LECTURE 8 - NUMERICAL METHODS (PART 1)

CE 225

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NUMERICAL METHODS

INTRODUCTION

* For "real" forcing (e.g. Earthquakes) $\longrightarrow$ analytical method is difficult (convolution ok)
* For "real" structures $\longrightarrow$ inelastic behavior (i.e. nonlinear, damage $\longrightarrow$ need new methods

Define: $\Delta t_i =$ time interval

Assume: linear damping $\longrightarrow EOM = m\ddot{u} + c\dot{u} + f_s(\dot{u}, u) = p(t)$

@ time $= t_i \longrightarrow$ assume we know the current 'state': $u_i,\ \dot{u}_i,\ \ddot{u}_i$ which satisfy $m\ddot{u}_i + c\dot{u}_i + (f_s)_i = p_i$

$\longrightarrow$ <u>GOAL</u>:

NUMERICAL METHODS

METHODS AND REQUIREMENTS

Groups of Methods:

1. "Simple" interpolation of force
2. Finite difference method
3. Assumed variation in acceleration

Requirements:

1. Convergence: As the time step decreases, the solution approaches the exact solution.
2. Numerical stability: Numerical errors don't grow unbounded.
3. Accuracy: Provide "suitable" results (close enough to exact solution for given application).

METHOD 1: SIMPLE INTERPOLATION OF FORCE

DERIVATION

Assume $p(t)$ is linear over time interval $t_i < t < t_{i+1}$

Now, define $\tau = 0$ at time t_i and solve for the force: $p(\tau) = p_i + \dfrac{\Delta p_i}{\Delta t_i}\tau$

METHOD 1: SIMPLE INTERPOLATION OF FORCE

DERIVATION

Solution:

$$u(\tau) = u_c(\tau) \longrightarrow \text{Complementary solution to IC's: } u(0) = u_i \;;\; \dot{u}(0) = \dot{u}_i$$
$$+ p_i g(\tau) \longrightarrow \text{Step function response for IC's = 0}$$
$$+ u_{ramp}(\tau) \longrightarrow \text{Response to ramp load of magnitude } \frac{\Delta p_i}{\Delta t_i} \text{ for IC's = 0}$$

Plug in previously derived equations (undamped case):

$$u(\tau) = u_i \cos\omega_n\tau + \frac{\dot{u}_i}{\omega_n}\sin\omega_n\tau \longrightarrow \text{Response to IC's}$$
$$+ \frac{p_i}{k}\left(1 - \cos\omega_n\tau\right) \longrightarrow \text{Step response}$$
$$+ \frac{p_{i+1} - p_i}{k}\left(\frac{\tau}{\Delta t_i} - \frac{\sin\omega_n\tau}{\omega_n\Delta t_i}\right) \longrightarrow \text{Ramp Response}$$

<u>GOAL</u>: Find $u(\tau),\ \dot{u}(\tau)$ at time $\tau = \Delta t_i \longrightarrow$

METHOD 1: SIMPLE INTERPOLATION OF FORCE

DERIVATION

Solution for $\tau = \Delta t_i$:

$$u(\Delta t_i) = u_i \cos\omega_n \Delta t_i + \dot{u}_i \frac{\sin\omega_n \Delta t_i}{\omega_n} + p_i \frac{(1-\cos\omega_n\Delta t_i)}{k} + (p_{i+1}-p_i)\frac{1}{k}\left(\frac{\Delta t_i}{\Delta t_i} - \frac{\sin\omega_n\Delta t_i}{\omega_n \Delta t_i}\right)$$

$$\boxed{u(\Delta t_i) = u_{i+1} = Au_i + B\dot{u}_i + Cp_i + Dp_{i+1}}$$

where:

$$A = \cos\omega_n\Delta t_i \; ; \quad B = \frac{\sin\omega_n\Delta t_i}{\omega_n} \; ; \quad C = \frac{1}{k}\left(-\cos\omega_n\Delta t_i + \frac{\sin\omega_n\Delta t_i}{\omega_n\Delta t_i}\right) \; ; \quad D = \frac{1}{k}\left(1 - \frac{\sin\omega_n\Delta t_i}{\omega_n\Delta t_i}\right)$$

What about $\dot{u}_{i+1}$?

