Geometry and dynamics of the Schur–Cohn stability algorithm for one variable polynomials

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Abstract

We provided a detailed study of the Schur–Cohn stability algorithm for Schur stable polynomials of one complex variable. A real analytic principal $(\mathbb{C}\times\mathbb{S}^1)$–bundle structure in the family of Schur stable polynomials of degree n is constructed. Secondly, we consider holomorphic Lie group $\mathbb{C}$–actions on the space of polynomials of degree n. For each orbit $\{s\cdot P(z) \mid s\in\mathbb{C}\}$, we study the general problem of the existence of a complex rational vector field on $\mathbb{C}_z$ such that its holomorphic s–flow describes the geometric change of the n–root configurations of the orbit $\{s\cdot P(z)=0\}$. In particular, we prove the existence of a complex rational vector field $\mathbb{X}$ on $\mathbb{C}_z$ describing the geometric change of the n–root configuration in the unitary disk $\mathbb{D}$ of a $\mathbb{C}$–orbit of Schur polynomials, here the orbits come from the above principal $\mathbb{C}\times\mathbb{S}^1$–bundle structure. We get analogous results in the framework of anti–Schur polynomials (those having all their roots in $\mathbb{C}\backslash\overline{\mathbb{D}}$).

Schur stable polynomials Schur–Cohn stability algorithm principal G–bundles complex rational vector fields Lie group actions

1 Introduction

The space of complex polynomials $\mathbb{C}[z]_{=n}=\{P(z)=\sum c_\iota z^\iota\}$ of degree n admits two natural parametrizations: coefficient coordinates $\mathscr{C}_n=\{(c_n,\dots,c_0)\}$ and root coordinates $\mathscr{R}_n=\{(c_n,[z_1,\dots,z_n])\}$ determined by unordered configurations of roots and c_n. The Viète map $\mathscr{V}_n:\mathscr{R}_n\longrightarrow\mathscr{C}_n$ given by the elementary symmetric functions is a natural translator, and the non–triviality of $\mathscr{V}_n^{-1}$ was done by N. H. Abel and É. Galois.

A polynomial $P(z)$ is called *Schur stable* (resp. *anti–Schur*) if all its roots lie in the unitary disk $\mathbb{D}$ (resp. in $\mathbb{C}\backslash\overline{\mathbb{D}}$). A classical problem in control theory and algebraic/analytic theory of polynomials is the construction of algorithms that from information in $\mathscr{C}_n$ decide when a polynomial is Schur stable, a conclusion in $\mathscr{R}_n$.

We regard the seminal work of I. Schur [29] and A. Cohn [11], currently known as the *Schur–Cohn stability algorithm*; see [27] Theorem 11.5.3, [8] section 1.4 for modern reviews, and [15] for computational aspects. The algorithm allows us to determine the number of roots of a polynomial $P(z)$ of degree n in the unitary disk $\mathbb{D}$, the boundary $\partial\mathbb{D}$ and the complement $\mathbb{C}\backslash\overline{\mathbb{D}}$. In differential topology language, the algorithm depends on four real polynomial maps on $\mathscr{C}_n$, denoted $\{R_{\alpha,n}\}_{\alpha=1}^4$. The first two maps

$$R_{1,n}:\mathscr{D}_{1,n}=\{\ |c_n|<|c_0|\ \}\subset\mathbb{C}[z]_{=n}\longrightarrow\mathbb{C}[z]_{\leqslant n-1},\quad R_{2,n}:\mathscr{D}_{2,n}=\{\ |c_n|>|c_0|\ \}\subset\mathbb{C}[z]_{=n}\longrightarrow\mathbb{C}[z]_{\leqslant n-1} \qquad (1)$$

deal with anti–Schur and Schur stable polynomials, respectively, enjoying the following crucial properties; see [8], [27]:
$P(z)$ is anti–Schur if and only if $R_{1,n}(P(z))$ is also anti–Schur.
$P(z)$ is Schur stable if and only if $R_{2,n}(P(z))$ is also Schur stable.

It is a very remarkable/rare fact, the existence of maps in the coefficient coordinates $\mathscr{C}_n$ enjoying the following two characteristics: under $R_{\alpha,n}$ the degree of $P(z)$ decreases and the position (respect to $\mathbb{D}$) of the roots of $R_{\alpha,n}(P(z))$ is preserved. Thus, the Schur–Cohn stability algorithm provides us conclusions for the root coordinates $\mathscr{R}_n$.

$\mathbb{C}^* \times \mathbb{S}^1$ **and** $\mathbb{C} \times \mathbb{S}^1$ **bundle geometry.** In [2] and [3], B. Aguirre–Hernández *et al.* introduce a vector bundle structure on the space of monic Schur stable polynomials of degree n. Our starting result enlarged this geometric structure as follows.

Theorem A. *For $\alpha = 1, 2$, the maps $R_{\alpha,n}$ are real analytic submersions and determine trivial principal bundles*

$$\begin{tikzcd} \mathbb{C}^* \times \mathbb{S}^1 \arrow[r] & \mathscr{D}_{1,n} \arrow[d, "R_{1,n}"] \\ & \mathbb{C}^{n-1} \times \mathbb{R}^+ \end{tikzcd} \qquad \begin{tikzcd} \mathbb{C} \times \mathbb{S}^1 \arrow[r] & \mathscr{D}_{2,n} \arrow[d, "R_{2,n}"] \\ & \mathbb{R}^+ \times \mathbb{C}^{n-1} \end{tikzcd} \tag{2}$$

having structural Lie groups $\mathbb{C}^* \times \mathbb{S}^1$ and $\mathbb{C} \times \mathbb{S}^1$.

What is the meaning in root coordinates $\mathscr{R}_n$ of these principal bundle geometries?

Lie group actions on the space of polynomials and Weierstrass polynomials. We consider the following *prototype dynamics*. Let $\{P(z) = 0\} = [z_1, \ldots, z_n] \subset \mathbb{C}$ be the n–root configuration of a monic polynomial. Assume that s varies in the additive Lie group $\mathbb{C} = \{s\}$:

How can we describe the geometric change of the n–root configurations $\{P(z) = s\} = [z_1(s), \ldots, z_n(s)]$?

see figure 5.1.b. in §5. Obviously the topology of the n–root configurations change when s crosses a critical value of $P(z)$. In a very rough analogy with the n–body problem, this is a n–particle dynamics, where s plays the role of the time. The *complex analytic vector field $\mathbb{X}(z) = (P(z)')^{-1}\frac{\partial}{\partial z}$ describes the n–root configuration dynamics*, this is the diagram

$$\begin{tikzcd} {[z_1, \ldots, z_n]} \arrow[r, "\mathscr{V}_n"] \arrow[d, "\varphi(s,\ )"'] & P(z) = c_n z^n + \ldots + c_1 z + c_0 \arrow[d, "\doteq \mathscr{A}(s,\ )"] \\ {[\varphi(s,z_1), \ldots, \varphi(s,z_n)]} & \mathrm{P}(s,z) \doteq c_n z^n + \ldots + c_1 z + (c_0 + s). \arrow[l, "\mathscr{V}_n^{-1}"] \end{tikzcd} \tag{3}$$

commutes, whenever the local holomorphic flow $\varphi(s,z) \doteq z(s)$ of $\mathbb{X}$ is well–defined. This is the prototype dynamics; see definition 5.1 and lemma 5.2 for its proof.

We enlarge this framework as follows. Let G be the Lie group $\mathbb{C}$ or $\mathbb{C}^*$, and consider

$$\mathscr{A} : G \times \mathbb{C}[z]_{=n} \longrightarrow \mathbb{C}[z]_{=n}, \quad (g, P) \longmapsto g \cdot P$$

a holomorphic Lie group action. Each orbit $\{g \cdot P \mid g \in G\}$ is a holomorphic Weierstrass polynomial, in the sense of V. L. Hansen [19], [20]. Note that $\mathscr{A}(s,\ )$ in (3) is a $\mathbb{C}$–action. The *general problem* is as follows:

Is there a rational vector field $\mathbb{X}$ on $\mathbb{C}_z$ describing the n–root configuration dynamics of the orbit $\{g \cdot P(z) \mid g \in \mathbb{C}\}$?

This is, the analogous diagram to (3) commutes using in the right column $\mathscr{A}(g,\ )$; see definition 5.1 again.

***n*–root configuration dynamics of the Schur and anti–Schur orbits.** We consider a Schur polynomial $P(z)$, the problem is describing the roots of all the polynomials in the fiber $\{s \cdot P(z) \mid s \in \mathbb{C}\}$ in $\mathbb{D}$, which originates from the $\mathbb{C}$–action in (respect the $\mathbb{C}^*$–action on the right in(2)). Since the Lie group $\mathbb{C}$ is simply connected, the Schur case is more simple than the anti–Schur case requiring $\mathbb{C}^*$. Our main dynamical results for Schur and anti–Schur polynomials are as follows; see sections 8, 9 respectively.

Theorem B. *(n–root configuration dynamics of the Schur and anti–Schur orbits)*
1. *Let $P(z)$ be a Schur stable polynomial. There exists a complex rational vector field $\mathbb{X}$ on $\mathbb{C}_z$ describing the n–root configuration dynamics of*

$$\{s \cdot P(z) = 0 \mid s \in \mathbb{C}\} \subset \mathscr{D}_{2,n},$$

up to a suitable reparametrization of its complex time. Here $\{s \cdot P(s)\}$ is a fiber (a Weierstrass polynomial) in the corresponding principal bundle (2).

2. *The analogous result is true for each anti–Schur stable polynomial $P(z)$, a suitable complex rational vector field $\mathbb{Y}$ on $\mathbb{C}_z$, and the fiber*

$$\{w \cdot P(z) = 0 \mid w \in \mathbb{C}^*\} \subset \mathscr{D}_{1,n},$$

in the corresponding principal bundle (2).

See figure 4 in §8 for several phase portraits of $\mathbb{X}$ on $\mathbb{D}$ and $\mathbb{Y}$ on $\mathbb{C}\backslash\mathbb{D}$; describing numerical examples.

The analogous versions of theorems A and B remain true for Hurwitz polynomials, considering the Möbius transformation T that sends $\mathbb{D}$ on to the left half–plane $\mathbb{H} = \{\mathfrak{Im}(z) < 0\}$.

The content of the paper is as follows. Section §2 describes several families of degree n polynomials according to the position of their roots. In §3, the Schur–Cohn stability algorithm is reviewed. Theorem A is proved in §4. The notion of Lie group actions on the space of polynomials of degree n and the prototype dynamics are done in section §5. The complex dynamics of singular complex analytic vector fields and their application to our general problem are given in §6. A dictionary between singular points of vector fields and n–root configuration dynamics is in definition 6.2, simple examples are provided. Section §7 explores complex rational vector fields that arose from several Lie group actions on $\mathbb{C}[z]_{=n}$. Theorem B for Schur–stable polynomials is proven in §8, the assertion for the anti–Schur case is done in §9.

2 Families of polynomials

Let $\mathbb{C}[z]_{=n} = \{P(z) = c_n z^n + \ldots + c_0 \mid c_n \in \mathbb{C}^*\}$ be the space of complex polynomials of degree n, which is a subset of the vector space of complex polynomials $\mathbb{C}^{n+1} \doteq \{(c_n, \ldots, c_0)\}$ of degree at most n. We introduce root coordinates, $\mathscr{R}_n$, using the coefficient $\{c_n\}$ and the unordered roots $\{[z_1, \ldots z_n]\}$ for each polynomial $P(z)$, as follows:

$$\begin{array}{ccccccc}
\mathbb{C}^* \times \mathbb{C}^n & \longrightarrow & \mathbb{C}^* \times \frac{\mathbb{C}^n}{S(n)} & \overset{\mathscr{V}_n}{\underset{\mathscr{V}_n^{-1}}{\rightleftarrows}} & \mathbb{C}^{n+1}\backslash\{c_n = 0\} \\
(c_n, z_1, \ldots, z_n) & \longmapsto & (c_n, [z_1, \ldots, z_n]) & \longmapsto & (c_n, c_{n-1}, \ldots, c_0) = \\
 & & & & \Big(c_n, -c_n(z_1 + \ldots + z_n), \ldots, (-1)^n c_n(z_1 \cdots z_n)\Big),
\end{array} \tag{4}$$

where the symmetric group of order n, $S(n)$, acts on the roots and $[\ldots]$ means a class under this action.
$\mathscr{V}_n$ is the n degree Viète map and the inverse map essentially $\mathscr{V}_n^{-1}$ sends a polynomial to its roots; see [1], [23].

Let $\mathbb{D} := \{z \in \mathbb{C} \mid |z| < 1\}$ be the unitary disk, having boundary $\partial\mathbb{D} \doteq \{z \in \mathbb{C} \mid |z| = 1\}$. It will be useful to consider families of polynomials of degree n, depending on the numbers of roots $\mathtt{p}, \mathtt{s}, \mathtt{q}$ in $\mathbb{D}$, $\partial\mathbb{D}$ and $\mathbb{C}\backslash\overline{\mathbb{D}}$ respectively ($n = \mathtt{p} + \mathtt{s} + \mathtt{q}$). The usual motivation is as follows, let F be a holomorphic germ map on $(\mathbb{C}^n, \overline{0})$ and $det(DF(\overline{0})) = P(z)$ its Jacobian polynomial having eigenvalues $[z_1, \ldots, z_n]$. It is well known that generically under the iteration of F; there appear stable, central and unstable local manifolds at the origin $\overline{0} \in \mathbb{C}^n$ of dimensions $\mathtt{p}$, $\mathtt{s}$ and $\mathtt{q}$, respectively. Other extensive references where interesting information about Schur stable polynomials are the books [10] and [22], also [4] and [14].

We describe the families in the table, depending on three attributes; the position of the roots, some inequality on the coefficients and the Schur–Cohn map required by the stability algorithm.

Table 1. Families of polynomials according to roots $\{\mathtt{p}, \mathtt{s}, \mathtt{q}\}$ and coefficients $\{|c_n|, |c_0|\}$.

		$\mathbb{D}$ $\mathtt{p}$	$\partial\mathbb{D}$ $\mathtt{s}$	$\mathbb{C}\backslash\overline{\mathbb{D}}$ $\mathtt{q}$		required Schur–Cohn map
Schur	$\mathscr{S}_n$	$\mathtt{p} = n$	0	0	$\lvert c_n\rvert > \lvert c_0\rvert$	$R_{2,n}$
semi–Schur	$s\mathscr{S}_n$	$\mathtt{p} \geqslant 1$	$\mathtt{s} \geqslant 1$	0		
saddle polynomials type 2	$us_2\mathscr{S}_n$	$\mathtt{p} \geqslant 1$	$\mathtt{s}$	$\mathtt{q} \geqslant 1$		
anti–Schur	$a\mathscr{S}_n$	0	0	$\mathtt{q} = n$	$\lvert c_n\rvert < \lvert c_0\rvert$	$R_{1,n}$
semianti–Schur	$sa\mathscr{S}_n$	0	$\mathtt{s} \geqslant 1$	$\mathtt{q} \geqslant 1$		
saddle polynomials type 1	$us_1\mathscr{S}_n$	$\mathtt{p} \geqslant 1$	$\mathtt{s}$	$\mathtt{q} \geqslant 1$		
balanced	$\mathscr{B}_n$	$\mathtt{p} \geqslant 1$	$\mathtt{s}$	$\mathtt{q} \geqslant 1$	$\lvert c_n\rvert = \lvert c_0\rvert$	$R_{3,n}$
self–inverse	$\mathscr{C}_n$	$\mathtt{p} \geqslant 0$	$\mathtt{s} \geqslant 0$	$\mathtt{q} = \mathtt{p}$	$\lvert c_n\rvert = \lvert c_0\rvert$	$R_{4,n}$

· *Schur*[1] is the family of polynomials that have all their roots in the open unitary disc.
· *Anti–Schur* is the family of polynomials that have all their roots in the exterior of the unitary disc.
· *Semi–Schur* is the topological closure of the Schur family, *i.e.* the family of polynomials that have all their roots in the closed unitary disc $\overline{\mathbb{D}}$.

