Solution Fourier Series Triangle Wave

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1 Introduction

1.1 General form of Fourier Series

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n Cos\left(\frac{2n\pi}{T}t\right) + b_n Sin\left(\frac{2n\pi}{T}t\right) \right] \tag{1}$$

$$a_0 = \frac{2}{T} \int_{\frac{-T}{2}}^{\frac{T}{2}} f(t)dt \tag{2}$$

$$a_n = \frac{2}{T} \int_{\frac{-T}{2}}^{\frac{T}{2}} f(t) Cos\left(\frac{2n\pi}{T}t\right) dt \tag{3}$$

$$b_n = \frac{2}{T} \int_{\frac{-T}{2}}^{\frac{T}{2}} f(t) Sin\left(\frac{2n\pi}{T}t\right) dt \tag{4}$$

1.2 Wave form

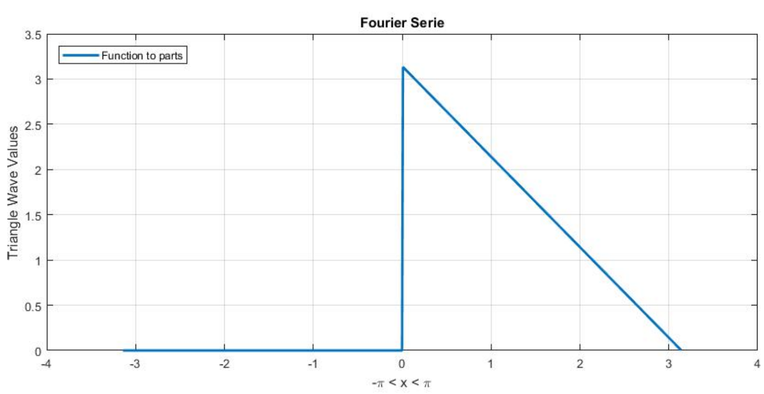

Figure 1: Triangle Wave

1.3 Function to parts

$$f(t)\begin{Bmatrix} 0 & -\pi < t < 0 \\ \pi - t & 0 < t < \pi \end{Bmatrix} \quad \text{T}= 2\pi \tag{5}$$

1.4 Solution a_0

$$a_0 = \frac{2}{T}\int_{\frac{-T}{2}}^{\frac{T}{2}} f(t)dt = \frac{2}{2\pi}\int_{\frac{-2\pi}{2}}^{0} 0dt + \frac{2}{2\pi}\int_{0}^{\frac{2\pi}{2}} (\pi - t)dt \tag{6}$$

$$a_0 = \frac{2}{2\pi}\int_{0}^{\frac{2\pi}{2}} (\pi - t)dt = \frac{1}{\pi}\int_{0}^{\pi} (\pi - t)dt$$

$$a_0 = \frac{1}{\pi}\left[\int_{0}^{\pi} \pi dt - \int_{0}^{\pi} tdt\right]$$

$$a_0 = \frac{1}{\pi}\left[\pi t - \frac{t^2}{2}\right]_0^{\pi} = \frac{1}{\pi}\left[\pi(\pi) - \frac{\pi^2}{2}\right] = \frac{1}{\pi}\left[\frac{2\pi^2 - \pi^2}{2}\right]$$

$$a_0 = \frac{1}{\pi}\left[\frac{\pi^2}{2}\right] = \frac{\pi}{2}$$

$$\boxed{a_0 = \frac{\pi}{2}} \tag{7}$$

1.5 Solution a_n

$$a_n = \frac{2}{T}\int_{\frac{-T}{2}}^{\frac{T}{2}} f(t)Cos\left(\frac{2n\pi}{T}t\right)dt \tag{8}$$

$$a_n = \frac{2}{2\pi}\int_{-\pi}^{0} 0 \cdot Cos\left(\frac{2n\pi}{2\pi}t\right)dt + \frac{2}{2\pi}\int_{0}^{\pi} (\pi - x)Cos\left(\frac{2n\pi}{2\pi}t\right)dt \tag{9}$$

$$a_n = \frac{1}{\pi}\int_{0}^{\pi} (\pi - x)Cos(nt)dt = \frac{1}{\pi}\left[\int_{0}^{\pi} \pi Cos(nt)dt - tCos(nt)dt\right]$$

According to integration tables

$$\int Cos(nt)dt = \frac{1}{n}Sin(nt)$$

$$\boxed{a_n = \frac{1}{\pi}\left\{\left[\frac{\pi}{n}Sin(nt)\right]_0^{\pi} - \int_{0}^{\pi} tCos(nt)dt\right\}}$$

