Problem Set 4
GR6413 [Ng]
Miguel Acosta
November 27, 2017

**Q1.**

i. $y_t \sim \text{poisson}(\theta)$ ; prior for $\theta$: $\theta \sim G(\underline{\alpha}, \underline{\beta})$

$\mathbb{P}(y_t = k) = \dfrac{\theta^k \exp(-\theta)}{k!}$ $\quad\quad f(\theta) = \dfrac{\underline{\beta}^{\underline{\alpha}}}{\Gamma(\underline{\alpha})} \theta^{\underline{\alpha}-1} \exp(-\underline{\beta}\,\theta)$

If $y_t$ i.i.d., then $P(y_1 = k_1, \dots, y_T = k_T) \propto \prod_{t=1}^T \theta^{k_t} \exp(-\theta)$
$= \exp(-\theta)\theta^{\Sigma k_t}$

So, $p(\theta \mid y_t) \propto p(y_t \mid \theta)\, p(\theta)$ (Bayes' Rule)

$$\begin{aligned}
&\propto \exp(-\theta)\theta^{\Sigma k_t}\theta^{\underline{\alpha}-1}\exp(-\underline{\beta}\,\theta) \\
&= \exp(-\theta(1+\underline{\beta}))\,\theta^{\underline{\alpha}-1+\Sigma k_t} \\
&= \text{Gamma}\left(\underline{\alpha} + \textstyle\sum_{t=1}^T k_t,\ \underline{\beta}+1\right) \\
&= \text{Gamma}\left(\underline{\alpha} + T\bar{k},\ \underline{\beta}+1\right).
\end{aligned}$$

$\bar{k} \equiv \frac{1}{T}\sum_{t=1}^T k_t$

ii. $P(y_t \mid \theta) = \begin{cases} \theta & y_t = 1 \\ 1-\theta & y_t = 0 \end{cases}$ $\quad\quad$ prior for $\theta$: $p(\theta) = \left[\dfrac{\Gamma(\underline{\alpha})\Gamma(\underline{\delta})}{\Gamma(\underline{\alpha}+\underline{\delta})}\right]^{-1}\theta^{\underline{\alpha}-1}(1-\theta)^{\underline{\delta}-1}$

$= \theta^{y_t}(1-\theta)^{1-y_t}$

The likelihood of the data, therefore, is

$$\begin{aligned}
P(\{y_t\}_{t=1}^T \mid \theta) &= \textstyle\prod_{t=1}^T \theta^{y_t}(1-\theta)^{1-y_t} \\
&= \theta^{\Sigma y_t}(1-\theta)^{\Sigma(1-y_t)} \\
&= \theta^{T\bar{y}}(1-\theta)^{T(1-\bar{y})}
\end{aligned}$$

So, $P(\theta \mid y_t) \propto P(y_t \mid \theta)\, p(\theta)$

$$\begin{aligned}
&= \theta^{T\bar{y}}(1-\theta)^{T(1-\bar{y})}\theta^{\underline{\alpha}-1}(1-\theta)^{\underline{\delta}-1} \\
&= \theta^{T\bar{y}+\underline{\alpha}-1}(1-\theta)^{T(1-\bar{y})+\underline{\delta}-1} \\
&\propto \text{Beta}(T\bar{y}+\underline{\alpha},\ T(1-\bar{y})+\underline{\delta})
\end{aligned}$$

**iii.** From the last problem we had

$$L(\theta \mid y) = P(\{y_t\}_{t=1}^T \mid \theta) = \theta^{\sum_t y_t}(1-\theta)^{\sum_t(1-y_t)}$$

If your sample size were $\underline{T} = \underline{\alpha} + \underline{\delta} - 2$ and there are $\underline{\alpha} - 1$ successes, then this becomes

$$L(\theta \mid y) = \theta^{\sum_t y_t} \times (1-\theta)^{\sum_t(1-y_t)} = \theta^{\underline{\alpha}-1}(1-\theta)^{\underline{\alpha}+\underline{\delta}-2-\underline{\alpha}+1} = \theta^{\underline{\alpha}-1}(1-\theta)^{\underline{\delta}-1}.$$

