Alvaro Alvarez-Parrilla* and Jesús Muciño-Raymundo

# Symmetries of complex analytic vector fields with an essential singularity on the Riemann sphere

DOI 10.1515/advgeom-2021-0002. Received 27 February, 2019; revised 7 April, 2020

**Abstract:** We consider the family $\mathscr{E}(s, r, d)$ of all singular complex analytic vector fields $X(z) = \frac{Q(z)}{P(z)} e^{E(z)} \frac{\partial}{\partial z}$ on the Riemann sphere $\widehat{\mathbb{C}}$, where $Q, P, E$ are polynomials with $\deg Q = s$, $\deg P = r$ and $\deg E = d \geq 1$. Using the pullback action of the affine group $\mathrm{Aut}(\mathbb{C})$ and the divisors for $X$, we calculate the isotropy groups $\mathrm{Aut}(\mathbb{C})_X$ of discrete symmetries for $X \in \mathscr{E}(s, r, d)$. The subfamily $\mathscr{E}(s, r, d)_{\mathrm{id}}$ of those $X$ with trivial isotropy group in $\mathrm{Aut}(\mathbb{C})$ is endowed with a holomorphic trivial principal $\mathrm{Aut}(\mathbb{C})$-bundle structure. A necessary and sufficient arithmetic condition on $s, r, d$ ensuring the equality $\mathscr{E}(s, r, d) = \mathscr{E}(s, r, d)_{\mathrm{id}}$ is presented. Moreover, those $X \in \mathscr{E}(s, r, d) \setminus \mathscr{E}(s, r, d)_{\mathrm{id}}$ with non-trivial isotropy are realized. This yields explicit global normal forms for all $X \in \mathscr{E}(s, r, d)$. A natural dictionary between analytic tensors, vector fields, 1-forms, orientable quadratic differentials and functions on Riemann surfaces $M$ is extended as follows. In the presence of non-trivial discrete symmetries $\Gamma < \mathrm{Aut}(M)$, the dictionary describes the correspondence between $\Gamma$-invariant tensors on $M$ and tensors on $M/\Gamma$.

**Keywords:** Complex analytic vector field, Riemann surface, essential singularity, discrete symmetry group.

**2010 Mathematics Subject Classification:** 34M35, 32S65, 30D20, 58D19

**Communicated by:** P. Eberlein

## 1 Introduction

Meromorphic vector fields on compact Riemann surfaces are well understood, at least in some aspects: see [27], [26], [10], [17], [30]. Essential singularities represent the next level of complexity. We study the holomorphic families consisting of singular complex analytic vector fields on the Riemann sphere $\widehat{\mathbb{C}}$ with a singular set composed of $s \geq 0$ zeros and $r \geq 0$ poles on $\mathbb{C}$, and an isolated essential singularity at $\infty \in \widehat{\mathbb{C}}$ of 1-order $d \geq 1$, namely

$$\mathscr{E}(s, r, d) = \left\{ X(z) = \frac{Q(z)}{P(z)} e^{E(z)} \frac{\partial}{\partial z} \;\middle|\; Q, P, E \in \mathbb{C}[z],\, Q \text{ and } P \text{ monic},\, \deg Q = s,\, \deg P = r,\, \deg E = d \geq 1 \right\}. \quad (1)$$

The naturally associated functions

$$\Psi_X(z) = \int^z \frac{P(\zeta)}{Q(\zeta)} e^{-E(\zeta)} d\zeta \quad \text{with } X \in \mathscr{E}(s, r, d) \quad (2)$$

and their Riemann surfaces $\mathcal{R}_X = \{(z, \Psi_X(z))\}$ belong to the transcendental functions described in Nevanlinna's seminal work; see [28] and [29, Ch. XI], in particular chapter XI. Taniguchi studied these families $\{\Psi_X\}$ from the viewpoint of deformation of functions, see [32] and [33]. Motivated by complex dynamics, Biswas and Pérez-Marco in [5] and [6] enrich the study of $\{\Psi_X\}$ and $\{\mathcal{R}_X\}$. In [3] the authors explored the families $\mathscr{E}(0, 0, d)$ obtaining an analytic classification when $d \geq 1$, and presenting analytic normal forms for $d \leq 3$.

The search for a natural and adequate notion of global normal form on $\widehat{\mathbb{C}}$ for vector fields in $\mathscr{E}(s, r, d)$ leads to novel paths. A characteristic of the study of vector fields on the Riemann sphere, or on the affine

***Corresponding author: Alvaro Alvarez-Parrilla,** Grupo Alximia SA de CV, México, email: alvaro.uabc@gmail.com
**Jesús Muciño-Raymundo,** Centro de Ciencias Matemáticas, Universidad Nacional Autónoma de México, México, email: muciray@matmor.unam.mx

plane, is that their groups of automorphisms are finite-dimensional complex Lie groups: rich enough and yet treatable. For $d \geq 1$ the essential singularity of $X \in \mathscr{E}(s, r, d)$ provides a marked point at $\infty \in \widehat{\mathbb{C}}$. We consider the canonical action

$$\mathcal{A} : \mathrm{Aut}(\mathbb{C}) \times \mathscr{E}(s, r, d) \longrightarrow \mathscr{E}(s, r, d), \quad (T, X) \longmapsto T^*X, \tag{3}$$

of the affine transformation group $\mathrm{Aut}(\mathbb{C})$ corresponding to those $T \in \mathrm{Aut}(\widehat{\mathbb{C}}) = \mathrm{PSL}(2, \mathbb{C})$ that fix $\infty$. Clearly $\mathcal{A}$ is a valuable and accurate tool for understanding the dynamics of the vector fields $X \in \mathscr{E}(s, r, d)$ and their associated families of functions $\{\Psi_X\}$. It leads to the following natural classification problems:

(AC) Characterize under which conditions $X_1$ and $X_2$ in $\mathscr{E}(s, r, d)$ are *complex analytically equivalent*, *i.e.* whether there exist $T \in \mathrm{Aut}(\mathbb{C})$ such that

$$X_2 \xrightarrow{T^*} X_1. \tag{4}$$

(MC) Consider the singular flat metric $(\widehat{\mathbb{C}}, g_X)$ associated to $X$, see Definition 2.4 and Proposition 4.1, and characterize under which conditions the metrics $g_{X_1}, g_{X_2}$ associated to $X_1, X_2$ in $\mathscr{E}(s, r, d)$ are *isometrically equivalent*, *i.e.* whether there exist $(T, e^{i\theta}) \in \mathrm{Aut}(\mathbb{C}) \times \mathbb{S}^1$ such that

$$(\widehat{\mathbb{C}}, g_{X_2}) \xrightarrow{T^*} (\widehat{\mathbb{C}}, g_{e^{i\theta}X_1}) \tag{5}$$

is an isometry, where $e^{i\theta} : X \mapsto e^{i\theta}X$ acts by rotations.

