Chapter 03

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Problem 3.4. *Proof.*

$$
\begin{aligned}
&\Pr[\mathrm{PrivK}^{\mathrm{eav}}_{\mathcal{A},\pi}(n)=1] \\
=&\Pr[b=0]\Pr[\mathrm{out}(\mathrm{PrivK}^{\mathrm{eav}}_{\mathcal{A},\Pi}(n,0))=0]+\Pr[b=1]\Pr[\mathrm{out}(\mathrm{PrivK}^{\mathrm{eav}}_{\mathcal{A},\Pi}(n,1))=1] \\
=&\Pr[b=0](1-\Pr[\mathrm{out}(\mathrm{PrivK}^{\mathrm{eav}}_{\mathcal{A},\Pi}(n,0))=1])+\Pr[b=1]\Pr[\mathrm{out}(\mathrm{PrivK}^{\mathrm{eav}}_{\mathcal{A},\Pi}(n,1))=1] \\
=&\frac{1}{2}(\Pr[\mathrm{out}(\mathrm{PrivK}^{\mathrm{eav}}_{\mathcal{A},\Pi}(n,1))=1]-\Pr[\mathrm{out}(\mathrm{PrivK}^{\mathrm{eav}}_{\mathcal{A},\Pi}(n,0))=1]+1)
\end{aligned}
$$

By DEFINITION 3.8, we have

$$\Pr[\mathrm{PrivK}^{\mathrm{eav}}_{\mathcal{A},\Pi}(n)=1]\leq\frac{1}{2}+\mathrm{negl}(n).$$

Additionally, if we have an adversary $\mathcal{A}$ such that $\Pr[\mathrm{PrivK}^{\mathrm{eav}}_{\mathcal{A},\Pi}(n)=1]<\frac{1}{2}-\mathrm{negl}(n)$, by simply inverse the b' $\mathcal{A}$ outputs, we get $\Pr[\mathrm{PrivK}^{\mathrm{eav}}_{\mathcal{A},\Pi}(n)=1]>\frac{1}{2}+\mathrm{negl}(n)$, a contradiction. Thus, DEFINITION 3.8 implies

$$\Pr[\mathrm{PrivK}^{\mathrm{eav}}_{\mathcal{A},\Pi}(n)=1]\geq\frac{1}{2}-\mathrm{negl}(n).$$

So

$$
\begin{aligned}
&\text{DEFINITION 3.8} \\
\Leftrightarrow&\left|\Pr[\mathrm{PrivK}^{\mathrm{eav}}_{\mathcal{A},\Pi}(n)=1]-\frac{1}{2}\right|\leq\mathrm{negl}(n) \\
\Leftrightarrow&\left|\frac{1}{2}\big(\Pr[\mathrm{out}(\mathrm{PrivK}^{\mathrm{eav}}_{\mathcal{A},\Pi}(n,1))=1]-\Pr[\mathrm{out}(\mathrm{PrivK}^{\mathrm{eav}}_{\mathcal{A},\Pi}(n,0))=1]+1\big)-\frac{1}{2}\right|\leq\mathrm{negl}(n) \\
\Leftrightarrow&\big|\Pr[\mathrm{out}(\mathrm{PrivK}^{\mathrm{eav}}_{\mathcal{A},\Pi}(n,1))=1]-\Pr[\mathrm{out}(\mathrm{PrivK}^{\mathrm{eav}}_{\mathcal{A},\Pi}(n,0))=1]\big|\leq\mathrm{negl}(n) \\
\Leftrightarrow&\text{DEFINITION 3.9},
\end{aligned}
$$

which finish the proof. □

Problem 3.6.

(a). Refute when n is odd:

If $|s|=n=2k+1$ and $l(n)=2n+1$, the length of the output of $G'(s)$ is $l(\lfloor\frac{n}{2}\rfloor)=2k+1=|s|$, which contradicts the condition that a pseudorandom generator satisfies $l(n)>n$.

Prove when n is even:

By the definition of pseudorandomness, we have: For any PPT algorithm D, there is a negligible function negl such that

$$| \Pr[D(G(s)) = 1] - \Pr[D(r) = 1] | \leq \text{negl}(n),$$

where $s \in \{0,1\}^n, r \in \{0,1\}^{l(n)}$ are drawn uniformly.

So the claim is the same when $s^* \in \{0,1\}^{\lfloor n/2 \rfloor}, r^* \in \{0,1\}^{l(\lfloor n/2 \rfloor)}$.

