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Problem Set 1

Due Tuesday, October 10, 2017 in class.

1 Preliminaries

There is a measure one continuum of agents, indexed by $i \in [0,1]$.

(a) There is an aggregate random variable θ drawn from the following distribution

$$\theta \sim \mathcal{N}\left(\mu, \sigma_\theta^2\right)$$

Let $\kappa_\theta = 1/\sigma_\theta^2$. Each agent receives an exogenous private signal about θ given by

$$x_i = \theta + \xi_i$$

where $\xi_i \sim \mathcal{N}(0, \sigma_x^2)$ is idiosyncratic noise (i.i.d. across agents). Let $\kappa_x = 1/\sigma_x^2$ denote the precision of this private signal.

(i) What is the agent's posterior over θ?

(ii) Suppose each agent chooses some action $a_i = E_i[\theta]$ where E_i denotes agent i's expectation conditional on his information set. Solve for the aggregate action $\bar{a} \equiv \int a_i di$.

(b) There is an aggregate random variable θ drawn from the improper uniform over the real line. Each agent receives an exogenous private signal about θ given by

$$x_i = \theta + \xi_i$$

and an exogenous public signal about θ given by

$$y = \theta + \varepsilon$$

where $\xi_i \sim \mathcal{N}(0, \sigma_x^2)$ is idiosyncratic noise (i.i.d. across agents) and $\varepsilon \sim \mathcal{N}(0, \sigma_y^2)$ is a common error. Let $\kappa_x = 1/\sigma_x^2$ and $\kappa_y = 1/\sigma_y^2$ denote the precisions of the private and public signals, respectively.

(i) What is the agent's posterior over θ?

(ii) Suppose each agent chooses some action $a_i = E_i[\theta]$ where E_i denotes agent i's expectation conditional on his information set. Solve for the aggregate action $\bar{a} \equiv \int a_i di$.

(c) Are there any differences between the equilibrium aggregate actions in parts (a) and (b)? If so, explain.

2 Currency attacks and Government Intervention

There is a measure one continuum of private-sector agents, indexed by $i \in [0,1]$. Each agent i chooses whether or not to attack a currency $a_i \in \{0,1\}$

$$a_i = \begin{cases} 0 & \text{if} \quad \text{not attack} \\ 1 & \text{if} \quad \text{attack} \end{cases}$$

Let A denote the mass of agents attacking.

$$A = \int a_i di$$

The regime can either change, $R = 1$, or not change, $R = 0$, depending on the size of the attack. A regime change in this game is the devaluation of the currency–this devaluation occurs ($R = 1$) if and only if

$$A > \theta + e.$$

Otherwise, there is no regime change, $R = 0$. Here $\theta \in \mathbb{R}$ represents the exogenous underlying economic fundamentals of the country and e is adjustment effort taken by the domestic government. Note that e increases the region of no devaluation, but this will come at a cost.

Payoffs. Here we describe the payoffs for (i) the private-sector agents, and (ii) the government

(i) Agents who attack get payoff $1 - c$. Agents who do not attack get payoff

$$v(\theta, A) = \begin{cases} 0 & \text{if} \quad A > \theta + e \\ 1 & \text{if} \quad A \leq \theta + e \end{cases}$$

Thus, the payoffs of an agent may be written as

	$R = 1$	$R = 0$
$a_i = 1$	$1 - c$	$1 - c$
$a_i = 0$	0	1

(ii) The government's payoffs are given by

$$v(\theta, A) - \chi(e)$$

Thus, there is a cost of adjustment effort.

Timing.

1. Nature draws θ from the improper uniform. This is unobservable.
2. All agents and the government observe a public signal about θ given by

$$y = \theta + \varepsilon \quad \text{with } \varepsilon \sim \mathcal{N}(0, 1/\alpha), \quad \text{i.i.d.}$$

3. The government chooses e. This parameter is then common knowledge.
4. Each agent observes a private signal about θ given by

$$x_i = \theta + \xi_i \quad \text{with } \xi_i \sim \mathcal{N}(0, 1/\beta), \quad \text{i.i.d.}$$

5. Each agent chooses whether or not to attack.

(a) Show that for strategic purposes, the agents' payoff matrix is equivalent to the usual one.

$$\begin{array}{ccc} & R=1 & R=0 \\ a_i = 1 & 1-c & -c \\ a_i = 0 & 0 & 0 \end{array}$$

(b) Given e and the public signal y, characterize the equilibrium in the last stage of the game. Let $x^*(y), \theta^*(y)$ denote the threshold functions of this equilibrium.

