Problem Set 1
Econ G6220 [La'O]
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1 Preliminaries

(a) **(i)** Upon seeing the realization of x_i, the agent's posterior is

$$\boxed{N\left(\frac{\kappa_\theta \mu + \kappa_x x_i}{\kappa_\theta + \kappa_x}, \frac{1}{\kappa_\theta + \kappa_x}\right)}$$

(ii) Since ξ_i is normal with mean 0, $\int \xi_i di = 0$. So, we can compute the aggregate action:

$$\begin{aligned}
\overline{a} &\equiv \int a_i di \\
&= \int \mathbb{E}[\theta \mid x_i] di \\
&= \int \frac{\kappa_\theta \mu + \kappa_x x_i}{\kappa_\theta + \kappa_x} di \\
&= \frac{\kappa_\theta \mu}{\kappa_\theta + \kappa_x} + \frac{\kappa_x}{\kappa_\theta + \kappa_x} \int (\theta + \xi_i) di \\
&= \boxed{\frac{\kappa_\theta \mu + \kappa_x \theta}{\kappa_\theta + \kappa_x}}
\end{aligned}$$

(b) **(i)** Upon observing the realizations of x_i and y, the private and public signals, the agent's posterior is

$$N\left(\frac{\kappa_x x_i + \kappa_y y}{\kappa_x + \kappa_y} \frac{1}{\kappa_x + \kappa_y}\right)$$

(ii) The aggregate action is:

$$\begin{aligned}
\overline{a} &= \int a_i di \\
&= \int \mathbb{E}_i[\theta] di \\
&= \int \frac{\kappa_x x_i + \kappa_y y}{\kappa_x + \kappa_y} di \\
&= \frac{\kappa_x \theta + \kappa_y (\theta + \varepsilon)}{\kappa_x + \kappa_y} \\
&= \theta + \frac{\kappa_y \varepsilon}{\kappa_x + \kappa_y}
\end{aligned}$$

(c) In the private-only setup (a), aggregate action will end up being a weighted average of the true mean of θ and the actual draw of θ. In the setup with a public signal, the action is θ plus a weighted-down amount of the public signal. In the private setup, however, aggregate action will tend to μ as the precision of the public signal deteriorates; in the public one, aggregate action will tend to the public signal. In the first setup, the signal pulls aggregate action towards the fundamental. With the public signal, the aggregate action is pulled away!

[1] I collaborated with Joe Saia.

2 Currency Attacks and Government Intervention

(a) When the agent is trying to decide between the two actions, what matters is the difference in the payoff between the two actions. Under the original payoff structure, this is

$$\begin{aligned} U(1,A,\theta) - U(0,A,\theta) &= \begin{cases} 1-c & \text{if } R=1 \\ 1-c-1 & \text{if } R=0 \end{cases} \\ &= \begin{cases} 1-c & \text{if } R=1 \\ -c & \text{if } R=0 \end{cases} \\ &= 1\cdot(\mathbb{1}_{R=1} - c) - 0\cdot(\mathbb{1}_{R=1} - c). \end{aligned}$$

That is, the relative payoffs under the original payoff structure are the same as they would be under the formula $a_i(\mathbb{1}_{R=1} - c)$, which results in the standard payoff matrix:

	$R=1$	$R=0$
$a_i = 1$	$1-c$	$-c$
$a_i = 0$	0	0

This may also have to do with the fact that each agent, being measure 0, cannot influence the probability that one of the states realizes. So, subtracting off a constant from the payoffs of both actions within a state will not affect the agents decision.

(b) Start by fixing e and a public signal y. I'll focus on a threshold equilibrium.[2] I'm looking for a strategy, $x^*(e,y)$, such that agents will attack iff

$$x \leq x^*(e,y).$$

This implies that everyone with a private signal above $x^*(e,y)$ doesn't attack, and vice versa. So, the aggregate size of the attack is decreasing in θ. This implies that there exists some threshold $\theta^*(e,y)$ such that there is a regime change iff

$$\theta \leq \theta^*(e,y).$$

Now, let's compute these. Start by characterizing θ^*, given x^*. Note that since $x \sim N(\theta, 1/\beta)$, we know that $\sqrt{\beta}(x-\theta) \sim N(0,1)$. So, the size of the attack is the mass of agents receiving a signal below x^*, or:

$$A(\theta,e,y) = \Phi\left(\sqrt{\beta}(x^*(e,y) - \theta)\right). \tag{1}$$

If the aggregate size of the attack is above $\theta + e$, then regime shift occurs. So, the regime-change threshold is the solution to:

$$A(\theta^*(e,y),e,y) = \theta^*(e,y) + e \tag{2}$$

So, combining (1) and (2), we have

$$\begin{aligned} &\Phi^{-1}(\theta^*(e,y)+e) = \sqrt{\beta}(x^*(e,y) - \theta)) \\ \implies\; & x^*(e,y) = \frac{1}{\sqrt{\beta}}\Phi^{-1}(\theta^*(e,y)+e) + \theta^*(e,y) \end{aligned} \tag{3}$$

[2] I suppose that one must further show existence of such a threshold equilibrium, but I leave that for another time.

Next, given θ^*, we want to characterize x^*. First, note that the posterior distribution of θ, having seen the two signals, is

$$\theta|_{x,y} \sim N\left(\frac{\beta}{\eta}x + \frac{\alpha}{\eta}y, \frac{1}{\eta}\right) \implies \sqrt{\eta}\left(\theta_{x,y} - \left(\frac{\beta}{\eta}x + \frac{\alpha}{\eta}y\right)\right) \sim \Phi$$

where $\eta \equiv \beta + \alpha$. Payoffs are of the form $a_i(\mathbb{1}_{R=1} - c)$, and the person is indifferent at the threshold—that is, the expected payoff from attacking is equal to the payoff from not attacking:

$$\begin{aligned}
\underbrace{0}_{\text{Not Attack}} &= \underbrace{\mathbb{E}[u(1, A(\theta, e, y), \theta) \mid x^*(e,y), y]}_{\text{Attack}} \\
&= a_i\left(\mathbb{P}[\theta \le \theta^*(e,y)] - c\right) \\
&= a_i\left(\Phi\left[\sqrt{\eta}\left(\theta^*(e,y) - \left(\frac{\beta}{\eta}x^*(e,y) + \frac{\alpha}{\eta}y\right)\right)\right] - c\right) \\
\implies c &= 1 - \Phi\left[\sqrt{\eta}\left(\frac{\beta}{\eta}x^*(e,y) + \frac{\alpha}{\eta}y - \theta^*(e,y)\right)\right] \\
&= 1 - \Phi\left[\sqrt{\eta}\left(\frac{\beta}{\eta}\left[\frac{1}{\sqrt{\beta}}\Phi^{-1}(\theta^*(e,y) + e) + \theta^*(e,y)\right] + \frac{\alpha}{\eta}y - \theta^*(e,y)\right)\right] && \text{by (3)} \\
\implies \frac{1}{\sqrt{\eta}}\Phi^{-1}(1-c) &= \theta^*(e,y)\left[\frac{\beta}{\eta} - 1\right] + \frac{\alpha}{\eta}y + \frac{\sqrt{\beta}}{\eta}\Phi^{-1}(\theta^*(e,y) + e) \\
&= \theta^*(e,y)\left[\frac{-\alpha}{\eta}\right] + \frac{\alpha}{\eta}y + \frac{\sqrt{\beta}}{\eta}\Phi^{-1}(\theta^*(e,y) + e) \\
\implies \underbrace{\left(\sqrt{\frac{\eta}{\beta}}\right)\Phi^{-1}(1-c)}_{g} &= \underbrace{\frac{\alpha}{\sqrt{\beta}}[y - \theta^*(e,y)] + \Phi^{-1}(\theta^*(e,y) + e)}_{G(\theta^*(e,y),y)} && \text{mult. by } \frac{\eta}{\sqrt{\beta}}
\end{aligned}$$

Now, I claim that there's a monotone equilibrium characterizes by $\theta^*(e,y)$ and $x^*(e,y)$ where θ^* is implicitly defined by $G(\theta^*(e,y), y) = g$ as in the previous expression, and x^* is as in (3). Let's verify that there is some point at which $G = g$. I claim that the point is somewhere between $[-e, 1-e]$. Because, for fixed e and y, G is continuous in θ, and we know that

$$\begin{aligned}
G(-e, y) &= \frac{\alpha}{\sqrt{\beta}}y + \Phi^{-1}(-e + e) = -\infty \\
\text{and } G(1-e, y) &= \frac{\alpha}{\sqrt{\beta}}y + \Phi^{-1}(1 - e + e) = \infty
\end{aligned}$$

that is, G goes off to negative infinity to the left, and positive infinity to the right, so by the IVT, it must intersect g at some point in between. (This is cool–the government can control the threshold!)

