Solved Examples 1

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1. Consider a town with n men and n women seeking to get married to one another. Each man has a preference list that ranks all the women, and each woman has a preference list that ranks all the men.

   The set of all $2n$ people is divided into two categories: good people and bad people. Suppose that for some number k, $1 \leq k \leq n - 1$, there are k good men and k good women; thus there are $n - k$ bad men and $n - k$ bad women.

   Everyone would rather marry any good person than any bad person. Formally, each preference list has the property that it ranks each good person of the opposite gender higher than each bad person of the opposite gender: its first k entries are the good people (of the opposite gender) in some order, and its next $n - k$ are the bad people (of the opposite gender) in some order.

   Show that in every stable matching, every good man is married to a good woman.

   Proof: By contradiction.

   Suppose that a good man, m_g, ended up married to a bad woman, w_b, in some stable matching S. If this is the case, then there remain $k - 1$ good men and k good women. Thus, there must exist another good woman, w_g, married to a bad man, m_b, for S to be perfect. Since S must be perfect to be stable, this must be the case.

   Then, there is an instability. Since w_g is good, she must rank higher than w_b in the preference list of m_g. By similar reasoning, m_g must rank higher than m_b in ther prefernece list of w_g. Thus, m_g prefers w_g and w_g prefers m_g, which is an instability.

   We have shown that there cannot exist a stable matching where a good man is married to a bad woman. Therefore, in every stable matching, every good man is married to a good woman.

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Collaborators: A, B, ...