

Maximum Likelihood Estimation: Numerical Solution for Weibull Distribution¹

The Weibull distribution has the probability density function:

$$f(y|\theta, \lambda) = \frac{\lambda y^{\lambda-1}}{\theta^\lambda} \exp \left\{ -\left(\frac{y}{\theta}\right)^\lambda \right\} \quad (1)$$

where: $y > 0$, θ is the scale parameter and λ is the shape parameter (nuisance parameter).

Maximum Likelihood Estimation

Let y_1, \dots, y_n , denote the data. Assume, $\forall i : y_i$ independent random variables, share the same parameters from a Weibull distribution described in (1).

$$f(y_i|\theta, \lambda) = \frac{\lambda y_i^{\lambda-1}}{\theta^\lambda} \exp \left\{ -\left(\frac{y_i}{\theta}\right)^\lambda \right\} \quad (2)$$

then, their join probability distribution is:

$$\begin{aligned} f(y_1, \dots, y_n|\theta, \lambda) &= \prod_{i=1}^n f(y_i|\theta, \lambda) \\ &= \prod_{i=1}^n \frac{\lambda y_i^{\lambda-1}}{\theta^\lambda} \exp \left\{ -\left(\frac{y_i}{\theta}\right)^\lambda \right\} \end{aligned} \quad (3)$$

The likelihood function is:

$$\mathcal{L}(\theta|y_1, \dots, y_n, \lambda) = f(y_1, \dots, y_n|\theta, \lambda) \quad (4)$$

The log-likelihood function is:

$$\begin{aligned} \ell(\theta|y_1, \dots, y_n, \lambda) &= \log(\mathcal{L}(\theta|y_1, \dots, y_n, \lambda)) \\ &= \log \prod_{i=1}^n \frac{\lambda y_i^{\lambda-1}}{\theta^\lambda} \exp \left\{ -\left(\frac{y_i}{\theta}\right)^\lambda \right\} \\ &= \sum_{i=1}^n \log \left(\frac{\lambda y_i^{\lambda-1}}{\theta^\lambda} \exp \left\{ -\left(\frac{y_i}{\theta}\right)^\lambda \right\} \right) \\ \ell(\theta|y_1, \dots, y_n, \lambda) &= \sum_{i=1}^n \left(\log \lambda + (\lambda - 1) \log y_i - \lambda \log \theta - \left(\frac{y_i}{\theta}\right)^\lambda \right) \end{aligned} \quad (5)$$

The maximum likelihood estimator (MLE), denoted by $\hat{\theta}$, is such that:

$$\hat{\theta} = \operatorname{argmax} \{ \ell(\theta|y_1, \dots, y_n, \lambda) \} \quad (6)$$

¹To refer this document and the implemented code, please cite as: Alvarado, M. (2020, August 8). Maximum Likelihood Estimation: Numerical Solution for Weibull Distribution (Version v1.0.0). Zenodo. <http://doi.org/10.5281/zenodo.3978591>. Also at GitHub: https://github.com/miguel-alvarado-stats/MLE_Weibull.

To maximize the log-likelihood function (5), requires the derivative with respect to θ ; which is called the Score function, denoted by $U(\theta|y_1, \dots, y_n, \lambda)$.

$$\begin{aligned} U(\theta|y_1, \dots, y_n, \lambda) &= \frac{\partial \ell(\theta|y_1, \dots, y_n, \lambda)}{\partial \theta} \\ &= \sum_{i=1}^n \left(-\frac{\lambda}{\theta} + \frac{\lambda y_i^\lambda}{\theta^{\lambda+1}} \right) \\ &= -\frac{\lambda n}{\theta} + \frac{\lambda}{\theta^{\lambda+1}} \sum_{i=1}^n y_i^\lambda \end{aligned} \quad (7)$$