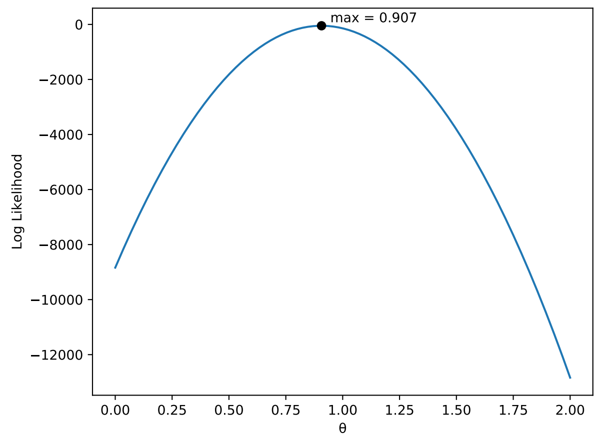
If your sample instead was $\underline{T} + T$, then the likelihood is

$$L(\theta \mid y) = \theta^{\underline{\alpha}-1+\sum y_t}(1-\theta)^{\underline{\delta}-1+\sum(1-y_t)}.\theta^{T\overline{y}+\underline{\alpha}-1}(1-\theta)^{\underline{\delta}-1+T(1-\overline{y})},$$

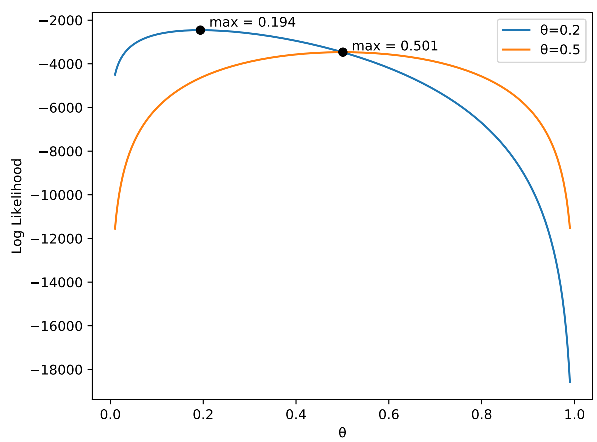
which is the exactly the posterior from the previous question.

**Q2.** In all the graphs that follow that are functions of $\theta$, I used a grid for $\theta$ with steps of 0.001. So, when I present the "max" or the "mode," these are just the points on the grid with the highest values.

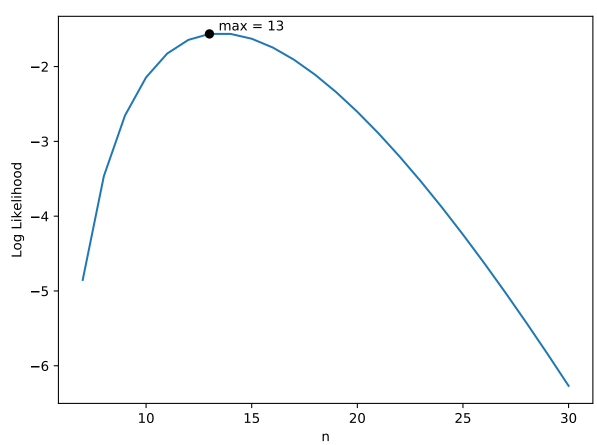
**i.** The likelihood for $\beta$ is maximized right around the true value, and doesn't appear to be very flat:




**ii.** These likelihoods appear to be a bit more flat, but that could be due to scaling. I don't know of a good way to say that a likelihood is "flat."




The likelihood for $n$ appears relatively well behaved, though it is skewed. I guess this would imply that a maximizer may tend to overestimate $n$, since the likelihood doesn't fall off as quickly to the right.



