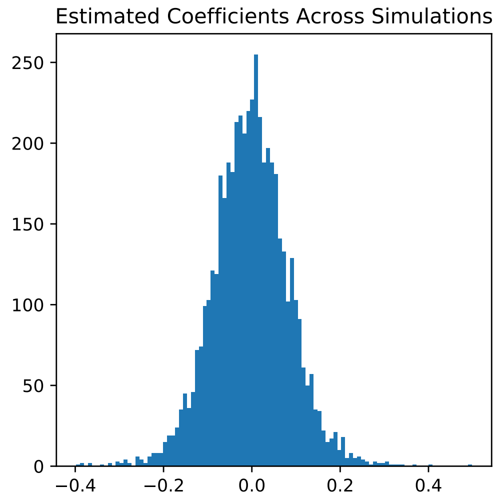
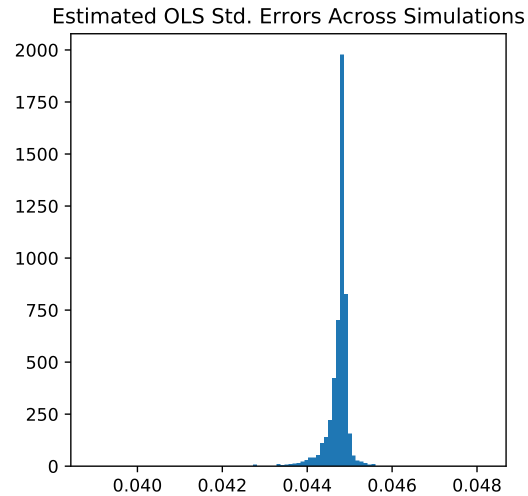
Problem Set 2
Econ 6413–Ng
Miguel Acosta
September 30, 2017

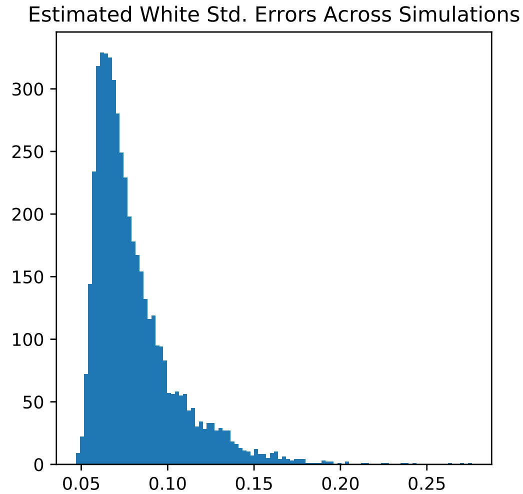
Q1. In the results below, I've assumed that $v_t \sim N(0,1)$, as in Hamilton (2008). I am able to replicate table 2 of Hamilton (results not presented). Here are the results for $T = 500$ and 5000 replications:

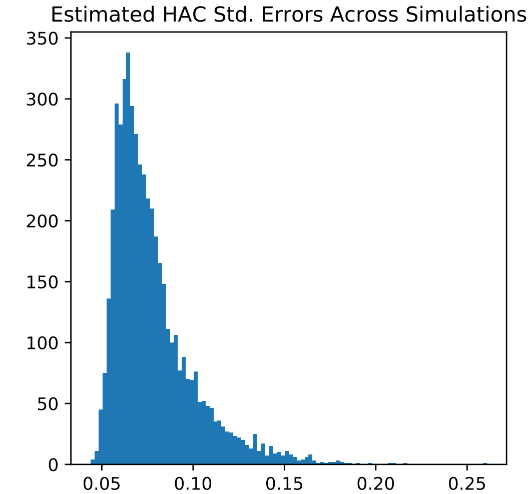
Standard Error	Size
OLS	0.280
White	0.062
HAC, $q = 5$	0.062

For fun, here are the distributions of the $\widehat{\beta}$ and standard errors across the simulations.





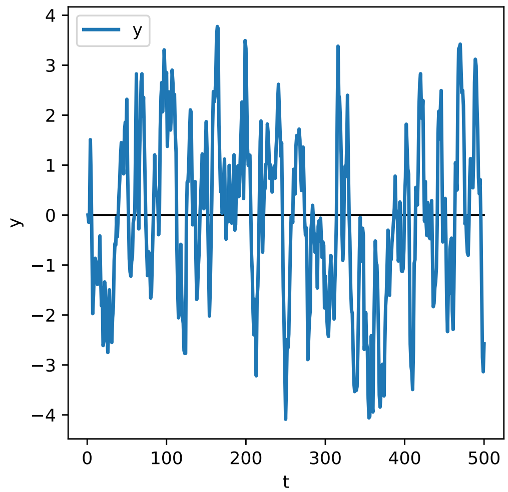




Q2. We are considering an ARMA(1, 1):

$$y_t = \alpha y_{t-1} + \beta e_{t-1} + e_t \qquad\qquad e_t \sim N(0, \sigma^2)$$

