CE 225: Dynamic of Structures **Fall 2025**

Discussion 2: Forced Vibration - Harmonic Excitation

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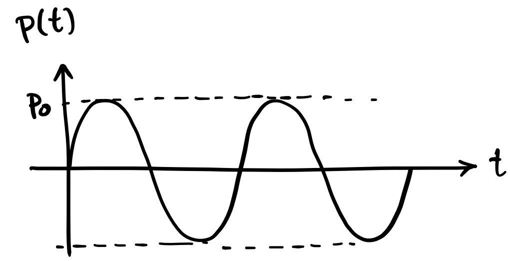
Objectives

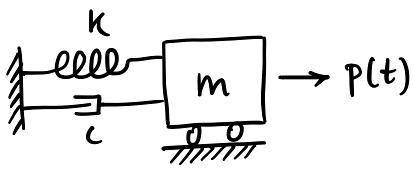
By the end of this discussion we'll be able to:

1. Find the lateral stiffness of a multi-column one-story shear building and its basic dynamic properties.
2. Find the damping coefficient of a structure from a resonance test.
3. Apply the equation for the transmissibility.



Harmonic excitation

$$\boxed{m\ddot{u} + c\dot{u} + ku = p(t) \quad \text{with} \quad p(t) = p_0 \sin \omega t}$$



; $p(t) = p_0 \cdot \sin(\omega t)$.

Cases

$\zeta = 0$ (UNDAMPED)

$$\boxed{m\ddot{u} + ku = p_0 \cdot \sin(\omega t)}$$

Homogeneous:

$$m\ddot{u} + ku = 0$$

$$\rightarrow \ddot{u} + \omega_n^2 u = 0$$

$$\Rightarrow u_h(t) = A \cdot \cos(\omega_n t) + B \sin(\omega_n t)$$

Particular: → for $p(t) = p_0 \cdot \sin(\omega_n t)$

↓

Depends

↓

ω/ω_n

$\frac{\omega}{\omega_n} \neq 1$: $u_p = C \cdot \sin(\omega t)$

$\frac{\omega}{\omega_n} = 1$ (Resonance): $u_p = Ct\cos(\omega t)$

$\zeta \neq 0$ (DAMPED) $\underline{\zeta < 1}$

$$\boxed{m\ddot{u} + c\dot{u} + ku = p_0 \cdot \sin(\omega t).}$$

Homogeneous:

$$m\ddot{u} + c\dot{u} + ku = 0$$

$$\rightarrow \ddot{u} + 2\zeta\omega_n\dot{u} + \omega_n^2 u = 0$$

$$\Rightarrow u_h(t) = e^{-\zeta\omega_n t}\left(A\cos(\omega_d t) + B\sin(\omega_d t)\right)$$

Particular.

$$u_p(t) = C\cos(\omega t) + D\sin(\omega t)$$

↓

Solve for C and D plugging this into E.O.M.

then do:

$$C\cos(\omega t) + D\sin(\omega t) = \sqrt{C^2 + D^2}\,\sin(\omega t - \phi)$$

(*) this can be done with trigonometric identity. –

$$\boxed{u_p = (u_{st})_0 \cdot R_d \cdot \sin(\omega t - \phi)}$$

(steady state response).

(a) Undamped system ($\zeta = 0$)