**iii.** The posterior for the conjugate prior is:

$$\begin{aligned} p(\theta \mid n, s) &= \text{prior} \times \text{likelihood} \\ &= \theta^{a-1}(1-\theta)^{b-1}\theta^{s}(1-\theta)^{n-s} \\ &= \theta^{a-1+s}(1-\theta)^{b-1+n-s}. \end{aligned}$$

For Jeffrey's prior, we first need the information:

$$\begin{aligned} I_\theta &= -\mathbb{E}\left[\frac{\partial^2 \log \ell(\theta; n, s)}{\partial \theta^2}\right] \\ &= -\mathbb{E}\left[\frac{\partial}{\partial \theta}\left(\frac{s}{\theta} - \frac{n-s}{1-\theta}\right)\right] \\ &= -\mathbb{E}\left[-\frac{s}{\theta^2} - \frac{n-s}{(1-\theta)^2}\right] \\ &= \frac{n\theta}{\theta^2} + \frac{n(1-\theta)}{(1-\theta)^2} \\ &= \frac{n}{\theta} + \frac{n}{1-\theta} \end{aligned}$$

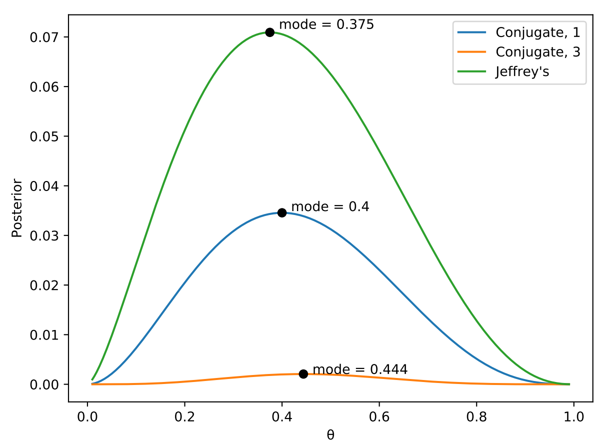
So the prior is proportional to the square root of this:

$$p(\theta) \propto \sqrt{\frac{1}{\theta} + \frac{1}{1-\theta}} = \sqrt{\frac{1}{\theta(1-\theta)}} = \theta^{-1/2}(1-\theta)^{-1/2}$$

So the posterior is:

$$p(\theta \mid n, s) \propto \theta^{s-0.5}(1-\theta)^{n-s-0.5}$$

The three posteriors are plotted below. When $a = b$, this conjugate prior collapses to $(\theta(1-\theta))^{a-1}$. In general, this is maximized at $\theta = 1/2$, so it is like saying that our prior is that $\theta = 1/2$. Setting $a = b = 1$ is just equivalent to setting a uniform prior; thuse, the posterior is just the likelihood of the data—and is therefore maximized at the true parameters. Setting $a = b = 3$ means that we are putting more weihgt on this $\theta = 1/2$ prior—and we see this in the posterior: the mode is farther to the right. The Jeffrey's prior pulls us in the other direction.



**Q3.** **i.** I set $\alpha = 1$. Practically, in order to be able to draw from the double exponential, I need the inverse CDF. But before that, I need the CDF. For $x < 0$:

$$F(x) = \int_{-\infty}^{x} \frac{1}{2}\exp(y)dy = \frac{1}{2}\exp(x)$$

For $x > 0$:

$$\begin{aligned} F(x) &= \int_{-\infty}^{y} \frac{1}{2}\exp(-|y|)dy \\ &= \int_{-\infty}^{0} \frac{1}{2}\exp(y)dy + \int_{0}^{y} \frac{1}{2}\exp(-y)dy \\ &= frac12 - \frac{1}{2}\exp(-y)\Big|_{0}^{x} \\ &= \frac{1}{2} - \left[\frac{1}{2}\exp(-x) - \frac{1}{2}\right] \\ &= 1 - \frac{1}{2}\exp(-x) \end{aligned}$$

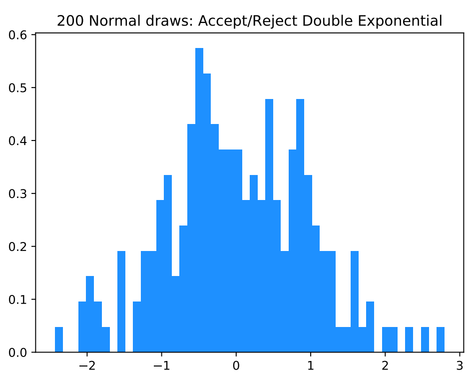
Now we can invert. If $x < 0$, then:

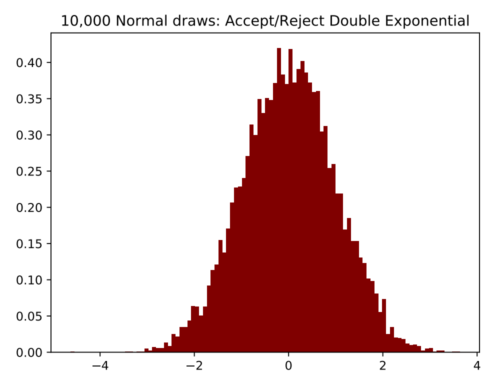
$$u = \frac{\exp(x)}{2} \implies x = \log(2u) \implies F^{-1}(u) = \log(2u) \text{ if } u < \frac{1}{2}.$$

On the other hand, if $x \geq 0$ then

$$u = 1 - \frac{\exp(-x)}{2} \implies x = -\log(2(1-u)) \implies F^{-1}(u) = -\log(2(1-u)) \text{ if } u < \frac{1}{2}.$$

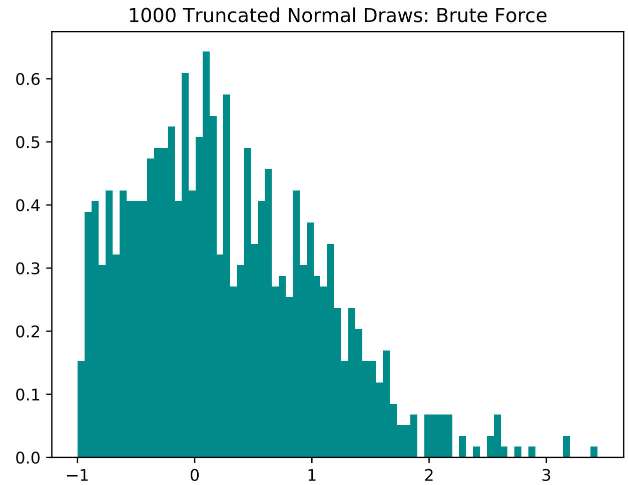
In order to draw 200 observations, it took 263 draws. I also drew 10,000 observations, which took 13142 draws. The (normalized) histograms of these draws look like so:



**ii.** Set $\mu = 0$, $\sigma = 1$, $\alpha = 1$ and $\underline{\mu} = -1$. I am drawing 1000 truncated normals.

**(a)** It took 1212 draws from a normal to get 1000 draws from the truncated normal. Here is a histogram of the draws:



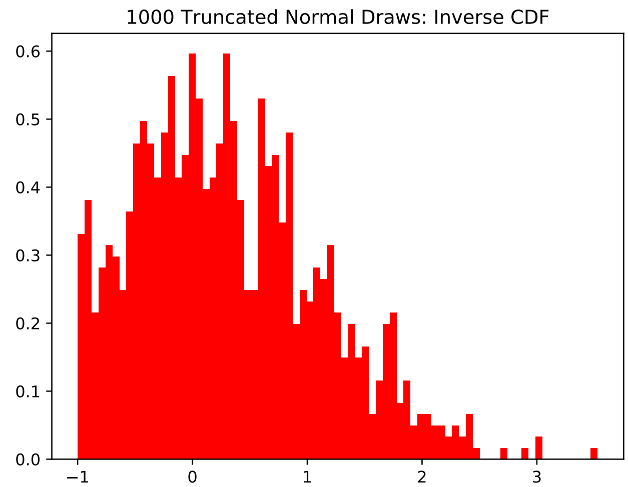

**(b)** According to Wikipedia, the CDF of the right-truncated normal is given by:

$$F(x) = \frac{\Phi\left(\frac{x-\mu}{\sigma}\right) - \Phi\left(\frac{\underline{\mu}-\mu}{\sigma}\right)}{1 - \Phi\left(\frac{\underline{\mu}-\mu}{\sigma}\right)}$$

So,

$$u = \frac{\Phi\left(\frac{x-\mu}{\sigma}\right) - \Phi\left(\frac{\underline{\mu}-\mu}{\sigma}\right)}{1 - \Phi\left(\frac{\underline{\mu}-\mu}{\sigma}\right)} \implies x = \sigma\Phi^{-1}\left[u\left(1 - \Phi\left(\frac{\underline{\mu}-\mu}{\sigma}\right)\right) + \Phi\left(\frac{\underline{\mu}-\mu}{\sigma}\right)\right] + \mu \equiv F^{-1}(u)$$