(c) Show that the condition

$$\frac{\alpha}{\sqrt{\beta}} \leq \sqrt{2\pi}$$

is sufficient for uniqueness of the equilibrium in the last stage subgame. From now on assume that this condition holds.

(d) Assume $\beta \to \infty$ and solve for θ^*. How does θ^* depend on y, e, and c? Give intuition for each of these.

(e) State the government's problem in the first stage. Continue to assume $\beta \to \infty$ and furthur assume $\chi(e) = e^2$. Solve for the government's optimal effort choice e. (For this part, if it makes it easier, assume $\alpha \to \infty$).

3 Typicality of Beliefs

There is a measure one continuum of agents, indexed by $i \in [0,1]$. Each agent i chooses whether or not to invest, $a_i \in \{0,1\}$. Payoffs are given by

$$u(a_i, A) = a_i(\theta + A)$$

where A denotes aggregate investment,

$$A = \int a_i di$$

and θ is a random variable with support on the real line.

(a) Assume θ is common knowledge and solve for the equilibrium/equilibria.

(b) Now assume θ is an unknown random variable. Agents observe a private signal x_i about θ. The agent's system of beliefs is such that for all x_i

$$\mathbb{E}[\theta | x_i] = x_i$$

and for any x_j,

$$\Pr[x_j < x_i | x_i] = \Pr[x_j > x_i | x_i] = \frac{1}{2}$$

We interpret this as each agent believes he is "typical". Solve for the equilibrium and prove uniqueness.

(c) Now assume the usual Morris Shin system of private and public signals. θ is an unknown random variable drawn from an improper uniform over the real line. Each agent observes a private signal

$$x_i = \theta + \nu_i \quad \text{with } \nu_i \sim \mathcal{N}(0, 1/\alpha), \text{ i.i.d.}$$

and a exogenous public signal

$$z = \theta + \varepsilon \quad \text{with } \varepsilon \sim \mathcal{N}(0, 1/\beta)$$

Compute the probabilities $\Pr[x_j < x_i | x_i]$ and $\Pr[x_j > x_i | x_i]$ for arbitrary precisions.

(d) Show that the two probabilities $\Pr[x_j < x_i | x_i]$ and $\Pr[x_j > x_i | x_i]$ found in part (c) converges to $1/2$ for any (x_i, z) when the precision of private information goes to infinity; that is, $\alpha \to \infty$. That is, we generate "typicality of beliefs". Relate this to the uniqueness result found in part (b).

(e) Finally, assume θ is an unknown random variable and agents observe a private signal x_i. The agent's system of beliefs is such that for all x_i

$$\mathbb{E}[\theta | x_i] = x_i$$

and for any x_j,

$$\begin{aligned} \Pr[x_j = x_i | x_i] &= \lambda \\ \Pr[x_j < x_i | x_i] &= \Pr[x_j > x_i | x_i] = \frac{1}{2}(1-\lambda) \end{aligned}$$

Show that you can construct two threshold equilibria: one with

$$x^* = -\lambda - \frac{1}{2}(1-\lambda)$$

and the other with

$$x^* = -\frac{1}{2}(1-\lambda)$$

What happens when $\lambda \to 0$? What happens when $\lambda \to 1$? Interpret your results.

4 Investment and Complementary

There are two types of agents. There is a measure one continuum of entrepreneurs indexed by $i \in [0, 1]$ and a measure one continuum of traders indexed by $j \in [0, 1]$. All agents consume at the end of the game.

There are two stages, a financial market and an investment game. In the first stage, the traders trade a financial asset. In the second stage, the entrepreneurs make an effort/investment choice. After both stages are completed, payoffs are realized and agents consume.

