

LME-MPM applied to quasi-brittle fracture.

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Abstract

The objective of this work is to introduce an alternative technique to address the fracture process of brittle and quasi-brittle materials under the Material Point Method (MPM) framework. With this purpose the eigensoftening algorithm, developed originally for the Optimal Transportation Meshfree (OTM) approximation scheme, is extended to the MPM with the aim of presenting a suitable alternative to the existing fracture algorithms developed for the MPM. The good fitting in the predictions made by the eigensoftening algorithm against both analytical and experimental results proofs the well performance of the method when challenging applications are to modeled.

Keywords: Quasi brittle fracture, Local-Max-Ent approximation, Material Point Method, Solid Dynamics

1. Introduction

The failure mechanisms of quasi-brittle materials and the crack propagation have been studied from different fields such as materials science, structural engineering, earth science and petroleum engineering . The prediction of crack propagation has been approached with different numerical techniques. The Finite Element Method (FEM) is widely used in industrial application. However, the presence of cracks is a violation of the continuity requirement of the FEM. To overcome this issue, numerous numerical artifacts have been proposed with the aim of reproducing such a complex behaviour. These techniques vary from the employment of cohesive approaches [1, 2], or to the gradual insertion of cohesive elements [3, 4, 5] at solid elements boundaries. To alleviate these shortcomings of the FEM, the Extended Finite Element Method (XFEM) in conjunction with level set

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technique was introduced by Belytschko *et al.* [6] to handle arbitrary crack paths level set. Some relevant contributions to the XFEM are the modeling micro-scale crack propagation [7], the introduction of a cohesive zone model [8], and more recently improving it capability to predict the crack path in 3D cases [9].

The simulation of fracture propagation in a more accurate and effective way can be considered as one of the ongoing goals of the development of novel spatial discretization methods like meshfree techniques. Some examples of it are the Material Point Method (MPM) [10, 11, 12, 13], the Element-Free Galerkin Method (EFGM) [14, 15, 16, 17], the Smoothed Particle Hydrodynamics (SPH) [18, 19], the Optimal Transportation Meshfree (OTM) [20, 21, 22, 23] or Peridynamics [24, 25] among others.

Concerning MPM, fracture has been treated numerically through two different ways. On the one hand, the “CRAcks with Material Points (CRAMP)” is proposed [12, 26]. The basis of this methodology consist of the removal of the restriction of the single-valued velocity field close to the crack through, at least, two sets of nodes. In this method, different labels are assigned to the material points and nodes to distinguish if they are in the same side of the crack or not. Under this approach, crack surface is described with line segments in 2D and triangle patches in 3D cases. The chosen criteria for crack propagation is based on such parameters as energy release rate analyzed by Tan & Nairn (2002)[27], and the stress intensity factor or the J-integral discussed by Guo & Nairn (2004)[28]. On the other hand, the foundation of the second approach is the introduction of failed material points to describe the crack evolution. In these methods, the formation of failed points describes the nucleation of cracks, and thereafter its propagation and branching. Consequently, the position of the crack does not need to be explicitly stated. This fact represents significant advantages over the “CRAMP”. Under this approach, the prediction of failure evolution is computed with a decohesion model, which has been discussed by Chen *et al.*[13] and Schreyer *et al.*[10]. Successful simulations are provided within the framework of the fracture of brittle materials by Chen *et al.* [11, 29] and Sulsky & Schreyer [30].

Similar to this approach, Schmidt *et al.* [31] introduced the concept of eigenfracture, where crack sets are approximated by means of eigen-deformations, which enable the material to develop displacement jumps at no cost of local elastic energy. Later, the eigenerosion approach to brittle fracture was carried out by Pandolfi *et al.* [32, 22]. In this technique, the concept of “erosion” of the material point is depicted, being the material point able to remain intact or become completely failed or eroded, when no loading capacity is

55 **bore**. This method has been successfully applied to simulate high complex phenomena such as dynamic fragmentation of metals [23]. Recently, Zhang *et al.* [33] adopted the eigenerosion to resolve the dynamic fracture of brittle materials in the MPM framework. Nonetheless, when quasi-brittle materials are simulated through this approach, the results exhibit an overestimation 60 of tensile stress and strain peaks. Furthermore, since the work of Zhang *et al.* [33] is made with conventional MPM/GIMP, stress oscillations are to be faced.

65 To overcome the limitations observed in the EigenMPM [33], the present research proposes the eigensoftening algorithm developed by Navas *et al.* [34, 35] for the OTM framework and engineered for quasi-brittle materials. Inspired in the concept of the crack band model [36], since energy dissipation is performed through the softened (or failed) volume, this methodology is able to capture the gradual rather than abrupt dissipation of the fracture 70 energy. Moreover, in order to mitigate stress oscillations as well as to deal with tensile instabilities, such those that occurs when particles crosses an element boundary, the work of Molinos *et al.* [37] is followed. The traditional MPM technique is enhanced through the Local Maximum-Entropy (LME) approximation technique [38] for the spatial discretization as well as 75 a explicitly MPM Newmark Predictor-Corrector (NPC) scheme is derived for the time discretization.

76 The paper is structured as follows. First meshfree methodology, eigenerosion and eigensoftening algorithms are presented in Section 2. Then, both 80 approaches are compared and verified by means of comparisons with analytical and experimental results in Section 3. Finally, relevant conclusions are exposed in Section 4.

2. The meshfree methodology

85 The aim of this section is to describe and introduce some special techniques required to face the fracture problem under the MPM framework. In consequence, this section is structured as follows: first, in 2.1, the Newmark Predictor-Corrector (NPC) algorithm for the MPM will be exposed; next the LME approximants are introduced in 2.2 as an accurate alternative technique to interpolate data between particles and nodes, and finally fracture 90 algorithms based on the eigendeformation concept are presented in 2.3.

2.1. The MPM time integration : A Newmark Predictor-Corrector scheme

The MPM [39] is a meshfree Lagrangian-Eulerian method where moving material points, henceforth in this research, particles (\square_p) will carry on all the physical information of the local state (σ_p, ε_p) and a set of fixed background nodes (\square_I) will be introduced to compute the balance of momentum equation. Since the MPM possesses the advantages of both Lagrangian and Eulerian descriptions, no element distortion takes place in the MPM. Therefore, it is an appropriate and efficient method to solve problems with moving discontinuities such as fracture evolution.