[1] We simplify Schur stable to say only Schur.

· *Semianti–Schur* is the topological closure of the anti–Schur family.
· *Saddle polynomials*: The term comes from the existence of non empty local stable and unstable manifolds under the iteration of F, as we remark above. Type 1 or 2 depends on the respective Schur–Cohn map that will be required.
· We define that $P(z)$ is *self–inverse* when $P(z) = \sigma P^*(z)$, for some point in $\{|\sigma| = 1\} = \mathbb{S}^1$, and

$$\begin{array}{rcl} (\ )^* : \mathbb{C}^{n+1} & \longrightarrow & \mathbb{C}^{n+1} \\ P(z) = (c_n, c_{n-1}, \ldots, c_0) & \longmapsto & P(z)^* = z^n \overline{P\left(\frac{1}{\bar{z}}\right)} = (\bar{c}_0, \ldots, \bar{c}_{n-1}, \bar{c}_n), \end{array} \tag{5}$$

here $\bar{z}$ denotes the conjugate. Equivalently each root $z_\ell \in \mathbb{D}$ of $P(z)$ comes with its reciprocal conjugate $1/\bar{z}_\ell$; see [27] pp. 375, [31] pp. 109, [6] and [9].
1. The operator $(\ )^*$ in (5) is an involution.
2. The polynomial $P(z)^*$ is Schur if and only if $P(z)$ is anti–Schur.
· We define that, $P(z)$ is *balanced* when $|c_n| = |c_0|$ and it is not self–inverse.
Looking at the eight families in Table 1, the intersection of two of them is empty.

Recall that $\{|c_n| > |c_0|\}$ is a necessary but no sufficient condition in order to characterize Schur polynomials. Following the fifth column in Table 1, let

$$\Sigma_n = \{P(z) \mid |c_n| = |c_0|\} \subset \mathbb{C}^{n+1} \tag{6}$$

be a real hypersurface having a singularity at the origin. Σ_n determines two open and connected domains

$$\mathscr{D}_{1,n} \doteq \{P(z) \mid |c_n| < |c_0|\}, \qquad \mathscr{D}_{2,n} = \{P(z) \mid |c_n| > |c_0|\}. \tag{7}$$

In addition, we define

$$\mathscr{D}_{4,n} = \{P(z) \mid |c_n| = |c_0|,\ P(z) = \sigma P^*(z) \text{ for } \sigma \in \mathbb{S}^1\}, \qquad \mathscr{D}_{3,n} = \Sigma_n \backslash \mathscr{D}_{4,n}, \tag{8}$$

whence

$$\mathbb{C}^{n+1} \backslash \{c_n = 0\} = \mathscr{D}_{1,n} \cup \cdots \cup \mathscr{D}_{4,n}.$$

Moreover, $\mathscr{D}_{\alpha,n}$ will be the domain of the Schur–Cohn map $R_{\alpha,n}$, as we will show in the next section.

3 The Schur–Cohn maps

We recall the Schur–Cohn maps for polynomial map families of degree n. In order to avoid double subindexes at the target, which is the space of polynomials of degree at most $n-1$, we use the notation

$$\mathbb{C}^n \doteq \{(b_{n-1}, \ldots, b_0)\} = \{b_{n-1} z^{n-1} + \ldots + b_1 z + b_0\}$$

and convene $\mathbb{R}^+ \subset \mathbb{C}$ as usual. Moreover, we follow the enumeration of Schur–Cohn maps as in [27] p. 375.
When $P(z) \in \mathscr{D}_{1,n} \cup \mathscr{D}_{2,n}$, which is an open and dense set of the polynomials of degree n, the maps are as follows.
Schur–Cohn map 1

$$\begin{array}{rcl} R_{1,n} : \mathscr{D}_{1,n} \subset \mathbb{C}^{n+1} & \longrightarrow & (\mathbb{C}^{n-1} \times \mathbb{R}^+) \subset \mathbb{C}^n \\ (c_n, \ldots, c_0) & \longmapsto & (c_{n-1}\bar{c}_0 - \bar{c}_1 c_n,\ c_{n-2}\bar{c}_0 - \bar{c}_2 c_n,\ \ldots,\ c_1\bar{c}_0 - \bar{c}_{n-1}c_n,\ |c_0|^2 - |c_n|^2) \\ & & \doteq (b_{n-1}, \ldots, b_0). \end{array} \tag{9}$$

Schur–Cohn map 2

$$\begin{array}{rcl} R_{2,n} : \mathscr{D}_{2,n} \subset \mathbb{C}^{n+1} & \longrightarrow & (\mathbb{R}^+ \times \mathbb{C}^{n-1}) \subset \mathbb{C}^n \\ (c_n, \ldots, c_0) & \longmapsto & (|c_n|^2 - |c_0|^2,\ c_{n-1}\bar{c}_n - c_0\bar{c}_1,\ \ldots,\ c_2\bar{c}_n - c_0\bar{c}_{n-2},\ c_1\bar{c}_n - c_0\bar{c}_{n-1}) \\ & & \doteq (b_{n-1}, \ldots, b_0). \end{array} \tag{10}$$

Now consider the nongeneric case $P(z) \in \Sigma_n = \{|c_0| = |c_n|\}$. In fact, $|c_n| = |c_0|$ if and only if $c_n - \sigma c_0 = 0$ for some $\sigma \in \mathbb{S}^1 = \{|\sigma| = 1\}$.

Case i. If $\sigma\bar{c}_{n-k} - c_k = 0$ for every $k \in 1, \ldots, n$, then we apply Schur–Cohn rule 4 below, in particular $|c_0|^2 - |c_n|^2 = 0$ holds.
Case ii. If there exists k_0 such that $\sigma\bar{c}_{n-k} - c_k \neq 0$, let $k_1 = min\{k_0\}$ be for some $0 < k_0 < n$, then we have the following.

Schur–Cohn map 3. This map has a very hard expression in terms of $(c_n, \ldots, c_0)$, for example if $P(z) = c_2 z^2 + c_1 z^2 + c_1 z + c_0$, then

$$R_{2,3}: P(z) \longmapsto P_1(z) = \left(\left| \frac{2\overline{c}_0\overline{b}c_1}{|b|} + 4\overline{c}_0 c_2 - c_2\overline{c}_0 - 2c_2\overline{c}_1 \frac{\overline{b}}{|b_0|} \right|^2 - \left| 4|c_0|^2 - |c_2|^2 \right|^2 \right) z$$
$$+ \left(\frac{2|c_0|^2\overline{b}}{|b|} + 4\overline{c}_0 c_1 - c_2\overline{c}_1 - \frac{2|c_2|^2\overline{b}}{|b|} \right) \left(\frac{2c_0 b\overline{c}_1}{|b|} + 4c_0\overline{c}_2 - \overline{c}_2 c_0 - \frac{2\overline{c}_2 c_1 b}{|b|} \right)$$
$$- \left(4|c_0|^2 - |c_2|^2\right) \left(\frac{2|c_0|^2 b}{|b|} + 4c_0\overline{c}_1 - \overline{c}_2 c_1 - \frac{2|c_2|^2 b}{|b|} \right)$$

where $b = \dfrac{c_1\overline{c}_2 - c_0\overline{c}_1}{|c_2|^2} \neq 0$. Furthermore, we prefer a synthetic expression

$$\begin{array}{rcl} R_{3,n}: \mathscr{D}_{3,n} \subset \mathbb{C}^{n+1} & \longrightarrow & \mathbb{C}^n \\ P(z) & \longmapsto & P_1(z) \doteq \frac{1}{z}\left[g_1^*(0)g_1(z) - g_1(0)g_1^*(z)\right], \end{array} \tag{11}$$

with

$$g(z) \doteq \left(z^k + \frac{2b}{b}\right) P(z), \quad g_1(z) \doteq \overline{g}(0)g(z) - \overline{g}^*(0)g^*(z),$$

here $b \doteq (c_{n-k} - \sigma\overline{c}_k)/c_n$.

Schur–Cohn map 4

$$\begin{array}{rcl} R_{4,n}: \mathscr{D}_{4,n} \subset \mathbb{C}^{n+1} & \longrightarrow & \mathbb{C}^n \\ (c_n, \ldots, c_0) & \longmapsto & (c_{n-1}, 2c_{n-2}, 3c_{n-3}, \ldots, (n-1)c_1, nc_0) \\ & & = (b_{n-1}, \ldots, b_0). \end{array} \tag{12}$$

The precise statement for the Schur–Cohn algorithm is stated in theorem 1. The original sources are [11] and [29], modern versions can be found in [8] pp. 55, [21] pp. 355, 368, and [27] pp. 375, 395, our redaction follows verbatim this last.

(Schur–Cohn stability algorithm) Let $P(z)$ be a polynomial of degree n, having zeros $\mathtt{p}, \mathtt{s}, \mathtt{q}$ as in table 1, and consider its image under the Schur–Cohn stability algorithm; thus

$$P_1(z) \doteq R_{\alpha,n}(P(z)),$$

where $\alpha \in \{1, \ldots, 4\}$ is determined by $P(z) \in \mathscr{D}_{\alpha,n}$, as in equations (7) and (8).
Let $\mathtt{p}_1, \mathtt{s}_1, \mathtt{q}_1$ be the corresponding zeros of $P_1(z)$ in $\mathbb{D}$, $\partial\mathbb{D}$, $\mathbb{C}\backslash\overline{\mathbb{D}}$.

1. If $|c_0| > |c_n|$, then $P_1(z) \doteq \overline{c}_0 P(z) - c_n P^*(z)$ is not identically zero and we have $\deg P_1 < \deg P$. In this case
$$\mathtt{p}_1 = \mathtt{p},\ \mathtt{s}_1 = \mathtt{s} \text{ and } 0 \leqslant \mathtt{q}_1 < \mathtt{q}.$$
2. If $|c_0| < |c_n|$, then $P_1(z) \doteq (\overline{c}_n P(z) - c_0 P^*(z))/z$ is of degree $n-1$. In this case
$$\mathtt{p}_1 = \mathtt{p} - 1,\ \mathtt{s}_1 = \mathtt{s} \text{ and } \mathtt{q}_1 = \mathtt{q}.$$
3. If there is an index $k \leqslant n/2$ such that
$$c_0 = \sigma\overline{c}_n,\ \ c_1 = \sigma\overline{c}_{n-1}, \ldots, c_{k-1} = \sigma\overline{c}_{n-k+1},\ \ c_k \neq \sigma\overline{c}_{n-k} \quad (|\sigma| = 1),$$
then define $b \doteq (c_{n-k} - \sigma\overline{c}_k)/c_n$,
$$g(z) \doteq \left(z^k + \tfrac{2b}{b}\right) P(z), \quad g_1(z) \doteq \overline{g}(0)g(z) - \overline{g}^*(0)g^*(z)$$
and
$$P_1(z) \doteq \frac{1}{z}\left[g_1^*(0)g_1(z) - g_1(0)g_1^*(z)\right],$$
which yields that P_1 is of degree $n-1$. In detail,
$$\mathtt{p}_1 = \mathtt{p} - 1,\ \mathtt{s}_1 = \mathtt{s}, \text{ and } \mathtt{q}_1 = \mathtt{q}.$$
4. If P is self–inversive, then $P_1(z) \doteq P(z) - zP'(z)$ is not identically zero, and we have $\deg P_1 < \deg P$ and
$$\mathtt{p}_1 = \mathtt{p},\ \mathtt{s} = n - 2\mathtt{p},\ \mathtt{s}_1 + \mathtt{q}_1 < \mathtt{s} + \mathtt{q}.$$

□

A qualitative root description of the Schur–Cohn maps is as follows.
· If $P(z) \in \mathscr{D}_{1,n}$ has $\mathtt{p}$, $\mathtt{s}$, $\mathtt{q}$ roots in $\mathbb{D}$, $\partial\mathbb{D}$, $\mathbb{C}\backslash\overline{\mathbb{D}}$, then the map $R_{1,n}$ removes $\ell \geqslant 1$ from the original $\mathtt{q}$ roots and relocates $\mathtt{p}$ roots in $\mathbb{D}$, $\mathtt{s}$ roots in $\partial\mathbb{D}$, $\mathtt{q} - \ell$ roots in $\mathbb{C}\backslash\overline{\mathbb{D}}$.
· If $P(z) \in \mathscr{D}_{2,n}$ has $\mathtt{p}$, $\mathtt{s}$, $\mathtt{q}$ roots, then the map $R_{2,n}$ removes one of the original $\mathtt{p}$ roots and relocates $\mathtt{p} - 1$ roots in $\mathbb{D}$, $\mathtt{s}$ roots in $\partial\mathbb{D}$, $\mathtt{q}$ roots in $\mathbb{C}\backslash\overline{\mathbb{D}}$.

· If $P(z) \in \mathscr{D}_{3,n}$ has $\mathtt{p}$, $\mathtt{s}$, $\mathtt{q}$ roots, then $R_{3,n}$ removes one of the $\mathtt{q}$ roots and relocates $\mathtt{p}$ roots in $\mathbb{D}$, $\mathtt{s}$ roots in $\partial\mathbb{D}$, $\mathtt{q}-1$ roots in $\mathbb{C}\backslash\overline{\mathbb{D}}$.
· $R_{4,n}$ acts on the roots as in the fourth assertion, and it is the restriction of a linear submersion.

1. $R_{\alpha,n}$ are real analytic maps in coefficient coordinates $\{(c_n,\dots,c_0)\}$, for $\alpha = 1,\dots,4$.

2. For $n \geqslant 2$, the Schur polynomials $\mathscr{S}_n \subset \mathscr{D}_{2,n} = \{|c_n| > |c_0|\}$ determine an open but not dense subset in this component. Similar properties are fullfiled by the anti–Schur $a\mathscr{S}_n$, polynomials in $\mathscr{D}_{2,n} = \{|c_n| > |c_0|\}$.
The set $us\mathscr{S}$ is open and has two connected components coming from $\{|c_n| > |c_0|\}$ and $\{|c_n| < |c_0|\}$.

3. The *topological boundary* $\partial\mathscr{S}_n$ of $\mathscr{S}_n$ has the following property; the intersection $\partial\mathscr{S}_n \cap \{c_n = 0\}$ is only the polynomial $P(z) \equiv 0$. In the whole $\mathbb{C}^{n+1}$ we have
$$\partial\mathscr{S}_n = s\mathscr{S}_n \cup \{\text{polynomials with all roots in the unitary circle}\}.$$

4 Geometry of Schur–Cohn maps in coefficient coordinates

Our first goal is the study of $R_{\alpha,n},\ \alpha \in 1,2$, on the domain where they are nonsingular maps. This domain of regular points corresponds to $\mathbb{C}^{n+1}\backslash\Sigma_n$, where we get associated geometric structures.

