Tessellations on surfaces from complex analytic functions

Jesús Muciño Raymundo

muciray@matmor.unam.mx
Centro de Ciencias Matemáticas,
Universidad Nacional Autónoma de México

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Index

Our scenario is the complex plane $\mathbb{C} = \{z = x + iy\}$ or the Riemann sphere $\widehat{\mathbb{C}} \doteq \mathbb{C} \cup \{\infty\}$,

our objects are the complex functions

$$\begin{array}{rcl} f : \widehat{\mathbb{C}}_z & \longrightarrow & \widehat{\mathbb{C}}_w \\ z & \longmapsto & f(z) = \mathfrak{Re}\,(f(z)) + i\mathfrak{Im}\,(f(z))\,. \end{array}$$

The simplest cases are

- monomials $P(z) = z^n$,
- polynomials $P(z) = a_0 + a_1 z + \ldots + a_{n-1} z^{n-1} + z^n$,
- rational functions $\dfrac{P(z)}{Q(z)} = \dfrac{a_0 + a_1 z + \ldots + a_{n-1} z^{n-1} + z^n}{b_0 + b_1 z + \ldots + b_{m-1} z^{m-1} + z^m}$

such that $P(z)$ has no common factors with $Q(z)$.

An elementary example:

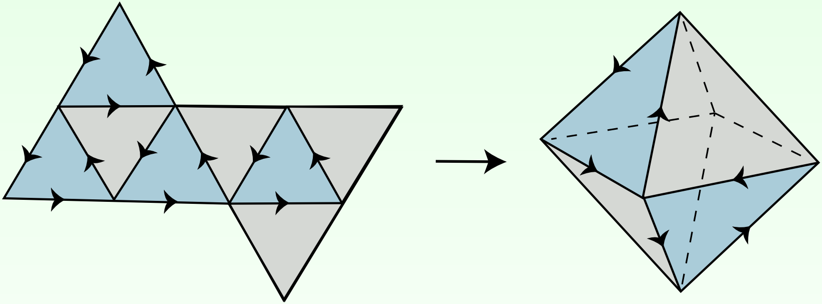

Figure: platonic-solids-22.pdf An elementary idea, the octahedron.

. . . From a famous book:

F. Klein,

Lectures on the Icosahedron and the Solution of the Equations of Fifth Degree, (1884).

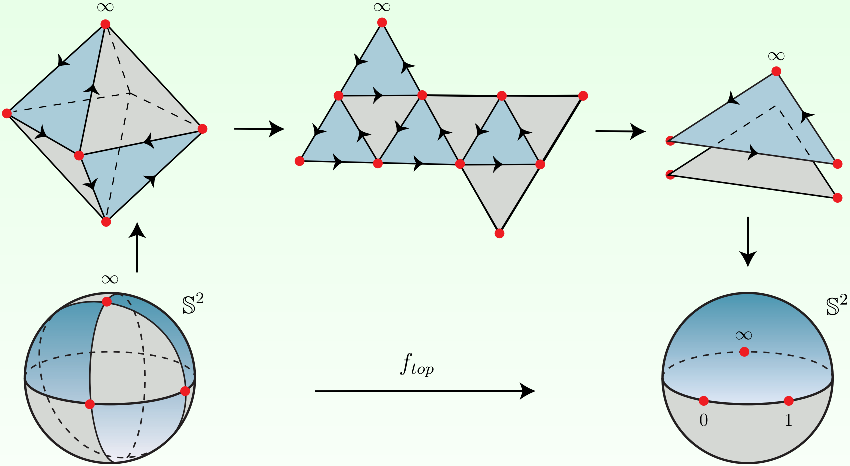

Figure: platonic-solids-3.pdf First assertion, the octahedron determines a **continuous function** $f_{top} : \mathbb{S}^2 \longrightarrow \mathbb{S}^2$.

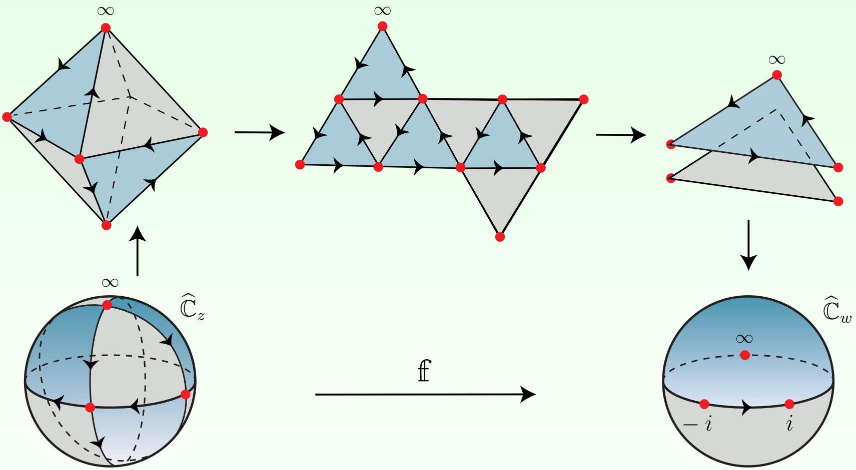

Figure: platonic-solids-31.pdf Second assertion; the octahedron determines a **complex rational function** $\mathbb{f}(z) : \widehat{\mathbb{C}}_z \longrightarrow \widehat{\mathbb{C}}_w$.

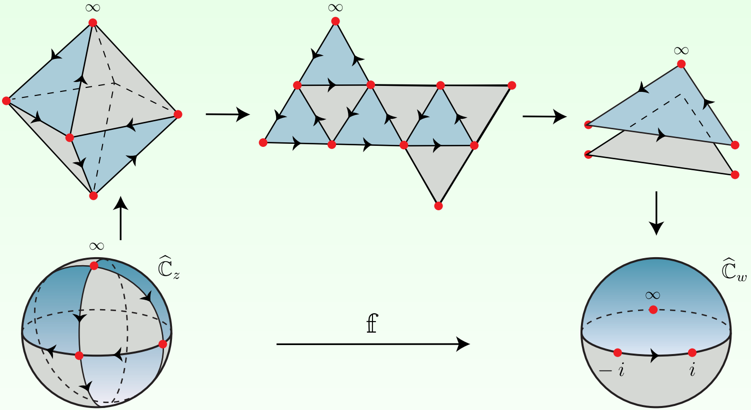

Figure: platonic-solids-31.pdf Third assertion; in fact the octahedron determines an **explicit** complex rational function $\mathbb{f}(z) = \dfrac{z^4 - 1}{2z^2} : \widehat{\mathbb{C}}_z \longrightarrow \widehat{\mathbb{C}}_w$.

Let $\Gamma \subset \widehat{\mathbb{C}}_z$ be the octahedron as **a graph in the sphere**.

Fortunate facts about the tessellation

$$\widehat{\mathbb{C}}_z \backslash \Gamma = \underbrace{T_1 \cup T_2 \cup T_3 \cup T_4}_{\text{blue tiles}} \cup \underbrace{T_1' \cup T_2' \cup T_3' \cup T_4'}_{\text{white tiles}} \doteq \mathfrak{M}_\Gamma .$$

1. A point $z \in \widehat{\mathbb{C}}_z$ is cloured in blue if and only if $f(z_1) \in \widehat{\mathbb{C}}_w$ is blue.
2. Along each edge of Γ, a blue tile and a white tile are glued in a continuous fashion.
3. The blue tiles and white tiles are adjacent at the edges of Γ, in an alternate way.
4. The number of blue tiles is equal to the algebraic degree of $\mathbb{f}$. The same holds for white tiles.
5. The number of tiles at the vertices of Γ is even.

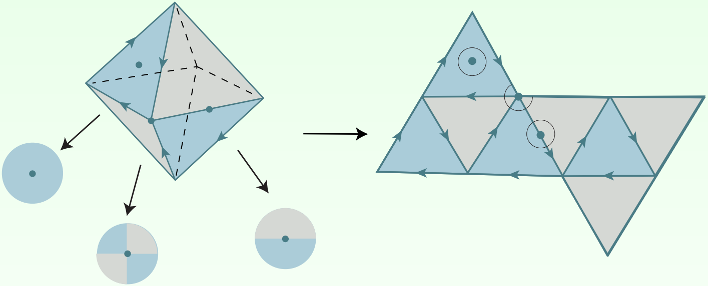

Figure: platonic-solids-2.pdf Fortunate facts about the octahedron as a tessellation $\mathfrak{M}_\Gamma$ of the sphere.

Which are the important/significative points of the function

$$\mathbb{f}(z) = \frac{z^4 - 1}{2z^2} \quad ?$$

- zeros $\quad 1,\ i,\ -1,\ -i,$
- poles $\quad 0,\ 0,\ \infty,\ \infty,$
- critical points

$$\begin{array}{ll} z_1 = \ \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}} & z_2 = -\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}} \\ z_3 = -\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}} & z_4 = \ \frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}} \\ z_5 = 0 & z_6 = \infty, \end{array}$$

- critical values

$$\begin{aligned} w_1 &= \mathbb{f}(z_1) = \mathbb{f}(z_3) = i \\ w_2 &= \mathbb{f}(z_2) = \mathbb{f}(z_4) = -i \\ w_3 &= \mathbb{f}(z_5) = \mathbb{f}(z_6) = \infty. \end{aligned}$$

Let us recall that $f'(z) = \frac{df}{dz}(z)$ denotes the complex derivative.

Lemma.

Let f be a polynomial (real or complex),
$f'(z_0) \neq 0$ if and only if f is a local bijection in a neighborhood of z_0.

Lemma.

For f a complex polynomial, if $f'(z_1) = 0$,
then f is k to 1 in a punctured neighborhood of z_1, where $k \geq 2$.

A **critical point** z_1 of f is such that $f'(z_1) = 0$.

The **critical value** of a critical point z_1 is $t_1 = f(z_1)$.

A **cocritical point** of f is a point z_{cc} such that

$$f'(z_{cc}) \neq 0 \text{ and } f(z_{cc}) = t_1 \text{ is a critical value.}$$

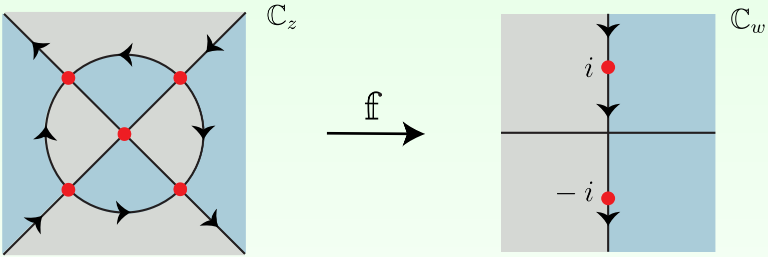

Figure: platonic-solids-4.pdf The important/significative points of the function $\mathbb{f}(z) = \dfrac{z^4 - 1}{2z^2}$ are just its critical points (since they are the vertices of Γ). Moreover, the critical points are the pull–back of the critical values under $\mathbb{f}$.

How lucky have you been?

In other words;
we can construct analogous tessellations for other functions/surfaces?

Our objective is explore the correspondence
complex holomorphic functions $f : M \longrightarrow \widehat{\mathbb{C}}_w \longleftrightarrow$ tessellations $\mathfrak{M}_\Gamma$ on M

A question[1] by William P. Thurston in 2010:

What is the shape of a complex rational function?

[1] See; *What's Next? The Mathematical Legacy of William P. Thurston,* Ed. by Dylan P. Thurston, Annals of Mathematics Studies vol. 205 (2020), page 215.

Figure: thurston-album.pdf William P. Thurston (1946 – 2012) was an American mathematician. In 1982 the International Mathematical Union awarded him the Fields Medal.

The above question is surprising, because in 2010, Thurston had already found very profound results for complex rational functions and their dynamics.

What is the shape of a

complex functions?

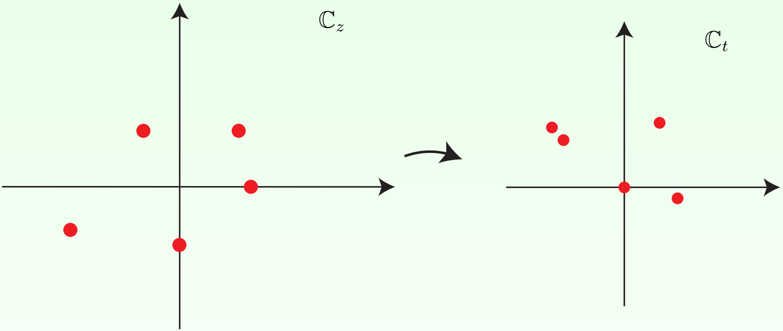

Figure: complex-functions1.pdf A complex rational function $f(z) = \frac{(z^2-1)(z-2-i)^2}{(z^2+2+2i)} : \mathbb{C}_z \longrightarrow \mathbb{C}_w$. Source: Wikipedia

Figura

Roberto

Figure:

A complex rational function $f(z) = \frac{(z^2-1)(z-2-i)^2}{(z^2+2+2i)} : \mathbb{C}_z \longrightarrow \mathbb{C}_w$. Source: Wikipedia

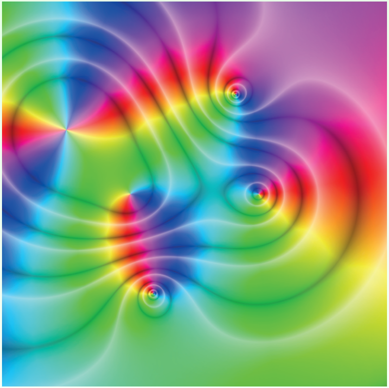

Figure: complex-functions.pdf A complex rational function $f(z) = \frac{(z^2-1)(z-2-i)^2}{(z^2+2+2i)} : \mathbb{C}_z \longrightarrow \mathbb{C}_w$. Source: Wikipedia

A surprise:

There are at least three equivalent ways (coordinate systems) to write complex monic polynomials of degree n

$$
\begin{tikzcd}
\text{coefficients} \arrow[rr, shift left] \arrow[dr, shift left] & & \text{roots} \arrow[ll, shift left] \arrow[dl, shift left] \\
& \text{critical points} \arrow[ul, shift left] \arrow[ur, shift left] &
\end{tikzcd}
\quad .
$$

Coefficients and roots lead us to algebra.
The critical points (those where the derivative vanishes) lead us to differential/integral calculus, and analysis.