1. Take the derivative of $u(\tau)$ (see previous slide)
2. Plug in $\tau = \Delta t_i$
3. Rearrange to: $\boxed{\dot{u}(\Delta t_i) = \dot{u}_{i+1} = A'u_i + B'\dot{u}_i + C'p_i + D'p_{i+1}}$ and find new set of coefficients A', B', C', D'

METHOD 1: SIMPLE INTERPOLATION OF FORCE

DAMPED CASE - SOLUTION

$$u_{i+1} = Au_i + B\dot{u}_i + Cp_i + Dp_{i+1}$$

$$\dot{u}_{i+1} = A'u_i + B'\dot{u}_i + C'p_i + D'p_{i+1}$$

$$A = e^{-\zeta\omega_n \Delta t}\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_D \Delta t + \cos\omega_D \Delta t\right)$$

$$B = e^{-\zeta\omega_n \Delta t}\left(\frac{1}{\omega_D}\sin\omega_D \Delta t\right)$$

$$C = \frac{1}{k}\left\{\frac{2\zeta}{\omega_n \Delta t} + e^{-\zeta\omega_n \Delta t}\left[\left(\frac{1-2\zeta^2}{\omega_D \Delta t} - \frac{\zeta}{\sqrt{1-\zeta^2}}\right)\sin\omega_D \Delta t - \left(1 + \frac{2\zeta}{\omega_n \Delta t}\right)\cos\omega_D \Delta t\right]\right\}$$

$$D = \frac{1}{k}\left[1 - \frac{2\zeta}{\omega_n \Delta t} + e^{-\zeta\omega_n \Delta t}\left(\frac{2\zeta^2-1}{\omega_D \Delta t}\sin\omega_D \Delta t + \frac{2\zeta}{\omega_n \Delta t}\cos\omega_D \Delta t\right)\right]$$

$$A' = -e^{-\zeta\omega_n \Delta t}\left(\frac{\omega_n}{\sqrt{1-\zeta^2}}\sin\omega_D \Delta t\right)$$

$$B' = e^{-\zeta\omega_n \Delta t}\left(\cos\omega_D \Delta t - \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_D \Delta t\right)$$

$$C' = \frac{1}{k}\left\{-\frac{1}{\Delta t} + e^{-\zeta\omega_n \Delta t}\left[\left(\frac{\omega_n}{\sqrt{1-\zeta^2}} + \frac{\zeta}{\Delta t\sqrt{1-\zeta^2}}\right)\sin\omega_D \Delta t + \frac{1}{\Delta t}\cos\omega_D \Delta t\right]\right\}$$

$$D' = \frac{1}{k\,\Delta t}\left[1 - e^{-\zeta\omega_n \Delta t}\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_D \Delta t + \cos\omega_D \Delta t\right)\right]$$

Notes:

1. Works well for linear systems
2. Not great for inelastic systems, particularly MDOF

METHOD 1: SIMPLE INTERPOLATION OF FORCE

REVIEW OF PROCEDURE

METHOD 1: SIMPLE INTERPOLATION OF FORCE

EXAMPLE

Given:

