Maximum Likelihood Estimation: Numerical Solution for Weibull Distribution[1]

The Weibull distribution has the probability density function:

$$f\left(y|\theta,\lambda\right) \;=\; \frac{\lambda y^{\lambda-1}}{\theta^{\lambda}}\text{exp}\left\{-\left(\frac{y}{\theta}\right)^{\lambda}\right\} \tag{1}$$

where: $y > 0$, θ is the scale parameter and λ is the shape parameter (nuisance parameter).

Maximum Likelihood Estimation

Let $y_1, \ldots, y_n$, denote the data. Assume, $\forall i : y_i$ independente random variables, share the same parameters from a Weibull distribution described in (1).

$$f\left(y_i|\theta,\lambda\right) \;=\; \frac{\lambda {y_i}^{\lambda-1}}{\theta^{\lambda}}\text{exp}\left\{-\left(\frac{y_i}{\theta}\right)^{\lambda}\right\} \tag{2}$$

then, their join probability distribution is:

$$\begin{aligned} f\left(y_1,\ldots,y_n|\theta,\lambda\right) &= \prod_{i=1}^{n} f\left(y_i|\theta,\lambda\right) \\ &= \prod_{i=1}^{n} \frac{\lambda {y_i}^{\lambda-1}}{\theta^{\lambda}}\text{exp}\left\{-\left(\frac{y_i}{\theta}\right)^{\lambda}\right\} \end{aligned} \tag{3}$$

The likelihood function is:

$$\mathscr{L}\left(\theta|y_1,\ldots,y_n,\lambda\right) \;=\; f\left(y_1,\ldots,y_n|\theta,\lambda\right) \tag{4}$$

The log-likelihood function is:

$$\begin{aligned} \ell\left(\theta|y_1,\ldots,y_n,\lambda\right) &= \text{log}\left(\mathscr{L}\left(\theta|y_1,\ldots,y_n,\lambda\right)\right) \\ &= \text{log}\prod_{i=1}^{n} \frac{\lambda {y_i}^{\lambda-1}}{\theta^{\lambda}}\text{exp}\left\{-\left(\frac{y_i}{\theta}\right)^{\lambda}\right\} \\ &= \sum_{i=1}^{n}\text{log}\left(\frac{\lambda {y_i}^{\lambda-1}}{\theta^{\lambda}}\text{exp}\left\{-\left(\frac{y_i}{\theta}\right)^{\lambda}\right\}\right) \\ \ell\left(\theta|y_1,\ldots,y_n,\lambda\right) &= \sum_{i=1}^{n}\left(\text{log }\lambda + (\lambda-1)\,\text{log }y_i - \lambda\,\text{log }\theta - \left(\frac{y_i}{\theta}\right)^{\lambda}\right) \end{aligned} \tag{5}$$

The maximum likelihood estimator (MLE), $\hat{\theta}$, is such that:

$$\hat{\theta} \;=\; \text{argmax}\left\{\ell\left(\theta|y_1,\ldots,y_n,\lambda\right)\right\} \tag{6}$$

[1] *To reefer this document and the code used, please cite as*: Alvarado, M. (2020, August 8). Maximum Likelihood Estimation: Numerical Solution for Weibull Distribution (Version v1.0.0). Zenodo. http://doi.org/10.5281/zenodo.3977819. Also at GitHub: https://github.com/miguel-alvarado-stats/MLE_Weibull.

To maximize the log-likelihood function (6), requires the derivative with respect to θ; which is called the Score function, denoted by $U(\theta|y_1,\ldots,y_n,\lambda)$.

$$\begin{aligned} U(\theta|y_1,\ldots,y_n,\lambda) &= \frac{\partial \ell(\theta|y_1,\ldots,y_n,\lambda)}{\partial\theta} \\ &= \sum_{i=1}^{n}\left(-\frac{\lambda}{\theta}+\frac{\lambda {y_i}^{\lambda}}{\theta^{\lambda+1}}\right) \\ &= -\frac{\lambda n}{\theta}+\frac{\lambda}{\theta^{\lambda+1}}\sum_{i=1}^{n}{y_i}^{\lambda} \end{aligned} \tag{7}$$

Then, the MLE, $\hat{\theta}$, is the solution of:

$$U\left(\theta=\hat{\theta}|y_1,\ldots,y_n,\lambda\right) = 0 \tag{8}$$

Maximum Likelihood Estimation: Newton-Raphson Method

Just for notation, (8) let write as:

$$U(\theta^*) = 0 \tag{9}$$

The equation (9), generally, is a nonlinear equation, that can be aproximate by Taylor Series:

$$U(\theta^*) \approx U\left(\theta^{(t)}\right)+U'\left(\theta^{(t)}\right)\left(\theta^*-\theta^{(t)}\right) \tag{10}$$

Then, using (10) into (9), and solving for θ^*:

$$\begin{aligned} U\left(\theta^{(t)}\right)+U'\left(\theta^{(t)}\right)\left(\theta^*-\theta^{(t)}\right) &= 0 \\ \theta^* &= \theta^{(t)}-\frac{U\left(\theta^{(t)}\right)}{U'\left(\theta^{(t)}\right)} \end{aligned} \tag{11}$$

This is the Newton-Raphson method: starting with an initial guess θ^1 successive approximations are obtained using (12), until the iterative process converges.

$$\theta^{(t+1)} = \theta^{(t)}-\frac{U\left(\theta^{(t)}\right)}{U'\left(\theta^{(t)}\right)} \tag{12}$$

where $U'\left(\theta^{(t)}\right)$ is the derivative of the Score function (7) respect of θ.

$$\begin{aligned} U'(\theta|y_1,\ldots,y_n,\lambda) &= \frac{\partial U(\theta|y_1,\ldots,y_n,\lambda)}{\partial\theta} \\ &= \frac{\lambda n}{\theta^2}-\frac{\lambda(\lambda+1)}{\theta^{\lambda+2}}\sum_{i=1}^{n}{y_i}^{\lambda} \end{aligned} \tag{13}$$

In order to example the use of Newton-Rapshon method, we use the data of lifetimes (times to failure in hours) of Kevlar epoxy strand pressure vessels at 70% stress level.

```
# load data
Y <- as.matrix(read.delim(file = "data.txt", header = FALSE))

# load the function to solve by Newton-Raphson
```

```r
load("MLE_NR_Weibull.RData")

# MLE by Newton-Raphson
MLE_NR_Weibull(Y)
```

```
##      ML Estimator    Likelihood Log-Likelihood
## [1,]     8805.694 3.515563e-210      -482.2857
## [2,]     9633.777 1.377755e-209      -480.9198
## [3,]     9875.898 1.477093e-209      -480.8502
## [4,]     9892.110 1.477486e-209      -480.8499
## [5,]     9892.177 1.477486e-209      -480.8499
## [6,]     9892.177 1.477486e-209      -480.8499
```

Maximum Likelihood Estimation: Fisher-Scoring Method