Here is a histogram of the draws:



**(c)** According to Wikipedia, the PDF of a left-truncated normal is:

$$f(x) = \frac{\phi\left(\frac{x-\mu}{\sigma}\right)}{\sigma\left(1 - \Phi\left(\frac{\underline{\mu}}{\sigma} - \mu\right)\right)} \equiv \frac{\phi\left(\frac{x-\mu}{\sigma}\right)}{\sigma(1-B)}$$

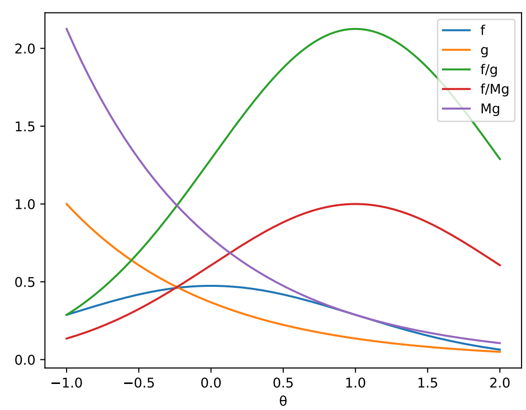
The enveloping pdf is

$$g(x) = \exp\left[-\left(\frac{x-\mu}{\sigma} - \underline{\mu}\right)\right] \mathbb{1}\left\{\frac{x-\mu}{\sigma} \geq \underline{\mu}\right\}$$

So, dividing the target, $f$, by the candidate, $g$, (and plugging-in $\mu = 0$ and $\sigma = 1$ for simplicity) produces a bound:

$$\begin{aligned}
\frac{f(x)}{g(x)} &= \frac{\frac{1}{\sigma(1-B)\sqrt{2\pi}} \exp(-x^2/2)}{\exp\left[-(x-\underline{\mu})\right] \mathbb{1}\{x \geq \underline{\mu}\}} \\
&= \frac{1}{\mathbb{1}\{x \geq \underline{\mu}\}\sigma(1-B)\sqrt{2\pi}} \exp\left(-\frac{x^2}{2} + (x - \underline{\mu})\right) \\
&\leq \frac{1}{\sigma(1-B)\sqrt{2\pi}} \exp(1.5) & \forall x \geq -1,
\end{aligned}$$

where the inequality follows from the fact that the exponential term is maximized when $x = 1$. Just so I have a better idea of what's going on, I've plotted a bunch of these quantities.




First, $Mg$ envelopes $f$, as desired. Next, let's think about what happens when we sample from $g$. When $\theta$ is small, $g$ is above $f$—but, $g$ is relatively large for small $\theta$, so we will draw from there often. But, we only keep a small fraction of these draws (those smaller than $f/Mg$). This aligns our draws closer to the distribution of $f$, where there are relatively few draws that would come from small $\theta$. Similarly, the relatively few draws taken for large $\theta$ are kept pretty often, reflecting the fact that $g < f$. Pretty cool!

Back to the problem at hand: we will accept $X$ if

$$U \leq \exp\left(-\frac{X^2}{2} + (X - \underline{\mu}) - 1.5\right)$$

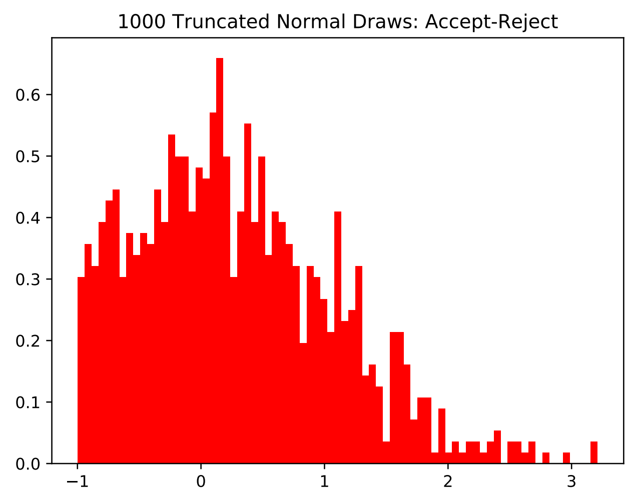
But before we do any of this, we need to calculate the inverse CDF associated with $g$. If $x \leq \underline{\mu}$, then $G(x) = 0$. If $x \geq \mu$, then