The object is Viète's map $\mathcal{V}_2$ for quadratic monic polynomials.

$$
\begin{array}{ccccc}
\text{roots} & \longmapsto & \text{roots without order} & \longmapsto & \text{coefficients} \\
\mathbb{C}^2 & \longrightarrow & \frac{\mathbb{C}^2}{Sym(2)} & \xrightarrow{\mathcal{V}_2} & \mathbb{C}^2 \\
 & & & \xleftarrow{\mathcal{V}_2^{-1}} & \\
(z_1, z_2) & \longmapsto & [z_1 : z_2] & \longmapsto & (c, b) \\
 & & & & = \big(z_1 z_2, -(z_1 + z_2)\big) \\
 & & & & = \underbrace{z_1 z_2}_{c} + \underbrace{(-z_1 - z_2)}_{b} z + \underbrace{1}_{a} z^2
\end{array}
$$

$Sym(n)$ is the symmetric group whose elements are the possible exchanges in n positions.
$\mathcal{V}^{-1}$ is the application of taking the roots.

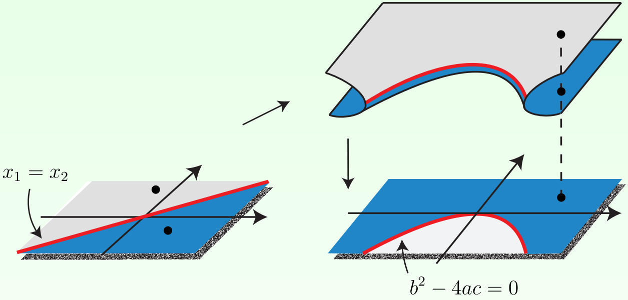

Figure: aplicacion-vieta.pdf Geometry of the Viète's map $\mathcal{V}_2$, for real quadratic monic polynomials.

$$\begin{array}{rcl} \mathcal{V}_2 : \mathbb{R}^2_{\textit{roots with order}} & \longrightarrow & \mathbb{R}^2_{\textit{coefficients}} \\ (x_1, x_2) & \longmapsto & \big(x_1 x_2,\ -x_1 - x_2\big) \doteq (c, b), \end{array}$$

using our elementary knowledge

$$(x - x_1)(x - x_2) = x_1 x_2 + (-x_1 - x_2)x + x^2.$$

The Viète's map $\mathcal{V}_n$ for monic polynomials of degree n.

$$
\begin{array}{ccccc}
\text{roots} & \longmapsto & \text{roots without order} & \longmapsto & \text{coefficients} \\
\mathbb{C}^n & \longrightarrow & \frac{\mathbb{C}^n}{Sym(n)} & \xrightarrow{\mathcal{V}_n} & \mathbb{C}^n \\
 & & & \xleftarrow{\mathcal{V}_n^{-1}} & \\
(z_1, \ldots, z_n) & \longmapsto & [z_1 : \cdots : z_n] & \longmapsto & (a_0, \ldots, a_{n-1})
\end{array}
$$

$$
= \Big((-1)^n (z_1 \cdots z_n), \quad \vdots \quad -(z_1 + \ldots + z_n) \Big)
$$

$$
= \underbrace{(-1)^n (z_1 \cdots z_n)}_{a_0} + \ldots \underbrace{-(z_1 + \ldots + z_n)}_{a_{n-1}} z^{n-1} + z^n
$$

From critical points to monic polynomials an ancient idea, ... R. Thom[2]:

critical points with order		critical points without order		coefficients
$\mathbb{C}^{n-1}$	$\to$	$\frac{\mathbb{C}^{n-1}}{Sym(n)}$	$\to$	$\mathbb{C}^n_{coeff}$
$(c_1, \ldots, c_{n-1})$	$\mapsto$	$[c_1, \ldots, c_{n-1}]$	$\mapsto$	$n \int_0^z (z - c_1) \cdots (z - c_{n-1}) dz$

$$= (\underbrace{a_0}_{=0}, a_1, \ldots, a_{n-1}) \,.$$

Note that, we only use $n-1$ critical points for degree n.

[2]See; *Polynomials with preassigned values at their branching points,* J. Mycielski, Amer. Math. Montly 77:8 (1970), page 853–855.

For rational functions $R(z) : \widehat{\mathbb{C}}_z \longrightarrow \widehat{\mathbb{C}}_w$, we have a complete diagram [3]

$$\begin{tikzcd}
\text{coefficients} \arrow[r, shift left] \arrow[d, shift right] & \text{roots} \arrow[l, shift left] \arrow[d, shift left] \\
\text{critical points} \arrow[r, shift left] \arrow[u, shift right] & \text{critical values} \arrow[l, shift left] \arrow[u, shift left]
\end{tikzcd} .$$

A polynomial is determined by

its critical points.

[3] See; *Polynomials with preassigned values at their branching points,* J. Mycielski, Amer. Math. Montly 77:8 (1970), page 853–855.

Example. Critical points, critical values and cocritical points.

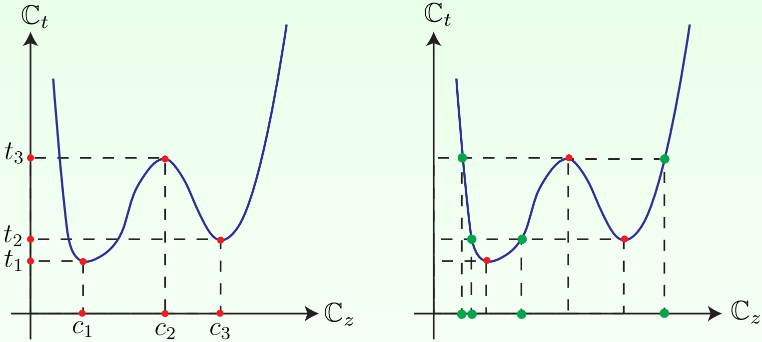

Figure: cuartica-real.pdf A quartic polynomial $P(z) : \mathbb{C}_z \longrightarrow \mathbb{C}_t$ with 3 critical points $c_1, c_2, c_3 \in \mathbb{C}_z$, 3 critical values $t_1, t_2, t_3 \in \mathbb{C}_t$ (all in red) and 4 cocritical points in $\mathbb{C}_z$ (in green).

Local normal form theorem meromorphic functions

Let us consider a meromorphic function

$$\Psi(z) : U \subset \mathbb{C}_z \longrightarrow \widehat{\mathbb{C}}_w$$

non identically constant At each point $z_j \in U$, up to local holomorphic change of coordinates $\{\mathfrak{z} \longmapsto h(\mathfrak{z}) = z\}$ (in the domain U) and Moebius transformation in $\widehat{\mathbb{C}}_z$, Ψ is reduced to

$$\mathfrak{z} \longmapsto \begin{cases} \mathfrak{z}^{\mu} & \mu \geq 1 \\ \mathfrak{z}^{\mu} = \frac{1}{\mathfrak{z}^{|\mu|}} & \mu \leq -1. \end{cases}$$

The integer number $\mu = \mu(z_j) \in \mathbb{Z} \backslash \{0\}$ locally controls Ψ near of z_j.

The critical points of Ψ occurs for $\mu \in \mathbb{Z} \backslash \{-1, 0, 1\}$.

1. $$\Psi(z) \equiv a_0, \quad \{\mu = 0\}.$$

2. $$\Psi(z) = \underbrace{a_1 z^1 + a_2 z^2 + a_3 z^3 + \cdots}_{\text{regular point, } a_1 \neq 0 \quad \{\mu=1\}.}$$

3. $$\Psi(z) = \underbrace{a_\mu z^\mu + a_{\mu+1} z^{\mu+1} + a_{\mu+2} z^{\mu+2} + \cdots}_{\text{zero of order, } \quad \{\mu \geq 1\}.}$$

4. $$\Psi(z) = \underbrace{\frac{a_{-\mu}}{z^\mu} + \frac{a_{-\mu+1}}{z^{\mu+1}} + \cdots + \frac{a_{-1}}{z^1} + a_0 + a_1 z^1 + \cdots}_{\text{order pole, } \quad \{\mu \leq 1\}.}$$

5. $$\Psi(z) = \underbrace{\cdots + \frac{a_{-k}}{z^k} + \cdots \frac{a_{-1}}{z^1} + a_0 + a_1 z^1 + \cdots}_{\text{essential singularity, } \quad \{\mu = -\infty\}.}$$

$$\text{The points} \begin{cases} \text{interesting,} \\ \text{important,} \\ \text{distinguished} \;\cdots \end{cases}$$

$\{z_j\}$ of a meromorphic function

$$\Psi : \mathbb{C}_z \longrightarrow \widehat{\mathbb{C}}_t$$

$\{z_j \subset \mathbb{C}_z\}$ are the critical points of Ψ.
$\{\Psi(z_j)\} \subset \widehat{\mathbb{C}}_t$ are the critical values of Ψ.

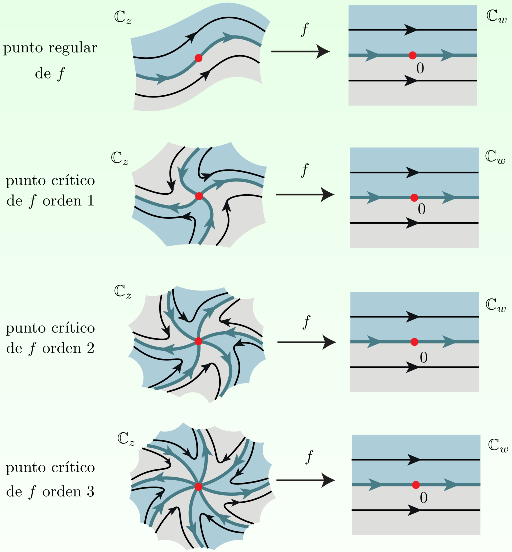

Figure: lista-de-ceros-mosaicos.pdf The critical points of Ψ (and their orders) are **visible** using γ and $\Gamma = \Psi^{-1}(\gamma)$.

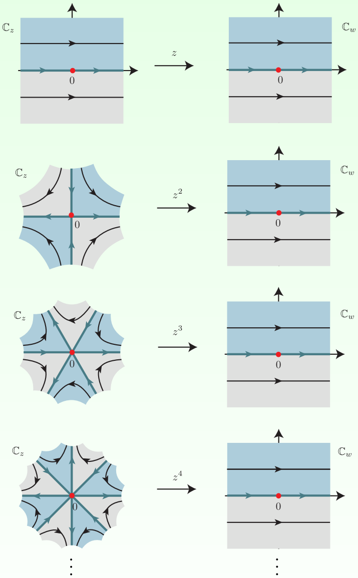

Figure: mosaicos-z2-z3-z4.pdf Critical points of Ψ in normal form.

Schwarz–Klein's algorithm:

Let $f : M \longrightarrow \widehat{\mathbb{C}}_w$ be a meromorphic function.

1. Localize the critical points of f, say
$$\{z_1, \ldots, z_m\} \subset M.$$
2. Localize the critical values of f,
$$\{w_1, \ldots, w_m\} \subset \widehat{\mathbb{C}}_w.$$
3. Construct a Jordan trajectory
$$\gamma \subset \widehat{\mathbb{C}}_w \text{ by } \{w_1, \ldots, w_m\}.$$
4. Compute
$$\Gamma = f^{-1}(\gamma) \text{ in } M.$$
5. The tessellation is
$$\mathfrak{M}_\Gamma = M \backslash \Gamma.$$

Schwarz–Klein's algorithm:

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Figure: klein-schwarz.pdf Felix Klein (1849 – 1925) and Hermann Schwarz (1843 – 1921), german mathematicians who pioneered the construction of tessellations for complex functions and for complex differential equations.

