CE 225: Dynamic of Structures **Fall 2024**

Discussion 2: Forced Vibration - Harmonic Excitation

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Announcements

- Homework #1.
- Solution for HW#1 is up in bCourses.

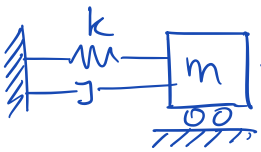
Objectives

By the end of this discussion we'll be able to:

1. Find the lateral stiffness of a multi-column one-story shear building and its basic dynamic properties.
2. Find the damping coefficient of a structure from a resonance test.
3. Apply the equation for the transmissibility.

Harmonic excitation

$$\boxed{m\ddot{u} + c\dot{u} + ku = p(t) \quad \text{with} \quad p(t) = p_0 \sin \omega t}$$



$\rightarrow p(t)$; $P(t) = p_0 \cdot \sin(\omega t)$

Cases

$\zeta = 0$ (undamped).

$m\ddot{u} + ku = p_0 \sin(\omega t)$.

homogeneous: $m\ddot{u} + ku = 0$

$\rightarrow \ddot{u} + \omega_n^2 u = 0$.

$\Rightarrow u_h(t) = A \cdot \cos(\omega_n t) + B \cdot \sin(\omega_n t)$

Particular solution: $\rightarrow$ depends $\rightarrow$ on what?

ω / ω_n

$\frac{\omega}{\omega_n} \neq 1$

$\frac{\omega}{\omega_n} = 1$ (resonance).

$u_p = C \cdot \sin(\omega t)$

$u_p = Ct \cdot \cos(\omega_n t)$.

$\zeta \neq 0$ (damped) $\underline{\underline{\zeta < 1}}$

$m\ddot{u} + c\dot{u} + ku = p_0 \cdot \sin(\omega t)$

homogeneous. $m\ddot{u} + c\dot{u} + ku = 0$

$\rightarrow \ddot{u} + (2\zeta\omega_n)\dot{u} + \omega_n^2 = 0$.

$u_h(t) = e^{-\zeta\omega_n t}\left(A \cdot \cos(\omega_D t) + B \cdot \sin(\omega_D t)\right)$

Particular soln, here:

$u_p(t) = C \cdot \sin(\omega t) + D \cdot \cos(\omega t)$.

plug into eq. of motion and solve for C and D.

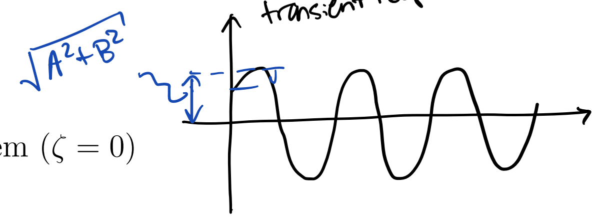
then, translate:

$C \cdot \sin(\omega t) + D \cdot \cos(\omega t)$

$= \underbrace{\sqrt{C^2 + D^2}}_{\text{amplitude of motion}} \cdot \sin(\omega t - \phi)$

phase shift

$\rightarrow$ Also can encounter "resonance", but same response equations are valid.



(a) Undamped system ($\zeta = 0$)

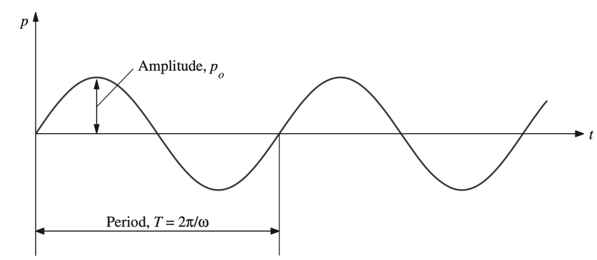
Case i: $\omega/\omega_n \neq 1$

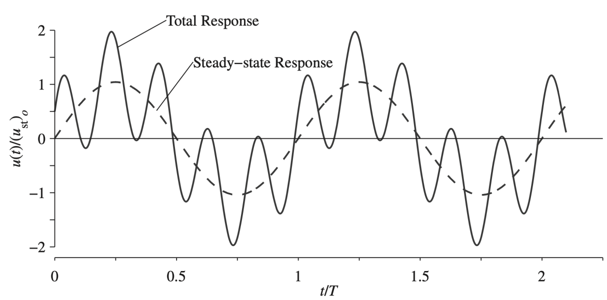
$$\boxed{u(t) = \underbrace{A\sin\omega_n t + B\cos\omega_n t}_{\text{transient}} + \underbrace{\frac{p_0}{k}R_d\sin(\omega t - \phi)}_{\text{steady-state}}} \quad \text{where} \quad \underbrace{R_d}_{\text{deformation.}} = \left|\frac{1}{1-\left(\dfrac{\omega}{\omega_n}\right)^2}\right| = \frac{u_0}{u_{st0}}$$

