STAT 3400
HW-4

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Oct/11th/2021

- Question # 1:

a) $\sum_{i=1}^{n}(X_i - \bar{X}) = 0 \Rightarrow \sum_{i=1}^{n}(x_i) - \sum_{i=1}^{n}(\bar{x}) = \sum_{i=1}^{n} x_i - \bar{x}\sum_{i=1}^{n}(1) = \sum_{i=1}^{n} X_i - \bar{X}(n)$

$= \sum_{i=1}^{n} X_i - n\bar{X} = \Rightarrow \bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \Rightarrow n\bar{x} = \sum_{i=1}^{n} x_i \Rightarrow = n\bar{x} - n\bar{x} = \underline{0}$ ✓

b) $\sum_{i=1}^{n}(x_i - \bar{x})^2 = \sum_{i=1}^{n} x_i^2 - n\bar{x}^2 \Rightarrow$ FOIL $\Rightarrow \sum_{i=1}^{n}(x_i - \bar{x})^2 = \sum_{i=1}^{n}(x_i^2 - 2x_i\bar{x} + \bar{x}^2)$

$= \sum_{i=1}^{n} x_i^2 - \sum_{i=1}^{n} 2x_i\bar{x} + \sum_{i=1}^{n}\bar{x} = \sum_{i=1}^{n} x_i^2 - 2\bar{x}\sum_{i=1}^{n} x_i + n\bar{x}^2 = \sum_{i=1}^{n} x_i^2 - 2\bar{x}n\bar{x} + n\bar{x}^2$

$= \sum_{i=1}^{n} x_i^2 - \cancel{2}n\bar{x}^2 + \cancel{n\bar{x}^2} = \underline{\sum_{i=1}^{n} x_i^2 - n\bar{x}^2}$ ✓

c) $\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y}) = \sum_{i=1}^{n} y_i x_i - n\bar{y}\bar{x} \Rightarrow \sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y}) = \sum_{i=1}^{n}(x_i - \bar{x})y_i - \sum_{i=1}^{n}(x_i - \bar{x})\bar{y}$

$= \sum_{i=1}^{n}(y_i x_i - y_i\bar{x}) - \sum_{i=1}^{n}(\bar{y}x_i - \bar{y}\bar{x}) = \sum_{i=1}^{n} y_i x_i - \sum_{i=1}^{n} y_i\bar{x} - \sum_{i=1}^{n}\bar{y}x_i + \sum_{i=1}^{n}\bar{y}\bar{x}$

$= \sum_{i=1}^{n} y_i x_i - n\bar{x}\sum_{i=1}^{n} y_i - n\bar{y}\sum_{i=1}^{n} x_i + n\bar{y}\bar{x} = \sum_{i=1}^{n} y_i x_i - n\bar{x}\bar{y} - \cancel{n\bar{y}}\bar{x} + \cancel{n\bar{y}}\bar{x}$

$= \underline{\sum_{i=1}^{n} y_i x_i - n\bar{y}\bar{x}}$ ✓

d) $\sum_{i=1}^{n}(\bar{x}\bar{y} - y_i\bar{x}) = 0 \Rightarrow \sum_{i=1}^{n}(\bar{x}\bar{y} - y_i\bar{x}) = \sum_{i=1}^{n}\bar{x}\bar{y} - \sum_{i=1}^{n} y_i\bar{x} = n\bar{x}\bar{y} - n\bar{x}\sum_{i=1}^{n} y_i = n\bar{x}\bar{y} - n\bar{x}\bar{y} = \underline{0}$ ✓

- Question # 2:

a) $f(\beta_0,\beta_1) = \sum_{i=1}^{n}\left(y_i - [\beta_0 + \beta_1 x_1]\right)^2$

- $\frac{d}{d\beta_0}(\beta_0,\beta_1) = \frac{d}{d\beta_0}\sum_{i=1}^{n}(y_i-\beta_0-\beta_1 x_i)^2 = 0 \Rightarrow 2\left[\sum_{i=1}^{n}(y_i-\beta_0-\beta_1 x_i)\right] = 0 \Rightarrow \sum_{i=1}^{n}(y_i-\beta_0-\beta_1 x_i) = 0 \Rightarrow$

$\Rightarrow \sum_{i=1}^{n} y_i - \sum_{i=1}^{n}\beta_0 - \sum_{i=1}^{n}\beta_1 x_i = 0 \Rightarrow \sum_{i=1}^{n} y_i = \sum_{i=1}^{n}\beta_0 + \sum_{i=1}^{n}\beta_1 x_1 \Rightarrow \boxed{\sum_{i=1}^{n} y_i = n\beta_0 + \beta_1\sum_{i=1}^{n} x_i}$

- $\frac{d}{d\beta_1}(\beta_0,\beta_1) = \frac{d}{d\beta_0}\sum_{i=1}^{n}(y_i-\beta_0-\beta_1 x_i)^2 = 0 \Rightarrow 2\left[\sum_{i=1}^{n}(y_i-\beta_0-\beta_1 x_i)\right]x_i = 0 \Rightarrow \sum_{i=1}^{n} y_i x_i - \sum_{i=1}^{n}\beta_0 x_i - \sum_{i=1}^{n}\beta_1 x_i^2 = 0 \Rightarrow$