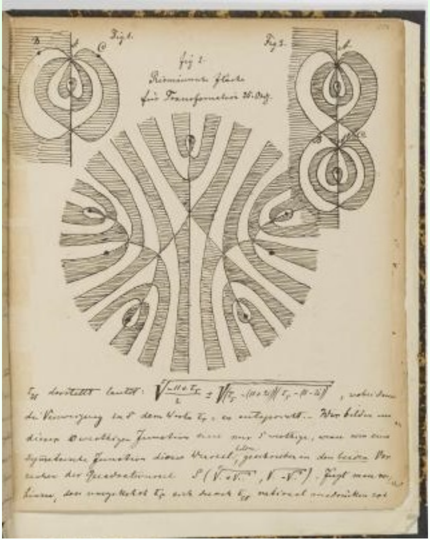

Figure: klein-protokolle.pdf Klein Protocols page. Chislenko E.; Tschinke Y.: *The Felix Klein protocols,* Notices of the AMS, vol. 54, núm. 8 (2007), 960–970.

Example. The monomial $P(z) = z^2$.

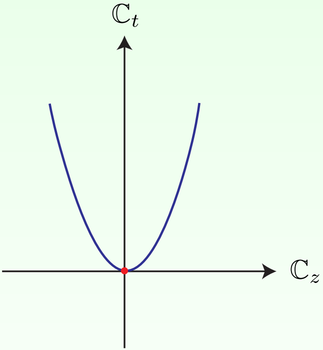

Figure: monomio-cuadratico.pdf The monomial $P(z) = z^2$ has a critical point at $0 \in \mathbb{C}_z$ and a critical value at $0 \in \mathbb{C}_t$.

Let us remember that $P(z) = z^2$ is a map of the plane in the plane, as shown in the diagram

$$
\begin{tikzcd}
\mathbb{C}_z \arrow[rr, "z^2"] & & \mathbb{C}_t \arrow[d, "T"] \\
\mathbb{R}^2 \arrow[u, "T^{-1}"] \arrow[rr, "{\left(\underbrace{x^2 - y^2}_{\mathfrak{Re}(z^2)},\ \underbrace{2xy}_{\mathfrak{Im}(z^2)}\right)}"'] & & \mathbb{R}^2
\end{tikzcd}
$$

where $T(x+iy) = (x, y)$ is the translator.

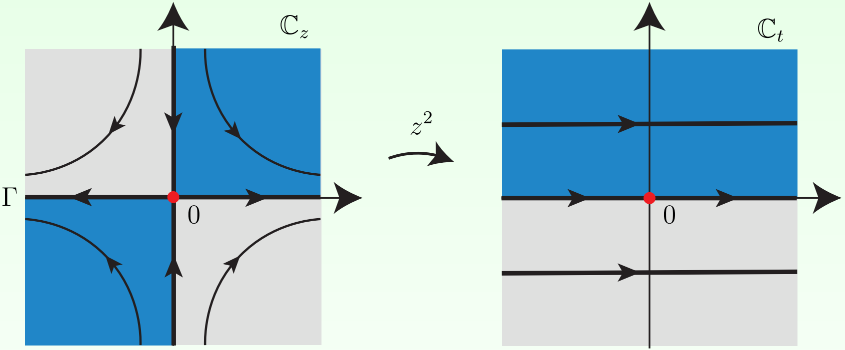

Figure: mosaico-monomio-cuadratico.pdf The “Theorem” is: a point $z_0 \in \mathbb{C}_z$ is blue if and only if $\mathfrak{Im}\left(z_0^2\right) > 0$, a point $z_0 \in \mathbb{C}_z$ is white if and only if $\mathfrak{Im}\left(z_0^2\right) < 0$. The four tiles in $\mathbb{C}_z$ determine a tessellation $\mathfrak{M}_\Gamma$.

The tessellation $\mathfrak{M}_\Gamma$ is the shape of $P(z) = z^2$!

Example.
What happens for a general monic quadratic polynomial?

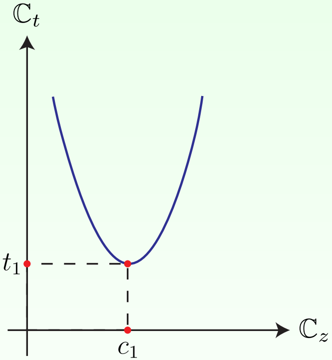

Figure: cuadratica-real.pdf The quadratic polynomial $P(z) = a_0 + a_1 z + z^2$ has a critical point $c_1 \in \mathbb{C}_z$ and a critical value $t_1 \in \mathbb{C}_t$ (both in red).

Two equivalent ways to write a monic quadratic polynomial $P(z)$.
The algebraist's form: using your coefficients (a_0, a_1), *i.e.*

$$\begin{aligned} P(z) : \mathbb{C}_z &\longrightarrow \mathbb{C}_t \\ z &\longmapsto a_0 + a_1 z + z^2 \,. \end{aligned}$$

The analyst's form, using critical points and a constant term (c_1, a_0), *i.e.*

$$\begin{aligned} P(z) : \mathbb{C}_z &\longrightarrow \mathbb{C}_t \\ z &\longmapsto 2\int_0^z (z - c_1) dz + a_0 = a_0 + \underbrace{(-2c_1)}_{a_1} z + z^2 \,. \end{aligned}$$

c_1 is the critical point from $P(z)$,
equivalently
c_1 is the zero of the derivative $P'(z)$.

Given a polynomial $P(z)$ we assign blue and white colors to the points of the plane $\mathbb{C}_z$ in a continuous and simple way.

- We consider the polynomial

$$P(z) = a_0 - 2c_1 z + z^2\,,$$

  with critical point $c_1 \doteq \mathfrak{c}_1 + i\mathfrak{c}_2$
  and critical value $t_1 = P(c_1) \doteq \mathfrak{t}_1 + i\mathfrak{t}_2$.

- We consider the horizontal line γ that passes through the critical value t_1, *i.e.*

$$\gamma = \left\{ t \mid \mathfrak{Im}\,(t) = \mathfrak{t}_2 \right\} \subset \mathbb{C}_t.$$

- γ determines a tessellation or mosaic $\mathfrak{M}_\gamma$ of the plane $\mathbb{C}_t$ as follows

$$\mathbb{C}_t \backslash \gamma = \underbrace{T}_{\substack{\text{blue}\\ \text{tessellation}}} \cup \underbrace{T'}_{\substack{\text{white}\\ \text{tessellation}}} \doteq \mathfrak{M}_\gamma. \tag{1}$$

A point $t_0 \in \mathbb{C}_t$ is blue if and only if $\mathfrak{Im}\,(t_0) > \mathfrak{t}_2$,

a point $t_0 \in \mathbb{C}_t$ is white if and only if $\mathfrak{Im}\,(t_0) < \mathfrak{t}_2$.

- The curve Γ in $\mathbb{C}_z$ that under $P(z)$ coincides with γ is called the **generating curve** of $P(z)$,

$$\begin{aligned} P : \mathbb{C}_z &\longrightarrow \mathbb{C}_t \\ \Gamma &\longmapsto \gamma. \end{aligned}$$

How is the curve Γ?, are there equations that describe Γ?

The generating curve Γ is algebraic[4]

$$
\begin{aligned}
\Gamma \doteq \left\{\mathfrak{Im}\,(P(z)) = \mathfrak{t}_2\right\} &= \left\{\mathfrak{Im}\left(a_0 - 2c_1 z + z^2\right) = \mathfrak{t}_2\right\} \\
&= \left\{\mathfrak{a}_2 + 2(xy - \mathfrak{c}_2 x - \mathfrak{c}_1 y) = \mathfrak{a}_2 - 2\mathfrak{c}_1\mathfrak{c}_2\right\} \\
&= \left\{xy - \mathfrak{c}_2 x - \mathfrak{c}_1 y + \mathfrak{c}_1\mathfrak{c}_2 = 0\right\} \\
&= \left\{\Big(\underbrace{x - \mathfrak{c}_1}_{\substack{\text{vertical}\\ \text{line}\\ \text{by } c_1}}\Big)\Big(\underbrace{y - \mathfrak{c}_2}_{\substack{\text{horizontal}\\ \text{line}\\ \text{by } c_1}}\Big) = 0\right\}.
\end{aligned}
$$

[4]This means that Γ is the zero of a polynomial of two variables x and y. Not every curve in $\mathbb{R}^2$ is algebraic, being algebraic is a simplicity condition.

The generating curve Γ is algebraic[4]

$$
\begin{aligned}
\Gamma \doteq \left\{\mathfrak{Im}\,(P(z)) = \mathfrak{t}_2\right\} &= \left\{\mathfrak{Im}\left(a_0 - 2c_1 z + z^2\right) = \mathfrak{t}_2\right\} \\
&= \left\{\mathfrak{a}_2 + 2(xy - \mathfrak{c}_2 x - \mathfrak{c}_1 y) = \mathfrak{a}_2 - 2\mathfrak{c}_1\mathfrak{c}_2\right\} \\
&= \left\{xy - \mathfrak{c}_2 x - \mathfrak{c}_1 y + \mathfrak{c}_1\mathfrak{c}_2 = 0\right\} \\
&= \Bigg\{\Big(\underbrace{x - \mathfrak{c}_1}_{\substack{\text{vertical}\\ \text{line}\\ \text{by } c_1}}\Big)\Big(\underbrace{y - \mathfrak{c}_2}_{\substack{\text{horizontal}\\ \text{line}\\ \text{by } c_1}}\Big) = 0\Bigg\}.
\end{aligned}
$$

[4] This means that Γ is the zero of a polynomial of two variables x and y. Not every curve in $\mathbb{R}^2$ is algebraic, being algebraic is a simplicity condition.

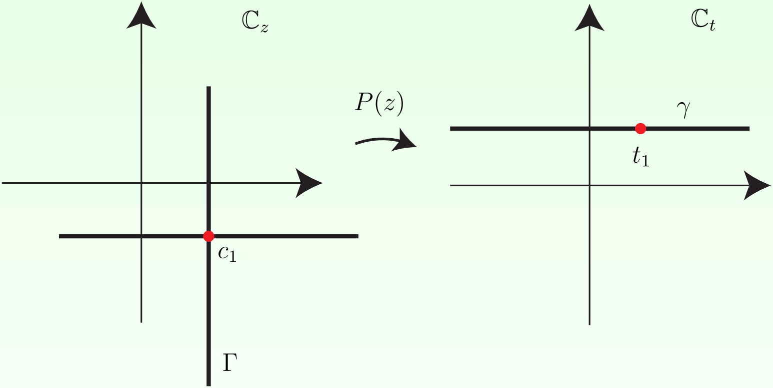

Figure: curva-generadora-grado-2.pdf The generating curve Γ of the polynomial $P(z) = a_0 - 2c_1 z + z^2$ is the union of the horizontal and vertical lines that pass by the critical point $c_1 \in \mathbb{C}_z$.

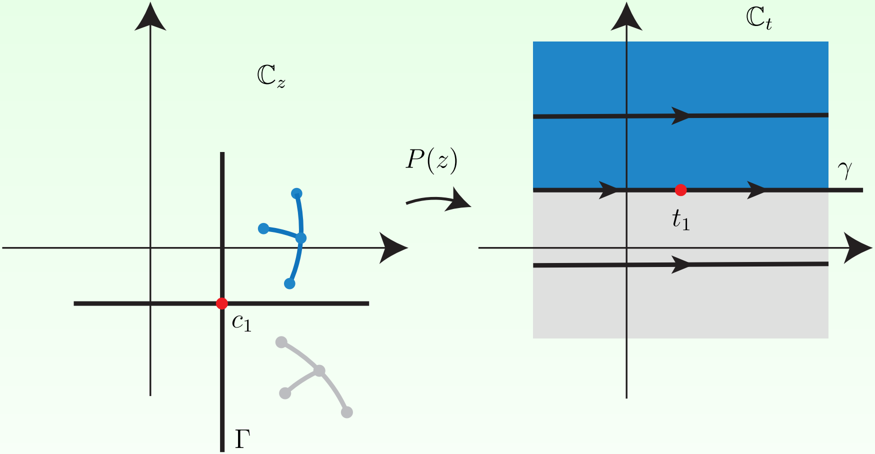

Figure: mosaico-cuadratico-suspenso-afin-aranas.pdf The polynomial $P(z)$ continuously determines the colors blue or white in $\mathbb{C}_z \backslash \Gamma$. Given a blue point in $\mathbb{C}_z$ the neighboring points are blue as long as there is a path between them that does not cross Γ.

- The generating curve Γ determines a tessellation $\mathfrak{M}_\Gamma$ of the $\mathbb{C}_z$ plane as follows

$$\mathbb{C}_z \backslash \Gamma = \underbrace{T_1 \cup T_2}_{\substack{\text{blue} \\ \text{tiles}}} \quad \cup \quad \underbrace{T_1' \cup T_2'}_{\substack{\text{white} \\ \text{tiles}}} \doteq \mathfrak{M}_\Gamma. \tag{2}$$

- A point $z_0 \in \mathbb{C}_z$ is blue if and only if $P(z_0)$ is blue in $\mathbb{C}_t$,

  a point $z_0 \in \mathbb{C}_z$ is white if and only if $P(z_0)$ is white in $\mathbb{C}_t$.

The tessellation $\mathfrak{M}_\Gamma$ is

the shape of $P(z) = a_0 - 2c_1 z + z^2$!