Case i: $\omega/\omega_n \neq 1$

steady state

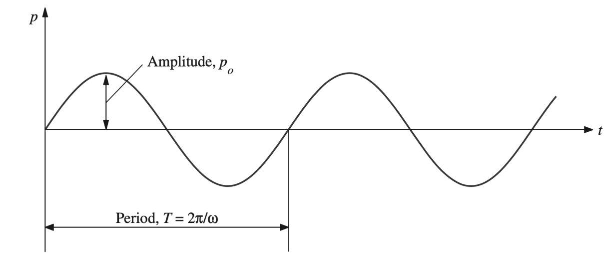
max. steady state response

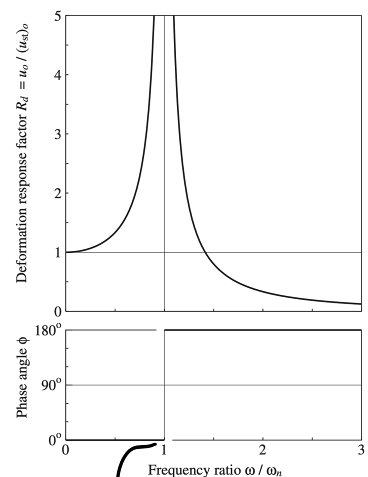
$$\boxed{u(t) = \underbrace{A\sin\omega_n t + B\cos\omega_n t}_{\text{transient, } \omega_n} + \overbrace{\frac{p_0}{k} R_d \sin(\omega t - \phi)}^{\text{steady state}}} \quad \text{where} \quad R_d = \left| \frac{1}{1 - \left(\frac{\omega}{\omega_n}\right)^2} \right| = \frac{u_0}{(u_{st})_0}$$

transient, ω_n

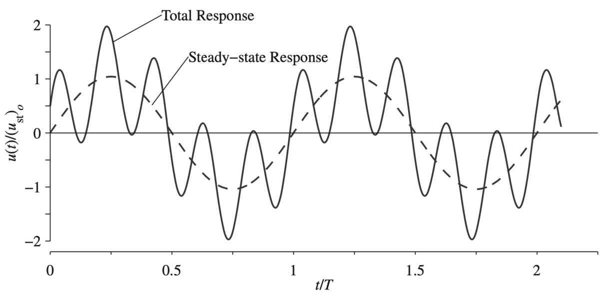
deformation

max static response





$(u_{st})_0 = \frac{p_0}{k}$.

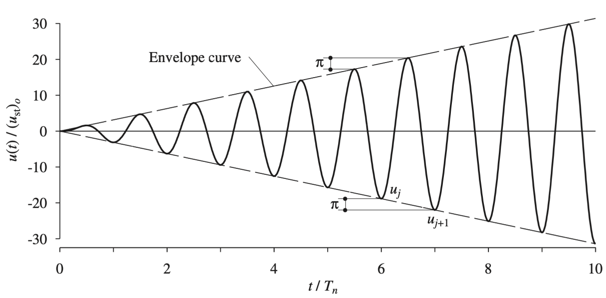


resonance.

Case ii: Resonance $\omega/\omega_n = 1$

$$\boxed{u(t) = A\cos\omega_n t + B\sin\omega_n t - \frac{p_0}{2k}\omega_n t\cos\omega_n t}$$

as $t \to \infty$
$u(t) \to \infty$



$$\frac{u(t)}{(u_{st})_0} = \frac{\frac{p_0}{2k} \cdot t}{p_0/k} = \frac{\omega_n}{2} t$$

$\Rightarrow$ every $\Delta t = T_n \rightarrow \frac{2\pi}{\omega_n}$ $\quad \underbrace{\Delta u}_{\text{change in displacement response.}} = \frac{\omega_n}{2} \cdot T_n = \frac{\omega_n}{2} \cdot \frac{2\pi}{\omega_n} = \pi$.

as $t \to \infty$, transient $\to 0$

(b) Damped system with $\zeta < 1$

transient.

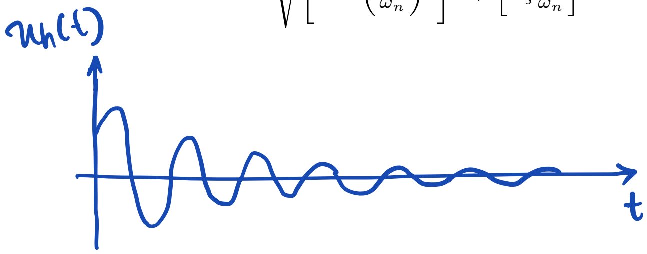
Case i: $\omega/\omega_n \neq 1$

$$\boxed{u(t) = e^{-\zeta\omega_n t}\left(A\cos\omega_d t + B\sin\omega_d t\right) + \frac{p_0}{k}R_d\sin\left(\omega t - \phi\right)}$$

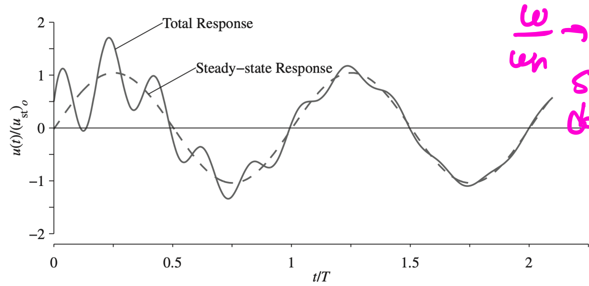
transient

Phase angle (time lag).