Without loosing generality, the MPM algorithm can be described with three main steps: (i) a variational recovery process, where particle data is projected to the grid nodes, (ii) an Eulerian step, where balance of momentum equation is expressed as a nodal equilibrium equation in a FEM-like procedure, and finally (iii) a Lagrangian advection of the particles. In the present research, an explicit predictor-corrector time integration scheme is adopted. The purpose of this choice is motivated due to its proved robustness and stability for dynamic computations. In the first stage, the nodal velocity predictor \vec{v}_I^{k+1} is computed through nodal velocities \vec{v}_I and accelerations \vec{a}_I and evaluated in the k time step,

$$\vec{v}_I^{k+1} = \frac{N_{Ip}^k m_p (\vec{v}_p^k + (1 - \gamma) \Delta t \vec{a}_p^k)}{\mathbf{m}_I^{k+1}} . \quad (1)$$

Where γ is an user-defined parameter (typically adopted as 0.5), \mathbf{m}_I represents the lumped mass matrix, and N_{Ip}^k is the nodal contribution of the shape function evaluated in the particle coordinates \vec{x}_p^k . This way of computing the nodal predictor is numerically stable as well as minimises the computational effort. Once nodal velocities are obtained, the essential boundary conditions are imposed. After that, an Eulerian phase is computed in the set of nodes in a FEM-like way, where nodal forces \vec{f}_I^{k+1} are computed through the equilibrium equation. Next the nodal velocities are corrected in a *corrector* stage:

$$\vec{v}_I^{k+1} = \vec{v}_I^{pred} + \gamma \Delta t \frac{\vec{f}_I^{k+1}}{\mathbf{m}_I^{k+1}} . \quad (2)$$

Finally, in order to update the particles, those are advected in the Lagrangian stage as

$$\vec{a}_p^{k+1} = \frac{N_{Ip}^k \vec{f}_I^k}{\mathbf{m}_I^k}, \quad (3)$$

$$\vec{v}_p^{k+1} = \vec{v}_p^n + \Delta t \frac{N_{Ip}^k \vec{f}_I^k}{\mathbf{m}_I^k}, \quad (4)$$

$$\vec{x}_p^{k+1} = \vec{x}_p^n + \Delta t N_{Ip}^k \vec{v}_I^k + \frac{1}{2} \Delta t^2 \frac{N_{Ip}^k \vec{f}_I^k}{\mathbf{m}_I^k}. \quad (5)$$

The complete pseudo-algorithm it is summarised in Appendix A.

2.2. Spatial interpolation technique : Local Maximum-Entropy approximants

Local Maximum-Entropy (LME) approximation scheme was introduced by Arroyo & Ortiz (2006)[38] as a bridge between finite elements and meshfree methods. The key idea under this interpolation technique is the interpretation of each nodal value N_I as a probability. Related with this definition, two important limits are introduced. First the maximum-entropy (*max-ent*) limit, which ensures a *unbiased statistical inference* based on the nodal data as states the Jayne's[40] principle of *maximum entropy*. And second the Delaunay triangulation that warranties the *least width* shape function support. To reach a compromise between two competing objectives, a Pareto set is defined as,

(LME) $_\beta$ For fixed \vec{x} minimise $f_\beta(\vec{x}_p, N_I) := \beta U(\vec{x}_p, N_I) - H(N_I)$

$$\text{subject to } \begin{cases} N_I \geq 0, \quad I=1, \dots, n \\ \sum_{I=1}^{N_n} N_I = 1 \\ \sum_{I=1}^{N_n} N_I \vec{x}_I = \vec{x} \end{cases}$$

where $H(N_I)$ is the entropy of the system of nodes following the definition given by Shannon (1948) [41], the shape function width is defined as $U(\vec{x}_p, N_I) := \sum_I N_I |\vec{x}_p - \vec{x}_I|^2$, and β is a regularization or *thermalization* parameter such that for Pareto optimal solutions $\beta \in (0, \infty)$. Notice that β has units of $[L]^{-2}$, therefore it can be controlled by adjusting a dimensionless parameter³, $\hat{\gamma} = \beta h^2$, where h is defined as a suitable measure of the nodal

³To avoid confusion with the γ parameter of the NPC, the dimensionless parameter defined in Arroyo & Ortiz [38] as γ will be represent by $\hat{\gamma}$ to preserve as much as possible

spacing. With the restrictions of the Pareto set, the unique solution of the local *max-ent* problem LME $_{\beta}$ is,

$$N_I^*(\vec{x}) = \frac{\exp \left[-\beta |\vec{x} - \vec{x}_I|^2 + \vec{\lambda}^* \cdot (\vec{x} - \vec{x}_I) \right]}{Z(\vec{x}, \vec{\lambda}^*)}, \quad (6)$$

where $Z(\vec{x}, \vec{\lambda}^*)$ is the *partition function* defined as,

$$Z(\vec{x}, \vec{\lambda}) = \sum_{I=1}^{N_n} \exp \left[-\beta |\vec{x} - \vec{x}_I|^2 + \vec{\lambda} \cdot (\vec{x} - \vec{x}_I) \right], \quad (7)$$

and a Lagrange multiplier $\vec{\lambda}^*$ such minimise the function $\log Z(\vec{x}, \vec{\lambda})$. The traditional way to compute the optimal value of $\vec{\lambda}^*$ is through a Newton-Raphson procedure, or in more challenging scenarios by a combination of the Newton-Raphson and the Nelder-Mead Simplex algorithms [35]. Nonetheless, since finite strains are not involved in the present research, Newton-Raphson is an enough efficient method. For an uniform nodal spacing, β can be considered constant, thus first derivatives of the interpolation technique ∇N_I^* can be obtained by evaluating the following expression

$$\nabla N_I^* = -N_I^* (\mathbf{J}^*)^{-1} (\vec{x} - \vec{x}_I) \quad (8)$$

where \mathbf{J} is the Hessian matrix, defined by

$$\mathbf{J}(\vec{x}, \vec{\lambda}, \beta) \equiv \frac{\partial \vec{r}}{\partial \vec{\lambda}}, \quad (9)$$

$$\vec{r}(\vec{x}, \vec{\lambda}, \beta) \equiv \frac{\partial \log Z(\vec{x}, \vec{\lambda})}{\partial \vec{\lambda}} = \sum_I^{N_n} p_I(\vec{x}, \vec{\lambda}, \beta) (\vec{x} - \vec{x}_I). \quad (10)$$

An additional remark concerning the support of the interpolation function is that, in practice, the value of N_I decay exponentially following $\exp(-\beta \vec{r})$. In this sense, a good practice is to truncate it under a tolerance (typically around 10^{-6}) that would ensure a reasonable range of neighbours, see [38] for details. This tolerance defines a limit value of the influence radius to find the neighbour nodes of a given integration point.

2.3. Fracture modelling approach

Within the context of MPM formulation, fracture can be modelled by failing particles according to a suitable failure criterion. When material points

the original notation.

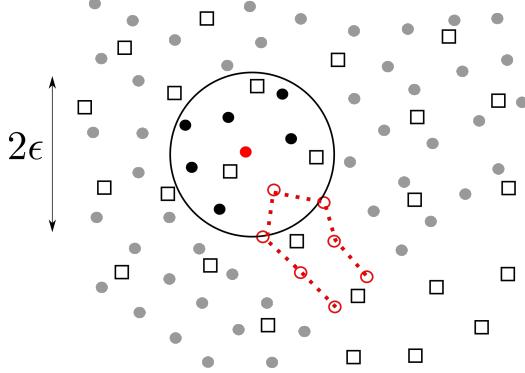


Figure 1: Scheme of a linear cohesive law, where the shaded area is G_f , f_t is the tensile strength, and w_c is the critical opening displacement.

are failed, they are assumed to have null stress tensor. To reproduce this behaviour in the present research, the eigensoftening algorithm is introduced in the MPM framework as an alternative approach to the decohesion model
115 [13, 10]. The eigensoftening concept was originally developed by Navas *et al.*
120 (2017)[34] as an extension for quasi-brittle materials of the eigenerosion proposed by Pandolfi & Ortiz (2012)[32] for fracture of brittle materials. **The evaluation of both approaches [34] against experimental measures show that the eigenerosion algorithm significantly overestimates the tensile stress and the strain peaks, meanwhile eigensoftening captures the forces and crack patterns accurately . Furthermore, the accuracy of this algorithm has been proven for complex fracture patterns such as the ones obtained with fiber reinforces concrete (FRC), [42].**