And $G'(s) = G(s_1 \cdots s_{\lfloor n/2 \rfloor})$, where $s = s_1 \cdots s_n$. The length of the output of $G'(s)$ is $l(\lfloor n/2 \rfloor) = |r^*|$. So

$$\begin{aligned}
&| \Pr[D(G'(s)) = 1] - \Pr[D(r^*) = 1] | \\
=& | \Pr[D(G(s^*)) = 1] - \Pr[D(r^*) = 1] | \\
\leq& \text{negl}(\lfloor n/2 \rfloor) \\
=& \text{negl}(n).
\end{aligned}$$

So $G'(s)$ is a pseudorandom generator.

(b). Refute:

Using the conclusion in **(a)**: $G'(s) = G(s_1 \cdots s_{\lfloor n/2 \rfloor})$ is a pseudorandom generator, where $s = s_1 \cdots s_n$.

Prove by **Contradiction**: Assume $G(0^{|s|} \| s)$ is a pseudorandom generator. Then Use the conclusion in **(a)**, we have $G'(s) = G(0^{|s|})$ also a pseudorandom generator. However, $G(0^{|s|})$ can be easily distinguished from uniform strings.

Specifically, just let adversary D compare its input with a fixed number $G(0^{n_0})$(Here n_0 is a fixed number.) Output 1 *if and only if* they are equal.

Uniformly choose $s_0 \in \{0,1\}^{n_0}$, we have

$$\Pr[D(G'(s_0)) = 1] = \Pr[D(G(0^{|s_0|}) = 1] = \Pr[D(G(0^{n_0})) = 1] = 1.$$

But randomly select $r_0 \in \{0,1\}^{l(n_0)}$, we have $r_0 = G(0^{n_0})$ with probability $2^{-l(n_0)}$. So

$$| \Pr[D(G'(s_0)) = 1] - \Pr[D(r_0) = 1] | = 1 - 2^{-l(n_0)} > \text{negl}(n_0),$$

a contradiction.

Thus $G(0^{|s|} \| s)$ is not a pseudorandom generator.

(c). Refute:

Prove by **Contradiction**: Assume $G(s)$ is a pseudorandom generator. Using the conclusion in **(a)**, we have $G''(s) = G(s_1 \cdots s_{\lfloor n/2 \rfloor})$ also a pseudorandom generator. However, $G'(s) = G''(s) \| G''(s+1)$ can be easily distinguished from uniform strings.

Construct D: it outputs 1 *if and only if* the first and second half of the input string is equal.

- If the input is $G'(s)$, we have $G'(s) = G''(s) \| G''(s+1)$. However, $G''(s) = G''(s+1)$ when the last $\lceil n/2 \rceil$ bits of s is not $11 \cdots 1$, so the probability is $1 - 2^{\lceil n/2 \rceil}$. That is

$$\Pr[D(G''(s)) = 1] = 1 - 2^{\lceil n/2 \rceil}.$$

- If the input is $r \in \{0,1\}^{2l(\lfloor n/2 \rfloor)}$, we have

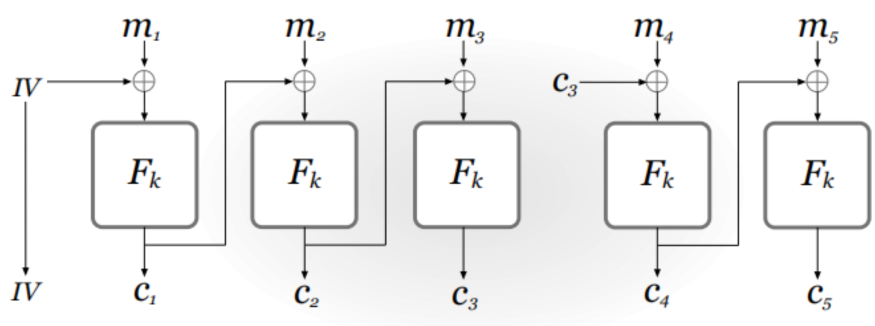
$$\Pr[D(r)=1] = \Pr[r_0 \cdots r_{l(\lfloor n/2 \rfloor)} = r_{l(\lfloor n/2 \rfloor)+1} \cdots r_{2l(\lfloor n/2 \rfloor)}] = 2^{\lfloor n/2 \rfloor} < \text{negl}(n).$$

- So

$$| \Pr[D(G''(s))=1] - \Pr[D(r)=1] | > 1 - 2^{\lceil n/2 \rceil} - \text{negl}(n) > \text{negl}(n),$$

  a contradiction.