The relation between (AC) and (MC) determines the diagram

$$\begin{tikzcd} \mathscr{E}(s, r, d) \arrow[r, "\pi_1"] & \dfrac{\mathscr{E}(s, r, d)}{\mathrm{Aut}(\mathbb{C})} \arrow[r, "\pi_2"] \arrow[d, leftrightarrow, "\doteq"] & \dfrac{\mathscr{E}(s, r, d)}{\mathrm{Aut}(\mathbb{C}) \times \mathbb{S}^1} \arrow[d, leftrightarrow, "\doteq"] \\ & \{\text{normal forms } [X]\} & \{\text{classes of flat metrics } (\widehat{\mathbb{C}}, g_X)\}, \end{tikzcd} \tag{6}$$

where $\pi_1, \pi_2$ are the natural projections to equivalence classes; see Lemma 2.5 for further details. Our purpose is the study of the quotient spaces in (6), mainly $\mathscr{E}(s, r, d)/\mathrm{Aut}(\mathbb{C})$.

As a first step in both classifications, we study the $\mathrm{Aut}(\mathbb{C})$-fibre bundle structure on $\mathscr{E}(s, r, d)$. Let

$$\mathscr{E}(s, r, d)_{\mathrm{id}} \subseteq \mathscr{E}(s, r, d)$$

denote those $X$ with *trivial isotropy group* $\mathrm{Aut}(\mathbb{C})_X \subset \mathrm{Aut}(\mathbb{C})$. A natural tool for understanding the geometry of $X$ is the *divisor*

$$\underbrace{[q_1, \dots, q_s]}_{\mathcal{Z}}, \underbrace{[p_1, \dots, p_r]}_{\mathcal{P}}, \underbrace{[e_1, \dots, e_d]}_{\mathcal{E}},$$

consisting of the roots of $Q(z)$, $P(z)$ and $E(z)$, see Definition 2.2. Three remarkable and novel features of $\mathscr{E}(s, r, d)$ are that:

- $(\mathcal{Z} \cup \mathcal{P}) \cap \mathcal{E}$ need not be empty.
- For a non-trivial subgroup $\Gamma$ of $\mathrm{Aut}(\mathbb{C})$, the $\Gamma$-invariance of the divisor of a vector field $X$ does not imply that $X$ is itself $\Gamma$-invariant, see Theorem 2 and Example 2.10.
- $\mathcal{E}$ is not part of the singular set of the phase portrait of $\mathfrak{Re}\,(X)$, in fact only $\mathcal{Z} \cup \mathcal{P} \cup \{\infty\}$ are visible singularities in $\widehat{\mathbb{C}}$.

In order to describe our results, using the last feature mentioned above, we denote the subfamily of vector fields having a pole or zero at a fixed point $C$ in $\mathbb{C}$ by

$$\mathscr{E}(s, r, d; C, \nu) = \{X \in \mathscr{E}(s, r, d) \mid C \text{ is a pole or zero of } X \text{ with multiplicity } \nu \neq 0\};$$

we convene that $\nu \leq -1$ if and only if $C$ is a pole, and $\nu \geq 1$ if and only if $C$ is a zero. Further, consider the sets

$$\mathscr{D}_{\mathrm{pole}} = \{\Bbbk \in \mathbb{N} \mid \Bbbk \text{ is a non-trivial common divisor of } s, r+1 \text{ and } d\},$$
$$\mathscr{D}_{\mathrm{zero}} = \{\Bbbk \in \mathbb{N} \mid \Bbbk \text{ is a non-trivial common divisor of } s-1, r \text{ and } d\}.$$

For each natural number $\mathtt{k}$ and each point $C$ we denote by $T_{\mathtt{k},C} \in \operatorname{Aut}(\mathbb{C})$ the rotation of angle $2\pi/\mathtt{k}$ around the center $C$, and by $T^{\ell}_{\mathtt{k},C}$ its $\ell$-fold composition. As will be shown, a simple geometric condition is that the isotropy groups for $X \in \mathscr{E}(s,r,d)$ are $\operatorname{Aut}(\mathbb{C})_X \cong \mathbb{Z}_{\mathtt{k}}$; the problem lies in determining the possible $\mathtt{k}$ in terms of $s$, $r$ and $d$. With all this in mind, we have

**Theorem 1** (Analytical and metric classification of $\mathscr{E}(s,r,d)$).

(1) *The family $\mathscr{E}(s,r,d) \setminus \mathscr{E}(s,r,d)_{\mathrm{id}}$ of vector fields with non-trivial isotropy group is*

$$\bigcup_{C\in\mathbb{C}}\Bigg(\bigcup_{\substack{\mathtt{k}\in\mathscr{D}_{\mathrm{pole}}\\ 1\le m\le\frac{r+1}{\mathtt{k}}}}\bigcup_{\ell=1}^{\mathtt{k}}(T^{\ell}_{\mathtt{k},C})^*\Big(\mathscr{E}\Big(\frac{s}{\mathtt{k}},\frac{r+1}{\mathtt{k}}-m,\frac{d}{\mathtt{k}};C,-(m\mathtt{k}-1)\Big)\Big)$$
$$\cup\bigcup_{\substack{\mathtt{k}\in\mathscr{D}_{\mathrm{zero}}\\ 0\le m\le\frac{s-1}{\mathtt{k}}}}\bigcup_{\ell=1}^{\mathtt{k}}(T^{\ell}_{\mathtt{k},C})^*\Big(\mathscr{E}\Big(\frac{s-1}{\mathtt{k}}-m,\frac{r}{\mathtt{k}},\frac{d}{\mathtt{k}};C,m\mathtt{k}+1\Big)\Big)\Bigg). \tag{7}$$