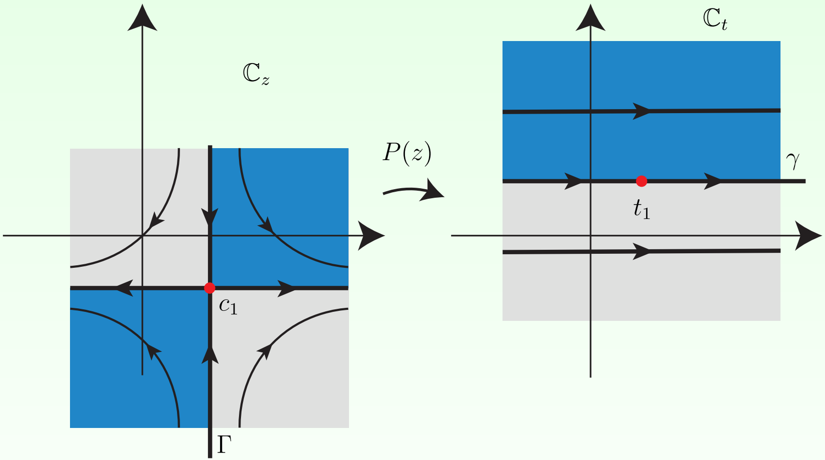

Figure: mosaico-cuadratico-afin.pdf
The tessellation $\mathfrak{M}_\Gamma$ in the plane $\mathbb{C}_z$ of the polynomial $P(z) = a_0 - 2c_1 z + z^2$; we coloured red the critical point $c_1 \in \mathbb{C}_z$ and the critical value $t_1 \in \mathbb{C}_t$.

Properties of the tessellation of $P(z)$.

Lemma

Given T_ℓ and T'_ℓ blue and white tiles of $\mathfrak{M}_\Gamma$, there exists

$$P^{-1}(t) : T \cup T' \subset \mathbb{C}_t \longrightarrow T_\ell \cup T'_\ell \subset \mathbb{C}_z$$

i.e. a branch of the inverse function of $P(z)$.

Do you remember $\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$?

Observation

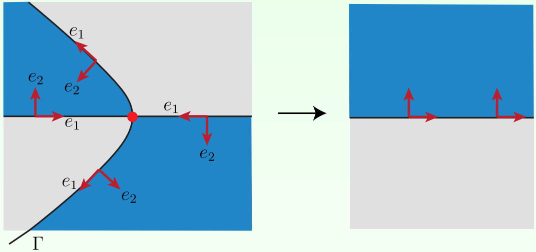

Figure: orientacion.pdf When we traversing an oriented trajectory, say γ or Γ, in $\mathbb{C}$, it makes sense to decide which is the region to its left; the one pointed to by e_2, the tangent vector of the trajectory. This left region is by definition a blue tile.

Example (object/experiment). A cubic polynomial $P(z)$.

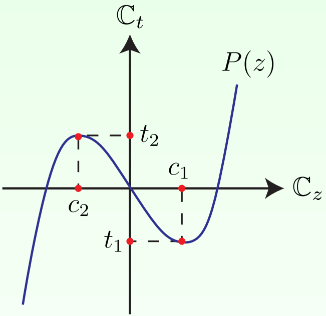

Figure: cubica-real.pdf The cubic polynomial $P(z) = z^3 - 3z$ has 2 critical points $c_1, c_2 \in \mathbb{C}_z$ and 2 critical values $t_1, t_2 \in \mathbb{C}_t$ (all in red). Additionally, it has 2 cocritical points in $\mathbb{C}_z$ (in green).

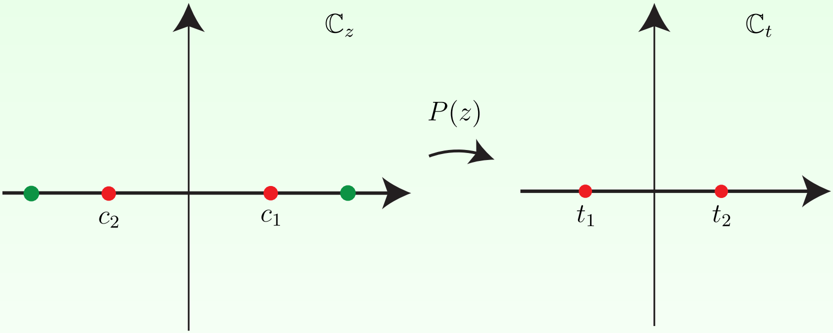

Figure: puntos-valores-criticos-grado-3.pdf The cubic polynomial $P(z) = z^3 - 3z$ has 2 critical points $c_1, c_2 \in \mathbb{C}_z$ and 2 critical values $t_1, t_2 \in \mathbb{C}_t$ (all in red). Additionally, it has 2 cocritical points in $\mathbb{C}_z$ (in green).

Two equivalent ways to write a monic cubic polynomial $P(z)$

The algebraist´s form: using your coefficients (a_0, a_1, a_2), *i.e.*

$$\begin{array}{rcl} P(z): \mathbb{C}_z & \longrightarrow & \mathbb{C}_t \\ z & \longmapsto & a_0 + a_1 z + a_2 z^2 + z^3 \,. \end{array}$$

The analyst form: using your critical point and your constant term (c_1, c_2, a_0), *i.e.*

$$\begin{array}{rcl} P(z): \mathbb{C}_z & \longrightarrow & \mathbb{C}_t \\ z & \longmapsto & 3\displaystyle\int_0^z (z-c_1)(z-c_2)dz + a_0 \\ & & = a_0 + \underbrace{3c_1c_2}_{a_1} z + \underbrace{\tfrac{-3(c_1+c_2)}{2}}_{a_2} z^2 + z^3 \,. \end{array}$$

$c_1,\ c_2$ are the critical point from $P(z)$,
equivalently
$c_1,\ c_2$ are the zeros of the derivative $P'(z)$.

We assign blue and white colors to the points of the plane $\mathbb{C}_z$ in a continuous and simple way.

- We consider the polynomial

$$P(z) = 3\int_0^z (z-1)(z+1)dz = z^3 - 3z\,,$$

  with critical points

$$c_1 = 1, \quad c_2 = -1$$

  and critical values

$$t_1 = P(1) = -2, \quad t_2 = P(-1) = 2.$$

- We consider

$$\gamma = \mathbb{R} \subset \mathbb{C}_t$$

  the horizontal line that passes by the critical values t_1 and t_2.

- γ determine a tessellation $\mathfrak{M}_\gamma$ of the $\mathbb{C}_t$ as follows

$$\mathbb{C}_t \backslash \gamma = \underbrace{T}_{\substack{\text{blue}\\ \text{tessellation}}} \cup \underbrace{T'}_{\substack{\text{white}\\ \text{tessellation}}} \doteq \mathfrak{M}_\gamma. \tag{3}$$

- A point $t_0 \in \mathbb{C}_t$ is blue if and only if $\mathfrak{Im}\,(t_0) > 0$,

  a point $t_0 \in \mathbb{C}_t$ is white if and only if $\mathfrak{Im}\,(t_0) < 0$.

- The trajectory Γ in $\mathbb{C}_z$ that under $P(z)$ coincides with γ is the generating trajectory of $P(z)$,

$$\begin{array}{rcl} P : \mathbb{C}_z & \longrightarrow & \mathbb{C}_t \\ \Gamma & \longmapsto & \gamma. \end{array}$$

Are there any equations that describe Γ?

The generating trajectory Γ is algebraic!

$$\Gamma \doteq \{\mathfrak{Im}\,(P(z)) = 0\} = \{\mathfrak{Im}\,(z^3 - 3z) = 0\}$$
$$= \{3x^2y - y^3 - 3y = 0\}$$
$$= \Big\{\underbrace{y}_{\substack{\text{real}\\ \text{axis}}}\underbrace{(3x^2 - y^2 - 3)}_{\substack{\text{hyperbola}\\ \text{centered}\\ \text{at } 0}} = 0\Big\}.$$

What lucky we are;
the cubic algebraic curve
$\Gamma = \{\mathfrak{Im}\,(P(z)) = 0\}$ is easy to describe!

Γ is the union of a curve of degree one and a curve of degree two.

The generating trajectory Γ is algebraic!

$$\begin{aligned}\Gamma \doteq \{\mathfrak{Im}\,(P(z)) = 0\} &= \{\mathfrak{Im}\,(z^3 - 3z) = 0\} \\ &= \{3x^2y - y^3 - 3y = 0\} \\ &= \Big\{ \underbrace{y}_{\substack{\text{real}\\ \text{axis}}} \underbrace{(3x^2 - y^2 - 3)}_{\substack{\text{hyperbola}\\ \text{centered}\\ \text{at } 0}} = 0 \Big\}.\end{aligned}$$

What lucky we are;
the cubic algebraic curve
$\Gamma = \{\mathfrak{Im}\,(P(z)) = 0\}$ is easy to describe!

Γ is the union of a curve of degree one and a curve of degree two.

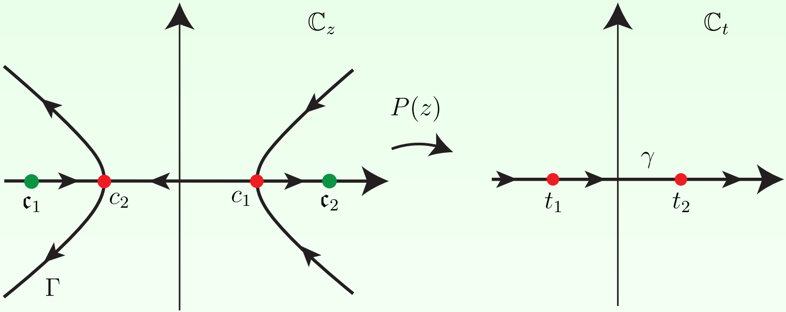

Figure: curva-generadora-grado-3.pdf The generating curve Γ of the polynomial $P(z) = z^3 - 3z$ is the unin of the real line $\{y = 0\}$ with the hyperbola $\{x^2 - y^2/3 = 1\}$.

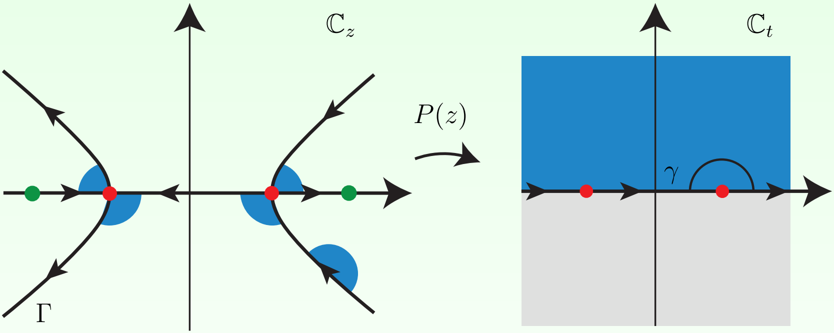

Figure: mosaico-colores-afin-grado-3.pdf The polynomil $P(z)$ determines the colors blue or white in $\mathbb{C}_z \backslash \Gamma$. Given a blue point in $\mathbb{C}_z$ the neighboring points are blue as long as there is a path between them that does not cross Γ.

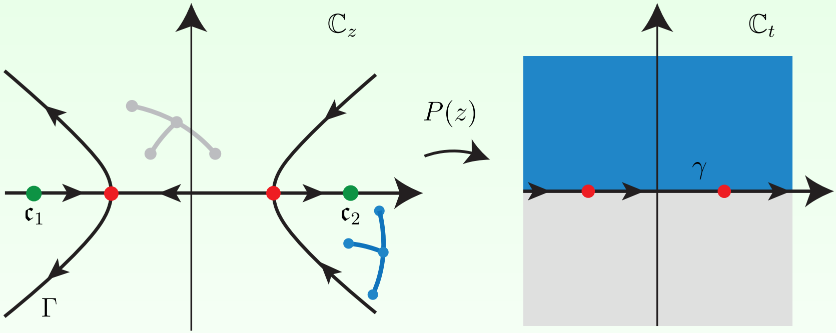

Figure: mosaico-cubico-suspenso-afin.pdf The polynomial $P(z)$ continuously determines the colors blue or white in $\mathbb{C}_z \backslash \Gamma$. Given a blue point in $\mathbb{C}_z$ the neighboring points are blue as longs as there is a path between them that does not cross Γ.

- The generating curve Γ determines a tessellation $\mathfrak{M}_\Gamma$ of the plane $\mathbb{C}_z$ as follow

$$\mathbb{C}_z \backslash \Gamma = \underbrace{T_1 \cup T_2 \cup T_3}_{\substack{\text{blue} \\ \text{tessellations}}} \cup \underbrace{T_1' \cup T_2' \cup T_3'}_{\substack{\text{white} \\ \text{tessellations}}} \doteq \mathfrak{M}_\Gamma. \tag{4}$$

- A point $z_0 \in \mathbb{C}_z$ is blue if and only if $P(z_0)$ is blue in $\mathbb{C}_t$,

  a point $z_0 \in \mathbb{C}_z$ is white if and only if $P(z_0)$ is white in $\mathbb{C}_t$.

The tessellation $\mathfrak{M}_\Gamma$ is the form of $P(z) = z^3 - 3z$!

