Problem Set 4
GR6413–Ng
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Q1.

i. $y_t \sim \text{poisson}(\theta)$; prior for θ: $\theta \sim G(\underline{\alpha}, \underline{\beta})$

$$\mathbb{P}(y_t = k) = \frac{\theta^k \exp(-\theta)}{k!} \qquad f(\theta) = \frac{\underline{\beta}^{\underline{\alpha}}}{\Gamma(\underline{\alpha})} \theta^{\underline{\alpha}-1} \exp(-\underline{\beta}\theta)$$

If y_t i.i.d., then $P(y_1 = k_1, \dots, y_T = k_T) \propto \prod_{t=1}^T \theta^{k_t} \exp(-\theta)$
$$= \exp(-\theta)\theta^{\Sigma k_t}$$

So, $p(\theta | y_t) \propto p(y_t|\theta)\, p(\theta)$ (Bayes' Rule)

$$\propto \exp(-\theta)\theta^{\Sigma k_t} \theta^{\underline{\alpha}-1} \exp(-\underline{\beta}\theta)$$
$$= \exp(-\theta(1+\underline{\beta}))\theta^{\underline{\alpha}-1+\Sigma k_t}$$
$$= \text{Gamma}\left(\underline{\alpha} + \Sigma_{t=1}^T k_t,\ \underline{\beta} + 1\right)$$
$$= \text{Gamma}\left(\underline{\alpha} + T\bar{k},\ \underline{\beta} + 1\right). \qquad \bar{k} \equiv \tfrac{1}{T}\Sigma_{t=1}^T k_t$$

ii. $P(y_t|\theta) = \begin{cases} \theta & y_t = 1 \\ 1-\theta & y_t = 0 \end{cases}$ prior for θ: $p(\theta) = \left[\frac{\Gamma(\underline{\alpha})\Gamma(\underline{\delta})}{\Gamma(\underline{\alpha}+\underline{\delta})}\right]^{-1} \theta^{\underline{\alpha}-1}(1-\theta)^{\underline{\delta}-1}$

$$= \theta^{y_t}(1-\theta)^{1-y_t}$$

The likelihood of the data, therefore, is

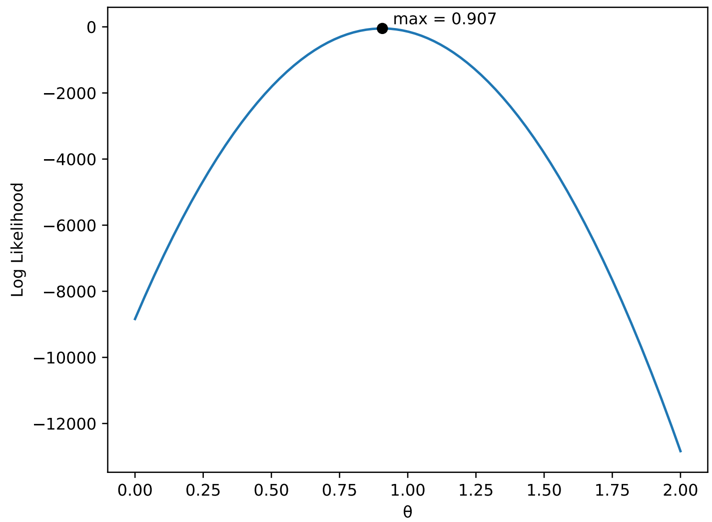
$$P(\{y_t\}_{t=1}^T | \theta) = \prod_{t=1}^T \theta^{y_t}(1-\theta)^{1-y_t}$$
$$= \theta^{\Sigma y_t}(1-\theta)^{\Sigma(1-y_t)}$$
$$= \theta^{T\bar{y}}(1-\theta)^{T(1-\bar{y})}$$