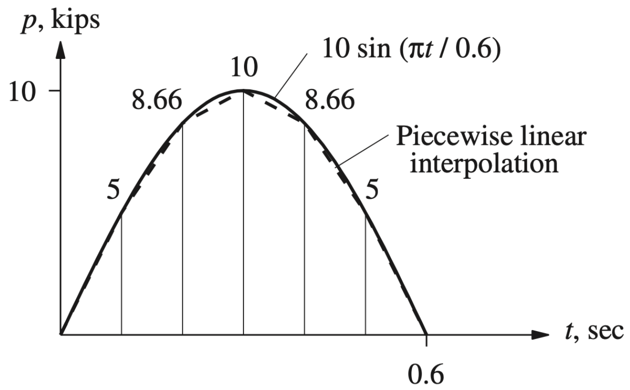
$m = 0.2533$ kip-sec^2/in

$k = 10$ kip/in

$T_n = 1$ sec

$\zeta = 0.05$

$p(t) = 10 \sin\left(\frac{2\pi}{1.2}t\right)$ for $t \in [0, 0.6]$ sec



Solution:

$$
\begin{aligned}
u_{i+1} &= Au_i + B\dot{u}_i + Cp_i + Dp_{i+1} \\
\dot{u}_{i+1} &= A'u_i + B'\dot{u}_i + C'p_i + D'p_{i+1}
\end{aligned}
$$

Where:

$$
\begin{array}{llll}
A = 0.8129 & B = 0.09067 & C = 0.01236 & D = 0.006352 \\
A' = -3.5795 & B' = 0.7559 & C' = 0.1709 & D' = 0.1871
\end{array}
$$

METHOD 1: SIMPLE INTERPOLATION OF FORCE

EXAMPLE (CONTINUED)

$$u_{i+1} = Au_i + B\dot{u}_i + Cp_i + Dp_{i+1}$$
$$\dot{u}_{i+1} = A'u_i + B'\dot{u}_i + C'p_i + D'p_{i+1}$$

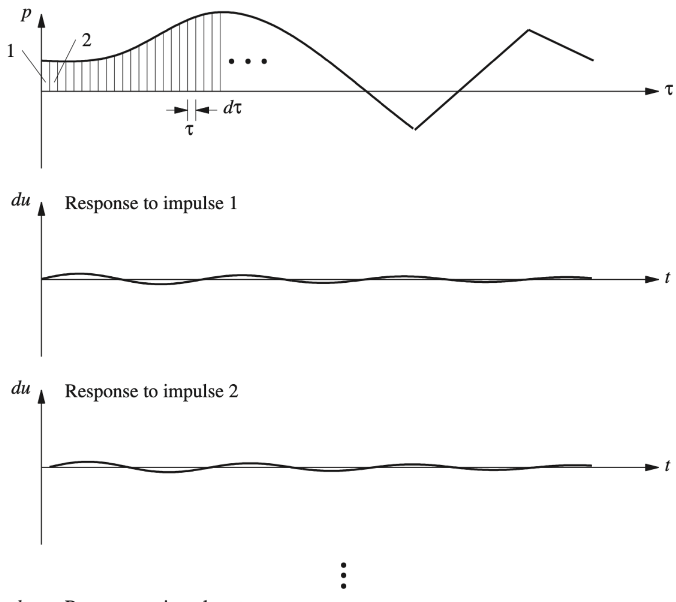
TABLE E5.1a NUMERICAL SOLUTION USING LINEAR INTERPOLATION OF EXCITATION

t_i	p_i	Cp_i	Dp_{i+1}	$B\dot{u}_i$	$\dot{u}_i$	Au_i	u_i	Theoretical u_i
0.0	0.0000	0.0000	0.0318	0.0000	0.0000	0.0000	0.0000	0.0000
0.1	5.0000	0.0618	0.0550	0.0848	0.9354	0.0258	0.0318	0.0328
0.2	8.6602	0.1070	0.0635	0.2782	3.0679	0.1849	0.2274	0.2332
0.3	10.0000	0.1236	0.0550	0.4403	4.8558	0.5150	0.6336	0.6487
0.4	8.6603	0.1070	0.0318	0.4290	4.7318	0.9218	1.1339	1.1605
0.5	5.0000	0.0618	0.0000	0.1753	1.9336	1.2109	1.4896	1.5241
0.6	0.0000	0.0000	0.0000	−0.2735	−3.0159	1.1771	1.4480	1.4814
0.7	0.0000	0.0000	0.0000	−0.6767	−7.4631	0.7346	0.9037	0.9245
0.8	0.0000	0.0000	0.0000	−0.8048	−8.8765	0.0471	0.0579	0.0593
0.9	0.0000	0.0000	0.0000	−0.6272	−6.9177	−0.6160	−0.7577	−0.7751
1.0	0.0000				−2.5171		−1.2432	−1.2718

