Category Theory

Summer of Science 2025 Project

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Part I.

Category Theory in Context

1. Categories, Functors, Natural Transformations

1.1. Abstract and concrete categories

1.1.i

Proof. Observe that $gfh : y \to x$. Also $gfh = (gf)h = h$ and $gfh = g(fh) = g$, thus $g = h$. This means that f has an inverse morphism g such that $fg = 1_x$ and $gf = 1_y$, so f must be an isomorphism. □

1.1.ii

Proof. First we show that the collection of isomorphisms in $\mathcal{C}$ with the objects of $\mathcal{C}$ forms a category. The identity morphism exists in the new category since it is its own inverse and therefore an isomorphism. Associativity is inherited from morphisms on $\mathcal{C}$. To prove composability, say $f : x \to y$ and $g : y \to z$ are isomorphisms, i.e., there exist f_* and g_* such that $ff_* = 1_y$, $f_*f = 1_x$ and $gg_* = 1_z$, $g_*g = 1_y$. Clearly gf is a morphism since so are f and g. We claim that f_*g_* is its inverse morphism, which is trivial to verify. Hence gf is an isomorphism. It is clear that the resultant category is a groupoid so let us prove its maximality. Suppose to the contrary that there is a larger groupoid $\mathcal{C}'$ containing it. Since they have the same objects, $\mathcal{C}'$ must contain an isomorphism not in $\mathcal{C}$. This is a contradiction because $\mathcal{C}$ contains all the isomorphisms between its objects. □

1.1.iii

Proof. Associativity follows in either case because the maps in $\mathcal{C}$ are associative.

(i) First note that there is an identity morphism for every $f : c \to x$ because we can take $g = f$. To show composability, let $h_1 : a \to b$ be from $f : c \to a$ to $g : c \to b$ and $h_2 : b \to d$ be from $g : c \to b$ to $h : c \to d$. Note that $g = h_1 f$ and $h = h_2 g$ implies $h = h_2(h_1 f) = (h_2 h_1)f$ and so the diagram with $h_2 h_1$ also commutes.

(ii) Identity morphism for f is found by taking $g = f$. To show composability, proceed as before $f = gh_1 = (hh_2)h_1 = h(h_2 h_1)$.

□

1.2. Duality

Part II.

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