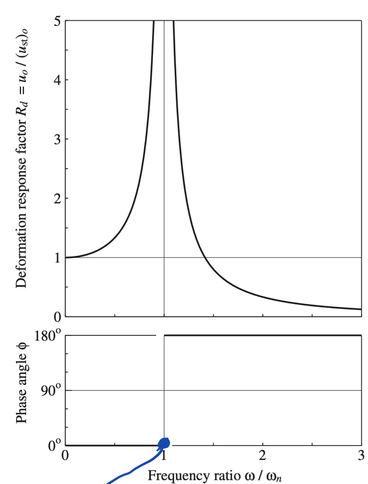
u_0: max steady-state response

u_{st0} → max. static response

$$u_{st0} = \frac{p_0}{k}$$





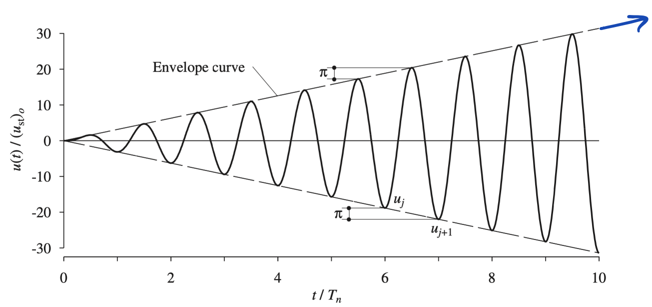


Case ii: Resonance $\omega/\omega_n = 1$

$$\boxed{u(t) = A\cos\omega_n t + B\sin\omega_n t - \frac{p_0}{2k}\omega_n t\cos\omega_n t}$$

at $t \to \infty$

response $\to \infty$



$$\frac{u(t)}{u_{st0}} = \frac{\frac{p_0}{2k}\cdot\omega_n\cdot t}{\frac{p_0}{k}} = \frac{\omega_n}{2}t$$

$\Rightarrow$ every $\Delta t = T_n = \dfrac{2\pi}{\omega_n}$ → change in $\dfrac{\omega_n}{2}\cdot\Delta t = \dfrac{\omega_n}{2}\cdot\dfrac{2\pi}{\omega_n} = \pi$.

(b) Damped system with $\zeta < 1$

Case i: $\omega/\omega_n \neq 1$

as $t \to \infty$ t.r. $\to 0$.

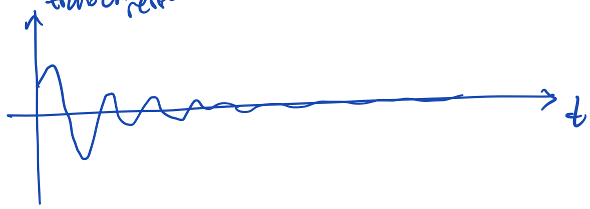
$$\boxed{u(t) = \underbrace{e^{-\zeta\omega_n t}\left(A\cos\omega_n t + \sin\omega_n t\right)}_{\text{transient response}} + \underbrace{\frac{p_0}{k} R_d \sin\left(\omega t - \phi\right)}_{\text{steady-state response}}}$$

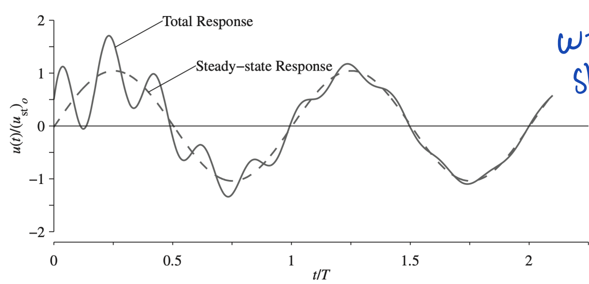
$$\text{where} \quad R_d = \frac{1}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)\right]^2 + \left[2\zeta\frac{\omega}{\omega_n}\right]^2}} \quad \text{and} \quad \phi = \tan^{-1}\left[\frac{2\zeta\frac{\omega}{\omega_n}}{1 - \left(\frac{\omega}{\omega_n}\right)^2}\right]$$

magnitude

phase angle (time lag).