$$\begin{aligned}
G(x) &\equiv \int_{-\infty}^{x} \exp\left[-(y - \underline{\mu})\right] \mathbb{1}\{y \geq \underline{\mu}\} dy \\
&= 0 + \int_{\underline{\mu}}^{x} \exp\left[-(y - \underline{\mu})\right] dy \\
&= -\exp(-(y - \underline{\mu}))\Big|_{\underline{\mu}}^{x} \\
&= 1 - \exp(-(x - \underline{\mu}))
\end{aligned}$$

The inverse CDF is therefore

$$u = 1 - \exp(-(x - \underline{\mu})) \Longrightarrow \;\; x = \underline{\mu} - \log(1 - u) \equiv G^{-1}(u).$$

In order to draw the 1000 draws, it took me 2103 draws. Here they are:



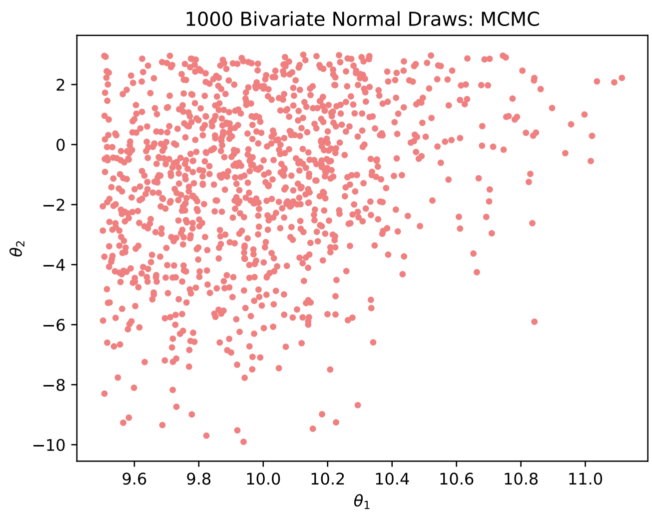
**iii.** Define

$$\Sigma \equiv \begin{bmatrix} \sigma_1^2 & \gamma \\ \gamma & \sigma_2^2 \end{bmatrix} \qquad \mu \equiv \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} \qquad \rho \equiv \frac{\gamma}{\sigma_1 \sigma_2}$$

By Wikipedia,

$$\theta_i \mid \theta_j \sim N\left(\mu_i + \frac{\sigma_i}{\sigma_j}\rho(\theta_j - \mu_j), (1-\rho^2)\sigma_i^2\right).$$

I kept all draws in which both $\theta_1$ and $\theta_2$ were within the bounds I set—just as in the "brute force" approach, above. I set $\sigma_1 = 0.5$, $\sigma_2 = 4$, $\gamma = 1$, $\mu_1 = 10$, $\mu_2 = 0$, and required $\theta_1 \in [9.5, 100.0]$ and $\theta_2 \in [-10, 3]$. It took 1960 draws to get the 250 burn-in draws and 1000 kept-draws. Here is a scatter plot of the retained draws:



**Q4.** The probit model specifies that[1]

$$y_i^* = \beta_0 + \beta_1 x_i + e_i \qquad\qquad e_i \sim N(0, \sigma^2)$$

We observe $y_i = 1$ if $y_i^* > 0$, and $y_i = 0$ otherwise.

$$P(y_i = 1 \mid x_i, \beta) = P(y_i^* > 0 \mid x_i, \beta) = P(\beta_0 + \beta_1 x_i + e_i > 0 \mid x_i, \beta) = \Phi(\beta_0 + \beta_1 x_i) \tag{1}$$

Our prior for $\beta$

$$\underline{\beta} \sim N\left(\underline{\mu}, \underline{g}\,(X'X)^{-1}\right)$$

where $\underline{\mu}$ and $\underline{g}$ are parameters to be specified—note that this is the $g$-prior for the variance. Now for the marginal of $y^*$:

$$y_i^* \mid \beta, x_i \overset{d}{=} \beta_0 + \beta_1 x_i + e_i \mid x_i, \beta \sim N(\beta_0 + \beta_1 x_i, \sigma^2).$$