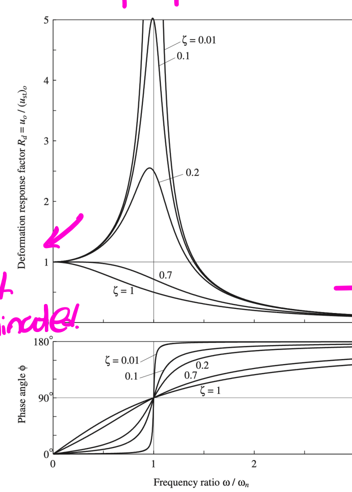
where $R_d = \dfrac{1}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\zeta\frac{\omega}{\omega_n}\right]^2}}$ and $\phi = \tan^{-1}\left[\dfrac{2\zeta\frac{\omega}{\omega_n}}{1-\left(\frac{\omega}{\omega_n}\right)^2}\right]$



damping dominated

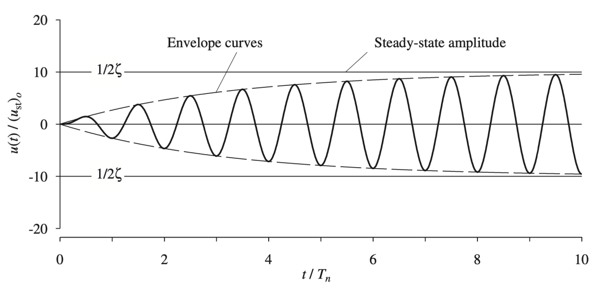


$\frac{\omega}{\omega_n} \to 0$ stiff dominated.



$\frac{\omega}{\omega_n} \to \infty$ mass dominated.

Case ii: Resonance $\omega/\omega_n = 1$



$\frac{\omega}{\omega_n} \to 0$: force varies slowly $\Rightarrow \phi = 0 \rightarrow u$ in phase with $p(t)$.

$\frac{\omega}{\omega_n} \sim 0$: resonance $\rightarrow \phi = \pi/2 \ \forall \zeta \Rightarrow u_o$ happens when $p(t) = 0$.

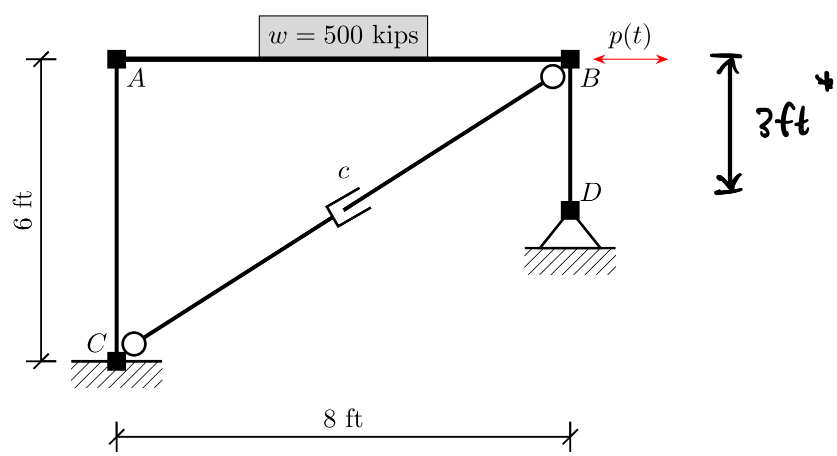
$\frac{\omega}{\omega_n} \to \infty$: force is rapidly varying $\Rightarrow \phi = \pi \rightarrow u$ out of phase from $p(t)$.