transient response

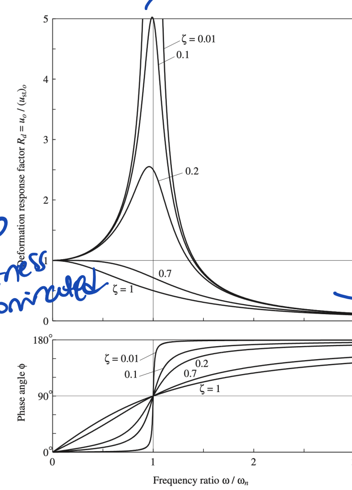




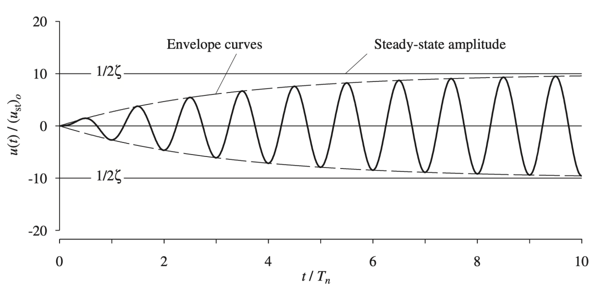
$\omega/\omega_n \sim 1.0 \to$ damping dominated

$\omega \to 0$ stiffness dominated

$\omega \to \infty$ mass dominated.



Case ii: Resonance $\omega/\omega_n = 1$



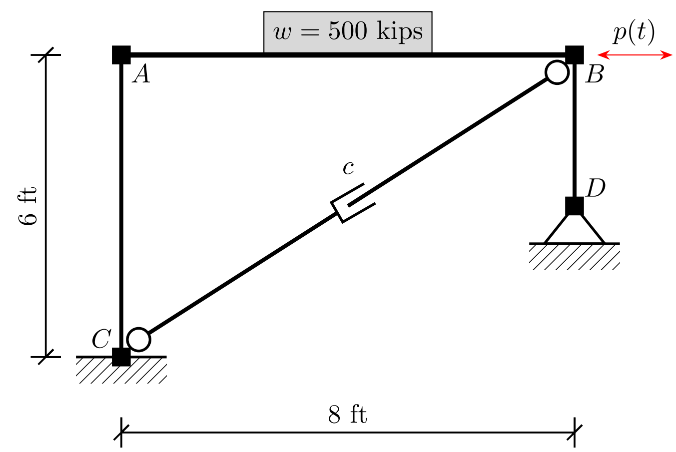
- $\omega/\omega_n \ll 1$: force varies slowly $\Rightarrow \phi = 0 \Rightarrow u$ in phase with $p(t)$. (same direction)
- $\omega/\omega_n \gg 1$: force is rapidly varying $\Rightarrow \phi = \pi \Rightarrow u$ in opposite phase from $p(t)$. (opposite direction).
- $\omega/\omega_n \sim 1.0$: same frequency $\Rightarrow \phi = \pi/2$, $\forall 3 \to u_0$ happens when $p(t)$ crosses zero. —

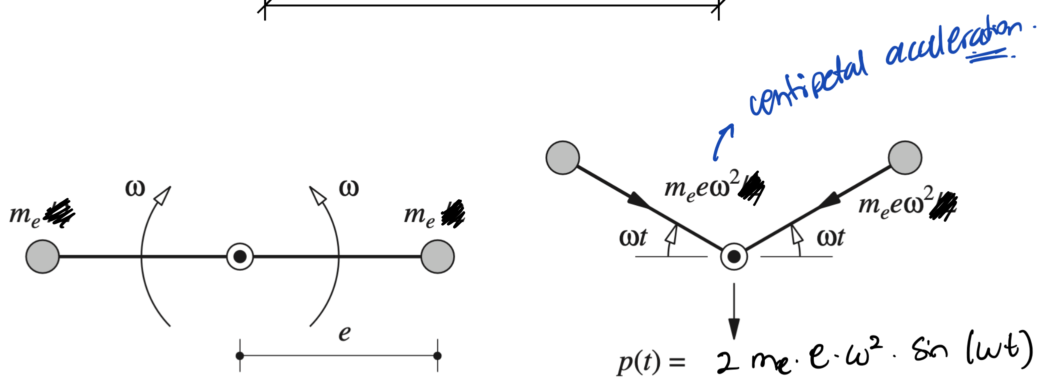
Vibration Generator

A one story reinforced concrete building has a roof weighing 500 kips, supported by two columns with $I = 448$ in^4 and $E = 29,000$ ksi (W16 × 36). The roof can be considered infinitely rigid ($EI = \infty$).