(c) Now, for uniqueness. We need to figure out when it's the case that G is strictly increasing, and when it's possible that it decreasesing. So, let's look at its derivative:

$$\frac{\partial G(\theta, y)}{\partial \theta} = \frac{1}{\phi(\Phi^{-1}(\theta + e))} - \frac{\alpha}{\sqrt{\beta}}$$

So, because

$$\phi(\omega) \in \left[0, \frac{1}{\sqrt{2\pi}}\right], \text{ then } \frac{1}{\phi(\omega)} \in [\sqrt{2\pi}, \infty],$$

and the slope is only guaranteed to be positive throughout the support of θ if

$$\frac{\alpha}{\sqrt{\beta}} < \sqrt{2\pi}$$

otherwise, there is a range of $\theta(e,y)$ where the solution is not unique.

(d) Letting $\beta \to \infty$ in the implicit definition for G, we have

$$\Phi^{-1}(1-c) = \Phi^{-1}(\theta^*(e,y)+e) \Longrightarrow \; 1-c-e = \theta^*(e,y).$$

By letting $\beta \to \infty$, we are saying that the distribution of x_i collapses to unit mass at θ; individuals perfectly know θ. Unsurprisingly, the threshold $x^* = \theta^*$.

The threshold no longer depends on the public signal. Everybody knows θ perfectly, so there's no reason to think that an imperfect measure of θ would change the equilibrium threshold value.

Now, this threshold is decreasing in e and c. Why? Well, if there were no costs to attacking and the government wasn't strengthening the regime, then agents would always attack.[3] By increasing the costs to attacking (increasing c), or by making it harder for an attack to succeed (increasing e), there are fewer values of the fundamental for which agents will attack. That is, if it's less likely that an attack will be profitable, agents will require seeing a weaker fundamental. Since there's no strategic complementarity effect here, we only need to think about fundamentals.

(e) The (benevolent) government wants to maximize the utility of non-attackers; it wants the attack to not go through. But it also has to pay a cost. And it knows what will happen in the second stage conditional on its own efforts, e. Remember that, since $\beta \to 0$, everyone either attacks or doesn't attack, based on their private signal. Unfortunately, the government only imperfectly measures θ. Therefore, the government's objective function is:

$$\begin{aligned}
\mathbb{E}[v(\theta, A) \mid y] - e^2 &= 0 \cdot \mathbb{P}[A > \theta + e \mid y] + 1 \cdot \mathbb{P}[A \leq \theta + e \mid y] - e^2 \\
&= \mathbb{P}[A \leq \theta + e \mid y] - e^2 \\
&= \mathbb{P}[\theta < \theta^* \mid y] - e^2 \\
&= \mathbb{P}[\theta < 1 - c - e \mid y] - e^2
\end{aligned}$$

Let's assume $\alpha \to \infty$. So, now we're assuming that the government knows θ. If people were still playing the threshold strategy, then the government would know that everyone will attack if $\theta < 1-c-e$. But now we're in this silly case where everyone knows θ, so it may actually me more-correct to say that everyone will attack if they know that $\theta + e < 1$, and ignore the cost. [4]

Let's assume that, if people are indifferent, they don't attack. We could handle the other case but it wouldn't be very enlightening.

- If $\theta \geq 1$, then the government can't do anything, but they are happy because nobody will attack. So, its best response is to set $e = 0$.
- If $\theta < 1$, then the cheapest way that the government can get people not to attack is to set $e = 1 - \theta$—that is, they prop up the regime to "full strength." If they do this, they spend e, but they get utility of 1. So, they should do this iff

$$\begin{aligned}
&1 - (1-\theta)^2 > 0 \\
\Longleftrightarrow\; &1 > |1-\theta| \\
\Longleftrightarrow\; &1 > 1-\theta \text{ and } 1 > -1+\theta \\
\Longleftrightarrow\; &\theta > 0 \text{ and } 2 > \theta
\end{aligned}$$

Recalling that we are in the case $\theta < 1$, this means it is worth spending the money if $\boxed{\theta \in (0,1)}$.

[3] Well, as long as it is profitable—that is, $\theta \leq 1$.