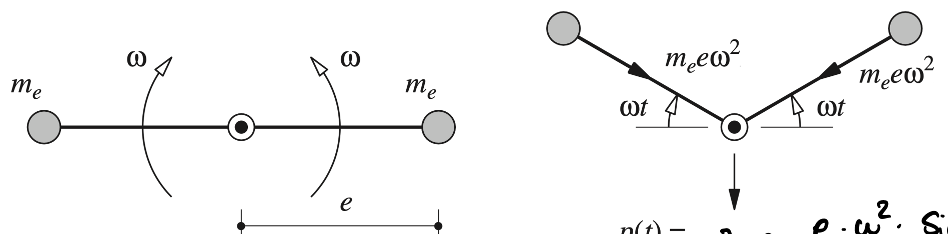
Vibration Generator

A one story reinforced concrete building has a roof weighing 500 kips, supported by two columns with $I = 448\ \text{in}^4$ and $E = 29,000$ ksi (W16 × 36). The roof can be considered infinitely rigid ($EI = \infty$).

The building is excited by a vibration generator with two weights, each 50 lb, rotating about a vertical axis at an eccentricity of 12 in. When the vibration generator runs at the natural frequency of the building, the amplitude of roof acceleration at steady-state is measured to be 0.02g.

Determine the damping ratio of the structure (ζ).





$p(t) = 2 m_e \cdot e \cdot \omega^2 \cdot \sin(\omega t)$.

Plan: want ξ

- @ resonance → $\dfrac{u_0}{(u_{st})_0} = R_d = \dfrac{1}{2\xi}$.

- know : $\ddot{u}_0 = 0.02g$.

- And $u(t) = (u_{st})_0 \cdot R_d \cdot \sin(\omega t - \phi)$

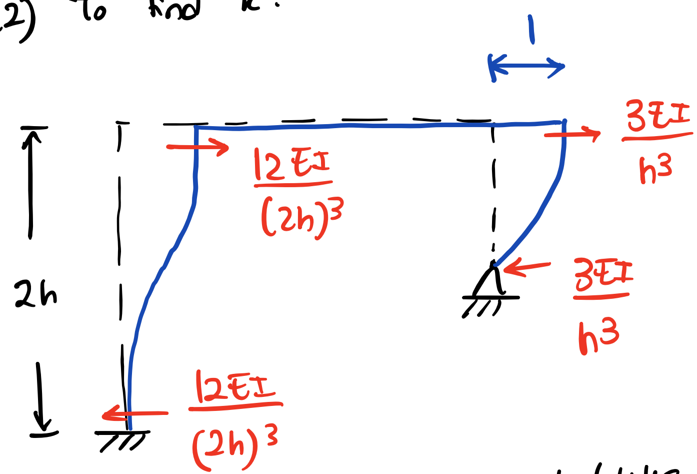
$\Rightarrow \ddot{u}(t) = -(u_{st})_0 \cdot R_d \cdot \omega^2 \cdot \sin(\omega t - \phi)$

solve for ξ.

$\Rightarrow \ddot{u}_0 = \dfrac{p_0}{k} \cdot R_d \cdot \omega^2 = \dfrac{p_0}{k} \cdot \dfrac{1}{2\xi} \cdot \omega^2 = 0.02g$

(1) $P_0 = 2 m_e \cdot e \cdot \omega^2$ $\quad \omega?$ $\quad \omega = \omega_n = \sqrt{\frac{k}{m}}$ $\rightarrow$ need k and m.

(2) To find k:



$\Rightarrow \quad k = \frac{12EI}{8h^3} + \frac{3EI}{h^3} = \frac{9EI}{2h^3}.$

$$k = \frac{9EI}{2h^3} = \frac{9\,(29{,}000\text{ ksi})\,(448\text{ in}^4)}{2\,(3 \cdot 12\text{ in})^3} = 1{,}253\ (\text{kips/in})$$

$w = 500 + 0.1 = 500.1\ (\text{kips}) \Rightarrow m = \frac{500.1\ (\text{kips})}{g}$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{1{,}253\ (\text{kips/in})}{500.1\ (\text{kips})} \cdot 386\ (\text{in/s}^2)} = 31.1\ (\text{rad/sec}).$$