(2) *The families $\mathscr{E}(s,r,d)$ and $\mathscr{E}(s,r,d)_{\mathrm{id}}$ coincide if and only if*

$$\gcd(s,r+1,d)=\gcd(s-1,r,d)=1. \tag{8}$$

(3) *For $s+r+d\ge 2$ and $d\ge 1$, the holomorphic (respectively real analytic) principal bundles*

$$\begin{array}{ccc} \operatorname{Aut}(\mathbb{C}) \longrightarrow \mathscr{E}(s,r,d)_{\mathrm{id}} & \qquad & \operatorname{Aut}(\mathbb{C})\times\mathbb{S}^1 \longrightarrow \mathscr{E}(s,r,d)_{\mathrm{id}} \\ \Big\downarrow \pi_1 & & \Big\downarrow \pi_2\circ\pi_1 \\ \dfrac{\mathscr{E}(s,r,d)_{\mathrm{id}}}{\operatorname{Aut}(\mathbb{C})} & & \dfrac{\mathscr{E}(s,r,d)_{\mathrm{id}}}{\operatorname{Aut}(\mathbb{C})\times\mathbb{S}^1} \end{array} \tag{9}$$

*are trivial. Moreover, $\mathscr{E}(s,r,d)_{\mathrm{id}}/\operatorname{Aut}(\mathbb{C})$ is a non-compact complex manifold of dimension $s+r+d-1$, and $\mathscr{E}(s,r,d)_{\mathrm{id}}/(\operatorname{Aut}(\mathbb{C})\times\mathbb{S}^1)$ is a non-compact real analytic manifold of dimension $2(s+r+d)-3$.*

The proof of (1) is found in § 3.1, while (2) and (3) are proved in § 2. The explicit global sections

$$\sigma : \mathscr{E}(s,r,d)_{\mathrm{id}}/\operatorname{Aut}(\mathbb{C}) \longrightarrow \mathscr{E}(s,r,d)_{\mathrm{id}}$$

described in the proof of Lemma 2.13 as (19), (20) and (21) imply the triviality of the bundles and provide *global normal forms* for vector fields $X \in \mathscr{E}(s,r,d)_{\mathrm{id}}$, see also Definition 2.14 and Corollary 2.15 (for the associated singular flat metrics in normal form see Remarks 2.17 and 3.3.4). They are global in the sense that the explicit expressions for $\sigma([X])$ are valid for the whole family $\mathscr{E}(s,r,d)_{\mathrm{id}}/\operatorname{Aut}(\mathbb{C})$, and on the whole Riemann sphere $\widehat{\mathbb{C}}$, when considering the phase portraits of $\mathfrak{Re}(X)$.

Noting that the singular locus of the quotient $\mathscr{E}(s,r,d)/\operatorname{Aut}(\mathbb{C})$ is localized at $\mathscr{E}(s,r,d)\setminus\mathscr{E}(s,r,d)_{\mathrm{id}}$, let $\Gamma$ be a non-trivial subgroup of $\operatorname{Aut}(\mathbb{C})$. This leads to a natural question: How can we construct $\Gamma$-invariant complex analytic vector fields $X \in \mathscr{E}(s,r,d)$?

**Theorem 2** ($\Gamma$-invariance). *Let $\{\mathrm{id}\} \ne \Gamma < \operatorname{Aut}(\mathbb{C})$. A vector field $X \in \mathscr{E}(s,r,d)$ is $\Gamma$-invariant if and only if*

1) *$\Gamma$ is a discrete rotation group of order $\mathtt{k} \in \mathscr{D}_{\mathrm{pole}} \cup \mathscr{D}_{\mathrm{zero}}$,*
2) *all three subsets of the divisor $[q_1,\dots,q_s]$, $[p_1,\dots,p_r]$, $[e_1,\dots,e_d]$ of $X$ are $\Gamma$-invariant.*

This result follows immediately from Proposition 2.19 and Corollary 2.21, where an additional equivalent characterization is given. It is clear that Condition (2) is necessary, however it comes as a (pleasant) surprise that Condition (1) provides sufficiency; compare with the case of $\Gamma$-invariant rational functions in [14, § 5] and $\Gamma$-invariant rational vector fields in [2].

As an application of Theorem 2 we realize those $X \in \mathscr{E}(s,r,d)$ with non-trivial isotropy, thus providing normal forms for $X \in \mathscr{E}(s,r,d)\setminus\mathscr{E}(s,r,d)_{\mathrm{id}}$, see Theorem 3.2 and Remark 3.3.3. All the above, together with Lemma 2.5, solves the problems of analytic (AC) and metric (MC) classification.

The above considerations lead to the following question: How does the $\Gamma$-invariance of the family $\{X\}$ translate to the associated family $\{\Psi_X\}$, for $X \in \mathscr{E}(s,r,d)$? The interplay between different singular complex analytic tensors is a general principle which is extensively used in the literature; partial statements appear as well known facts. Consider an arbitrary Riemann surface $M$ (not necessarily compact) and a discrete arbitrary $\Gamma < \mathrm{Aut}(M)$, its complex automorphism group. As far as we known and in accordance with [31], [22], [27], [26], we review a dictionary explaining the naturality and the richness of the theory: a statement regarding symmetry in the context of vector fields can be restated in any of the other contexts as follows.