[4] Before I realized this "more correct" thing, this was my answer: If $\theta \geq 1-c$, then the government can't do anything because everyone will attack regardless—their best response is to set $e = 0$. If $\theta < 1-c$, then the cheapest way to people to not attack is to set $e = 1-c-\theta$. If they do this, they spend e, but get utility of 1. So, this is worth it if

$$\begin{aligned}
&1 - (1-c-\theta)^2 > 0 \\
\Longleftrightarrow\; &1 > |1-c-\theta| \\
\Longleftrightarrow\; &1 > 1-c-\theta \text{ and } 1 > -1+c+\theta \\
\Longleftrightarrow\; &\theta > -c \text{ and } 2-c > \theta
\end{aligned}$$

So, they should spend the money for $\theta \in (-c, 2-c) \cap (\infty, 1-c)$, i.e., $\boxed{\theta \in (-c, 1-c)}$.

3 Typicality of Beliefs

(a) An agent choosing $a_i = 0$ is guaranteed to get 0. It's individually rational to choose $a_i = 1$ iff

$$\theta + A > 0 \Longrightarrow \theta > -A$$

Since $A \in [0, 1]$, then it is rational to choose $a_i = 1$ iff $\theta > -1$. So, two sets of equilibria are characterized by:

$$\begin{aligned} \theta &\leq -1 & a_i &= 0, \ \forall i \\ \theta &> -1 & a_i &= 1, \ \forall i \end{aligned}$$

There are a few cases to consider when $\theta \in [-1, 0]$ (for $\theta > 0$, people will always attack). First, if everyone believes that $A > -\theta$, then attacking is a BR, so everyone will attack and, indeed, we will have $A = 1 > -\theta$. Symmetrically for everyone believes $A < -\theta$. If $A = -\theta$, then there is an equilibrium in which agents can choose anything as long as the measure of agents attacking equals θ.

(b) I've got a hunch that there's going to be some threshold rule in which the threshold is $\frac{1}{2}$, but let's see. I'm going to posit that there's some threshold x^* such that if $x_i < x^*$, then agent i will choose to invest. So, at the threshold, the agent is indifferent between investing and not investing. That is,

$$\begin{aligned} 0 &= \mathbb{E}\left[\theta + A \mid x_i^*\right] \\ &= x_i^* + \mathbb{E}\left[\int a_i di \middle| x_i^*\right] \\ &= x_i^* + \frac{1}{2} \\ \Longrightarrow &\ \boxed{x_i^* = -\frac{1}{2}} \end{aligned}$$

The agent is following the threshold, and she believes that everyone else is. When she gets x_i^*, she things "well, half of the people got above x_i^*, and half got below. So, half are going to invest, half aren't. That's what happens with the integral.

Suppose you received $x_i > x^*$, but decided not to invest. Then you would get 0. Your expected value from investing would be

$$\mathbb{E}[A + \theta] = \mathbb{E}[A] + x_i > \frac{1}{2} + x_i > \frac{1}{2} + x^* = 0$$

positive.[5] So, there's no incentive to deviate.

(c) The posterior distribution of θ is

$$\theta \mid x_i, z \sim N\left(\frac{\alpha}{\eta}x_i + \frac{\beta}{\eta}z, \frac{1}{\eta}\right)$$

where $\eta \equiv \alpha + \beta$. The distribution of $\theta + \nu_j \mid x_i$ is

$$\theta + \nu_j \mid x_i \sim N\left(\frac{\alpha}{\eta}x_i + \frac{\beta}{\eta}z, \frac{1}{\eta} + \frac{1}{\alpha}\right) \stackrel{d}{=} \left(\frac{\alpha}{\eta}x_i + \frac{\beta}{\eta}z, \frac{2\alpha + \beta}{\eta\alpha}\right)$$

So that

$$\sqrt{\frac{\eta\alpha}{2\alpha + \beta}}\left[\theta + \nu_j - \left(\frac{\alpha}{\eta}x_i + \frac{\beta}{\eta}z\right)\right] \sim N(0, 1)$$

[5] You think half of the people are above you and following the threshold strategy, so $\mathbb{E}[A] > \frac{1}{2}$

Thus,

$$\begin{aligned}\mathbb{P}[x_j < x_i \mid x_i] &= \mathbb{P}[\theta + \nu_j < x_i \mid x_i] \\ &= \Phi\left\{\sqrt{\frac{\eta\alpha}{2\alpha+\beta}}\left[x_i - \left(\frac{\alpha}{\eta}x_i + \frac{\beta}{\eta}z\right)\right]\right\}\end{aligned}$$

The other is symmetric.