Here is a simulated series with 500 time periods and $\theta \equiv (\alpha, \beta, \sigma^2) = (0.8, 0.5, 0.5)$.



i. In problem set 1, I calculated γ_0 and γ_1 as:

$$\gamma_0 = \frac{\sigma^2(2\alpha\beta + \beta^2 + 1)}{1 - \alpha^2} \qquad\qquad \gamma_1 = \sigma^2 \cdot \frac{\alpha^2\beta + \beta^2\alpha + \beta + \alpha}{1 - \alpha^2}. \tag{1}$$

Remember that the series is mean-0. So, the second-order autocovariance is:

$$\gamma_2 = \mathbb{E}[y_t y_{t-2}] = \alpha y_{t-1} y_{t-2} + \beta e_{t-1} y_{t-2} + e_t y_{t-2}] = \alpha\gamma_1 \tag{2}$$

ii. The GMM objective function, therefore, is:

$$\overline{g}(\theta) \equiv \frac{1}{T-3}\sum_{t=3}^{T} \widehat{\psi} - \gamma(\theta) = \frac{1}{T-3}\sum_{t=3}^{T} \begin{bmatrix} y_t^2 - \frac{\sigma^2(2\alpha\beta+\beta^2+1)}{1-\alpha^2} \\ y_t y_{t-1} - \sigma^2 \cdot \frac{\alpha^2\beta+\beta^2\alpha+\beta+\alpha}{1-\alpha^2} \\ y_t y_{t-2} - \alpha\sigma^2 \cdot \frac{\alpha^2\beta+\beta^2\alpha+\beta+\alpha}{1-\alpha^2} \end{bmatrix}$$

Results follow:

α	0.8312
β	0.4470
σ^2	0.4546

iii. The third and fourth centered moments of an ARMA$(1,1)$ will be difficult to calculate, as you can see here (thanks to SymPy for these expressions):

$$\begin{aligned}
\mathbb{E}[y_t^3] &= \mathbb{E}[\alpha^3 y_{t-1}^3 + 3\alpha^2\beta e_{t-1} y_{t-1}^2 + 3\alpha^2 e_t y_{t-1}^2 + 3\alpha\beta^2 e_{t-1}^2 y_{t-1} \\
&\quad + 6\alpha\beta e_{t-1} e_t y_{t-1} + 3\alpha e_t^2 y_{t-1} + \beta^3 e_{t-1}^3 + 3\beta^2 e_{t-1}^2 e_t + 3\beta e_{t-1} e_t^2 + e_t^3] \\
\mathbb{E}[y_t^4] &= \mathbb{E}[\alpha^4 y_{t-1}^4 + 4\alpha^3\beta e_{t-1} y_{t-1}^3 + 4\alpha^3 e_t y_{t-1}^3 + 6\alpha^2\beta^2 e_{t-1}^2 y_{t-1}^2 \\
&\quad + 12\alpha^2\beta e_{t-1} e_t y_{t-1}^2 + 6\alpha^2 e_t^2 y_{t-1}^2 + 4\alpha\beta^3 e_{t-1}^3 y_{t-1} + 12\alpha\beta^2 e_{t-1}^2 e_t y_{t-1} \\
&\quad + 12\alpha\beta e_{t-1} e_t^2 y_{t-1} + 4\alpha e_t^3 y_{t-1} + \beta^4 e_{t-1}^4 + 4\beta^3 e_{t-1}^3 e_t + 6\beta^2 e_{t-1}^2 e_t^2 \\
&\quad + 4\beta e_{t-1} e_t^3 + e_t^4]
\end{aligned}$$

Luckily, most of these terms are 0, or simplify substantially. In particular, note that

$$\mathbb{E}[f(e_t) y_{t-1}] = \mathbb{E}[e_t e_{t-1} y_{t-1}] = \mathbb{E}[e_t f(y_{t-1})] = 0 \qquad \mathbb{E}[e_t^3] = 0 \qquad \mathbb{E}[e_t^4] = 3\sigma^4$$

The first expression is therefore:

$$\mathbb{E}[y_t^3] = \alpha^3\, \mathbb{E}[y_{t-1}^3] \Longrightarrow \mathbb{E}[y_{t-1}^3] = 0$$

(assuming $\mathbb{E}[y_0^3] = 0$, which is consistent with the assumption that $y_0 = 0$). The second expression reduces to

$$\mathbb{E}[y_t^4] = \alpha^4\, \mathbb{E}[y_{t-1}^4] + 3\sigma^4(\beta^4 + 4\alpha\beta^3 + 6\alpha^2\beta^2 + 4\alpha^3\beta + 1).$$

Since we only have covariance stationarity, the 4th moment depends on time, and can be calculated recursively assuming $\mathbb{E}[y_t^4] = 0$.

With all this, I present results that didn't work.