Problem 3.11. Construct a encryption theme like this:



In this picture, the last block of the previous ciphertext is used as the IV when encrypting the next message.

First, it has indistinguishable multiple encryptions in the presence of an eavesdropper.

Given any adversary $\mathcal{A}$, we can construct a distinguisher D which access an oracle $\mathcal{O}$:$\{0,1\}^n \to \{0,1\}^n$. In detail:

1. Run $\mathcal{A}(1^n)$. When $\mathcal{A}$ outputs $(m_1^0, \cdots, m_d^0)$ and $(m_1^1, \cdots, m_d^1)$, choose a uniform bit $b \in \{0,1\}$ and then:

   (a) Get the last ciphertext c_0 in the previous encryption(If this is the first encryption, randomly choose $c_0 = IV \in \{0,1\}^n$ with uniform distribution.)

   (b) Query $\mathcal{O}(c_0 \oplus m_1)$ and obtain response c_1, then Query $\mathcal{O}(c_1 \oplus m_2)$ and obtain response c_2,

   ...

   until it gets $(c_1, \cdots, c_d)$.

   (c) Return $(c_0, c_1, \cdots, c_d)$ to $\mathcal{A}$.

2. A outputs a bit b'. Output 1 if $b' = b$, and 0 otherwise.

Thus, D outputs 1 *if and only if* $\mathcal{A}$ succeeds. Let Π denotes our construction, and $\widetilde{\Pi}$ denotes the theme when we replace F_k with a truly random function f. Then:

$$\Pr[\text{PrivK}^{mult}_{\mathcal{A},\Pi}(n)=1] = \Pr_{k \leftarrow \{0,1\}^n}[D^{F_k(\cdot)}(1^n)=1],$$

$$\Pr[\text{PrivK}^{mult}_{\mathcal{A},\widetilde{\Pi}}(n)=1] = \Pr_{f \leftarrow \text{Func}_n}[D^{f(\cdot)}(1^n)=1].$$

By the definition of pseudorandom function, we have

$$|\Pr[D^{F_k(\cdot)}(1^n) = 1] - \Pr[D^{f(\cdot)}(1^n) = 1]| \leq \text{negl}(n).$$

Thus

$$|\Pr[\text{PrivK}^{mult}_{\mathcal{A},\Pi}(n) = 1] - \Pr[\text{PrivK}^{mult}_{\mathcal{A},\widetilde{\Pi}}(n) = 1]| \leq \text{negl}(n).$$

To prove $\Pr[\text{PrivK}^{mult}_{\mathcal{A},\Pi}(n) = 1] \leq \frac{1}{2} + \text{negl}(n)$, we only need to prove:

$$\Pr[\text{PrivK}^{mult}_{\mathcal{A},\widetilde{\Pi}}(n) = 1] \leq \frac{1}{2} + \text{negl}(n)$$

We should know that $\text{Enc}_{c_0}(m_1, \cdots, m_d) = f(m_1 \oplus c_0, m_2 \oplus c_1, \cdots, m_d \oplus c_{d-1}) = f(m'_1, \cdots, m'_d) = (c_1, \cdots, c_d)$. Since f is a random function, given $c \in \{0,1\}^n$, the probability of $f(\cdot) = c$ is 2^{-n}. (f is uniformly drawn from the set of all functions of $\{0,1\}^n \to \{0,1\}^n$.)

Define event Repeat which denotes that there exits $i \neq j \in \{1, 2 \cdots, n\}$ such that $c_i \oplus m_{i+1} = c_j \oplus m_{j+1}$. If there is no Repeat, then answer $(c_1, \cdots, c_d)$ is just a stream of uniform bits. So

$$\begin{aligned}
&\Pr[\text{PrivK}^{mult}_{\mathcal{A},\widetilde{\Pi}}(n) = 1] \\
=&\Pr[\text{PrivK}^{mult}_{\mathcal{A},\widetilde{\Pi}}(n) = 1 \wedge \text{Repeat}] + \Pr[\text{PrivK}^{mult}_{\mathcal{A},\widetilde{\Pi}}(n) = 1 \wedge \overline{\text{Repeat}}] \\
\leq&\frac{\mathcal{O}(d)}{2^n} + \frac{1}{2} \\
=&\frac{1}{2} + \text{negl}(n). \qquad (d \text{ is polynomial of n.})
\end{aligned}$$

Thus,

$$\Pr[\text{PrivK}^{mult}_{\mathcal{A},\Pi}(n) = 1] \leq \frac{1}{2} + \text{negl}(n).$$

which means the theme has indistinguishable multiple encryptions in the presence of an eavesdropper.