Information. Before stage 1, aggregate productivity θ is drawn from the improper uniform over the real line. Productivity is unobserved by the agents.

Each entrepreneur observes a private signal about θ

$$x_i = \theta + \xi_i \quad \text{with } \xi_i \sim N\left(0, 1/\kappa_x\right)$$

and each trader observes a private signal about θ

$$x_j = \theta + \xi_j \quad \text{with } \xi_j \sim N\left(0, 1/\kappa_x\right)$$

Note that the noises ξ_i and ξ_j are i.i.d. across all agents and all drawn from the $N\left(0, 1/\kappa_x\right)$ distribution.

Once these signals are observed, the game transitions to stage 1 and then 2.

Stage 1: The financial market. In the first stage, the traders trade in a financial asset. Traders have CARA utility with coefficient of absolute risk aversion γ

$$v_j = -\mathbb{E}_j \exp\left\{-\gamma c_j\right\}$$

Consumption of trader j is given by

$$c_j = w_0 + (R - p)\, k_j$$

where w_0 is their initial endowment of wealth and $k_j \in \mathbb{R}$ denotes the trader's demand for the asset. The financial asset has dividend R and is traded at price p.[1] The dividend R is determined in stage 2.

The exogenous supply of the asset is ε and is drawn from $N\left(0, \sigma_\varepsilon^2\right)$. Let $\kappa_\varepsilon = 1/\sigma_\varepsilon^2$. Market clearing is given by

$$K = \varepsilon$$

where $K = \int k_i di$ denotes the aggregate demand for the asset.

Each trader's information set consists of his private signal x_j about θ, and the equilibrium price p of the financial asset.

Stage 2: The investment game. This investment game in the second stage is as follows. Each entrepreneur chooses his own effort (investment) level ℓ_i. The payoff for the entrepreneur is

$$u_i = R\ell_i - \frac{1}{2}\ell_i^2$$

where $\ell_i \in \mathbb{R}$ denotes his effort (investment) level. The payoff of the entrepreneur is thus the return R to effort, minus the cost $\ell_i^2/2$ of effort. The return to effort is given by

$$R = (1 - \beta)\,\theta + \beta L$$

where θ is the exogenous productivity shock (i.e. the fundamental) and L is the aggregate amount of effort given by

$$L = \int \ell_i di$$

The parameter $\beta \in (0, 1)$ is thus the degree of strategic complementarity.

When making his effort decision, each entrepreneur's information set consists of his private signal x_i about θ, and the equilibrium price p of the financial asset that prevailed in stage 1.

[1] This consumption just comes from normalizing the risk-free rate to zero.

(a) Give a formal definition of equilibrium for the entire game.

Notation: Let $\ell = \ell(x, p)$ denote the effort strategy for an entrepreneur, let $k = k(x, p)$ denote the investment strategy for a trader, let $L(\theta, p)$ and $K(\theta, p)$ denote the corresponding aggregates, and let $p = P(\theta, \varepsilon)$ denote the equilibrium price function.

(b) Suppose that

$$P(\theta, \varepsilon) = \theta + \lambda\varepsilon$$

for some $\lambda \neq 0$ and let $\kappa_p \equiv \kappa_\varepsilon/\lambda^2$. Consider the equilibrium in the second stage for a given realization of p. Solve for the equilibrium strategy $\ell(x, p)$ and aggregate $L(\theta, p)$.

(c) Show that, in equilibrium, $R = R(\theta, p)$ and solve for the function $R(\theta, p)$. Interpret the dependence of R on p.

(d) Now consider the equilibrium in the first stage. Solve for the equilibrium demand function $k(x, p)$ and the aggregate $K(\theta, p)$.

(e) Show that market clearing indeed implies that

$$p = P(\theta, \varepsilon) = \theta + \lambda\varepsilon$$

for some λ. Find an equation that (implicitly) determines the equilibrium value for λ as a function of the exogenous parameters.

What are the comparative statics of λ with respect to κ_ε and with respect to γ?

(f) Compute the volatility of the price and the volatility of the aggregate effort (investment) level for given fundamentals,

$$Var(p|\theta) \quad \text{and} \quad Var(L|\theta)$$

in terms of the exogenous parameters.