The building is excited by a vibration generator with two weights, each 50 lb, rotating about a vertical axis at an eccentricity of 12 in. When the vibration generator runs at the natural frequency of the building, the amplitude of roof acceleration at steady-state is measured to be 0.02g.

Determine the damping ratio of the structure (ζ).





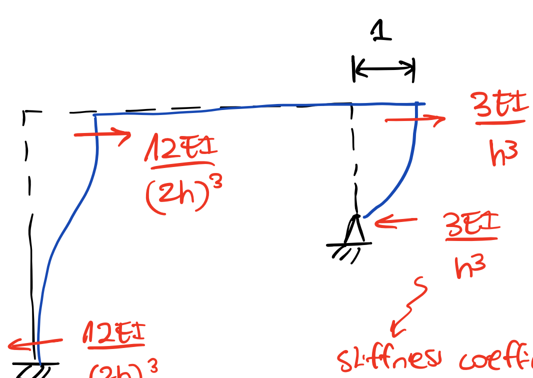
$p(t) = 2\, m_e \cdot e \cdot \omega^2 \cdot \sin(\omega t)$

$\Rightarrow \; P_0 = 2 m_e \cdot e \cdot \omega^2$

what is ω? $\rightarrow$ $\omega = \omega_n = \sqrt{k/m}$ ⇝ need k and m.

→ to find k

$h = 3$ft



$\Rightarrow \; k = \frac{12EI}{8h^3} + \frac{3EI}{h^3}$

$k = \frac{3EI}{2h^3} + \frac{3EI}{h^3} = \frac{9EI}{2h^3}.$

stiffness coefficients → arise as a result of the unit displacement.

$$k = \frac{9EI}{2h^3} = \frac{9\,(29{,}000\ ksi)\,(448\ in^4)}{2\,(3\cdot 12\ in)^3} = 1{,}253\ (kips/in).$$

$$mg = 500 + 0.1 = 500.1\ (kips).$$

$$\Rightarrow\ m = \frac{500.1\ (kips)}{g}$$

and
$$\omega_n = \sqrt{\frac{1{,}253\ (kips/in)\cdot 386\ (in/s^2)}{500.1\ (kips)}} = 31.1\ (rad/sec).$$

$$f_n = \frac{\omega_n}{2\pi} = 4.95\ (Hz)$$

$$\cdot\ T_n = \frac{1}{f_n} = 0.202\ (sec).$$

— · —

Steady-state response to harmonic forcing:

$$u(t) = (u_{st})_0 \cdot R_d \cdot \sin(\omega t - \phi)$$

$$\Rightarrow\ \dot{u}(t) = (u_{st})_0 \cdot \omega \cdot R_d \cdot \cos(\omega t - \phi)$$

$$\ddot{u}(t) = -\underbrace{(u_{st})_0 \cdot \omega^2 \cdot R_d}\cdot \sin(\omega t - \phi).$$

$$\Rightarrow\ \ddot{u}_0 = \frac{p_0}{k}\cdot \omega^2 \cdot R_d\ .$$

Also $\rightarrow$ $\omega = \omega_n \Rightarrow R_d = 1/2\zeta\ .$

$$\Rightarrow\ \ddot{u}_0 = \frac{p_0}{k}\cdot \omega^2 \cdot \frac{1}{2\zeta} = 0.02g$$

and $p_0 = 2\, m_e \cdot e \cdot \omega^2$

$p_0 = 2$

$$\Rightarrow\ \frac{2\, m_e \cdot e \cdot \omega^2}{m \cdot \cancel{\omega_n^2}} \cdot \cancel{\omega^2} \cdot \frac{1}{2\zeta} = 0.02g$$

$$\Rightarrow\ \frac{2\cdot 0.05\ (kips/g)\ (12\ in)\cdot (31.1\ 1/s)^2}{500.1\ (kips/g)} \cdot \frac{1}{2\zeta} = 0.02 \cdot 386\ (in/s^2)$$