Second, it's not CPA-secure.
Construct an adversary $\mathcal{A}$:

1. $\mathcal{A}$ randomly select two different string $m_0, m_1 \in \{0,1\}^n$ and output them.
2. A uniform bit $b \in \{0,1\}$ is chosen, then a ciphertext $c = \text{Enc}_k(m_b) = (IV, c)$ is computed and given to $\mathcal{A}$.
3. $\mathcal{A}$ compute $m_2 = m_0 \oplus IV \oplus c$, and get access to $c_2 = \text{Enc}_k(m_2)$.
4. If $c = c_2$, $\mathcal{A}$ outputs 0; otherwise outputs 1.

When $b = 0$, then $m_0 \oplus IV = m_2 \oplus c$. Thus the input of F_k is the same, and $c = c_2$ with probability 1. When $b = 1$, $c \neq c_2$ with high probability$(1 - 2^{-|c|})$. So $\mathcal{A}$ succeeds with probability near 1, which indicates that this theme is not CPA-secure.

Problem 3.18. How to decrypt: Given $k \in \{0,1\}^n, c$, compute $m' = F_k^{-1}(c)$. Message m is the second half of m'.

PART 1: CPA-secure

Proof. Without loss of generality, assume the permutation is fixed length Given any adversary $\mathcal{A}$, we can construct a distinguisher D which access an oracle $\mathcal{O}$:$\{0,1\}^n \to \{0,1\}^n$. In detail:

1. Run $\mathcal{A}(1^{2n})$. When $\mathcal{A}$ queries its encryption oracle on a message $m \in \{0,1\}^n$, answer this query in the following way:
   (a) choose uniform $r \in \{0,1\}^n$.
   (b) Query $\mathcal{O}(r\|m)$ and obtain response y.
   (c) Return the ciphertext y to $\mathcal{A}$.
2. When $\mathcal{A}$ outputs messages $m_0, m_1 \in \{0,1\}^n$, choose a uniform bit $b \in \{0,1\}$ and then:
   (a) choose uniform $r_0 \in \{0,1\}^n$.
   (b) Query $\mathcal{O}(r_0\|m_b)$ and obtain response y.
   (c) Return the ciphertext y to $\mathcal{A}$.
3. Continue answering encryption-oracle queries of $\mathcal{A}$ as before until $\mathcal{A}$ outputs a bit b'. Output 1 if $b' = b$, and 0 outherwise.

Thus, D outputs 1 *if and only if* $\mathcal{A}$ succeeds. Let Π denotes our construction, and $\widetilde{\Pi}$ denotes the theme when we replace F_k with a uniform permutation $f \in \mathrm{Perm}_n$. Then:

$$\Pr[\mathrm{PrivK}^{cpa}_{\mathcal{A},\Pi}(2n) = 1] = \Pr_{k \leftarrow \{0,1\}^{2n}}[D^{F_k(\cdot)}(1^{2n}) = 1],$$

$$\Pr[\mathrm{PrivK}^{cpa}_{\mathcal{A},\widetilde{\Pi}}(2n) = 1] = \Pr_{f \leftarrow \mathrm{Perm}_{2n}}[D^{f(\cdot)}(1^{2n}) = 1].$$

By the definition of pseudorandom permutation, we have

$$|\Pr[D^{F_k(\cdot)}(1^{2n}) = 1] - \Pr[D^{f(\cdot)}(1^{2n}) = 1]| \leq \mathrm{negl}(2n).$$

Thus

$$|\Pr[\mathrm{PrivK}^{cpa}_{\mathcal{A},\Pi}(2n) = 1] - \Pr[\mathrm{PrivK}^{cpa}_{\mathcal{A},\widetilde{\Pi}}(2n) = 1]| \leq \mathrm{negl}(2n),$$

that is

$$|\Pr[\mathrm{PrivK}^{cpa}_{\mathcal{A},\Pi}(n) = 1] - \Pr[\mathrm{PrivK}^{cpa}_{\mathcal{A},\widetilde{\Pi}}(n) = 1]| \leq \mathrm{negl}(n).$$

