[STAT 4540] HW-2

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Problem 1

$X_t + \frac{5}{4}X_{t-1} = \frac{1}{2}\tilde{Z}_t + \frac{3}{4}\tilde{Z}_{t-1}$

let $Z_t = \frac{1}{2}\tilde{Z}_t$

$\tilde{Z}_t \sim wn(0, \sigma^2) \rightarrow Z_t \sim wn(0, \frac{1}{4}\sigma^2)$

$\rightarrow \frac{1}{2}\tilde{Z}_t + \frac{3}{4}\tilde{Z}_{t-1} = Z_t + \frac{\frac{3}{4}}{2}Z_{t-1} = Z_t + \frac{3}{8}Z_{t-1}$

$\rightarrow X_t + \frac{5}{4}X_{t-1} = Z_t + \frac{3}{8}Z_{t-1}$

Let $\frac{5}{4}X_{t-1} = -(-\frac{5}{4})X_{t-1}$

$\rightarrow X_t - (-\frac{5}{4})X_{t-1} = Z_t + \frac{3}{8}Z_{t-1}$

(a)

Given the below points, we can conclude that there exists a unique and stationary solution for $\{X_t\}$:

1. $X_t - (-\frac{5}{4})X_{t-1} = Z_t + \frac{3}{8}Z_{t-1}$
2. $\phi = -\frac{5}{4}$ and $\theta = \frac{3}{8} \rightarrow \phi + \theta = \frac{3}{8} - \frac{5}{4} = \frac{3-10}{8} = -\frac{7}{8} \neq 0$
3. $\phi = -\frac{5}{4} \rightarrow |\phi| \neq 1$

(b)

Given that $\phi = -\frac{5}{4} \rightarrow |\phi| > 1$, thus the process is non-causal.

(c)

Given that $\theta = \frac{3}{8} \rightarrow |\theta| < 1$, thus the process is invertible.

(d)

$X_t - (-\frac{5}{4})X_{t-1} = Z_t + \frac{3}{8}Z_{t-1}$

$$
\begin{aligned}
X_t &= -(\frac{\frac{3}{8}}{-\frac{5}{4}})Z_t - (\frac{3}{8} + (-\frac{5}{4}))\sum_{j=0}^{1}\frac{1}{\phi^{j+1}}Z_{t+j} \\
&= -(-\frac{40}{12})Z_t - (\frac{3}{8} - \frac{5}{4})\sum_{j=0}^{1}\frac{1}{(\frac{5}{4})^{j+1}}Z_{t+j} \\
&= \frac{40}{12}Z_t - (\frac{3-10}{8}) \cdot (\frac{Z_{t+0}}{(\frac{5}{4})^{0+1}} + \frac{Z_{t+1}}{(\frac{5}{4})^{1+1}}) \\
&= \frac{10}{3}Z_t - (-\frac{7}{8}) \cdot (\frac{Z_t}{(\frac{5}{4})^1} + \frac{Z_{t+1}}{(\frac{5}{4})^2}) \\
&= \frac{10}{3}Z_t + \frac{7}{8} \cdot (\frac{Z_t}{\frac{5}{4}} + \frac{Z_{t+1}}{\frac{25}{26}}) \\
&= \frac{10}{3}Z_t + \frac{7}{8} \cdot (\frac{4}{5}Z_t + \frac{25}{16}Z_{t+1}) \\
&= \frac{10}{3}Z_t + \frac{28}{40}Z_t + \frac{175}{128}Z_{t+1} \\
&= (\frac{100+21}{30})Z_t + \frac{175}{128}Z_{t+1} \\
&= \frac{121}{30}Z_t + \frac{175}{128}Z_{t+1} \\
&= \frac{121}{30} \cdot (\frac{1}{2}\tilde{Z}_t) + \frac{175}{128} \cdot (\frac{1}{2}\tilde{Z}_{t+1}) \\
&= \frac{121}{60}\tilde{Z}_t + \frac{175}{256}\tilde{Z}_{t+1}
\end{aligned}
$$

Thus: $X_t = \frac{121}{60}\tilde{Z}_t + \frac{175}{256}\tilde{Z}_{t+1}$.

