解

$$\int_0^{\frac{\pi}{4}} \sqrt{\tan x}\, dx = \int_0^{\frac{\pi}{4}} \tan^{\frac{1}{2}} x\, dx$$

$$= \frac{\pi}{4\cos\left(\frac{1}{2}\cdot\frac{\pi}{2}\right)} - \frac{1}{2}\sum_{k=1}^{2} \cos\left[(2+1)\cdot\frac{2k-1}{2\cdot 2}\pi\right] \log\left(2\left(1-\cos\frac{2k-1}{2\cdot 2}\pi\right)\right)$$

$$= \frac{\pi}{2\sqrt{2}} - \frac{1}{2}\left(\cos\frac{3}{4}\pi \log\left(2-\sqrt{2}\right) + \cos\frac{9}{4}\pi \log\left(2+\sqrt{2}\right)\right)$$

$$= \frac{\sqrt{2}}{4}\pi - \frac{\sqrt{2}}{4}\log\left(1+\sqrt{2}\right)^2$$

$$= \frac{\sqrt{2}}{4}\pi - \frac{\sqrt{2}}{2}\log\left(1+\sqrt{2}\right) \cdots\cdots (答)$$

例題

次の積分を求めよ。

$$\int_0^{\frac{\pi}{4}} \tan^{\frac{3}{2}} \theta\, d\theta$$

解法；偶数に寄せて次数を減らす！

$$\int_0^{\frac{\pi}{4}} \tan^{\frac{3}{2}} \theta\, d\theta$$

$$= \int_0^{\frac{\pi}{4}} \left(1+\tan^2\theta\right)\tan^{-\frac{1}{2}}\theta - \tan^{-\frac{1}{2}}\theta\, d\theta$$

$$= \left[2\tan^{\frac{1}{2}}\theta\right]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \tan^{-\frac{1}{2}}\theta\, d\theta$$

$$= 2 - \frac{\pi}{2\sqrt{2}} + \frac{1}{2}\sum_{k=1}^{2}\cos\left[(2-1)\frac{2k-1}{2\cdot 2}\pi\right]\log\left[2\left(1-\cos\frac{2k-1}{2\cdot 2}\right)\pi\right]$$

$$= 2 - \frac{\sqrt{2}}{4}\pi + \frac{1}{2}\left(\cos\frac{\pi}{4}\log\left[2\left(1-\frac{1}{\sqrt{2}}\right)\right] + \cos\frac{3}{4}\pi\log\left[2\left(1+\frac{1}{\sqrt{2}}\right)\right]\right)$$

$$= 2 - \frac{\sqrt{2}}{4}\pi + \frac{1}{2}\left\{\frac{1}{\sqrt{2}}\log\left(2-\sqrt{2}\right) - \frac{1}{\sqrt{2}}\log\left(2+\sqrt{2}\right)\right\}$$

$$= 2 - \frac{\sqrt{2}}{4}\pi - \frac{\sqrt{2}}{4}\log\left|\frac{\sqrt{2}+1}{\sqrt{2}-1}\right|$$

$$= 2 - \frac{\sqrt{2}}{4}\pi - \frac{\sqrt{2}}{2}\log\left(1+\sqrt{2}\right) \cdots\cdots (答)$$