To prove $\Pr[\mathrm{PrivK}^{cpa}_{\mathcal{A},\Pi}(n) = 1] \leq \frac{1}{2} + \mathrm{negl}(n)$, we only need to prove:

$$\Pr[\mathrm{PrivK}^{cpa}_{\mathcal{A},\widetilde{\Pi}}(n) = 1] \leq \frac{1}{2} + \mathrm{negl}(n)$$

Define event Repeat which denotes r_0 was selected in some query. (There are totally $d = q(n)$ queries.) If there is no Repeat, then answer c is just uniform bits. So

$$\begin{aligned}
&\Pr[\mathrm{PrivK}^{cpa}_{\mathcal{A},\widetilde{\Pi}}(n) = 1]\\
=&\Pr[\mathrm{PrivK}^{cpa}_{\mathcal{A},\widetilde{\Pi}}(n) = 1 \wedge \mathrm{Repeat}] + \Pr[\mathrm{PrivK}^{cpa}_{\mathcal{A},\widetilde{\Pi}}(n) = 1 \wedge \overline{\mathrm{Repeat}}]
\end{aligned}$$

Separately,

1. $\Pr[\mathrm{PrivK}^{cpa}_{\mathcal{A},\widetilde{\Pi}}(n) = 1 \wedge \mathrm{Repeat}]$:

   Since r is drawn uniformly, the above probability is $\frac{\mathcal{O}(d)}{2^n} = \mathrm{negl}(n)$.

2. Let R_{asked} denote the set of r used by the experiment during the queries. And R denotes the event that which r is chosen.

   $$\begin{aligned}
   &\Pr[\mathrm{PrivK}^{cpa}_{\mathcal{A},\widetilde{\Pi}}(n) = 1 \wedge \overline{\mathrm{Repeat}}]\\
   =&\Pr[\mathrm{PrivK}^{cpa}_{\mathcal{A},\widetilde{\Pi}}(n) = 1 \mid \overline{\mathrm{Repeat}}] \times \Pr[\overline{\mathrm{Repeat}}]\\
   =&\Pr[\overline{\mathrm{Repeat}}] \sum_{r \notin R_{asked}} \Pr[R = r](\Pr[\mathrm{Enc}(r \| m_0) = c]\Pr[b = 0] + \Pr[\mathrm{Enc}(r \| m_1) = c]\Pr[b = 1])\\
   =&\Pr[\overline{\mathrm{Repeat}}] \sum_{r \notin R_{asked}} \Pr[R = r](\Pr[\mathrm{Enc}(r \| m_1) = c]\Pr[b = 0] + \Pr[\mathrm{Enc}(r \| m_0) = c]\Pr[b = 1])\\
   =&\Pr[\mathrm{PrivK}^{cpa}_{\mathcal{A},\widetilde{\Pi}}(n) = 0 \wedge \overline{\mathrm{Repeat}}]
   \end{aligned}$$

   So
   $$\Pr[\mathrm{PrivK}^{cpa}_{\mathcal{A},\widetilde{\Pi}}(n) = 1 \wedge \overline{\mathrm{Repeat}}] = \frac{1}{2}$$

Thus,

$$\begin{aligned}
&\Pr[\mathrm{PrivK}^{cpa}_{\mathcal{A},\widetilde{\Pi}}(n) = 1]\\
=&\Pr[\mathrm{PrivK}^{cpa}_{\mathcal{A},\widetilde{\Pi}}(n) = 1 \wedge \mathrm{Repeat}] + \Pr[\mathrm{PrivK}^{cpa}_{\mathcal{A},\widetilde{\Pi}}(n) = 1 \wedge \overline{\mathrm{Repeat}}]\\
\leq&\frac{1}{2} + \mathrm{negl}(n)
\end{aligned}$$

which means CPA-secure.

$\square$

PART 2: The proof of CCA-secure is the same with problem 4.25.