Let us consider $\mathbb{C}_s = \{s\}$, $\mathbb{C}^*_w = \{w\}$ and $\mathbb{S}^1 = \{e^{i\theta}\}$ the Lie groups aditive, multiplicative and unitary (sometimes we will use symply $\mathbb{C}$, $\mathbb{C}^*$). In [2] and [3], Aguirre–Hernández *et al.* introduce a vector bundle structure on the space of monic Schur polynomials of n degree. This geometric structure is enlarged on $\mathbb{C}^{n+1}\backslash\Sigma_n$ as follows.

1. The map
$$R_{1,n} : \mathscr{D}_{1,n} \subset \mathbb{C}^{n+1} \longrightarrow \mathbb{C}^{n-1} \times \mathbb{R}^+ \tag{13}$$
is a submersion and determines a trivial principal $(\mathbb{C}^* \times \mathbb{S}^1)$–bundle.

2. The map
$$R_{2,n} : \mathscr{D}_{2,n} \subset \mathbb{C}^{n+1} \longrightarrow \mathbb{R}^+ \times \mathbb{C}^{n-1} \tag{14}$$
is a submersion and determines a trivial principal $(\mathbb{C} \times \mathbb{S}^1)$–bundle.

These submersions and bundle structures are in the real analytic category.

We note that the submersion $R_{2,n}$ in (14) is well–defined in the whole component $\mathscr{D}_{2,n}$, even if Schur polynomials $\mathscr{S}_n$ are a strict, proper subset for $n \geqslant 3$, similarly for anti–Schur polynomials in (13).

Step 1. $R_{2,n}$ is a submersion in the domain $\mathscr{D}_{2,n}$. By simple inspection, using (10) we note that
$$R_{2,n}(\mathscr{D}_{2,n}) = \mathbb{R}^+ \times \mathbb{C}^{n-1} \subsetneq \mathscr{D}_{2,n-1}.$$
For an example of $P(z) \in \mathscr{D}_{2,2}\backslash\mathscr{S}_2$, see example 8. We introduce the following map
$$\begin{array}{rcl}
\Xi_{2,n} : \mathbb{C} \times \mathbb{S}^1 \times \mathbb{R}^+ \times \mathbb{C}^{n-1} & \longrightarrow & \mathscr{D}_{2,n} \\
(s, e^{i\theta}, b_{n-1}, b_{n-2}, \dots, b_0) & \longmapsto & (c_n, \dots, c_0) \\
& \doteq & \left(\sqrt{b_{n-1}+|s|^2}e^{i\theta}, \frac{\sqrt{b_{n-1}+|s|^2}e^{i\theta}b_{n-2}}{b_{n-1}} + \frac{\overline{b}_0 s}{b_{n-1}}, \dots, \frac{\sqrt{b_{n-1}+|s|^2}e^{i\theta}b_0}{b_{n-1}} + \frac{\overline{b}_{n-2}s}{b_{n-1}}, s\right) \\
& = & \frac{e^{i\theta}}{b_{n-1}}\sqrt{b_{n-1}+|s|^2}\Big(b_{n-1}, b_{n-2}, \dots, b_0, 0\Big) + \frac{s}{b_{n-1}}\Big(0, \overline{b}_0, \dots, \overline{b}_{n-2}, \overline{b}_{n-1}\Big)
\end{array} \tag{15}$$
by assumption $b_{n-1} \in \mathbb{R}^+$. Hence the condition $|c_n| > |c_0|$ matches (the last line will be very illustrative).
Thus, $\Xi_{2,n}$ is a real analytic map, having inverse
$$\begin{array}{rcl}
\Xi^{-1}_{2,n} : \mathscr{D}_{2,n} & \longrightarrow & \mathbb{C} \times \mathbb{S}^1 \times \mathbb{R}^+ \times \mathbb{C}^{n-1} \\
(c_n, \dots, c_0) & \longmapsto & \big(c_0, e^{i\theta_n}, b_{n-1}, \dots, b_1, b_0\big) \\
& \doteq & \big(c_0, e^{i\theta_n}, |c_n|^2 - |c_0|^2, c_{n-1}\overline{c}_n - c_0\overline{c}_1, \dots, c_2\overline{c}_n - c_0\overline{c}_{n-2}, c_1\overline{c}_n - c_0\overline{c}_{n-1}\big),
\end{array} \tag{16}$$
here we use the retraction
$$\rho : \mathbb{C}^* \longrightarrow \mathbb{S}^1 \cong \frac{i\mathbb{R}}{i2\pi\mathbb{Z}}, \quad c_n \longmapsto \frac{c_n}{|c_n|} \doteq e^{i\theta_n}. \tag{17}$$
Observe that $\Xi^{-1}_{2,n}$ restricted to the last n coordinates is $R_{2,n}$. In fact,
$$\Xi^{-1}_{2,n} \circ \Xi_{2,n} = Id, \quad \Xi_{2,n} \circ \Xi^{-1}_{2,n} = Id. \tag{18}$$

We compute the first composition as follows;

$$
\begin{aligned}
&\Xi_{2,n}^{-1}\circ\Xi_{2,n}(s,\mathrm{e}^{i\theta},b_{n-1},b_{n-2},\dots,b_1,b_0)\\
&\quad=\Xi_{2,n}^{-1}\left(\sqrt{b_{n-1}+|s|^2}\mathrm{e}^{i\theta},\frac{\sqrt{b_{n-1}+|s|^2}\mathrm{e}^{i\theta}b_{n-2}}{b_{n-1}}+\frac{\overline{b}_0}{b_{n-1}}s,\dots,\frac{\sqrt{b_{n-1}+|s|^2}\mathrm{e}^{i\theta}b_0}{b_{n-1}}+\frac{\overline{b}_{n-2}}{b_{n-1}}s,s\right)\\
&\quad=\left(s,\mathrm{e}^{i\theta},b_{n-1},\left[\frac{\sqrt{b_{n-1}+|s|^2}\mathrm{e}^{i\theta}b_{n-2}}{b_{n-1}}+\frac{\overline{b}_0}{b_{n-1}}s\right]\sqrt{b_{n-1}+|s|^2}\mathrm{e}^{-i\theta}-s\left[\frac{\sqrt{b_{n-1}+|s|^2}\mathrm{e}^{-i\theta}\overline{b}_0}{b_{n-1}}+\frac{b_{n-2}}{b_{n-1}}\overline{s}\right],\right.\\
&\quad\left.\dots,\left[\frac{\sqrt{b_{n-1}+|s|^2}\mathrm{e}^{i\theta}b_0}{b_{n-1}}+\frac{\overline{b}_{n-2}}{b_{n-1}}s\right]\sqrt{b_{n-1}+|s|^2}\mathrm{e}^{-i\theta}-s\left[\frac{\sqrt{b_{n-1}+|s|^2}\mathrm{e}^{-i\theta}\overline{b}_{n-2}}{b_{n-1}}+\frac{b_0}{b_{n-1}}\overline{s}\right]\right)\\
&\quad=\left(s,\mathrm{e}^{i\theta},b_{n-1},\frac{b_{n-2}}{b_{n-1}}(b_{n-1}|s|^2)\frac{b_{n-2}}{b_{n-1}}|s|^2,\dots,\frac{b_0}{b_{n-1}}(b_{n-1}+|s|^2)-\frac{b_0}{b_{n-1}}|s|^2\right)\\
&\quad=(s,\mathrm{e}^{i\theta},b_{n-1},b_{n-2},\dots,b_1,b_0).
\end{aligned}
$$

Looking at the other composition,

$$
\begin{aligned}
&\Xi_{2,n}\circ\Xi_{2,n}^{-1}(c_n,c_{n-1},\dots,c_0)\\
&\quad=\Xi_n(s,\mathrm{e}^{i\theta_n},|c_n|^2-|c_0|^2,c_{n-1}\overline{c}_n-c_0\overline{c}_1,c_{n-2}\overline{c}_n-c_0\overline{c}_2,\dots,c_1\overline{c}_n-c_0\overline{c}_{n-1})\\
&\quad=\left(\sqrt{|c_n|^2-|c_0|^2+|s|^2}\mathrm{e}^{i\theta_n},\frac{\sqrt{|c_n|^2-|c_0|^2+|s|^2}\mathrm{e}^{i\theta_n}(c_{n-1}\overline{c}_n-c_0\overline{c}_1)}{|c_n|^2-|c_0|^2}+\frac{\overline{c_1\overline{c}_n-c_0\overline{c}_{n-1}}}{|c_n|^2-|c_0|^2}s,\right.\\
&\quad\left.\frac{\sqrt{|c_n|^2-|c_0|^2+|s|^2}\mathrm{e}^{i\theta_n}(c_{n-2}\overline{c_n}-c_0\overline{c}_2)}{|c_n|^2-|c_0|^2}+\frac{\overline{c_2\overline{c}_n-c_0\overline{c}_{n-2}}}{|c_n|^2-|c_0|^2}s,\dots,s\right)\\
&\quad=\left(|c_n|\mathrm{e}^{i\theta},\frac{|c_n|\mathrm{e}^{i\theta_n}(c_{n-1}\overline{c}_n-c_0\overline{c}_1)}{|c_n|^2-|c_0|^2}+\frac{(\overline{c}_1c_n-\overline{c}_0c_{n-1})c_0}{|c_n|^2-|c_0|^2},\dots,s\right)\\
&\quad=\left(c_n,\frac{c_{n-1}|c_n|^2-|c_0|^2c_{n-1}}{|c_n|^2-|c_0|^2},\frac{c_{n-2}|c_n|^2-|c_0|^2c_{n-2}}{|c_n|^2-|c_0|^2},\dots,s\right)\\
&\quad=(c_n,c_{n-1},\dots,c_0).
\end{aligned}
$$

Step 2. All the fibers $\{R_{2,n}^{-1}(b_{n-1},\dots,b_0)\}$ of (10) are diffeomorphic to the Lie group $\mathbb{C}\times\mathbb{S}^1$ provided with a natural action. Let $P(z)\in\mathscr{D}_{2,n}$ be a polynomial, we recall that $P_1(z)=R_{2,n}(P(z))$ and $b_{n-1}=|c_n|^2-|c_0|^2$. Using (17), (18) and (5), we get the following decomposition (by abusing notation);

$$
\begin{aligned}
P(z)=c_nz^n+\dots+c_1z+c_0\leftrightarrow&(c_n,\dots,c_0)\\
=&\frac{\mathrm{e}^{i\theta_n}}{b_{n-1}}\sqrt{b_{n-1}+|c_0|^2}\Big(|c_n|^2-|c_0|^2,c_{n-1}\overline{c}_n-c_0\overline{c}_1,\dots,c_1\overline{c}_n-c_0\overline{c}_{n-1},0\Big)+\\
&\frac{c_0}{b_{n-1}}\Big(0,\overline{c}_1c_n-\overline{c}_0c_{n-1},\dots,\overline{c}_{n-1}c_n-\overline{c}_0,c_1,|c_n|^2-|c_0|^2\Big)\\
\leftrightarrow&\frac{\mathrm{e}^{i\theta_n}}{b_{n-1}}\sqrt{b_{n-1}+|c_0|^2}zP_1(z)+\frac{c_0}{b_{n-1}}P_1^*(z).
\end{aligned}
$$

The fiber of $P_1(z)=b_{n-1}z^{n-1}+\dots+b_0\in\mathbb{R}^+\times\mathbb{C}^{n-1}$ is

$$
R_{2,n}^{-1}(b_{n-1},\dots,b_0)=\left\{P(z)=\frac{\mathrm{e}^{i\theta_n}}{b_{n-1}}\sqrt{b_{n-1}+|s|^2}zP_1(z)+\frac{s}{b_{n-1}}P_1^*(z)\;\middle|\;s\in\mathbb{C}_s,\ \mathrm{e}^{i\theta_n}\in\mathbb{S}^1\right\}. \tag{19}
$$

Moreover, new coordinates on $\mathscr{D}_{2,n}$ are as follows;

$$
\mathscr{D}_{2,n}=\left\{P(z)=\frac{\mathrm{e}^{i\theta_n}}{b_{n-1}}\sqrt{b_{n-1}+|c_0|^2}zP_1(z)+\frac{c_0}{b_{n-1}}P_1^*(z)\;\middle|\;c_0\in\mathbb{C},\ \ \theta_n\in\mathbb{S}^1,P_1(z)\in\mathbb{R}^+\times\mathbb{C}^{n-1}\right\}. \tag{20}
$$

Consider $s=c_0$ and by choosing fixed $\theta_n=0$, we have

$$
\mathscr{X}=\left\{\mathrm{P}(s,z)=\frac{1}{b_{n-1}}\sqrt{b_{n-1}+|s|^2}zP_1(z)+\frac{s}{b_{n-1}}P_1^*(z)=0\;\middle|\;s\in\mathbb{C}\right\}\subset\mathbb{D}. \tag{21}
$$

On these coordinates, the Lie group action admits a plain expression;

$$\begin{array}{rcl} \mathscr{A}_{2,n} : \mathbb{C} \times \mathbb{S}^1 \times \mathscr{D}_{2,n} & \longrightarrow & \mathscr{D}_{2,n} \\ \left(s, \mathrm{e}^{i\theta}, \frac{\mathrm{e}^{i\theta_n}}{b_{n-1}} \sqrt{b_{n-1} + |c_0|^2} z P_1(z) + \frac{c_0}{b_{n-1}} P_1^*(z)\right) & \longmapsto & \frac{\mathrm{e}^{i(\theta_n+\theta)}}{b_{n-1}} \sqrt{b_{n-1} + |c_0 + s c_n|^2} z P_1(z) + \frac{c_0 + s c_n}{b_{n-1}} P_1^*(z). \end{array} \tag{22}$$

Recalling that $P_1(z) = R_{2,n}(P(z))$, the associative rule is as follows (for any $\mathrm{e}^{i\theta_1},\ \mathrm{e}^{i\theta_2} \in \mathbb{S}^1$);

$$\begin{aligned} &\left(s_1 + s_2, \mathrm{e}^{i\theta_1 + i\theta_2}, \frac{\mathrm{e}^{i\theta_n}}{b_{n-1}} \sqrt{b_{n-1} + |c_0|^2} z P_1(z) + \frac{c_0}{b_{n-1}} P_1^*(z)\right) \longmapsto \\ &\qquad \frac{\mathrm{e}^{i(\theta_n + \theta_1 + \theta_2)}}{b_{n-1}} \sqrt{b_{n-1} + |c_0 + (s_1 + s_2) c_n|^2} z P_1(z) + \frac{c_0 + (s_1 + s_2) c_n}{b_{n-1}} P_1^*(z) \\ &\qquad = \left(s_2, \mathrm{e}^{i\theta_2}, \frac{\mathrm{e}^{i(\theta_n + \theta_1)}}{b_{n-1}} \sqrt{b_{n-1} + |c_0 + s_1 c_n|^2} z P_1(z) + \frac{c_0 + s_1 c_n}{b_{n-1}} P_1^*(z)\right). \end{aligned}$$