Problem 2

(a)

The big picture is filtering the estimated trend from the data through filtering out until we are left with nothing but residuals that are stationary.

(b)

No, the shift by re-indexing should have not affect any significant change.

(c)

The condition should not depend on the indexing choice, however the output would change if we redefined the indexing sequence.

Problem 3

(a)

```
library(lubridate)
```

```
##
## Attaching package: 'lubridate'
```

```
## The following objects are masked from 'package:base':
##
##     date, intersect, setdiff, union
```

```
library(gridExtra)
library(ggplot2)
library(dplyr)
```

```
## Warning: package 'dplyr' was built under R version 4.1.2
```

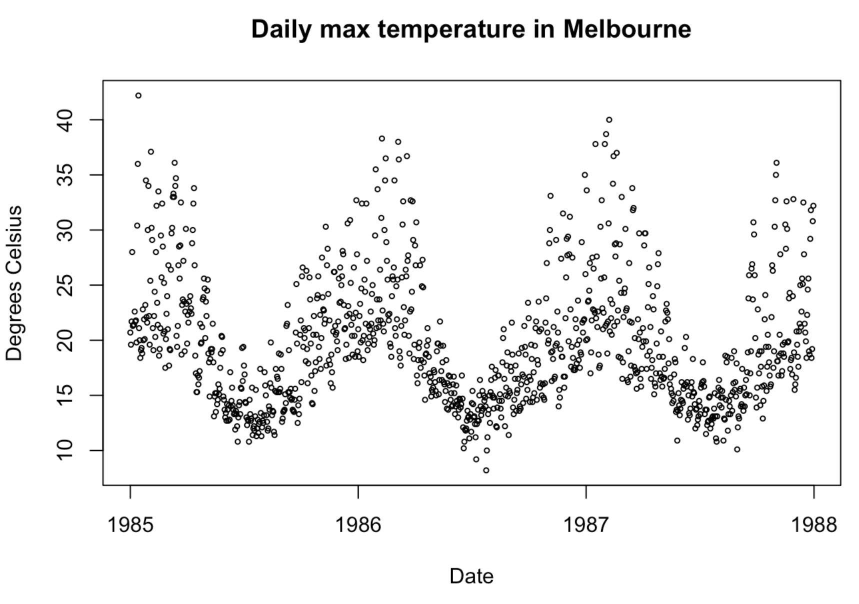
```
##
## Attaching package: 'dplyr'
```

```
## The following object is masked from 'package:gridExtra':
##
##     combine
```

```
## The following objects are masked from 'package:stats':
##
##     filter, lag
```

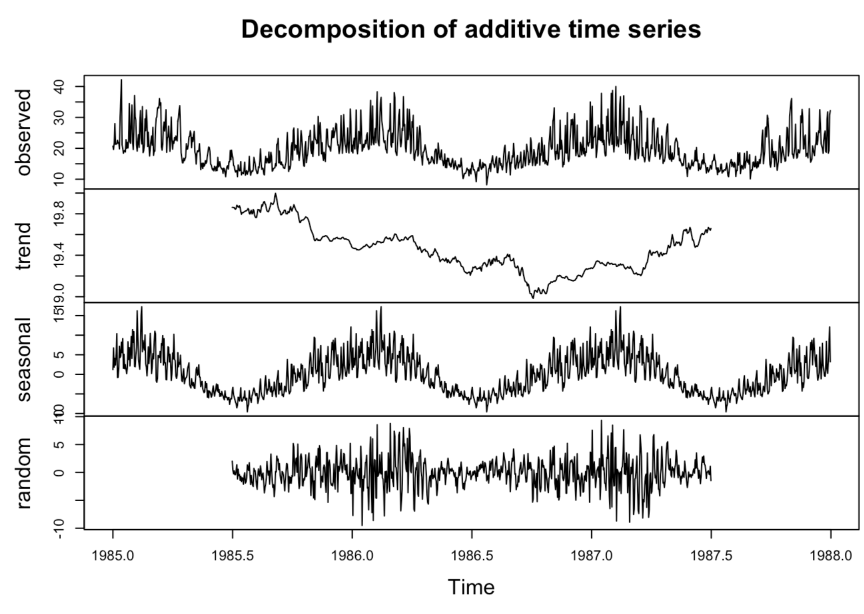
```
## The following objects are masked from 'package:base':
##
##     intersect, setdiff, setequal, union
```

```
data <- load("/Users/Home/Documents/Michael_Ghattas/School/CU_Boulder/2022/Spring 2022/STAT - 4540/HW/2/DailyMaxM
elbourne19851987.RData")
dates = as.Date(dates,format = "%m/%d/%y")
df = data.frame(dates, temp)
plot(dates, temp, xlab = "Date", ylab = "Degrees Celsius", main = "Daily max temperature in Melbourne", pch = 01,
cex = 0.5)
```



