CSCI 3104 Fall 2021
Instructors: Profs. Grochow and Waggoner

Midterm 1- Standard 11

Due Date .... October / 11^{th} / 2021
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Contents

1 Instructions

- The solutions **should be typed**, using proper mathematical notation. We cannot accept hand-written solutions. Here's a short intro to LaTeX.
- You should submit your work through the **class Canvas page** only. Please submit one PDF file, compiled using this LaTeX template.
- You may not need a full page for your solutions; pagebreaks are there to help Gradescope automatically find where each problem is. Even if you do not attempt every problem, please submit this document with no fewer pages than the blank template (or Gradescope has issues with it).
- You **may not collaborate with other students**. **Copying from any source is an Honor Code violation. Furthermore, all submissions must be in your own words and reflect your understanding of the material.** If there is any confusion about this policy, it is your responsibility to clarify before the due date.
- Posting to **any** service including, but not limited to Chegg, Discord, Reddit, StackExchange, etc., for help on an assignment is a violation of the Honor Code.
- You **must** virtually sign the Honor Code (see Section 2). Failure to do so will result in your assignment not being graded.

2 Honor Code (Make Sure to Virtually Sign)

Problem 1.

- My submission is in my own words and reflects my understanding of the material.
- I have not collaborated with any other person.
- I have not posted to external services including, but not limited to Chegg, Discord, Reddit, StackExchange, etc.
- I have neither copied nor provided others solutions they can copy.

I agree to the above, Michael Ghattas. □

3 Standard 11- Network Flows: Reductions

Problem 2. In this problem we will consider a new reduction from finding a maximum matching in a unweighted, undirected bipartite graph to finding a maximum (s,t)-flow in a flow network. Let $G = (L \dot\cup R, E)$ be a bipartite graph with the vertices partitioned into L and R. We construct an (s,t)-flow network $\mathcal{N} = (H, c, s, t)$ from G as follows:

- Let $V(H) = V(G) \cup \{s,t\}$.
- For each edge $\{u,v\}$ in $E(G)$ with $u \in L$ and $v \in R$, add a directed edge (u,v) to $E(H)$. For each of these edges, set $c(u,v) = 1$.
- For each $v \in L$, add a directed edge (s,v) to $E(H)$. For each of these edges, set $c(s,v) = 2$.
- For each $v \in R$, add a directed edge (v,t) to $E(H)$. For each of these edges, set $c(v,t) = 1$.

To help you understand the reduction we have drawn in Figure 1 an example of a bipartite graph and the resulting flow network. If you need to draw a graph as part of your answer you may copy and modify the tikzpictures or submit hand-drawn examples.

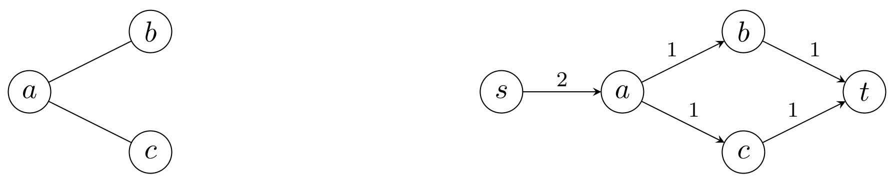

Figure 1: A bipartite graph and the flow network we get by applying the above reduction. The number next to each edge represents the capacity of that edge.

Do the following. Let $G(L \dot\cup R, E)$ be a bipartite graph, and let $\mathcal{N}$ the corresponding flow network obtained by applying the reduction.

(a) Let $\mathcal{M}$ be a matching of G, where $|\mathcal{M}| = k$. Is it the case that $\mathcal{N}$ has a feasible flow f where $\text{val}(f) = k$? If so, explain how to construct such a flow f from $\mathcal{M}$. If not, carefully explain your reasoning.

Answer:

No. We note that according to Figure 1, $\mathcal{M} : \{(a,b),(a,c)\}$, with $a \in L$, $b \in R$, and $c \in R$. Thus, we can confirm $|\mathcal{M}| = 2$, thus $k = 2$. However, no matter what algorithm we apply to find our answer, our flow will either be $a \to b$, or $a \to c$, where $c(a,b) = 1$ and $c(a,c) = 1$. Therefore, we will always get $\text{val}(f) = 1 < 2 = k = |\mathcal{M}|$.

$\square$

(b) Let f' be a feasible flow of $\mathcal{N}$ where for each edge $(u,v) \in E(H)$, $f'(u,v)$ is an integer. Suppose that $\text{val}(f') = k$. Does the existence of f' imply that there is a matching $\mathcal{M}$ of size k in G? That is, can we necessarily recover a matching of size k of G from f'? If so, explain how to construct such a matching $\mathcal{M}$ from f'. If not, give a counterexample.

Answer:

Yes. We note that according to Figure 1 and part (a), $\mathcal{M} : \{(a,b),(a,c)\}$, with $a \in L$, $b \in R$, and $c \in R$. Thus, we can confirm $|\mathcal{M}| = 2$, thus $k = 2$. Now, if we apply a Ford-Fulkerson algorithm with Minimum-Capacity cut of $(f^*) = c(X,Y)$ on f', we get that $X : \{s,a\}$ and $Y : \{b,c,t\}$. Thus, $c(X,Y) = c(a,b) + c(a,c) = 1 + 1 = 2$. Since $MaximumFlow = MinimumCut$, we have it that $val(f') = |\mathcal{M}| = 2 = k$. Once we transform H back to G by removing s and t we will have our bipartite graph, if we sum the edges connecting L and R we will get $|\mathcal{M}|$, which is defined as $\mathcal{M} : \{(a,b),(a,c)\}$.

$\square$