There exists a global section

$$\Xi_{2,n} : \mathbb{R}^+ \times \mathbb{C}^{n-1} \longrightarrow \mathscr{D}_{2,n}, \qquad \Xi_{2,n}(0, 1, b_{n-1}, \dots, b_0) = \left(\sqrt{b_{n-1}}, \frac{b_{n-2}}{\sqrt{b_{n-1}}}, \dots, \frac{b_0}{\sqrt{b_{n-1}}}, 0\right),$$

by abusing notation, this section and the map $\Xi_{2,n}(0, 1, \dots, b_0)$ coincide. Hence, the principal fiber bundle in (14) is trivial.
The action in coefficient coordinates is as follows;

$$\begin{aligned} (s, \mathrm{e}^{i\theta}, c_n, c_{n-1}, \dots, c_0) = \Bigg(&\sqrt{|c_n|^2 - |c_0|^2 + |(c_0 + s c_n)|^2} \frac{c_n}{|c_n|} \mathrm{e}^{i\theta}, \\ &\frac{c_n}{|c_n|} \frac{\sqrt{|c_n|^2 - |c_0|^2 + |(c_0 + c_n s)|^2} \mathrm{e}^{i\theta} (c_{n-1} \bar{c}_n - c_0 \bar{c}_1)}{|c_n|^2 - |c_0|^2} + \frac{\overline{c_1 \bar{c}_n - c_0 \bar{c}_{n-1}}}{|c_n|^2 - |c_0|^2} (c_0 + s c_n), \\ &\frac{c_n}{|c_n|} \frac{\sqrt{|c_n|^2 - |c_0|^2 + |(c_0 + s c_n)|^2} \mathrm{e}^{i\theta} (c_{n-2} \bar{c}_n - c_0 \bar{c}_2)}{|c_n|^2 - |c_0|^2} + \frac{\overline{c_2 \bar{c}_n - c_0 \bar{c}_{n-2}}}{|c_n|^2 - |c_0|^2} (c_0 + s c_n), \\ &\dots, \frac{c_n}{|c_n|} \frac{\sqrt{|c_n|^2 - |c_0|^2 + |(c_0 + s c_n)|^2} \mathrm{e}^{i\theta} (c_1 \bar{c}_n - c_0 \bar{c}_{n-1})}{|c_n|^2 - |c_0|^2} + \frac{\overline{c_{n-1} \bar{c}_n - c_0 \bar{c}_1}}{|c_n|^2 - |c_0|^2} (c_0 + s c_n), \\ &(c_0 + s c_n) \Bigg). \end{aligned}$$

Step 3. The description of $R_{1,n}$ follows the analogous steps. In order to show that $R_{1,n}$ is a submersion on the domain $\mathbb{C}^{n+1} \backslash \Sigma_n$, we introduce the following map:

$$\begin{array}{rcl} \Xi_{1,n} : \mathbb{C}^* \times \mathbb{S}^1 \times \mathbb{R}^+ \times \mathbb{C}^{n-1} & \longrightarrow & \mathscr{D}_{1,n} \\ (w, e^{i\theta}, b_{n-1}, \dots, b_0) & \longmapsto & (c_n, \dots, c_0). \end{array} \tag{23}$$

The formula in the right side is

$$\doteq \left(w, \frac{w \bar{b}_1}{b_0} + \frac{\sqrt{b_0 + |w|^2} \mathrm{e}^{i\theta} \bar{b}_{n-1}}{b_0}, \dots, \frac{w \bar{b}_{n-\ell}}{b_0} + \frac{\sqrt{b_0 + |w|^2} \mathrm{e}^{i\theta} \bar{b}_\ell}{b_0}, \dots, \sqrt{b_0 + |w|^2} \mathrm{e}^{i\theta}\right),$$

here $c_n \doteq s$. By assumption $(b_0 + |w|^2) \in \mathbb{R}^+$, and $\Xi_{1,n}$ is a real analytic map.

Recall that if $P_1(z) = R_{1,n}(P(z))$. As a consequence of (18), we have that $\mathscr{D}_{1,n}$ is parametrized as follows:

$$\mathscr{D}_{1,n} = \left\{ P(z) = \frac{c_n}{b_0} z P_1^*(z) + \frac{e^{i\theta_0} \sqrt{b_0 + |c_n|^2}}{b_0} P_1(z) \;\middle|\; c_n \in \mathbb{C}^*,\ \ \theta_0 \in \mathbb{S}^1 \right\}. \tag{24}$$

We consider $w = c_n$ and $\theta_0 = 0$, whence

$$\mathscr{X} = \left\{ \mathtt{P}(w, z) = \frac{w}{b_0} z P_1^*(z) + \frac{\sqrt{b_0 + |w|^2}}{b_0} P_1(z) = 0 \;\middle|\; w \in \mathbb{C}^* \right\} \subset \mathbb{C} \backslash \overline{\mathbb{D}}, \tag{25}$$

note that $\rho(c_0) = \mathrm{e}^{i\theta_0}$ in according to (17). The action is

$$\begin{array}{rcl} \mathscr{A}_{1,n} : \mathbb{C}^* \times \mathbb{S}^1 = \mathscr{D}_{1,n} & \longrightarrow & \mathscr{D}_{1,n} \\ \left(w, \theta, \frac{c_n}{b_0} z P_1^*(z) + \frac{e^{i\theta_0}\sqrt{b_0+|c_n|^2}}{b_0} P_1(z) \right) & \longmapsto & \frac{c_n w}{b_0} z P_1^*(z) + \frac{e^{i(\theta_0+\theta)}\sqrt{b_0+|c_n w|^2}}{b_0} P_1(z). \end{array} \tag{26}$$

The computation follows as above, and the trivial principal $(\mathbb{C}^* \times \mathbb{S}^1)$–bundle in (13) is done. $\square$

5 n–root configuration dynamics: a prototype

Our goal in this section is to explore Lie group actions on the space of complex polynomials of n degree, regarding the respective n–root configuration dynamics. We consider two ingredients in order to construct our main definition 5.1.

5.1 Lie group actions on the space of polynomials $\mathbb{C}[z]_{\leqslant n}$ and n–root configurations

Let G be a real (resp. complex) Lie group, we are mainly interested when G is $(\mathbb{S}^1 = \{\mathrm{e}^{i\theta}\}, \cdot)$, $(\mathbb{C}_s, +)$ and $(\mathbb{C}^*_w, \cdot)$. Let $\mathscr{A}$ be a real (resp. complex) analytic action of G on the complex manifold of polynomials of degree at most n,

$$\begin{array}{rcl} \mathscr{A} : G \times \mathbb{C}[z]_{\leqslant n} & \longrightarrow & \mathbb{C}[z]_{\leqslant n} \\ (g, P) & \longmapsto & g \cdot P(z) \doteq \mathtt{P}(s,z). \end{array} \tag{27}$$

The associated principal G–bundle is

$$\begin{array}{ccc} G & \xrightarrow{\mathtt{i}} & \mathscr{D}^0 \subseteq \mathbb{C}[z]_{=n} \\ & & \Big\downarrow \Pi \\ & & \mathscr{D}^0/G, \end{array}$$

here $\mathscr{D}^0$ is the open manifold where $\mathscr{A}$ is proper and free from fixed points, see [13] theorem 1.11.4, chapter 1. Now we look at the fibers of the G–bundle. The Weierstrass preparation theorem [18] pp. 8 and the work of V. L. Hansen [19], [20] give origin to the following concept.

A *Weierstrass polynomial of degree $n \geqslant 1$ over a Lie group G* is a map

$$\mathtt{P}(g,z) = c_n(g)z^n + \ldots + c_0(g) : G \times \mathbb{C} \longrightarrow \mathbb{C}, \tag{28}$$

where $c_\iota(g) : G \to \mathbb{C}$ are real analytic functions, $\iota \in \{0, \ldots, n\}$, and $c_n(g)$ is nonidentically zero. Furthermore, if G is a complex Lie group, then we require $c_\iota(g)$ to be complex analytic.

a) Let

$$\{\mathtt{P}(\theta,z) = z^3 - \mathrm{e}^{i\theta} \mid \theta \in \mathbb{S}^1\}$$

be a Weierstrass polynomial of degree 3 over a Lie group $\mathbb{S}^1$. If θ varies over the whole circle, the 3–root configurations of $\{\mathtt{P}(\theta,z) = 0\}$ describe three arcs of angle $2\pi/3$ in $\mathbb{C}_z$. See figure 5.1.a and [28] for a general study of these kinds of knots and braids in $\mathbb{C}^2$.

b) Consider $P(z)$ a complex polynomial of degree $n \geqslant 2$, and let

$$\{\mathtt{P}(s,z) = P(z) - s \mid s \in \mathbb{C}\} \tag{29}$$

be a Weierstrass polynomial of degree $n \geqslant 1$ over the additive Lie group $\mathbb{C}$. If s varies in a neighborhood of regular values of $P(z)$, then the n–root configurations of $\{P(z) - s = 0\}$ do not change topologically. If s varies in a neighborhood of a critical value s_0 of $P(z)$, then the n–root configuration of $\{P(z) - s = 0\}$ changes topologically at s_0. See figure 5.1.b. We will describe both situations accurately in lemma 5.2.

In this work, we convene that each orbit of an action $\mathscr{A}$ determines a Weierstrass polynomial over G,

$$\mathtt{P}(g,z) = \{g \cdot P(z) \mid g \in G\} \quad \longrightarrow \quad \{G \times \mathbb{C} \longrightarrow \mathbb{C}, \ (g,s) \longmapsto \mathtt{P}(g,z) = c_n(g)z^n + \ldots + c_0(g)\}, \tag{30}$$

in the sense of (28). Clearly the converse is not true; a Weierstrass polynomial over G as in (28) does not always come from the orbit of a suitable G–action.

Let $\{\mathtt{P}(g,z)\}$ be a Weierstrass polynomial. For $c_n(g_0) \neq 0$, the *n–root configuration* is

$$\{\mathtt{P}(g_0,z) = 0\} = [z_1(g_0), \ldots, z_n(g_0)]$$

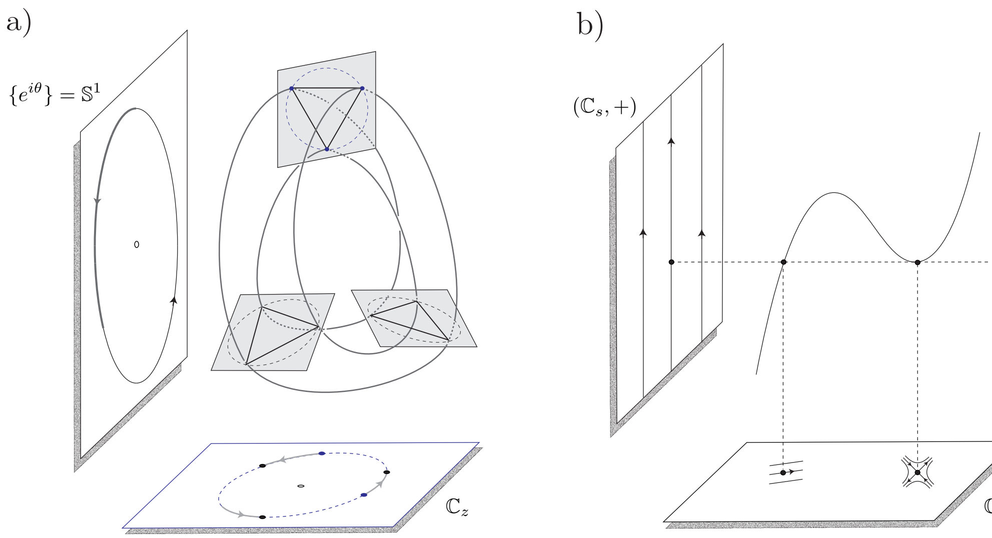

Figure 1: Two families of polynomials. a) A family over the circle $\mathbb{S}^1$. b) A family over the plane $(\mathbb{C}_s, +)$. The phase portraits sketched in $\mathbb{C}_z$ describe the n–root configuration dynamics, respectively.

or simply $[z_1, \ldots, z_n]$. The associated *zero locus* is

$$\mathscr{Z} = \{\mathtt{P}(g,z) = 0\} \subset G \times \mathbb{C}.$$

As in [19], [20], the induced *n–fold branched polynomial covering map* Π is

$$\begin{array}{ccccc} \mathscr{Z} & \xhookrightarrow{\ \ I\ \ } & G \times \mathbb{C} & \xrightarrow{proj_2} & \mathbb{C}. \\ & \Pi \searrow & \swarrow proj_1 & & \\ & & G & & \end{array} \tag{31}$$

Let $P(z)$ be a polynomial of degree $n \geqslant 1$. For $s \in \mathbb{C}$, the zero locus $\mathscr{Z} = \{P(z) - s = 0\}$ is the graph of $P(z)$.

The second ingredient is the dynamics of vector fields. Let $\mathbb{X}(z) = f(z)\frac{\partial}{\partial z}$ be a *singular complex analytic vector field* on $\mathbb{C}_z$ or $\widehat{\mathbb{C}}$; the word singular means that $f(z)$ can admit as singularities zeros, poles $\mathscr{P}$ and isolated essential singularities $\mathscr{E}$; see [5]. If $\mathbb{X}$ is holomorphic at z, then the vector field $\mathbb{X}$ determines a (local) holomorphic flow;

$$\varphi : \mathbb{D}^2\big((s_0,z_0),\varepsilon\big) \subset \big(\mathbb{C}_s \times \mathbb{C}_z \backslash (\mathscr{P} \cup \mathscr{E})\big) \longrightarrow \mathbb{C}_z, \quad (s,z) \longmapsto \varphi(s,z), \tag{32}$$

here s is the complex time, $\mathbb{D}^2\big((s_0,z_0),\varepsilon\big)$ is an open bidisk having center at (s_0,z_0) and radius $\varepsilon > 0$.
Given a continuous time path $\gamma(\tau) : [0,1] \to \mathbb{C}_s$ with $\varphi(\gamma(0),z_0) = z_0 \in \mathbb{C}_z \backslash \{\mathscr{P} \cup \mathscr{E}\}$, we consider the analytic continuation of the flow $\varphi(\gamma(\tau),z_0)$ starting at the initial condition z_0. Depending on the path and the initial condition, the analytic continuation of φ can be well–defined or not.
By abusing notation, $\varphi(s,z)$ will denote the flow[2] of $\mathbb{X}$ over the maximal domain, using the analytic continuation process.

The coupling between the two ingredients, an action $\mathscr{A}$ (their orbits as Weierstrass polynomials $\mathtt{P}(s,z)$) and the local flow $\varphi(s,\ )$ from a vector field $\mathbb{X}$, is given by the Viète map as follows.

Let $\mathscr{A} : G \times \mathbb{C}[z]_{=n} \longrightarrow \mathbb{C}[z]_{=n}$ be a $\mathbb{C}_s$ or $\mathbb{C}^*_w$–action and a Weierstrass polynomial and their roots $\big\{\mathtt{P}(id,z) = 0\big\} \doteq [z_1, \ldots, z_n]$, coming from $\mathscr{A}$. A complex analytic vector field $\mathbb{X}$ on $\mathbb{C}_z$ *describes the n–root configuration dynamics* of $\big\{\mathtt{P}(g,z) = 0\big\}$ if

$$\mathtt{P}(g,z) = c_n(g)\big(z - \varphi(g,z_1)\big) \cdots \big(z - \varphi(g,z_n)\big),$$

whenever the analytic continuation of the local flow $\varphi(g,\ )$ of $\mathbb{X}$ is well–defined, for a suitable function $c_n(g)$.