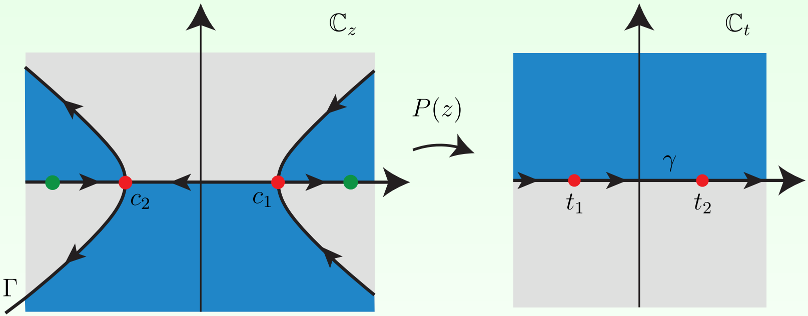

Figure: mosaico-cubico-afin.pdf The "Theorem" is: tessellation $\mathfrak{M}_\Gamma$ in the plane $\mathbb{C}_z$ for the polynomial $P(z) = z^3 - 3z$, critical points and critical values in red, cocritical points in green.

Example (object/experiment). A quartic polynomial $P(z)$.

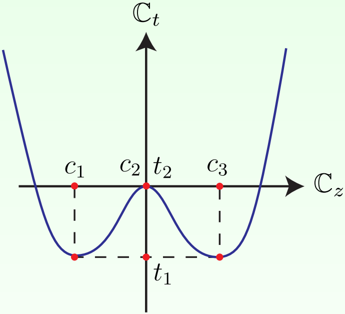

Figure: cuartica-real-sin-cocriticos.pdf A quartic polynomial $P(z) : \mathbb{C}_z \longrightarrow \mathbb{C}_t$; with 3 critical points $c_1, c_2, c_3 \in \mathbb{C}_z$ and 2 critical values $t_1, t_2 \in \mathbb{C}_t$ (all in red), without cocritical points.

Observation. A general quartic polynomial $P(z)$

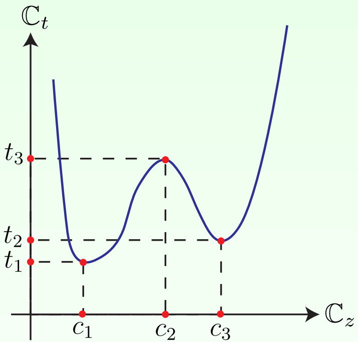

Figure: cuartica-real.pdf A quartic polynomial $P(z) : \mathbb{C}_z \longrightarrow \mathbb{C}_t$ has 3 critical points $c_1, c_2, c_3 \in \mathbb{C}_z$ y 3 critical values $t_1, t_2, t_3 \in \mathbb{C}_t$ (all in red). Furthermore, there can be up 6 cocritcal points in $\mathbb{C}_z$ (in green).

Two equivalent ways to write a monic quartic polynomial $P(z)$.

The algebraist´s form: using your coefficients (a_0, a_1, a_2, a_3), *i.e.*

$$\begin{aligned} P(z) : \mathbb{C}_z &\longrightarrow \mathbb{C}_t \\ z &\longmapsto a_0 + a_1 z + a_2 z^2 + a_3 z^3 + z^4 \,. \end{aligned}$$

The analyst form: using your critical point and your constant term (c_1, c_2, c_3, a_0), *i.e.*

$$\begin{aligned} P(z) : \mathbb{C}_z &\longrightarrow \mathbb{C}_t \\ z &\longmapsto 4\int_0^z (z-c_1)(z-c_2)(z-c_3)dz + a_0 \\ &= a_0 + \underbrace{(-4c_1c_2c_3)}_{a_1} z + \underbrace{2(c_1c_2 + c_1c_3 + c_2c_3)}_{a_2} z^2 + \underbrace{\tfrac{-4(c_1+c_2+c_3)}{3}}_{a_3} z^3 + z^4 \,. \end{aligned}$$

$c_1,\ c_2,\ c_3$ are the critical point from $P(z)$,
equivalently
$c_1,\ c_2,\ c_3$ are the zeros of the derivative $P'(z)$.

We assign blue and white colors to the points of the plane $\mathbb{C}_z$ in a continuous and simple way.

- We consider the polynomial[5]

$$P(z) = 4\int_0^z z(z-1)(z+1)dz = z^4 - 2z^2\,,$$

  with critical points

$$c_1 = 0,\quad c_2 = 1,\quad c_3 = -1$$

  and critical values

$$t_1 = P(0) = 0,\quad t_2 = P(1) = P(-1) = -1.$$

- We consider

$$\gamma = \mathbb{R} \subset \mathbb{C}_t$$

  the horizontal line that passes by the critical values t_1 and t_2.

[5] simple, symmetrical, to facilitate calculations.

- γ determine a tessellation $\mathfrak{M}_\gamma$ of the $\mathbb{C}_t$ as follows

$$\mathbb{C}_t \backslash \gamma = \underbrace{T}_{\substack{\text{blue}\\ \text{tessellation}}} \cup \underbrace{T'}_{\substack{\text{white}\\ \text{tessellation}}} \doteq \mathfrak{M}_\gamma. \tag{5}$$

- A point $t_0 \in \mathbb{C}_t$ is blue if and only if $\mathfrak{Im}\,(t_0) > 0$,

  a point $t_0 \in \mathbb{C}_t$ is white if and only if $\mathfrak{Im}\,(t_0) < 0$.

- The curve Γ in $\mathbb{C}_z$ that under $P(z)$ coincides with γ is the generating curve of $P(z)$,

$$\begin{array}{rcl} P : \mathbb{C}_z & \longrightarrow & \mathbb{C}_t \\ \Gamma & \longmapsto & \gamma. \end{array}$$

Are there any equations that describe Γ?

The generating curve Γ is algebraic!

$$\begin{aligned}\Gamma \doteq \{\mathfrak{Im}\,(P(z)) = 0\} &= \{\mathfrak{Im}\,(z^4 - 2z^2)\} \\ &= \{x^3y - xy^3 - xy = 0\} \\ &= \Big\{ \underbrace{x}_{\substack{\text{imaginary}\\ \text{line}}} \underbrace{y}_{\substack{\text{real}\\ \text{line}}} \underbrace{(x^2 - y^2 - 1)}_{\substack{\text{hyperbola}\\ \text{centered}\\ \text{at } 0}} = 0 \Big\}.\end{aligned}$$

The generating curve Γ is algebraic!

$$\begin{aligned}\Gamma \doteq \left\{\mathfrak{Im}\,(P(z)) = 0\right\} &= \left\{\mathfrak{Im}\left(z^4 - 2z^2\right)\right\} \\ &= \left\{x^3y - xy^3 - xy = 0\right\} \\ &= \left\{ \underbrace{x}_{\substack{\text{imaginary} \\ \text{line}}} \; \underbrace{y}_{\substack{\text{real} \\ \text{line}}} \underbrace{\left(x^2 - y^2 - 1\right)}_{\substack{\text{hyperbola} \\ \text{centered} \\ \text{at } 0}} = 0 \right\}.\end{aligned}$$

Figure: curva-generadora-grado-4.pdf The generating curve Γ of the polynomial $P(z) = z^4 - 2z^2$ is the union of the real line $\{y = 0\}$, the imaginary line $\{x = 0\}$ and the hyperbola $\{x^2 - y^2 = 1\}$.

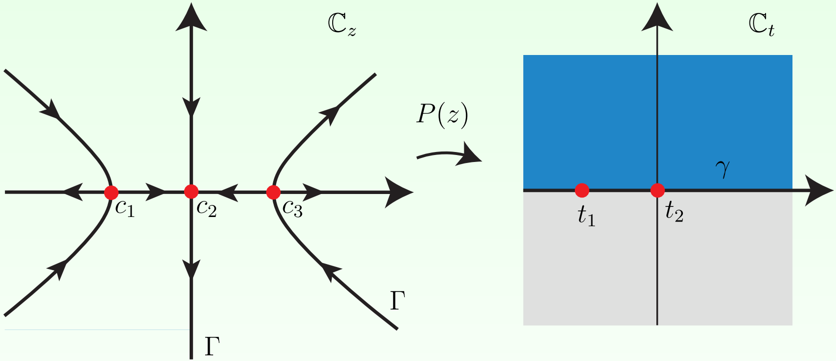

Figure: mosaico-cuartico-suspenso-afin.pdf The polynomial $P(z)$ continuously determines the colors blue or white in $\mathbb{C}_z \backslash \Gamma$.

- The generating Γ determines a tessellation $\mathfrak{M}_\Gamma$ of the plane $\mathbb{C}_z$ as follow

$$\mathbb{C}_z \backslash \Gamma \;=\; \underbrace{T_1 \cup T_2 \cup T_3 \cup T_4}_{\substack{\text{blue}\\ \text{tessellation}}} \cup \underbrace{T_1' \cup T_2' \cup T_3' \cup T_4'}_{\substack{\text{white}\\ \text{tessellation}}} \;\doteq\; \mathfrak{M}_\Gamma \tag{6}$$

- A point $z_0 \in \mathbb{C}_z$ is blue if and only if $P(z_0)$ is blue in $\mathbb{C}_t$,

  a point $z_0 \in \mathbb{C}_z$ is white if and only if $P(z_0)$ is white in $\mathbb{C}_t$.

The tessellation $\mathfrak{M}_\Gamma$ is the form of $P(z) = z^4 - 2z^2$!

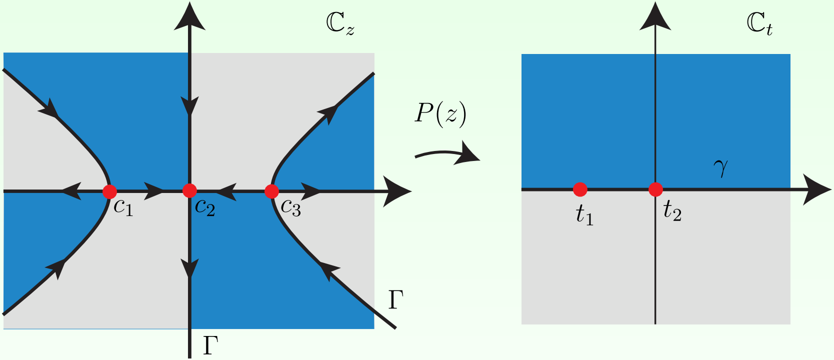

Figure: mosaico-cuartica-afin.pdf The "Theorem" is: tesselation $\mathfrak{M}_\Gamma$ in the plane $\mathbb{C}_z$ for the polynomial $P(z) = z^4 - 2z^2$, critical points and critical values in red, critical points in green.

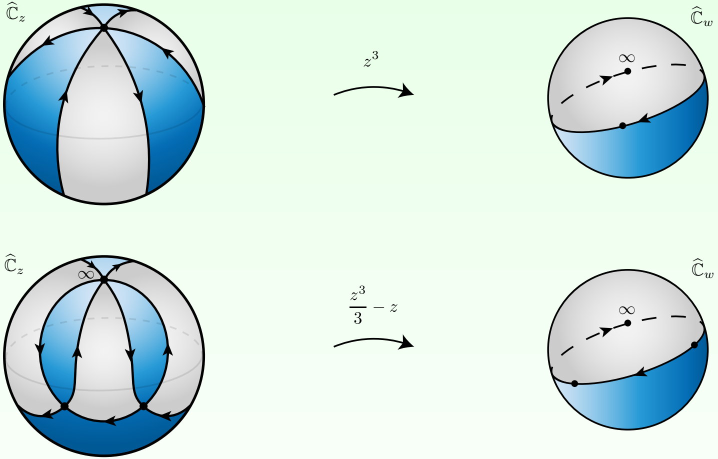

Figure: polinomios-grado-3-mosaicos.pdf Topological zoo for degree three polynomials.

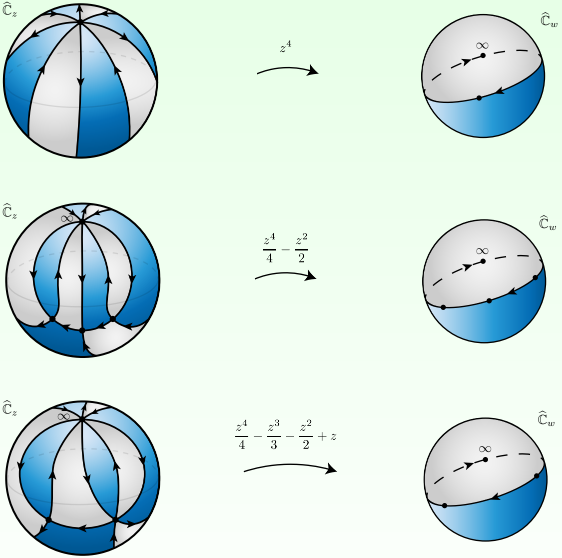

Figure: polinomios-grado-4-mosaicos.pdf Topological zoo for degree four polynomials.

On the topology of polynomial tessellations.

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Conclusions/problems:

1. Consider the correspondence

$$\text{polynomial } P(z) \longrightarrow \gamma \longrightarrow \Gamma \longrightarrow \text{tessellation } \mathfrak{M}_\Gamma.$$

   Which theorem/result can be done?

2. For a fixed degree n, can we topologically characterize the graphs $\{\Gamma \subset \widehat{\mathbb{C}}_z\}$ that origin from polynomials?

3. What happens for rational functions $R(z)$?

4. What happens for transcendental functions $\Psi(z)$?

.