So, $P(\theta|y_t) \propto P(y_t|\theta)\, P(\theta)$

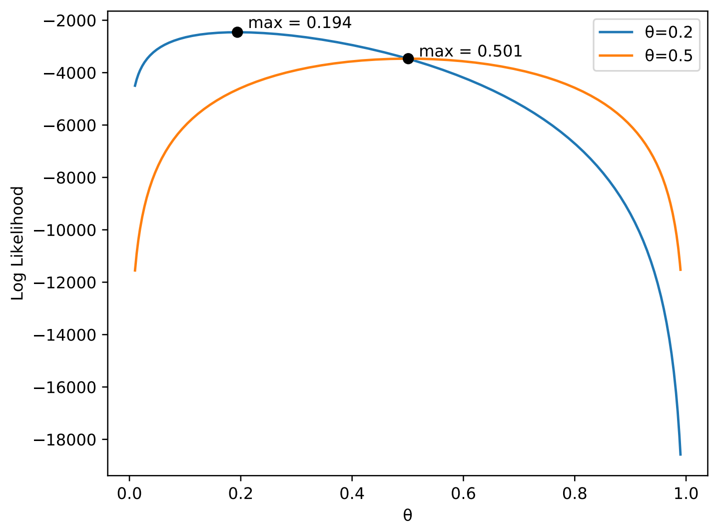
$$= \theta^{T\bar{y}}(1-\theta)^{T(1-\bar{y})}\theta^{\underline{\alpha}-1}(1-\theta)^{\underline{\delta}-1}$$
$$= \theta^{T\bar{y}+\underline{\alpha}-1}(1-\theta)^{T(1-\bar{y})+\underline{\delta}-1}$$
$$\propto \text{Beta}\left(T\bar{y} + \underline{\alpha},\ T(1-\bar{y}) + \underline{\delta}\right)$$

Q2. In all the graphs that follow that are functions of θ, I used a grid for θ with steps of 0.001. So, when I present the "max" or the "mode," these are just the points on the grid with the highest values.

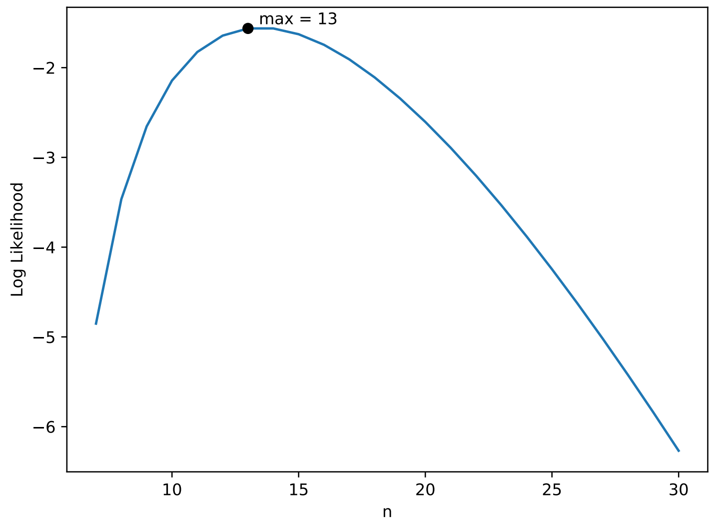
i. The likelihood for β is maximized right around the true value, and doesn't appear to be very flat:



ii. These likelihoods appear to be a bit more flat, but that could be due to scaling. I don't know of a good way to say that a likelihood is "flat."



The likelihood for n appears relatively well behaved, though it is skewed. I guess this would imply that a maximizer may tend to overestimate n, since the likelihood doesn't fall off as quickly to the right.



iii. The posterior for the conjugate prior is:

$$\begin{aligned} p(\theta \mid n, s) &= \text{prior} \times \text{likelihood} \\ &= \theta^{a-1}(1-\theta)^{b-1}\theta^{s}(1-\theta)^{n-s} \\ &= \theta^{a-1+s}(1-\theta)^{b-1+n-s}. \end{aligned}$$