[2] As usual, an *action* of a Lie group G on a manifold, $G \times M \longrightarrow M$, is well–defined for all the pairs (g,p), whereas a *local action* is defined only for a certain open subset of pairs $\{(g,p)\}$; the analogous notation is suitable for real and complex flows as $\mathbb{R}$ and $\mathbb{C}$ actions.

In the case of monic polynomials, definition 5.1 is equivalent to the fact that the diagram

$$
\begin{array}{ccc}
[z_1,\dots,z_n] & \xrightarrow{\mathscr{V}_n} & \mathtt{P}(id,z)=c_n(id)z^n+\ldots+c_0(id)\\
\Big\downarrow \varphi(g,\ \ ) & & \Big\downarrow \mathscr{A}(g,\ \ )\\
[\varphi(g,z_1),\dots,\varphi(g,z_n)] & \xleftarrow[\mathscr{V}_n^{-1}]{} & \mathtt{P}(g,z)=c_n(g)z^n+\ldots+c_0(g)
\end{array}
\tag{33}
$$

commutes. Here $\mathscr{V}_n$ is the Viète map and $\{\mathtt{P}(id,z)=0\}=[z_1,\dots,z_n]$ is considered as initial condition for $\varphi(g,\ \ )$. The diagram (3) is a particular case.

5.2 The prototype

Reviewing example 5.1.2 and definition 5.1, we want to show that each n–point configuration $[z_1,\dots,z_n]\subset\mathbb{C}$ determines the n–root configuration dynamics of a suitable Weierstrass polynomial $\{P(g,z)=0\}$.

(The prototype) Let

$$
\begin{array}{rcl}
\mathscr{A}:\mathbb{C}_s\times\mathbb{C}[z]_{=n} & \longrightarrow & \mathbb{C}[z]_{=n}\\
(s,c_nz^n+\ldots+c_1z+c_0) & \longmapsto & c_nz^n+\ldots+c_1z+(c_0+s)
\end{array}
\tag{34}
$$

be a holomorphic action. For each orbit

$$\{\mathtt{P}(s,z)=c_n\Pi_{\iota=1}^n(z-z_\iota)-s \mid s\in\mathbb{C}\},$$

the rational vector field

$$\mathbb{X}(z)=\tfrac{1}{P(z)'}\tfrac{\partial}{\partial z}\quad\text{on } \mathbb{C}_z$$

describes the n–root configuration dynamics of $\big\{\mathtt{P}(s,z)=0\big\}=[z_1,\dots,z_n]$.

Let $P(z)=c_n\Pi_{\iota=1}^n(z-z_\iota):\mathbb{C}_z\longrightarrow\mathbb{C}_s$ be a polynomial determined by a configuration of simple roots $[z_1,\dots,z_n]$ and $c_n\in\mathbb{C}^*$, by construction 0 is a regular value of $P(z)$.
Let $\{\mathtt{P}(s,z)=\mathtt{P}(z)+s=0\}$ be the n–root configurations of the orbit of $P(z)$ under $\mathscr{A}$.
We associate with $P(z)$ a polynomial 1–form and a complex rational vector field, as follows;

$$P(z)\longleftrightarrow\ \ \omega(z)=P'(z)dz\ \ \longleftrightarrow\ \ \mathbb{X}(z)=\frac{1}{P'(z)}\frac{\partial}{\partial z}.$$

In addition, ω and $\mathbb{X}$ enjoy the two (equivalent) properties

$$\omega(\mathbb{X})\equiv 1\quad\text{and}\quad P_*(\mathbb{X})=\tfrac{\partial}{\partial s},$$

here P_* denotes the pushforward and s is the time of the associted ordinary differential equation $\frac{dz}{ds}=(P'(z))^{-1}$.
Consider n open disks $\mathbb{D}(z_\iota,\varepsilon)\subset\mathbb{C}_z$ with center at z_ι and a small enough radius $\varepsilon>0$, $\iota\in\{1,\dots,n\}$ such that

$$P(z):\mathbb{D}(z_\iota,\varepsilon)\subset\mathbb{C}_z\longrightarrow\mathbb{C}_s$$

are local biholomorphisms. The local flows $\{\varphi(s,z_\iota)\mid\iota\in1,\dots,n\}$ of $\mathbb{X}$, for complex time $\{s\,\big|\,|s|<\varepsilon\}$, are well–defined.
Using $\omega(\mathbb{X})\equiv1$, follows that

$$P(\varphi(s,z_\iota))=s\quad\text{for all }\iota\in\{1,\dots,n\}.$$

The diagram (33) commutes whenever the analytic continuation of $\varphi(s,\ \ )$ is well–defined. □

Let us recall a principal fiber bundle interpretation of the prototype. Given the Lie group action $\mathscr{A}$ in (34), the associated holomorphic trivial principal $\mathbb{C}$–bundle is

$$
\begin{array}{ccc}
\mathbb{C}_s & \xrightarrow{I} & \mathbb{C}[z]_{=n}\\
 & & \Big\downarrow \Pi\\
 & & \mathbb{C}^n=\{(c_n,\dots,c_1)\},
\end{array}
$$

here $\Pi:(c_n,\dots,c_0)\longmapsto(c_n,\dots,c_1)$, and I denotes the embedding of a fiber in the total space.

6 n–root configuration dynamics: the rational case

6.1 Complex rational vector fields on the Riemann sphere $\widehat{\mathbb{C}}$

We enlarge the description of $\mathbb{X}$ in the prototype in order to consider rational vector fields. Let

$$\mathbb{X}(z)=\frac{P(z)}{Q(z)}\frac{\partial}{\partial z}=\frac{\lambda(z-p_1)^{\mu_1}\cdots(z-p_\nu)^{\mu_\nu}}{(z-q_1)^{\kappa_1}\cdots(z-q_m)^{\kappa_m}}\frac{\partial}{\partial z}\quad\text{on}\quad\widehat{\mathbb{C}}=\mathbb{C}_z\cup\{\infty\}\tag{35}$$

be a rational vector field; here $P(z)$, $Q(z)$ are polynomials without common factors and $\kappa,\ \mu \leqslant 0$ their respective degrees. The vector field $\mathbb{X}$ extends rationally to the whole Riemann sphere $\widehat{\mathbb{C}}$. Moreover, a simple calculation shows that

$$\mu_1 + \ldots + \mu_\nu - \kappa_1 - \ldots - \kappa_m = \mu - \kappa = 2$$

on $\widehat{\mathbb{C}}$. This is equivalent to the fact that the point $\infty \in \widehat{\mathbb{C}}$ is of multiplicity $2 - \kappa + \mu \in \mathbb{Z}$ for $\mathbb{X}$ (we convene that the multiplicity of poles of $\mathbb{X}$ is negative).

Following [7] pp. 579, a singular complex analytic (probably multivalued) function $\psi : \mathbb{C} \longrightarrow \widehat{\mathbb{C}}$ is called *additively automorphic* if $d\psi$ is a univalued singular complex analytic 1–form on $\mathbb{C}$, *i.e.* for ψ_α, ψ_β any two branches of ψ,

$$\psi_\alpha(z) = \psi_\beta(z) + a_{\alpha\beta}, \quad a_{\alpha\beta} \in \mathbb{C}.$$

A concrete example is $\psi(z) = ln(z)$.

[25], [26], [30] On the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C}_z \cup \{\infty\}$, there are canonical correspondences between the following:

1. Complex analytic vector fields $\mathbb{X}(z)$.
2. Complex analytic 1–forms $\omega(z)$.
3. Multivalued additively automorphic complex analytic functions $\Psi(z)$ having a rational derivative.
4. Weierstrass polynomials $\{\psi(z) - s \mid s \in \mathbb{C}\}$.

Using (35), we define $\Psi(z) \doteq \int_{z_0}^{z} \omega : \widehat{\mathbb{C}} \backslash \{p_\iota\}_{\iota=1}^{\nu} \longrightarrow \mathbb{C}_t$. Diagrammatically, we have

$$\begin{tikzcd}
 & \mathbb{X}(z) = \frac{P(z)}{Q(z)}\frac{\partial}{\partial z} \arrow[dl, leftrightarrow] \arrow[dr, leftrightarrow] & & \\
\omega(z) = \frac{P(z)}{Q(z)}dz \arrow[rr, leftrightarrow] & & \Psi(z) = \int^z \frac{P(\zeta)}{Q(\zeta)}d\zeta \arrow[r, leftrightarrow] & \{\Psi(z) = s\}.
\end{tikzcd} \tag{36}$$

□

Recall that $\Psi(z) = ln(z)$ provides an example with $\mathbb{X}(z) = z\frac{\partial}{\partial z}$ rational, hence the enlarged hypothesis, multivalued additively automorphic complex analytic functions, for $\Psi(z)$ is useful.

Recall that a singular complex analytic vector field $\mathbb{X}$ on $\mathbb{C}$ is *complete* when its flow is well–defined for all initial conditions and for all complex time s. However, it is a very restrictive condition; as is well–known, $\mathbb{X}$ is complete on $\mathbb{C}$ if and only if $\mathbb{X}(z) = (bz + c)\frac{\partial}{\partial z}$; see [24]. Moreover, the comprehension of rational vector fields (and their flows) at poles is required.

(Complex analytic normal forms at poles and zeros of vector fields) Let $\mathbb{X}$ be a rational vector field germ on $(\mathbb{C}, 0)$.

1. If 0 is a pole of order $-\kappa \leqslant -1$ for $\mathbb{X}$, then it is holomorphically equivalent to
$$z^{-\kappa}\frac{\partial}{\partial z}.$$
2. If 0 is a zero of order one for $\mathbb{X}$, then it is holomorphically equivalent to
$$\lambda z\frac{\partial}{\partial z}, \quad \text{where } \lambda = \mathbb{X}'(0).$$
3. If 0 is a zero of order $\mu \geqslant 2$ for $\mathbb{X}$, then it is holomorphically equivalent to
$$\frac{z^\mu}{1+\lambda z^{\mu-1}}\frac{\partial}{\partial z},$$
here $\lambda \in \mathbb{C}$ is the residue of the associated rational differential 1–form $\omega_{\mathbb{X}}$ at 0.

The result is well–known [16], [17], [30]. See also figure 2. □

Now we introduce how the flow singularities of a rational vector field $\mathbb{X}$ on $\widehat{\mathbb{C}}$ can be related to the n–root configuration dynamics of a Weierstrass polynomial $\{\mathtt{P}(s,z) = 0\}$.

6.2 A dictonary

A dictionary between singular points of vector fields and n–root configuration dynamics.

Let $\mathbb{X}$ be a complex analytic vector field on the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C}_z \cup \{\infty\}$ describing the n–root configuration dynamics of a Weierstrass polynomial $\{\mathtt{P}(s,z) = 0\}$ of degree at most n.

1. A *fixed root* of $\{\mathtt{P}(s,z) = 0\}$ is a finite zero $p \in \mathbb{C}$ of $\mathbb{X}$ such that is root of $\{\mathtt{P}(s_0, p) = 0\}$ for a s_0.
2. An *unattainable root* of $\{\mathtt{P}(s,z) = 0\}$ is a finite zero $p \in \mathbb{C}$ of $\mathbb{X}$ such that $\mathtt{P}(s_0, p) \neq 0$ for all $s_0 \in \mathbb{C}$.
3. A *collision of* $(\kappa+1)$*–roots* of $\{\mathtt{P}(s,z) = 0\}$ is a finite pole $q \in \mathbb{C}$ of $\mathbb{X}$ having order $-\kappa \leqslant -1$.
4. $\infty \in \widehat{\mathbb{C}}$ a regular point ($\kappa = 0$) or a pole of order $-\kappa \leqslant -1$ of $\mathbb{X}$ determines *a* $1+\kappa$ *reduction of the degree of* $\{\mathtt{P}(s,z) = 0\}$.

The four concepts in definition 6.2 are justified by using the normal forms of $\mathbb{X}$ as follows.

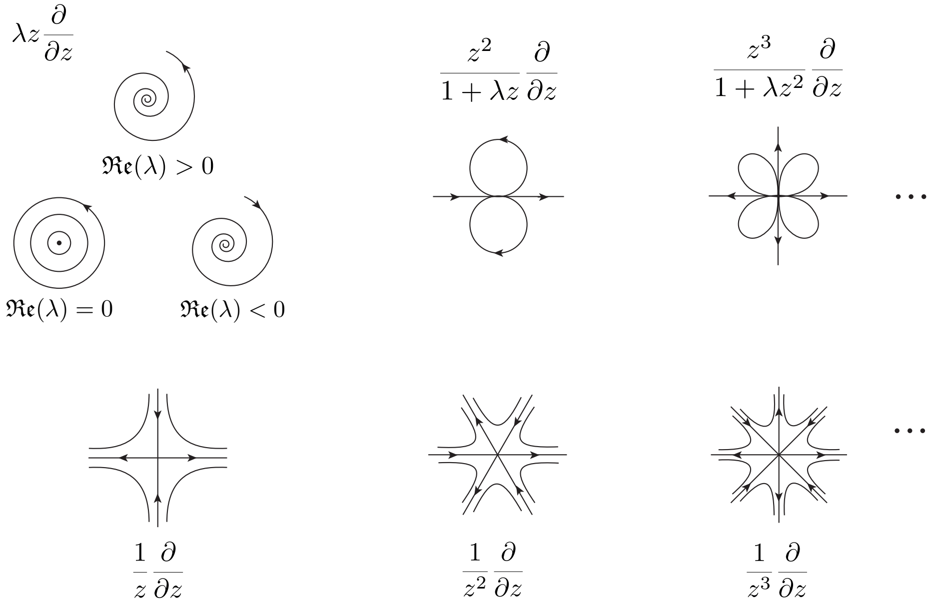

Figure 2: Topological phase portraits of $\mathfrak{Re}\,(\mathbb{X}) \doteq \mathfrak{Re}\,(f(z))\frac{\partial}{\partial x} + \mathfrak{Im}(f(z))\frac{\partial}{\partial y}$ for the complex analytic normal forms of poles and zeros of $\mathbb{X} = f(z)\frac{\partial}{\partial x}$.

Let $[z_1,\dots,z_n] \subset \mathbb{C}$ be the zeros of $\{\mathtt{P}(s_0,z)=0\}$ for a value s_0. Assume that p is a zero of $\mathbb{X}$ and is a zero of $\{\mathtt{P}(s_0,z)=0\}$. Then p is *fixed* under the flow of $\mathbb{X}$.
Let $[z_1,\dots,z_n] \subset \mathbb{C}$ be the zeros of $\{\mathtt{P}(s_0,z)=0\}$ for a value s_0. Assume that p is a zero of $\mathbb{X}$ and does not a zero of $\{\mathtt{P}(s_0,z)=0\}$. Clearly for all initial condition $z_0 \notin \{z_1,...,z_n\}$, we have that

$$\varphi(s_1,z_0) \neq p$$

for every analytic continuation of the flow of $\mathbb{X}$ along a continuous time path $\gamma(\tau):[0,1] \longrightarrow \mathbb{C}_s$ with $\gamma(0)=z$.
Hence, p_ℓ is *an unattainable root* of $\{\mathtt{P}(s,z)=0\}$.
Let $\{\mathtt{P}(g,z)=0\}$ be Weierstrass polynomial. When exists a complex analytic vector field $\mathbb{X}$ on $\mathbb{C}_z$ describing the n–root configurations of $\{\mathtt{P}(g,z)=0\}$?

