CE 225: Dynamic of Structures **Fall 2025**

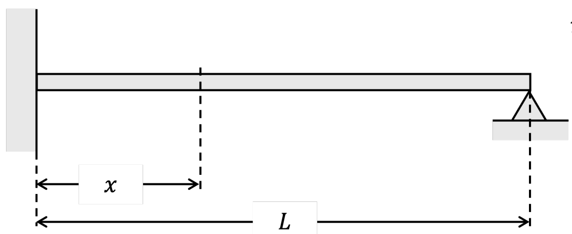
Discussion 8: Generalized SDOF systems

Instructor: Matthew DeJong *GSI: Miguel A. Gomez*

Generalized SDOF systems: understanding the equations

Main Goal: understand vibrations of systems with distributed mass and stiffness.

Let's think of the beam below:



Problem: the exact solution of the vibrations of this beam (using Euler-Bernoulli Beam theory) require solving the following PDE:

$$\rho A \frac{\partial^2 u}{\partial t^2} + EI \frac{\partial^4 u}{\partial x^4} = f(x,t)$$

Approach: Let's not solve for all the motion, but for a 'main' vibration mode. Let's assume that the vibrations can be decoupled in time and space:

$$u(x,t) = \psi(x) z(t)$$

Well, this is actually the same thing you'd do to solve the PDE above. But, let's add something else...

- Let's say we have a good guess of what $\psi(x)$ looks like. The main thing here is getting the boundary conditions right.
- Now, we are guessing what $\psi(x)$ is, so we can't enforce equilibrium directly. We enforce an **energy balance** instead (or a virtual work balance, see textbook). This is sometimes referred to as a **'weak form'** of equilibrium (connection to finite element method).
- Once we find a shape function that satisfies the boundary conditions that we care about, we can solve for the dynamics of the system, assuming **all the response happens in this vibration mode**. Let's see a few examples of how this works.

The formulas:

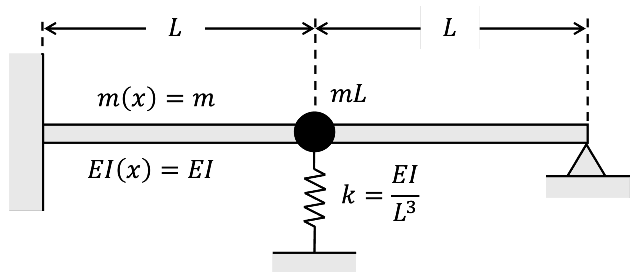
Term	Continuous	Discrete
Stiffness	$\tilde{k} = \int_0^L EI(x)\left(\psi(x)''\right)^2 dx$	$\tilde{k} = \sum_{j=1}^n k_j(\psi_j - \psi_{j-1})^2$
Mass	$\tilde{m} = \int_0^L m(x)\left(\psi(x)\right)^2 dx$	$\tilde{m} = \sum_{j=1}^n m_j(\psi_j)^2$
External Force	$\tilde{p} = \int_0^L p(x,t)\psi(x)dx$	$\tilde{p} = \sum_{j=1}^n p_j\psi_j$
Earthquake	$\tilde{L} = \int_0^L m(x)\psi(x)dx$	$\tilde{L} = \sum_{j=1}^n m_j\psi_j$

Table 0.1: Formulas for Rayleigh method for continuous and discrete systems

Also, a small math preliminary, in case you haven't seen this:

$$\int_{-\infty}^{\infty} \delta(x-a)f(x)dx = f(a)$$

Problem 1: Continuous system.

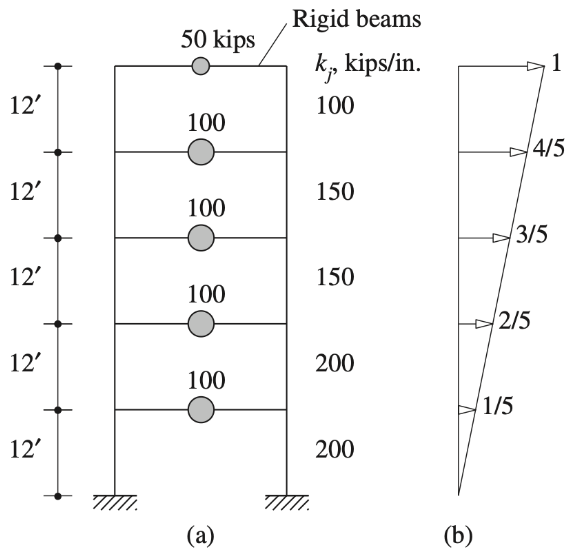


The structure in the picture above is a propped cantilever beam of length $2L$, with distributed mass per unit length $m(x)$, and a concentrated mass at the center $M = mL$. The system is also connected to a spring at the center, with stiffness $k = EI/L^3$. Determine an expression for the natural frequency of the system using Rayleigh method.