For Jeffrey's prior, we first need the information:

$$\begin{aligned} I_\theta &= -\mathbb{E}\left[\frac{\partial^2 \log \ell(\theta; n, s)}{\partial \theta^2}\right] \\ &= -\mathbb{E}\left[\frac{\partial}{\partial \theta}\left(\frac{s}{\theta} - \frac{n-s}{1-\theta}\right)\right] \\ &= -\mathbb{E}\left[-\frac{s}{\theta^2} - \frac{n-s}{(1-\theta)^2}\right] \\ &= \frac{n\theta}{\theta^2} + \frac{n(1-\theta)}{(1-\theta)^2} \\ &= \frac{n}{\theta} + \frac{n}{1-\theta} \end{aligned}$$

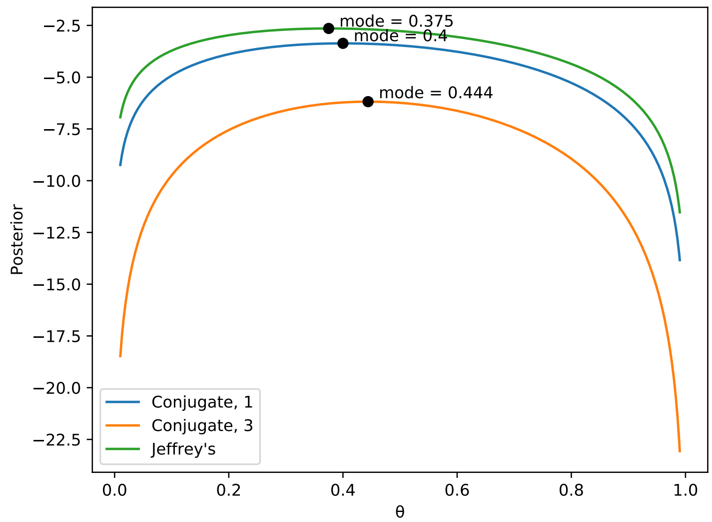
So the prior is proportional to the square root of this:

$$p(\theta) \propto \sqrt{\frac{1}{\theta} + \frac{1}{1-\theta}} = \sqrt{\frac{1}{\theta(1-\theta)}} = \theta^{-1/2}(1-\theta)^{-1/2}$$

So the posterior is:

$$p(\theta \mid n, s) \propto \theta^{s-0.5}(1-\theta)^{n-s-0.5}$$

The three posteriors are plotted below. When $a = b$, this conjugate prior collapses to $(\theta(1-\theta))^{a-1}$. In general, this is maximized at $\theta = 1/2$, so it is like saying that our prior is that $\theta = 1/2$. Setting $a = b = 1$ is just equivalent to setting a uniform prior; thuse, the posterior is just the likelihood of the data—and is therefore maximized at the true parameters. Setting $a = b = 3$ means that we are putting more weihgt on this $\theta = 1/2$ prior—and we see this in the posterior: the mode is farther to the right. The Jeffrey's prior pulls us in the other direction.



Q3. **i.** I set $\alpha = 1$. Practically, in order to be able to draw from the double exponential, I need the inverse CDF. But before that, I need the CDF. For $x < 0$:

$$F(x) = \int_{-\infty}^{x} \frac{1}{2}\exp(y)dy = \frac{1}{2}\exp(x)$$

For $x > 0$:

$$\begin{aligned}
F(x) &= \int_{-\infty}^{y} \frac{1}{2}\exp(-|y|)dy \\
&= \int_{-\infty}^{0} \frac{1}{2}\exp(y)dy + \int_{0}^{y} \frac{1}{2}\exp(-y)dy \\
&= frac12 - \frac{1}{2}\exp(-y)\Big|_{0}^{x} \\
&= \frac{1}{2} - \left[\frac{1}{2}\exp(-x) - \frac{1}{2}\right] \\
&= 1 - \frac{1}{2}\exp(-x)
\end{aligned}$$

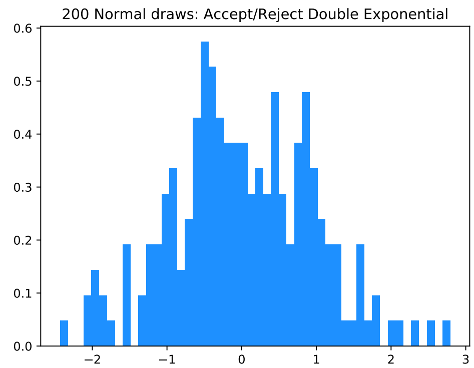
Now we can invert. If $x < 0$, then:

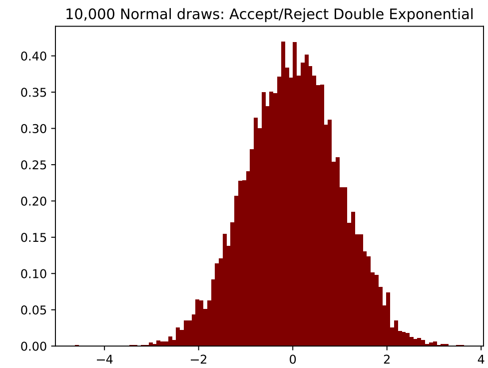
$$u = \frac{\exp(x)}{2} \implies x = \log(2u) \implies F^{-1}(u) = \log(2u) \text{ if } u < \frac{1}{2}.$$

On the other hand, if $x \geq 0$ then

$$u = 1 - \frac{\exp(-x)}{2} \implies x = -\log(2(1-u)) \implies F^{-1}(u) = -\log(2(1-u)) \text{ if } u < \frac{1}{2}.$$

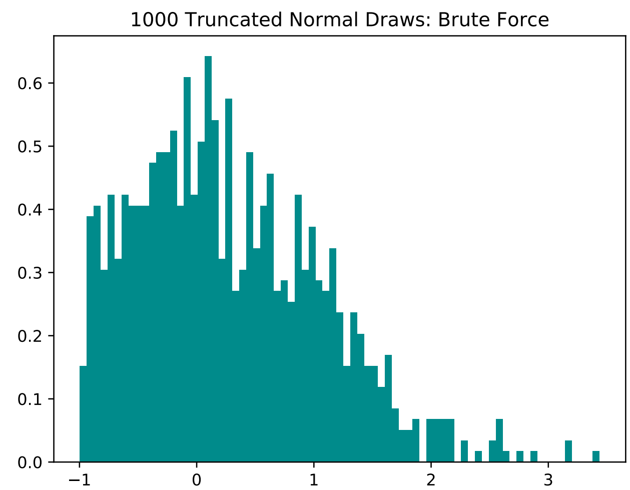
In order to draw 200 observations, it took 263 draws. I also drew 10,000 observations, which took 13142 draws. The (normalized) histograms of these draws look like so:





ii. Set $\mu = 0$, $\sigma = 1$, $\alpha = 1$ and $\underline{\mu} = -1$. I am drawing 1000 truncated normals.

(a) It took 1212 draws from a normal to get 1000 draws from the truncated normal. Here is a histogram of the draws:



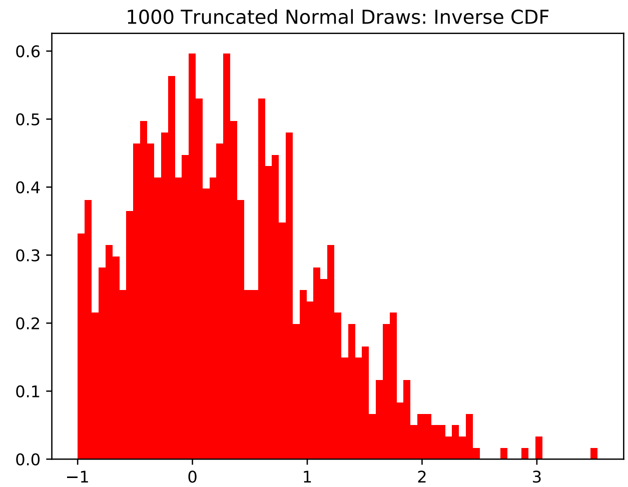
(b) According to Wikipedia, the CDF of the right-truncated normal is given by:

$$F(x) = \frac{\Phi\left(\frac{x-\mu}{\sigma}\right) - \Phi\left(\frac{\underline{\mu}-\mu}{\sigma}\right)}{1 - \Phi\left(\frac{\underline{\mu}-\mu}{\sigma}\right)}$$