**Dictionary under invariance.** *Let $\Gamma$ be a subgroup of $\mathrm{Aut}(M)$ having quotient $\mathrm{proj} : M \longrightarrow M/\Gamma$ to a Riemann surface. On $M$ there is a canonical one to one correspondence between*

1) *$\Gamma$-invariant singular complex analytic vector fields $X$.*
2) *$\Gamma$-invariant singular complex analytic differential forms $\omega_X$ satisfying $\omega_X(X) \equiv 1$.*
3) *$\Gamma$-invariant singular complex analytic orientable quadratic differentials $\omega_X \otimes \omega_X$.*
4) *$\Gamma$-invariant singular flat metrics $(M, g_X)$ and the $\Gamma$-invariant (real) geodesic unitary vector fields $\mathfrak{Re}(X)$ with suitable singularities.*
5) *$\Gamma$-invariant global singular complex analytic (possibly multivalued) distinguished parameters $\Psi_X$, see* (2).
6) *Pairs $(\mathcal{R}_X, \pi^*_{X,2}(\frac{\partial}{\partial t}))$ consisting of branched Riemann surfaces $\mathcal{R}_X = \{(z, \Psi_X(z))\}$, associated to the $\Gamma$-invariant maps $\Psi_X$.*

For completeness two versions of a *dictionary/correspondence* between tensors are presented. These are stated below as Proposition 4.1 and Proposition 4.2. Both apply to $X$ (and $\Psi_X$) in the family $\mathscr{E}(s, r, d)$. Examples of $\Gamma$-invariant rational vector fields and $\Gamma$-invariant vector fields $X \in \mathscr{E}(s, r, d)$ are presented in Figures 1–4.

In Proposition 4.3 and Remark 4.4, the geometrical meaning of the subgroups $\Gamma < \mathrm{Aut}(\mathbb{C})$ that leave invariant $X \in \mathscr{E}(s, r, d)$ is studied by considering the natural projection $\mathrm{proj} : \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{C}}/\Gamma$ and the associated vector fields $\mathrm{proj}_* X$ on $\widehat{\mathbb{C}}/\Gamma$, allowing the calculation of the singularities of $Y = \mathrm{proj}_* X$; see also Table 1.

The groups $\Gamma$ of symmetries of Riemann surfaces and their $\Gamma$-invariant holomorphic tensors have been the subject of study in different works from their own perspective. Klein was a pioneer, see [23]; for more recent work see [1], [16, Ch. V], [11] for the general theory of automorphism groups $\mathrm{Aut}(M)$ and spaces of differentials, [14, § 5] for invariant rational functions on $\widehat{\mathbb{C}}$, and references therein.

## 2 Aut($\mathbb{C}$)-fibre bundle structure on $\mathscr{E}(s, r, d)$

We work in the singular complex analytic category. Recall Definition 2.1 of [3], for our present purpose: a *singular analytic vector field $X$* is holomorphic on $\widehat{\mathbb{C}}_z \setminus \mathrm{Sing}(X)$, with *singular set* $\mathrm{Sing}(X)$ consisting of *zeros* denoted by $\mathcal{Z}$, *poles* denoted by $\mathcal{P}$, and an *isolated essential singularity* at $\infty \in \widehat{\mathbb{C}}$.

Because of Picard's theorem, even the local description of essential singularities of functions leads to a global study, see for instance [3] p. 129. Due to the diversity and wildness of essential singularities, a first step in understanding them is to restrict to the tame family $\mathscr{E}(s, r, d)$.

This section is devoted to the proof of assertions (2) and (3) of Theorem 1. In § 2.1 we provide explicit coordinates for $\mathscr{E}(s, r, d)$ that facilitate the work to be done. In § 2.2 we present the action of $\mathrm{Aut}(\mathbb{C})$ on $\mathscr{E}(s, r, d)$ and prove that $\mathscr{E}(s, r, d)_{\mathrm{id}}$ is a trivial principal $\mathrm{Aut}(\mathbb{C})$-bundle, proving assertion (3). In § 2.3 the simple arithmetic condition (8) is proved by first considering the more natural condition *"if $\mathtt{k} \nmid q$ and $\mathtt{k} \nmid r$ for all nontrivial common divisors $\mathtt{k}$ of $d$ and $(s - r - 1)$, then $\mathscr{E}(s, r, dr) = \mathscr{E}(s, r, d)_{\mathrm{id}}$".*

### 2.1 Coordinates for $\mathscr{E}(s, r, d)$

Viète's map $\mathscr{V}_s : (\mathbb{C}^s_{\mathrm{roots}}/\,\mathrm{Sym}(s)) \longrightarrow \mathbb{C}[z]_{=s}$ sends each unordered set $\{q_i\}_{i=1}^s$ of $s \geq 1$ roots to the respective monic polynomial of degree $s$, where $\mathrm{Sym}(s)$ is the symmetric group. Hence the polynomials describing $\mathscr{E}(s, r, d)$ as in (1) can be expressed as

$$
\begin{aligned}
Q(z) &= (z - q_1)\cdots(z - q_s) := z^s + a_1 z^{s-1} + \cdots + a_s,\\
P(z) &= (z - p_1)\cdots(z - p_r) := z^r + b_1 z^{r-1} + \cdots + b_r,\\
E(z) &= \mu(z - e_1)\cdots(z - e_d) := \mu(z^d + c_1 z^{d-1} + \cdots + c_d).
\end{aligned}
$$

For $d \geq 1$, $s, r \geq 0$, this provides the following diagram

$$
\begin{tikzcd}[column sep=small]
& \begin{array}{c} \mathscr{E}(s,r,d) \\ \dfrac{z^s + a_1 z^{s-1} + \cdots + a_s}{z^r + b_1 z^{r-1} + \cdots + b_r} \exp(\mu(z^d + c_1 z^{d-1} + \cdots + c_d))\frac{\partial}{\partial z} \\ = \dfrac{(z-q_1)\cdots(z-q_s)}{(z-p_1)\cdots(z-p_r)} \exp(\mu(z-e_1)\cdots(z-e_d))\frac{\partial}{\partial z} \end{array} \arrow[dl, leftrightarrow] \arrow[dr, shift left] & \\
\begin{array}{c} \mathcal{V} \subset \mathbb{C}^* \times \mathbb{C}^{s+r+d}_{\text{coef}} \\ (\mu, a_1, \ldots, a_s, b_1, \ldots, b_r, c_1, \ldots, c_d) \end{array} \arrow[rr, leftrightarrow] & & \begin{array}{c} \mathcal{U} \subset \mathbb{C}^* \times \left(\dfrac{\mathbb{C}^s_{\text{roots}}}{\mathrm{Sym}(s)}\right) \times \left(\dfrac{\mathbb{C}^r_{\text{roots}}}{\mathrm{Sym}(r)}\right) \times \left(\dfrac{\mathbb{C}^d_{\text{roots}}}{\mathrm{Sym}(d)}\right) \\ (\mu, \underbrace{[q_1, \ldots, q_s]}_{\mathcal{Z}}, \underbrace{[p_1, \ldots, p_r]}_{\mathcal{P}}, \underbrace{[e_1, \ldots, e_d]}_{\mathcal{E}}). \end{array} \arrow[ul, dashed, shift left]
\end{tikzcd}
\tag{10}
$$

