A clique graph based merging strategy for decomposable SDPs

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(virtual) Journal Club, Control Group
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Semidefinite programming

- Given matrices $C, A_1, \dots, A_m \in \mathbb{S}^n$ and $b \in \mathbb{R}^m$, find X:

$$\begin{array}{ll} \text{minimize} & \langle C, X \rangle \\ \text{subject to} & \langle A_i, X \rangle = b_i,\ i = 1, \dots, m \\ & X \in \mathbb{S}^n_+ \end{array}$$

$$\begin{array}{ll} \text{maximize} & b^\top \nu \\ \text{subject to} & A^\top \nu + Y = C \\ & Y \in \mathbb{S}^n_+. \end{array}$$

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$\mathcal{O}(n^3)$

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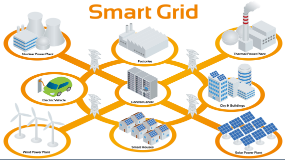
$n = 10^4 \rightarrow \mathcal{O}(10^{12})$ operations at each step

Semidefinite programming

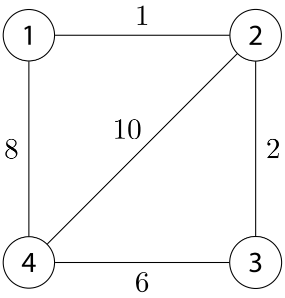
Where do we find positive semidefinite matrices?

- Lyapunov functions
- Linear Matrix Inequalities / S-procedure
  [Martin S Andersen et al. 2014]
- Kernel matrices [Lanckriet et al. 2004]
- Covariance matrices [Bertsimas and Nino-Mora 1999]
- Graph Laplacian
- Sum-of-Squares [Lasserre 2009]
- Semidefinite relaxation of
  - cardinality constraints (sparse PCA) [d'Aspremont et al. 2004]
  - QCQPs
  - mixed-integer constraints [Goemans and Williamson 1995]



The MAXCUT problem

- Weighted graph $G(V,E)$ with weights $w_{ij} \geq 0$, find $S \subset V$ such that the edge weights between S and $\bar{S} = V \setminus S$ are maximized



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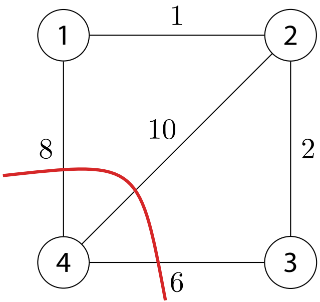

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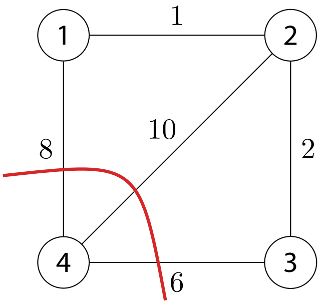

$$\begin{array}{ll} \text{maximize} & \frac{1}{4} \sum_{i=1}^{n} \sum_{j=1}^{n} w_{ij}(1 - y_i y_j) \\ \text{subject to} & y_i \in \{-1, 1\}, \; \forall i \in V \end{array}$$

- NP-hard, part of Karp's 21 NP-complete problems [Karp 1972]
- best approximation until 1995: $0.5p^*$

The MAXCUT problem: A semidefinite relaxation

- SDP relaxation that guarantees $0.87856\, p^*$ [Goemans and Williamson 1995]

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$\longrightarrow$ Lifting: $X = yy^\top$

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Primal SDP

$$\begin{array}{ll} \text{maximize} & \frac{1}{4}\langle L, X\rangle \\ \text{subject to} & X_{ii} = 1,\ i = 1, \dots, n \\ & X \succeq 0 \end{array}$$

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\text{minimize} & \sum_i \nu_i \\
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- SDP relaxation that guarantees $0.87856\, p^*$ [Goemans and Williamson 1995]

maximize $\frac{1}{4}\sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij}(1 - y_i y_j)$

subject to $y_i \in \{-1, 1\},\ \forall i \in V$

Lifting: $X = yy^\top$

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$X \succeq 0$

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$Y \succeq 0$

Overview

Matrix Sparsity and Graphs

Chordal decomposition

Clique merging

- Clique tree-based merging strategies
- Clique graph-based merging strategy

Benchmarks

Conclusion

Matrix sparsity and graphs

$$
\begin{array}{c|c}
 & \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{matrix} \\
\begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{matrix} &
\begin{bmatrix}
\bullet & & & & & & & & \\
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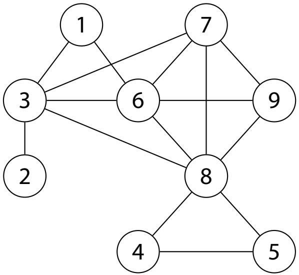
\end{bmatrix}
\end{array}
\xrightarrow{G(V,E)}
$$

vertex set V

edge set $E \subseteq V \times V$

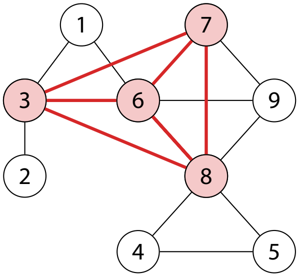


*Survey paper [L. Vandenberghe and M. S. Andersen 2015]