$f_n = \frac{\omega_n}{2\pi} = 4.95\ (\text{Hz})$

$T_n = \frac{1}{f_n} = 0.2\ (\text{sec}).$

now: $\frac{P_0}{k} \cdot w^2 \cdot R_d = 0.02g \qquad P_0 = 2 m_e \cdot e \cdot w^2$

$k = m \cdot \omega_n^2$

@ resonance.

$$\Rightarrow \quad \frac{2 m_e \cdot e \cdot w^2}{m \cdot \cancel{\omega_n^2}} \cdot \cancel{w^2} \cdot \frac{1}{2\zeta} = 0.02g$$

$\Rightarrow$ solve for $\zeta = 0.15 \rightarrow \boxed{\zeta = 15\%}$

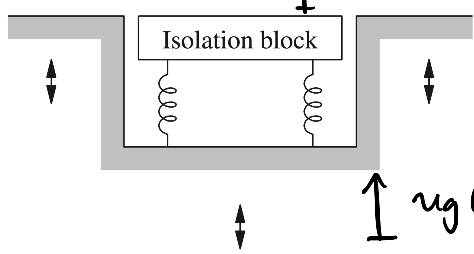
Transmissibility

A vibration isolation block is to be installed in a laboratory so that the vibration from adjacent factory operations will not disturb certain experiments. If the isolation block weighs 2000 lb and the surrounding floor and foundation vibrate at 1500 cycles per minute, determine the stiffness of the isolation system such that the absolute motion of the isolation block is limited to 10% of the floor vibration; neglect damping

$u^t(t) = u(t) + u_g(t) \rightarrow u_o^t$

want:

$$\frac{u_o^t}{u_{go}} \le 10\%$$



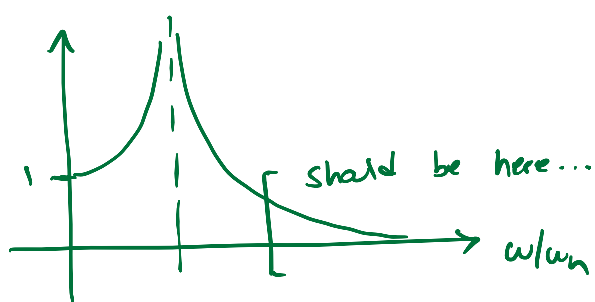
$u_g(t) = u_{go} \cdot \sin(\omega t)$ (floor vibration).

Hint: classic transmissibility problem. (Chopra, section 3.6).

$$TR = \frac{\sqrt{1 + (2\zeta\omega/\omega_n)^2}}{\sqrt{(1-(\omega/\omega_n)^2)^2 + (2\zeta\frac{\omega}{\omega_n})^2}} = \frac{\ddot{u}_o^t}{\ddot{u}_{go}} = \boxed{\frac{u_o^t}{u_{go}}} = \frac{(f_T)_o}{p_o}$$

for undamped case:

$$TR = \left| \frac{1}{1-\left(\frac{\omega}{\omega_n}\right)^2} \right| \le 0.1$$



$r > 1$

$\downarrow$

$1 - r^2 < 0$

Define $r = \omega/\omega_n$

$$\Rightarrow \left| \frac{1}{1-r^2} \right| \le 0.1 \Rightarrow -0.1 \le \frac{1}{1-r^2} \le 0.1$$

(a) $\frac{1}{1-r^2} \le \frac{1}{10} \Rightarrow 10 \ge 1 - r^2 \Rightarrow 9 \ge -r^2 \Rightarrow r^2 \ge -9 \ldots$

(b) $-\frac{1}{10} \leq \frac{1}{1-r^2} \rightarrow -(1-r^2) \geq 10 \rightarrow r^2 - 1 \geq 10$

$\Rightarrow \boxed{r^2 \geq 11} \rightarrow$ good.