In order to accurately describe the above diagram, note that in the set-theoretic category there is a bijection between $\mathscr{E}(s, r, d)$ and the open set

$$
\mathcal{U} = \{(\mu, \mathcal{Z}, \mathcal{P}, \mathcal{E}) \mid \mathcal{Z} \cap \mathcal{P} = \varnothing\} \subset \mathbb{C}^* \times \left(\frac{\mathbb{C}^s_{\text{roots}}}{\mathrm{Sym}(s)}\right) \times \left(\frac{\mathbb{C}^r_{\text{roots}}}{\mathrm{Sym}(r)}\right) \times \left(\frac{\mathbb{C}^d_{\text{roots}}}{\mathrm{Sym}(d)}\right).
$$

The resultant $\mathrm{Res}(Q, P)$ of the polynomials $Q(z)$ and $P(z)$ is equal to zero if and only if the polynomials have a common root, see [19] Lecture 3 pp. 35–36. For the open set $\mathcal{V} = \mathbb{C}^* \times (\mathbb{C}^{s+r+d}_{\text{coef}} \setminus \{\mathrm{Res}(Q, P) = 0\})$, the map $\mathcal{V} \subset \mathbb{C}^* \times \mathbb{C}^{s+r+d}_{\text{coef}} \longleftrightarrow \mathscr{E}(s, r, d)$ is also a bijection. This provides the natural definition.

**Definition 2.1.** For $d \geq 1$ and $s, r \geq 0$, the space of vector fields $\mathscr{E}(s, r, d)$ is a non-compact complex manifold of dimension $s + r + d + 1$, whose *complex structure* is inherited from that of $\mathbb{C}^* \times \mathbb{C}^{s+r+d+1}_{\text{coef}}$ via the map

$$
\begin{aligned}
\mathcal{V} \subset \mathbb{C}^* \times \mathbb{C}^{s+r+d+1}_{\text{coef}} \quad &\longleftrightarrow \quad \mathscr{E}(s, r, d)\\
(\mu, a_1, \ldots, a_s, b_1, \ldots, b_r, c_1, \ldots, c_d) \quad &\longleftrightarrow \quad \frac{z^s + a_1 z^{s-1} + \cdots + a_s}{z^r + b_1 z^{r-1} + \cdots + b_r} \exp(\mu(z^d + c_1 z^{d-1} + \cdots + c_d))\frac{\partial}{\partial z}.
\end{aligned}
\tag{11}
$$

If $sr = 0$ then we assume $a_s$ and/or $b_r$ to be 1.

Regarding Diagram (10), with the complex structure assigned by the above definition, the left hand side is an analytic diffeomorphism between $\mathcal{V}$ and $\mathscr{E}(s, r, d)$, and the right side (dashed arrow) provides an analytic submersion from $\mathcal{U}$ onto $\mathscr{E}(s, r, d)$; note that $\mathcal{U}$ is a singular complex analytic space.

**Definition 2.2.** The *divisor of* $X \in \mathscr{E}(s, r, d)$ is $\underbrace{[q_1, \ldots, q_s]}_{\mathcal{Z}}, \underbrace{[p_1, \ldots, p_r]}_{\mathcal{P}}, \underbrace{[e_1, \ldots, e_d]}_{\mathcal{E}}$, the unordered configuration of the roots of $Q(z)$, $P(z)$ and $E(z)$.

**Remark 2.3.** In order to account for the multiplicity of the roots of $Q$, $P$ and $E$, we allow the repetition of points in $\mathcal{Z}$, $\mathcal{P}$ and $\mathcal{E}$. For $X \in \mathscr{E}(s, r, d)$, another possibility for a *divisor* would be to consider $\mathcal{Z}$, $\mathcal{P}$ and the finite asymptotic values of $\int^z \omega_X$. However, as shown in Example 8.12 of [3], the finite asymptotic values do not in a unique way determine $X \in \mathscr{E}(s, r, d)$ for $d \geq 3$.

Obviously we assume $\mathcal{Z} \cap \mathcal{P} = \varnothing$, as in $\mathcal{U}$, however $(\mathcal{Z} \cup \mathcal{P}) \cap \mathcal{E}$ need not be empty. Different versions of the moduli space of $n$ points on the Riemann sphere under the action of $\mathrm{SL}(2, \mathbb{C})$ are currently considered in the literature; by using Mumford's geometric invariant theory GIT, see for instance [12], [20] and references therein or by Lie theory as in [24]. In our case we consider $s + r + d$ unordered points with three "flavors".

The naturality of the divisors should come as no surprise: for $X \in \mathscr{E}(s, r, d)$ there is an identification between the zero-dimensional object (the divisor) and the one-dimensional object (the singular analytic vector field); see [34] for other examples of the same phenomena.

## 2.2 The action of Aut($\mathbb{C}$) on $\mathscr{E}(s, r, d)$

The group Aut($\mathbb{C}$) of complex automorphisms determines the complex analytic equivalence (AC) and the isometric equivalence (MC) for $\mathscr{E}(s, r, d)$, as in the Introduction.

Compare the dimension of Aut($\mathbb{C}$) with the case of the group of smooth automorphisms of the sphere, $\mathrm{Diff}^\infty(\mathbb{S}^2)$, which is infinite dimensional, or with the case of a compact Riemann surface $M_g$ of genus $g \geq 2$ that has finite automorphism group, see [16, Ch. V]. The case $g = 1$ does admit a large automorphism group for $M_g$, however, in this work we only consider the Riemann sphere.

**Definition 2.4** ([31] §5, [27], [26], [3]). For $X \in \mathscr{E}(s, r, d)$, the *singular flat Riemannian metric* $g_X$ on $\mathbb{C} \setminus (\mathcal{P} \cup \mathcal{Z})$ is the pullback under $\Psi_X$ of the usual flat metric $\delta$ on $\mathbb{C}_t$.