Matrix sparsity and graphs

$$
\begin{array}{c|ccccccccc|}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
\hline
1 & \bullet & & & & & & & & \\
2 & & \bullet & & & & & & & \\
3 & \bullet & \bullet & \bullet & & & & & & \\
4 & & & & \bullet & & & & & \\
5 & & & & \bullet & \bullet & & & & \\
6 & \bullet & & \bullet & & & \bullet & & & \\
7 & & & \bullet & & & \bullet & \bullet & & \\
8 & & & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \\
9 & & & & & & \bullet & \bullet & \bullet & \bullet
\end{array}
$$

Complete subgraph



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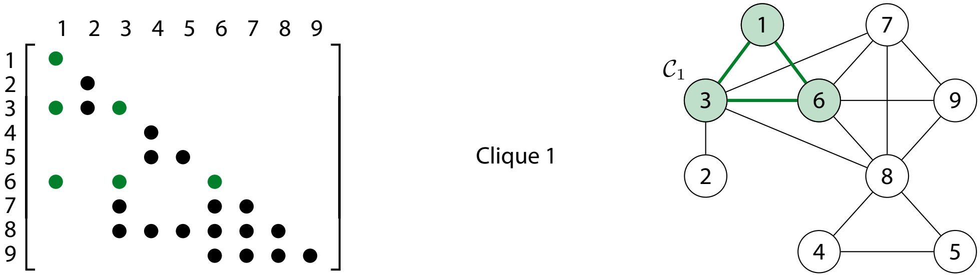

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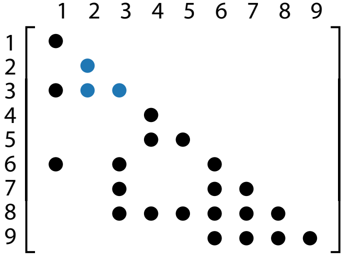
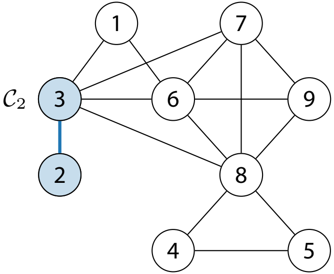


Clique 2



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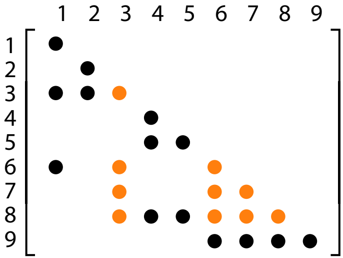
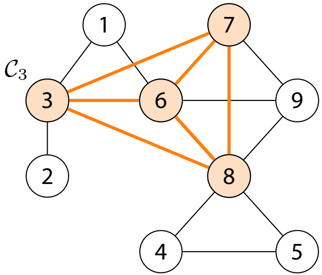


Clique 3



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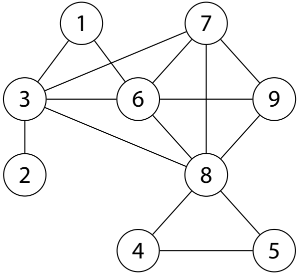
\end{array}\right]
\end{array}
$$

Chordal graph



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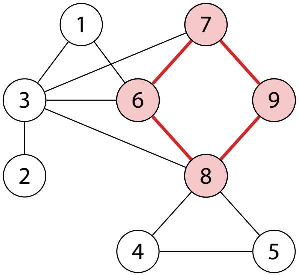
~~Chordal graph~~



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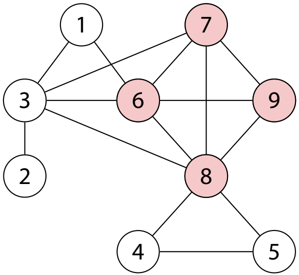

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Chordal decomposition

Dual SDP*

$$\begin{array}{ll} \text{maximize} & b^\top y \\ \text{subject to} & \sum_{i=1}^m A_i y_i + S = C \\ & \boxed{S \in \mathbb{S}_+^n} \end{array}$$

Sparse matrix cones

$$\mathbb{S}^n(E,0) := \{S \in \mathbb{S}^n \mid S_{ij} = S_{ji} = 0, \text{ if } i \neq j,\ (i,j) \notin E\}$$

$$\mathbb{S}_+^n(E,0) := \{S \in \mathbb{S}^n(E,0) \mid S \succeq 0\}$$

*equivalent decomposition result for primal form SDPs exists

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Agler's theorem

Agler's theorem*

Let $G(V,E)$ be a chordal graph with a set of maximal cliques $\{\mathcal{C}_1,\dots,\mathcal{C}_p\}$. Then $S \in \mathbb{S}^n_+(E,0)$ if and only if there exist matrices $S_\ell \in \mathbb{S}^{|\mathcal{C}_\ell|}_+$ for $\ell = 1,\dots,p$ such that

$$S = \sum_{\ell=1}^{p} T_\ell^\top S_\ell T_\ell.$$

*[Agler et al. 1988], Grone's theorem for primal form SDPs [Grone et al. 1984]

