LECTURE 3 - HARMONIC FORCING

CE 225

Prof DeJong

UC Berkeley

September 4, 2024

Harmonic Forcing - Undamped

Equation of Motion & Solution

EOM : $m\ddot{u} + ku = p_0 \sin \omega t$ where ω = driving frequency

Particular Solution : $u_p(t) = C \sin \omega t \quad \rightarrow \ddot{u}_p(t) =$

Plug into EOM: $\longrightarrow m[-C\omega^2 \sin \omega t] + k[C \sin \omega t] = p_0 \sin \omega t$

Solve for C : $\longrightarrow C(k - \omega^2 m) = p_0 \quad \rightarrow \quad C =$

$$\boxed{u_p(t) = \frac{p_0}{k} \frac{1}{1 - \left(\frac{\omega}{\omega_n}\right)^2} \sin(\omega t)} \tag{1}$$

Complementary Solution: $u_c(t) = A \cos(\omega_n t) + B \sin(\omega_n t)$ (2)

HARMONIC FORCING - UNDAMPED

EQUATION OF MOTION & SOLUTION

Total response: $u_c(t) + u_p(t) = (2) + (1)$ $\longrightarrow$ Solve A,B using initial conditions: $u_0, \dot{u}_0$

$$u(t) = u_0 \cos \omega_n t + \left[\frac{\dot{u}_0}{\omega_n} - \frac{p_0}{k} \frac{\omega/\omega_n}{1-(\omega/\omega_n)^2} \right] \sin \omega_n t + \frac{p_0}{k} \frac{1}{1-(\omega/\omega_n)^2} \sin \omega t \tag{3}$$

Assume: $u = \dot{u}_0 = 0;\quad \dfrac{\omega}{\omega_n} = \dfrac{T_n}{T} = 0.25;\quad (u_{st})_0 = \dfrac{p_0}{k}$

HARMONIC FORCING - UNDAMPED

RESPONSE PLOT

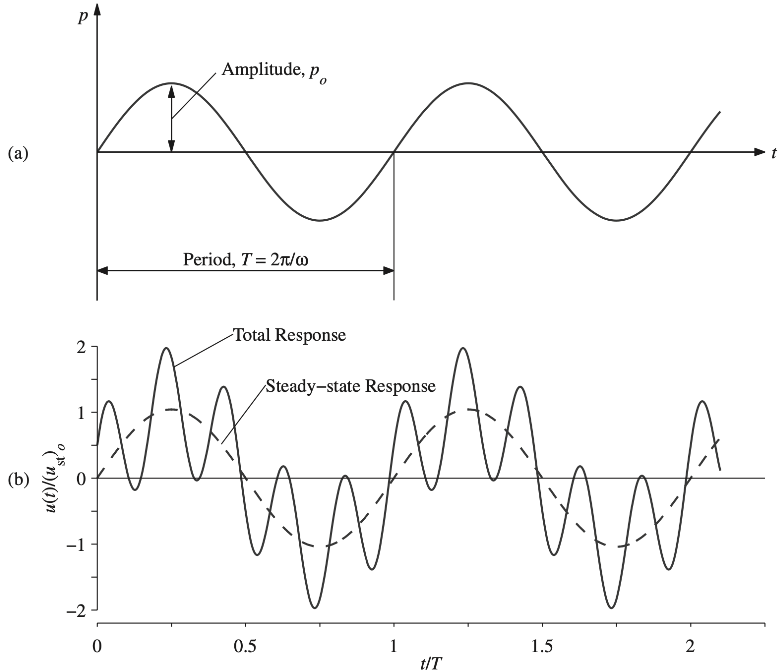

Figure 3.1.1 (a) Harmonic force; (b) response of undamped system to harmonic force; $\omega/\omega_n = 0.2$, $u(0) = 0.5p_o/k$, and $\dot{u}(0) = \omega_n p_o/k$.