Paste

Definition

An *admissible paste of the collection of triangles* $\{\triangle_\alpha \mid \alpha \in \mathcal{I}\} \subset \mathbb{R}^2$ satisfies the following conditions,

1. A side $\ell_\alpha \subset \triangle_\alpha$ is identified with at most one side $\ell_\beta \subset \triangle_\beta$ for $\alpha \neq \beta$, or is not identified with any other side of $\triangle_\beta$ in the collection.
2. Each identification $I_{\alpha\beta} : \ell_\alpha \longrightarrow \ell_\beta$ is a bijective linear function between the line segments $\ell_\alpha,\ \ell_\beta$.
3. Side identifications are made in such a way that the surface is locally homeomorphic.
   to an open disk $D(0,\epsilon) \subset \mathbb{R}^2$ ó
   to an half disk $D(0,\epsilon) \cap \overline{\mathbb{H}}^2 = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 \geq 0\}$.

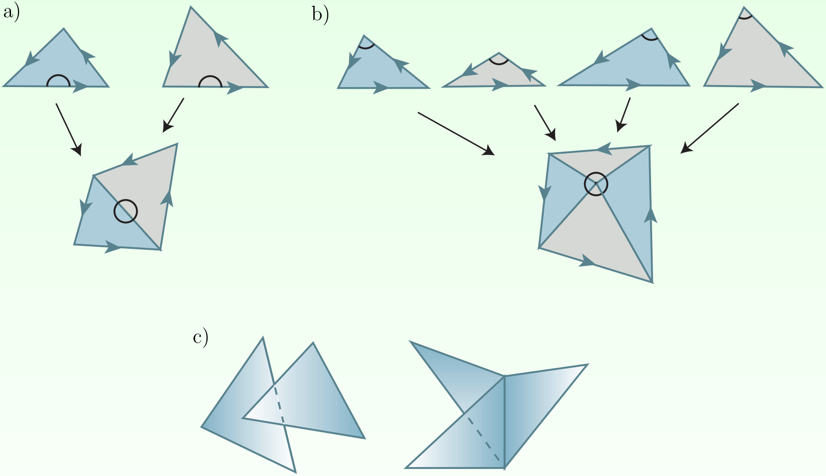

Figure: pegados.pdf (a)–(b) admissibles pasting of triangles, (c) pasted thar the not admissible of triangles.

Lemma

(Paste for sides) *Given two sides* $\ell_\alpha \subset \partial\triangle_1$ *and* $\ell_\beta \subset \partial\triangle_2$*. Si* ℓ_α, ℓ_β *are identified by* $I_{\alpha\beta} : \ell_\alpha \longrightarrow \ell_\beta$*, then the resulting space S is homeomorphic to a quadriateral.*

Lemma

(Paste for vertices.) *Given a finite collection of triangles $\triangle_1, \ldots, \triangle_k$. If $2k$ sides are identified as follows:*

1. *A vertice is selected from each triangle $v_\alpha \in \triangle_\alpha$ and identify $v_1, \ldots, v_k$ to a single point.*
2. *Additionally, one side is identified $\ell_\alpha \subset \triangle_\alpha$ (where v_α is one endpoint of ℓ_α) with one side $\ell_{\alpha+1} \subset \triangle_{\alpha+1}$ (where $v_{\alpha+1}$ is an endpoint of $\ell_{\alpha+1}$).*
3. *The identification is done in a "cyclical way to from a disk", i.e. the side $\ell_k \subset \triangle_k$ (where v_k is an extremun of ℓ_k) is identified with the side $a_1 \subset \triangle_1$ (where v_1 is an extremun of a_1);*

then the resuling space S is homeomorphic to a k–agono with its border.

The role of the

Uniformization Theorem (Poincaré, Koebe)

Let M be a simply connected Riemann surface.
Then M is biholomorphic to one of the following

- the complex plane $\mathbb{C}$,
- the Riemann sphere $\widehat{\mathbb{C}}$,
- the half plane $\mathbb{H} = \{z \mid \mathfrak{Im}\,(z) > 0\}$.

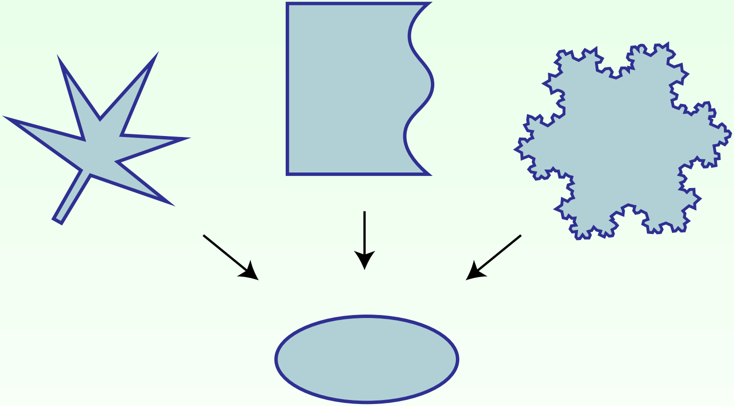

Figure: aplicacion-riemann.pdf

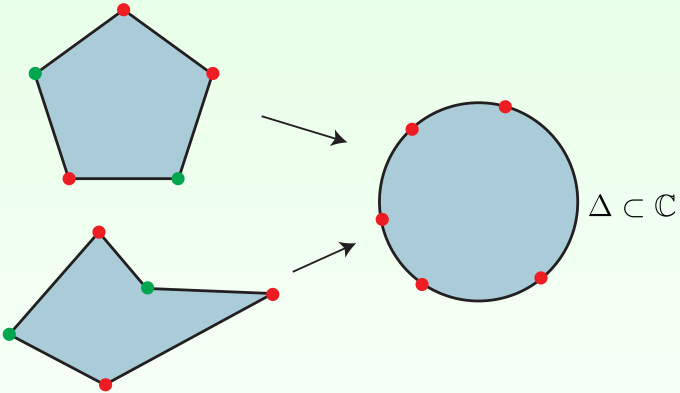

Figure: uniformizacion.pdf Meaning of **r**–gonality. In complex analysis any **r**–gon is conformally equivalent to a disk in the Riemann sphere (the corners of the **r**–gon disappear under a holomorphic map).

Conjecture

There exists a "correspondence" between

$$\left\{ \begin{array}{c} \text{rational functions} \\ \mathbb{f} : \widehat{\mathbb{C}}_z \longrightarrow \widehat{\mathbb{C}}_t \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{certain kind of} \\ \text{embeded graphs} \\ \Gamma \subset \widehat{\mathbb{C}}_z \end{array} \right\}.$$

The correspondence depends on the choice of γ.

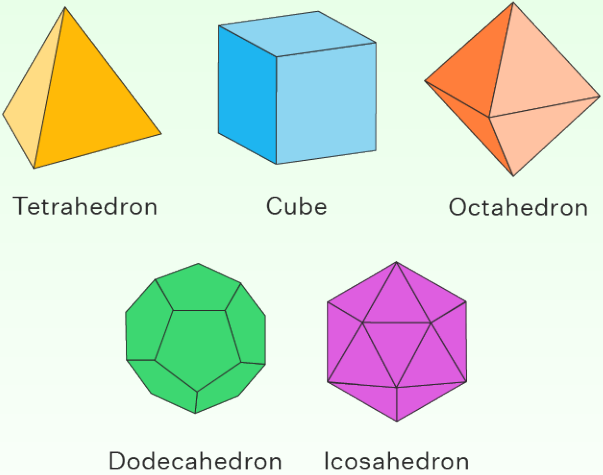

Figure: Platonic-solids.pdf Platonic solids; which of they are maps Γ?

Applications

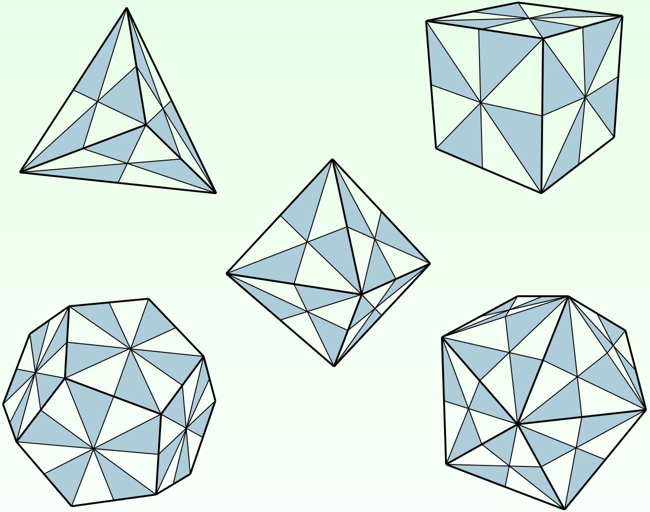

Figure: solidos-platonico.pdf The platonic tessellations determine rational Belyĭ functions.

Theorem (B. Riemann, H. A. Schwartz, XIX century)

1. A rational function $\mathbb{f} : \widehat{\mathbb{C}}_z \longmapsto \widehat{\mathbb{C}}_t$ and a suitable γ determine a map $\Gamma = (V, E)$ as above.

2. A map $\Gamma = (V, E)$ determines a rational map

$$\mathbb{f} : \widehat{\mathbb{C}}_z \longrightarrow \widehat{\mathbb{C}}_t$$

having vertices,

$$\{\text{critical points of } \mathbb{f}\} \cup \{\text{cocritical points of } \mathbb{f}\} = V.$$

In assertion (2) the vertices V of Γ are canonical.

However, the edges E of Γ are not canonical, they depend of γ.

The two families of objects are of a very different nature:

$$\Gamma = (V, E) \qquad\qquad \mathbb{f}(z) = \frac{a_\alpha z^\alpha + \ldots + a_1 z + a_0}{b_\beta z^\beta + \ldots + b_1 z + b_0}$$

$$\mathbf{d} = \max\{\alpha,\, \beta\} \qquad\qquad d = \max\{\alpha,\, \beta\}$$

$$2 \leq \mathbf{r} \leq 2d - 1 \qquad\qquad 2 \leq \{\#critcal\ values\} \leq 2d - 1$$

$\mathbf{d} = d$ the number of blue tiles,

$\mathbf{r}$ number of corners or vertices of Γ in the boundary of each tile.

. . . the role of the cocritical points.

Corollary

A polynomial or rational tessellation $\mathfrak{M}_\Gamma$ has tiles T_ℓ with the same number $\mathbf{r}$ (= the number of critical values in $\widehat{\mathbb{C}}_w$) of edges.

We are considering the cocritical points as edges of the tiles T_ℓ in $\mathfrak{M}_\Gamma$.

Example. A cubic polynomial $\mathbb{f}$.

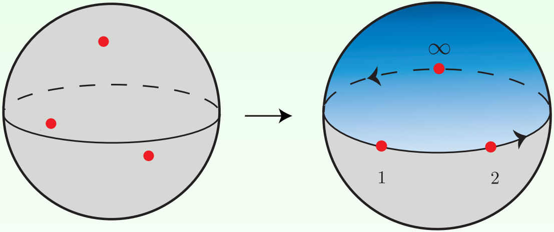

Figure: cubica-0.pdf A cubic polynomial $\mathbb{f} : \widehat{\mathbb{C}}_z \longrightarrow \widehat{\mathbb{C}}_t$ Generically, $\mathbb{f}'$ has two zeros in $\mathbb{C}_z$, and two respective values in $\mathbb{C}_t$ (both in red). In this case $\mathbf{d} = d = 3$, $\mathbf{r} = 3$.

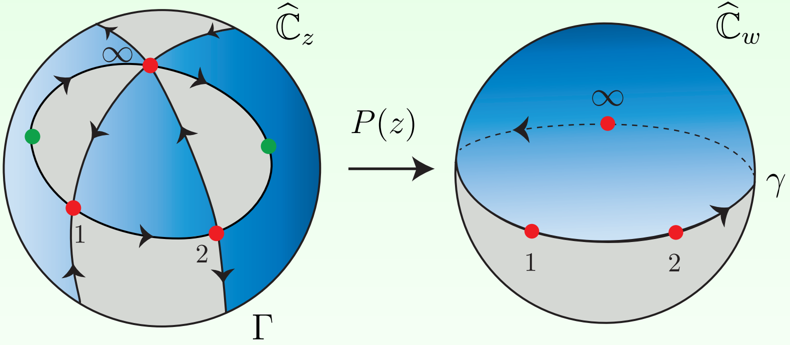

Figure: cubica-compleja.pdf Let $\mathbb{f}$ be a generic cubic polynomial. The Schwarz–Klein's algortihm recognize $\widehat{\mathbb{C}}_t$ as the union of two **3**–gons . . . The Klein tessellation $\mathfrak{M}_\Gamma$ for a generic cubic polynomial $\mathbb{f}$, in red critical points, in green cocritical points.

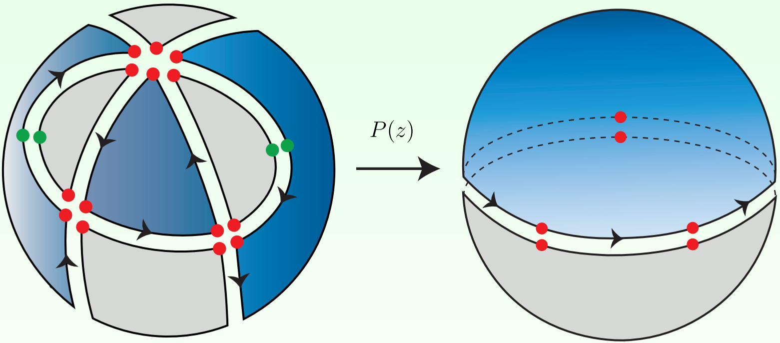

Figure: cubica-compleja-1.pdf Meaning of $\mathbf{r} = 3$: the critical and cocritical points allow us to identify each tile of $\mathfrak{M}_\gamma$ and $\mathfrak{M}_\Gamma$ as triangles.