Example: Chordal Decomposition

$$\begin{array}{ll} \text{maximize} & b^\top y \\ \text{subject to} & \sum_{i=1}^m A_i y_i + S = C \\ & S \in \mathbb{S}_+^n(E, 0) \end{array}$$

$$\begin{bmatrix} S_{11} & S_{12} & 0 & 0 & 0 & S_{16} \\ S_{21} & S_{22} & S_{23} & S_{24} & 0 & S_{26} \\ 0 & S_{32} & S_{33} & S_{34} & 0 & 0 \\ 0 & S_{42} & S_{43} & S_{44} & S_{45} & S_{46} \\ 0 & 0 & 0 & S_{54} & S_{55} & S_{56} \\ S_{61} & S_{62} & 0 & S_{64} & S_{65} & S_{66} \end{bmatrix}$$

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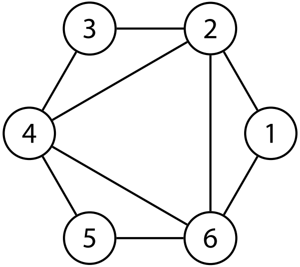

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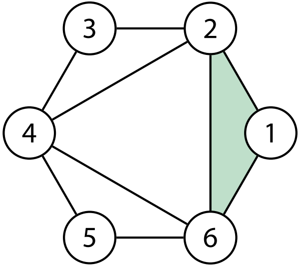

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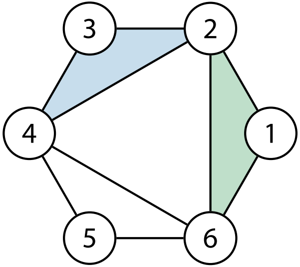

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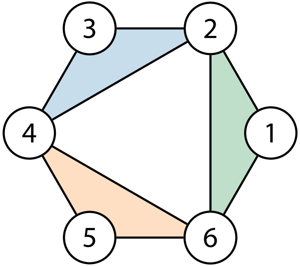

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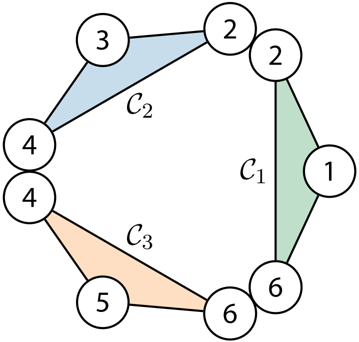
$$\begin{array}{ll} \text{maximize} & b^\top y \\ \text{subject to} & \sum_{i=1}^{m} A_i y_i + S = C \\ & S \in \mathbb{S}_+^n(E, 0) \end{array}$$

$$\begin{array}{ll} \text{maximize} & b^\top y \\ \text{subject to} & \sum_{i=1}^{m} A_i y_i + \sum_{\ell=1}^{3} T_\ell^\top S_\ell T_\ell = C \\ & S_1 \in \mathbb{S}_+^{|\mathcal{C}_1|},\ S_2 \in \mathbb{S}_+^{|\mathcal{C}_2|},\ S_3 \in \mathbb{S}_+^{|\mathcal{C}_3|} \end{array}$$

$$\begin{bmatrix} S_{11} & S_{12} & 0 & 0 & 0 & S_{16} \\ S_{21} & S_{22} & S_{23} & S_{24} & 0 & S_{26} \\ 0 & S_{32} & S_{33} & S_{34} & 0 & 0 \\ 0 & S_{42} & S_{43} & S_{44} & S_{45} & S_{46} \\ 0 & 0 & 0 & S_{54} & S_{55} & S_{56} \\ S_{61} & S_{62} & 0 & S_{64} & S_{65} & S_{66} \end{bmatrix}$$

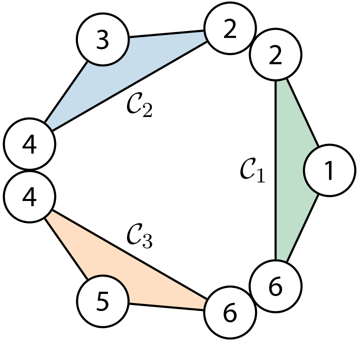


Example: Chordal Decomposition

$$\begin{array}{ll} \text{maximize} & b^\top y \\ \text{subject to} & \sum_{i=1}^m A_i y_i + S = C \\ & S \in \mathbb{S}_+^n(E, 0) \end{array}$$

$$\begin{array}{ll} \text{maximize} & b^\top y \\ \text{subject to} & \sum_{i=1}^m A_i y_i + \sum_{\ell=1}^3 T_\ell^\top S_\ell T_\ell = C \\ & S_1 \in \mathbb{S}_+^{|\mathcal{C}_1|},\ S_2 \in \mathbb{S}_+^{|\mathcal{C}_2|},\ S_3 \in \mathbb{S}_+^{|\mathcal{C}_3|} \end{array}$$

$$\begin{bmatrix} S_{11} & S_{12} & 0 & 0 & 0 & S_{16} \\ S_{21} & S_{22} & S_{23} & S_{24} & 0 & S_{26} \\ 0 & S_{32} & S_{33} & S_{34} & 0 & 0 \\ 0 & S_{42} & S_{43} & S_{44} & S_{45} & S_{46} \\ 0 & 0 & 0 & S_{54} & S_{55} & S_{56} \\ S_{61} & S_{62} & 0 & S_{64} & S_{65} & S_{66} \end{bmatrix}$$