Harmonic Forcing - Undamped

Key Points

- Transient response is at ω_n
- Steady State (S.S.) response is at ω
- If $\omega_n \approx \infty \rightarrow \frac{\omega}{\omega_n} \approx 0$ (i.e. structure is essentially rigid), structure responds instantly!

$$\rightarrow u(t) = u_{st}(t) = \frac{p_0}{k} \sin(\omega t) = (u_{st})_o \sin(\omega t)$$

HARMONIC FORCING - UNDAMPED

DYNAMIC AMPLIFICATION

Define 'Dynamic Amplification':

$$DA = \frac{u_{p,max}}{u_{st,max}} = \frac{1}{1 - \left(\frac{\omega}{\omega_n}\right)^2}$$

Alternative Steady State solution: $u_p(t) = \frac{p_0}{k} R_d \sin(\omega t - \phi)$ where: $R_d = |DA|$

HARMONIC FORCING - UNDAMPED

PHASE ANGLE & RESONANCE

* What does $\phi = \pi$ mean?

$$\rightarrow \text{ for } \frac{\omega}{\omega_n} = 1.3 \quad \rightarrow$$

@ resonance: $\frac{\omega}{\omega_n} = 1.0 \quad \rightarrow \quad u_p(t) = \frac{-p_0}{2k}\omega_n t\cos(\omega_n t)$

HARMONIC FORCING - DAMPED

EQUATION OF MOTION & SOLUTION

EOM: $m\ddot{u} + c\dot{u} + ku = p_0 \sin \omega t$

Particular Solution: $u_p(t) = C \sin \omega t + D \cos \omega t$

$$\rightarrow \dot{u}_p(t) = C\omega \cos \omega t - D\omega \sin \omega t$$

$$\rightarrow \ddot{u}_p(t) = -C\omega^2 \sin \omega t - D\omega^2 \cos \omega t$$

Solve for C & D:

$$C = \frac{p_0}{k} \frac{1 - (\frac{\omega}{\omega_n})^2}{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\zeta\left(\frac{\omega}{\omega_n}\right)\right]^2} \qquad D = \frac{p_0}{k} \frac{-2\zeta\left(\frac{\omega}{\omega_n}\right)^2}{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\zeta\left(\frac{\omega}{\omega_n}\right)\right]^2}$$

Total solution: $\longrightarrow u(t) = u_c(t) + u_p(t) = e^{-\zeta\omega_n t}(A \cos \omega_D t + B \sin \omega_D t) + C \sin \omega t + D \cos \omega t$

HARMONIC FORCING - DAMPED

EXAMPLE RESPONSE

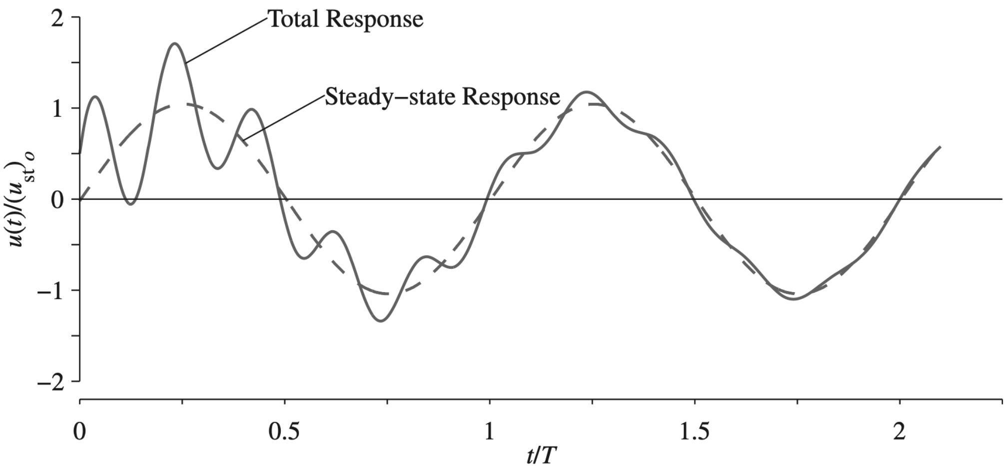

Figure 3.2.1 Response of damped system to harmonic force; $\omega/\omega_n = 0.2$, $\zeta = 0.05$, $u(0) = 0.5p_o/k$, and $\dot{u}(0) = \omega_n p_o/k$.