Example. A quartic polynomial $\mathbb{f}$.

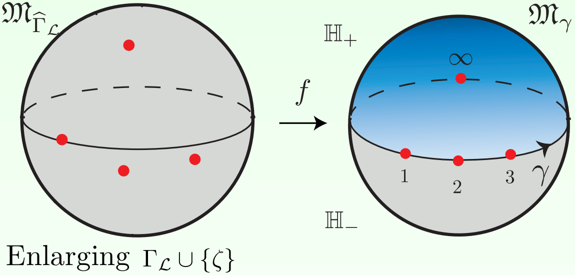

Figure: cuartica-0.pdf A generic quartic polynomial $\mathbb{f} : \widehat{\mathbb{C}}_z \longrightarrow \widehat{\mathbb{C}}_t$ is a map between spheres, $\mathbb{f}(\infty) = \infty$.

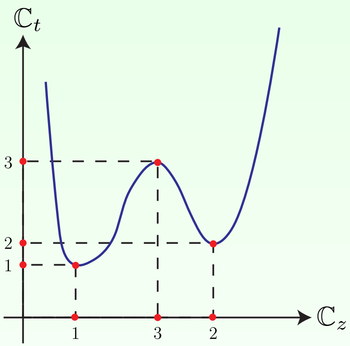

Figure: cuartica-real-2.pdf For a quartic polynomial $\mathbb{f} : \mathbb{C}_z \longrightarrow \mathbb{C}_t$; in red, 3 critical points in $\mathbb{C}_z$, 3 critical values in $\mathbb{C}_t$; in green cocritical points in $\mathbb{C}_z$.

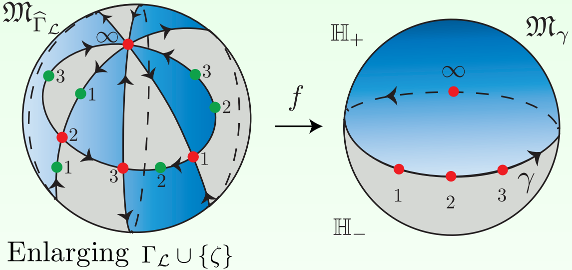

Figure: cuartica-compleja.pdf Klein tessellation for a generic cubic polynomial $\mathbb{f}$, in red critical points, in green cocritical points. The degree is 4 and **4**–gonality.

• **Bad news**:
the tessellation $\mathfrak{M}_\Gamma$ depends of the choice of γ in an strong way.

• **Good news**:
the algorithm of Schwarz–Klein applies for
many other functions (with essential singularities) and
on any Riemann surface, not just on $\widehat{\mathbb{C}}_z$.

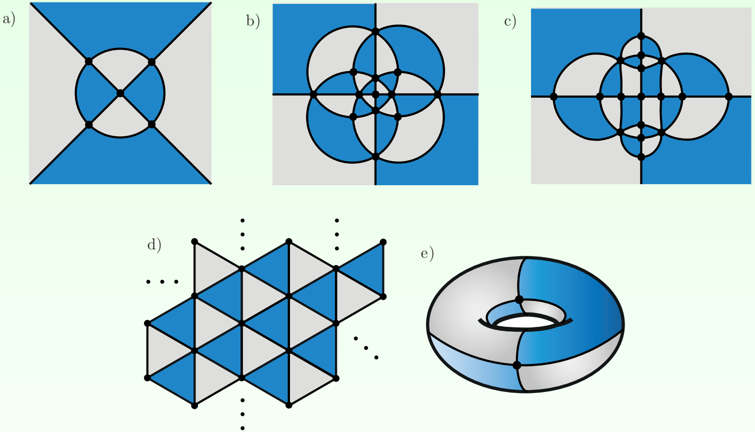

Figure: mosaicos-2.pdf Three Schwarz–Klein tessellations (a), (b), (c) from rational functions on $\mathbb{C}_z$, and two tessellations on the plane (d) and the torus (e) from the Weierstrass $\wp$ function.

Theorem of Belyĭ, Grothendieck et al.

There are correspondences between:

- Rational functions $\mathbb{f} : \widehat{\mathbb{C}}_z \longrightarrow \widehat{\mathbb{C}}_t$ having three critical values, up to a Mobius transformation, $0, 1, \infty \in \widehat{\mathbb{C}}_t$.
- Dessin d' enfants $\mathbb{f}^{-1}[0, 1] \subset \widehat{\mathbb{C}}_z$.
- Tessellations by triangles with blue & white alternated colors $\mathfrak{M}_{\mathbb{f}^*\gamma} = \widehat{\mathbb{C}}_z \backslash \{\mathbb{f}^*\gamma\}$, for γ teh circle $\mathbb{R} \cup \{\infty\} \subset \widehat{\mathbb{C}}_t$ by $0,\ 1,\ \infty$.

1[6]

[6] See; Belyĭ G. V.: *On Galois extensions of a maximal cyclotonic field,* Mah. USSR Izvestija, vol. 193, núm. 14 (1980), 247–256.

Where are problems for more general hypothesis?

The correspondence
from tessellations (without cocritical point information) to rational functions
is not fully understood.

For tessellations having as tiles $\mathbf{r}$–gons, $\mathbf{r} \geq 4$.
The cocritical points and critical points in the $\mathbf{r}$–gons must be match globally.

The labelling problem

Starting with a two color tessellation $\mathfrak{M}$ of $\widehat{\mathbb{C}}_z$ or M_g.

- How can we compute the $\mathbf{r}$–gonality?
- How can we attach new vertices at the boundary of the tiles and label them using $\{1, \ldots, \mathbf{r}\}$, in order to match them globally?

Example

A polynomial of degree four

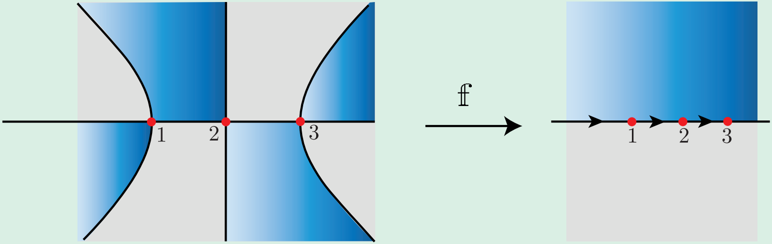

Figure: Contraejemplo.pdf Case $\mathbf{r} = 4$. A bad choice for the sequences of critical points (and cocritical points) in the boundary of the tiles ... A re–labelling is necessary.

It illustrated a forbidden distribution of the cocritical points.

Thuston's example 2010

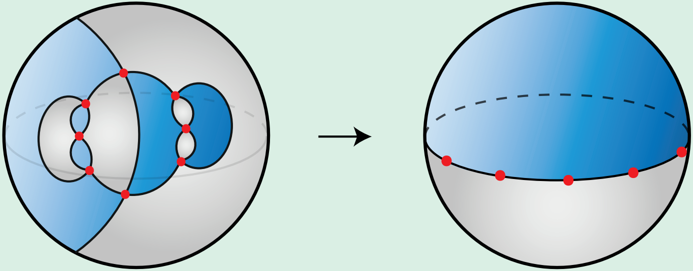

Figure: Thurston.pdf The above tessellation can not be realized by any a rational function $\mathbb{f} : \widehat{\mathbb{C}}_z \longmapsto \widehat{\mathbb{C}}_t$. In fact the sequences of critical points (and cocritical points) in the boundary of the tiles become always contradictory. Degree $d = 5$, and polygons with $2 \leq \mathbf{r} \leq 4 \cdot 5$?

1[7]

[7] See; Koch S. ; T. Lei: *On balanced planar graphs, following W. P. Thurston,* In What's Next The Mathematical Legacy of William P. Thurston, D. Thurston Ed. Ann. of Math. Studies, 205, Princeton Univ. Press U. S. A. (2020), 215–232.

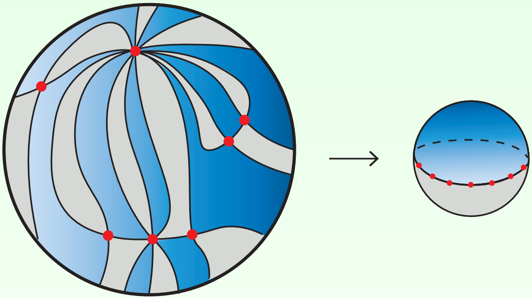

Figure: ejemplo-complicado.pdf A non trivial example of the labelling problem; starting with the tesellation; how can label the tiles in order to get 8 vertices in their boundaries? Degree $d = 8$ and 8–gonality.

Theorem (L. Johanna Gonzalez Cely, 2019.)

If a tesellation $\mathfrak{M}$ has only saddles in $\mathbb{C}$ and every tile has ∞ in its boundary, then it is realized by a polynomial function.

Now, we search some applications . . .

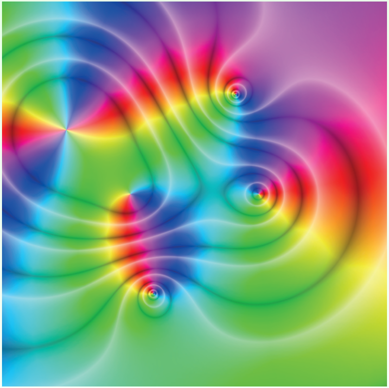

Figure: complex-functions.pdf A complex rational function $f(z) = \frac{(z^2-1)(z-2-i)^2}{(z^2+2+2i)} : \mathbb{C}_z \longrightarrow \mathbb{C}_w$. (Source Wikipedia). Can we read this figure?

Proposition. The dictionary (Klein, ..., Ahlfors, Strebel, Kerkoff ...)

On any Riemann surface M and for any complex analytic function $h(z)$ (i.e. meromorphic, with essential isolated singularities, accumulation of poles, zeros, essential singularities),
there exists one to one correspondences

$$
\begin{tikzcd}
 & X(z) = \frac{1}{h(z)} \frac{\partial}{\partial z} \arrow[dl, leftrightarrow] \arrow[dr, leftrightarrow] & \\
\omega_X(z) = h(z)dz \arrow[d, leftrightarrow] & & \Psi_X(z) = \int^z h(z)dz \arrow[d, leftrightarrow] \\
\omega_X \otimes \omega_X(z) = h^2(z)dz^2 \arrow[rr, leftrightarrow] & & ((\widehat{\mathbb{C}}, g_X), \mathfrak{Re}\,(X)).
\end{tikzcd}
$$

,

1[8]

[8] See J. Mucino–Raymundo, *Complex structures adapted to smooth vector fields*, Mathematische Annalen, vol. 322 (2002) 229–265.

We can apply tessellations in order to study differential equations and problems of integration.

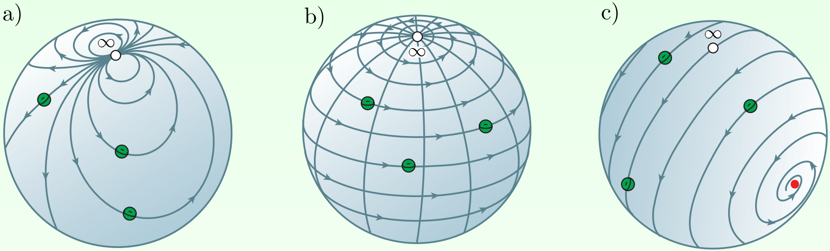

Figure: Holomorphic vector fields on $\widehat{\mathbb{C}}$.

a) $\frac{\partial}{\partial z}$ corresponding to the function $\Psi(z) = z$.
b) $z\frac{\partial}{\partial z}$ corresponding to the function $\Psi(z) = log(z)$.
c) $(z-a)(z-b)\frac{\partial}{\partial z}$.
They generate the Lie algebra $\mathfrak{psl}(2,\mathbb{C})$ of the Moebius transformation group $PSL(2,\mathbb{C})$.

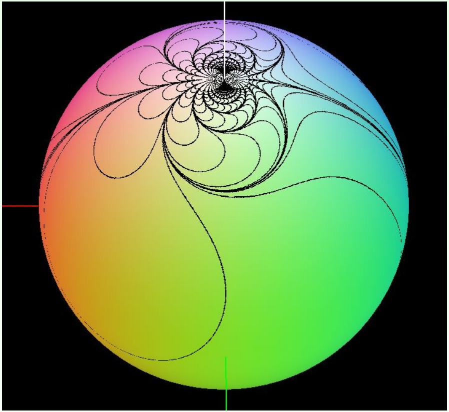

Figure: EsferaExpFlujoSinCampo.pdf Tessellation for the function $f(z) = e^z$ at ∞.