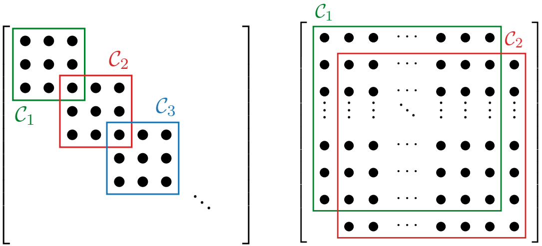


- Interior point method [Fukuda et al. 2001]
- First-order method [Sun, M. S. Andersen, and L. Vandenberghe 2014]
- ADMM-HSDE [Zheng et al. 2019]

Clique merging

- Combine cliques by introducing new edges in the graph
- One merge operation:
  - replaces two PSD constraints by one larger PSD constraint
  - removes equality constraints

  $\rightarrow$ trade-off depends on the employed solver algorithm
- Obvious cases:



Algorithm: First-order solver

Factor constraint matrix;
while not converged:
  [...]
  Eigenvalue decomposition of PSD decision variables;

Clique tree-based merging strategies

$$
\begin{array}{c}
\\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9
\end{array}
\begin{array}{c}
\begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array} \\
\left[\begin{array}{ccccccccc}
\bullet & & & & & & & & \\
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\end{array}\right]
\end{array}
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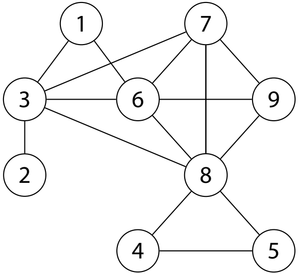

Algorithm: Clique tree-based merging

*Available packages: `SparseCoLO` [Fujisawa et al. 2009], `Chompack` [M. Andersen and Lieven Vandenberghe 2015]

Clique tree-based merging strategies

$$
\begin{array}{c}
\begin{array}{c} \\ 1\\2\\3\\4\\5\\6\\7\\8\\9 \end{array}
\begin{array}{c}
\begin{array}{ccccccccc} 1&2&3&4&5&6&7&8&9 \end{array}\\
\left[\begin{array}{ccccccccc}
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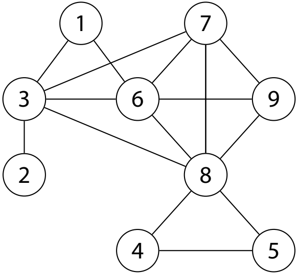
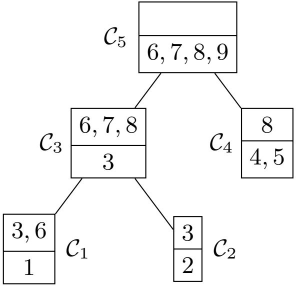
\end{array}\right]
\end{array}
\end{array}
$$



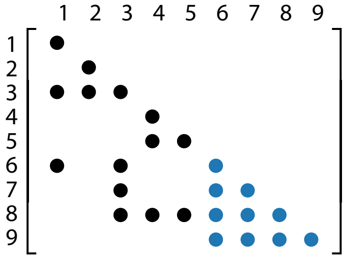


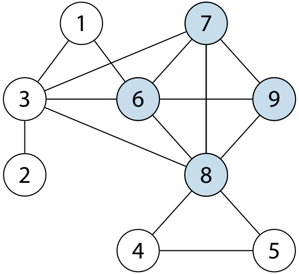
Algorithm: Clique tree-based merging

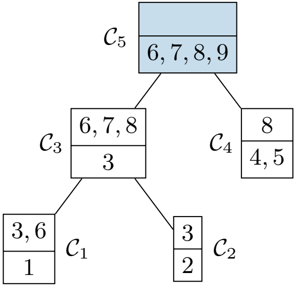
Compute clique tree;

*Available packages: `SparseCoLO` [Fujisawa et al. 2009], `Chompack` [M. Andersen and Lieven Vandenberghe 2015]

Clique tree-based merging strategies





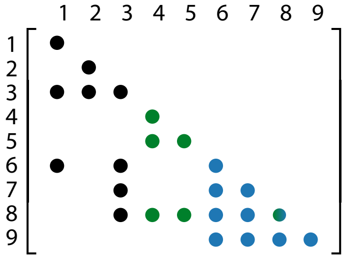


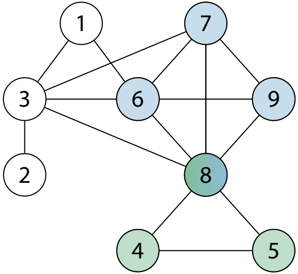
Algorithm: Clique tree-based merging

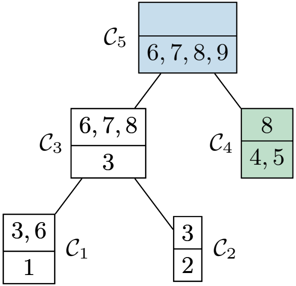
Compute clique tree;
Traverse tree depth-first: $\mathcal{C}_i$:

