Homework 4
MATH/STAT 4540/5540 Spr 2022 Time Series

Due date: Friday, March 18, before midnight, on Canvas/Gradescope **Instructor**: Prof. Becker
Theme: Estimation and Multivariate Normal distributions.

Instructions Collaboration with your fellow students is OK and in fact recommended, although direct copying is not allowed. The internet is allowed for basic tasks (e.g., looking up definitions on wikipedia) but it is not permissible to search for proofs or to *post* requests for help on forums such as `http://math.stackexchange.com/` or to look at solution manuals. Please write down the names of the students that you worked with.

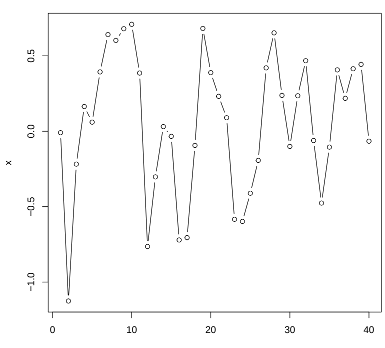
An arbitrary subset of these questions will be graded.

Reading You are responsible for chapters 5.1–5.2 in our textbook (Brockwell and Davis, "Intro to Time Series and Forecasting", 3rd ed).

Homework Formatting As this is mid-semester in a 4/5000 level class, we now expect homework to be nicely formatted. Code should *not* be screen-shots; PDFs should be letter paper sized. It *is* acceptable to work out theory problems by hand (like Problems 2 and 3) and then take a photo or scan your work. However, all computer generated output (figures, code, etc.) should be professional looking. Ask the instructor or TA if you are unsure if your output is well-formatted.

Problem 1:

a) Download the dataset `HW4_problem1_data.RData` (or if using Python, `HW4_problem1_data.csv`) from the class website github.com/stephenbeckr/time-series-class/tree/main/Data. It should have 40 observations and look like this:



Use the Yule-Walker method to find the coefficients ϕ assuming this is an AR(3) process. Return your estimates of the three coefficients (with 4 digits of precision), as well as showing your code. You can use any method you want to solve the linear system (i.e., no need to use Durbin-Levinson).

Hint: in `R`, you may find `acf` useful for estimating sample covariance, and `toeplitz` useful for constructing the covariance matrix.

Do *not* use the `R` function `ar` or `ar.yw` for this subproblem. You can use these in (1c), and you can use them to check your answer, but to get full credit you must provide code that sets up the Yule-Walker estimation.

b) Using the same data and assumptions as in (1a), also estimate the covariance $\widehat{\sigma}^2$ of the underlying white noise process.

c) Simulate your own causal AR(2) process using $\phi_1 = -1$ and $\phi_2 = -0.1$, for 101 time points (and use the first $n = 100$ observations for the estimation). Fit this via Yule-Walker estimation using $p = 1$ (an underestimate), $p = 2$ (correct value), and $p = 99$ (overestimate); you can use builtin `R` estimation tools such as `ar` or `ar.yw` [note: we use `ar` since we know it is an AR process, but in general you'd use `arima` if you thought it was an ARMA model]. Then predict the 101^{th} value (again, you can use builtin `R` tools, like `predict`). To make some conclusions, repeat this simulation/estimation/prediction process 1000 times (each time with a new random simulation), and report (1) the average prediction error for each of the three methods, (2) the standard deviation of the prediction error for each of the three methods, (3) brief comments on the results, and (4) your code.

Problem 2: We've discussed estimating X_{n+h} ($h \geq 1$) from $(X_1, \ldots, X_n)$, but we can apply similar techniques to estimate missing values. In this problem, let (X_t) be a MA(1) process, $X_t = Z_t + \theta Z_{t-1}$, with $Z_t \sim \text{WN}(0, \sigma^2)$. Let $\widehat{X}_3$ denote the best (in terms of mean squared error) linear estimate of X_3 using the data (X_1, X_2, X_4, X_5); see section 2.5.2 in the book for details.

a) Find an expression for $\widehat{X}_3$

b) Using the values $\sigma^2 = 1/3$ and $\theta = \sqrt{2}$, what is the mean squared error of the estimate?

Problem 3: Multivariate normal

a) Let $\boldsymbol{X} \sim \mathcal{N}(0, \Sigma)$ be a n-dimensional multivariate normal (MVN) with nonsingular $n \times n$ covariance matrix Σ. Let $\boldsymbol{Y} \stackrel{\text{def}}{=} \Sigma^{-\frac{1}{2}} \boldsymbol{X}$ [the notation $A^{-\frac{1}{2}}$ means the inverse of the matrix square root; the matrix square root $A^{\frac{1}{2}}$ is the matrix that satisfies $A^{\frac{1}{2}} A^{\frac{1}{2}} = A$, and always exists if A is positive definite]. Prove that the entries of $\boldsymbol{Y}$ are *independent*.

Note: this is called "whitening", and it doesn't guarantee independence unless $\boldsymbol{X}$ is multivariate normal.

Hint: note that affine transformations of a MVN is just another MVN with possibly new parameters. See the class notes or Appendix A.3 in the book to review MVN properties.

b) Let $X_t = Z_t + \theta_1 Z_{t-1} + \theta_2 Z_{t-2}$ be a MA(2) process and assume $(Z_t) \sim \mathcal{N}(0, \sigma^2)$ are iid, so (X_t) is a Gaussian process. What is the joint distribution of $\boldsymbol{X} \stackrel{\text{def}}{=} (X_1, X_2, X_3, X_4)$?

c) (**Graduate students only**) Let $X_t = Z_t + \theta Z_{t-1}$ be a MA(1) process and assume $(Z_t) \sim \mathcal{N}(0, \sigma^2)$ are iid, so (X_t) is a Gaussian process. Let $\sigma^2 = 1/3$ and $\theta = \sqrt{2}$. What is the conditional distribution $X_3 \mid X_1, X_2, X_4, X_5$?