

# Extra Maximum Likelihood Estimate Practice

Michael Ghattas

## Introduction

These are additional, optional practice problems to supplement that material covered in Week 3 and Problem Set 3 on Maximum Likelihood Estimation (MLE). These are intended to help you get extra practice with these concepts and prepare for the midterm exam.

These practice problems are also an opportunity to improve your score for Problem Set 3 which originally covered MLE. I will add these on as bonus points to improve your Problem Set 3 grade. If you would like to submit these practice questions for a grade, they are due at the same time as Problem Set 5.

## Problem 1 (4 points)

Suppose we have data points  $x_1, x_2, \dots, x_n$  where each  $x_i$  is an i.i.d. random variable generated from a distribution with the following pdf:

$$f(x|\theta)(1+\theta)x^\theta, \text{ for } x \in [0, 1]$$

and  $f(x|\theta) = 0$  for  $x \notin [0, 1]$ . Using maximum likelihood estimate, estimate the parameter  $\theta$  from the data.

### Answer

To find the maximum likelihood estimate (MLE) for  $\theta$ , we first write the likelihood function  $L(\theta)$  based on the given data points:

$$L(\theta) = \prod_{i=1}^n f(x_i|\theta) = \prod_{i=1}^n (1+\theta)x_i^\theta$$

Next, we take the natural logarithm of the likelihood function to obtain the log-likelihood function  $\ell(\theta)$ :

$$\ell(\theta) = \log L(\theta) = \log \left( \prod_{i=1}^n (1+\theta)x_i^\theta \right)$$

$$\ell(\theta) = \sum_{i=1}^n \log ((1+\theta)x_i^\theta)$$

$$\ell(\theta) = \sum_{i=1}^n (\log(1+\theta) + \log x_i^\theta)$$

$$\ell(\theta) = \sum_{i=1}^n (\log(1+\theta) + \theta \log x_i)$$

$$\ell(\theta) = n \log(1 + \theta) + \theta \sum_{i=1}^n \log x_i$$

To find the MLE, we take the derivative of  $\ell(\theta)$  with respect to  $\theta$  and set it to zero:

$$\frac{d\ell(\theta)}{d\theta} = \frac{d}{d\theta} \left( n \log(1 + \theta) + \theta \sum_{i=1}^n \log x_i \right)$$

$$\frac{d\ell(\theta)}{d\theta} = \frac{n}{1 + \theta} + \sum_{i=1}^n \log x_i$$

Setting the derivative to zero:

$$\frac{n}{1 + \theta} + \sum_{i=1}^n \log x_i = 0$$

$$\frac{n}{1 + \theta} = - \sum_{i=1}^n \log x_i$$

$$1 + \theta = - \frac{n}{\sum_{i=1}^n \log x_i}$$

$$\theta = -1 - \frac{n}{\sum_{i=1}^n \log x_i}$$

Therefore, the maximum likelihood estimate for  $\theta$  is:

$$\hat{\theta} = -1 - \frac{n}{\sum_{i=1}^n \log x_i}$$

## Problem 2 (4 points)

Suppose we have data points  $x_1, x_2, \dots, x_n$  where each  $x_i$  is an i.i.d. random variable drawn from a Poisson distribution with rate parameter  $\lambda$ . Find the maximum likelihood estimate for  $\lambda$ .

### Answer

Given data points  $x_1, x_2, \dots, x_n$  where each  $x_i$  is an i.i.d. random variable drawn from a Poisson distribution with rate parameter  $\lambda$ . The probability mass function of a Poisson distribution is given by:

$$P(X = x|\lambda) = \frac{\lambda^x e^{-\lambda}}{x!}$$

To find the MLE for  $\lambda$ , we write the likelihood function  $L(\lambda)$ :

$$L(\lambda) = \prod_{i=1}^n \frac{\lambda^{x_i} e^{-\lambda}}{x_i!}$$

Taking the natural logarithm of the likelihood function, we obtain the log-likelihood function  $\ell(\lambda)$ :

$$\ell(\lambda) = \log L(\lambda) = \log \left( \prod_{i=1}^n \frac{\lambda^{x_i} e^{-\lambda}}{x_i!} \right)$$

$$\ell(\lambda) = \sum_{i=1}^n \log \left( \frac{\lambda^{x_i} e^{-\lambda}}{x_i!} \right)$$

$$\ell(\lambda) = \sum_{i=1}^n (x_i \log \lambda - \lambda - \log x_i!)$$

$$\ell(\lambda) = \left( \sum_{i=1}^n x_i \right) \log \lambda - n\lambda - \sum_{i=1}^n \log x_i!$$

To find the MLE, we take the derivative of  $\ell(\lambda)$  with respect to  $\lambda$  and set it to zero:

$$\frac{d\ell(\lambda)}{d\lambda} = \frac{d}{d\lambda} \left( \left( \sum_{i=1}^n x_i \right) \log \lambda - n\lambda \right)$$

$$\frac{d\ell(\lambda)}{d\lambda} = \frac{\sum_{i=1}^n x_i}{\lambda} - n$$

Setting the derivative to zero:

$$\frac{\sum_{i=1}^n x_i}{\lambda} - n = 0$$

$$\frac{\sum_{i=1}^n x_i}{\lambda} = n$$

$$\lambda = \frac{\sum_{i=1}^n x_i}{n}$$

Therefore, the maximum likelihood estimate for  $\lambda$  is:

$$\hat{\lambda} = \frac{1}{n} \sum_{i=1}^n x_i$$

**End.**