*Available packages: `SparseCoLO` [Fujisawa et al. 2009], `Chompack` [M. Andersen and Lieven Vandenberghe 2015]

Clique tree-based merging strategies





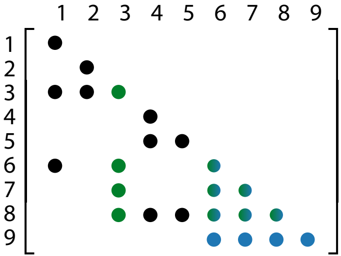


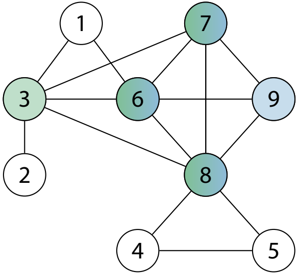
Algorithm: Clique tree-based merging

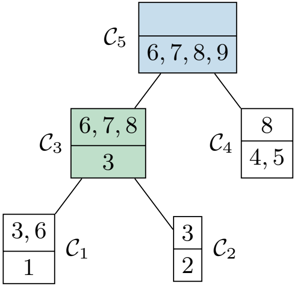
Compute clique tree;
Traverse tree depth-first: $\mathcal{C}_i$:
  Find child node: $\mathcal{C}_j$;
  if heuristic condition $f(\mathcal{C}_i, \mathcal{C}_j) \geq \gamma$ holds:

* Available packages: `SparseCoLO` [Fujisawa et al. 2009], `Chompack` [M. Andersen and Lieven Vandenberghe 2015]

Clique tree-based merging strategies







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Compute clique tree;
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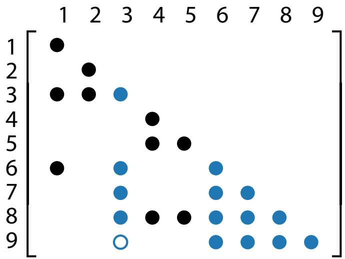
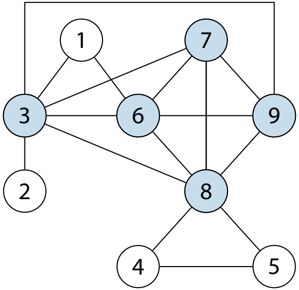
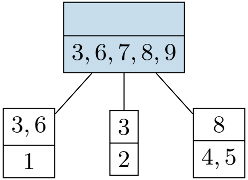
Clique tree-based merging strategies







Algorithm: Clique tree-based merging

Compute clique tree;
Traverse tree depth-first: $\mathcal{C}_i$:
  Find child node: $\mathcal{C}_j$;
  if heuristic condition $f(\mathcal{C}_i, \mathcal{C}_j) \geq \gamma$ holds:
    $\mathcal{C}_m \leftarrow \mathcal{C}_i \cup \mathcal{C}_j$

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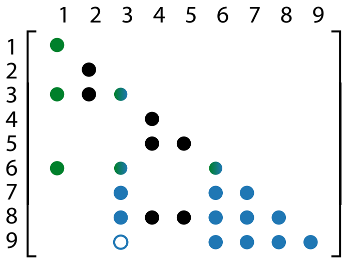
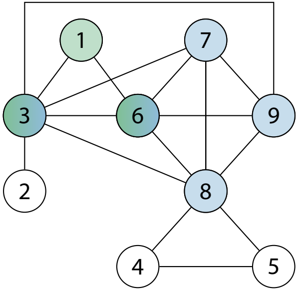
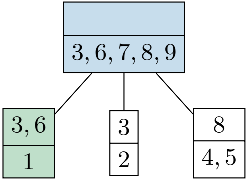
Clique tree-based merging strategies







Algorithm: Clique tree-based merging

Compute clique tree;
Traverse tree depth-first: $\mathcal{C}_i$:
  Find child node: $\mathcal{C}_j$;
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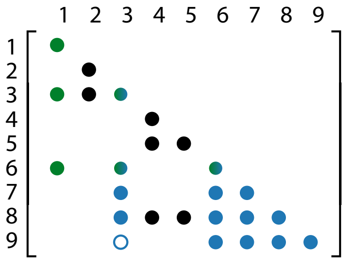
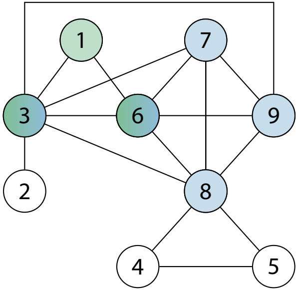
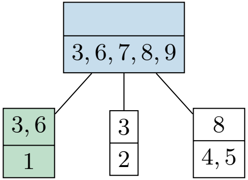
Clique tree-based merging strategies







- Designed for interior-point solvers

+ Clique tree cheap to compute and evaluate

– Disregards distant merge candidates

– Relies on heuristic parameters

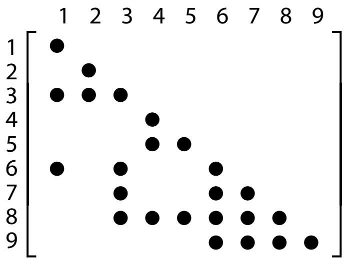
*Available packages: `SparseCoLO` [Fujisawa et al. 2009], `Chompack` [M. Andersen and Lieven Vandenberghe 2015]