The key idea behind the eigenerosion algorithm is the computation of the energy-release rate attendant to the failure of material point p ,

$$G_p^{k+1} = \frac{C_\epsilon}{m_p^{k+1}} \sum_{x_q^{k+1} \in B_\epsilon(x_p^{k+1})} m_q W_q^{k+1} \quad (11)$$

$$m_p^{k+1} = \sum_{x_q^{k+1} \in B_\epsilon(x_p^{k+1})} m_q \quad (12)$$

125 where $B_\epsilon(x_p^{k+1})$ is a n -dimensional sphere of radius ϵ centered at x_p^{k+1} . The particles which lie under this sphere are known as the ϵ -neighborhood of the material point (see [32]). This concept is conveniently sketched in Figure 1. Other parameters are the mass of the neighborhood m_p^{k+1} , the current free-energy density per unit mass W_q^{k+1} and the normalizing constant C_ϵ , **which also defines the ϵ -neighborhood size, being $\epsilon = C_\epsilon h$, and h is a measure of**

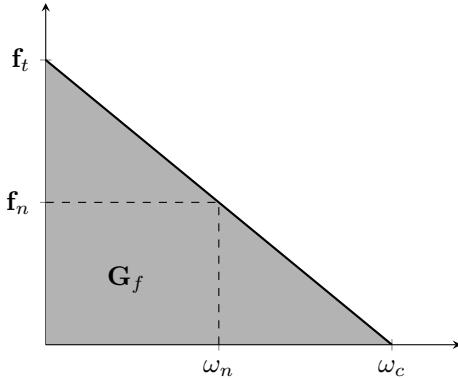


Figure 2: Scheme of a linear cohesive law, where the shaded area is G_f , f_t is the tensile strength, and w_c is the critical opening displacement.

the distance between nodes.

The failure criterion consist of considering that a material point is failed when G_p^{k+1} surpasses the critical energy release rate that measures the material-specific energy, G_F . The convergence of this approach has been analyzed by Schmidt *et al.* (2009)[31], who proved that G_F converges to the Griffith fracture when the discretization size tends to zero. It is necessary to point out that, when a material point overpass the critical energy, its contribution to the internal forces vector is set to zero, but its contribution to the mass matrix is preserved.

140

As can be noticed, the eigenerosion algorithm relies over an energetic failure criterion. Because of this, unrealistic stress concentration (higher than tensile strength) appears in quasi-brittle materials [34]. To overcome this limitation, the aforementioned authors proposed the concept of eigensoftening to take into account the gradual failure in quasi-brittle materials. The idea behind this concept is inspired in the cohesive fracture. This gradual failure criterion is plotted in figure , where a linear decreasing cohesive law is presented to illustrate the concept earlier described. In the picture, the shaded region represents the static fracture energy per unit of area, G_F . Notice how a cohesive crack appears when the maximum tensile strength, f_t , is reached. Once the crack, w_n , reaches the value of the critical crack opening, w_c , a stress-free crack is attained. For the eigensoftening algorithm, a strength criterion for crack initialization was adopted. Particularly the maximum principal stress theory for brittle fracture was considered in [34], this allows to quantify the variation of the averaged strain energy density in

the ϵ -neighborhood of the material point \vec{x}_p^{k+1} as

$$\delta W_{\epsilon,p} = \frac{\partial G_p}{C_\epsilon} = \frac{1}{m_p} \sum_{x_q^{k+1} \in B_\epsilon(x_p^{k+1})} m_q \sigma_{q,I} \delta \varepsilon_q, \quad (13)$$

where $\sigma_{q,I}$ is the maximum principal stress of each material point in the ϵ -neighborhood. At this point, an *effective* strain ε_q is introduced in order to obtain the variation of the local strain energy as $\delta W_q = \sigma_{q,I} \delta \varepsilon_q$. Now, with the assumption that the effective strain of each material point at every time step is constant in the neighborhood of \vec{x}_p^{k+1} , the equation (13) can be simplified to

$$\delta W_{\epsilon,p} = \frac{\delta \epsilon_p}{m_p} \sum_{x_q^{k+1} \in B_\epsilon(x_p^{k+1})} m_q \sigma_{q,I}. \quad (14)$$

Consequently, it is possible to define an equivalent critical stress at the material point \vec{x}_p^{k+1} as

$$\sigma_{\epsilon,p} = \frac{1}{m_p} \sum_{x_q^{k+1} \in B_\epsilon(x_p^{k+1})} m_q \sigma_{q,I}, \quad (15)$$

where m_p is the total mass of the ϵ -neighborhood, defined similarly to the eigenerosion method (Eq. (12)). The equivalent critical stress leads to a definition of the averaged strain energy in terms of the averaged strain as $\delta W_{\epsilon,p} = \sigma_{\epsilon,p} \delta \varepsilon_p$. The softening behaviour is activated once $\sigma_{\epsilon,p}^{k+1}$ surpasses the tensile strength, f_t . This consists of a reduction of the internal forces as,

$$f_I^{int} = \sum_p (1 - \chi_p) \sigma_p^{k+1} \cdot \text{grad}(N_{Ip}) \Omega_p \quad (16)$$

where χ_p and Ω_p are respectively the damage or softening variable and the volume for each material point p . χ_p is any function, taking values between zero (an intact material) and one (completely failed material points), that relates the crack opening and the residual strength. To calculate it, typical shapes of cohesive laws such as linear, bilinear or exponential, may be employed. For the case of a linear softening such the sketched one in the Figure 2.3, χ_p is computed as,

$$1 - \chi = \frac{f_n}{f_t} = 1 - \frac{w_n}{w_c} \rightarrow \chi = \frac{w_n}{w_c}. \quad (17)$$

Analogous to the crack band model presented by Bazant [36], Navas *et al.* [42] [34] introduced a band width parameter h_ϵ in order to relate the crack opening with the strain. Concerning this parameter, a typical value between two and four times the maximum size of the aggregates is adopted in the case of concrete as brittle material. The effective fracture strain $\varepsilon_{\epsilon,f}$ is defined as the difference between the strain at crack initialization, $\varepsilon_1(\bar{x}_p^0)$, and the current strain, $\varepsilon_1(\bar{x}_p^{k+1})$, for a material point p . Also, $\varepsilon_{\epsilon,f}$ can be represented as the current crack opening w_n within the band width h_ϵ . Therefore,

$$\varepsilon_{\epsilon,f} = \varepsilon_1(\bar{x}_p^{k+1}) - \varepsilon_1(\bar{x}_p^0) = \frac{w_n}{h_\epsilon} \quad (18)$$

Introducing (18) in (17), the damage variable can be computed as,

$$\chi = \frac{\varepsilon_{\epsilon,f} h_\epsilon}{w_c}. \quad (19)$$

The function of χ presented in (19) represents a linear softening behaviour. For a general case, the damage variable can be expressed in terms of the following variables,

$$\chi = \chi(\varepsilon_{\epsilon,f}, h_\epsilon, f_t, w_c, G_f) \quad (20)$$

Implementation details can be consulted in Appendix B.