Integration Methods (Integration by parts)

$$\int tCos(nt)dt \rightarrow uV - \int Vdu \tag{10}$$

$$\begin{array}{cc} u = t & dv = Cos(nt)dt \\ \frac{du}{dt} = 1 & \int dv = Cos(nt)dt \\ du = dt & V = \frac{1}{n}Sin(nt) \end{array}$$

$$uV - \int Vdu \rightarrow t \cdot \frac{1}{n}Sin(nt) - \int \frac{1}{n}Sin(nt)dt$$

$$\frac{t}{n}Sin(nt) - \frac{1}{n}\int Sin(nt)d \rightarrow \frac{t}{n}Sin(nt) + \frac{1}{n^2}Cos(nt)$$

$$a_n = \frac{1}{\pi}\left\{\left[\frac{\pi}{n}Sin(nt)\right]_0^{\pi} - \left[\frac{1}{n^2}Cos(nt) + \frac{t}{n}Sin(nt)\right]_0^{\pi}\right\}$$

$$a_n = \frac{1}{\pi}\left\{\frac{\pi}{n}Sin(n\pi) - \frac{1}{n^2}Cos(n\pi) - \frac{\pi}{n}Sin(n\pi) + \frac{1}{n^2}\right\}$$

$Sin(n\pi) = 0$

$$a_n = \frac{1}{\pi}\left\{-\frac{1}{n^2}Cos(n\pi) + \frac{1}{n^2}\right\} = -\frac{1}{\pi n^2}Cos(n\pi) + \frac{1}{\pi n^2}$$

$Cos(n\pi) = (-1)^n$

$$a_n = \frac{1}{\pi n^2} - \frac{(-1)^n}{\pi n^2}$$

$$\boxed{a_n = \frac{1-(-1)^n}{\pi n^2}} \tag{11}$$

1.6 Solution b_n

$$b_n = \frac{2}{T}\int_{\frac{-T}{2}}^{\frac{T}{2}} f(t)Sin\left(\frac{2n\pi}{T}t\right)dt \tag{12}$$

$$b_n = \frac{2}{2\pi}\int_0^{\frac{2\pi}{2}} (\pi - t)Sin\left(\frac{2n\pi}{2\pi}t\right)dt = \frac{1}{\pi}\left[\int_0^{\pi} \pi Sin(nt)dt - tSin(nt)dt\right] \tag{13}$$

$$b_n = \frac{1}{\pi}\left[\int_0^\pi \pi Sin(nt)dt - \int_0^\pi tSin(nt)dt\right]$$

$$\int tSin(nt)dt = \frac{1}{n^2}Sin(nt) - \frac{t}{n}Cos(nt)$$

$$b_n = \frac{1}{\pi}\left[\frac{-\pi}{n}Cos(nt) - \frac{1}{n^2}Sin(nt) + \frac{t}{n}Cos(nt)\right]_0^\pi$$

$$b_n = \frac{1}{\pi}\left[\frac{-\pi}{n}Cos(n\pi) + \frac{\pi}{n} - \frac{1}{n^2}Sin(n\pi) + \frac{\pi}{n}Cos(n\pi)\right]$$

$$Sin(n\pi) = 0$$

$$\boxed{b_n = \frac{1}{\pi}\left[\frac{\pi}{n}\right] = \frac{1}{n}}$$

1.7 Coefficients

$$a_0 = \tfrac{\pi}{2} \qquad a_n = \tfrac{1-(-1)^n}{\pi n^2} \qquad b_n = \tfrac{1}{n} \tag{14}$$

1.8 Replacing coefficients

$$f(t) = \frac{\pi}{4} + \sum_{n=1}^{\infty}\left[\frac{1-(-1)^n}{\pi n^2}Cos\left(\frac{2n\pi}{T}t\right) + \frac{1}{n}Sin\left(\frac{2n\pi}{T}t\right)\right] \tag{15}$$

1.8.1 Evaluation of variable n

$$With\ \ n = 1$$

$$f(t)_{n=1} = \frac{\pi}{4} + \sum_{n=1}^{\infty}\left[\frac{1-(-1)^1}{(\pi)1^2}Cos\left(\frac{2(1)\pi}{2\pi}t\right) + \frac{1}{1}Sin\left(\frac{2(1)\pi}{2\pi}t\right)\right]$$

$$With\ \ n = 2$$

$$f(t)_{n=2} = \frac{\pi}{4} + \sum_{n=1}^{\infty}\left[f(t)_{n=1} + \frac{1-(-1)^2}{(\pi)2^2}Cos\left(\frac{2(2)\pi}{2\pi}t\right) + \frac{1}{2}Sin\left(\frac{2(2)\pi}{2\pi}t\right)\right]$$

$$With\ \ n = 3$$

$$f(t)_{n=3} = ...$$

$$With\ \ n = 4$$

$$f(t)_{n=4} = ...$$