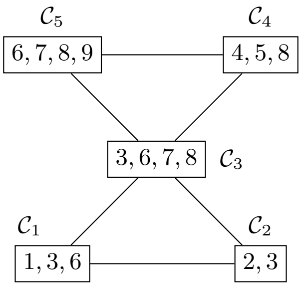
A new clique graph-based merging strategy

$$\begin{array}{cc} & \begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{array} & \begin{bmatrix} \bullet & & & & & & & & \\ & \bullet & & & & & & & \\ \bullet & \bullet & \bullet & & & & & & \\ & & & \bullet & & & & & \\ & & & \bullet & \bullet & & & & \\ \bullet & & \bullet & & & \bullet & & & \\ & & \bullet & & & \bullet & \bullet & & \\ & & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \\ & & & & & \bullet & \bullet & \bullet & \bullet \end{bmatrix} \end{array}$$

Algorithm: Clique graph-based merging

A new clique graph-based merging strategy





Algorithm: Clique graph-based merging

Compute reduced clique graph;

A new clique graph-based merging strategy

$$
\begin{array}{c}
\begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array} \\
\begin{array}{c} 1\\2\\3\\4\\5\\6\\7\\8\\9 \end{array}
\begin{bmatrix}
\bullet & & & & & & & & \\
 & \bullet & & & & & & & \\
\bullet & \bullet & \bullet & & & & & & \\
 & & & \bullet & & & & & \\
 & & & \bullet & \bullet & & & & \\
\bullet & & \bullet & & & \bullet & & & \\
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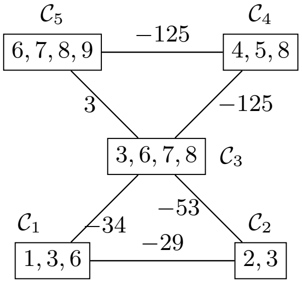
 & & & & & \bullet & \bullet & \bullet & \bullet
\end{bmatrix}
\end{array}
$$

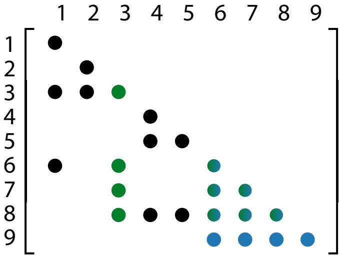


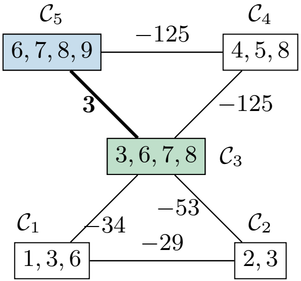
in this example:
$e(\mathcal{C}_i, \mathcal{C}_j) = |\mathcal{C}_i|^3 + |\mathcal{C}_j|^3 - |\mathcal{C}_i \cup \mathcal{C}_j|^3$

Algorithm: Clique graph-based merging

Compute reduced clique graph;
Compute edge weights $w_{ij} = e(\mathcal{C}_i, \mathcal{C}_j)$;

A new clique graph-based merging strategy





in this example:

$e(\mathcal{C}_i, \mathcal{C}_j) = |\mathcal{C}_i|^3 + |\mathcal{C}_j|^3 - |\mathcal{C}_i \cup \mathcal{C}_j|^3$

Algorithm: Clique graph-based merging

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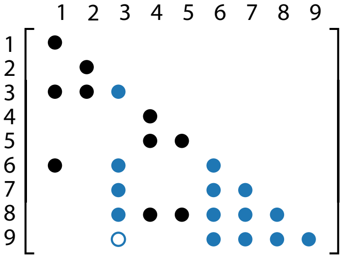
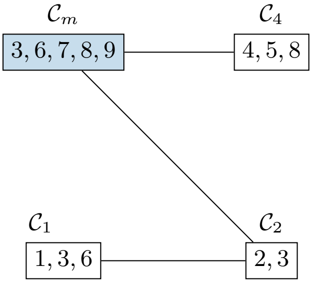
while $w_{ij} > 0$ exists:

A new clique graph-based merging strategy



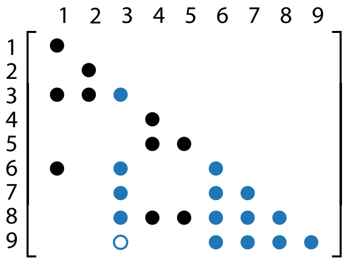


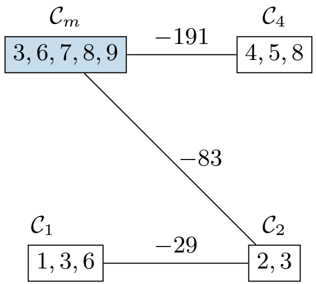
in this example:
$e(\mathcal{C}_i, \mathcal{C}_j) = |\mathcal{C}_i|^3 + |\mathcal{C}_j|^3 - |\mathcal{C}_i \cup \mathcal{C}_j|^3$