Conditioning on $y_i$ gives:

$$y_i^* \mid y_i, x_i, \beta = \begin{cases} TN_{(0,\infty)}(\beta_0 + \beta_1 x_i, \sigma^2) & \text{if } y_i = 1 \\ TN_{(-\infty,0]}(\beta_0 + \beta_1 x_i, \sigma^2) & \text{if } y_i = 0. \end{cases}$$

We know from class that, if the variance of $e_i$ is known then

$$\beta \mid y_i, x_i, \sigma^2, g \sim N\left(\overline{\beta}, \sigma^2 \overline{B}\right) \qquad \overline{B} = (g(X'X)^{-1} + X'X)^{-1} \qquad \overline{\beta} = \overline{B}(g^{-1}(X'X)\underline{\mu} + X'y^*).$$

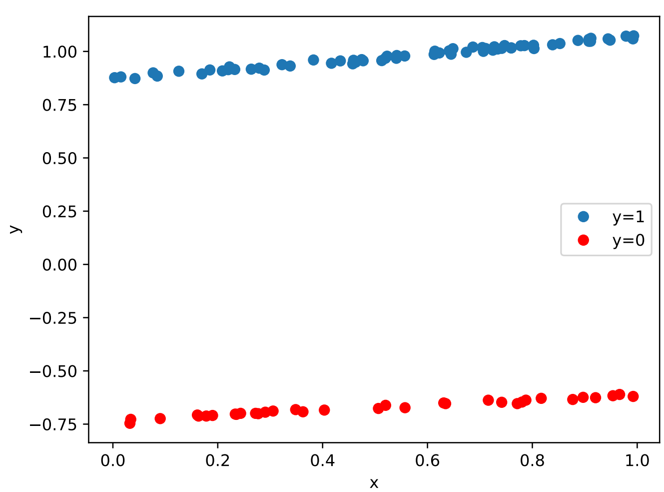
So, using the following parameters:

$$\underline{\mu} = [0, 0] \qquad g = 10 \qquad \underline{y}_i^* = 0 \qquad \sigma^2 = 1$$

And drawing $\beta$ then $y_i^*$, I get the following esimates:

| | Prior | Mean | Standard Deviation |
|---|---|---|---|
| $\beta_0$ | 0 | 0.19 | 0.24 |
| $\beta_1$ | 0 | 0.42 | 0.38 |

And the probability of $y_i = 1$ given $x_i = 2$ is 0.85, by (1). For fun, here are the estimated $y_i^*$'s and $x_i$'s in a scatter plot.




[1]Following the lecture notes, the Hanbook of Econometrics chapter quoted therein, and Joe.