(b)

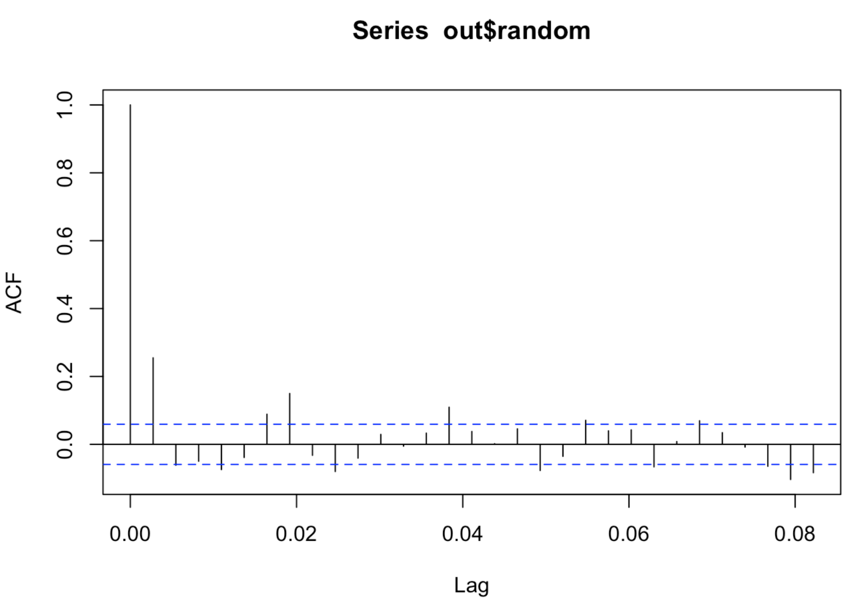
```
df.ts <- ts(data = df$temp, start = c(1985, 1), end = c(1987, 365), frequency = 365)
out <- decompose(df.ts, type = "additive")
plot(out)
```



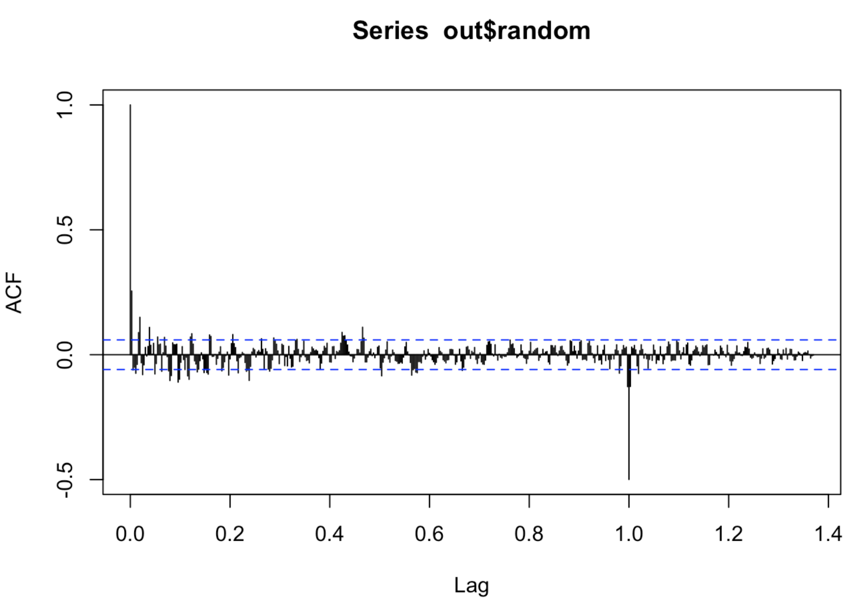
We can see that the decomposition did not work well, as the results are to noisy. An appropriate filter is needed prior to the decompose function.