Algorithm: Clique graph-based merging

Compute reduced clique graph;
Compute edge weights $w_{ij} = e(\mathcal{C}_i, \mathcal{C}_j)$;
while $w_{ij} > 0$ exists:
  Merge permissible $(\mathcal{C}_i, \mathcal{C}_j)$ with max weight $\rightarrow \mathcal{C}_m$;

A new clique graph-based merging strategy





in this example:

$$e(\mathcal{C}_i, \mathcal{C}_j) = |\mathcal{C}_i|^3 + |\mathcal{C}_j|^3 - |\mathcal{C}_i \cup \mathcal{C}_j|^3$$

Algorithm: Clique graph-based merging

Compute reduced clique graph;
Compute edge weights $w_{ij} = e(\mathcal{C}_i, \mathcal{C}_j)$;
while $w_{ij} > 0$ exists:
  Merge permissible $(\mathcal{C}_i, \mathcal{C}_j)$ with max weight $\to \mathcal{C}_m$;
  Update edge weights connected to $\mathcal{C}_m$;

Benchmarks

- **Goal:** Reduce the projection time of our first-order ADMM solver `COSMO`
- **Problem set:** Large sparse SDPs from the SDPLib collection and SuiteSparse Matrix Library
- **Setup:** Compare different merge strategies with our solver

a) No decomposition	b) No merging
c) `SparseCoLO` merging	d) Parent-child merging
e) Clique graph merging (nominal)	f) Clique graph merging (estimated)

Benchmarks

- **Goal:** Reduce the projection time of our first-order ADMM solver `COSMO`
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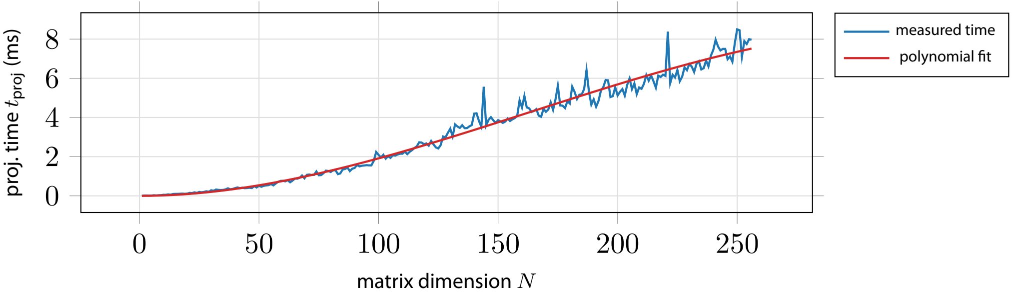
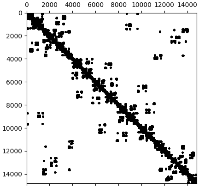
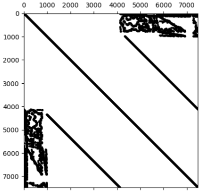


Benchmark sparsity patterns

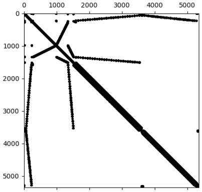
- 500 - 21M nonzeros



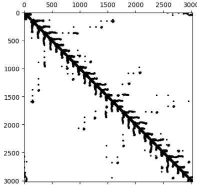
(a)



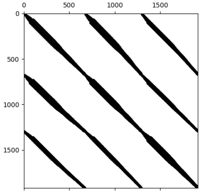
(b)



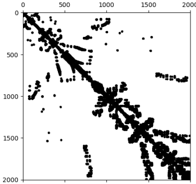
(c)



(d)

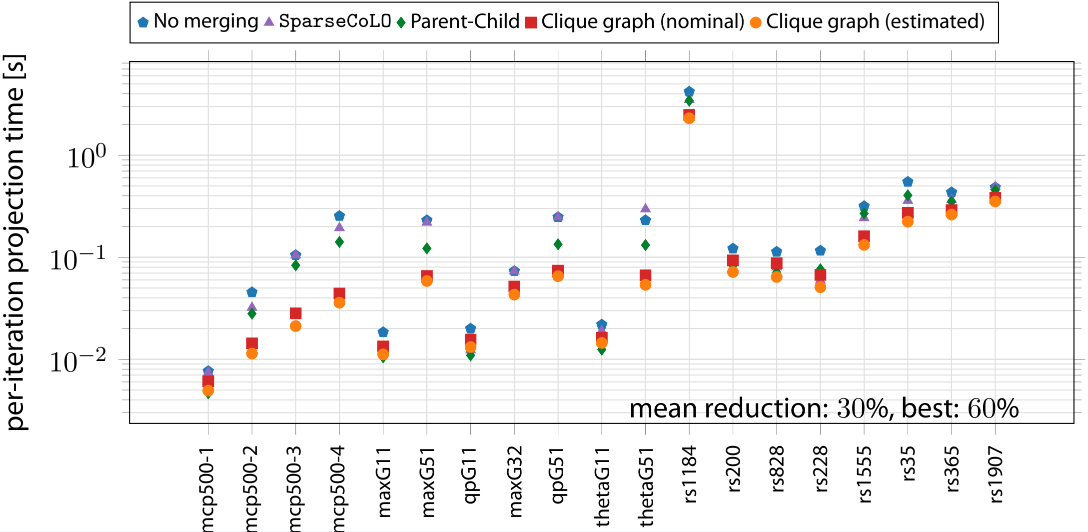


(e)

