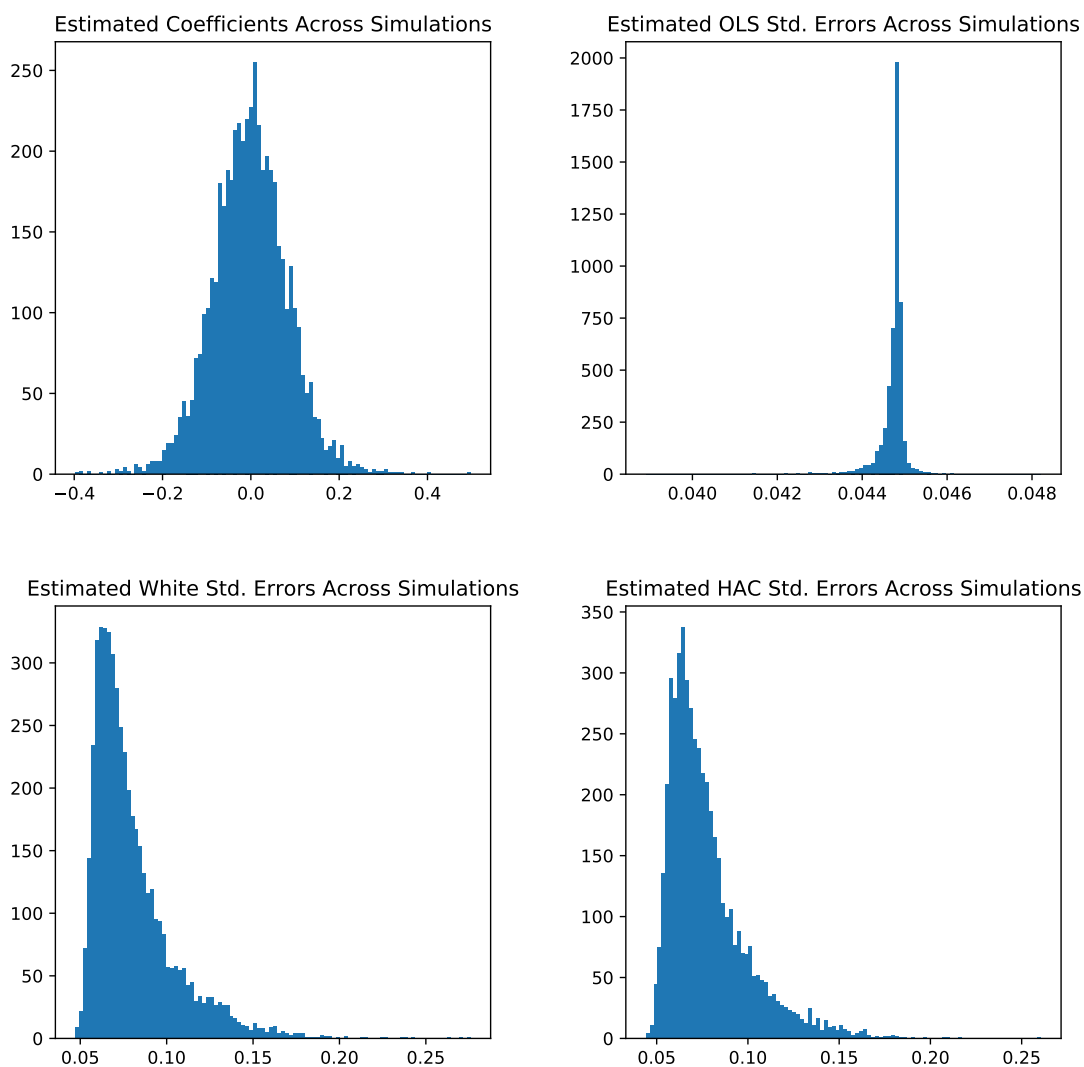Problem Set 2
Econ 6413–Ng
Miguel Acosta
September 22, 2017

Q1. In the results below, I've assumed that $v_t \sim N(0,1)$, as in Hamilton (2008). I am able to replicate table 2 of Hamilton (2008). Here are the results for $T = 500$ and 5000 replications:

Standard Error	Size
OLS	0.280
White	0.062
HAC, $q = 5$	0.062

For fun, here are the distributions of the $\widehat{\beta}$ and standard errors across the simulations.