So,

$$u = \frac{\Phi\left(\frac{x-\mu}{\sigma}\right) - \Phi\left(\frac{\underline{\mu}-\mu}{\sigma}\right)}{1 - \Phi\left(\frac{\underline{\mu}-\mu}{\sigma}\right)} \implies x = \sigma\Phi^{-1}\left[u\left(1 - \Phi\left(\frac{\underline{\mu}-\mu}{\sigma}\right)\right) + \Phi\left(\frac{\underline{\mu}-\mu}{\sigma}\right)\right] + \mu \equiv F^{-1}(u)$$

Here is a histogram of the draws:



(c) According to Wikipedia, the PDF of a left-truncated normal is:

$$f(x) = \frac{\phi\left(\frac{x-\mu}{\sigma}\right)}{\sigma\left(1 - \Phi\left(\frac{\underline{\mu}}{\sigma} - \mu\right)\right)} \equiv \frac{\phi\left(\frac{x-\mu}{\sigma}\right)}{\sigma(1-B)}$$

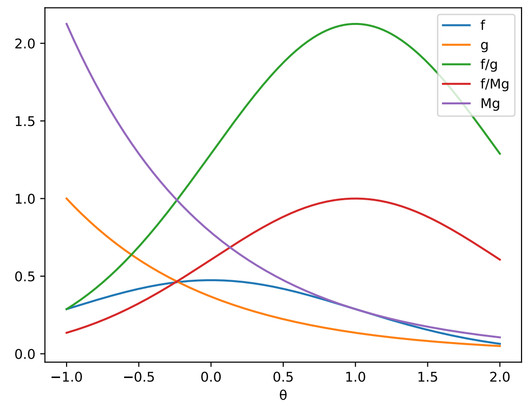
The enveloping pdf is

$$g(x) = \exp\left[-\left(\frac{x-\mu}{\sigma} - \underline{\mu}\right)\right] \mathbb{1}\left\{\frac{x-\mu}{\sigma} \geq \underline{\mu}\right\}$$

So, dividing the target, f, by the candidate, g, (and plugging-in $\mu = 0$ and $\sigma = 1$ for simplicity) produces a bound:

$$\begin{aligned}
\frac{f(x)}{g(x)} &= \frac{\frac{1}{\sigma(1-B)\sqrt{2\pi}} \exp(-x^2/2)}{\exp\left[-(x-\underline{\mu})\right] \mathbb{1}\{x \geq \underline{\mu}\}} \\
&= \frac{1}{\mathbb{1}\{x \geq \underline{\mu}\}\sigma(1-B)\sqrt{2\pi}} \exp\left(-\frac{x^2}{2} + (x - \underline{\mu})\right) \\
&\leq \frac{1}{\sigma(1-B)\sqrt{2\pi}} \exp(1.5) && \forall x \geq -1,
\end{aligned}$$

where the inequality follows from the fact that the exponential term is maximized when $x = 1$. Just so I have a better idea of what's going on, I've plotted a bunch of these quantities.



First, Mg envelopes f, as desired. Next, let's think about what happens when we sample from g. When θ is small, g is above f—but, g is relatively large for small θ, so we will draw from there often. But, we only keep a small fraction of these draws (those smaller than f/Mg). This aligns our draws closer to the distribution of f, where there are relatively few draws that would come from small θ. Similarly, the relatively few draws taken for large θ are kept pretty often, reflecting the fact that $g < f$. Pretty cool!