Problem 3.29. Define when given 1^n to Gen, the message is $l_{in}(n)$ in length, which is polynomial in n. For that reason, we can assume $l_{in}(n) = n$, which don't affect the result of $\mathrm{negl}(n)$. Construct a theme $\Pi = (\mathrm{Gen}, \mathrm{Enc}, \mathrm{Dec})$ as followed:

1. Gen: on input 1^n, choose (k_1, k_2) based on the $\mathrm{Gen}_1(1^n), \mathrm{Gen}_2(1^n)$ in Π_1, Π_2.

2. Enc: Given $m \in \{0,1\}^n$, uniformly select a $m_l \in \{0,1\}^n$. Then compute $m_r = m \oplus m_l$. Use $\mathrm{Enc}_1^{k_1}$ to encrypt m_l and $\mathrm{Enc}_2^{k_2}$ to encrypt m_r. The ciphertext we get is (c_1, c_2).

3. Dec: $m = \mathrm{Dec}_1^{k_1}(c_1) \oplus \mathrm{Dec}_2^{k_2}(c_2)$.

Proof. Now we prove it's CPA-secure.

Assume there is an adversary $\mathcal{A}$ for Π. Construct an adversary $\mathcal{A}_2$ for Π as followed:

1. $\mathcal{A}_2$ has oracle $\mathrm{Enc}_{k_2}(\cdot)$, with k_2 unknown to $\mathcal{A}_2$. But $\mathcal{A}_2$ can choose $k_1 \leftarrow \mathrm{Gen}_1(1^n)$, then construct a new oracle

$$\mathrm{Enc}_{k_1,k_2}(\cdot) = (\mathrm{Enc}_{k_1}(m_l), \mathrm{Enc}_{k_2}(m_l \oplus \cdot)).$$

   Here $m_l \in \{0,1\}^n$ is uniformly selected by the oracle.

2. $\mathcal{A}$ queries the oracle $\mathrm{Enc}_{k_1,k_2}(\cdot)$ and get the ciphertexts.

3. $\mathcal{A}$ asks m_0', m_1'. Then $\mathcal{A}_2$ uniformly choose $m_l \in \{0,1\}^n$, construct $(m_0, m_1) = (m_l \oplus m_0', m_l \oplus m_1')$ and ask the CPA-experiment. Then $\mathcal{A}_2$ get c, and pass $(\mathrm{Enc}_{k_1}(m_l), c)$ to $\mathcal{A}$.

4. $\mathcal{A}$ asks some queries to $\mathrm{Enc}_{k_1,k_2}(\cdot)$ and output $b' \in \{0,1\}$.

5. $\mathcal{A}_2$ output b'.

In this experiment, both $\mathcal{A}$ and $\mathcal{A}_2$ don't know b, and what $\mathcal{A}_2$ guesses is the same as what $\mathcal{A}$ guesses. Also,

$$\begin{aligned}
&\mathrm{PrivK}^{cpa}_{\mathcal{A}_2,\Pi}(n) = 1 \\
\Leftrightarrow & c = \mathrm{Enc}_{k_2}(m_{b'}) \\
\Leftrightarrow & (\mathrm{Enc}_{k_1}(m_l), c) = (\mathrm{Enc}_{k_1}(m_l), \mathrm{Enc}_{k_2}(m_l \oplus m_{b'}')) \\
\Leftrightarrow & \mathrm{PrivK}^{cpa}_{\mathcal{A},\Pi}(n) = 1.
\end{aligned}$$

So

$$\Pr[\mathrm{PrivK}^{cpa}_{\mathcal{A}_2,\Pi}(n) = 1] = \Pr[\mathrm{PrivK}^{cpa}_{\mathcal{A},\Pi}(n) = 1].$$

Similarly, we can construct an adversary $\mathcal{A}_1$ such that

$$\Pr[\mathrm{PrivK}^{cpa}_{\mathcal{A}_1,\Pi}(n) = 1] = \Pr[\mathrm{PrivK}^{cpa}_{\mathcal{A},\Pi}(n) = 1].$$

So if Π is not CPA-secure, then $\exists \mathcal{A}, s.t. \Pr[\mathrm{PrivK}^{cpa}_{\mathcal{A},\Pi}(n) = 1] > \frac{1}{2} + \mathrm{negl}(n)$, so

$$\Pr[\mathrm{PrivK}^{cpa}_{\mathcal{A}_2,\Pi}(n) = 1] = \Pr[\mathrm{PrivK}^{cpa}_{\mathcal{A}_1,\Pi}(n) = 1] > \frac{1}{2} + \mathrm{negl}(n),$$

which means that each of Π_1 and Π_2 is not CPA-secure, a contradiction.

Thus Π is CPA-secure. $\square$