Estimated Coefficients Across Simulations

Estimated OLS Std. Errors Across Simulations

Estimated White Std. Errors Across Simulations

Estimated HAC Std. Errors Across Simulations

Q2. We are considering an ARMA(1, 1):

$$y_t = \alpha y_{t-1} + \beta e_{t-1} + e_t \qquad\qquad e_t \sim N(0, \sigma^2)$$

Here is a simulated series with 500 time periods and $\theta \equiv (\alpha, \beta, \sigma^2) = (0.8, 0.5, 0.5)$.
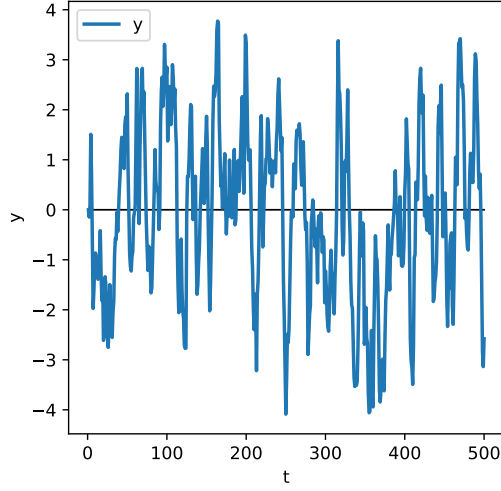
i. In problem set 1, I calculated γ_0 and γ_1 as:

$$\gamma_0 = \frac{\sigma^2(2\alpha\beta + \beta^2 + 1)}{1 - \alpha^2} \qquad\qquad \gamma_1 = \sigma^2 \cdot \frac{\alpha^2\beta + \beta^2\alpha + \beta + \alpha}{1 - \alpha^2}. \qquad (1)$$

Remember that the series is mean-0. So, the second-order autocovariance is:

$$\gamma_2 = \mathbb{E}[y_t y_{t-2}] = \alpha y_{t-1} y_{t-2} + \beta e_{t-1} y_{t-2} + e_t y_{t-2}] = \alpha \gamma_1 \qquad (2)$$

ii. The GMM objective function, therefore, is:

$$\bar{g}(\theta) \equiv \frac{1}{T-3} \sum_{t=3}^{T} \widehat{\psi} - \gamma(\theta) = \frac{1}{T-3} \sum_{t=3}^{T} \begin{bmatrix} y_t^2 - \frac{\sigma^2(2\alpha\beta + \beta^2 + 1)}{1-\alpha^2} \\ y_t y_{t-1} - \sigma^2 \cdot \frac{\alpha^2\beta + \beta^2\alpha + \beta + \alpha}{1-\alpha^2} \\ y_t y_{t-2} - \alpha\sigma^2 \cdot \frac{\alpha^2\beta + \beta^2\alpha + \beta + \alpha}{1-\alpha^2} \end{bmatrix}$$

Results follow:

α	0.8315
β	0.4478
σ^2	0.4546

2

iii. The third and fourth centered moments of an $\text{ARMA}(1,1)$ will be difficult to calculate, as you can see here (thanks to SymPy for these expressions):