Problem 1 (cont'd)

Problem 2: Discrete system.

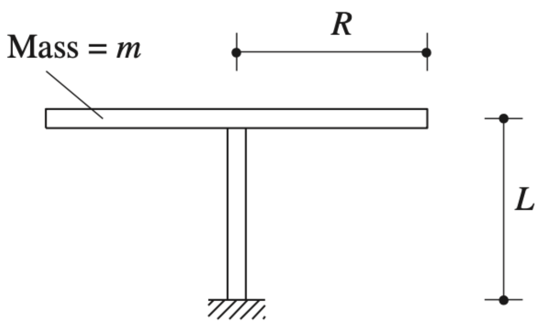
The five-story frame with rigid beams shown below is subjected to ground acceleration $\ddot{u}_g(t)$; k_j are story stiffnesses. Assuming the displacements increase linearly with height, formulate the equation of motion for the system and determine its natural frequency. Determine the floor displacements to ground motion characterized by the design spectrum below, scaled to a peak ground acceleration of 0.2g.



Problem 2 (cont'd)

Problem 3: Umbrella structure.

The umbrella structure shown below consists of a uniform column of flexural rigidity EI supporting a uniform slab of radius R and mass m. By Rayleigh's method determine the natural vibration frequency of the structure. Neglect the mass of the column and the effect of axial force on column stiffness. Assume that the slab is rigid in flexure and that the column is axially rigid.



Problem 3 (cont'd)

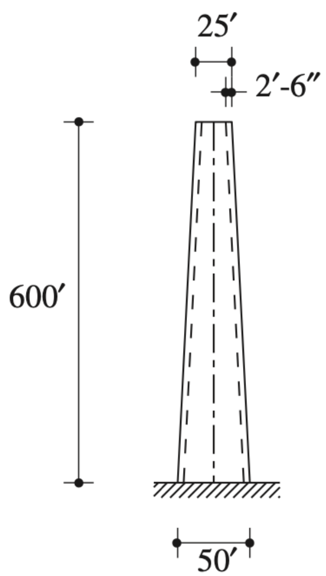
Problem 4: A more complex continuous system.

A reinforced-concrete chimney 600 ft high has a hollow circular cross section with outside diameter 50 ft at the base and 25 ft at the top; the wall thickness is 2 ft 6 in., uniform over the height. Using the approximation that the wall thickness is small compared to the radius, the mass and flexural stiffness properties are computed from the gross area of concrete (neglecting reinforcing steel).

The chimney is assumed to be clamped at the base, and its damping ratio is estimated to be 5%. The unit weight of concrete is 150 lb/ft^3, and its elastic modulus $E_c = 3600$ ksi. Assuming that the shape function is:

$$\psi(x) = 1 - \cos\left(\frac{\pi}{2L}x\right)$$

where L is the length of the chimney and x is measured from the base, calculate the following quantities: (a) the shear forces and bending moments at the base and at the midheight, and (b) the top deflection due to ground motion defined by the design spectrum below, which is scaled to a peak acceleration of 0.25g.



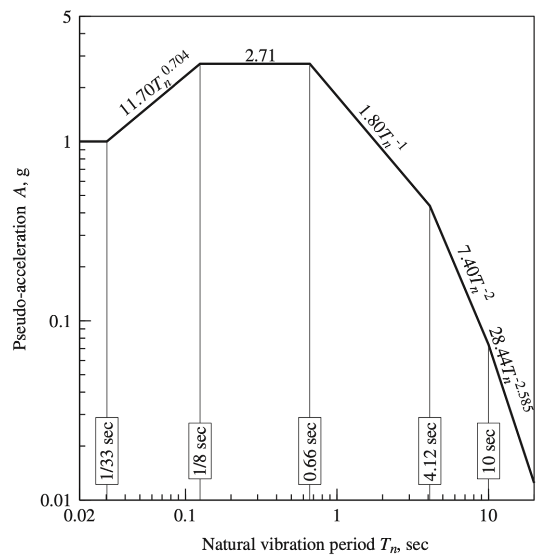


Figure 6.9.5 Elastic pseudo-acceleration design spectrum (84.1th percentile) for ground motions with $\ddot{u}_{go} = 1$g, $\dot{u}_{go} = 48$ in./sec, and $u_{go} = 36$ in.; $\zeta = 5\%$.

Problem 1 (cont'd)

Problem 4 (cont'd)