General hazard structures

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Abstract

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1 The hazard structures

Consider the following excess hazard structures (see Chen and Wang (2000) and Chen et al. (2003) for extensive discussions on these structures). We express the different structures below with the instantaneous hazard $h()$ and the cumulative hazard $H()$, respectively, according to time t and a vector of covariables $\mathbf{x}_j$. The vector β denotes the regression parameters.

(i) Proportional hazards model (PH).

$$\begin{aligned} h^{\text{PH}}(t; \mathbf{x}_j) &= h_0(t) \exp\left(\mathbf{x}_j^T \beta\right) \\ H^{\text{PH}}(t; \mathbf{x}_j) &= H_0(t) \exp\left(\mathbf{x}_j^T \beta\right) \end{aligned} \tag{1}$$

(ii) Accelerated hazards model (AH).

$$\begin{aligned} h^{\text{AH}}(t; \mathbf{x}_j) &= h_0\left(t \exp(\mathbf{x}_j^T \beta)\right) \\ H^{\text{AH}}(t; \mathbf{x}_j) &= H_0\left(t \exp(\mathbf{x}_j^T \beta)\right) \exp(-\mathbf{x}_j^T \beta) \end{aligned}$$

(iii) Hybrid hazards model (HH).

$$\begin{aligned} h^{\text{H}}(t; \mathbf{x}_j) &= h_0\left(t \exp(\mathbf{x}_{1j}^T \beta_1)\right) \exp(\mathbf{x}_{2j}^T \beta_2) \\ H^{\text{H}}(t; \mathbf{x}_j) &= H_0\left(t \exp(\mathbf{x}_{1j}^T \beta_1)\right) \exp(-\mathbf{x}_{1j}^T \beta_1 + \mathbf{x}_{2j}^T \beta_2) \end{aligned} \tag{2}$$

where $\beta^T = (\beta_1^T, \beta_2^T)$ and $\mathbf{x}_j^T = (\mathbf{x}_{1j}^T, \mathbf{x}_{2j}^T)$

(iv) Accelerated failure time model (AFT).

$$\begin{aligned} h^{\text{AFT}}(t; \mathbf{x}_j) &= h_0\left(t \exp(\mathbf{x}_j^T \beta)\right) \exp(\mathbf{x}_j^T \beta), \\ H^{\text{AFT}}(t; \mathbf{x}_j) &= H_0\left(t \exp(\mathbf{x}_j^T \beta)\right). \end{aligned} \tag{3}$$

(v) General hazards model (GH).

$$\begin{aligned} h^{\text{G}}(t; \mathbf{x}_j) &= h_0\left(t \exp(\mathbf{x}_j^T \beta_1)\right) \exp(\mathbf{x}_j^T \beta_2), \\ H^{\text{G}}(t; \mathbf{x}_j) &= H_0\left(t \exp(\mathbf{x}_j^T \beta_1)\right) \exp(-\mathbf{x}_j^T \beta_1 + \mathbf{x}_j^T \beta_2). \end{aligned} \tag{4}$$

Notice that the AFT, PH, and AH models coincide for the case when the baseline hazard is Weibull (Chen and Jewell, 2001). The GH model (4) represents a general hazard structure that contains, as particular cases, the PH, AH, HH, and AFT models. More specifically, if $\beta_1 = 0$, then GH = PH; if $\beta_2 = 0$, then GH = AH; and if $\beta_1 = \beta_2$, then GH = PH. Moreover, in all of these hazard structures but the PH, the covariates are time dependent.

1.1 The Exponentiated Weibull distribution

The Exponentiated Weibull density and cumulative distribution functions with scale, shape,and power parameters (σ, κ, α) are given, respectively, by (Mudholkar and Srivastava, 1993):

$$\begin{aligned} f_{\text{EW}}(t;\sigma,\kappa,\alpha) &= \alpha\frac{\kappa}{\sigma}\left(\frac{t}{\sigma}\right)^{\kappa-1}\left[1-\exp\left\{-\left(\frac{t}{\sigma}\right)^{\kappa}\right\}\right]^{\alpha-1}\exp\left\{-\left(\frac{t}{\sigma}\right)^{\kappa}\right\}, \\ F_{\text{EW}}(t;\sigma,\kappa,\alpha) &= \left[1-\exp\left\{-\left(\frac{t}{\sigma}\right)^{\kappa}\right\}\right]^{\alpha}, \end{aligned} \tag{5}$$

where $t, \sigma, \kappa, \alpha > 0$. This distribution reduces to the Weibull distribution for $\alpha = 1$. The corresponding hazard function is obtained, by definition, as $h_{\text{EW}}(t;\sigma,\kappa,\alpha) = f_{\text{EW}}(t;\sigma,\kappa,\alpha)/\left[1 - F_{\text{EW}}(t;\sigma,\kappa,\alpha)\right]$. This hazard function can capture the basic shapes: flat, increasing, decreasing, unimodal and bathtub.

1.2 The likelihood

Let $(t_j, \mathbf{x}_j, \delta_j)$, $j = 1, \ldots, n$ be a sample of times to event from a population of cancer patients $t_j > 0$, with covariates $\mathbf{x}_j \in \mathbb{R}^p$, and vital status indicators δ_j (1-alive, 0-dead). The likelihood function of the full vector of parameters $\boldsymbol{\psi}$ is then given by

$$\begin{aligned} \mathscr{L}_0(\boldsymbol{\psi};\text{Data}) &= \prod_{j=1}^{n} h(t_j;\mathbf{x}_j)^{\delta_j} S(t_j;\mathbf{x}_j), \\ &\propto \prod_{j=1}^{n} h(t_j;\mathbf{x}_j)^{\delta_j} \exp\left\{-H(t_j;\mathbf{x}_j)\right\}. \end{aligned}$$

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