(c)

```
acf(out$random, na.action = na.pass)
```



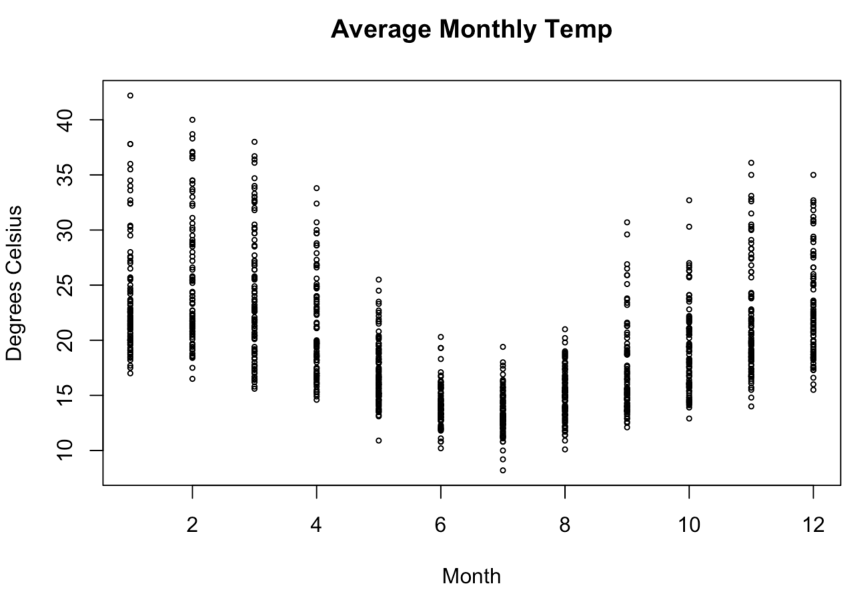

```
acf(out$random, na.action = na.pass, lag.max = 500)
```



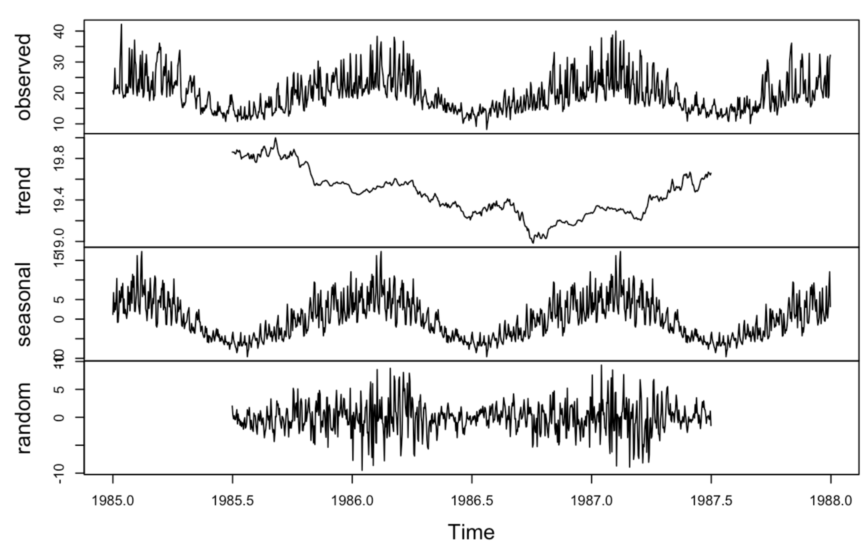
The results present moderate correlation, thus further filtering is needed to be able to identify white noise. A low-degree polynomial based regression model might be helpful.