2.4. ϵ -neighbourhood reconstruction : A node-linked method

The construction of the ϵ -neighbourhood is a major issue of the proposed methodology. Since this operation could be extremely demanding, optimal numerical implementation should be employed. In opposite with the Cell-linked method proposed by Allen & Tildesley [43] where the definition of a numerical tolerance is required for auxiliar algorithms like Crossing Number [44] or Correct Even-Odd [45], the proposed methodology adopted herein is linked to the shape function neighbourhood. In order to avoid the aforementioned error prone techniques, we endeavour to exploit the meshfree benefits of the LME approximants to introduce a node-linked method. Due to the global support of this interpolation technique, the classical interpretation of the FEM element typically adopted in MPM simulations is not required anymore. Instead, a linked list with the surrounding nodes for each node is defined at the initial time. Also, each particle has to be associated with the closest node instead with the belonging element as heretofore has been done. Now lets define the concept of a particle *tributary nodes* as the list of nodes close to it among which the information transfers occurs. Once the closest node are defined, the particle posses automatically a list of *tributary*

160 *nodes*. This has two immediate benefits, the first is that shape functions
 with global support as LME are easier to implement, and the second is the
 increase of the efficiency of the ϵ -neighbourhood reconstruction. Regarding
 this last benefit, since the current particle has a *tributary nodes* list assigned
 to it, and consequently each node has a linked list with a set of *tributary*
 165 *particles*⁴ close to it, is possible to define efficiently a search list with those
 particles close to the current particle. In the end, particles out of the area of
 influence, Figure 1, are pop out of the final ϵ -neighbourhood. A final remark
 about this new approach is its ability to reconstruct easily a search list with
 the support of the initial connectivity of the mesh. It provides a suitable
 170 basis for an easy implementation of other interpolation techniques with min-
 imum coding effort. A detailed explanation of the proposed algorithm can
 be found in Appendix C.

3. Cases of study and discussion

175 The proposed approach to overcome the two main shortcomings of Eighen-
 MPM [33] has been evaluated with “three benchmark tests”. The first one
 deals with the presence of stress instabilities provoked by the spatial dis-
 cretization, appearing even when the GIMP shape functions are employed.
 To overcome it, LME approximants are introduced as an alternative to the
 existing interpolation techniques. Its performance in mitigating spurious
 180 stress oscillations under the MPM [46, 37] framework helps to enhance no-
 toriously the quality of the results as it is observed in Section 3.1, where
 both interpolation techniques are compared by carrying out an eigenerosion
 simulation. The second limitation is concerned to its inability to simulate
 185 properly quasi-brittle fracture [42]. It can be solved through the eigensoft-
 ening algorithm described in Section 2.3. A proof of it is exposed in Sections
 3.2 3.3, where experimental results are compared with experimental ones
 as well as eigensoftening computations. In the first case comparing with a
 semicircular bending test and in the second case with a drop-weight impact
 test.

190 3.1. Edge-cracked square panel in mode I

The aim of the problem presented hereinafter is to assess the capability
 of the LME approximants to improve the result *versus* the standard linear
 interpolation. The application consists of a square plate of size $H = 1$ [L]
 containing an initial edge crack of length $0.25 \cdot H$ loaded in a pure mode I by

⁴Notice that the list of the particles close to each node is unique for each node and is updated in each time step.

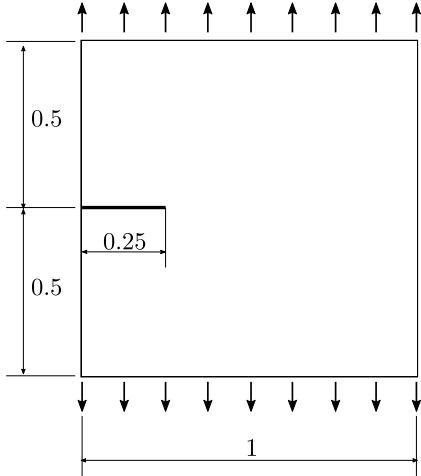


Figure 3: Geometry and boundary condition of the drop-weight impact test.

195 displacement control on the outer flanks of the plate (See Figure 3 for details).
 The constitutive model considered in the numerical experiment is a linear-
 elastic Hookean material, whose Young's modulus $E = 1.06 [M][L]^{-1}[T]^{-2}$,
 the Poisson's ratio $\nu = 0.333$, and critical energy-release rate $G_F = 0.0001$
 $[L]^2[T]^{-2}$. The absence of units is due to the current simulation will not
 200 be validated against experimental results, where scale factor is relevant, but
 will be against an analytical solution provided by Pandolfi & Ortiz [32]. In
 MPM, two different discretization are required. On one hand, a cartesian
 grid of nodes is considered with a nodal spacing value of 0.025. On the other
 hand, the plate will be modeled with a initial layout of four particles per
 205 element and occupying the Gauss quadrature positions.

210 Figure 4 clearly shows the LME $_{\gamma=3.5}$ solution has a reaction peak value
 of $0.03 [M][L][T]^{-2}$ for a imposed displacement $0.015 [L]$ which agrees with
 the analytical solution [32] . It also shows the presence of wiggles in the
 reaction-displacements curve, in both loading and post-failure stages, when
 linear interpolation technique is used. In contrast, LME simulation does not
 exhibit these spurious oscillations and remains linear until the failure. About
 the strength, linear interpolation produces stiffer results than LME. This can
 215 be attributed to imprecision in the stress field due to grid-crossing phenom-
 ena.

216 Additionally, Figure 5 exhibits more clearly the difference between linear
 interpolation and LME, since numerical instability becomes rather signifi-
 cant when particles crosses the boundary of the element in the linear case,
 which is observed at 60 seconds. **Although, apparently, the differences**