$\Rightarrow \sum_{i=1}^{n} y_i x_i = \sum_{i=1}^{n}\beta_0 x_i + \sum_{i=1}^{n}\beta_1 x_i^2 \Rightarrow \boxed{\sum_{i=1}^{n} y_i x_i = \beta_0\sum_{i=1}^{n} x_i + \beta_1\sum_{i=1}^{n} x_i^2}$

b) $\sum_{i=1}^{n} y_i = n\beta_0 + \beta_1\sum_{i=1}^{n} x_i \Rightarrow$ from Part (a) $\Rightarrow \frac{1}{n}\sum_{i=1}^{n} y_i = \frac{n}{n}\beta_0 + \beta_1\left[\frac{1}{n}\sum_{i=1}^{n} x_i\right] \Rightarrow \bar{y} = \beta_0 + \beta_1\bar{x} \Rightarrow \boxed{\beta_0 = \bar{y} - \beta_1\bar{x}}$ ✓

$\sum_{i=1}^{n} y_i x_i = \beta_0\sum_{i=1}^{n} x_i + \beta_1\sum_{i=1}^{n} x_i^2 \Rightarrow$ from Part (a) $\Rightarrow \frac{1}{n}\sum_{i=1}^{n} y_i x_i = \overbrace{(\bar{y} - \beta_1\bar{x})}^{\to \beta_0}\frac{1}{n}\sum_{i=1}^{n} x_i + \beta_1\frac{1}{n}\sum_{i=1}^{n} x_i^2 \Rightarrow$

$$\Rightarrow \boxed{\beta_1 = \frac{\sum_{i=1}^{n}(x_i-\bar{x})(y_i-\bar{y})}{\sum_{i=1}^{n}(x_i-\bar{x})^2}}$$

- Question # 3:

a) Advantage: • Considers all observations, and easy to derive.

Disadvantage: • Can not be used for assesments that are qualitative in nature, such as characteristics of conceptual concepts. i.e. love, luck, etc...

b) Advantage: It is not affected by input values, and can be easily calculated for known distributions.

Disadvantage: It is affected by the fluctuations of sampling.

c) $\hat{\beta}_0 = \bar{y}$ $\Rightarrow y = \beta_0 + \varepsilon \xRightarrow{d.s.} y_i = \beta_0 + \varepsilon_i \Rightarrow \varepsilon_i = y_i - \beta_0 \Rightarrow \varepsilon = \overset{RSS}{\sum_{i=1}^{n} \varepsilon_i^2} = \sum_{i=1}^{n} (y_i - \beta_0)^2$

$\Rightarrow \frac{\partial \varepsilon}{\partial \beta_0} = 0 \Rightarrow \frac{\partial \sum_{i=1}^{n} (y_i - \beta_0)}{\partial \beta_0} = 0 \Rightarrow 2 \sum_{i=1}^{n} (y_i - \beta_0)(1) = 0 \Rightarrow \sum_{i=1}^{n} (y_i - \beta_0) = 0$

$\Rightarrow \sum_{i=1}^{n} y_i - \sum_{i=1}^{n} \beta_0 = 0 \Rightarrow \sum_{i=1}^{n} y_i - n\hat{\beta}_0 = 0 \Rightarrow n\bar{y} - n\hat{\beta}_0 = 0 \Rightarrow n(\bar{y} - \hat{\beta}_0) = 0 \Rightarrow \bar{y} - \hat{\beta}_0 = 0$

$\Rightarrow \underline{\bar{y} = \hat{\beta}_0}$ ✓

- Question # 4: • P := Parameter • C := Constant • I.V. := Independent Variable

a) $Y_i = \beta_0 + \overset{P}{\beta_1} \overset{I.V.}{\log(x_i)}$ ⇒ Yes, the model is linear in parameters.

b) $Y_i = \beta_0 + \overset{P}{e^{\beta_1}} \overset{I.V.}{X_i}$ ⇒ No, the model is not linear in parameters, β_1 is not linearly related.

c) $Y_i = \beta_0 + \overset{P}{\beta_1} \overset{I.V.}{\sin(x_i)}$ ⇒ Yes, the model is linear in parameters.

d) $Y_i = \beta_0 + \sin(\overset{P}{\beta_1} \overset{I.V.}{X_i})$ ⇒ No, the model is not linear in parameters, β_1 is not linearly related.

- Question # 5:

a) $\beta^T X^T Y = Y^T X \beta$ ⇒ $(\beta^T X^T Y)^T = [(\beta^T X^T) Y]^T = Y^T(\beta^T X^T)^T = Y^T (X^T)^T (\beta^T)^T = \underline{Y^T X \beta}$ ✓

b) $Y = Ax$, $\frac{\partial Y}{\partial X} = A$ ⇒ $A = \overset{m \times n}{\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}}$ ⇒ $X = \overset{n \times 1}{\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}}$ ⇒ $y = \overset{m \times n}{\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}} \cdot \overset{n \times 1}{\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}}$ ⇒

⇒ $y = \overset{m \times 1}{\begin{bmatrix} a_{11}x_1 + \cdots + a_{1n}x_n \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n \end{bmatrix}}$ ⇒ $y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \left\} \begin{matrix} y_1 = a_{11}x_1 + \cdots + a_{1n}x_1 \\ \vdots \\ y_n = a_{n1}x_1 + \cdots + a_{mn}x_n \end{matrix} \right\} \frac{\partial x}{\partial y} = \overset{m \times n}{\begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial y_n}{\partial x_1} & \cdots & \frac{\partial y_n}{\partial x_n} \end{bmatrix}} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} = \underline{A}$ ✓