(d)

$$\begin{aligned}\mathbb{P}[x_j < x_i \mid x_i] &= \Phi\left\{\sqrt{\frac{\eta\alpha}{2\alpha+\beta}}\left[x_i - \left(\frac{\alpha}{\eta}x_i + \frac{\beta}{\eta}z\right)\right]\right\} \\ &= \Phi\left\{\sqrt{\frac{\alpha^2+\alpha\beta}{2\alpha+\beta}}\left[x_i - \left(\frac{\alpha}{\alpha+\beta}x_i + \frac{\beta}{\alpha+\beta}z\right)\right]\right\} \\ &= \Phi\left\{\underbrace{\infty}_{\text{at rate }\sqrt{\alpha}}\left[\underbrace{x_i - x_i}_{\text{at rate }\alpha}\right]\right\} \\ &= \Phi(\underbrace{\infty}_{\text{slow}}\cdot\underbrace{0}_{\text{fast}}) \\ &= \Phi(0) \\ &= \frac{1}{2}.\end{aligned}$$

As related Morris and Shin, we definitely have arbitrarily small private noise, so we definitely have uniqueness. So it's good that I found uniqueness above.

(e) If the agent believes that those at the threshold will not break their indifference in favor of not investing, then we have one threshold equilibrium characterized by the following indifference point:

$$\begin{aligned}0 &= \mathbb{E}\left[\theta + A \mid x_i^*\right] \\ &= x_i^* + \mathbb{E}[A \mid x_i^*] \\ &= x_i^* + \frac{1}{2}(1-\lambda) \\ \implies x_i^* &= -\frac{1}{2}(1-\lambda).\end{aligned}$$

If, on the other hand, the agent believes that those at the threshold will invest when they're indifferent, then the other threshold equilibrium will be characterized by:

$$\begin{aligned}0 &= \mathbb{E}\left[\theta + A \mid x_i^*\right] \\ &= x_i^* + \underbrace{\lambda}_{\text{measure of indifferent agents}} + \frac{1}{2}(1-\lambda) \\ \implies x_i^* &= -\lambda - \frac{1}{2}(1-\lambda).\end{aligned}$$

When $\lambda \to 0$, there is only one equilibrium with the threshold strategy equal to that from part (b): people think they're exactly in the middle of the signals.

With $\lambda \to 1$, the person thinks that everyone is like them. When this is the case, there are two equilibria: one with $x^* = -1$, and one with $x^* = 0$. In the first equilibrium, people invest with $x_i > -1$, and they believe that everyone else has the same signal and is investing, so they must either all invest and $A = 1$. But, with $A = 1$, expected payoffs are 0. This seems less stable than the other threshold equilibrium.

4 Investment and Complementary

(a) An equilibrium in this game is given by a demand function for the asset, $k_j(x_j, p)$; an effort function $\ell_i(x_i, p)$; and a price function $p = P(\theta, \varepsilon)$ such that:

1. traders maximize utility,

$$k_j(x_j, p) \in \underset{k_j}{\operatorname{argmax}} - \mathbb{E}\left[\exp\left\{-\gamma(w_0 + (R-p)k_j)\right\} \middle| x_j, p\right], \ \ \forall j,$$

2. entrepreneurs maximize utility,

$$\ell_i(x_i, p) \in \underset{\ell_i}{\operatorname{argmax}}\, \mathbb{E}\left[R\ell_i - \frac{1}{2}\ell_i^2 \middle| x_i, p\right], \ \ \forall i,$$

3. and markets clear,

$$K = \varepsilon\,.$$

$K \equiv \int k_i di$ is aggregate demand; aggregate supply is denoted by $\varepsilon \sim N(0, \sigma_\varepsilon^2)$; the return to effort is $R \equiv (1-\beta)\theta + \beta L$; and aggregate effort is $L = \int \ell_i di$. Let's impose a symmetric equilibrium whereby

$$\begin{aligned} k(x,p) &\equiv k_j(x,p), \ \ \forall j \\ \ell(x,p) &\equiv \ell_i(x,p), \ \ \forall i. \end{aligned}$$

(b) We conjecture that $p = P(\theta, \varepsilon) = \theta + \lambda\,\varepsilon$.