Then, the MLE $\hat{\theta}$, is the solution of:

$$U(\theta = \hat{\theta}|y_1, \dots, y_n, \lambda) = 0 \quad (8)$$

Maximum Likelihood Estimation: Newton-Raphson Method

Just for notation, let write equation (8) as:

$$U(\theta^*) = 0 \quad (9)$$

The equation (9), generally, is a nonlinear equation, that can be approximate by Taylor Series:

$$U(\theta^*) \approx U(\theta^{(t)}) + U'(\theta^{(t)})(\theta^* - \theta^{(t)}) \quad (10)$$

Then, using (10) into (9), and solving for θ^* :

$$\begin{aligned} U(\theta^{(t)}) + U'(\theta^{(t)})(\theta^* - \theta^{(t)}) &= 0 \\ \theta^* &= \theta^{(t)} - \frac{U(\theta^{(t)})}{U'(\theta^{(t)})} \end{aligned} \quad (11)$$

where U' is the derivative of the Score function (7) respect of θ .

$$\begin{aligned} U'(\theta|y_1, \dots, y_n, \lambda) &= \frac{\partial U(\theta|y_1, \dots, y_n, \lambda)}{\partial \theta} \\ &= \frac{\lambda n}{\theta^2} - \frac{\lambda(\lambda+1)}{\theta^{\lambda+2}} \sum_{i=1}^n y_i^\lambda \end{aligned} \quad (12)$$

Then, with the Newton-Raphson method: starting with an initial guess $\theta^{(1)}$ successive approximations are obtained using (13), until the iterative process converges.

$$\theta^{(t+1)} = \theta^{(t)} - \frac{U(\theta^{(t)})}{U'(\theta^{(t)})} \quad (13)$$

In order to example the use of Newton-Rapshon method, we use the data of lifetimes (times to failure in hours) of Kevlar epoxy strand pressure vessels at 70% stress level².

²This data was taken from Dobson, A. Barnett, A. (2018) An Introduction to Generalized Linear Models. Texts in Statistical Science. Chapman Hall/CRC.

```
# load data
Y <- as.matrix(read.delim(file = "data.txt", header = FALSE))
```

We load the code developed into our R function `MLE_NR_Weibull`, stored in the R object with the same name.

```
# load the function to solve by Newton-Raphson
load("MLE_NR_Weibull.RData")
```

The function `MLE_NR_Weibull`, besides some other default parameters that can be modified such as $\lambda = 2$, only needs the data vector Y .

```
# MLE by Newton-Raphson (NR) for Weibull distribution
MLE_NR_Weibull(Y)
```

##	ML Estimator	Likelihood	Log-Likelihood
## [1,]	"8805.693878"	"3.51556268514669e-210"	"-482.285669935176"
## [2,]	"9633.777408"	"1.37775513478331e-209"	"-480.919828975047"
## [3,]	"9875.898292"	"1.47709267871969e-209"	"-480.850208686227"
## [4,]	"9892.110004"	"1.47748579671734e-209"	"-480.849942578555"
## [5,]	"9892.176818"	"1.47748580332318e-209"	"-480.849942574083"
## [6,]	"9892.176819"	"1.47748580332318e-209"	"-480.849942574084"

Then, the MLE by Newton-Raphson method: $\hat{\theta} = 9892.176819$.

Maximum Likelihood Estimation: Fisher-Scoring Method

A distribution belongs to the exponential family if it can be written in the form:

$$f(y|\theta) = \exp \left\{ \frac{a(y)b(\theta) - c(\theta)}{\phi} + d(y, \phi) \right\} \quad (14)$$

Since (1) can be written as a member of exponential family as in (14):

$$\begin{aligned} f(y|\theta, \lambda) &= \exp \left\{ \log \left(\frac{\lambda y^{\lambda-1}}{\theta^\lambda} \exp \left\{ -\left(\frac{y}{\theta}\right)^\lambda \right\} \right) \right\} \\ &= \exp \left\{ y^\lambda (-\theta^{-\lambda}) - (\lambda \log \theta - \log \lambda) + (\lambda - 1) \log y \right\} \end{aligned} \quad (15)$$

where, $a(y) = y^\lambda$, $b(\theta) = -\theta^{-\lambda}$, $c(\theta) = \lambda \log \theta - \log \lambda$, $\phi = 1$, and $d(y, \phi) = (\lambda - 1) \log y$.