HARMONIC FORCING - DAMPED

RESONANCE

At resonance: $\dfrac{\omega}{\omega_n} = 1.0$

→ Total solution: $u(t) = \dfrac{p_0}{k}\dfrac{1}{2\zeta}\left[e^{-\zeta\omega_n t}\left(\cos\omega_D t + \dfrac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_D t\right) - \cos\omega_n t\right]$

→ Max amplification $= DA = \dfrac{1}{2\zeta}$

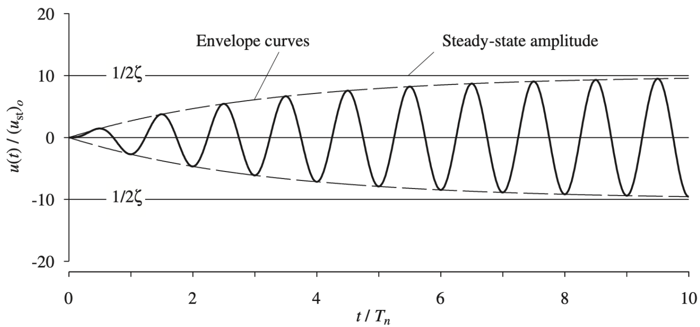

Figure 3.2.2 Response of damped system with $\zeta = 0.05$ to sinusoidal force of frequency $\omega = \omega_n$; $u(0) = \dot{u}(0) = 0$.

HARMONIC FORCING - DAMPED

STEADY STATE - DYNAMIC AMPLIFICATION & PHASE

Alternative form of steady state solution (i.e. $u_p(t)$):

$$u_p(t) = (u_{st})_0 R_d \sin(\omega t - \phi) \qquad \text{where:} \quad \begin{cases} R_d = \dfrac{1}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\zeta\left(\frac{\omega}{\omega_n}\right)\right]^2}} \\ \phi = \tan^{-1}\left[\dfrac{2\zeta\frac{\omega}{\omega_n}}{1 - \left(\frac{\omega}{\omega_n}\right)^2}\right] \end{cases}$$

Plot:

HARMONIC FORCING - DAMPED

STEADY STATE - EXAMPLE RESPONSES

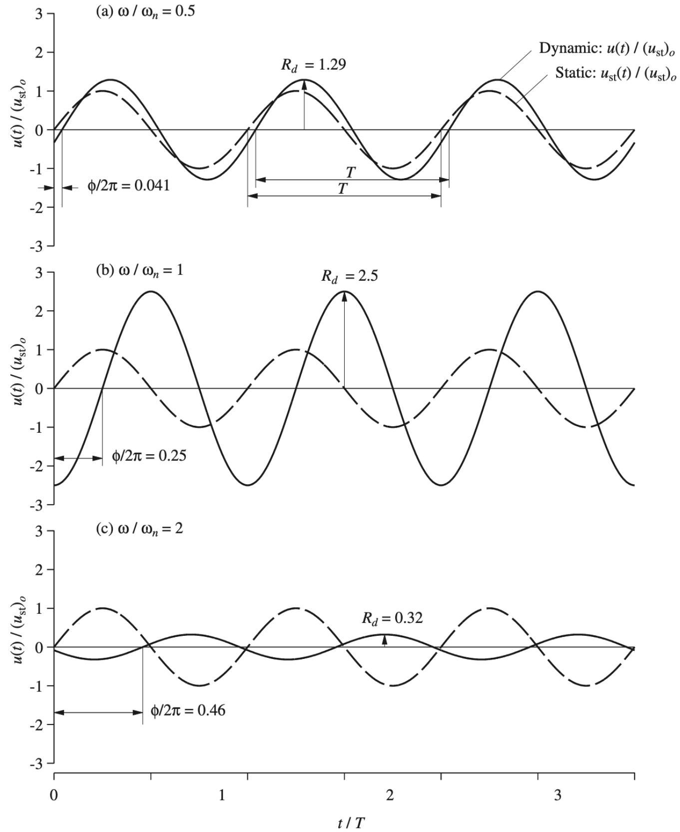

Figure 3.2.5 Steady-state response of damped systems ($\zeta = 0.2$) to sinusoidal force for three values of the frequency ratio: (a) $\omega/\omega_n = 0.5$, (b) $\omega/\omega_n = 1$, (c) $\omega/\omega_n = 2$.

