LECTURE 2 - FREE VIBRATION

CE 225

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UNDAMPED FREE VIBRATION

Initial Conditions (IC's): $u(0) = u_0 \quad ; \quad \dot{u}(0) = \dot{u}_0$

EOM:

$$m\ddot{u} + ku = 0$$
$$\ddot{u} + \frac{k}{m}u = 0$$
$$\ddot{u} + \omega_n^2 u = 0$$

Solution: $u(t) = A\cos\omega_n t + B\sin\omega_n t$

$$@t = 0 \qquad \rightarrow u_0 = A(1) + B(0) \quad \rightarrow \quad A = u_0$$

$$\rightarrow \dot{u}(t) = -A\omega_n \sin\omega_n t + B\omega_n \cos\omega_n t \quad \rightarrow \quad \dot{u}_0 = B\omega_n$$

$$\boxed{u(t) = u_0 \cos\left(\omega_n t\right) + \frac{\dot{u}_0}{\omega_n}\sin\left(\omega_n t\right)}$$

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natural frequency [rad/s]

$$\omega_n = \sqrt{\frac{k}{m}}$$

natural frequency [Hz] = cycles/sec

$$f_n = \frac{\omega_n}{2\pi}$$

natural period [s] = sec/cycle

$$T_n = \frac{1}{f_n} = \frac{2\pi}{\omega_n}$$

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Alternatively: $u(t) = \sqrt{u_0^2 + \left(\frac{\dot{u}_0}{\omega_n}\right)^2} \cos\left(\omega_n t - \phi\right)$ $\phi =$ phase angle $= f\left(u_0, \dot{u}_0\right)$

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EXAMPLE

Find EOM and free vibration response when released from rest (horizontal)

$$\boxed{u(t) = \frac{g}{\omega_n^2} \cos\left(\omega_n t\right) + 0}$$

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EQUATION OF MOTION

$$m\ddot{u} + c\dot{u} + ku = 0$$

$$\ddot{u} + \frac{c}{m}\dot{u} + \frac{k}{m}u = 0$$

define $\longrightarrow \zeta = \dfrac{c}{2m\omega_n} = \dfrac{c}{c_v}$

$$\longrightarrow \ddot{u} + 2\zeta\omega_n\dot{u} + \omega_n^2 u = 0$$

Underdamped (subcritical)	Critically Damped	Overdamped (hypercritical)
$\zeta < 1$	$\zeta = 1$	$\zeta > 1$

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SOLUTION TO EOM

$\zeta < 1 \rightarrow$ Solution to EOM: $u(t) = e^{-\zeta\omega_n t}(A\cos\omega_d t + B\sin\omega_d t) \quad \rightarrow$ where $\omega_d = \omega_n\sqrt{1-\zeta^2}$

Initial Conditions: $u(0) = u_0\,; \quad \dot{u}(0) = \dot{u}_0$

$$u(t) = e^{-\zeta\omega_n t}\left(u_0\cos\omega_d + \left(\frac{\dot{u}_0 + \zeta\omega_n u_0}{\omega_d}\right)\sin\omega_d t\right) = e^{-\zeta\omega_n t}\sqrt{A^2+B^2}\cos\left(\omega_d t - \phi\right)$$

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LOGARITHMIC DECREMENT

One way to find damping:

At time t : $$u(t) = e^{-\zeta\omega_n t}\left(u_0 \cos\omega_d t + \left(\frac{\dot{u}_0 + \zeta\omega_n u_0}{\omega_d}\right)\sin\omega_d t\right)$$

j cycles later : $$u(t+jT_D) = e^{-\zeta\omega_n (t+jT_D)}\left(u_0 \cos\omega_d (t+jT_D) + \left(\frac{\dot{u}_0 + \zeta\omega_n u_0}{\omega_d}\right)\sin\omega_d (t+jT_D)\right)$$

$$\longrightarrow \frac{u(t)}{u(t+jT_D)} = \frac{e^{-\zeta\omega_n t}}{e^{-\zeta\omega_n t}\cdot e^{-\zeta\omega_n jT_D}} = e^{\zeta\omega_n jT_D}$$

Recall: $$T_D = \frac{2\pi}{\omega_D} = \frac{2\pi}{\omega_n\sqrt{1-\zeta^2}}$$

$$\longrightarrow \ln\left(\frac{u_i}{u_{i+j}}\right) = \frac{2\pi\zeta j}{\sqrt{1-\zeta^2}} \approx 2\pi\zeta j$$

$$\boxed{\zeta = \frac{\ln\frac{u_i}{u_{i+j}}}{2\pi j} \qquad \text{for } \zeta < 0.2}$$

Note on Energy

Undamped: $E_{total} = KE + PE = E_{initial} = \frac{1}{2}ku_0^2 + \frac{1}{2}m\dot{u}_0^2$

Damped: $E_{total} = E_{initial} - E_{Damping}$

$$E_{Damping} = \int f_D du = \int (c\dot{u})(\dot{u}dt) \longrightarrow \boxed{E_d = \int_0^t c\dot{u}^2 dt}$$