Homework 2
MATH/STAT 4540/5540 Spr 2022 Time Series

Due date: Friday, Feb 11, before midnight, on Canvas/Gradescope **Instructor**: Prof. Becker
Theme: Basic ARMA processes.

Instructions Collaboration with your fellow students is OK and in fact recommended, although direct copying is not allowed. The internet is allowed for basic tasks (e.g., looking up definitions on wikipedia) but it is not permissible to search for proofs or to *post* requests for help on forums such as `http://math.stackexchange.com/` or to look at solution manuals. Please write down the names of the students that you worked with.

An arbitrary subset of these questions will be graded.

Reading You are responsible for chapters 2.1–2.3 and 3.1 in our textbook (Brockwell and Davis, "Intro to Time Series and Forecasting", 3rd ed).

Problem 1: Let $\{\widetilde{Z}_t\} \sim \mathrm{WN}(0, \widetilde{\sigma}^2)$ and let $\{X_t\}$ satisfy the equation

$$X_t + \frac{5}{4}X_{t-1} = \frac{1}{2}\tilde{Z}_t + \frac{3}{4}\widetilde{Z}_{t-1}.$$

Hint: Is this an ARMA(1,1) process? Or if it isn't, can you make it into one?

a) Does there exist a stationary solution $\{X_t\}$? If so, is it unique? Explain how you reached your conclusion.

b) Is this a causal process? Explain how you reached your conclusion.

c) Is this an invertible process? Explain how you reached your conclusion.

d) Represent X_t in terms of $\{\widetilde{Z}_t\}$, i.e., an explicit equation for X_t in terms of $\widetilde{Z}_t$.

Problem 2: The **linear filter** associated with a sequence of real numbers $(a_j)_{j\in J}$ is the operator $\sum_{j\in J} a_j B^j$, where B is the back shift operator. (Here, B^j with $j < 0$ is interpreted as a forward shift. For example, $B^{-3}x_t = x_{t+3}$.) The set of indices J may be finite or infinite; in the latter case we assume $\sum_{j\in J} |a_j| < \infty$.

It is a fact that the linear filter associated with $(a_j)_{j\in J}$ **passes polynomials of degree** k **without distortion** (i.e., $p_t = \sum_j a_j m_{t-j}$, for all polynomial p_t of degree k) if and only if $\sum_{j\in J} a_j = 1$, and $\sum_{j\in J} j^r a_j = 0$ for $r = 1, \ldots, k$. For example, the sequence $(a_{-1}, a_0, a_1) = (1, 1, 1)/3$ passes all linear functions without distortion; the sequence $(a_{-2}, a_{-1}, a_0, a_1, a_2) = (-1, 4, 3, 4, -1)/9$ passes all polynomials of degree 3 or less without distortion.

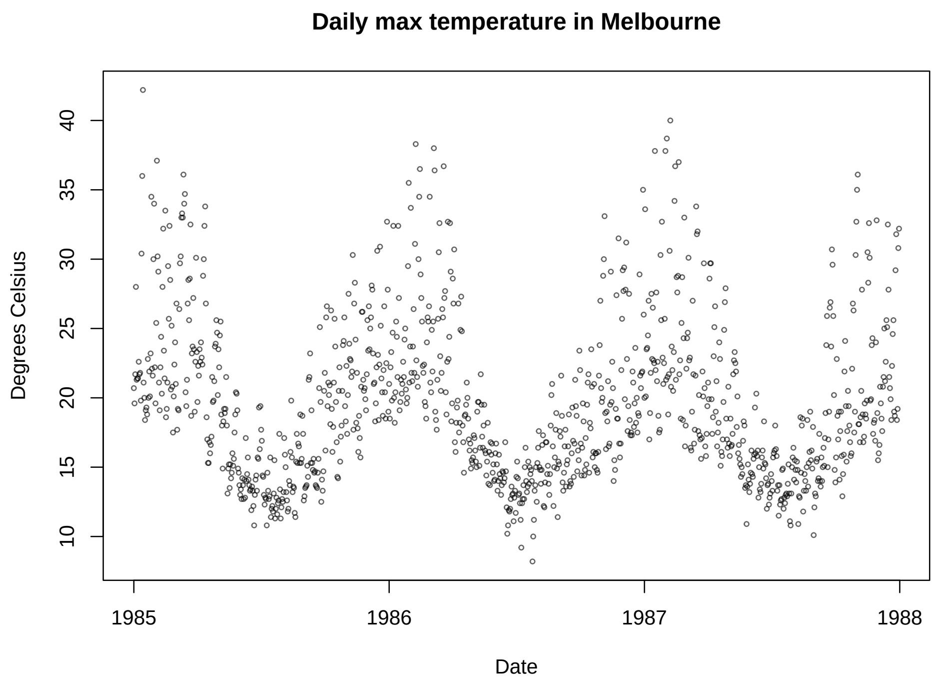
This is a **conceptual question** so it is extra important that even if you are working with other students, you should try this problem on your own first before discussing it with others.