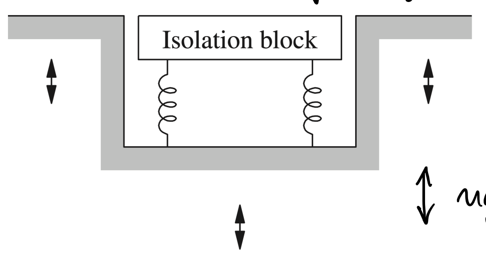
solve for $\underline{\zeta}$

$$\zeta = 0.15 \rightarrow \zeta = 15\%.$$

Transmissibility

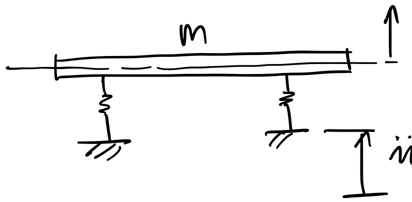
A vibration isolation block is to be installed in a laboratory so that the vibration from adjacent factory operations will not disturb certain experiments. If the isolation block weighs 2000 lb and the surrounding floor and foundation vibrate at 1500 cycles per minute, determine the stiffness of the isolation system such that the absolute motion of the isolation block is limited to 10% of the floor vibration; neglect damping

$u(t) = u_0 \cdot \sin(\omega t - \phi)$ in steady-state

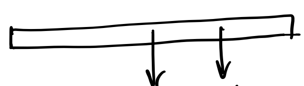


$u_g = u_{go} \cdot \sin(\omega t)$.

$u^t(t)$ (measured from the position of static equilibrium).



$u = u^t - u_g$



$\Rightarrow \ddot{u} = \ddot{u}^t - \ddot{u}_g \Rightarrow \ddot{u}^t = \ddot{u} + \ddot{u}_g$

$p_0 = -m\ddot{u}_{go}$

$m\ddot{u} + m\ddot{u}_g + ku = 0 \Rightarrow m\ddot{u} + ku = -m\ddot{u}_g$
$= -m \cdot \ddot{u}_{go} \cdot \sin(\omega t)$.

$\Rightarrow u(t) = -\frac{m \cdot \ddot{u}_{go}}{k} \cdot R_d \cdot \sin(\omega t - \phi)$ → need $\frac{\omega}{\omega_n} > \sqrt{2}$. → $\phi = \pi$ (for undamped)

$\ddot{u}(t) = R_d \frac{m}{k} \cdot \ddot{u}_{go}\, \omega^2 \cdot \sin(\omega t - \pi) = R_d \ddot{u}_{go} \cdot \sin(\omega t - \pi)$

$\ddot{u}(t) = R_d \left(\frac{\omega}{\omega_n}\right)^2 \ddot{u}_{go} \cdot \sin(\omega t - \pi)$

$\Rightarrow \ddot{u}^t(t) = R_d \left(\frac{\omega}{\omega_n}\right)^2 \ddot{u}_{go} \sin(\omega t - \pi) + \ddot{u}_{go} \cdot \sin(\omega t)$.

But $\sin(\omega t - \pi) = \sin(\omega t)\cdot\cos(-\pi) + \cos(\omega t)\cdot\cancel{\sin(-\pi)}^{0}$

$= \sin(\omega t)$

$$\Rightarrow \ddot{u}^{t}(t) = \left(R_d\cdot\left(\frac{\omega}{\omega_n}\right)^2 + 1\right)\ddot{u}_{g0}\cdot\sin(\omega t).$$

$$\Rightarrow \frac{\ddot{u}^{t}_{0}}{\ddot{u}_{g0}} = \left|\left(\frac{\omega}{\omega_n}\right)^2 R_d + 1\right| \qquad R_d = \left|\frac{1}{1-(\omega/\omega_n)^2}\right|$$

$$= \left|\frac{\omega^2/\omega_n^2 + 1 - \omega^2/\omega_n^2}{1-\omega^2/\omega_n^2}\right| = \left|\frac{1}{1-(\omega/\omega_n)^2}\right| = \underline{\underline{TR}} \text{ trans}\underline{\underline{\text{missi}}}\text{bility.}$$

in general.

$\Rightarrow$ Want:

$$\left|\frac{1}{1-r^2}\right| \leq 0.1 \Rightarrow \frac{1}{1-r^2} = \pm 0.1$$

$r^2 = -9$ ✗

$r^2 = 11$ ✓

$$\Rightarrow \left(\frac{\omega}{\omega_n}\right)^2 = 11 \Rightarrow \omega^2\cdot\frac{m}{k} = 11 \Rightarrow k = \frac{\omega^2\cdot m}{11}$$

$$k = \left[(2\pi \text{ rad/cycle})(1500 \text{ cycle/min})(1/60 \text{ min/sec})\right]^2\cdot\frac{2000 \text{ lb}}{386 \text{ in/s}^2} \Big/ 11$$

$k = 11.61$ kip/in.