$$p = \theta + \lambda\,\varepsilon \sim N\left(\theta, \frac{\lambda^2}{\kappa_\varepsilon}\right) \equiv N\left(\theta, \frac{1}{\kappa_p}\right)$$

The entrepreneur's posterior distribution of θ is

$$\theta | x_i, p \sim N\left(\frac{\kappa_x}{\kappa_x + \kappa_p}x_i + \frac{\kappa_p}{\kappa_x + \kappa_p}p, \frac{1}{\kappa_x + \kappa_p}\right) \tag{4}$$

Let's further guess some linear forms for the equilibrium strategies:

$$\begin{aligned} L(\theta, p) &= \Phi_\theta\theta + \Phi_p p \\ \ell(x_i, p) &= \phi_x x_i + \phi_p p \end{aligned}$$

The FOC of the entrepreneur utility maximization is

$$\mathbb{E}_i[R] = \ell_i \iff (1-\beta)\,\mathbb{E}[\theta \mid p, x_i] + \beta\Phi_\theta\,\mathbb{E}[\theta \mid p, x_i] + \beta\Phi_p p = \ell_i$$

Plugging in expectations yields:

$$\begin{aligned} \ell_i &= (1-\beta+\beta\Phi_\theta)\left(\frac{\kappa_x}{\kappa_x+\kappa_p}x_i + \frac{\kappa_p}{\kappa_x+\kappa_p}p\right) + \beta\Phi_p p \\ &= \underbrace{(1-\beta+\beta\Phi_\theta)\left(\frac{\kappa_x}{\kappa_x+\kappa_p}\right)}_{\phi_x} x_i + \underbrace{\left(\frac{(1-\beta+\beta\Phi_\theta)\kappa_p}{\kappa_x+\kappa_p} + \beta\Phi_p\right)}_{\phi_p} p \end{aligned} \tag{5}$$

Aggregate effort is then given by

$$L(\theta, p) = \int \ell_i(x,p)f(x)dx = \int (\phi_x x_i + \phi_p p)\phi\left(\frac{x-\theta}{\sigma_x}\right)dx = \phi_x\theta + \phi_p p$$

So, $\phi_x = \Phi_\theta$ and $\phi_p = \Phi_p$. Using this in (5) reveals

$$\begin{aligned}
\phi_x &= (1-\beta+\beta\phi_x)\left(\frac{\kappa_x}{\kappa_x+\kappa_p}\right) \\
\implies \phi_x\left(1-\beta\left(\frac{\kappa_x}{\kappa_x+\kappa_p}\right)\right) &= (1-\beta)\left(\frac{\kappa_x}{\kappa_x+\kappa_p}\right) \\
\implies \phi_x\left(\frac{(1-\beta)\kappa_x+\kappa_p}{\kappa_x+\kappa_p}\right) &= (1-\beta)\left(\frac{\kappa_x}{\kappa_x+\kappa_p}\right) \\
\implies &\boxed{\phi_x = \frac{(1-\beta)\kappa_x}{(1-\beta)\kappa_x+\kappa_p}}
\end{aligned}$$

Again, using (5) to solve for ϕ_p reveals:

$$\begin{aligned}
\phi_p &= \frac{(1-\beta+\beta\Phi_\theta)\kappa_p}{\kappa_x+\kappa_p} + \beta\phi_p \\
\implies \phi_p &= \frac{1}{1-\beta}\left(\frac{(1-\beta)\kappa_p}{\kappa_x+\kappa_p} + \frac{\beta\kappa_p}{\kappa_x+\kappa_p}\phi_x\right) \\
\implies \phi_p &= \frac{\kappa_p}{\kappa_x+\kappa_p} + \frac{\beta\kappa_p}{\kappa_x+\kappa_p}\cdot\frac{\kappa_x}{(1-\beta)\kappa_x+\kappa_p} \\
\implies &\frac{\kappa_p}{\kappa_x+\kappa_p}\left(\frac{(1-\beta)\kappa_x+\kappa_p+\beta\kappa_x}{(1-\beta)\kappa_x+\kappa_p}\right) \\
\implies &\boxed{\phi_p = \frac{\kappa_p}{(1-\beta)\kappa_x+\kappa_p}}
\end{aligned}$$

(c) Plugging in and asking Matlab for some algebra help reveals:

$$\begin{aligned}
R &= (1-\beta)\theta + \beta(\phi_x\theta + \phi_p p) \\
&= \frac{\theta(1-\beta)(\kappa_p+\kappa_x) + \beta\kappa_p p}{(1-\beta)\kappa_x+\kappa_p}
\end{aligned} \tag{6}$$

R depends on p with coefficient $\frac{\beta\kappa_p}{(1-\beta)\kappa_x+\kappa_p}$ If p becomes arbitrarily precise, then $R = \beta p$ If p becomes infinitely variable, then R does not depend on it.