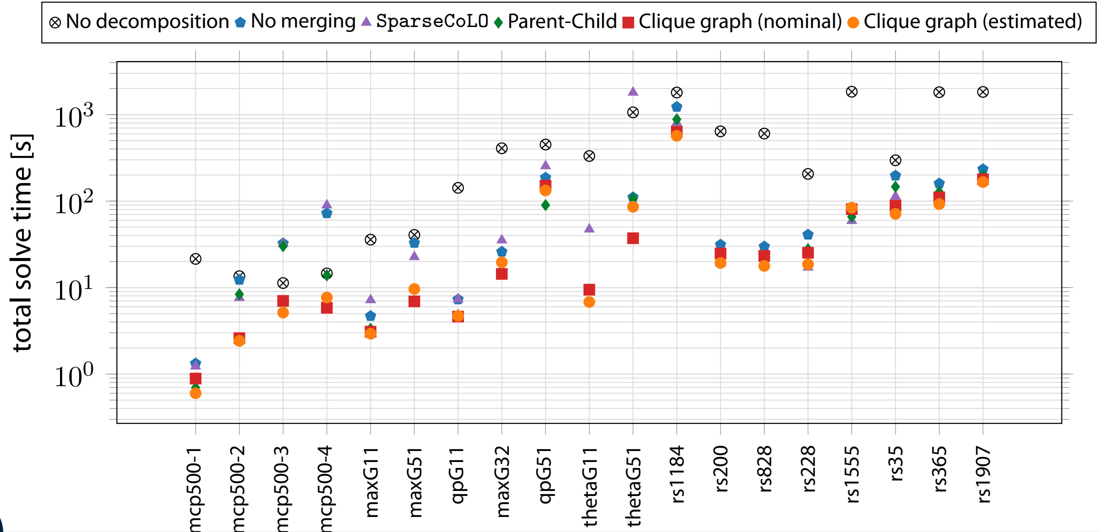


(f)

Benchmark results - projection time



Benchmark results - solve time



Benchmark results

- Hardware: Oxford ARC-HTC 16 logical Intel Xeon E5-2560 cores, 64GB RAM

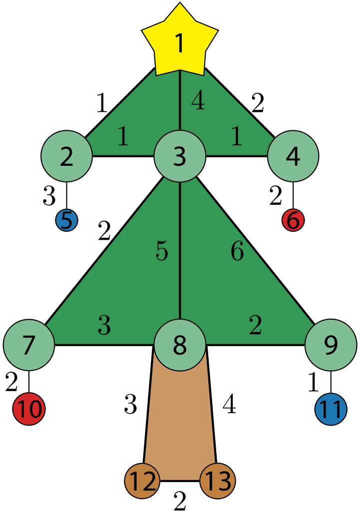
Table: Solve times for different SDP solvers.

problem	COSMO	Mosek	SCS	problem	COSMO	Mosek	SCS
maxG11	**1.47**	4.45	131.8	rs1184	**224.86**	$***^m$	$***^m$
maxG32	**6.25**	50.84	840.79	rs1555	**66.6**	$***^\dagger$	$***^m$
maxG51	**8.09**	9.92	36.56	rs1907	**104.61**	$***^\dagger$	$***^m$
mcp500-1	**0.24**	1.7	29.28	rs200	**12.47**	752.27	$***^\dagger$
mcp500-2	**1.68**	1.75	17.36	rs228	**12.86**	395.24	982.5
mcp500-3	4.41	**1.68**	8.36	rs35	**54.88**	919.19	$***^\dagger$
mcp500-4	8.2	**1.76**	7.4	rs365	**62.65**	$***^\dagger$	$***^\dagger$
qpG11	**2.36**	26.23	734.7	rs828	**10.84**	825.03	$***^\dagger$
qpG51	121.6	**96.42**	527.55	thetaG51	71.21	**50.08**	967.43
thetaG11	**2.32**	8.53	142.53				

^{m} out-of-memory error, † 30min timelimit

Code example

What is the maximum cut through this Christmas tree?



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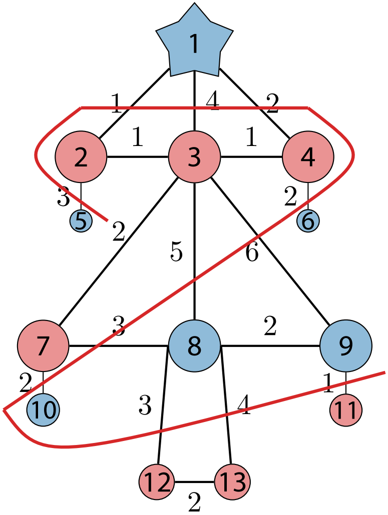

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- Considers many pair-wise merge candidates
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- Michael Garstka, Mark Cannon, and Paul Goulart (2020b). "COSMO: A conic operator splitting method for convex conic problems". In: *arXiv*. arXiv: 1901.10887 [math.OC]. URL: https://arxiv.org/abs/1901.10887

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Questions?

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