Then, since the Weibull distribution belongs to the exponential family, it can be show that the variance of U , denoted by \mathcal{J} , is:

$$\mathcal{J} = \text{Var} \{U\} = -E \{U'\} \quad (16)$$

where:

$$E \{U'\} = -\frac{1}{\phi} \left(b''(\theta) \frac{c'(\theta)}{b'(\theta)} - c''(\theta) \right) \quad (17)$$

For MLE, it is common to approximate U' by its expected value $E\{U'\}$. In this case:

$$\begin{aligned}
\mathcal{J} &= -E\{U'\} \\
&= E\{-U'\} \\
&= E\left\{-\sum_{i=1}^n U'_i\right\} \\
&= \sum_{i=1}^n -E\{U'_i\} \\
&= \sum_{i=1}^n -\frac{1}{\phi} \left(b''(\theta) \frac{c'(\theta)}{b'(\theta)} - c''(\theta) \right)
\end{aligned} \tag{18}$$

where, using (1), the previous derivaties:

$$\begin{aligned}
b'(\theta) &= \lambda \theta^{-(\lambda+1)} \\
b''(\theta) &= -\lambda(\lambda+1) \theta^{-(\lambda+2)} \\
c'(\theta) &= \lambda \theta^{-1} \\
c''(\theta) &= -\lambda \theta^{-2} \\
\frac{1}{\phi} &= 1
\end{aligned}$$

Then, replacing them into (18):

$$\begin{aligned}
\mathcal{J} &= \sum_{i=1}^n - \left(-\lambda(\lambda+1) \theta^{-(\lambda+2)} \frac{\lambda \theta^{-1}}{\lambda \theta^{-(\lambda+1)}} + \lambda \theta^{-2} \right) \\
&= \sum_{i=1}^n \frac{\lambda^2}{\theta^2} \\
&= n \left(\frac{\lambda}{\theta} \right)^2
\end{aligned} \tag{19}$$

Finally:

$$\begin{aligned}
\mathcal{J} &= -E\{U'\} = n \left(\frac{\lambda}{\theta} \right)^2 \\
-\mathcal{J} &= E\{U'\} = n \left(\frac{\lambda}{\theta} \right)^2
\end{aligned} \tag{20}$$

Then, approximating U' by its expected value $E\{U'\}$, the equation (13) results into:

$$\theta^{(t+1)} = \theta^{(t)} + \frac{U(\theta^{(t)})}{\mathcal{J}(\theta^{(t)})} \tag{21}$$

In order to example the use of Fisher-Scoring method, we use the same data used in the Newton-Rapshon method. We load the code developed into our R function `MLE_FS_Weibull`, stored in the R object with the same name.

```
# load the function to solve by Fisher-Scoring
load("MLE_FS_Weibull.RData")
```

The function `MLE_FS_Weibull`, besides some other default parameters that can be modified such as $\lambda = 2$, only needs the data vector Y .

```
# MLE by Fisher-Scoring (FS) for Weibull distribution
MLE_FS_Weibull(Y)
```

##	ML Estimator	Likelihood	Log-Likelihood
## [1,]	"8805.693878"	"3.51556268514669e-210"	"-482.285669935176"
## [2,]	"9959.204199"	"1.47092711455647e-209"	"-480.854391543592"
## [3,]	"9892.402373"	"1.47748572804852e-209"	"-480.849942625031"
## [4,]	"9892.176822"	"1.47748580332318e-209"	"-480.849942574083"
## [5,]	"9892.176819"	"1.47748580332319e-209"	"-480.849942574084"

Then, the MLE by Fisher-Scoring method: $\hat{\theta} = 9892.176819$.