(d) Now we solve for equilibrium in the first stage. Again we conjecture guesses for the equilibrium demand function and aggregate demand:

$$\begin{aligned}
k(x,p) &= \delta_x x + \delta_p p \\
K(\theta,p) &= \Delta_\theta\theta + \Delta_p p
\end{aligned}$$

The trader's problem is

$$\begin{aligned}
&\max_{k_j} \ \mathbb{E}\left[\exp\{-\gamma(w_0+(R-p)k_j)\} \mid x_j\right] \\
\iff &\max_{k_j} \ \mathbb{E}\left[\exp\{-\gamma(R-p)k_j\} \mid x_j\right] \\
\iff &\max_{k_j} \ \mathbb{E}[-\gamma(R-p)k_j \mid x_j] + \frac{1}{2}\operatorname{var}[-\gamma(R-p)k_j \mid x_j]
\end{aligned}$$

Taking the FOC of this puppy yields:

$$\begin{aligned}\gamma p - \gamma\,\mathbb{E}[R \mid x_j] &= k_j \operatorname{var}[-\gamma(R-p) \mid x_j] \\ &= k_j \gamma^2 \operatorname{var}[R \mid x_j] \\ \Longrightarrow \quad \boxed{k_j = \frac{p - \mathbb{E}[R \mid x_j]}{\gamma \operatorname{var}[R \mid x_j]}} & \end{aligned}$$

Now, we need to do something about these expectations. Having observed the price and their signal, and using logic similar to what led us to (4), the trader's posterior over θ is given by

$$\theta | x_j, p \sim N\left(\frac{\kappa_x}{\kappa_x+\kappa_p} x_j + \frac{\kappa_p}{\kappa_x+\kappa_p} p, \frac{1}{\kappa_x+\kappa_p}\right)$$

Traders know what R will prevail in the second round: it's given by (6).[6] So,

$$\begin{aligned}\mathbb{E}[R \mid x_j] &= \mathbb{E}[(1-\beta+\beta\phi_x)\theta + \beta\phi_p p \mid x_j] \\ &= (1-\beta+\beta\phi_x)\left[\frac{\kappa_x}{\kappa_x+\kappa_p} x_j + \frac{\kappa_p}{\kappa_x+\kappa_p} p\right] + \beta\phi_p p\end{aligned}$$

The variance, on the other hand, is:

$$\begin{aligned}\operatorname{var}[R \mid x_j] &= \operatorname{var}[(1-\beta+\beta\phi_x)\theta + \beta\phi_p p \mid x_j] \\ &= (1-\beta+\beta\phi_x)^2 \left[\frac{1}{\kappa_x+\kappa_p}\right]\end{aligned}$$

Combining these with Matlab, we get

$$k_j = \underbrace{\frac{\kappa_x(\kappa_p + (1-\beta)\kappa_x)}{\gamma(\kappa_p+\kappa_x)(\beta-1)}}_{\delta_x = -\delta_p}(x-p) \tag{7}$$

This is the equilibrium demand function. Equilibrium aggregate demand integrates this:

$$K = \int k_j dj = \int \delta_x x_j + \delta_p p dF_x = \int (\delta_x x_j + \delta_p p)\phi\left(\frac{x-\theta}{\sigma_x}\right) dx = \delta_x\theta + \delta_p p$$

(e) Market clearing equates $K = \varepsilon$. Taking the previous expression reveals

$$\begin{aligned}\varepsilon &= k \\ &= \delta_x\theta + \delta_p p \\ \Longrightarrow \quad p &= \frac{1}{\delta_p}\,\varepsilon - \frac{\delta_x}{\delta_p}\cdot\theta \\ &= \underbrace{\frac{1}{\delta_p}}_{\equiv\lambda}\,\varepsilon + \theta.\end{aligned}$$

