step.* Steps 2.1 to 2.4 are repeated for successive time steps and are summarized in Table E5.3, where the theoretical result (from Table E5.1a) is also included.

METHOD 3: ASSUMED VARIATION OF ACCELERATION

AVERAGE ACCELERATION METHOD - SOLUTION PROCEDURE

t_i	p_i	$\hat{p}_i$ (Step 2.1)	$\ddot{u}_i$ (Step 2.4)	$\dot{u}_i$ (Step 2.3)	u_i (Step 2.2)	Theoretical u_i
0.0	0.0000		0.0000	0.0000	0.0000	0.0000
0.1	5.0000	5.0000	17.4666	0.8733	0.0437	0.0328
0.2	8.6603	26.6355	23.1801	2.9057	0.2326	0.2332
0.3	10.0000	70.0837	12.3719	4.6833	0.6121	0.6487
0.4	8.6603	123.9535	−11.5175	4.7260	1.0825	1.1605
0.5	5.0000	163.8469	−38.1611	2.2421	1.4309	1.5241
0.6	0.0000	162.9448	−54.6722	−2.3996	1.4230	1.4814
0.7	0.0000	110.1710	−33.6997	−6.8182	0.9622	0.9245
0.8	0.0000	21.8458	−2.1211	−8.6092	0.1908	0.0593
0.9	0.0000	−69.1988	28.4423	−7.2932	−0.6043	−0.7751
1.0	0.0000	−131.0066	47.3701	−3.5026	−1.1441	−1.2718

NEWMARK'S METHOD

SOLUTION PROCEDURE

- Average (constant) Acceleration Method is special case of Newmark's Method with: $\beta = \frac{1}{4}$, $\gamma = \frac{1}{2}$

Special cases

(1) Constant average acceleration method ($\gamma = \frac{1}{2}$, $\beta = \frac{1}{4}$)

(2) Linear acceleration method ($\gamma = \frac{1}{2}$, $\beta = \frac{1}{6}$)

1.0 *Initial calculations*

1.1 $\ddot{u}_0 = \dfrac{p_0 - c\dot{u}_0 - ku_0}{m}$.

1.2 Select Δt.

1.3 $a_1 = \dfrac{1}{\beta(\Delta t)^2}m + \dfrac{\gamma}{\beta \Delta t}c$; $\quad a_2 = \dfrac{1}{\beta \Delta t}m + \left(\dfrac{\gamma}{\beta} - 1\right)c$; and

$$a_3 = \left(\frac{1}{2\beta} - 1\right)m + \Delta t\left(\frac{\gamma}{2\beta} - 1\right)c.$$

1.4 $\hat{k} = k + a_1$.

2.0 *Calculations for each time step,* $i = 0, 1, 2, \ldots$

2.1 $\hat{p}_{i+1} = p_{i+1} + a_1\, u_i + a_2\, \dot{u}_i + a_3\, \ddot{u}_i$.

2.2 $u_{i+1} = \dfrac{\hat{p}_{i+1}}{\hat{k}}$.

2.3 $\dot{u}_{i+1} = \dfrac{\gamma}{\beta \Delta t}(u_{i+1} - u_i) + \left(1 - \dfrac{\gamma}{\beta}\right)\dot{u}_i + \Delta t\left(1 - \dfrac{\gamma}{2\beta}\right)\ddot{u}_i$.

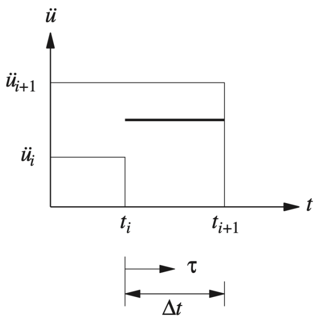
2.4 $\ddot{u}_{i+1} = \dfrac{1}{\beta(\Delta t)^2}(u_{i+1} - u_i) - \dfrac{1}{\beta \Delta t}\dot{u}_i - \left(\dfrac{1}{2\beta} - 1\right)\ddot{u}_i$.

3.0 *Repetition for the next time step*. Replace i by $i + 1$ and implement steps 2.1 to 2.4 for the next time step.

