より分母を通分して

$$\frac{z^{n+m-1}}{1+z^{2n}} = -\frac{1}{2n}\sum_{k=1}^{n}\frac{2z\cos(n+m)\theta_{n,\,k}-2\cos(n+m-1)\theta_{n,\,k}}{z^2-2\cos\theta_{n,\,k}z+1}$$
$$= -\frac{1}{2n}\sum_{k=1}^{n}\left[\ \frac{(2z-2\cos\theta_{n,\,k})\cos(n+m)\theta_{n,\,k}}{z^2-2\cos\theta_{n,\,k}z+1}+2\ \frac{\cos\theta_{n,\,k}\cos(n+m)\theta_{n,\,k}-\cos(n+m-1)\theta_{n,\,k}}{\left\{\left(\dfrac{z-\cos\theta_{n,\,k}}{\sin\theta_{n,\,k}}\right)^2+1\right\}\sin^2\theta_{n,\,k}}\ \right]$$

ここで加法定理より

$$\cos(n+m-1)\theta_{n,\,k}=\cos\theta_{n,\,k}\cos(n+m)\theta_{n,\,k}+\sin\theta_{n,\,k}\sin(n+m)\theta_{n,\,k}$$

であるから第 2 項の分子は

$$\cos\theta_{n,\,k}\cos(n+m)\theta_{n,\,k}-\cos(n+m-1)\theta_{n,\,k}=-\sin\theta_{n,\,k}\sin(n+m)\theta_{n,\,k}$$

と変形できる。z を x に戻して，見やすいようにいくつか積の順番を入れ替えると (1) と合わせて

$$\int\tan^{\frac{m}{n}}\theta\,d\theta \overset{x=\tan^{\frac{1}{n}}\theta}{=\!=\!=\!=} -\frac{n}{2n}\int\sum_{k=1}^{n}\left[\cos(n+m)\theta_{n,\,k}\frac{2x-2\cos\theta_{n,\,k}}{x^2-2x\cos\theta_{n,\,k}+1}\right.$$
$$\left.+2\frac{-\sin\theta_{n,\,k}\sin(n+m)\theta_{n,\,k}}{\sin^2\theta_{n,\,k}}\frac{1}{\left(\dfrac{z-\cos\theta_{n,\,k}}{\sin\theta_{n,\,k}}\right)^2+1}\ \right]dx$$

と変形できる。積分を $\sum$ の中に入れると

$$（上式）= -\frac{1}{2}\sum_{k=1}^{n}\left[\cos(n+m)\theta_{n,\,k}\int\frac{\left(x^2-2x\cos\theta_{n,\,k}+1\right)'}{x^2-2x\cos\theta_{n,\,k}+1}dx\right.$$
$$\left.-2\sin(n+m)\theta_{n,\,k}\int\ \frac{1}{1+\left(\dfrac{x-\cos\theta_{n,\,k}}{\sin\theta_{n,\,k}}\right)^2}\ \frac{dx}{\sin\theta_{n,\,k}}\right]$$
$$= -\frac{1}{2}\sum_{k=1}^{n}\left[\cos(n+m)\theta_{n,\,k}\log\left(x^2-2x\cos\theta_{n,\,k}+1\right)\right.$$
$$\left.-2\sin(n+m)\theta_{n,\,k}\arctan\left(\frac{x-\cos\theta_{n,\,k}}{\sin\theta_{n,\,k}}\right)\right]+C$$
$$= \sum_{k=1}^{n}\left[-\frac{1}{2}\cos(n+m)\theta_{n,\,k}\log\left(\tan^{\frac{2}{n}}\theta-2\cos\theta_{n,\,k}\tan^{\frac{1}{n}}\theta+1\right)\right.$$
$$\left.+\sin(n+m)\theta_{n,\,k}\arctan\left(\frac{\tan^{\frac{1}{n}}\theta-\cos\theta_{n,\,k}}{\sin\theta_{n,\,k}}\right)\right]+C$$

が導かれた。

（証明終）

以降，この不定積分から積分定数 C を除いたものを $f(\theta)$ とおく。