Q2)

a)

Both are support vectors. Let w be $\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$

Constraints: $y^{(1)}\left(\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \cdot x^{(1)}\right) = 1 \quad -(1)$

$y^{(2)}\left(\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \cdot x^{(2)}\right) = 1 \quad -(2)$

$\therefore$ (1): $w_1 + w_2 = 1$

(2): $w_1 = -1$

$\therefore$ if $w_1 = -1$,

$w_2 = 2$

$$\boxed{\therefore \; w = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \quad \therefore \; \gamma = \frac{1}{\sqrt{(-1)^2 + 2^2}} = \frac{1}{\sqrt{5}}}$$

b) Since offset is now non-zero, let w be $\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$

Both are still support vectors.

$y^{(1)}\left(\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \cdot x^{(1)} + b\right) = 1$

$y^{(2)}\left(\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \cdot x^{(2)} + b\right) = 1$

$\therefore \; w_1 + w_2 + b = 1 \quad -(1)$

$w_1 + b = -1 \quad -(2)$

$\therefore \; w_2 = 2, \quad w_1 = -1 - b.$

Differentiating Lagrange: (α_1 and α_2) > 0

$$\frac{dL}{dw} = 0 = \begin{bmatrix} -1-b \\ 2 \end{bmatrix} + (-1)\alpha_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\therefore \quad -1 - b - \alpha_1 + \alpha_2 = 0 \qquad -(3)$$

$$2 - \alpha_1 = 0 \qquad -(4)$$

$$\alpha_1 = 2$$

$\therefore$ Additional constraint for bias:

$$\sum_{i \in R} a_i y_i = 0$$

$$(2)(1) - (\alpha_2) = 0$$

$$\therefore \alpha_2 = 2.$$

$\therefore$ Using (3):

$$-1 - b - 2 + 2 = 0$$

$$\therefore b = -1 \quad \#$$

Since $b = -1$, $w_1 = -1 - b = 0$

$$\boxed{\begin{array}{l} \therefore w = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \quad b = -1 \\ \gamma = \dfrac{1}{\sqrt{2}} = \dfrac{1}{2} \ \# \end{array}}$$

Q3) Let $k_1(x, z)$ corr. $\phi^{(1)}(x)$

a) $k_2(x, z)$ corr $\phi^{(2)}(x)$

$$\therefore K(x, z) = k_1(x, z)\, k_2(x, z)$$
$$= \left(\phi^{(1)}(x) \cdot \phi^{(1)}(z)\right)\left(\phi^{(2)}(x)\ \phi^{(2)}(z)\right)$$
$$= \left(\sum_{i \in R} \phi_i^{(1)}(x)\ \phi_i^{(1)}(z)\right)\left(\sum_{J \in R} \phi_J^{(2)}(x)\ \phi_J^{(2)}(z)\right)$$
$$= \sum_i \sum_J \phi_i^{(1)}(x)\ \phi_J^{(2)}(x)\ \phi_i^{(1)}(z)\ \phi_J^{(2)}(z)$$

By letting $\psi(x) = \phi^{(1)}(x) \cdot \phi^{(2)}(x)$,

$$K(x, z) = \sum_{iJ} \psi(x)\,\psi(z) = \psi(x) \cdot \psi(z)$$

$\therefore$ $K(x, z)$ is a kernel.

b) Vector addition results into symmetrical vectors from inputs. Using above definitions,

$$\therefore K(x, z) = \delta(x) \cdot \delta(z)$$
$$= \begin{bmatrix} \phi^{(1)}(x) \\ \phi^{(2)}(x) \end{bmatrix} \cdot \begin{bmatrix} \phi^{(1)}(z) \\ \phi^{(2)}(z) \end{bmatrix}$$
$$= \phi^{(1)}(x)\ \phi^{(1)}(z) + \phi^{(2)}(x)\ \phi^{(2)}(z)$$
$$= k_1(x, z) + k_2(x, z)$$

$\therefore$ $K(x, z)$ is a kernel.

c) Vector subtraction results into symmetrical vectors too.

$\therefore$ Proving positive semidefinite:

$$\underline{z^T K z \geq 0 \text{ for all } z.}$$

$$z^T K z = z^T (K_1 - K_2) z$$

$$= z^T K_1 z - z^T K_2 z$$

$z^T K z \geq 0$ if and only if $z^T K_1 z \geq z^T K_2 z$

$\therefore$ $\underline{K(x,z) \text{ is not a kernel.}}$

d) $K(x, z) = f(x) f(z)$

$f(x)$ will return a scalar.

$\therefore$ $f(x) = \phi^1(x) \cdot \phi^2(x)$ (Similar to Q1)

$\therefore$ $K(x, z) = [\phi^1(x) \cdot \phi^2(x)][\phi^1(x) \cdot \phi^2(z)]$

$$= \psi(x) \cdot \psi(z)$$

$$= K(x, z)$$

$\therefore$ $\underline{K(x,z) \text{ is a kernel.}}$