We note that $\Psi_X : (\mathbb{C} \setminus (\mathcal{P} \cup \mathcal{Z}), g_X) \longrightarrow (\mathbb{C}, \delta)$ is a local isometry and that the trajectories of $\mathfrak{Re}\,(X)$, $\mathfrak{Im}\,(X)$ are orthonormal (unitary) geodesics. In fact, $(\widehat{\mathbb{C}}, g_X)$ has singular set consisting of $\mathcal{P} \cup \mathcal{Z} \cup \{\infty\}$.

**Lemma 2.5.** *Let $X_1, X_2 \in \mathscr{E}(s, r, d)$ be two vector fields.*

(1) $X_2 = T^* X_1$ *if and only if* $\Psi_{X_2} = T^* \Psi_{X_1}$.

(2) *If $X_2 = T^* X_1$ then the associated singular flat metrics $g_{X_1}$ and $g_{X_2}$ are orientation-preserving isometrically equivalent.*

(3) *Conversely, if $(\widehat{\mathbb{C}}, g_{X_1})$ and $(\widehat{\mathbb{C}}, g_{X_2})$ are orientation-preserving isometrically equivalent, then necessarily $e^{i\theta} X_1 = T^* X_2$, for $(T, e^{i\theta}) \in \mathrm{Aut}(\mathbb{C}) \times \mathbb{S}^1$.*

*Proof.* For assertions (2) and (3) use the ideas in [3] p. 137. □

Denote the *stabilizer* or *isotropy group* of $X \in \mathscr{E}(s, r, d)$ by

$$\mathrm{Aut}(\mathbb{C})_X = \{T \in \mathrm{Aut}(\mathbb{C}) \mid T^* X = X\}.$$

We say that $\Gamma < \mathrm{Aut}(\mathbb{C})$ *leaves invariant* $X \in \mathscr{E}(s, r, d)$ if $\Gamma$ is a subgroup of $\mathrm{Aut}(\mathbb{C})_X$. Further, let

$$\mathscr{E}(s, r, d)_{\mathrm{id}} = \{X \in \mathscr{E}(s, r, d) \mid \mathrm{Aut}(\mathbb{C})_X = \{\mathrm{id}\}\}$$

be the family consisting of those $X$ with trivial isotropy. It is immediate that $\mathscr{E}(s, r, d)_{\mathrm{id}}$ is open and dense in $\mathscr{E}(s, r, d)$. Finding necessary and sufficient conditions for the equality is a central question.

The action (3) of $\mathrm{Aut}(\mathbb{C}) = \{T : w \mapsto \mathtt{a}w + \mathtt{b} = z\}$ on $\mathscr{E}(s, r, d)$ by pullback is

$$\begin{array}{rcl} \mathcal{A} : \mathrm{Aut}(\mathbb{C}) \times \mathscr{E}(s, r, d) & \longrightarrow & \mathscr{E}(s, r, d) \\ (\mathtt{a}w + \mathtt{b}, X(z)) & \longmapsto & T^* X(w). \end{array}$$

According to Definition 2.1 and (10), the expression for $T^* X(w)$ is

$$\begin{aligned} &T^*\Big(\frac{(z - q_1)\cdots(z - q_s)}{(z - p_1)\cdots(z - p_r)} \exp(\mu(z - e_1)\cdots(z - e_d))\frac{\partial}{\partial z}\Big)(w) \\ &\quad = \frac{(w - T^{-1}(q_1))\cdots(w - T^{-1}(q_s))}{(w - T^{-1}(p_1))\cdots(w - T^{-1}(p_r))} \exp\Big(\mathtt{a}^d \mu(w - T^{-1}(e_1))\cdots(w - T^{-1}(e_d)) + \log\frac{\mathtt{a}^s}{\mathtt{a}^{r+1}}\Big)\frac{\partial}{\partial w}. \end{aligned} \tag{12}$$

With this expression for the action and the convention that $\Gamma$ is a non-trivial discrete rotation group, i.e.

$$\Gamma = \{T^\ell_{\mathtt{k},C}(w) = e^{i2\pi\ell/\mathtt{k}} w + \mathtt{b} \mid \ell = 1, \ldots, \mathtt{k}\} \cong \mathbb{Z}_{\mathtt{k}}, \quad \text{with center of rotation } C \doteq \frac{\mathtt{b}}{1 - e^{i2\pi/\mathtt{k}}} \in \mathbb{C} \tag{13}$$

for some $\mathtt{k} \in \mathbb{N} \setminus \{1\}$, we prove the following.

**Lemma 2.6.** *Let $X \in \mathscr{E}(s, r, d)$ and consider the set*

$$\mathscr{D} = \{\mathrm{k} \in \mathbb{N} \mid \mathrm{k} \text{ is a non-trivial common divisor of } d \text{ and } (s - r - 1)\}. \tag{14}$$

*A non-trivial subgroup $\Gamma < \mathrm{Aut}(\mathbb{C})$ leaves invariant $X$ if and only if the following two conditions hold:*

(1) *$\Gamma$ is a discrete rotation group of some order $\mathrm{k} \in \mathscr{D}$, i.e. $\Gamma$ is as in (13).*

(2) *All three subsets $\mathcal{Z}$, $\mathcal{P}$ and $\mathcal{E}$ of the divisor of $X$ are $\Gamma$-invariant, in particular each subset is evenly distributed on concentric circles about $C$.*

Of course $\mathrm{Aut}(\mathbb{C})_X$ denotes the biggest subgroup $\Gamma$ that leaves invariant $X$, so we immediately have:

**Corollary 2.7.** *The isotropy group $\mathrm{Aut}(\mathbb{C})_X$ of $X \in \mathscr{E}(s, r, d)$ is non-trivial if and only if the following two conditions hold:*

(1) (*arithmetic condition*) *$\mathrm{Aut}(\mathbb{C})_X$ is a discrete rotation group of order $\mathrm{k} \in \mathscr{D} \neq \varnothing$,*