a) If you are given a dataset $(x_t)_{t=1}^n$, how would you use a filter like $(a_{-2}, a_{-1}, a_0, a_1, a_2) = (-1, 4, 3, 4, -1)/9$ to detrend the data? That is, describe in a few sentences what the high-level procedure is. Specifically, which part is the estimate of the *trend*: the filtered data or the residual?

b) Does the condition $\sum_{j\in J} a_j = 1$ depend on how you index the filter (a_j)? For example, if instead of defining $(a_{-1}, a_0, a_1) = (1, 1, 1)/3$ we defined $(a_1, a_2, a_3) = (1, 1, 1)/3$, would anything change?

c) Repeat the previous question but this time considering the condition $\sum_{j\in J} j^r a_j = 0$ for $r = 1, \ldots, k$. Does this condition depend on our indexing choice? Interpret your result (i.e., how would your output sequence change if you redefined $(a_{-1}, a_0, a_1) = (-1, 1, 1)/3$ to be $(a_1, a_2, a_3) = (1, 1, 1)/3$)?

Problem 3: The class website github.com/stephenbeckr/time-series-class/tree/main/Data has a daily maximum temperature dataset for Melbourne, Australia during the years 1985-1987 (`DailyMaxMelbourne19851987.RData` or, if you're using Python or something other than `R`, use `DailyMaxMelbourne19851987.csv`). To actually download it from github, there's either a "Download" or "Raw" button that will let you save the raw file to your computer.



a) Plot the time series for the full three year period, using a suitable format. You should label the plot and include information about the x- and y-axes, i.e., your plot should look similar to the plot above!

   Note: `R` likes dates that are in the "date" format. You can covert a string like `'1/1/85'` to the date format using `as.Date`, e.g., `as.Date('1/1/85',format="%m/%d/%y")`

b) Apply a suitable variant of the classical decomposition algorithm for seasonal models with trend *(any method we've talked about in class is valid)*. Estimate and plot the trend component, the seasonal component and the random component (i.e., the residuals); or for a method that estimates the seasonal and trend components together, you may plot those combined. Present the results graphically, and discuss (e.g., do you think the decomposition worked well?)

c) Compute and plot the sample autocovariance function of the estimated random component. Can the residuals from your decomposition plausibly be modeled as white noise? If not, what model might be appropriate?

d) Repeat steps (a)-(c) for the monthly averages (i.e., produce 3 plots), and write a few sentences comparing to your results for the daily data.

   Hint: There are many ways to do compute the monthly average. You can create a filter by hand, or write a `for` loop, or use `aggregate` if you package the data as a `data.frame` object, or use `as_period` if you package the data as a `tbl_time` object, etc. Any method is valid as long as you get the correct output (in fact, you don't even have to use `R`).

Problem 4: Let $\{X_t\}$ be a stationary process with mean zero and ACVF $\gamma_X(h)$, and let a and b be constants. Let

$$Y_t = a + bt + s(t) + X_t$$

where $s(t)$ is a seasonal component with period 12. Define

$$Z_t = (1 - B)(1 - B^{12})Y_t.$$

Hint for this problem: use results from section 2.2 "Linear Processes" in our Brockwell and Davis textbook.

a) Show that $\{Z_t\}$ is stationary.

b) (**Graduate students only**) Find the autocovariance function γ_Z in terms of γ_X (you may leave this unsimplified), then find a more explicit form for $\gamma_Z(1)$ assuming $\{X_t\}$ is a MA(1) process with parameter $\theta = \frac{1}{2}$ and $\sigma^2 = 1$ (i.e., for the second part of this problem, your answer should be a number).