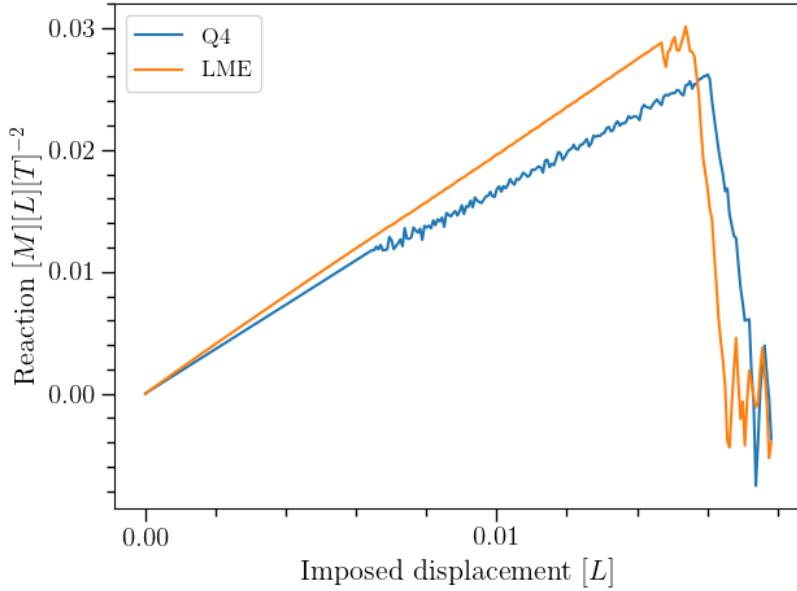


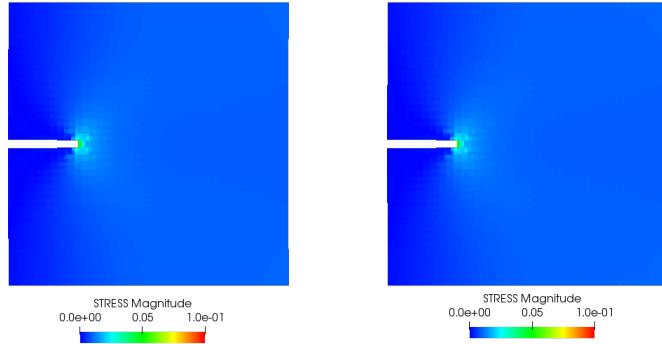
Figure 4: Evolution of the reaction forces plotted *versus* the imposed displacement for linear interpolation technique and the LME approximants.

220 in the obtained peaks are not extremely significant, it can be owing to the
 221 aforementioned hookean material employed [33]. When more sophisticated
 222 constitutive models will be employed, severe inaccuracies may appear in the
 223 stress field that could affect dramatically to the final result. Focusing on
 224 the post failure behavior, it can be seen how LME produces soft stress field
 225 evolution even after breaking. An important consideration regarding the
 226 presence of oscillations once both parts of the panel are separated is observed
 227 in Figure 4. These phenomena should not be attributed to the eigenerosion
 228 algorithm since the fracture process is over. They are due to the dynamic
 229 nature of the solver: once the energy is release dinamically by the crack,
 230 the waves propagate along the domain.

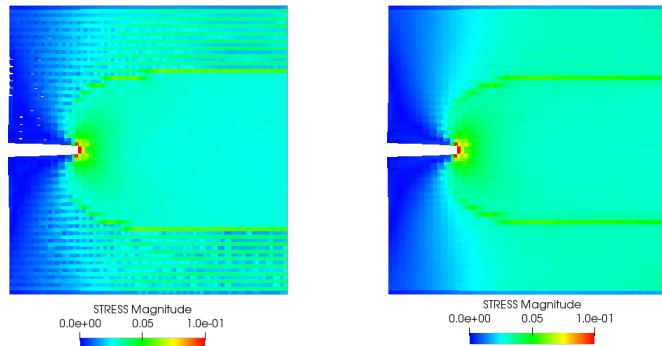
3.2. Semicircular bending test

The second application to be considered is two semicircular bending tests on Johnstone, which have been carried out by Lim *et al.* [47] and numerically reproduced by Wang *et al.* [18] under SPH framework within an implicit
 235 algorithm. The aim of the test is to examine the predictive capability of the
 236 eigensoftening algorithm under different kind of mixed-mode loading conditions.
 237 This capability was previously within a OTM implementation [42].

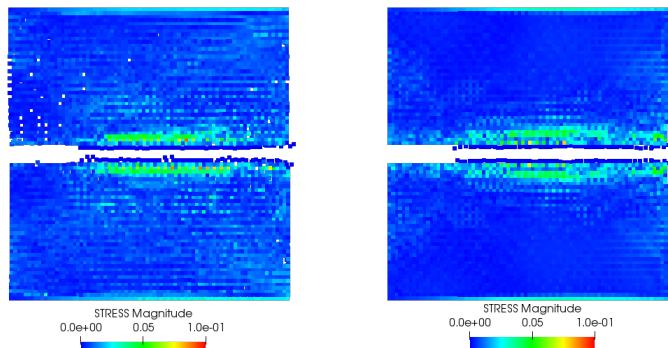
The geometry and boundary conditions are sketched in Figure 6. The experiment consist in two semicircular specimens with a radius of 47.5 mm



(a) $t = 25$



(b) $t = 60$



(c) $t = 100$

Figure 5: Evolution of the stress tensor magnitude for linear interpolation (pictures in the left side), and LME (pictures in the right side). Both simulations performed with an eigenerosion algorithm.

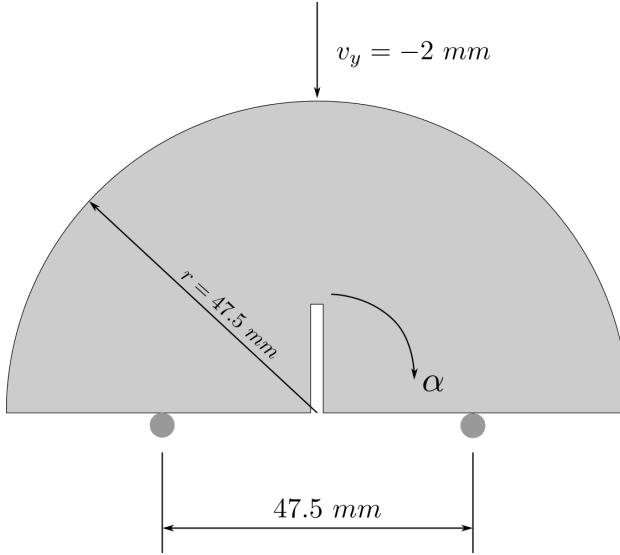


Figure 6: Geometry and boundary condition of the semicircular bending test.

Young's modulus (E)	0.4 GPa
Poisson ratio (ν)	0.25
Density (ρ)	1.54 g/cm^3
Tensile strength (f_t)	0.6 MPa
Critical opening displacement (w_c)	0.015 mm

Table 1: Material properties of Johnstone.

and a thickness of 20 mm are supported by two rollers at a span of 47.5 mm and loaded by another roller on its mid-span. Notches with a length of 16.6 mm are created with different notch angles of 0 and 60 with respect to the vertical axis to investigate the influences of the notch angles on the peak load 8. The material properties of Johnstone are listed in Table 1. In the tests, particles were uniformly distributed with an almost constant separation of 1 mm between them, resulting in a total number of 4672 MPM particles for generating the specimen. Additionally a regular cartesian background grid of 1917 nodes with 1.5 mm of nodal distance was generated. The velocities of the top boundary particles are $v_x = 0$, $v_y = 2 \text{ mm/s}$, whereas the velocities of the bottom boundary particles are $v_y = 0 \text{ mm/s}$. To reduce dynamic effects due to a sudden loading, it was preformed within a soft loading ramp. Finally, the model was calibrated with a regularization parameter $C_\epsilon = 1.7$ and a bandwidth value $h_\epsilon = 15$.