(d)

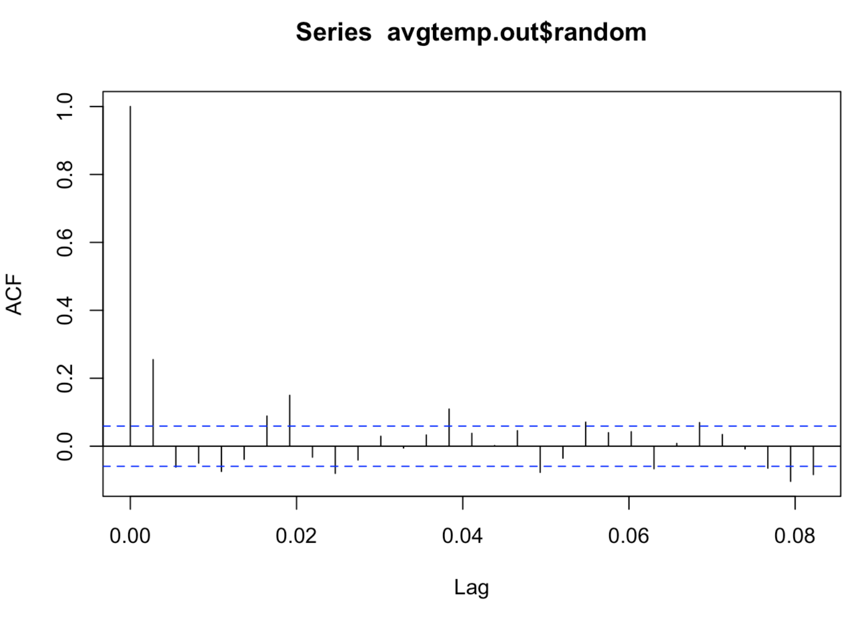
```
df$year <- year(df$dates)
df$month <- month(df$dates)

avgtemp <- aggregate(temp ~ year + month, df, mean)
avgtemp$date = as.Date(paste(avgtemp$year, avgtemp$month, 01), "%y %m %d")
avgtemp = arrange(avgtemp, date)

avgtemp.ts <- ts(data = avgtemp$temp, start = c(1985, 1), end = c(1987, 12), frequency = 12)
avgtemp.out <- decompose(df.ts, type = "additive")

plot(df$month, temp, xlab = "Month", ylab = "Degrees Celsius", main = "Average Monthly Temp", pch = 01, cex = 0.5
)
```




```
plot(avgtemp.out)
```




```
acf(avgtemp.out$random, na.action = na.pass)
```



From the data we can see a correlation between the month and average temperature. We can also see a relationship between the the average temperature of each month in relation to the previous month. Additional filtering is needed to extract trends, seasonality and noise.

Problem 4

(a)

$$
\begin{aligned}
Z_t &= (1-B)(1-B^{12})Y_t \\
&= (1 - B - B^{12} + B^{13})Y_t \\
&= Y_t - Y_{t-1} - Y_{t-12} + Y_{t-13} \\
&= (Y_t - Y_{t-12}) - (Y_{t-1} - Y_{t-13}) \\
&= (a + bt + s_t + X_t - a - b(t-12) - s_{t-12} - X_{t-12}) - (a + b(t-1) + s_{t-1} + X_{t-1} - a - b(t-13) - s_{t-13} - X_{t-13}) \\
&= (bt + X_t - b(t-12) - X_{t-12}) - (b(t-1) + X_{t-1} - b(t-13) - X_{t-13}) \\
&= (X_t - X_{t-12} + 12b) - (X_{t-1} - X_{t-13} + 13b) \\
&= X_t - X_{t-12} + 12b - X_{t-1} + X_{t-13} - 13b) \\
&= X_t - X_{t-1} - X_{t-12} + X_{t-13} - b \\
&= X_t - X_{t-1} - X_{t-12} + X_{t-13}
\end{aligned}
$$

Thus: $Z_t = X_t - X_{t-1} - X_{t-12} + X_{t-13}$ satisfies an AR(p) stationary process.