（補題 1 証明終）

引き続き z のまま積分の中身を変形する。

$$\frac{z^{n+m-1}}{1+z^{2n}} = -\frac{1}{2n}\sum_{k=1}^{2n}\frac{z_{n,\,k}^{n+m}}{z-z_{n,\,k}} = -\frac{1}{2n}\sum_{k=1}^{n}\left(\frac{z_{n,\,k}^{n+m}}{z-z_{n,\,k}}+\frac{z_{n,\,2n+1-k}^{n+m}}{z-z_{2n+1-k}}\right)$$

ここで

$$\theta_{n,\,2n+1-k} = \frac{2(2n+1-k)-1}{2n}\pi = 2\pi-\frac{2k-1}{2n} = 2\pi-\theta_{n,\,k}$$

より $z_{n,\,2n+1-k} = z_{n,\,k}^{-1}$ であり

$$\begin{cases} z_{n,\,k}+z_{n,\,2n+1-k} = 2\cos\theta_{n,\,k} \\ z_{n,\,k}z_{n,\,2n+1-k} = 1 \end{cases}$$

である。また

$$\begin{aligned} &z_{n,\,k}^{n+m}z_{n,\,2n+1-k}+z_{n,\,2n+1-k}^{n+m}z_{n,\,k} = z_{n,\,k}^{n+m-1}+z_{n,\,k}^{-(n+m-1)} \\ &= e((n+m-1)\theta_{n,\,k})+e(-(n+m-1)\theta_{n,\,k}) = 2\cos(n+m-1)\theta_{n,\,k} \end{aligned}$$

より分母を通分して

$$\begin{aligned} \frac{z^{n+m-1}}{1+z^{2n}} &= -\frac{1}{2n}\sum_{k=1}^{n}\frac{2z\cos(n+m)\theta_{n,\,k}-2\cos(n+m-1)\theta_{n,\,k}}{z^2-2\cos\theta_{n,\,k}z+1} \\ &= -\frac{1}{2n}\sum_{k=1}^{n}\left[\frac{(2z-2\cos\theta_{n,\,k})\cos(n+m)\theta_{n,\,k}}{z^2-2\cos\theta_{n,\,k}z+1}+2\,\frac{\cos\theta_{n,\,k}\cos(n+m)\theta_{n,\,k}-\cos(n+m-1)\theta_{n,\,k}}{\left\{\left(\dfrac{z-\cos\theta_{n,\,k}}{\sin\theta_{n,\,k}}\right)^2+1\right\}\sin^2\theta_{n,\,k}}\right] \end{aligned}$$

ここで加法定理より

$$\cos(n+m-1)\theta_{n,\,k} = \cos\theta_{n,\,k}\cos(n+m)\theta_{n,\,k}+\sin\theta_{n,\,k}\sin(n+m)\theta_{n,\,k}$$

であるから第 2 項の分子は

$$\cos\theta_{n,\,k}\cos(n+m)\theta_{n,\,k}-\cos(n+m-1)\theta_{n,\,k} = -\sin\theta_{n,\,k}\sin(n+m)\theta_{n,\,k}$$

と変形できる。z を x に戻して，見やすいようにいくつか積の順番を入れ替えると (1) と合わせて

$$\begin{aligned} \int\tan^{\frac{m}{n}}\theta\,d\theta \overset{x=\tan^{\frac{1}{n}}\theta}{=\!=\!=} -\frac{n}{2n}\int\sum_{k=1}^{n}\Bigg[&\cos(n+m)\theta_{n,\,k}\frac{2x-2\cos\theta_{n,\,k}}{x^2-2x\cos\theta_{n,\,k}+1} \\ &+2\frac{-\sin\theta_{n,\,k}\sin(n+m)\theta_{n,\,k}}{\sin^2\theta_{n,\,k}}\frac{1}{\left(\dfrac{z-\cos\theta_{n,\,k}}{\sin\theta_{n,\,k}}\right)^2+1}\Bigg]dx \end{aligned}$$