Method 1: Simple Interpolation of Force

Side Note on Convolution

- Also a method that breaks the force into small 'increments'
- However, does not only solve over Δt_i. Solve over all time and then superpose results.
- Also provides solution for $\Delta t_i \longrightarrow 0$, so eliminates 'approximation'.

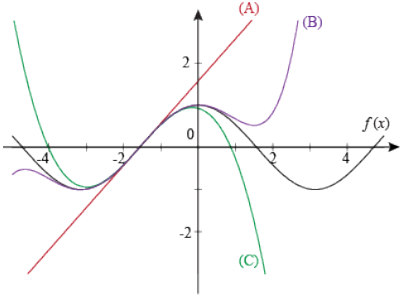


METHOD 2: FINITE DIFFERENCE METHODS

CENTRAL DIFFERENCE METHOD

Uses current derivatives (e.g. velocity and acceleration) to estimate the dynamic response.

Central Difference Method: Use Taylor Series to represent a function as an ∞ sum of derivatives at a single point.



$$f(x+h) = f(x) + f'(x)h + f''(x)\frac{h^2}{2} + f'''(x)\frac{h^3}{6} + ... \quad (1)$$

$$f(x-h) = f(x) - f'(x)h + f''(x)\frac{h^2}{2} - f'''(x)\frac{h^3}{6} ... \quad (2)$$

$$(1) - (2) \rightarrow f(x+h) - f(x-h) = 2f'(x)h + 0 + 2f'''(x)\frac{h^3}{6} ... \rightarrow \boxed{f'(x) = \frac{f(x+h) - f(x-h)}{2h}}$$

$$(1) + (2) \rightarrow f(x+h) + f(x-h) = 2f(x) + 0 + 2f''(x)\frac{h^2}{2} ... \rightarrow \boxed{f''(x) = \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}}$$

Method 2: Finite Difference Methods

Central Difference Method

$$\dot{u}_i = \frac{u_{i+1} - u_{i-1}}{2\Delta t}$$

$$\ddot{u}_i = \frac{u_{i+1} - 2u_i + u_{i-1}}{(\Delta t)^2}$$

**Basically, using local derivatives at t_i to predict at t_{i+1}

Enforce "equilibrium" (EOM) at time t_i :

**Note: We are solving equilibrium at t_i to find determine u_{i+1} @ time t_{i+1} ! This is an explicit method of numerical integration.

METHOD 2: FINITE DIFFERENCE METHODS

CENTRAL DIFFERENCE METHOD

Re-arrange: $\left[\frac{m}{(\Delta t)^2} + \frac{c}{2\Delta t}\right] u_{i+1} = p_i - \left[\frac{m}{(\Delta t)^2} - \frac{c}{2\Delta t}\right] u_{i-1} - \left[k - \frac{2m}{(\Delta t)^2}\right] u_i$

$$\longrightarrow \boxed{u_{i+1} = \frac{p_i - au_{i-1} - bu_i}{\hat{k}} = \frac{\hat{p}_i}{\hat{k}}}$$

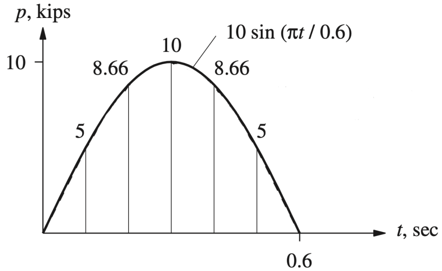
At $t = 0 \longrightarrow u_{-1}$ doesn't exist. Need to fix this:

Central Difference at $t = 0$: $\dot{u}_0 = \frac{u_1 - u_{-1}}{2\Delta t}$; $\ddot{u}_0 = \frac{u_1 - 2u_0 + u_{-1}}{(\Delta t)^2}$

$$\boxed{u_{-1} = u_0 - \Delta t(\dot{u}_0) + \frac{(\Delta t)^2}{2}\ddot{u}_0}$$

METHOD 2: FINITE DIFFERENCE METHODS

CENTRAL DIFFERENCE METHOD - SOLUTION PROCEDURE (EXAMPLE)



Solution

1.0 *Initial calculations*

$$m = 0.2533 \qquad k = 10 \qquad c = 0.1592$$

$$u_0 = 0 \qquad \dot{u}_0 = 0$$

1.1 $\ddot{u}_0 = \dfrac{p_0 - c\dot{u}_0 - ku_0}{m} = 0.$

1.2 $u_{-1} = u_0 - (\Delta t)\dot{u}_0 + \dfrac{(\Delta t)^2}{2}\ddot{u}_0 = 0.$

1.3 $\hat{k} = \dfrac{m}{(\Delta t)^2} + \dfrac{c}{2\Delta t} = 26.13.$

1.4 $a = \dfrac{m}{(\Delta t)^2} - \dfrac{c}{2\Delta t} = 24.53.$

1.5 $b = k - \dfrac{2m}{(\Delta t)^2} = -40.66.$

2.0 *Calculations for each time step*

2.1 $\hat{p}_i = p_i - au_{i-1} - bu_i = p_i - 24.53u_{i-1} + 40.66u_i.$

2.2 $u_{i+1} = \dfrac{\hat{p}_i}{\hat{k}} = \dfrac{\hat{p}_i}{26.13}.$

3.0 Computational steps 2.1 and 2.2 are repeated for $i = 0, 1, 2, 3, \ldots$ leading to Table E5.2, wherein the theoretical result (from Table E5.1a) is also included.

Method 2: Finite Difference Methods

Central Difference Method - Solution Procedure (Example)

t_i	p_i	u_{i-1}	u_i	$\hat{p}_i$ [Eq. (2.1)]	u_{i+1} [Eq. (2.2)]	Theoretical u_{i+1}
0.0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0328
0.1	5.0000	0.0000	0.0000	5.0000	0.1914	0.2332
0.2	8.6602	0.0000	0.1914	16.4419	0.6293	0.6487
0.3	10.0000	0.1914	0.6293	30.8934	1.1825	1.1605
0.4	8.6603	0.6293	1.1825	41.3001	1.5808	1.5241
0.5	5.0000	1.1825	1.5808	40.2649	1.5412	1.4814
0.6	0.0000	1.5808	1.5412	23.8809	0.9141	0.9245
0.7	0.0000	1.5412	0.9141	−0.6456	−0.0247	0.0593
0.8	0.0000	0.9141	−0.0247	−23.4309	−0.8968	−0.7751
0.9	0.0000	−0.0247	−0.8968	−35.8598	−1.3726	−1.2718
1.0	0.0000	−0.8968	−1.3726	−33.8058	−1.2940	−1.2674

METHOD 2: FINITE DIFFERENCE METHODS

CENTRAL DIFFERENCE METHOD - SOLUTION PROCEDURE (EXAMPLE)

Stability: Conditionally stable $\longrightarrow \Delta t < \frac{T_n}{\pi} = \Delta t_{max}$

Essentially:

$\Delta t > \Delta t_{max}$ ... error grows unbounded

Typically use: $\boxed{\Delta t \approx \frac{1}{10} \Delta t_{max}}$

**MDOF systems: critical timestep must capture the shortest T_n in the model

3DEC Example: