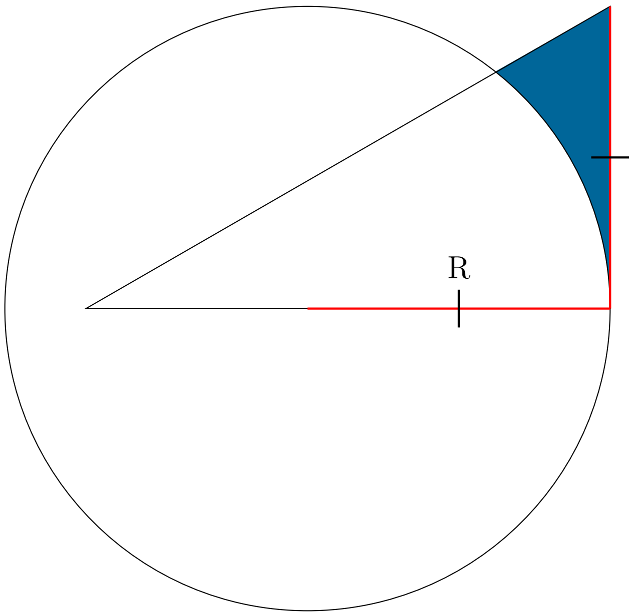
1 Question

Find the area shaded in blue.



2 Solution

2.1 Setup

I can't do geometry, so I graphed it on a Cartesian plane, so that I can just use integrals to find the area.

First, is the equation of the triangle, which we can just find by doing rise over run.

$$\frac{rise}{run} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

We can just assume it starts at the origin, so we get that the hypotenuse of the triangle can be modelled by (the restriction on the domain becomes important later):

$$f_1(x) = \frac{\sqrt{3}}{3}x \quad \left\{0 \leq x \leq \sqrt{3}\right\}$$

Next, we need to find the equation of the arc that the circle makes.

First, we need the equation of the circle. Easy enough:

$$(x-d)^2 + y^2 = 1 \qquad d = \sqrt{3} - 1$$

Now, we need to isolate for y (we can just safely ignore the positive and negative branches, you'll see why later):

$$y = f_2(x) = \sqrt{1 - (x-d)^2}$$

To represent the area bounded by the circle and triangle, we can use a piecewise function:

$$f(x) = \begin{cases} \frac{\sqrt{3}}{3}x & x < i \\ \sqrt{1-(x-d)^2} & x \geq i \end{cases}$$

So, what should i be? If we graph it, we can see that i should be the x value where $f_1(x)$ and $f_2(x)$ intersect.

$$\frac{\sqrt{3}}{3}x = \sqrt{1-(x-d)^2}$$

$$\frac{1}{3}x^2 = 1 - (x-d)^2 = 1 - \left(x^2 - 2dx + d^2\right)$$

$$0 = -\frac{4}{3}x^2 + 2dx - \left(d^2 - 1\right)$$

Now we can solve for the roots, which will give us two values.

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad a = -\frac{4}{3} \qquad b = 2d \qquad c = -\left(d^2 - 1\right)$$

$$\frac{-(2d) \pm \sqrt{(2d)^2 - 4\left(-\frac{4}{3}\right)\left(-\left(d^2-1\right)\right)}}{2\left(-\frac{4}{3}\right)}$$

$$\frac{(2d) \pm \sqrt{4d^2 - 4\left(\frac{4}{3}\right)\left(d^2-1\right)}}{\frac{8}{3}}$$

$$\frac{2d \pm \sqrt{4d^2 - \frac{16}{3}d^2 + \frac{16}{3}}}{\frac{8}{3}}$$

$$\frac{2d \pm \sqrt{-\frac{4}{3}d^2 + \frac{16}{3}}}{\frac{8}{3}}$$

Now we can use the fact that $d = \sqrt{3} - 1$ to simplify further:

$$\frac{2\left(\sqrt{3}-1\right) \pm \sqrt{-\frac{4}{3}\left(\sqrt{3}-1\right)^2 + \frac{16}{3}}}{\frac{8}{3}}$$

$$\frac{6\sqrt{3} - 6 \pm 3\sqrt{-\frac{4}{3}\left(3 - 2\sqrt{3} + 1\right) + \frac{16}{3}}}{8}$$

$$\frac{6\sqrt{3} - 6 \pm 3\sqrt{-\frac{12}{3} + \frac{8\sqrt{3}}{3} - \frac{4}{3} + \frac{16}{3}}}{8}$$