Figure 7 shows the comparison of mixed-mode fracture envelope between experiments and MPM simulations for different notch inclination angles $\alpha =$

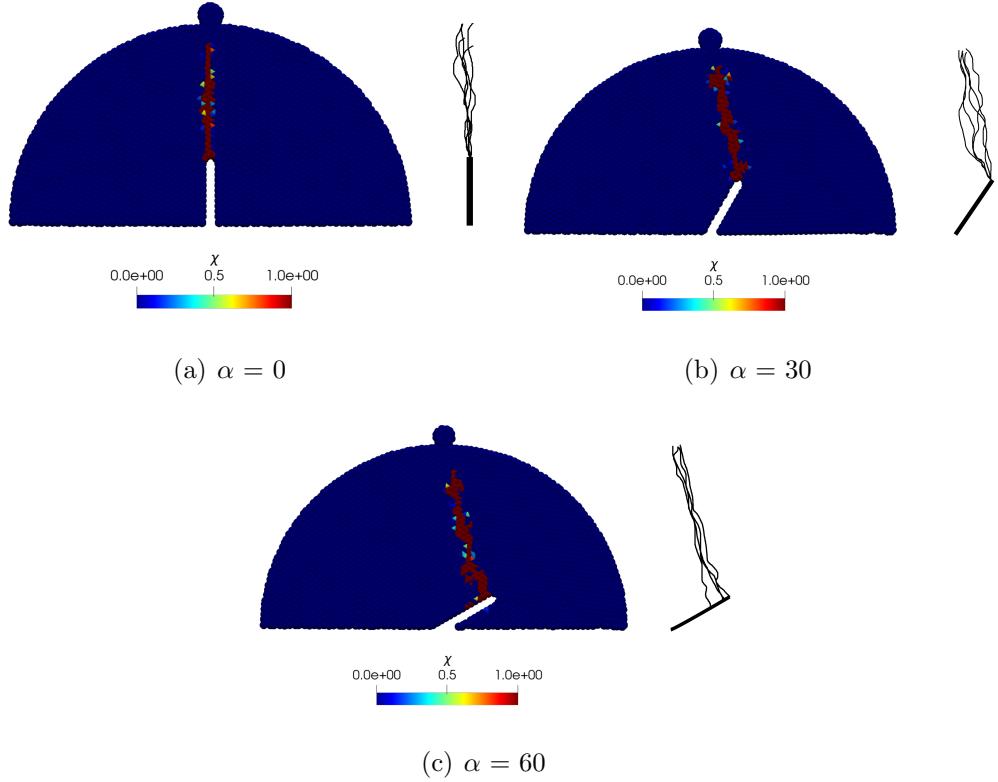


Figure 7: Crack pattern for different notch inclination angle, and comparison with the experimental results [47].

0° , 30° and 60° . The final fracture pattern is a vertical straight line when $\alpha = 0^\circ$ (pure mode I). When $\alpha > 0^\circ$, the sample is subjected to mixed-mode. For simulation cases 0° and 30° , the fractures initiate from the notch tip and then propagate towards upper loading point. Experiments show that a few fractures tend to initiate behind notch tip when the notch angle is beyond 50° , figure 7 demonstrate that eigensoftening combined with MPM is able to reproduce this behavior. Therefore, the simulation results predict successfully the fracture envelope in experiments and demonstrate the capability of the eigensoftening-MPM model for capturing the fracture development under various complex loading conditions. Additionally, 8 compare the peak load between experiments and simulations for the notch inclination angles 0° , 30° and 60° . This figure proofs how the eigensoftening-MPM simulation can well capture the increasing trend of the peak load with the increasing notch angles in the experiment.

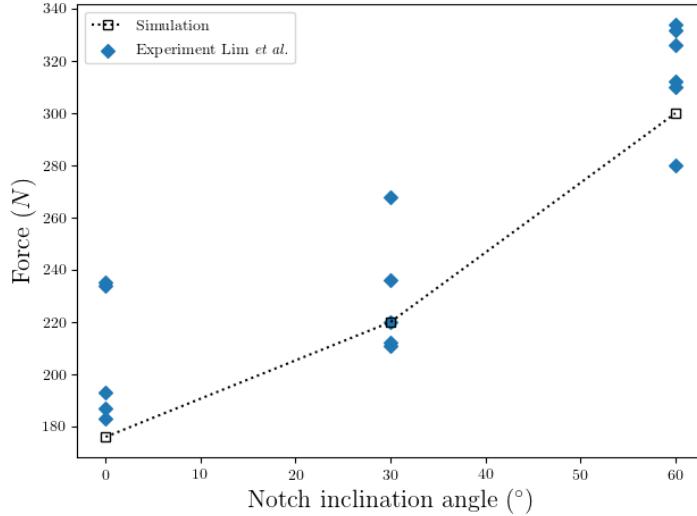


Figure 8: Numerical results compared with the experimental results [47].

270 3.3. Drop-weight impact test

This section is devoted to proof the accuracy of the eigensoftening algorithm within the MPM framework when the behavior of quasi-brittle materials under dynamic loading is simulated. One of the most interesting examples of this loading case is the three-point bending test on a concrete 275 notched beam when it is conducted under impact loading. Indeed, this experimental test is appropriate to be reproduced numerically with the explicit solver proposed in this paper since fracture occurs in a period of milliseconds and waves propagate fast along the high strength materials involved in the aforementioned test. Accordingly, small time steps are required to achieve 280 a numerically stable simulation.

An interesting case of this experimental test was reported by Zhang *et al.* 285 [48, 49]. As in the aforementioned study, an impact hammer of 120.6 kg has been employed in the proposed simulations to drop it at an impact speed of 2640 mm/s. The beam dimensions were 100 mm x 100 mm (B x D) in cross section, and 420 mm in total lenght (L). The initial notch-depth ratio was approximately 0.5, and the span, S, was fixed at 300 mm during the tests (see Figure 9). The material adopted for the simulation was characterised by 290 Navas *et al.* [34], being the material properties, such as the material density, ρ , the compressive strenght, f_c , the tensile strenght f_t , the specific fracture energy, G_F , and the elastic modulus E, provided in table 2.

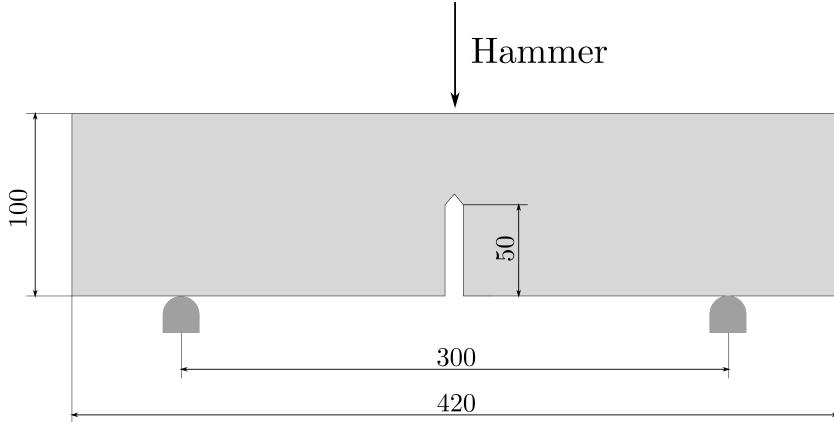


Figure 9: Geometry and boundary condition of the drop-weight impact test.

	ρ (kg/m ³)	f_c (MPa)	f_t (MPa)	G_F (N/m)	E (GPa)	d_{max} (mm)
Value	2368	102.7	5.4	141	31	12

Table 2: Mechanical properties of the high strength concrete.