The following result gives an answer.

Let $\{\mathtt{P}(s,z)\}$ be a holomorphic Weierstrass polynomial on the complex Lie group $\mathbb{C}_s$.
If the zero locus $\mathscr{Z} = \{\mathtt{P}(s,z)=0\}$ can be described as the graph of a rational function

$$s = \Psi(z) : \mathbb{C}_z \longrightarrow \mathbb{C}_s,$$

then there exists a rational vector field $\mathbb{X}$ on $\mathbb{C}_z$ that describes the n–root configuration dynamics of $\{\mathtt{P}(s,z)=0\}$, even if the flow of $\mathbb{X}$ is non holomorphic.

In this result, the existence of a Lie group action is unnecessary.

Starting with $s=\Psi(z)$, the correspondence (36) provides a vector field $\mathbb{X}$ that is not necessarily holomorphic.

For the study at the poles of $\mathbb{X}$ using local holomorphic coordinates $(\mathbb{C}=\{z\},0)$ provided by the local normal forms, we can assume that the Weierstrass polynomial is $\{\mathtt{P}(z,s)=z^n-s=0\}$, locally. By using the correspondence (36) (that is well–defined up to change of coordinates), we have

$$\mathbb{X}(z) = \frac{1}{nz^{n-1}}\frac{\partial}{\partial z} \quad\longleftrightarrow\quad \omega(z) = nz^{n-1}dz \quad\longleftrightarrow\quad \Psi(z) = z^n. \tag{37}$$

The n–th roots of the unity $[\mathrm{e}^{2\pi i/n},\mathrm{e}^{4\pi i/n},\dots,\mathrm{e}^{2n\pi i/n}=1]$ collide at z_0 for time $s_0 = \int_1^0 n\zeta^{n-1}d\zeta$.

Despite the non–holomorphicity of the vector field $\mathbb{X}$ and its flow in (37), the Weierstrass polynomial $\{\mathtt{P}(s,z) \mid s \in \mathbb{C}\}$ is holomorphic on both variables and the n–root dynamics is well–defined for all complex time $s \in \mathbb{C}$. See figure 6.2. $\square$

Recall that the study of collisions in the n–body problem is a hard subject; in flat billiards the trajectories arriving to the singularities are usually removed, since they have zero measure. In our case, root collisions are natural and easy to be described.

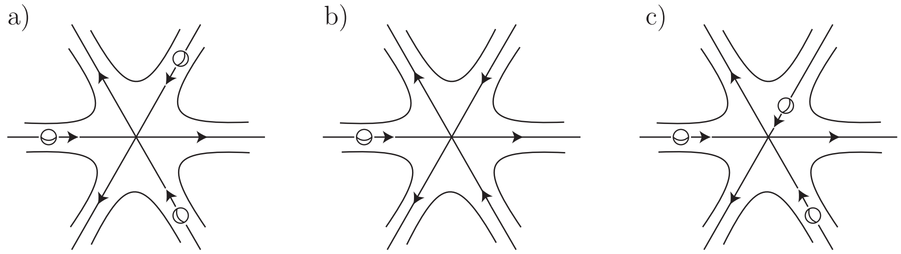

Figure 3: Let $\varphi(s,z)$ be the holomorphic flow of $\mathbb{X}(z) = \frac{1}{3z^2}\frac{\partial}{\partial z}$. a) The three points $[-1, \mathrm{e}^{5\pi i/3}, \mathrm{e}^{2\pi i/6}]$ from $\{z^3 - 1 = 0\}$ collide at $z_0 = 0$ for real positive time $s_0 = 1$ under $\varphi(s,\ )$ of $\mathbb{X}$, and the polynomial $\mathtt{P}(s,z) = \prod(z - \varphi(s,z_i))$ is holomorphic for all complex time $s \in \mathbb{C}$. b) However for only one starting point $z_1 = -1$ the flow $\varphi(s,\ )$ and the respective polynomial $(z - \varphi(s,z_1)$ do not make sense for real time $s > 1$. c) For three starting points in arbitrary positions (different from 0), the resulting polynomial $\prod(z - \varphi(s,z_i))$ is not well–defined for certain time values $\{s_j\}$.

Let

$$\mathscr{A} : \mathbb{C}_s \times \mathbb{C}[z]_{=2} \longrightarrow \mathbb{C}[z]_{=2}, \quad (s, c_2 z^2 + c_1 z + c_0) \longmapsto c_2 z^2 + (c_1 + s)z + c_0$$

be a holomorphic action. Consider the orbit $\{\mathtt{P}(s,z) = z^2 - sz + 1\}$. We apply the correspondence (36);

$$\mathbb{X}(z) = \frac{z^2}{z^2-1}\frac{\partial}{\partial z} \quad \longleftrightarrow \quad \omega(z) = \frac{z^2-1}{z^2}dz \quad \longleftrightarrow \quad \Psi(z) = \frac{z^2+1}{z} = s.$$

This vector field describes the 2–root configuration dynamics of $\{\mathtt{P}(s,z) = 0\}$.
The point $z_1 = 0$ is an unattainable root of $\{\mathtt{P}(s,z) = 0\}$.
Two collisions of 2–root appear at $z = 1, -1$; they correspond to the poles of $\mathbb{X}$.
The point ∞ is a double zero of $\mathbb{X}$, an the degree of $\{\mathtt{P}(s,z) = 0\}$ remains 2 for all complex time s.
On the other hand, if we consider the orbit $\{\mathtt{P}(s,z) = z^2 - sz\}$. Then the vector field $\mathbb{X}(z) = \frac{\partial}{\partial z}$ describes the 2–root configuration dynamics of it; that is $z_1 = 0$ is a fixed root and the second root $z_2 = s_0$ moves linearly with respect to the s–time flow of $\mathbb{X}$.
Let $\mathbb{X}$ be a vector field describing the n–root configuration dynamics of a Weierstrass polynomial. Recall the figure 2, in a small enough neighborhood of a pole q of order $-\kappa$ of X, there are $\{z_1, \ldots, z_{\kappa+1}\}$ roots (*i.e.* initial conditions) such that

$$\varphi(s_0, z_1) = \ldots = \varphi(s_0, z_{k+1}) = q \quad \text{for a suitable time } s_0 \in \mathbb{C},$$

whence the pole determines the collision of $(\kappa+1)$–roots of $\{\mathtt{P}(s,z) = 0\}$. In other words, $\kappa + 1$ simple roots give origin to a root of multiplicity $\kappa + 1$.

A Weierstrass polynomial (coming from a holomorphic Lie group action) such that does not exist a univalued complex analytic vector field $\mathbb{X}$ describing its roots. The Weierstrass polynomial $\{\mathrm{e}^s z + s = 0\}$ is an orbit of the action

$$\begin{array}{rcl} \mathscr{A} : (\mathbb{C}, s) \times \mathbb{C}[z]_{\leqslant 1} & \longrightarrow & \mathbb{C}[z]_{\leqslant 1} \\ (s, c_1, c_0) & \longmapsto & (\mathrm{e}^{-s} c_1, c_0 - s). \end{array}$$

The orbit of $(c_1, c_0) = (1, 0)$ has zero locus

$$\mathscr{X} = \{\mathrm{e}^{-s} z - s = 0\} = \{s\mathrm{e}^s - z = 0\},$$

which is the graph of the multivalued non additively automorphic Lambert–W function $s = W(z)$. We would want to settle $W(z) = \Psi(z)$; see R. M. Corless *et al.* [12]. The hypothesis in proposition 6.2 does not hold, since

$$\frac{dW}{dz} = \frac{W(z)}{z(1 + W(z))} \quad \text{for } z \notin 0,\ -1/\mathrm{e}.$$

The diagram (36) produces a multivalued complex analytic 1–form $\omega = d\Psi(z)$, whence the vector field $\mathbb{X}$ on $\mathbb{C}_z$ shares the mutivaluedness.

(Transcendental functions $s = \Psi(z)$).
1. The Weierstrass polynomial $\{\mathrm{e}^s z + \mathrm{e}^{-s} = 0\}$ comes from a holomorphic $\mathbb{C}$–action on $\mathbb{C}[z]_{\leqslant 1}$. Moreover, by using correspondence (36), there exists a complex analytic vector field $\mathbb{X}(z) = -2z\frac{\partial}{\partial z}$ on $\mathbb{C}_z$, which describes its roots dynamics.
2. Let $s = \Psi(z) = \ln(z)$ be a multivalued additively complex analytic function. Using the correspondence (36), the rational vector field is $\mathbb{X}(z) = z\frac{\partial}{\partial z}$ and the Weierstrass polynomial is $\{\mathrm{e}^s - z = 0\}$.
See proposition 7.2 for a generalization of both examples.

7 Vector fields describing n–root dynamics from Lie group actions of $\mathbb{C}$ and $\mathbb{C}^*$

Suitable tools for the study of Schur and anti–Schur polynomials are given in the following two subsections.

7.1 Actions on $\mathbb{C}[z]_{\leqslant n}$ by translations

(Action by translations on $\mathbb{C}[z]_{\leqslant n}$) Let $Q \in \mathbb{C}[z]_{\leqslant n}$ be a polynomial. There exists a correspondence between the following objects.
1. A holomorphic Lie group action

$$\begin{array}{rcl} \mathscr{A} : \mathbb{C}_s \times \mathbb{C}[z]_{\leqslant n} & \longrightarrow & \mathbb{C}[z]_{\leqslant} n \\ (s, P(z)) & \longmapsto & P(z) - sQ(z). \end{array}$$

2. A family of Weierstrass polynomials (obits of $\mathscr{A}$)

$$\Big\{ \mathtt{P}(s,z) = P(z) - sQ(z) \;\Big|\; s \in \mathbb{C} \Big\}$$

over the complex Lie group $\mathbb{C}_s$.
3. A family of rational maps

$$\Big\{ \Psi(z) = \frac{P}{Q}(z) + s : \mathbb{C}_z \longrightarrow \mathbb{C}_s \;\Big|\; P(z) \in \mathbb{C}[z]_{=n}, s \in \mathbb{C} \Big\}.$$

4. A family of rational vector fields

$$\Big\{ \mathbb{X}(z) = \frac{Q^2(z)}{P'(z)Q(z) - Q'(z)P(z)} \frac{\partial}{\partial z} \;\Big|\; P(z) \in \mathbb{C}[z]_{=n} \Big\} \quad \text{on} \quad \widehat{\mathbb{C}}_z. \tag{38}$$

In (2), the polynomial $P(z)$ must be considered as the initial condition for an orbit $\{P(z) - sQ(z) \mid s \in \mathbb{C}\}$ of the action. For each rational function $\Psi(z) \doteq \frac{P}{Q}(z) = s$, the diagram (36) provides the correspondence (2)–(4). □

Note that a rational vector field $\mathbb{X}$ on $\widehat{\mathbb{C}}$ as in (38) has a local flow $\varphi : \Omega \times \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{C}}$ such that

$$\frac{\Pi_\iota(z - \varphi(s, z_\iota(s)))}{Q(z)} = \frac{P(z)}{Q(z)} = s.$$

Recalling definition (6.2), the zeros $\{Q(z) = 0\}$ are fixed points under the local flow of $\mathbb{X}$.

Let us consider the action

$$\mathscr{A} : \mathbb{C}_s \times \mathbb{C}[z]_{\leqslant 2} \longrightarrow \mathbb{C}[z]_{\leqslant} 2, \quad (s, c_2z^2 + c_1z + c_0) \longmapsto (c_2 + s)z^2 + c_1z + c_0.$$

For $c_0 = 0$, $\mathscr{A}$ gives origin to the subfamily of Weierstrass polynomials $\{\mathtt{P}(s,z) = (c_2 + s)z^2 + c_1z \mid s \in \mathbb{C}\}$, and the associated vector fields are

$$\mathbb{X} = \tfrac{z^2}{c_1} \tfrac{\partial}{\partial z}.$$

Note that $z = 0$ is a fixed root of all $\{\mathtt{P}(s,z) = 0\}$. For $s = -c_2$, the moving root escapes to infinity.

7.2 Linear actions on $\mathbb{C}[z]_{\leqslant n}$

Let us consider a linear action $\mathscr{A}$ of $(\mathbb{C}, +)$ on $\mathbb{C}[z]_{\leqslant n} = \{(c_n, \ldots, c_1, c_0)\}$, with $\{z^n, \ldots, z, 1\}$ as a basis, such that its infinitesimal generator is the linear holomorphic vector field

$$\sum_{j=0}^{n} \lambda_j c_j \frac{\partial}{\partial c_j}, \qquad \lambda_j \in \{\lambda_\alpha, \lambda_\beta\} \subset \mathbb{C}^*, \tag{39}$$

which has two different eigenvalues, at least one of them nonzero. All polynomial in $\mathbb{C}[z]_{\leqslant n}$ can be written as

$$P(z) = \sum_{j=0}^{n} c_j z^j = \sum_j c_{j(\alpha)} z^{j(\alpha)} - \sum_j c_{j(\beta)} z^{j(\beta)} = R(z) - Q(z),$$

by definition $R(z),\ Q(z)$ belong to the eigenspaces of $\lambda_1,\ \lambda_2$. Under (39), each orbit is $\{\mathrm{e}^{\lambda_\alpha s} R(z) - \mathrm{e}^{\lambda_\beta s} Q(z) \mid s \in \mathbb{C}\}$.

(Linear actions with two different nonzero eigenvalues) The following objects are equivalent.
1. A holomorphic Lie group action

$$\begin{array}{rcl}\mathscr{A}:\mathbb{C}_s\times\mathbb{C}[z]_{\leqslant n} & \longrightarrow & \mathbb{C}[z]_{\leqslant n}\\ \big(s,P(z)\big) & \longmapsto & \mathrm{e}^{\lambda_\alpha s}R(z)-\mathrm{e}^{\lambda_\beta s}Q(z),\end{array}$$

with an infinitesimal generator as in (39).
2. Families of Weierstrass polynomials

$$\Big\{\mathrm{P}_{R,Q}(s,z)=\mathrm{e}^{\lambda_\alpha s}R(z)-\mathrm{e}^{\lambda_\beta s}Q(z)\ \Big|\ s\in\mathbb{C},\ Q(z),R(z)\in\mathbb{C}[z]_{\leqslant n}\Big\}$$

over the complex Lie group $\mathbb{C}_s$.
3. Families of multivalued additively automorphic functions

$$\left\{\Psi_{R,Q}(z)=\frac{1}{\lambda_\alpha-\lambda_\beta}ln\Big(\frac{R(z)}{Q(z)}\Big)\ \Big|\ s\in\mathbb{C},\ Q(z),R(z)\in\mathbb{C}[z]_{\leqslant n}\right\}.$$

4. A family of rational vector fields

$$\left\{\mathbb{X}(z)=\frac{R(z)Q(z)}{Q'(z)R(z)-R'(z)Q(z)}\frac{\partial}{\partial z}\quad\text{on}\quad\mathbb{C}_z\ \Big|\ R(z),\ Q(z)\in\mathbb{C}[z]_{\leqslant n}\right\}.$$

For each multivalued additively automorphic function $\Psi_{R,Q}(z)$, we have the diagram

$$\begin{tikzcd}[column sep=small]
 & \mathbb{X}=\frac{RQ}{Q'R-R'Q}\frac{\partial}{\partial z} \arrow[dl, leftrightarrow] \arrow[dr, leftrightarrow] & & \\
\omega=\Big(\frac{1}{\lambda_\alpha-\lambda_\beta}ln\Big(\frac{R(z)}{Q(z)}\Big)\Big)'dz \arrow[rr, leftrightarrow] & & \Psi(z)=\int^z\Big(\frac{1}{\lambda_\alpha-\lambda_\beta}ln\Big(\frac{R(z)}{Q(z)}\Big)\Big)'dz \arrow[r, leftrightarrow] & \Big\{\frac{1}{\lambda_\alpha-\lambda_\beta}ln\Big(\frac{R(z)}{Q(z)}\Big)-s=0\Big\}.
\end{tikzcd} \tag{40}$$

□

Let $\lambda_\alpha=1$ and $\lambda_\beta=0$ be eigenvalues in (39). The families of Weierstrass polynomials are

$$\{\mathrm{P}(s,z)=\mathrm{e}^sR(z)+Q(z)\mid \mathrm{e}^s\in\mathbb{C}^*\},$$

and the family of vector fields is

$$\mathbb{X}(z)=\frac{R(z)Q(z)}{Q'(z)R(z)-R'(z)Q(z)}\frac{\partial}{\partial z}\quad\text{on}\quad\mathbb{C}_z. \tag{41}$$

Assume that there are points in $\{R(z)=0\}\cap\{Q(z)=0\}\subset\mathbb{C}_z$, then they are fixed under the local flow of $\mathbb{X}$ and roots of the Weierstrass polynomial. The points $\{R(z)=0\}\cup\{Q(z)=0\}\backslash\big(\{R(z)=0\}\cap\{Q(z)=0\}\big)\subset\mathbb{C}_z$ are invariant under the local flow of $\mathbb{X}$. They are unattainable roots under the analytic extension of the flow of $\mathbb{X}$.