$$\frac{6\sqrt{3} - 6 \pm 3\sqrt{\frac{8\sqrt{3}}{3}}}{8}$$

$$\frac{6\sqrt{3} - 6 \pm \sqrt{24\sqrt{3}}}{8}$$

$$\frac{6\sqrt{3} - 6 \pm 2\sqrt{6\sqrt{3}}}{8}$$

After simplifying, we can see that we have 2 solutions:

$$x_1 = \frac{3}{4}\sqrt{3} + \frac{1}{4}\sqrt{6\sqrt{3}} - \frac{3}{4} \qquad x_2 = \frac{3}{4}\sqrt{3} - \frac{1}{4}\sqrt{6\sqrt{3}} - \frac{3}{4}$$

Numerically evaluating them, we get that

$$x_1 \approx 1.355 \qquad x_2 \approx -0.257$$

Now since we set a domain on $f_1(x)$, such that $\left\{0 \leq x \leq \sqrt{3}\right\}$, x_2 is an invalid solution, since it is outside that domain. This must mean that the intersection point, and therefore $i = x_1$

Finding the area under the curve is now a (somewhat) trivial task.

2.2 Integration

In order to find the area that the circle and triangle cover, we can take the integral of the piecewise function defined earlier, from the two zeroes, which must necessarily be 0 and $\sqrt{3}$ (it is also helpful to know that $0 < i < \sqrt{3}$)

$$A_C = \int_0^{\sqrt{3}} f\left(x\right) dx = \int_0^i f_1\left(x\right) dx + \int_i^{\sqrt{3}} f_2\left(x\right) dx$$

The integral of $f_1\left(x\right)$ will be as follows:

$$\int f_1\left(x\right) dx = \int \frac{\sqrt{3}}{3} x \; dx$$

$$F_1(x) = \frac{\sqrt{3}}{6} x^2 + C$$

The integral of $f_2(x)$ will be as follows:

$$\int f_2(x)\,dx = \int \sqrt{1-(x-d)^2}\ dx$$

$$\int \sqrt{1-u^2}\ du \qquad u = x-d \qquad du = dx$$

Use trigonometric substitution and pythagorean identity:

$$\int \sqrt{1-\sin^2 v}\ \cos v\ dv \qquad u = \sin v \qquad du = \cos v\ dv$$

$$\int \sqrt{\cos^2 v}\ \cos v\ dv$$

$$\int \cos^2 v\ dv$$

Rewriting the entire thing using the cosine double angle formula:

$$\cos 2x = \cos^2 x - \sin^2 x$$

$$\cos 2x = 2\cos^2 x - 1$$

$$\cos^2 x = \frac{1}{2}\cos 2x + \frac{1}{2}$$

Picking up from where we left off:

$$\int \cos^2 v\ dv = \int \frac{1}{2}\cos 2v + \frac{1}{2}\ dv$$

$$\frac{1}{2}\int \cos 2v\ dv\ +\ \frac{1}{2}\int 1\ dv$$

$$\frac{1}{2}\int \cos w\ \frac{1}{2}dw \qquad w = 2v \qquad dw = 2dv$$

$$\frac{1}{4}\int \cos w\ dw$$

$$\frac{1}{4}\sin w = \frac{1}{4}\sin 2v$$

$$\frac{1}{4}\sin 2v + \frac{1}{2}v$$

Now we need to express everything in terms of u again (oh how fun...). Note that $v = \arcsin u$, since $u = \sin v$

$$\frac{1}{2}\sin v\cos v + \frac{1}{2}v$$

$$\frac{1}{2}\sin v\sqrt{\cos^2 v}+\frac{1}{2}v$$

$$\frac{1}{2}\sin v\sqrt{1-\sin^2 v}+\frac{1}{2}v$$

$$\frac{1}{2}\sin v\sqrt{1-\sin v\sin v}+\frac{1}{2}v$$

$$\frac{1}{2}u\sqrt{1-u^2}+\frac{1}{2}\arcsin u$$

Finally, $u=x-d$, which gives us this:

$$F_2(x)=\frac{1}{2}\left(x-d\right)\sqrt{1-\left(x-d\right)^2}+\frac{1}{2}\arcsin\left(x-d\right)+C$$

