Problem Set 1
ECON 6413–Ng
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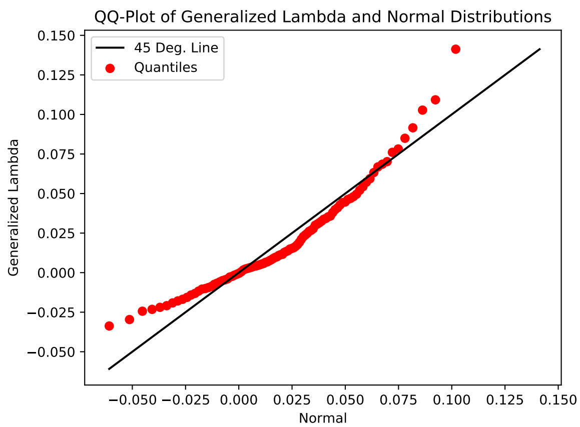
Q1. **a.** The sample moments and qq-plot are presented here:



Moment	Value
mean	0.0205
variance	0.0012
skewness	1.34
kurtosis	6.11
Bera-Jarque	351.25
5% Critical Value	5.99

So, we can reject at the 1% level that the data are normal.

b. Let μ_k be the k-th centered moment of x_t ; that is, $\mu_k = \mathbb{E}[(x_t - \mu)^k]$. Under normality of x_t, it is well established[1] that its centered moments are given by:

$$\mu_k = \begin{cases} 0 & \text{if } k \text{ is odd} \\ \sigma^k (k-1)!! & \text{if } k \text{ is even} \end{cases}$$

i. First, calculate $\frac{\partial \overline{g}}{\partial \theta}$, then take limits:

$$\frac{\partial\, g(\theta)}{\partial\, \theta} = \frac{1}{n}\sum_{t=1}^{T} \begin{bmatrix} -1 & 0 \\ -2(x_t-\mu) & -1 \\ -3(x_t-\mu)^2 & 0 \\ -4(x_t-\mu)^3 & -6\sigma^2 \end{bmatrix} \xrightarrow{p} \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ -3\sigma^2 & 0 \\ -4\mu_3 & -6\sigma^2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ -3\sigma^2 & 0 \\ 0 & -6\sigma^2 \end{bmatrix} \equiv G_0$$

where we've relied on the fact tht $\mu_3 = 0$.

[1] On Wikipedia.

ii. We'll need the following quantity:

$$\mathbb{E}[\overline{g}(\theta)] = \begin{bmatrix} 0 \\ 0 \\ \mu_3 \\ \mu_4 - 3\sigma^4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 3\sigma^4 - 3\sigma^4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

So that

$$\begin{aligned}
\text{Avar}(\overline{g}(\theta)) &= \mathbb{E}[(\overline{g} - 0)(\overline{g} - 0)'] \\
&= \begin{bmatrix} \sigma^2 & 0 & \mu_4 & \mu_5 \\ \bullet & \mu_4 - 2\sigma^+\sigma^4 & \mu_5 - \mu_3\sigma^2 & \mu_6 - 3\sigma^6 - \mu_4\sigma^2 + 3\sigma^6 \\ \bullet & \bullet & \mu_6 & \mu_7 - 3\mu_3\sigma^4 \\ \bullet & \bullet & \bullet & \mu_8 - 6\mu_4\sigma^4 + 9\sigma^8 \end{bmatrix} \\
&= \begin{bmatrix} \sigma^2 & 0 & 3\sigma^4 & 0 \\ \bullet & 2\sigma^4 & 0 & 12\sigma^6 \\ \bullet & \bullet & 15\sigma^6 & 0 \\ \bullet & \bullet & \bullet & 96\sigma^8 \end{bmatrix}
\end{aligned}$$

Using Matlab's symbollic algebra yields

$$\text{Avar}(\widehat{\theta}) = (G_0' S^{-1} G_0)^{-1} = \begin{bmatrix} \sigma^2 & 0 \\ 0 & 2\sigma^4 \end{bmatrix}$$

c. Under the null, the value of J (which I define in part **d**) should be close to 0. This will test all moments. The Bera-Jarque test only cares about third and fourth moments.