For the present research, a 2D setting of the experiment has been adopted. Both the hammer and the concrete beam are explicitly represented in order to properly capture the impact. Several levels of discretization were required to assess the objectiveness of the obtained results. The results here exposed are from a discretization of 21576 material points and 3943 nodes. To achieve better results, an unstructured grid layout has been adopted focusing the minimum nodal spacing, 1.2 mm, in the middle of the beam and in the impact surface of the hammer. For this simulation, $\hat{\gamma}$ was fixed to 1.8, this low value of the regularization parameter smooth high stress overshoots after the impact. Here the stabilization properties of the LME described in [37] becomes crucial to achieve non-spurious results. In fact, standard linear element were tested with unsatisfactory results since grid-crossing occurs in the hammer head producing non realistic forces.

First, the reaction and impact forces are validated against their experimental counterparts. Since the impact forces applied by the hammer and the reaction forces at the two supports were experimentally measured, they are compared with numerical ones in Figure 10. Note that the general trend of the impact forces was correctly captured. Observe that in experimental solution the delay between impact peak and reactions peak does not agree with the numerical ones. A feasible explanation to this unexpected result

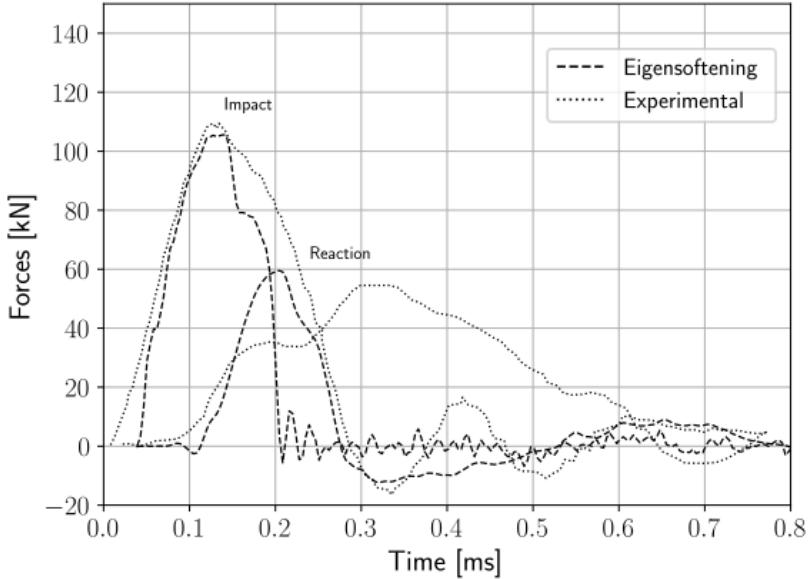


Figure 10: Evolution of the reaction forces plotted *versus* the imposed displacement for linear interpolation technique and the LME approximants.

can due to a over-rigidity in the supports. Although the traditional MPM implementation allows collision of two bodies, inaccurate energy conservation is obtained without the implementation of a conservative contact algorithm to reproduce the hammer impact in the beam. Despite the observed discrepancies, this is out of the scope of this document, although future research of authors will overcome this limitation. Similarly, results of support forces will be more realistic when modelled by a suitable contact algorithm instead of avoiding any vertical displacement . Navas *et al.* (2017)[34] demonstrate the importance of the stiffness of the support in order to obtain accurate results of the reaction forces. Interesting contact algorithms within the MPM methodology are available in the literature [53, 54].

Finally, longitudinal stress distribution and crack propagation are depicted in Figure 11. Stress evolution agrees with the results obtained by [34] under the OTM framework. Two additional considerations about this results are concerning the interpolation technique. On the one hand, in the opposite to the uGIMP shape function, LME allows unstructured grids. This is a remarkable benefit for this kind of simulations since the majority of the computational effort is clearly localised in a tiny region of the domain. On the other hand, there is the possibility to reduce the shape function support to avoid a material point in one side of the crack is affected by those in the other side of the crack.

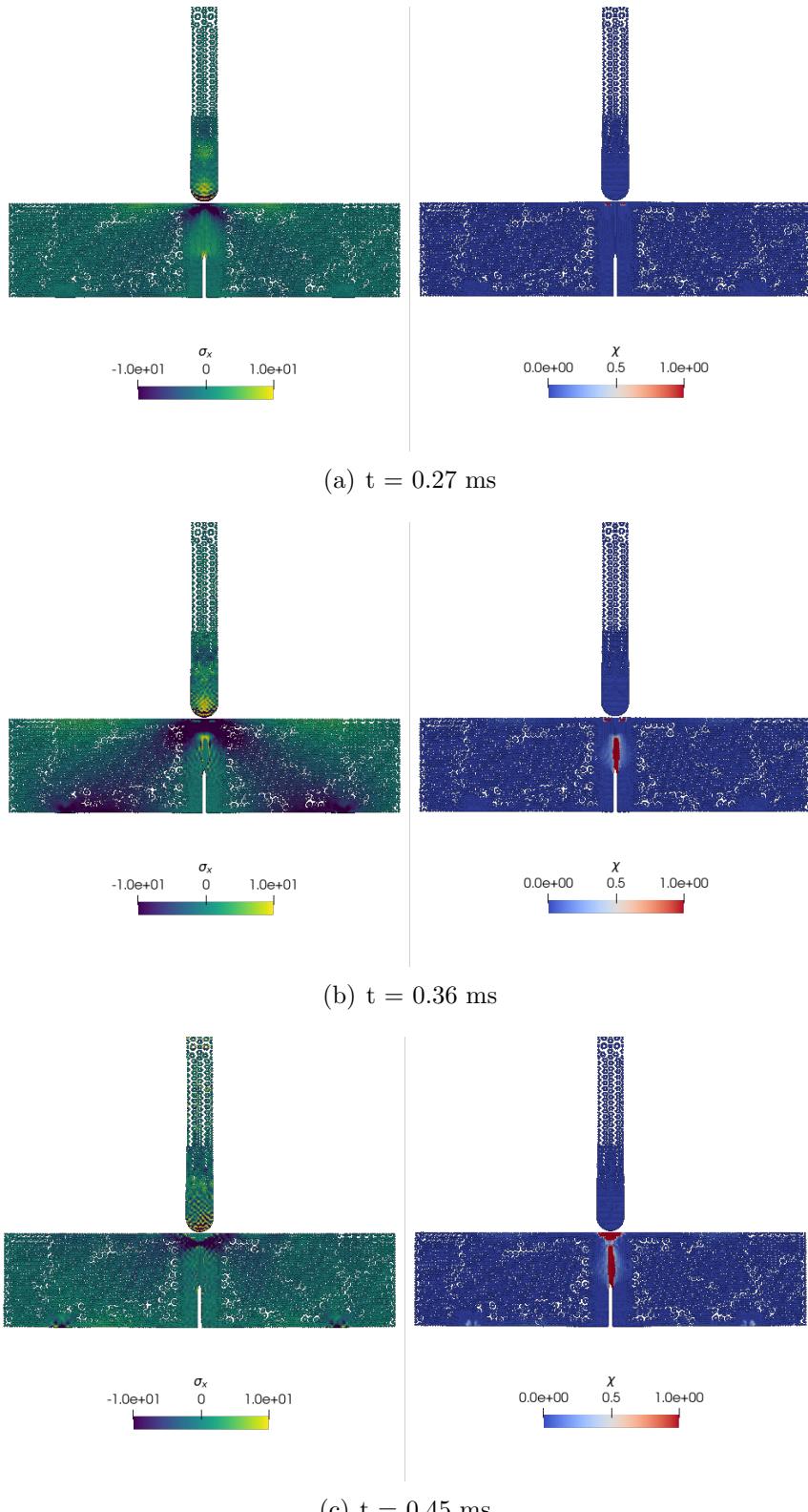


Figure 11: Evolution of the stress tensor magnitude for linear interpolation (pictures in the left side), and LME (pictures in the right side). Both simulations performed with an eigenerosion algorithm.