Recalling that $\kappa_p = \kappa_\varepsilon/\lambda^2$ and substituting this fact into (7), we have an implicit definition for λ:

$$\lambda = \frac{\gamma(\kappa_p+\kappa_x)(1-\beta)}{\kappa_x(\kappa_p+(1-\beta)\kappa_x)} = \frac{\gamma\left(\frac{\kappa_\varepsilon}{\lambda^2}+\kappa_x\right)(1-\beta)}{\kappa_x\left(\frac{\kappa_\varepsilon}{\lambda^2}+(1-\beta)\kappa_x\right)} \Longrightarrow \boxed{m(\kappa_\varepsilon,\gamma,\lambda(\kappa_\varepsilon,\gamma)) \equiv \lambda - \frac{\gamma\left(\frac{\kappa_\varepsilon}{\lambda^2}+\kappa_x\right)(1-\beta)}{\kappa_x\left(\frac{\kappa_\varepsilon}{\lambda^2}+(1-\beta)\kappa_x\right)} = 0}$$

[6] Good ol' backward inductioners.

Rewriting this gives:

$$\frac{1}{\lambda}\kappa_e\kappa_x + \lambda(1-\beta)\kappa_x - \frac{1}{\lambda^2}\kappa_e\gamma(1-\beta) - \kappa_x\gamma(1-\beta) = m(\kappa_e, \gamma, \lambda(\kappa_e, \gamma))$$

So, the implicit function theorem tells us that

$$\frac{\partial\,\lambda}{\partial\,\kappa_\varepsilon} = -\frac{\frac{\partial\,m}{\partial\,\kappa_\varepsilon}}{\frac{\partial\,m}{\partial\,\lambda}} = -\frac{\lambda^{-1}\kappa_x - \lambda^{-2}\gamma(1-\beta)}{-\lambda^{-2}\kappa_e\kappa_x + (1-\beta)\kappa_x + 2\lambda^{-3}\kappa_e\gamma(1-\beta)}$$

Which is difficult to sign; an upward sloping demand curve would imply $\lambda > 0$. The sign of this will depend on parameters, since both the top and bottom are ambiguous. Similarly:

$$\frac{\partial\,\lambda}{\partial\,\gamma} = -\frac{\frac{\partial\,m}{\partial\,\gamma}}{\frac{\partial\,m}{\partial\,\lambda}} = -\frac{-\lambda^{-2}\kappa_e(1-\beta) - \kappa_x(1-\beta)}{-\lambda^{-2}\kappa_e\kappa_x + (1-\beta)\kappa_x + 2\lambda^{-3}\kappa_e\gamma(1-\beta)}$$

Similarly difficult to sign. The top is likely positive (multiplying by the leading $-$ sign). But, again, the bottom is difficulty to sign. I imagine that we would get some type of "something cannot be too noisy" result, if I worked this through.

(f) For p:

$$\mathrm{var}[p \mid \theta] = \mathrm{var}[\lambda\,\varepsilon + \theta \mid \theta] = \frac{\lambda^2}{\kappa_\varepsilon}.$$

For Ł:

$$\mathrm{var}[L \mid \theta] = \mathrm{var}[\Phi_\theta\theta + \Phi_p p \mid \theta] = \Phi_p^2\,\mathrm{var}[p \mid \theta] = \left(\frac{\kappa_p}{(1-\beta)\kappa_x + \kappa_p}\right)^2 \frac{\lambda^2}{\kappa_\varepsilon}$$

So, these things can fluctuate because of beliefs.

Here's the famed Matlab code for #4.

```
clear all
syms phix phip beta kx kp p theta gamma x lambda ke

PHIX = solve(phix == (1-beta + beta * phix)*(kx/(kx+kp)));

PHIP = solve(phip == (1-beta+beta*PHIX)*kp/(kx+kp) + beta * phip);

R    = (1-beta)*theta + beta*(PHIX*theta + PHIP*p);

ER = (1-beta+beta*PHIX)*(kx*x + kp*p)/(kx+kp) + beta *PHIP*p;

VR = (1-beta + beta*PHIX)^2/(kx+kp);

kj = (p-ER)/(gamma*VR);

m = lambda - gamma*(ke/(lambda^2) + kx)*(1-beta)/(kx*ke/(lambda^2) + (1-beta)*kx);

pretty(simplify(-diff(m,ke)/diff(m,lambda)))
pretty(simplify(-diff(m,gamma)/diff(m,lambda)))
```