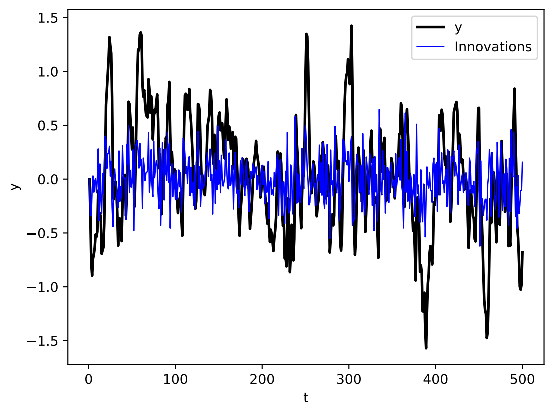
d. I'll perform this estimation via GMM. For simplicity, I'll weight by the identity matrix, so that the objective function is just the sum of squared moments:

$$J_n(\mu, \sigma^2) = n\overline{g}(\mu, \sigma^2)' \, \mathbf{I} \, \overline{g}(\mu, \sigma^2)$$

The Hessian for even this simple formula gets a little out of control, so I take derivatives using finite differences. The estimates for my algorithm and Julia's `LBFGS` in the `Optim` package yield similar results:

	Miguel	Julia
μ	0.0205	0.0205
σ^2	0.0012	0.0012

Q2. **i.** Below is a picture of the simulated series, and the estimated parameters.



Moment	Value
α	0.8020
θ	0.5151
σ^2	0.0463

These estimates were similar regardless of starting value and minimizer (once I got rid of errors…). I again used Julia's `LBFGS` in the `Optim` package, with an analytic gradient. The objective function was just the sum of squared errors, assuming the first innovation was 0.

ii. First, the variance and autocovariance of an ARMA$(1,1)$ are given by

$$\gamma_0 = \frac{\sigma^2(2\alpha\theta + \theta^2 + 1)}{1-\alpha^2} \qquad \gamma_1 = \sigma^2 \cdot \frac{\alpha^2\theta + \theta^2\alpha + \theta + \alpha}{1-\alpha^2}$$

(My calculations are attached at the bottom of this page.) Now, back to the question at hand. Suppose that we have two ARMA$(1,1)$s:

$$y_t = \alpha y_{t-1} + \theta e_{t-1} + e_t \qquad \widetilde{y}_t = \alpha\widetilde{y}_{t-1} + \widetilde{\theta}\widetilde{e}_{t-1} + \widetilde{e}_t.$$

For fixed $\neq \theta$, we want to find a pair $\widetilde{\theta}$ and $\widetilde{\sigma}^2$ that would yield the same variance and autocovariance. Equating the variances gives:

$$\widetilde{\sigma}^2 \cdot \frac{(2\alpha\widetilde{\theta} + \widetilde{\theta}^2 + 1)}{1-\alpha^2} = \sigma^2 \cdot \frac{(2\alpha\theta + \theta^2 + 1)}{1-\alpha^2} \Longrightarrow \frac{\widetilde{\sigma}^2}{\sigma^2} = \frac{2\alpha\theta + \theta^2 + 1}{2\alpha\widetilde{\theta} + \widetilde{\theta}^2 + 1}$$

Similarly, equating the 1st order AC yields:

$$\frac{\widetilde{\sigma}^2}{\sigma^2} = \frac{\alpha^2\widetilde{\theta} + \widetilde{\theta}^2\alpha + \widetilde{\theta} + \alpha}{\alpha^2\theta + \theta^2\alpha + \theta + \alpha}$$

Equating our two expressions for $\frac{\widetilde{\sigma}^2}{\sigma^2}$ and asking Matlab to do the algebra reveals that $\widetilde{\theta} = \frac{1}{\theta}$. Then we can just plug this back in to one of the expressions:

$$\widetilde{\sigma}^2 = \sigma^2 \cdot \frac{2\alpha\theta + \theta^2 + 1}{2\frac{\alpha}{\theta} + \frac{1}{\theta^2} + 1} = \sigma^2\theta^2 \cdot \frac{2\alpha\theta + \theta^2 + 1}{2\alpha\theta + 1 + \theta^2} \Longrightarrow \widetilde{\sigma}^2 = \sigma^2\theta^2.$$

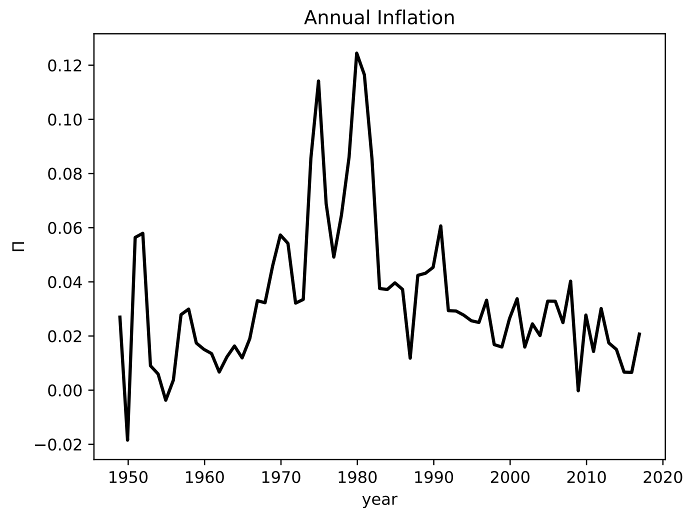
And for sanity's sake, here is a numerical verification of the estimates of the new simulated series, and a comparison of the variances and covariances:

Moment	Value
α	0.8032
θ	0.5144
σ^2	0.0463

	y_t	$\widetilde{y}_t$
γ_0	0.2734	0.2732
γ_1	0.2429	0.2425

So, we had two processes generated by two different sets of parameters that look exactly the same.

iii. Inflation on FRED is monthly, so I take the log difference of the price level from December to December of each year. This results in 69 observations, and the series looks like so:



These results are more sensitive than others on this problem set, though the signs and magnitudes are the same across starting values and minimizers. For CLS, I use the same criterion as in part **i** above. Here are the estimates using two minimizers in Julia's `Optim` package, using starting values of $(0.5, 0.5)$

	Simulated Annealing	LBFGS; numerical derivative
α	0.9262	0.8992
θ	-0.2072	-0.0898
σ^2	0.0004	0.0004

For GMM, I use the following implied moments

$$g(\alpha, \theta) = \mathbb{E}\begin{bmatrix} e_t y_{t-1} \\ e_t e_{t-1} \end{bmatrix} = 0$$

where, recall,

$$y_t = \alpha y_{t-1} + e_t + \theta e_{t-1}.$$

I use optimal GMM (weighting by the inverse of the variance-covariance matrix attained from estimtion using identity weighting), with my own Newton-Raphson algorithm, and Julia's `LBFGS` in the `Optim` package with numerical derivatives.

	Miguel	LBFGS
α	0.8953	0.8991
θ	-0.0776	-0.0900
σ^2	0.0004	0.0004

$y_t = \alpha y_{t-1} + \theta e_{t-1} + e_t$ (mean 0!)

$$\gamma_1 = \mathbb{E}\left[y_t y_{t-1}\right]$$

$$= \mathbb{E}\left(\alpha y_{t-1}^2 + \theta e_{t-1} y_{t-1} + e_t y_{t-1}\right)$$

$$= \alpha \mathbb{E}\left(y_{t-1}^2\right) + \theta \mathbb{E}\left(e_{t-1}\left(\alpha y_{t-2} + \theta e_{t-2} + e_{t-1}\right)\right)$$

$$= \alpha \gamma_0 + \theta \sigma^2$$

$\gamma_0 =$

$$\mathbb{E}(y_t y_t) = \mathbb{E}\left(y_t\left(\alpha y_{t-1} + \theta e_{t-1} + e_t\right)\right)$$

$$= \alpha \gamma_1 + \theta \mathbb{E}\left[(\alpha y_{t-1} + \theta e_{t-1} + e_t) e_{t-1}\right] + \mathbb{E}\left[e_t\left(\alpha y_{t-1} + \theta e_{t-1} + e_t\right)\right]$$

$$= \alpha \gamma_1 + \theta \alpha \sigma^2 + \theta^2 \sigma^2 + \sigma^2$$

$$\gamma_0 = \alpha \gamma_1 + \sigma^2(\theta\alpha + \theta^2 + 1)$$

$$\gamma_0 = \alpha(\alpha \gamma_0 + \theta \sigma^2) + \sigma^2(\theta\alpha + \theta^2 + 1)$$

$$\Rightarrow \gamma_0(1 - \alpha^2) = \alpha \theta \sigma^2 + \sigma^2(\theta\alpha + \theta^2 + 1)$$

$$\Rightarrow \boxed{\gamma_0 = \frac{\sigma^2(2\alpha\theta + \theta^2 + 1)}{1 - \alpha^2}} \qquad \boxed{\gamma_1 = \frac{\sigma^2}{1-\alpha^2}\left[\alpha^2\theta + \alpha\theta^2 + \alpha + \theta\right]}$$

$$\gamma_1 = \frac{\alpha\sigma^2(2\alpha\theta + \theta^2 + 1)}{1 - \alpha^2} + \frac{\theta\sigma^2(1 - \alpha^2)}{1-\alpha^2} = \frac{\sigma^2}{1-\alpha^2}\left[2\alpha^2\theta + \alpha\theta^2 + \alpha + \theta - \alpha^2\theta\right]$$