Problem Set 1
ECON G6220 [La'O]
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October 10, 2017

1 Preliminaries

(a) **(i)** Upon seeing the realization of x_i, the agent's posterior is

$$\boxed{N\left(\frac{\kappa_\theta \mu + \kappa_x x_i}{\kappa_\theta + \kappa_x}, \frac{1}{\kappa_\theta + \kappa_x}\right)}$$

(ii) Since ξ_i is normal with mean 0, $\int \xi_i di = 0$. So, we can compute the aggregate action:

$$\begin{aligned}
\overline{a} &\equiv \int a_i di \\
&= \int \mathbb{E}[\theta \mid x_i] di \\
&= \int \frac{\kappa_\theta \mu + \kappa_x x_i}{\kappa_\theta + \kappa_x} di \\
&= \frac{\kappa_\theta \mu}{\kappa_\theta + \kappa_x} + \frac{\kappa_x}{\kappa_\theta + \kappa_x} \int (\theta + \xi_i) di \\
&= \boxed{\frac{\kappa_\theta \mu + \kappa_x \theta}{\kappa_\theta + \kappa_x}}
\end{aligned}$$

(b) **(i)** Upon observing the realizations of x_i and y, the private and public signals, the agent's posterior is

$$N\left(\frac{\kappa_x x_i + \kappa_y y}{\kappa_x + \kappa_y} \frac{1}{\kappa_x + \kappa_y}\right)$$

(ii) The aggregate action is:

$$\begin{aligned}
\overline{a} &= \int a_i di \\
&= \int \mathbb{E}_i[\theta] di \\
&= \int \frac{\kappa_x x_i + \kappa_y y}{\kappa_x + \kappa_y} di \\
&= \frac{\kappa_x \theta + \kappa_y (\theta + \varepsilon)}{\kappa_x + \kappa_y} \\
&= \theta + \frac{\kappa_y \, \varepsilon}{\kappa_x + \kappa_y}
\end{aligned}$$

(c) In the private-only setup (a), aggregate action will end up being a weighted average of the true mean of θ and the actual draw of θ. In the setup with a public signal, the action is θ plus a weighted-down amount of the public signal. In the private setup, however, aggregate action will tend to μ as the precision of the public signal deteriorates; in the public one, aggregate action will tend to the public signal. In the first setup, the signal pulls aggregate action towards the fundamental. With the public signal, the aggregate action is pulled away!

2 Currency Attacks and Government Intervention

(a) When the agent is trying to decide between the two actions, what matters is the difference in the payoff between the two actions. Under the original payoff structure, this is

$$\begin{aligned} U(1, A, \theta) - U(0, A, \theta) &= \begin{cases} 1 - c & \text{if } R = 1 \\ 1 - c - 1 & \text{if } R = 0 \end{cases} \\ &= \begin{cases} 1 - c & \text{if } R = 1 \\ -c & \text{if } R = 0 \end{cases} \\ &= 1 \cdot (\mathbb{1}_{R=1} - c) - 0 \cdot (\mathbb{1}_{R=1} - c). \end{aligned}$$

That is, the relative payoffs under the original payoff structure are the same as they would be under the formula $a_i(\mathbb{1}_{R=1} - c)$, which results in the standard payoff matrix:

	$R = 1$	$R = 0$
$a_i = 1$	$1 - c$	$-c$
$a_i = 0$	0	0

(b) Ok, let's start by fixing e and a public signal y. I'll focus on a threshold equilibrium like we did in class. I'm looking for a strategy, $x^*(e, y)$, such that agents will attack iff

$$x \leq x^*(e, y).$$

This implies that everyone with a private signal above $x^*(e, y)$ doesn't attack, and vice versa. So, the aggregate size of the attack is decreasing in θ. This implies that there exists some threshold $\theta^*(e, y)$ such that there is a regime change iff

$$\theta \leq \theta^*(e, y).$$

Now, let's compute these. Start by characterizing θ^*, given x^*. Note that since $x \sim N(\theta, 1/\beta)$, we know that $\sqrt{\beta}(x - \theta) \sim N(0, 1)$. So, the size of the attack is the mass of agents receiving a signal below x^*, or:

$$A(\theta, e, y) = \Phi\left(\sqrt{\beta}(x^*(e, y) - \theta)\right). \tag{1}$$

If the aggregate size of the attack is above $\theta + e$, then regime shift occurs. So, the regime-change threshold is the solution to:

$$A(\theta^*(e, y), e, y) = \theta^*(e, y) + e \tag{2}$$

So, combining (1) and (2), we have

$$\begin{aligned} & \Phi^{-1}(\theta^*(e, y) + e) = \sqrt{\beta}(x^*(e, y) - \theta)) \\ \implies & x^*(e, y) = \frac{1}{\sqrt{\beta}} \Phi^{-1}(\theta^*(e, y) + e) + \theta^*(e, y) \end{aligned} \tag{3}$$

Next, given θ^*, we want to characterize x^*. First, note that the posterior distribution of θ, having seen the two signals, is

$$\theta|_{x,y} \sim N\left(\frac{\beta}{\eta}x + \frac{\alpha}{\eta}y, \frac{1}{\eta}\right) \implies \sqrt{\eta}\left(\theta_{x,y} - \left(\frac{\beta}{\eta}x + \frac{\alpha}{\eta}y\right)\right) \sim \Phi$$

where $\eta \equiv \beta + \alpha$. Payoffs are of the form $a_i(\mathbb{1}_{R=1} - c)$, and the person is indifferent at the threshold—that is, the expected payoff from attacking is equal to the payoff from not attacking:

$$\begin{aligned}
\underbrace{0}_{\text{Not Attack}} &= \underbrace{\mathbb{E}[u(1, A(\theta, e, y), \theta) \mid x^*(e, y), y]}_{\text{Attack}} \\
&= a_i\left(\mathbb{P}[\theta \le \theta^*(e, y)] - c\right) \\
&= a_i\left(\Phi\left[\sqrt{\eta}\left(\theta^*(e, y) - \left(\frac{\beta}{\eta}x^*(e, y) + \frac{\alpha}{\eta}y\right)\right)\right] - c\right) \\
\implies c &= 1 - \Phi\left[\sqrt{\eta}\left(\frac{\beta}{\eta}x^*(e, y) + \frac{\alpha}{\eta}y - \theta^*(e, y)\right)\right] \\
&= 1 - \Phi\left[\sqrt{\eta}\left(\frac{\beta}{\eta}\left[\frac{1}{\sqrt{\beta}}\Phi^{-1}(\theta^*(e, y) + e) + \theta^*(e, y)\right] + \frac{\alpha}{\eta}y - \theta^*(e, y)\right)\right] && \text{by (3)} \\
\implies \frac{1}{\sqrt{\eta}}\Phi^{-1}(1 - c) &= \theta^*(e, y)\left[\frac{\beta}{\eta} - 1\right] + \frac{\alpha}{\eta}y + \frac{\sqrt{\beta}}{\eta}\Phi^{-1}(\theta^*(e, y) + e) \\
&= \theta^*(e, y)\left[\frac{-\alpha}{\eta}\right] + \frac{\alpha}{\eta}y + \frac{\sqrt{\beta}}{\eta}\Phi^{-1}(\theta^*(e, y) + e) \\
\implies \underbrace{\left(\sqrt{\frac{\eta}{\beta}}\right)\Phi^{-1}(1 - c)}_{g} &= \underbrace{\frac{\alpha}{\sqrt{\beta}}[y - \theta^*(e, y)] + \Phi^{-1}(\theta^*(e, y) + e)}_{G(\theta^*(e, y), y)} && \text{mult. by } \frac{\eta}{\sqrt{\beta}}
\end{aligned}$$

Now, I claim that there's a monotone equilibrium characterizes by $\theta^*(e, y)$ and $x^*(e, y)$ where θ^* is implicitly defined by $G(\theta^*(e, y), y) = g$ as in the previous expression, and x^* is as in (3). Let's verify that there is some point at which $G = g$. I claim that the point is somewhere between $[-e, 1 - e]$. Because, for fixed e and y, G is continuous in θ, and we know that

$$\begin{aligned}
G(-e, y) &= \frac{\alpha}{\sqrt{\beta}}y + \Phi^{-1}(-e + e) = -\infty \\
\text{and } G(1 - e, y) &= \frac{\alpha}{\sqrt{\beta}}y + \Phi^{-1}(1 - e + e) = \infty
\end{aligned}$$

that is, G goes off to negative infinity to the left, and positive infinity to the right, so by the IVT, it must intersect g at some point in between. (This is cool–the government can control the threshold!)