335 **4. Conclusions**

The paper proposes the eigensoftening algorithm in a framework for quasi-brittle materials to overcome the limitations observed in the EigenMPM [33]. The overestimation of tensile stress and strain peaks in the fracture mechanism of quasi-brittle materials and the stress oscillations observed
340 with conventional MPM/GIMP are faced. Based on the EigenMPM approach to fracture, two additional improvements have been proposed in the present research in order to solve the shortcomings of the original concept pointed out in the aforementioned paper [33].

Firstly, by introducing the concept of eigensoftening to the MPM framework,
345 which allows to consider gradual failure process of each material point. This concept is analogous the softening law which is traditionally employed in the context of cohesive fracture in FEM. Furthermore, this failure criterion admits the design of more elaborated damage curves, which can be employed in order to describe the failure process in more sophisticated materials.
350 Secondly to achieve better results, three new numerical algorithms has been implemented. The LME approximants are employed to remove from the solution those wiggles provoked by inaccuracies in the stress integration. The traditional time scheme of the MPM has been improved with a Newmark Predictor Corrector scheme as well. And the traditional concept of element connectivity has been removed to allow a more flexible and efficient implementation of search algorithm. In this research, the aforementioned techniques have helped to improve the results in fracture propagation through the eigensoftening and eigenerosion algorithms.

Finally, further improvements of the present research are exposed. The
360 algorithm would have the ability to simulate of crack patterns in composite material such as steel or carbon fiber reinforced concretes[42, 50]. In addition, the eigensoftening approach can be extrapolated to rock fracture in the same manner the eigenerosion was [51], taking into account that the volumetric stiffness remains in order to support bulk loads. Also the benefits of LME can
365 be exploited by adapting it through the deformation gradient [52]. *A priori* this would enhance the localisation properties of the proposed algorithms in more sophisticated domains. And the implementation of a contact algorithm to model other impact cases such as bullets collisions.

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375 **Appendix A. Explicit Newmark Predictor-Corrector algorithm**

1 Update mass matrix : $\mathbf{m}_I = N_{Ip}^k m_p$

2 Explicit Newmark Predictor :

$$\bar{v}_I^{pred} = \frac{N_{Ip}^k m_p (\bar{v}_p^k + (1 - \gamma) \Delta t \bar{a}_p^k)}{m_I}$$

3 Impose essential boundary conditions : At the fixed boundary, set $\bar{v}_I^{pred} = 0$.

4 Deformation tensor increment calculation :

$$\Delta \varepsilon_p^{k+1} = \Delta t \dot{\varepsilon}_p^{k+1} = \Delta t \left[\bar{v}_I^{pred} \otimes \text{grad}(N_{Ip}^{k+1}) \right]^s$$

5 Update the density field : $\rho_p^{k+1} = \frac{\rho_p^k}{1 + tra[\Delta \varepsilon_p^{k+1}]}$.

6 Compute stress field and update damage parameter

7 Balance of forces calculation : Calculate the total grid nodal force $\vec{f}_I^{k+1} = \vec{f}_I^{int,k+1} + \vec{f}_I^{ext,k+1}$.

8 Explicit Newmark Corrector :

$$\bar{v}_I^{k+1} = \bar{v}_I^{pred} + \gamma \Delta t \frac{\vec{f}_I^{k+1}}{\mathbf{m}_I^{k+1}}$$

9 Update particles lagrangian quantities :

$$\bar{a}_p^{k+1} = \frac{N_{Ip}^k \vec{f}_I^k}{\mathbf{m}_I^k}$$

$$\bar{v}_p^{k+1} = \bar{v}_p^n + \Delta t \frac{N_{Ip}^k \vec{f}_I^k}{\mathbf{m}_I^k}$$

$$\bar{x}_p^{k+1} = \bar{x}_p^n + \Delta t N_{Ip}^k \bar{v}_I^k + \frac{1}{2} \Delta t^2 \frac{N_{Ip}^k \vec{f}_I^k}{\mathbf{m}_I^k}$$

10 Reset nodal values

Algorithm 1: Explicit Newmark Predictor-Corrector scheme

Appendix B. Eigensoftening Algorithm

```

Input: For each  $p$ ,  $\epsilon$ -neighbourhood,  $f_{t,p}$ ,  $h_{\epsilon,p}$ ,  $w_c$ 
Output: Return damage parameter  $\chi := \{\chi_p\}$ 

1  $\chi_p \leftarrow \chi_p^k$ 
2 for  $p$  to  $N_p$  do
3   if  $\chi_p = 0$   $\epsilon_{f,p} = 0$  then
4     for  $q \in B_{\epsilon,p}$  do
5        $\sigma_{q,I} \leftarrow \text{getEigenvaluesOf}(\sigma_q)$ 
6       if  $\chi_q < 1$  then
7          $\sum m_p \sigma_{p,I} \leftarrow \sum m_p \sigma_{p,I} + m_q \sigma_{q,I}$ 
8       end
9        $m_p \leftarrow m_p + m_q$ 
10      end
11       $\sigma_{p,\epsilon} \leftarrow \frac{1}{m_p} \sum m_p \sigma_{p,I}$ 
12      if  $\sigma_{p,\epsilon} > f_{t,p}$  then
13         $\epsilon_{f,p} = \epsilon_{I,p}$ 
14      end
15      else if  $\chi_p \neq 1$   $\epsilon_{f,p} > 0$  then
16         $\chi_p^{k+1} \leftarrow \min \left\{ 1, \max \left\{ \chi_p^k, \frac{(\epsilon_{I,p} - \epsilon_{f,p}) h_{\epsilon,p}}{w_c} \right\} \right\}$ 
17      end
18    end
19 end

```

Algorithm 2: Compute damage parameter χ_p^{k+1}

Appendix C. Node-linked method for ϵ -neighbourhood reconstruction

380

```

Input:  $I_0$  : Closest node for each particle
Input:  $I_{Nodes}$  : List of nodes close to each node
Input:  $Particles_I$  : List of particles close to each node
Output: neighbourhood $_\epsilon$  :  $\epsilon$ -neighbourhood for each particle

1 for  $p$  to  $N_p$  do
2   For each  $p$  get  $I_0$ 
3   Eval  $I_{Nodes}(I_0)$ 
4   for  $k \in I_{Nodes}(I_0)$  do
5     | neighbourhood $_\epsilon^{k+1}$  = neighbourhood $_\epsilon^k \cup Particles_I(k)$ 
6   end
7   for  $p_\epsilon \in neighbourhood_\epsilon$  do
8     | if  $distance(p_\epsilon, p) > \epsilon$  then
9       |   | pop  $p_\epsilon$  from neighborhood $_\epsilon$ 
10      | end
11    end
12 end

```

Algorithm 3: ϵ -neighborhood reconstruction algorithm

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