NEWMARK'S METHOD

ASSUMED LINEAR ACCELERATION

Constant Average Acceleration



$$\ddot{u}(\tau) = \frac{1}{2}(\ddot{u}_{i+1} + \ddot{u}_i)$$

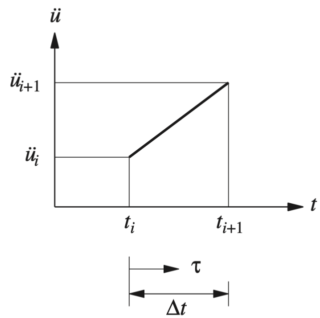
$$\dot{u}(\tau) = \dot{u}_i + \frac{\tau}{2}(\ddot{u}_{i+1} + \ddot{u}_i)$$

$$\dot{u}_{i+1} = \dot{u}_i + \frac{\Delta t}{2}(\ddot{u}_{i+1} + \ddot{u}_i)$$

$$u(\tau) = u_i + \dot{u}_i\tau + \frac{\tau^2}{4}(\ddot{u}_{i+1} + \ddot{u}_i)$$

$$u_{i+1} = u_i + \dot{u}_i\,\Delta t + \frac{(\Delta t)^2}{4}(\ddot{u}_{i+1} + \ddot{u}_i)$$

Linear Acceleration



$$\ddot{u}(\tau) = \ddot{u}_i + \frac{\tau}{\Delta t}(\ddot{u}_{i+1} - \ddot{u}_i)$$

$$\dot{u}(\tau) = \dot{u}_i + \ddot{u}_i\tau + \frac{\tau^2}{2\Delta t}(\ddot{u}_{i+1} - \ddot{u}_i)$$

$$\dot{u}_{i+1} = \dot{u}_i + \frac{\Delta t}{2}(\ddot{u}_{i+1} + \ddot{u}_i)$$

$$u(\tau) = u_i + \dot{u}_i\tau + \ddot{u}_i\frac{\tau^2}{2} + \frac{\tau^3}{6\Delta t}(\ddot{u}_{i+1} - \ddot{u}_i)$$

$$u_{i+1} = u_i + \dot{u}_i\,\Delta t + (\Delta t)^2\left(\frac{1}{6}\ddot{u}_{i+1} + \frac{1}{3}\ddot{u}_i\right)$$

NEWMARK'S METHOD

STABILITY

For Stability: $$\frac{\Delta t}{T_n} \leq \frac{1}{\pi\sqrt{2}}\frac{1}{\sqrt{\gamma - 2\beta}}$$

- Constant Acceleration: $\beta = \frac{1}{4}$, $\gamma = \frac{1}{2}$ $\longrightarrow$

  $\longrightarrow$ Unconditionally stable!!

- Linear Acceleration: $\beta = \frac{1}{6}$, $\gamma = \frac{1}{2}$ $\longrightarrow$ $\frac{\Delta t}{T_n} \leq 0.551$

NUMERICAL METHODS - LINEAR SYSTEMS

ERROR

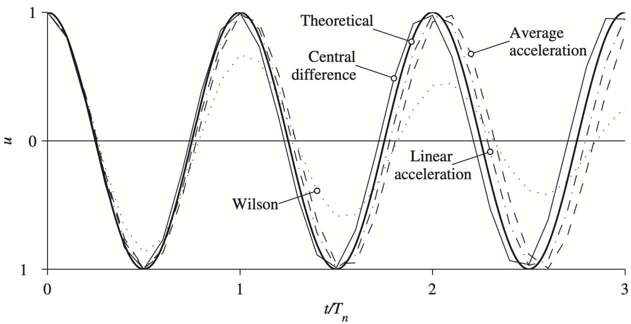


Figure 5.5.1 Free vibration solution by four numerical methods ($\Delta t / T_n = 0.1$) and the theoretical solution.

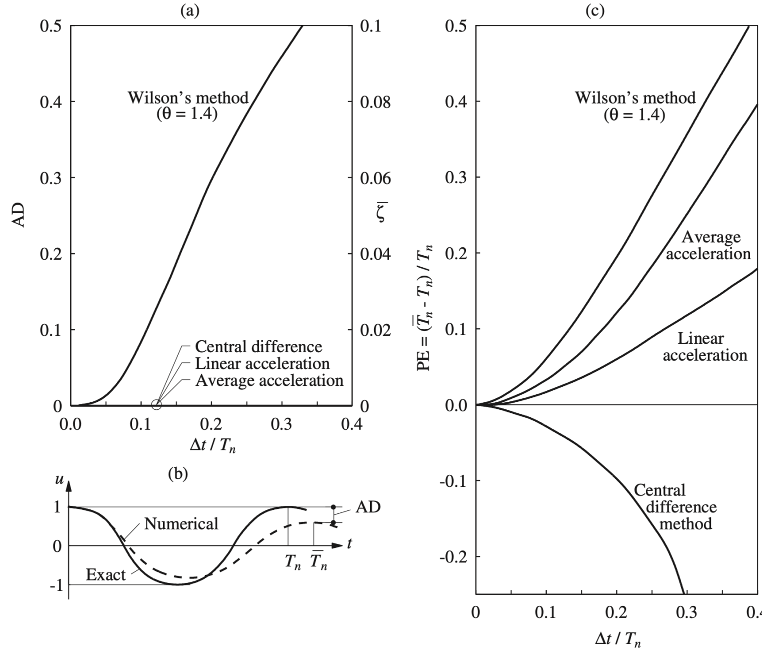


Figure 5.5.2 (a) Amplitude decay versus $\Delta t / T_n$; (b) definition of AD and PE; (c) period elongation versus $\Delta t / T_n$.