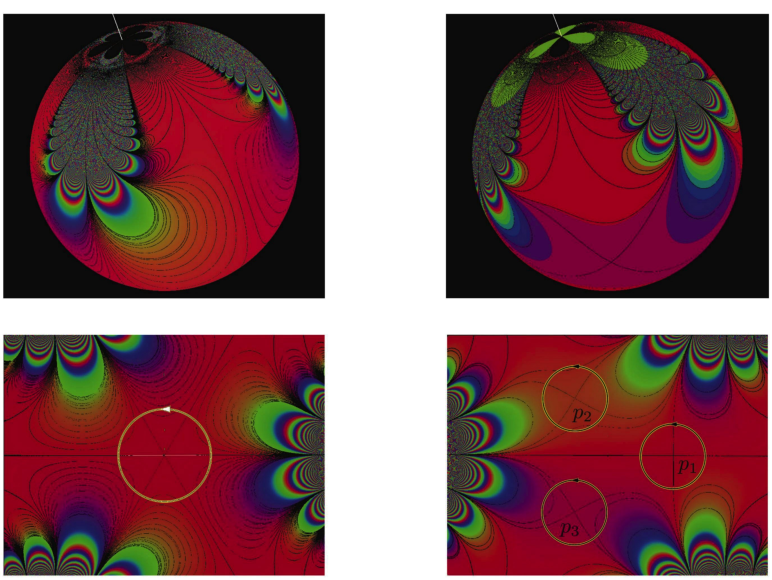
(2) (*geometric condition*) *all three subsets $\mathcal{Z}$, $\mathcal{P}$ and $\mathcal{E}$ of the divisor of $X$ are $\mathrm{Aut}(\mathbb{C})_X$-invariant.* □

**Remark 2.8.** By Definition 2.2, the geometric condition (2) implies that $C \in \mathbb{C}$ coincides with the *barycenters* $\mathrm{Z}$ of $\mathcal{Z}$, $\mathrm{P}$ of $\mathcal{P}$ and $\mathrm{E}$ of $\mathcal{E}$. This is a necessary but not sufficient condition for having non-trivial isotropy group.

In order to gain some intuition, consider the following simple examples.

**Example 2.9.** Consider $X(z) = -\frac{e^{z^3}}{3z^2}\frac{\partial}{\partial z} \in \mathscr{E}(0, 2, 3)$. Its divisor is $\mathcal{Z} = \varnothing$, $\mathcal{P} = [0, 0]$, $\mathcal{E} = [0, 0, 0]$ which is clearly invariant by $\mathbb{Z}_3$. Moreover the non-trivial common divisor of $d = 3$ and $s - r - 1 = 0 - 2 - 1 = -3$ is $\mathscr{D} = \{3\}$. Hence, by Corollary 2.7 it follows that the isotropy group of $X$ is $\mathbb{Z}_3$, see Figure 1 (A).

**Example 2.10.** Consider $X(z) = \frac{e^{z^3}}{3z^3-1}\frac{\partial}{\partial z} \in \mathscr{E}(0, 3, 3)$. Its divisor is $\mathcal{Z} = \varnothing$, $\mathcal{P} = [1/3, e^{i2\pi/3}/3, e^{-i2\pi/3}/3]$, $ES = [0, 0, 0]$ which is clearly invariant by $\mathbb{Z}_3$. However the common divisor of $d = 3$ and $s - r - 1 = 0 - 3 - 1 = -4$ is $1$, hence $\mathscr{D} = \varnothing$. So, even though $X$ satisfies the geometric condition of Corollary 2.7, it does not satisfy the arithmetic condition, which implies that its isotropy group $\mathrm{Aut}(\mathbb{C})_X$ is the identity, see Figure 1 (B).




(A) (B)

**Figure 1:** Phase portrait of Examples 2.9 and 2.10. Borders of the strip flows correspond to streamlines of $\mathfrak{Re}(X)$. (A) The vector field $X(z) = -\frac{e^{z^3}}{3z^2}\frac{\partial}{\partial z} \in \mathscr{E}(0, 2, 3)$ with isotropy group isomorphic to $\mathbb{Z}_3$. (B) The vector field $X(z) = \frac{e^{z^3}}{3z^3-1}\frac{\partial}{\partial z} \in \mathscr{E}(0, 3, 3)$ with trivial isotropy group. On the top we have the view on $\widehat{\mathbb{C}}$ and on the bottom the affine view.

**Remark 2.11.** All the figures of vector fields were obtained by the visualization techniques presented in [4]. In particular, the streamlines of $\mathfrak{Re}\,(X)$ are represented as the borders of the *strip flows* (represented as bands of the same color) or, in particular cases that need to be emphasized, as individual trajectories. See § 6.2 of [4] for further explanation of the numerical behaviour at zeros, poles and essential singularities.

*Proof of Lemma* 2.6. Let $X \in \mathscr{E}(s,r,d)$ be a singular complex analytic vector field. From (12) it is clear that $T^*X = X$ for some $T(w) \in \mathrm{Aut}(\mathbb{C})$ if and only if

(a) both $\mathcal{Z}$ and $\mathcal{P}$ (with multiplicities) are $T$-invariant, and

(b) $\mu(z-e_1)\cdots(z-e_d) = \mathrm{a}^d\mu(w-T^{-1}(e_1))\cdots(w-T^{-1}(e_d)) + \log(\frac{\mathrm{a}^s}{\mathrm{a}^{r+1}})\pmod{2\pi i}$.

From (a) and since $s, r < \infty$, it immediately follows that the affine transformation $T$ is a rational rotation as in (13), for some $\mathtt{k} \in \mathbb{N}\setminus\{1\}$, having two fixed points: the center of rotation $C \in \mathbb{C}$ and $\infty \in \widehat{\mathbb{C}}$. From this point we convene that $\mathrm{a} = e^{i2\pi/\mathtt{k}}$, so $T(w) = \mathrm{a}w + \mathrm{b}$ is a generator of the group of said rational rotations. Noting that $d < \infty$, if $\mathtt{k}$ is a common divisor of $(s-r-1)$ and $d$, and $\mathcal{E}$ is $T$-invariant, then $T^*X = X$.