α	-0.8072
β	0.9985
σ^2	2.5612

Perhaps this question wants us to *add* the third and fourth moments to the conditions from part **ii**. I present that below, along with an estimation with $[\gamma_0, \gamma_1, \gamma_2, \kappa_4]$, since $\kappa_3 = 0$ certainly cannot help with the estimation. These do not help with estimation—I'm starting the optimizer from the true (α, β).

Using	$[\gamma_0, \gamma_1, \gamma_2, \kappa_3, \kappa_4]$
α	0.7598
β	0.1047
σ^2	0.5352

Using	$[\gamma_0, \gamma_1, \gamma_2, \kappa_4]$
α	0.7587
β	0.1163
σ^2	0.5313

Q3. **i.** Remember what OLS does:

$$\begin{bmatrix}\widehat{\phi}_1\\ \widehat{\phi}_2\end{bmatrix} = \begin{bmatrix}\widehat{\gamma}_0 & \widehat{\gamma}_1\\ \widehat{\gamma}_1 & \widehat{\gamma}_0\end{bmatrix}^{-1}\begin{bmatrix}\widehat{\gamma}_1\\ \widehat{\gamma}_2\end{bmatrix}$$

We can take advantage of the expressions in (1) and (2) to back out the estimates for $(\alpha, \beta, \sigma^2)$. Multiply both sides by the projection matrix to get

$$\begin{bmatrix}\widehat{\gamma}_0 & \widehat{\gamma}_1\\ \widehat{\gamma}_1 & \widehat{\gamma}_0\end{bmatrix}\begin{bmatrix}\widehat{\phi}_1\\ \widehat{\phi}_2\end{bmatrix} = \begin{bmatrix}\widehat{\gamma}_1\\ \widehat{\gamma}_2\end{bmatrix}$$

We can estimate the RHS easily. So, this identifies α since $\frac{\gamma_2}{\gamma_1} = \alpha$. Next, taking the ratio of γ_0 to γ_1 in (1), we have one equation to solve for $\widehat{\beta}$. I asked SymPy to do this. Finally, we can use the γ_0 equation, (1), to get σ^2.

ii. The OLS estimates are:

	Estimate	Newey-West S.E.
ϕ_1	1.2144	0.0394
ϕ_2	-0.3424	0.0393

The estimate of $\phi(1)$ is 0.87 [0.017]. The number in brackets is the Newey-West standard error with $q = 5$. Inverting this to get the parametets of the ARMA(1, 1) model as discussed in the previous subquestion results in:

α	0.8315
β	0.4168
σ^2	0.4717

iii. As an auxiliary model, we will estimate the AR(2). We will compare these estimates to what the AR(2) estimates *should be* under an ARMA with the supplied parameters. Start by noticing the relationship between an ARMA(1, 1) and an AR(2):

$$\begin{aligned} y_t &= \alpha y_{t-1} + e_t + \beta e_{t-1}\\ &= \alpha y_{t-1} + e_t + \beta(y_{t-1} - \alpha y_{t-2} - \beta e_{t-2})\\ &= \underbrace{(\alpha+\beta)}_{\phi_1} y_{t-1} + \underbrace{(-\alpha\beta)}_{\phi_2} y_{t-2} + \underbrace{e_t - \beta^2 e_{t-2}}_{u_t}, \end{aligned}$$

with

$$\sigma_u^2 \equiv \text{var}(u_t) = \text{var}(e_t - \beta^2 e_{t-2}) = \sigma_e^2(1+\beta^4).$$

The minimum distance objective function, therefore, is sum of squared entries of the following:

$$\begin{aligned} \overline{g}(\alpha,\beta,\sigma_e^2) &= \widehat{\psi} - \psi(\beta)\\ &\equiv \begin{bmatrix}\widehat{\phi}_1\\ \widehat{\phi}_2\\ \widehat{\sigma}_u^2\end{bmatrix} - \begin{bmatrix}\phi_1(\alpha,\beta,\sigma_e^2)\\ \phi_2(\alpha,\beta,\sigma_e^2)\\ \sigma_u^2(\alpha,\beta,\sigma_e^2)\end{bmatrix}\\ &= \begin{bmatrix}\widehat{\phi}_1\\ \widehat{\phi}_2\\ \widehat{\sigma}_u^2\end{bmatrix} - \begin{bmatrix}\alpha+\beta\\ -\alpha\beta\\ \sigma_e^2(1+\beta^4).\end{bmatrix} \end{aligned}$$

The results of estimation are:

α	0.7694
β	0.4450
σ^2	0.4456

Q4. I went with qustion (i). Here are the results for SMM, using autocovariances of order 0, 1, and 2 as the moment conditions:

α	0.8383
β	0.4365
σ^2	0.4573

Here are the results using indirect inference, with an AR(2) as the auxiliary model:

α	0.8405
β	0.4379
σ^2	0.5000

For each of the results above, I used 501 simulations, and kept $T = 500$. I used the same sample of iid shocks in both simulation estimations.