Basic Definitions and Equations

$$\omega_n = \sqrt{\frac{k}{m}} \qquad T_n = \frac{2\pi}{\omega_n} \qquad f_n = \frac{1}{T_n} \qquad (u_{st})_o = \frac{p_o}{k}$$

Damping

$$\zeta = \frac{c}{2m\omega_n} = \frac{c}{c_{cr}}$$

$\zeta > 1$ overdamped
$\zeta = 1$ critically damped
$\zeta < 1$ underdamped

$$\omega_D = \omega_n\sqrt{1-\zeta^2}$$

Free Vibration $\quad m\ddot{u} + c\dot{u} + ku = 0$

For $\zeta = 0$:

$$u(t) = u(0)\cos\omega_n t + \frac{\dot{u}(0)}{\omega_n}\sin\omega_n t$$

For $0 < \zeta < 1$:

$$u(t) = e^{-\zeta\omega_n t}\left(u(0)\cos\omega_D t + \frac{\dot{u}(0) + \zeta\omega_n u(0)}{\omega_D}\sin\omega_D t\right)$$

Decay of motion

$$\frac{u_i}{u_{i+j}} = \exp\left(\frac{2\pi j\zeta}{\sqrt{1-\zeta^2}}\right) \qquad \zeta = \frac{1}{2\pi j}\ln\frac{u_i}{u_{i+j}}$$

Harmonic Excitation $\quad m\ddot{u} + c\dot{u} + ku = p_o\sin\omega t$

Steady State Response

$$u(t) = u_o\sin(\omega t - \phi)$$

$$R_d = \frac{u_o}{(u_{st})_o} = \frac{1}{\sqrt{[1-(\omega/\omega_n)^2]^2 + [2\zeta(\omega/\omega_n)]^2}}$$

$$\varphi = \tan^{-1}\frac{2\zeta(\omega/\omega_n)}{1-(\omega/\omega_n)^2}$$

Resonance at $\omega_n\sqrt{1-2\zeta^2}$ with $R_d = \dfrac{1}{2\zeta\sqrt{1-\zeta^2}}$

Half-Power Bandwidth

$$\frac{\omega_b - \omega_a}{\omega_n} = 2\zeta$$

Vibration Generator

$$p(t) = (m_e e\omega^2)\sin\omega t$$

Transmissibility: $\quad TR = (f_T)_o/p_o = \ddot{u}_o^t/\ddot{u}_{go} = u_o^t/u_{go}$

$$TR = R_d\sqrt{1 + [2\zeta(\omega/\omega_n)]^2}$$

Equivalent Viscous Damping

$$\zeta_{\text{eq}} = \frac{1}{4\pi}\frac{E_D}{E_{so}}$$

Arbitrary Excitation $\quad m\ddot{u} + c\dot{u} + ku = p(t)$

Response to unit impulse: $p(t) = \delta(t-\tau)$

$$h(t-\tau) \equiv u(t) = \frac{1}{m\omega_n}\sin(\omega_n(t-\tau)) \qquad (\zeta = 0)$$

Duhamel's integral

$$u(t) = \int_o^t p(\tau)h(t-\tau)d\tau$$

Response to step force ($\zeta = 0$)

$$u(t) = (u_{st})_o(1-\cos\omega_n t)$$

Response to ramp force ($\zeta = 0$)

$$p(t) = p_o\frac{t}{t_r} \qquad u(t) = (u_{st})_o\left(\frac{t}{t_r} - \frac{\sin\omega_n t}{\omega_n t_r}\right)$$

Response to rectangular pulse ($\zeta = 0$)

$$R_d = \begin{cases} 2\sin\pi t_d/Tn & t_d < T_n/2 \\ 2 & t_d \geq T_n/2 \end{cases}$$

Short pulse

$$I = \int p(t)dt \qquad u(t) = I\left(\frac{1}{m\omega_n}\sin\omega_n t\right)$$

Earthquake Response $\quad p(t) = p_{\text{eff}}(t) = -m\ddot{u}_g(t)$

$$u_o \equiv u_o(T_n, \zeta) \equiv \max(u(t))$$

$$D \equiv u_o \qquad V = \omega_n D \qquad A = \omega_n^2 D$$

$$\frac{A}{\omega_n} = V = \omega_n D \qquad f_{so} = kD = mA$$

For One Story Structure

$$V_{bo} = f_{so} = \frac{A}{g}w \qquad M_{bo} = hV_{bo}$$

EQ Response of Inelastic Systems

Equation of Motion

$$m\ddot{u} + c\dot{u} + f_s(u,\dot{u}) = p_{\text{eff}}(t) = -m\ddot{u}_g(t)$$

Normalized Yield Strength

$$\bar{f}_y = \frac{f_y}{f_o} = \frac{u_y}{u_o}$$

f_y and u_y are yield strength and yield displacement.
f_o and u_o are peak force and deformation in corresponding linear system.

Yield Strength Reduction Factor

$$R_y = \frac{f_o}{f_y} = \frac{u_o}{u_y} \qquad f_o = ku_o$$

f_o is min strength required for structure to remain elastic.

Ductility Factor

$$\mu = \frac{u_m}{u_y}$$ (u_m is peak deformation of elastoplastic system)

Response Spectrum for Inelastic Systems

$$D_y = u_y \qquad V_y = \omega_n D_y \qquad A_y = \omega_n^2 D_y$$

$$f_y = \frac{A_y}{g}w \qquad u_m = \mu\left(\frac{T_n}{2\pi}\right)^2 A_y$$

Elastic pseudo-acceleration design spectrum for ground motions with $\ddot{u}_{go} = 1$ g, $\dot{u}_{go} = 48$ in/s, and $u_{go} = 36$ in; $\zeta = 5\%$