Category Theory

Summer of Science 2025 Project

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Contents

Preface

This report is a collection of things I have learnt and problems that I have solved during my Summer of Science 2025 project on *Category Theory* under Aryaman Maithani.

If some advanced example from *Category Theory in Context* took some time to process, I add a corresponding chapter in the "Crash Courses" part so that it is easier to understand. Nevertheless, it is supposed to be quite concise (and therefore may prove to be insufficient). In my defence, we can't keep reading a new book to understand an example in a category theory book; the program is called Summer of Science and not Sages of Science.

More Details to be added later.

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Part I.

Introduction to Category Theory

1. Introduction to Categories

1.1. What is a Category?

Definition 1 (Category)**.** A *category* consists of mathematical objects and morphisms (aka functions, arrows) between them. Arrows have a direction and point from domain to codomain[1]. For every object $\mathfrak{o}$ there is an identity arrow $\mathrm{Id} : \mathfrak{o} \to \mathfrak{o}$. Arrows are composable with each other associatively, with the caveat that the "resultant" arrow is already present in the category; composition does not create "new" arrows.

$$f \,\mathrm{Id}_{\mathfrak{o}} = \mathrm{Id}_{\mathfrak{o}'} \, f = f \text{ for all objects } \mathfrak{o}, \mathfrak{o}' \text{ and } f : \mathfrak{o} \to \mathfrak{o}'$$

What differentiates category theory from set theory is the focus of category theory on relations between objects[2] rather than trying to describe "more complex" objects in terms of "more atomic" objects.

Remark (Non-Trivial or Trivial?)**.** Unlike most other mathematical objects we encounter, a category with one object need not be as trivial as its mundane siblings. For example, consider $\mathbf{Z}$ with functions $\mathbf{Z} \to \mathbf{Z}$ being the morphsims. An infinite number of morphisms are possible even if there is only one object!

An **endomorphism** is an arrow from an object to itself. By the last remark, you know that endomorphisms on even one object can be quite elaborate. However not all endomorphisms are born equal.

Definition 2 (Isomorphism)**.** We can now describe a purely category theoretic definition of an isomorphism. An arrow $f : \mathfrak{o} \to \mathfrak{o}'$ is an *isomorphism* if there exists an arrow $g : \mathfrak{o}' \to \mathfrak{o}$ satisfying the relations,

$$\begin{aligned} fg &= \mathrm{Id}_{\mathfrak{o}'} \\ gf &= \mathrm{Id}_{\mathfrak{o}} \,. \end{aligned}$$

If an isomorphism exists between $\mathfrak{o}$ and $\mathfrak{o}'$, then they are *isomorphic*, i.e., $\mathfrak{o} \cong \mathfrak{o}'$.

An endomorphism that is also an *isomorphism* is called an **automorphism**.

[1] This terminology may be a misnomer. See Concrete and Abstract. Also note that specifying only the domain and codomain does not necessarily specify the arrow.

[2] [Rie17] pp. 3 asks to consider the object "in tandem". However, I feel it is more of an obligation to consider the object, as they appear only to define what morphisms are (cf. nLab). For instance, just look at the number of axioms morphisms are supposed to satisfy.

Definition 3 (Groupoid)**.** A category where every morphism is an isomorphism is called a groupoid. For example, a groupoid with one object is precisely a group (also, every element of a group $g \in G$ is associated with a canonical isomorphism $x \mapsto gx$; in a particular category though, you may wish to declare a subset of these possible isomorphisms as arrows – there you have a *quotient group*).

Definition 4 (Concrete and Abstract)**.** Time for a surprise! A category where all arrows are functions on the domain (as a set of values) and take values in codomain (as a set of values) is known as *concrete*. The category is said to be *abstract* otherwise.

Definition 5 (Smallness)**.** A category is said to be *locally small* if the collection of morphisms between any given pair of objects, forms a set. The category is *small* if the collection of all arrows inside it forms a set.

Definition 6 (Hom-set)**.** The collection of morphisms between a fixed pair of objects in a category, say $X \to Y$, is denoted by $\mathrm{Hom}(X, Y)$. In a locally small category, this is called the Hom-set (even if it may not be a set of homomorphism).

1.2. Functors

Part II.

Problems and Solutions

TO BE ADDED FROM ANOTHER FILE DEDICATED TO PROBLEMS AND THEIR SOLUTIONS.

Part III.

Crash Courses

1. Group Theory

Normal Subgroup G is a normal subgroup of H, denoted by

$$G \lhd H \iff hgh^{-1} \in G$$

for every $g \in G$ and $h \in H$. The map $x \mapsto hxh^{-1}$ is known as a *conjugation*. They are kernels of some homomorphism. Quotient groups can only be defined for normal subgroups.

Short exact sequences A sequence of groups with homomorphisms between them, i.e.,

$$\cdots \to G_{-1} \to G_0 \to G_1 \to G_2 \to \dots$$

is said to be *exact* if the image of the map $\psi : G_{k-1} \to G_k$ is the kernel of the map $\phi : G_k \to G_{k+1}$ for each k.

Group Extension A group extension of an abelian group H by an abelian group G consists of a group E so that

$$0 \to G \to E \to H \to 0$$

is a short exact sequence. Note that this embeds G as a normal subgroup of E such that $H \approx E/G$. The inclusion $\psi : G \hookrightarrow E$ and surjection $\phi : E \twoheadrightarrow H$ are a part of the group extension.

Prufer p-group It is the Sylow p-subgroup of $\mathbf{Q}/\mathbf{Z}$, i.e., $\mathbf{Z}[1/p]/\mathbf{Z}$. This is so because all elements having their order as a prime power of p satisfy,

$$p^k \cdot \frac{a}{b} \in [0] \implies \frac{a}{b} = \frac{n}{p^k} \text{ for some } n \in \mathbf{Z}.$$

Bibliography

[Rie17] Emily Riehl. *Category theory in context.* Courier Dover Publications, 2017.