8 Schur–Cohn map 2 in root coordinates

Recall the $(\mathbb{C},+)$–action $\mathscr{A}_{2,n}(0,\theta\equiv1,P(z))=P(z)$ in (22);

$$\begin{array}{rcl}\mathscr{A}_{2,n}:\mathbb{C}\times\mathscr{D}_{2,n} & \longrightarrow & \mathscr{D}_{2,n}\\ (s,P(z)) & \longmapsto & \frac{\mathrm{e}^{i(\theta_n+\theta)}}{b_{n-1}}\sqrt{b_{n-1}+|c_0+sc_n|^2}zP_1(z)+\frac{c_0+sc_n}{b_{n-1}}P_1^*(z),\end{array} \tag{42}$$

here $R_{2,n}(P(z))=P_1(z)$. The associated principal fiber bundle is

$$\begin{tikzcd}
\mathbb{C}_s \arrow[r, "\mathtt{i}"] & \mathscr{D}_{2,n}\subset\mathbb{C}[z]_{=n} \arrow[d, "R_{2,n}"] \\
 & \mathbb{R}^+\times\mathbb{C}^{n-1}\subsetneq\mathscr{D}_{2,n-1}.
\end{tikzcd}$$

For each polynomial $P(z) \in \mathscr{D}_{2,n}$, the action $\mathscr{A}_{2,n}$ provides a Weierstrass polynomial of degree n

$$\mathtt{P}(s,z) \doteq \Xi_{2,n}(s, P_1(z)) : \mathbb{C}_s \times \mathbb{C} \longrightarrow \mathbb{C},$$

over the additive Lie group $(\mathbb{C}_s, +)$. A very useful expression for the $(\mathbb{C}, +)$–fiber of the $R_{2,n}$ bundle and its zeros is given by

$$\mathscr{Z} = \left\{ \mathtt{P}(s,z) = \frac{\mathrm{e}^{i\theta_n}}{b_{n-1}} \sqrt{b_{n-1} + |c_0 + sc_n|^2} z P_1(z) + \frac{c_0 + sc_n}{b_{n-1}} P_1^*(z) = 0 \;\Big|\; s \in \mathbb{C} \right\} \subset \mathbb{D}. \tag{43}$$

For each $\mathtt{P}(s,z)$, we require the real analytic diffeomorphism

$$\varepsilon : \mathbb{C}_s \longrightarrow \mathbb{D} \subset \mathbb{C}_t, \;\; s \longmapsto \frac{c_0 + c_n s}{\frac{c_n}{|c_n|} \sqrt{b_{n-1} + |c_0 + c_n s|^2}}. \tag{44}$$

In equation (43), dividing by the coefficient of $zP_1(z)$, the equation of $\mathscr{Z}$ assumes the form

$$\mathscr{Z} = \{ \mathtt{P}(s,z) = 0 \mid s \in \mathbb{C} \} = \left\{ \hat{\mathtt{P}}(t,z) = zP_1(z) + tP_1^*(z) = 0 \mid t \in \mathbb{D} \right\}. \tag{45}$$

As in proposition (7.1), we recognize the right side as a Weierstrass polynomial over $\mathbb{C}_t$ (only for $t \in \mathbb{D}$).
Recall two properties:
Using the identity element $s = 0 \in \mathbb{C}_s$ in (42) and (43), we recover $P(z) = \mathtt{P}(0,z)$.
For each $s_0 \in \mathbb{C}$, there exists a $t_0 \in \mathbb{D}$ such that $\{\mathtt{P}(s_0,z)\} = \{\hat{\mathtt{P}}(t_0,z) = 0\}$.

Furthermore, if in addition $P(z)$ is Schur–stable, then we distingish three cases.

Let $P(z) = \hat{\mathtt{P}}(0,z)$ be a Schur stable polynomial and consider the associated zeros of the Weierstrass polynomial $\{\hat{\mathtt{P}}(t,z) \mid t \in \mathbb{C}\}$ as in (45).
1. For $|t| < 1$, $\hat{\mathtt{P}}(t,z)$ is Schur stable.
2. For $|t| > 1$, $\hat{\mathtt{P}}(t,z)$ is anti–Schur.
3. For $|t| = 1$, $\hat{\mathtt{P}}(t,z)$ has all its zeros in the unitary circle $\partial\mathbb{D}$.

For $|t| < 1$, we recall that $\{\hat{\mathtt{P}}(t,z) = 0\}$ already appeared in equation (20), so the assertion follows.

For $|t| > 1$, the family (45) is extended to $\{\hat{\mathtt{P}}(t,z) \mid t \in \mathbb{C} \backslash \mathbb{D}\}$. Since it is equal to (24) when we interchange the roles of $P_1^*(z)$ y $P_1(z)$ and by theorem 3.1 for anti–Schur polynomials, we have assertion 2.

For $|t| = 1$, the assertion follows by a continuity argument and the assertions 1 and 2. □

Summing up

(n–root configuration dynamics of Schur stable polynomials) Let

$$\mathscr{A}_{2,n} : \mathbb{C} \times \mathscr{D}_{2,n} \longrightarrow \mathscr{D}_{2,n}$$

be the real analytic action (42) from the bundle defined by the Schur–Cohn map $R_{2,n}$ and let $P(z)$ be a Schur stable polynomial. For the respective orbit

$$\left\{ \mathtt{P}(s,z) = \frac{\mathrm{e}^{i\theta_n}}{b_{n-1}} \sqrt{b_{n-1} + |c_0 + c_n s|^2} z P_1(z) + \frac{c_0 + c_n s}{b_{n-1}} P_1^*(z) \;\Big|\; s \in \mathbb{C} \right\} \tag{46}$$

over $\mathbb{C}_s$, the rational vector field

$$\mathbb{X}(z) = \frac{P_1^{*2}(z)}{zP_1(z)P_1^{*'}(z) - (zP_1(z)' + P_1(z))P_1^*(z)} \frac{\partial}{\partial z} \tag{47}$$

describes the n–root configuration dynamics of the Weierstrass polynomial $\left\{\mathtt{P}(s,z) = 0\right\}$ on $\mathbb{D}$.

Let $P(z)$ be a Schur stable polynomial. We recognize that each Weierstrass polynomial (46) fills up the conditions in proposition 7.1 family 2 (where t plays the role of the parameter s in this proposition). Hence, the vector field $\mathbb{X}$ in (47) is well–defined, as in (38).
The flow of the vector field $\mathbb{X}$ for time $t \in \mathbb{D}$ describes the n–root configuration dynamics of $\left\{\hat{\mathtt{P}}(t,z) = 0\right\}$, by (45). Recall that s and t are related by (51) and we can interchange the zero locus $\{\hat{\mathtt{P}}(t,z) = 0\}$ by $\{\mathtt{P}(s,z) = 0\}$. Hence, the equation

$$\mathtt{P}(t,z) = c_n(t)\big(z - \varphi(t,z_1)\big) \cdots \big(z - \varphi(t,z_n)\big)$$

holds true whenever the flow is well–defined, as in definition 5.1. □

An advantage of $\mathbb{X}$ in (47) is that it is rational, having an explicit expression using $P_1(z)$ and $P_1^*(z)$. The next result describes the dynamics of $\mathbb{X}$ in the sense of definition 6.2.

1. The unattainable points of $\{\mathtt{P}(s,0)=0\}$ are $\{P_1^*(z)=0\}\subset\mathbb{C}\backslash\mathbb{D}$.
2. The poles of $\mathbb{X}$ are $2n-2$ counted with multiplicity, and there are at most $n-1$ inside $\mathbb{D}$, they produce collisions of the roots of $\{\mathtt{P}(s,0)=0\}$.

For 1, we note that the zeros of $\mathbb{X}$ are in $\mathbb{C}\backslash\mathbb{D}$, and by lemma 8, it follows that $\{\mathtt{P}(t,z)=0\}$ has its unattainable points outside of the unitary disk.
For 2, Recall that the vector field (47) has an anti–Schur numerator and a self–inverse denominator, thus

$$\begin{aligned} zP_1(z)P_1^{*\prime}(z)-(zP_1'(z)+P_1(z))P_1^*(z) = \quad & \overline{b}_0 b_{n-1}z^{2n-2}+\ldots+\left(\textstyle\sum_{k=0}^{n-3}(n-2-2k)\overline{b}_{n-(k+3)}b_{n-(k+1)}\right)z^{n+1} \\ & +\left(\textstyle\sum_{k=0}^{n-2}(n-1-2k)\overline{b}_{n-(k+2)}b_{n-(k+1)}\right)z^{n}+ \\ & \left(\textstyle\sum_{k=0}^{n-1}(n-2k)|b_{n-(k+1)}|\right)z^{n-1}+\left(\textstyle\sum_{k=0}^{n-2}(n-1-2k)b_{n-(k+2)}\overline{b}_{n-(k+1)}\right)z^{n-2}+ \\ & \left(\textstyle\sum_{k=0}^{n-3}(n-2-2k)b_{n-(k+3)}\overline{b}_{n-(k+1)}\right)z^{n+1}+\ldots+b_0b_{n-1}, \end{aligned}$$

which is self–inverse by using [27] definition 11.5.1, p. 375. The vector field $\mathbb{X}$ has at most $n-1$ poles inside of $\mathbb{D}$. $\square$

Let $P(z)=z^3$ be a Schur polynomial. Its associated polynomial is $P_1(z)=R_{2,n}(P(z))=z^2$ and the respective anti–Schur polynomial is $P_1^*(z)=1$. By (45), the Weierstrass polynomials are

$$\{\mathtt{P}(s,z)=s+\sqrt{1+|s|^2}z^3\} \quad \text{and} \quad \{\widehat{\mathtt{P}}(t,z)=t+z^3\}.$$

According to (47), the rational vector field

$$\mathbb{X}(z)=\frac{-1}{3z^2}\frac{\partial}{\partial z}$$

describes the 3–root configuration dynamics of $\{\mathtt{P}(s,z)=0\}$ over $\mathbb{C}$. The roots are inside $\mathbb{D}$ and there is a triple collision in $z=0$; see figure 4.a. Note that the expected number of poles of $\mathbb{X}$ is four; however, only a triple pole appears.

Let $P(z)=z(z-1/2)(z-(1+i)/2)$ be a Schur polynomial. Its associated polynomial is $P_1(z)=R_{2,n}(P(z))=(z-1/2)(z-(1+i)/2)$ and the respective anti–Schur polynomial is $P_1^*(z)=\frac{1-i}{4}z^2-(1-i/2)z+1$. By (45), the Weierstrass polynomials are

$$\{\mathtt{P}(s,z)=s(\tfrac{1-i}{4}(z-2)(z-(1+i)))+\sqrt{1+|s|^2}z(z-1/2)(z-(1+i)/2)\}$$

and

$$\{\widehat{\mathtt{P}}(t,z)=t(\tfrac{1-i}{4}(z-2)(z-(1+i)))+z(z-1/2)(z-(1+i)/2)\}.$$

According to (47), the vector field

$$\mathbb{X}(z)=-\frac{\left((2+2i)-(3+i)z+z^2\right)^2}{(-2+2i)+(8-16i)z+33iz^2-(8+16i)z^3+(2+2i)z^4}\frac{\partial}{\partial z}$$

describes the 3–root configuration dynamics of $\{\mathtt{P}(s,z)=0\}$ over $\mathbb{C}$. The number of poles of $\mathbb{X}$ is four. The root collisions appear in $z_1=0.25+0.03i$, $z_2=0.42+0.32i$, $z_3=1.48+1.14i$ and $z_4=3.83+0.49i$, but only z_1 and z_2 are in the unitary disk, so the other collisions appear in $\mathbb{C}\backslash\mathbb{D}$; see figure 4.b.
Furthermore, we verify numerically lemma 8.3 for some $|t|=1$, obtaining figures similar to those sketched in example 1.a. on $\mathbb{C}_z$.

Let $P(z)=(-1+i)-2iz+8z^3$ be a Schur polynomial. Its associated polynomial is $P_1(z)=R_{2,n}(P(z))=-16i+(2+2i)z+62z^2$ and the respective anti–Schur polynomial is $P_1^*(z)=62+(2-2i)z+16iz^2$. By (45), the Weierstrass polynomial are

$$\{\mathtt{P}(s,z)=\tfrac{1}{62}((-1+i)+8s)\left(62+(2-2i)z+16iz^2\right)+\tfrac{1}{62}z\left(-16i+(2+2i)z+62z^2\right)\sqrt{62+|(-1+i)+8s|^2}\}$$

and

$$\{\widehat{\mathtt{P}}(t,z)=t(\tfrac{1-i}{4}(z-2)(z-(1+i)))+z(z-1/2)(z-(1+i)/2)\}.$$

According to (47), the vector field

$$\mathbb{X}(z)=-\frac{i\left(31i+(1+i)z-8z^2\right)^2}{31\left(-8+(2-2i)z-91iz^2-(2+2i)z^3+8z^4\right)}\frac{\partial}{\partial z}$$

describes the 3–root configuration dynamics of $\{\mathtt{P}(s,z)=0\}$ over $\mathbb{C}$. The number of poles of $\mathbb{X}$ is four. The root collisions appear in $z_1=0.2+0.2i$, $z_2=-0.22-0.22i$, $z_3=2.24-2.24i$ and $z_4=2.51+2.51i$, but only z_1 and z_2 are in the unitary disk, so the other collisions appear in $\mathbb{C}\backslash\mathbb{D}$; see figure 4.c.