Harmonic Forcing - Damped

Comments on R_d Plot

- For $\frac{\omega}{\omega_n} << 1 \quad \longrightarrow$

- For $\frac{\omega}{\omega_n} \approx 1 \quad \longrightarrow$

- For $\frac{\omega}{\omega_n} >> 1 \quad \longrightarrow$

$$\left\{ \begin{aligned} & R_d \approx \left(\frac{\omega_n}{\omega}\right)^2 \\ & u(t) = \frac{p_0}{k}\left(\frac{\omega_n}{\omega}\right)^2 = \frac{p_0}{k}\frac{k}{m}\frac{1}{\omega^2} = \frac{p_0}{m\omega^2} \end{aligned} \right\}$$

HARMONIC FORCING - DAMPED

DYNAMIC RESPONSE FACTORS (NORMALIZED)

$$\boxed{u(t)} = \frac{p_0}{k} R_d \sin(\omega t - \phi) \quad \longrightarrow$$

$$\boxed{\dot{u}(t)} = \frac{p_0}{k} R_d \omega \cos(\omega t - \phi) \quad \xrightarrow{\text{multiply by } \frac{1}{\omega_n}\sqrt{\frac{k}{m}}=1}$$

$$\longrightarrow \frac{\dot{u}(t)}{p_0/\sqrt{km}} = R_d \frac{\omega}{\omega_n}[\cos(\omega t - \phi)] \quad \longrightarrow \quad \text{Define:} \quad \boxed{R_v = \frac{\omega}{\omega_n} R_d}$$

$$\boxed{\ddot{u}(t)} = \frac{-p_0}{k} R_d \omega^2 \sin(\omega t - \phi) \quad \xrightarrow{\text{multiply} \frac{1}{\omega_n^2}\frac{k}{m}=1}$$

$$\longrightarrow \frac{\ddot{u}(t)}{p_0/m} = -R_d \left(\frac{\omega}{\omega_n}\right)^2 [\sin(\omega t - \phi)] \longrightarrow \quad \text{Define:} \quad \boxed{R_a = \left(\frac{\omega}{\omega_n}\right)^2 R_d}$$

HARMONIC FORCING - DAMPED

DYNAMIC RESPONSE FACTORS (NORMALIZED)

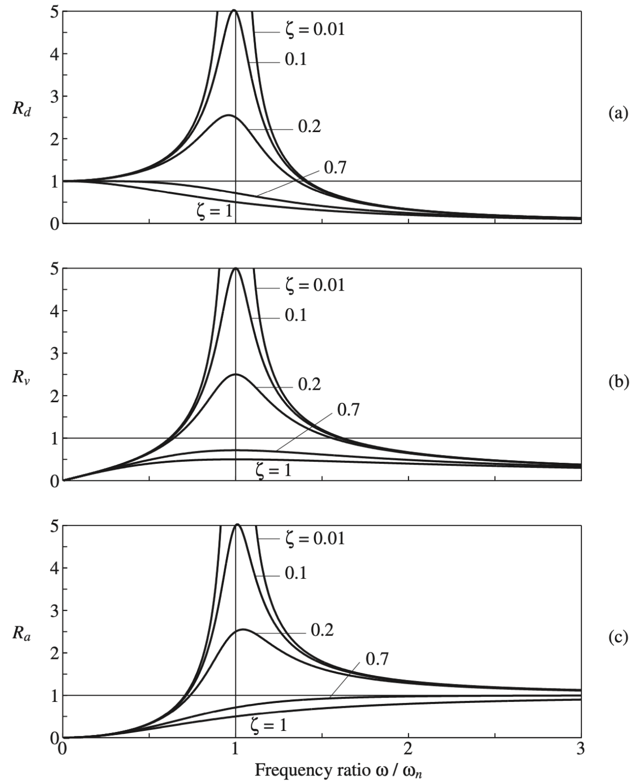

Figure 3.2.7 Deformation, velocity, and acceleration response factors for a damped system excited by harmonic force.