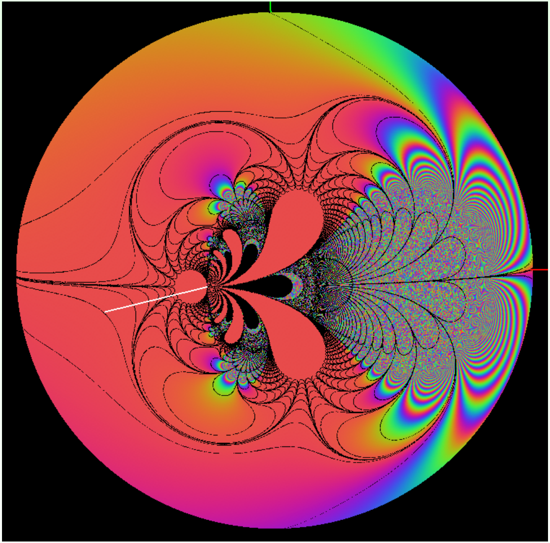

Figure: EjemploNoTipoFinito1.pdf Tessellation for the function $f(z) = \int^z e^{e^z} dz$ at ∞

The transcendental function cosine.

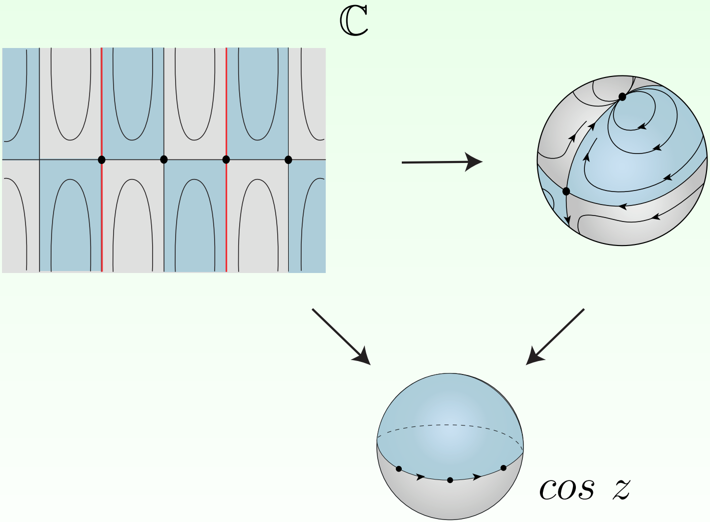

Figure: funcion-1.pdf Tessellation for the function $cos(z)$.

The transcendental function $\wp$.

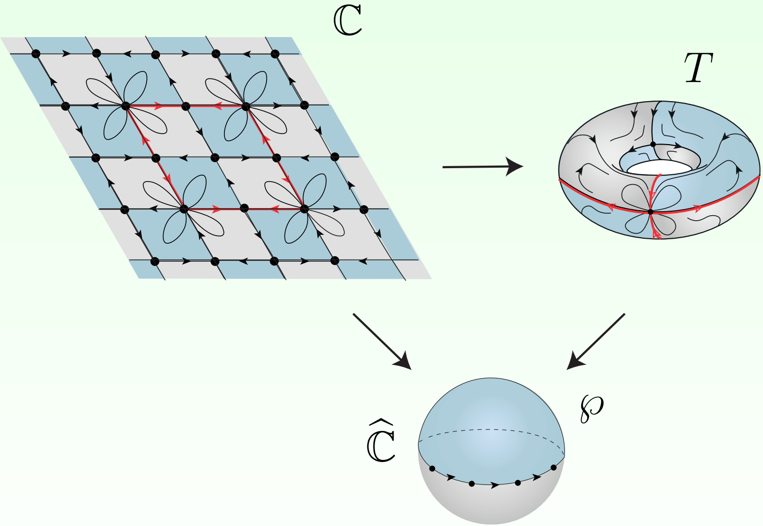

Figure: funcion-2.pdf Tessellation for the function $\wp$.

The Weierstrass's function

$$\wp(z) = \frac{1}{z} + \sum_{\lambda \in \Lambda \setminus 0} \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2}\right) : T \longrightarrow \widehat{\mathbb{C}}_w$$

and its tessellation in $T = \dfrac{\mathbb{C}}{\Lambda}$**.**

.

The transcendental function $\wp'$.

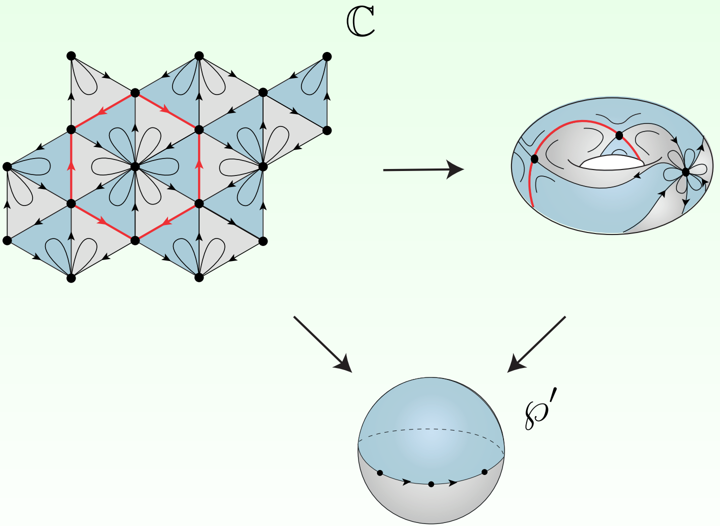

Figure: funcion-3.pdf Tessellation for the function $\wp'$

The elliptical integral.

A meromorphic function $\Psi : M_2 \longrightarrow \widehat{\mathbb{C}}_w$ and its tessellation.

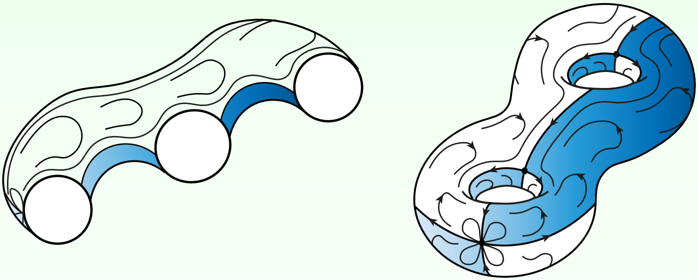

Figure: genero-dos.pdf

The elliptical integral.

A tessellation of the hyperelliptic Riemann surface

$$M_2 = \{y^2 - (x - p_1)(x -_2)(x - p_3) = 0\} \subset \mathbb{C}^2_{x,y},$$

it is related to the famous integral

$$\int^x \frac{dx}{(x - p_1)(x -_2)(x - p_3)}$$

according to Jacobi and Abel.

.

New roads after Riemann–Schwarz–Klein:

- 80's G. V. Belyĭ, A. Grothendieck et al.
  study functions $\mathbb{f} : M_g \longrightarrow \widehat{\mathbb{C}}_t$ with 3 critical values ($\mathbf{r} = 3$).

- 2010 W. Thurston et al.
  study tessellations $\mathfrak{M}$ on $\widehat{\mathbb{C}}_z$ (without vertex information), in order to decide which tessellations come from rational functions.

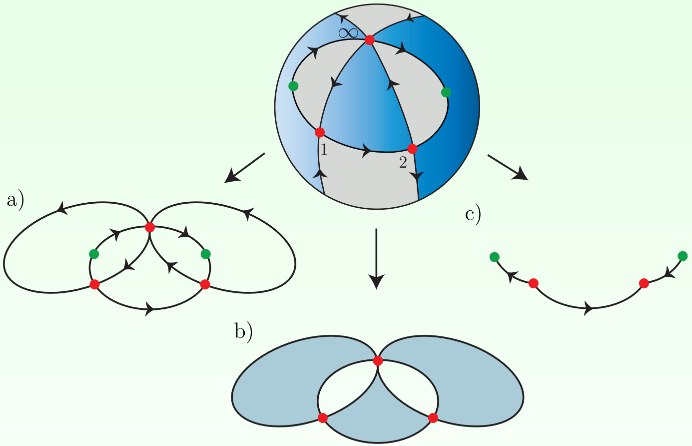

Figure: cubica-compleja-2.pdf Three graphs associated to $\mathbb{f}$. a) The map Γ Riemann–Schwarz–Klein. b) The map Γ deleting the cocritical points Thurston. c) The dessin d'enfant (by definition $\mathbb{f}^{-1}[1, 2]$), Belyĭ–Grothendieck .

Future projects.
Open as far as we know.

Project 1.

Given a realizable tessellation $\mathfrak{M}$ in $\widehat{\mathbb{C}}_z$ (Thurston 2011 et al. provide the combinatorial conditions):
compute explicitly the corresponding rational function, finding suitable coefficients $\mathbb{f}(z) = \dfrac{a_\alpha z^\alpha + \ldots + a_1 z + a_0}{b_\beta z^\beta + \ldots + b_1 z + b_0}$.

Project 2.

Given a two color tessellation $\mathfrak{M}$ in a compact Riemann surface of genus $g \geq 1$. Extended the results of

- Thurston et al. (affirmative conditions) and/or
- J. González–Cely (constructive for polynomials),

asserting under which conditions a good labelling exists, i.e. $\mathfrak{M}$ determines a meromorphic map $\mathbb{f} : M_g \longrightarrow \widehat{\mathbb{C}}_t$.

Project 3.

If all the critical values are in a line on a circle, then $\mathbb{f}^*\gamma$ has an algebraic equation (up to Mobius transformation)

$$\{\mathfrak{Im}\,(\mathbb{f}) = 0\} \subset \widehat{\mathbb{C}}_z.$$

Study these algebraic curves.
Classify the rational functions $\mathbb{f}$ with the above property.

• Relate clearly, the monodromy of a rational function to its tessellation.

• Relate the zoo's of topological tessellations to other enumerative questions for polynomial and rational functions.

Project 4.

Use triangulations or **r**–gonality to construct/classify rational or transcendental vector fields on $\widehat{\mathbb{C}}_z$ or M_g.

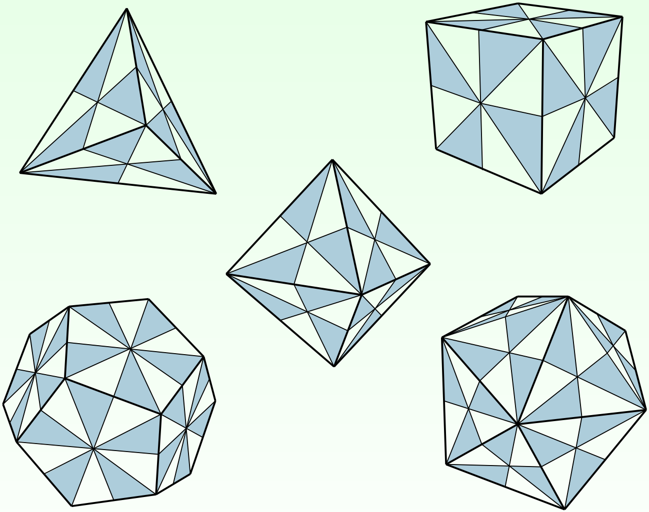

Figure: solidos-platonico.pdf

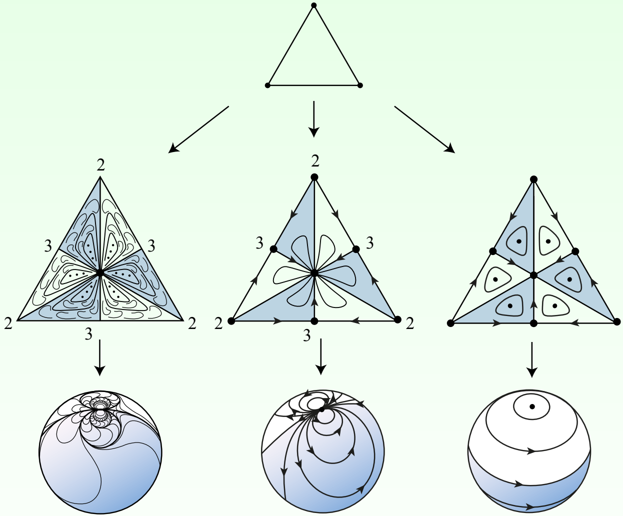

Figure: triangulos-mitad-4.pdf

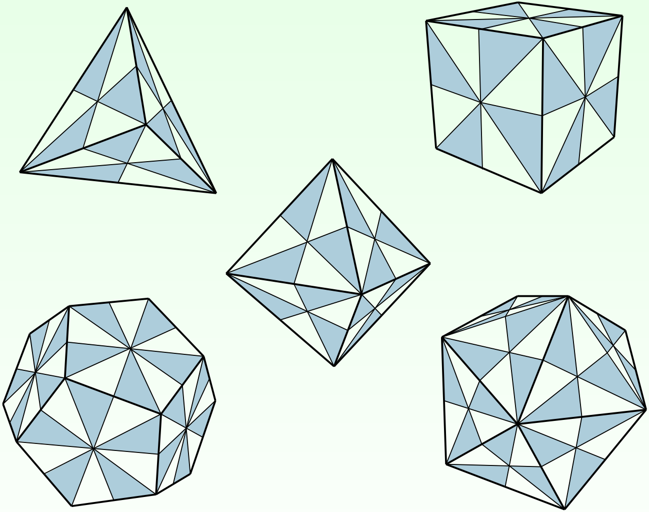

Figure: solidos-platonico.pdf

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