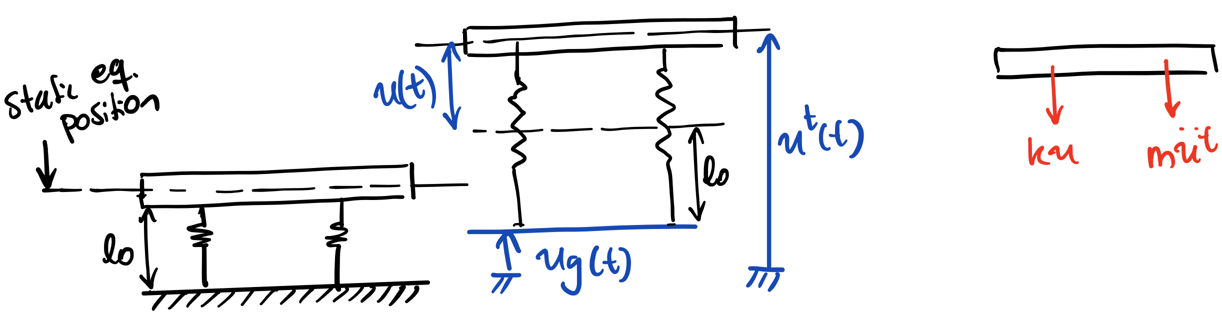
$$\left(\frac{\omega}{\omega_n}\right)^2 \geq 11 \rightarrow \omega^2 \cdot \frac{m}{k} \geq 11 \rightarrow k \leq \frac{\omega^2 . m}{11}$$

plugging in the values:

$$k \leq \left[(2\pi \text{ rad/cycle})(1500 \text{ cycle/min}) \cdot (1/60 \text{ min/sec})\right]^2 \cdot \frac{2{,}000 \text{ lb}}{386 \text{ (in/s}^2)} \Big/ 11$$

$$\Rightarrow k \leq 11.61 \text{ (kip / in)}.$$

For those who don't believe in the formula ;



@ $t=0$

$$u^t(t) = u(t) + u_g(t) + l_0 \Rightarrow u(t) = u^t(t) - u_g(t) - l_0$$

$$\Rightarrow \ddot{u}(t) = \ddot{u}^t(t) - \ddot{u}_g(t)$$

EOM: $m\ddot{u}^t + ku = 0$

$$\Rightarrow m(\ddot{u} + \ddot{u}_g) + ku = 0 \Rightarrow \boxed{m\ddot{u} + ku = -m\ddot{u}_g}$$

$p(t) = -m \cdot \ddot{u}_g$

the solution is (steady state) for $\ddot{u}_g = \ddot{u}_{go} \cdot \sin(\omega t)$

$p_o = m \cdot \ddot{u}_{go}$

$$\rightarrow \quad u(t) = \frac{p_o}{k} \cdot R_d \cdot \sin(\omega t - \phi)$$

$$\ddot{u}(t) = -\underbrace{\frac{p_o}{k} \cdot R_d \cdot \omega^2}_{} \sin(\omega t - \phi)$$

$$\ddot{u}_o = \frac{p_o \cdot R_d \cdot \omega^2}{k} = \frac{m \cdot \ddot{u}_{go} \cdot R_d \cdot \omega^2}{k} = R_d \left(\frac{\omega}{\omega_n}\right)^2 \ddot{u}_{go}$$

so $$\ddot{u}^t(t) = -R_d \left(\frac{\omega}{\omega_n}\right)^2 \ddot{u}_{go} \cdot \sin(\omega t - \phi) + \ddot{u}_{go} \cdot \sin(\omega t).$$

$$= -\frac{1}{1-(\omega/\omega_n)^2} \left(\frac{\omega}{\omega_n}\right)^2 \cdot \ddot{u}_{go} \cdot \sin(\omega t) + \ddot{u}_{go} \sin(\omega t)$$

$$= \left(1 - \frac{(\omega/\omega_n)^2}{(\omega/\omega_n)^2 - 1}\right) \ddot{u}_{go} \sin(\omega t)$$

$$= \frac{\cancel{(\omega/\omega_n)^2} - 1 - \cancel{(\omega/\omega_n)^2}}{(\omega/\omega_n)^2 - 1} \cdot \ddot{u}_{go} \sin(\omega t)$$

$$\ddot{u}^t(t) = \frac{1}{1-(\omega/\omega_n)^2} \cdot \ddot{u}_{go} \cdot \sin(\omega t).$$

and: $$\left|\frac{\ddot{u}^t_o}{\ddot{u}_{go}}\right| = \left|\frac{1}{1-(\omega/\omega_n)^2}\right| = TR.$$