$$\mathbb{E}[y_t^3] = \mathbb{E}[\alpha^3 y_{t-1}^3 + 3\alpha^2\beta e_{t-1}y_{t-1}^2 + 3\alpha^2 e_t y_{t-1}^2 + 3\alpha\beta^2 e_{t-1}^2 y_{t-1}$$
$$+ 6\alpha\beta e_{t-1}e_t y_{t-1} + 3\alpha e_t^2 y_{t-1} + \beta^3 e_{t-1}^3 + 3\beta^2 e_{t-1}^2 e_t + 3\beta e_{t-1}e_t^2 + e_t^3]$$
$$\mathbb{E}[y_t^4] = \mathbb{E}[\alpha^4 y_{t-1}^4 + 4\alpha^3\beta e_{t-1}y_{t-1}^3 + 4\alpha^3 e_t y_{t-1}^3 + 6\alpha^2\beta^2 e_{t-1}^2 y_{t-1}^2$$
$$+ 12\alpha^2\beta e_{t-1}e_t y_{t-1}^2 + 6\alpha^2 e_t^2 y_{t-1}^2 + 4\alpha\beta^3 e_{t-1}^3 y_{t-1} + 12\alpha\beta^2 e_{t-1}^2 e_t y_{t-1}$$
$$+ 12\alpha\beta e_{t-1}e_t^2 y_{t-1} + 4\alpha e_t^3 y_{t-1} + \beta^4 e_{t-1}^4 + 4\beta^3 e_{t-1}^3 e_t + 6\beta^2 e_{t-1}^2 e_t^2$$
$$+ 4\beta e_{t-1}e_t^3 + e_t^4]$$

Luckily, most of these terms are 0, or simplify substantially. In particular, note that

$$\mathbb{E}[f(e_t)y_{t-1}] = \mathbb{E}[e_t e_{t-1}y_{t-1}] = \mathbb{E}[e_t f(y_{t-1})] = 0 \qquad \mathbb{E}[e_t^3] = 0 \qquad \mathbb{E}[e_t^4] = 3\sigma^4$$

The first expression is therefore:

$$\mathbb{E}[y_t^3] = \alpha^3\,\mathbb{E}[y_{t-1}^3] \implies \mathbb{E}[y_{t-1}^3] =$$

(assuming $\mathbb{E}[y_0^3] = 0$, which is consistent with the assumption that $y_0 = 0$). The second expression reduces to

$$\mathbb{E}[y_t^4] = \alpha^4\,\mathbb{E}[y_{t-1}^4] + 3\sigma^4(\beta^4 + 4\alpha\beta^3 + 6\alpha^2\beta^2 + 4\alpha^3\beta + 1).$$

Since we only have covariance stationarity, the 4th moment depends on time, and can be calculated recursively assuming $\mathbb{E}[y_t^4] = 0$.

With all this, I present results that didn't work.

α	-0.8128
β	0.9982
σ^2	2.5513

Q3. **i.** Remember what OLS does:

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} \gamma_0 & \gamma_1 \\ \gamma_1 & \gamma_0 \end{bmatrix}^{-1} \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix}$$

We can take advantage of the expressions in 1 and 2 to back out the estimates for $(\alpha, \beta, \sigma^2)$. Multiply both sides by the projection matrix to get

$$\begin{bmatrix} \gamma_0 & \gamma_1 \\ \gamma_1 & \gamma_0 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix}$$

We can estimate the RHS easily. So, this identifies α since $\frac{\gamma_2}{\gamma_1} = \alpha$. Next, taking the ratio of γ_0 to γ_1, we have one equation to solve for $\widehat{\beta}$. I asked SymPy to do this. Finally, we can use the γ_0 equation to get σ^2.

ii. The OLS estimates are:

	Estimate	Newey-West S.E.
ϕ_1	1.2144	0.0394
ϕ_2	-0.3424	0.0393

The estimate of $\phi(1)$ is 0.87 [0.017]. The number in brackets is the Newey-West standard error with $q = 5$. Inverting this to get the parametets of the $\text{ARMA}(1,1)$ model as discussed in the previous subquestion results in:

α	0.8315
β	0.4168
σ^2	0.4717

iii. I don't know what this means...

Q4. I went with qustion (i). Here are the results for SMM, using autocovariances of order 0, 1, and 2 as the moment conditions:

α	0.8383
β	0.4365
σ^2	0.4573

Here are the results using indirect inference, with an AR(2) as the auxiliary model:

α	0.8405
β	0.4379
σ^2	0.5000

For each of the results above, I used 501 simulations, and kept $T = 500$. I used the same sample of iid shocks in both simulation estimations.