NUMERICAL METHODS - NONLINEAR SYSTEMS

CENTRAL DIFFERENCE METHOD

Recall (last lecture): $u_{i+1} = \dfrac{\hat{p}_i}{\hat{k}}$ where: $\begin{cases} \hat{p}_i = f_1(m,\ c,\ \Delta t,\ u_i,\ u_{i-1}) - ku_i \\ \hat{k} = f_2(m,\ c,\ \Delta t) \end{cases}$

For nonlinear model, substitute:

Assume that stiffness force is constant over Δt, but Δt is small, and update every step.

a) <u>Nonlinear elastic</u>

NUMERICAL METHODS - NONLINEAR SYSTEMS

CENTRAL DIFFERENCE METHOD (CONTINUED)

b) Inelastic (e.g. elastoplastic spring):

- Need both u and $\dot{u}$ to define the stiffness.
- Still lookup and update in each step.
- In general, Δt must be small

Problems:

1) Assuming constant $(f_s)_i$ over one time step may not be "good enough".
2) Can require very small Δt_i to obtain acceptable error

NUMERICAL METHODS - NONLINEAR SYSTEMS

NEWMARK'S METHOD

Recall: Enforced EOM at each step $\longrightarrow$ $m(\Delta\ddot{u}_i) + c(\Delta\dot{u}_i) + k\Delta u_i = \Delta p_i$

Assume, to start, that the effective stiffness is constant over the time step:

$\longrightarrow (\Delta f_s)_i = (k_i)_T \Delta u_i$, where $(k_i)_T =$ tangent stiffness

Plot:

We would actually want:

NUMERICAL METHODS - NONLINEAR SYSTEMS

NEWMARK'S METHOD (CONTINUED)

Two options:

i) Use very small Δt and assume 'good enough'
Problem: need very small Δt to capture "sharp" changes in system:

ii) Iterate:

(a) Modified Newton Raphson Method (e.g. Constant Acceleration Solution from before)

$$\Delta u_i = \frac{\Delta \hat{p}_i}{\hat{k}} \qquad \text{where:} \qquad \begin{cases} \Delta \hat{p}_i = \Delta p_i + \left(\dfrac{4m}{\Delta t} + 2c\right) + 2m\ddot{u}_i \\ \hat{k} = k + \dfrac{2c}{\Delta t} + \dfrac{4}{(\Delta t)^2}m = \end{cases}$$

NUMERICAL METHODS - NONLINEAR SYSTEMS

NEWMARK'S METHOD (CONTINUED)

Plot:

(1) Find $\Delta u_i^{(j)} = \dfrac{\Delta \hat{p}_i}{\hat{k}_{T,i}}$

(2) Use $\Delta u_i^{(j)}$ to find $\Delta f^{(j)}$:

$$\Delta f^{(j)} = f_s^j - f_s^{(j-1)} + (\hat{k}_{Ti} - k_{Ti})\Delta u_i^{(j)}$$

Numerical Methods - Nonlinear Systems

Newmark's Method (continued)

(b) Newton-Raphson Method ("original"): Same procedure, but use new k_T at every iteration

- Converges faster than Modified Newton-Raphson.
- However, must recalculate stiffness (matrix) in each iteration (costly for MDOF systems).
- More iterations required for Modified N-R can be faster than recalculating k_T

NUMERICAL METHODS - SUMMARY

- Linear Systems:
  1. Interpolation of $p(t)$
     - assumes linear $p(t)$ over time step.
  2. Central Difference Method
     - Uses Taylor series, i.e. local derivatives ($\dot{u},\ \ddot{u}$), to approximate "curve"
     - Explicit method, i.e. enforces EOM at time t_i
  3. Newmark's Method
     - E.g. Assumed variation in acceleration (e.g. constant or linear)
     - Implicit method, i.e. enforces EOM at time t_{i+1}
     - No iteration required

- Nonlinear Systems:
  1. Central Difference Method
     - Assume f_s is constant in each time step, but update each step
     - Each step has small computation time, but Δt is very small
     - Explicit method, i.e. enforces EOM at time t_i
  2. Newmark: Two options (both implicit)
     2.1 Use k_T and use small steps but don't iterate.
     2.2 Iterate and enforce correct f_s at u_{i+1}