2.3 Solving for Area

I'm too lazy to write out all the steps to simplify, but trust me, it works out. Substituting 0 for x in $F_1(x)$, we get:

$$F_1(0)=\frac{\sqrt{3}}{6}(0)^2=0$$

Substituting i for x in $F_1(x)$, we get:

$$F_1(i)=\frac{\sqrt{3}}{6}i^2=\frac{\sqrt{3}}{6}\left(\frac{3}{4}\sqrt{3}+\frac{1}{4}\sqrt{6\sqrt{3}-\frac{3}{4}}\right)\left(\frac{3}{4}\sqrt{3}+\frac{1}{4}\sqrt{6\sqrt{3}-\frac{3}{4}}\right)$$

which will simplify to this:

$$\frac{1}{8}\left(-3+3\sqrt{3}+4\sqrt{\frac{9\sqrt{3}}{8}-\frac{27}{16}}\right)$$

which will simplify further to this (trust me, a will be important later):

$$-\frac{1}{16}\left(a-2\right)\sqrt{6\sqrt{3}-\frac{3}{8}}\qquad a=\sqrt{3}-\sqrt{6\sqrt{3}-1}$$

The area of the first half of the piecewise function is $F_1(i)-F_1(0)$, which will just be $F_1(i)$

Now for $F_2(x)$, if we substitute in $\sqrt{3}$, and $d=\sqrt{3}-1$

$$\sqrt{3}-d=\sqrt{3}-\sqrt{3}+1=1$$

$$F_2(\sqrt{3})=\frac{1}{2}\left(\sqrt{3}-d\right)\sqrt{1-\left(\sqrt{3}-d\right)^2}+\frac{1}{2}\arcsin\left(\sqrt{3}-d\right)$$

$$F_2(\sqrt{3}) = \frac{1}{2}(1)\sqrt{1-(1)^2} + \frac{1}{2}\arcsin(1)$$

$$F_2(\sqrt{3}) = \frac{1}{2}\left(\frac{\pi}{2}\right) = \frac{\pi}{4}$$

Substituting in i and simplifying a lot, we get this:

$$F_2(i) = -\frac{1}{32}\left(\sqrt{3}-\sqrt{6\sqrt{3}}-1\right)\sqrt{16-\left(\sqrt{3}-\sqrt{6\sqrt{3}}-1\right)^2} - \frac{1}{2}\arcsin\left(\frac{1}{4}\left(\sqrt{3}-\sqrt{6\sqrt{3}}-1\right)\right)$$

And now, we can just substitute in a to get this:

$$F_2(i) = -\frac{1}{32}a\sqrt{16-a^2} - \frac{1}{2}\arcsin\left(\frac{1}{4}a\right)$$

The area of the second half of the piecewise function is $F_2(\sqrt{3}) - F_2(i)$

$$\frac{\pi}{4} + \frac{1}{32}a\sqrt{16-a^2} + \frac{1}{2}\arcsin\left(\frac{1}{4}a\right)$$

Combining these together, we get that the area of the piecewise function between it's zeroes is:

$$A_C = \frac{\pi}{4} + \frac{1}{32}a\sqrt{16-a^2} + \frac{1}{2}\arcsin\left(\frac{1}{4}a\right) - \frac{1}{16}(a-2)\sqrt{6\sqrt{3}} - \frac{3}{8}$$

The area of the triangle is simply $\frac{1}{2}bh$

$$A_T = \frac{1}{2}\left(\sqrt{3}\right)(1) = \frac{\sqrt{3}}{2}$$

The area of the shaded part, A_S is given by $A_T - A_C$

$$\frac{\sqrt{3}}{2} - \left(\frac{\pi}{4} + \frac{1}{32}a\sqrt{16-a^2} + \frac{1}{2}\arcsin\left(\frac{1}{4}a\right) - \frac{1}{16}(a-2)\sqrt{6\sqrt{3}} - \frac{3}{8}\right)$$

Moving some things around, we get the area of the shaded section is (with $a = \sqrt{3} - \sqrt{6\sqrt{3}} - 1$):

$$A_S = -\frac{1}{32}a\sqrt{16-a^2} - \frac{1}{2}\arcsin\left(\frac{1}{4}a\right) + \frac{1}{16}(a-2)\sqrt{6\sqrt{3}} + \frac{1}{2}\sqrt{3} + \frac{3}{8} - \frac{\pi}{4}$$

$$A_S \approx 0.13052$$