Let

$$P(z)=z(z-(1+i)) \ \in\mathscr{D}_{2,2}\backslash\mathscr{S}_2 \tag{48}$$

be a non Schur polynomial, but in the domain of $R_{2,2}$. Its associated polynomial is

$$P_1(z)=R_{2,2}(P(z))=z-(1+i) \ \notin\mathscr{D}_{2,1}$$

and the respective anti–Schur polynomial is $P_1^*(z)=(-1+i)z+1$. By (45), the Weierstrass polynomials are

$$\{\mathtt{P}(s,z)=\sqrt{1+|s|^2}z(z-(1+i))+s((-1+i)z+1)\} \quad \text{and} \quad \left\{\widehat{\mathtt{P}}(t,z)=z(z-(1+i))+t((-1+i)z+1)\right\}.$$

According to (47), the rational vector field

$$\mathbb{X}(z)=\frac{(2-2i)(z-\frac{1+i}{2})^2}{2(z-1)(z-i)}\frac{\partial}{\partial z}$$

describes the 2–root configuration dynamics of $\{\mathtt{P}(s,z)=0\}$ over $\mathbb{C}$. Since the initial $P(z)$ is not Schur, for $|t|=1$ the roots $\{\widehat{\mathtt{P}}(t,z)=0\}$ do not belong to the unitary circle $\partial\mathbb{D}$; see figure 4.d.

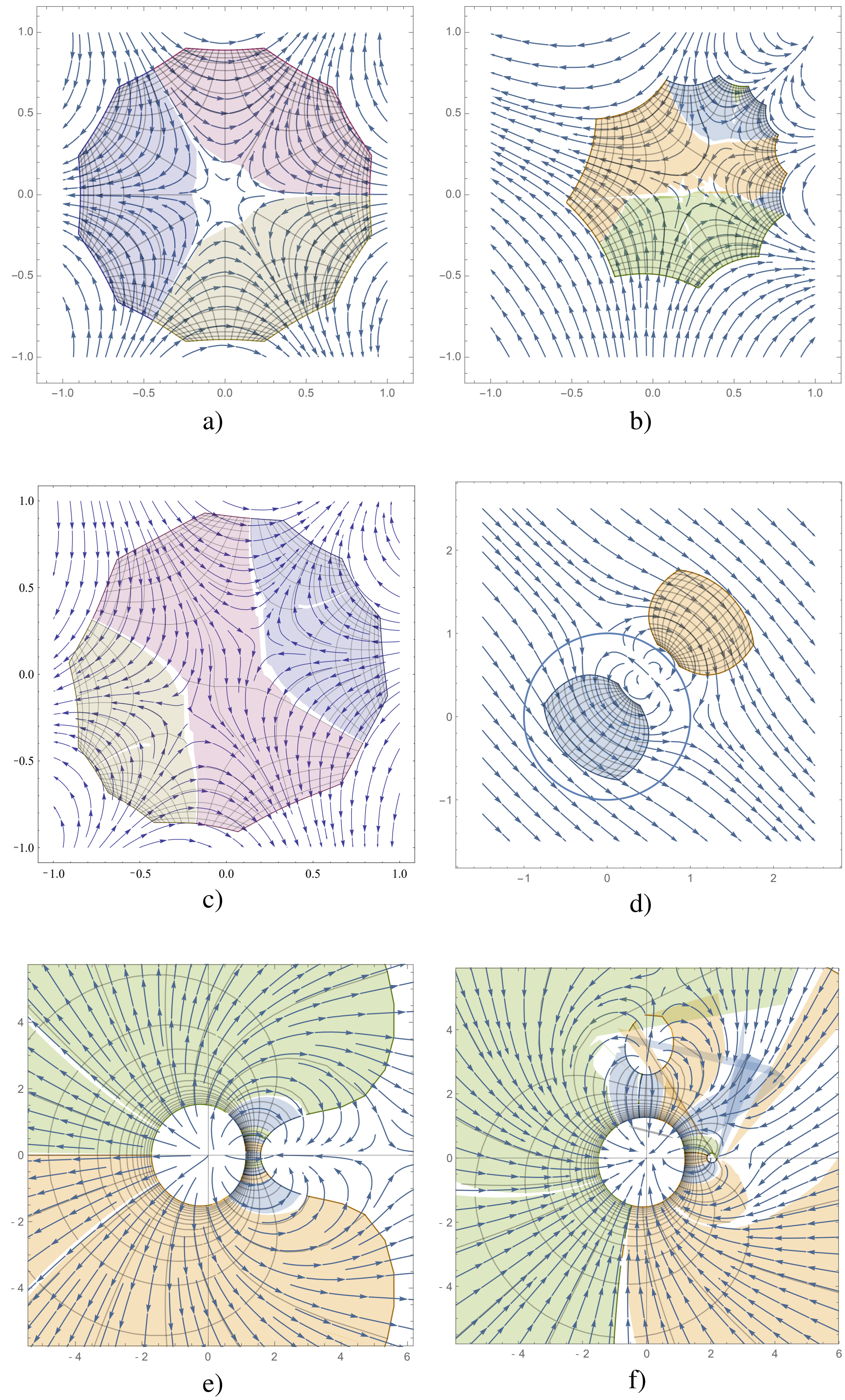

Figure 4: Three phase portraits of vector fields $\mathbb{X}$ describing the 3–root dynamics of Schur polynomials in $\mathbb{D}$ for time $\{s \mid -1 \leqslant \mathfrak{Re}(s), \mathfrak{Im}(s) \leqslant 1\}$; a), b) and c) correspond to examples 8, 8 and 8 respectively. d) Phase portrait of a vector field $\mathbb{X}$ describing the 2–root dynamics of Weierstrass polynomial in $\mathscr{D}_{2,2}$, not Schur or anti–Schur, see example 8. Two phase portraits (e)–(f) of vector fields $\mathbb{Y}$ describing the 3–root dynamics of anti–Schur polynomals in $\mathbb{C}\backslash\mathbb{D}$ for time $\{w \mid 0.1 \leqslant |w| \leqslant 0.8,\ 0 \leqslant Arg(w) \leqslant 2\pi\}$; e) and f) correspond to examples 9 and 9. In the six figures, the color islands correspond to the behavior under the complex flow (of $\mathbb{X}$ or $\mathbb{Y}$) using one root of the respective $\{P(z) = 0\}$ as an initial condition.

9 Schur–Cohn map 1 in root coordinates

Recall the $(\mathbb{C}^*,\cdot)$–action $\mathscr{A}_{1,n}(1,\theta\equiv 0,P(z))=P(z)$ in (26);

$$\begin{array}{rcl} \mathscr{A}_{1,n}:\mathbb{C}\times\mathscr{D}_{1,n} & \longrightarrow & \mathscr{D}_{1,n} \\ \left(w,\frac{c_n}{b_0}zP_1^*(z)+\frac{e^{i\theta_0}\sqrt{b_0+|c_n|^2}}{b_0}P_1(z)\right) & \longmapsto & \frac{c_nw}{b_0}zP_1^*(z)+\frac{e^{i(\theta_0+\theta)}\sqrt{b_0+|c_nw|^2}}{b_0}P_1(z), \end{array} \tag{49}$$

here $R_{1,n}(P(z))=P_1(z)$. The associated principal fiber bundle is

$$\begin{array}{ccc} \mathbb{C}_w^* & \xrightarrow{\;\mathtt{i}\;} & \mathscr{D}_{1,n}\subset\mathbb{C}[z]_{=n} \\ & & \Big\downarrow R_{1,n} \\ & & \mathbb{C}^{n-1}\times\mathbb{R}^+\subsetneq\mathscr{D}_{1,n-1}. \end{array}$$

For each polynomial $P(z)\in\mathscr{D}_{1,n}$, the action $\mathscr{A}_{1,n}$ provides a Weierstrass polynomial of degree at most n

$$\mathtt{P}(w,z)\doteq\Xi_{1,n}(w,P_1(z)):\mathbb{C}_w^*\times\mathbb{C}\longrightarrow\mathbb{C},$$

over the multiplicative Lie group $(\mathbb{C}_w^*,\cdot)$. Note that since we are interested in the dynamics of the roots without loss of generality we can start with $P_1(z)=R_{1,n}(P(z))$ having $b_0\in\mathbb{R}^+$.
A very useful expression for the $(\mathbb{C}_w^*,\cdot)$–fiber of the $R_{1,n}$ bundle and its zeros is given by

$$\mathscr{Z}=\left\{\mathtt{P}(w,z)=\frac{c_nw}{b_0}zP_1^*(z)+\frac{\mathrm{e}^{i\theta_0}\sqrt{b_0+|c_nw|^2}}{b_0}P_1(z)=0\;\Big|\;w\in\mathbb{C}^*\right\}\subset\mathbb{C}\backslash\overline{\mathbb{D}}, \tag{50}$$

where $\theta_0=arg(c_0)$ is given as in (17).

For each $\mathtt{P}(s,z)$, we require the real analytic diffeomorphism

$$\varepsilon:\mathbb{C}_w^*\longrightarrow\mathbb{C}_{\mathrm{e}^t}\backslash\mathbb{D}\qquad\varepsilon:w\longmapsto\frac{\mathrm{e}^{i\theta_0}\sqrt{b_0+|c_nw|^2}}{c_nw}\doteq\mathrm{e}^t, \tag{51}$$

where $b_0\in\mathbb{R}^+$ and $\mathfrak{Re}(t)>0$. In equation (50), dividing by the coefficient, $\left(\mathrm{e}^{i\theta_0}\sqrt{b_0+|c_nw|^2}\right)/b_0$, the equation of its zero locus $\mathscr{Z}$ assumes the form

$$\mathscr{Z}=\{\mathtt{P}(w,z)=0\mid w\in\mathbb{C}^*\}=\left\{\hat{\mathtt{P}}(t,z)=zP_1^*(z)+\mathrm{e}^tP_1(z)=0\mid\mathfrak{Re}(t)>0\right\}. \tag{52}$$

As in proposition 7.2, we recognize the right side as a Weierstrass polynomial over $\mathbb{C}^*$.
Recall two properties:
Using the identity element $w=1\in\mathbb{C}_w^*$ by (49) and (50), we recover $P(z)=\mathtt{P}(1,z)$.
For each $w_0\in\mathbb{C}$, there exists $t_0\in\mathbb{D}$ such that $\{\mathtt{P}(w_0,z)\}=\{\hat{\mathtt{P}}(t_0,z)=0\}$.

We assume that $P(z)$ is anti–Schur.

Let $P(z)$ be an anti–Schur polynomial, consider the associated Weierstrass polynomial, $\hat{\mathtt{P}}(t,z)$ as in (52).
1. For $\mathfrak{Re}(t)>0$, $\hat{\mathtt{P}}(t,z)$ is anti–Schur.
2. For $\mathfrak{Re}(t)<0$, $\hat{\mathtt{P}}(t,z)$ is Schur.
3. For $\mathfrak{Re}(t)=0$, $\hat{\mathtt{P}}(t,z)$ has all its zeros in the unitary circle $\partial\mathbb{D}$. It is analogous to the proof in lemma 8. □

Using proposition 7.2, we have an auxiliary rational vector field $\mathbb{Y}(z)$ on $\widehat{\mathbb{C}}$, associated to with the family (52).

Summing up

(n–root configuration dynamics of anti–Schur stable polynomials) Let

$$\mathscr{A}_{1,n}:\mathbb{C}^*\times\mathscr{D}_{1,n}\longrightarrow\mathscr{D}_{1,n}$$

be the real analytic action (49) from the bundle defined by the Schur–Cohn map $R_{1,n}$, and let $P(z)$ be an anti–Schur polynomial. For the respective orbit

$$\left\{\mathtt{P}(w,z)=\frac{c_nw}{b_0}zP_1^*(z)+\frac{e^{i\theta_0}\sqrt{b_0+|c_nw|^2}}{b_0}P_1(z)=0\;\Big|\;w\in\mathbb{C}^*\right\}\subset\mathbb{C}\backslash\overline{\mathbb{D}}, \tag{53}$$

the rational vector field

$$\mathbb{Y}(z)=-\frac{zP_1(z)P_1^*(z)}{(P_1^*(z)+zP_1^{*'}(z))P_1(z)-zP_1^*(z)P_1'(z)}\frac{\partial}{\partial z}\quad\text{on}\;\;\widehat{\mathbb{C}} \tag{54}$$

describes the n–root configuration dynamics of the Weierstrass polynomial $\{\mathtt{P}(w,z)=0\}\subset\mathbb{D}$. $\square$

The next lemma describes the dynamics of $\mathbb{Y}$ in the sense of definition 6.2.

Consider the vector field $\mathbb{Y}$ in (54).

1. The unattainable points are the roots of $zP_1^*(z)P_1(z)$, and there are $n-1$ unattainable points in $\mathbb{C}\backslash\mathbb{D}$.
2. The poles of $\mathbb{Y}$ are at most $2n-2$ counted with multiplicity, and there are at most $n-1$ in $\mathbb{C}\backslash\mathbb{D}$, which produce a root collisions of $\mathtt{P}(w,z)$.

It is analogous to the proof for corollary 8. $\square$

Let

$$P(z)=z(2z-1)^2+2(z-2)^2$$

be anti–Schur polynomial. Its image is $P_1(z)=R_{1,n}(P(z))=12(z-2)^2$ and the associated Schur polynomial is $P_1^*(z)=12(2z-1)^2$. The Weierstrass polynomials are given by

$$\{\mathtt{P}(w,z)=\tfrac{4w}{48}12z(2z-1)^2+\tfrac{\sqrt{48+|4w|^2}}{48}12(z-2)^2\}$$

and

$$\{\hat{\mathtt{P}}(t,z)=z(2z-1)^2+e^t(z-2)^2\}.$$

According to (54), the vector field

$$\mathbb{Y}(z)=-\tfrac{(-2+z)z(-1+2z)}{2-11z+2z^2}\tfrac{\partial}{\partial z}$$

describes the 4–root configuration dynamics of $\{\mathtt{P}(w,z)=0\}$ over $\mathbb{C}^*$. In $\mathbb{C}\backslash\overline{\mathbb{D}}$ there is a collision of 2–roots at $z=5.31$, and $z=2$ is an unattainable root. See figure 4.e.

Let

$$P(z)=-6iz^3+3iz^2-(3+6i)z+12i$$

be an anti–Schur polynomial. Its associated polynomial is $P_1(z)=-18i(z-2)(z-3i)$, which is an anti–Schur polynomial, the associated Schur polynomial is $P_1^*(z)=108z^2-(54+36i)z+18i$. From here the Weierstrass polynomial is

$$\{\mathtt{P}(w,z)=\{\tfrac{-6iw}{108}z(108z^2-(54+36i)z+18i)+\tfrac{i\sqrt{108+|-6iw|^2}}{108}(-18i)(z-2)(z-3i)\}.$$

According to (54), the vector field

$$\mathbb{Y}(z)=\tfrac{z\left(6i-(2+3i)z+z^2\right)\left(i-(3+2i)z+6z^2\right)}{6\left(-1+(4-6i)z+20iz^2-(4+6i)z^3+z^4\right)}\tfrac{\partial}{\partial z}$$

describes the root dynamics of $\{\mathtt{P}(w,z)=0\}$. The roots belong to $\mathbb{C}\backslash\overline{\mathbb{D}}$, and there is a collision of 2–roots in $z_1=0.55+5.47i$, $z_2=3.11+0.31i$. The Weierstrass polynomial $\mathtt{P}(w,z)$ has unattainable roots in 0 and $3i$. See figure 4.f.

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