Back to the problem at hand: we will accept X if

$$U \leq \exp\left(-\frac{X^2}{2} + (X - \underline{\mu}) - 1.5\right)$$

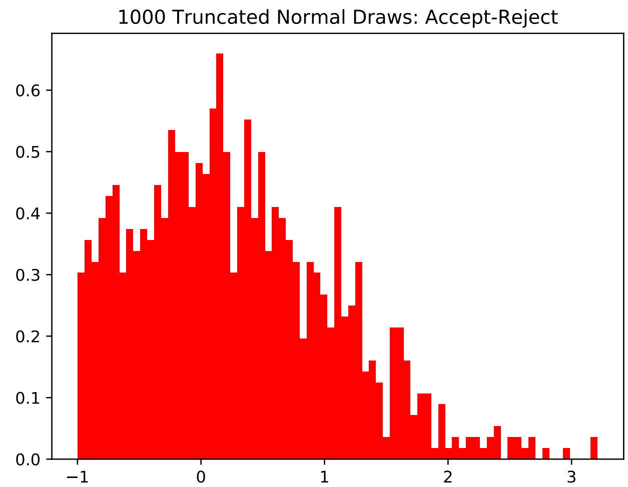
But before we do any of this, we need to calculate the inverse CDF associated with g. If $x \leq \underline{\mu}$, then $G(x) = 0$. If $x \geq \mu$, then

$$\begin{aligned}
G(x) &\equiv \int_{-\infty}^{x} \exp\left[-(y - \underline{\mu})\right] \mathbb{1}\{y \geq \underline{\mu}\} dy \\
&= 0 + \int_{\underline{\mu}}^{x} \exp\left[-(y - \underline{\mu})\right] dy \\
&= -\exp(-(y - \underline{\mu}))\Big|_{\underline{\mu}}^{x} \\
&= 1 - \exp(-(x - \underline{\mu}))
\end{aligned}$$

The inverse CDF is therefore

$$u = 1 - \exp(-(x - \underline{\mu})) \Longrightarrow \; x = \underline{\mu} - \log(1 - u) \equiv G^{-1}(u).$$

In order to draw the 1000 draws, it took me 2103 draws. Here they are:



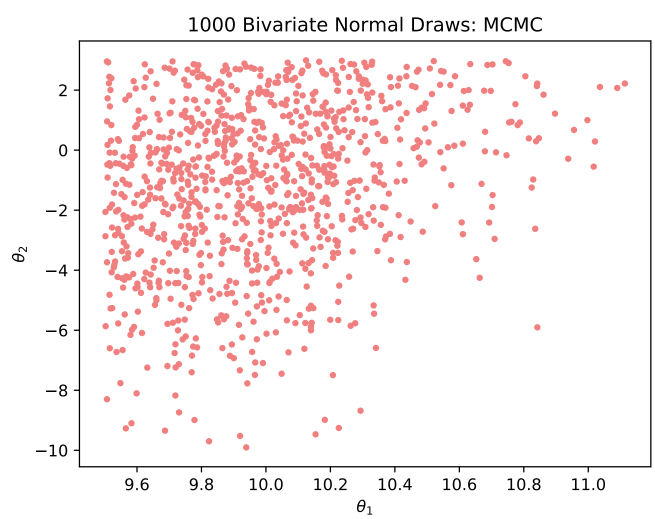
iii. Define

$$\Sigma \equiv \begin{bmatrix} \sigma_1^2 & \gamma \\ \gamma & \sigma_2^2 \end{bmatrix} \qquad \mu \equiv \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} \qquad \rho \equiv \frac{\gamma}{\sigma_1 \sigma_2}$$

By Wikipedia,

$$\theta_i \mid \theta_j \sim N\left(\mu_i + \frac{\sigma_i}{\sigma_j}\rho(\theta_j - \mu_j), (1-\rho^2)\sigma_i^2\right).$$

I kept all draws in which both θ_1 and θ_2 were within the bounds I set—just as in the "brute force" approach, above. I set $\sigma_1 = 0.5$, $\sigma_2 = 4$, $\gamma = 1$, $\mu_1 = 10$, $\mu_2 = 0$, and required $\theta_1 \in [9.5, 100.0]$ and $\theta_2 \in [-10, 3]$. It took 1960 draws to get the 250 burn-in draws and 1000 kept-draws. Here is a scatter plot of the retained draws:



Q4.

Q5.