(c) Now, for uniqueness. We need to figure out when it's the case that G is strictly increasing, and when it's possible that it decreasesing. So, let's look at its derivative:

$$\frac{\partial G(\theta, y)}{\partial \theta} = \frac{1}{\phi(\Phi^{-1}(\theta + e))} - \frac{\alpha}{\sqrt{\beta}}$$

So, because

$$\phi(\omega) \in \left[0, \frac{1}{\sqrt{2\pi}}\right] \text{ and } \frac{1}{\phi(\omega)} \in [\sqrt{2\pi}, \infty],$$

the slope is only guaranteed to be positive throughout the support of θ if

$$\frac{\alpha}{\sqrt{\beta}} < \sqrt{2\pi}$$

otherwise, there is a range of $\theta(e, y)$ where the solution is not unique.

(d) Letting $\beta \to \infty$ in the implicit definition for G, we have

$$\Phi^{-1}(1-c) = \Phi^{-1}(\theta^*(e,y)+e) \Longrightarrow \; 1-c-e = \theta^*(e,y).$$

By letting $\beta \to \infty$, we are saying that the distribution of x_i collapses to unit mass at θ; individuals perfectly know θ. Unsurprisingly, the threshold $x^* = \theta^*$.

The threshold no longer depends on the public signal. Everybody knows θ perfectly, so there's no reason to think that an imperfect measure of θ would change the equilibrium threshold value.

Now, this threshold is decreasing in e and c. Why? Well, if there were no costs to attacking and the government wasn't strengthening the regime, then agents would always attack.[1] By increasing the costs to attacking (increasing c), or by making it harder for an attack to succeed (increasing e), there are fewer values of the fundamental for which agents will attack. That is, if it's less likely that an attack will be profitable, agents will require seeing a weaker fundamental. Since there's no strategic complementarity effect here, we only need to think about fundamentals.

(e) The (benevolent) government wants to maximize the utility of non-attackers; it wants the attack to not go through. But it also has to pay a cost. And it knows what will happen in the second stage conditional on its own efforts, e. Remember that, since $\beta \to 0$, everyone either attacks or doesn't attack, based on their private signal. Unfortunately, the government only imperfectly measures θ. Therefore, the government's objective function is:

$$\begin{aligned} \mathbb{E}\, v(\theta, A) - e^2 &= \mathbb{P}[A \le \theta + e] - e^2 \\ &= \mathbb{P}[\theta < 1-c-e] - e^2 \end{aligned}$$

Ok, I see why it is easier to assume $\alpha \to \infty$—I don't know how to find the posterior distribution of $\theta \mid y$. So, now we're assuming that the government knows θ. They know that everyone will attack if $\theta < 1-c-e$. But now we're in this silly case where everyone knows θ.

If $\theta \ge 1-c$, then the government can't do anything because everyone will attack regardless—their best response is to set $e = 0$. If $\theta < 1-c$, then the cheapest way to people to not attack is to set $e = 1-c-\theta$. If they do this, they spend e, but get utility of 1. So, this is worth it if

$$\begin{aligned} & 1-(1-c-\theta)^2 > 0 \\ \Longleftrightarrow\; & 1 > |1-c-\theta| \\ \Longleftrightarrow\; & 1 > 1-c-\theta \text{ and } 1 > -1+c+\theta \\ \Longleftrightarrow\; & \theta > -c \text{ and } 2-c > \theta \end{aligned}$$

So, they should spend the money for $\theta \in (-c, 2-c)$.

[1] Well, as long as it is profitable—that is, $\theta \le 1$.

3 Typicality of Beliefs

4 Investment and Complementary