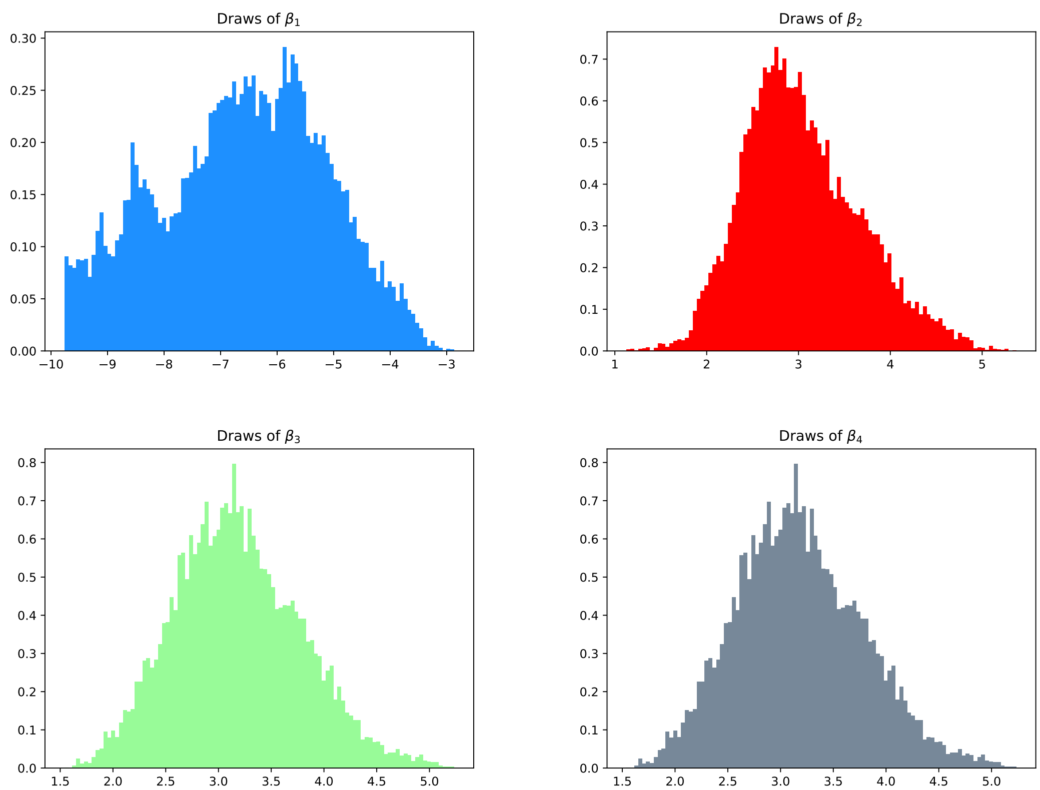
**Q5.** The model from Chernozhukov and Hong (CH) is:

$$
\begin{aligned}
Y^* &= \beta_0 + X'\beta + u \\
X &\overset{d}{=} N(0, I_3) \\
u &= X_1^2 N(0,1) \\
Y &= \max(0, Y^*) \\
(\beta_0, \beta_1, \beta_2, \beta_3) &= (-6, 3, 3, 3)
\end{aligned}
$$

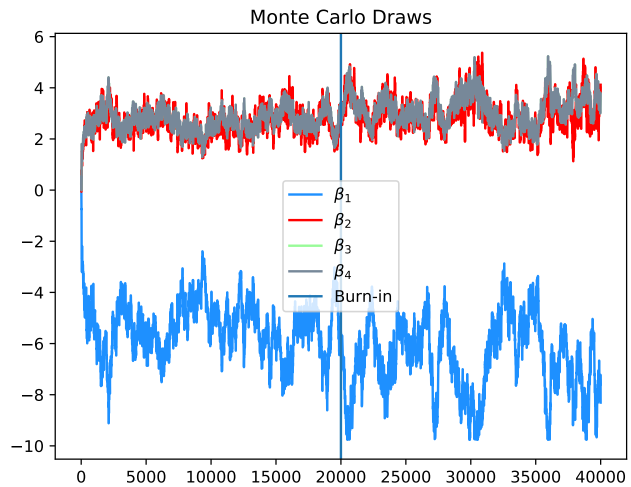
I made a few slight modifications to the code based on the reading of the paper. The first is that there are four (not three) parameters to estimate. I also had to adjust the parameter `lv` since I was getting an acceptance ratio that was too low. This helped, though my draws look much more correlated now. Finally, I changed the bounds for the paramters to be the OLS estiamtes $\pm 10$, following CH. Finally, I specified a uniform prior over this range of acceptable estimates, following CH. The results are as follows:[2]

| | $\beta_0$ | $\beta_1$ | $\beta_2$ | $\beta_3$ |
|---|---|---|---|---|
| Mean | $-6.6$ | 3 | 3.2 | 3.2 |
| Median | $-6.5$ | 3 | 3.1 | 3.2 |
| SD | 1.5 | 0.63 | 0.6 | 0.59 |
| 95 CI Low | $-9.2$ | 2.1 | 2.2 | 2.3 |
| 95 CI High | $-4.3$ | 4.2 | 4.2 | 4.2 |
| Acceptance | 0.56 | 0.54 | 0.55 | 0.52 |

Next are some pictures of the results. The histograms show the draws after burn in. The second picture shows the data in its "time-series" form.




[2] The "acceptance" in the table is the acceptance ratio of the last 200 draws—I update as in the code, though it looks slightly different.




**Q6.** At some point in the future I would really like to do the optional problem, but I can't do it right now because of time constraints.