We now prove the converse statement. Condition (b) imposes geometric/algebraic conditions on the divisor $\mathcal{E}$ and on $\mathrm{a}$, which are equivalent to the following system of $d+1$ equations:

$$\begin{cases} \mu = \mathrm{a}^d\mu \\ \mu(e_1+\cdots+e_d) = \mathrm{a}^d\mu(T^{-1}(e_1)+\cdots+T^{-1}(e_d)) \\ \vdots \\ \mu(\Sigma_j e_1\cdots\widehat{e_j}\cdots e_d) = \mathrm{a}^d\mu(\Sigma_j T^{-1}(e_1)\cdots\widehat{T^{-1}(e_j)}\cdots T^{-1}(e_d)) \\ \mu(e_1\cdots e_d) = \mathrm{a}^d\mu(T^{-1}(e_1)\cdots T^{-1}(e_d)) + (s-r-1)\log|a| + i(s-r-1)\arg(\mathrm{a}) \pmod{2\pi i}. \end{cases} \tag{15}$$

Since $\mathrm{a} = e^{i2\pi/\mathtt{k}}$, the last equation of (15) is

$$\mu(e_1\cdots e_d) = e^{i2\pi d/\mathtt{k}}\mu(T^{-1}(e_1)\cdots T^{-1}(e_d)) + i2\pi(s-r-1)/\mathtt{k} \pmod{2\pi i}. \tag{16}$$

We first show that $\mathtt{k}$ is a common divisor of $(s-r-1)$ and $d$. Considering the first equation of (15), we immediately obtain that $\mathtt{k}$ divides $d$. On the other hand, since $T$ is a rational rotation,

$$\begin{cases} \left|\mu(T^{-1}(e_1)\cdots T^{-1}(e_d))\right| = \left|\mu(e_1\cdots e_d)\right| \\ \arg(\mu(T^{-1}(e_1)\cdots T^{-1}(e_d))) = \arg(\mu(e_1\cdots e_d)) - d(2\pi/\mathtt{k}). \end{cases} \tag{17}$$

Moreover, arg is a $2\pi$-periodic function and $\mathtt{k} \mid d$, hence $\mu(e_1\cdots e_d) = \mu(T^{-1}(e_1)\cdots T^{-1}(e_d))$. Thus (16) simplifies to $i2\pi(s-r-1)/\mathtt{k} \pmod{2\pi i} = 0$. So $\mathtt{k}$ must also divide $(s-r-1)$.

Considering once again Condition (b) and since $\mathtt{k}$ is a common divisor of $(s-r-1)$ and $d$, it is clear that $\mathcal{E}$ must also be $T$-invariant. □

In particular, if $\mathtt{k} = \gcd(s-r-1, d) = 1$ then $\mathrm{Aut}(\mathbb{C})_X = \{\mathrm{id}\}$. As usual, the triviality of the isotropy group of $X \in \mathscr{E}(s,r,d)$ has geometric implications for the quotient spaces.

**Remark 2.12.** From the description (12) of the action $\mathcal{A}$ of $\mathrm{Aut}(\mathbb{C})$ on $\mathscr{E}(s,r,d)_{\mathrm{id}}$ in terms of the divisor of $X$, it is clear that $\mathcal{A}$ is a proper map for $s+r+d \geq 2$.

It is well known, see e.g. [15] p. 53, that the quotient $\mathscr{E}(s,r,d)_{\mathrm{id}}/\mathrm{Aut}(\mathbb{C})$ for the action (4) is a manifold of dimension $\dim(\mathscr{E}(s,r,d)_{\mathrm{id}}) - \dim(\mathrm{Aut}(\mathbb{C}))$. Naturally $\mathscr{E}(s,r,d)_{\mathrm{id}}$ is open and dense in $\mathscr{E}(s,r,d)$, thus $\dim(\mathscr{E}(s,r,d)_{\mathrm{id}}) = \dim(\mathscr{E}(s,r,d))$. The analogous fact holds for the action (5) of $\mathrm{Aut}(\mathbb{C})\times\mathbb{S}^1$. From this it follows that

$$\pi_1 : \mathscr{E}(s,r,d)_{\mathrm{id}} \longrightarrow \frac{\mathscr{E}(s,r,d)_{\mathrm{id}}}{\mathrm{Aut}(\mathbb{C})} \quad \text{and} \quad (\pi_2\circ\pi_1) : \mathscr{E}(s,r,d)_{\mathrm{id}} \longrightarrow \frac{\mathscr{E}(s,r,d)_{\mathrm{id}}}{\mathrm{Aut}(\mathbb{C})\times\mathbb{S}^1}$$

in (9) are holomorphic and real-analytic principal $\mathrm{Aut}(\mathbb{C})$ and $(\mathrm{Aut}(\mathbb{C})\times\mathbb{S}^1)$-bundles, respectively.

**Lemma 2.13.** *If $s+r+d \geq 2$ and $d \geq 1$, then $\mathscr{E}(s,r,d)_{\mathrm{id}}$ is a holomorphic trivial principal* $\mathrm{Aut}(\mathbb{C})$*-bundle over a non-compact complex manifold.*

When $d = 0$ the isotropy group $\mathrm{Aut}(\mathbb{C})_X$ for $X \in \mathscr{E}(s, r, 0)$ does not generically fix $\infty \in \widehat{\mathbb{C}}$, see § 4.3 for further details.

*Proof.* On $\mathscr{E}(s, r, d)_{\mathrm{id}}$, every fiber is a copy of $\mathrm{Aut}(\mathbb{C})$. We shall explicitly exhibit three choices of *global* sections. We recall that $X \in \mathscr{E}(s, r, d)$ can be expressed as

$$X(z) = \frac{z^s + a_1 z^{s-1} + \cdots + a_s}{z^r + b_1 z^{r-1} + \cdots + b_r} \exp(\mu(z^d + c_1 z^{d-1} + c_2 z^{d-2} + \cdots + c_d))\frac{\partial}{\partial z}$$

and consider a “gauge transformation prospect”

$$\mathcal{G} : \mathscr{E}(s, r, d)_{\mathrm{id}}/\mathrm{Aut}(\mathbb{C}) \longrightarrow \mathrm{Aut}(\mathbb{C}), \quad [X] \longmapsto G(w) = \mathtt{a}w + \mathtt{b} \tag{18}$$

with suitable $\mathtt{a}$ and $\mathtt{b}$ that will depend on the specific representative $X$ of the class $[X]$. We now choose appropriate $\mathtt{a}$ and $\mathtt{b}$. The choice $\mathtt{a} = \mu^{-1/d}$ forces the polynomial that appears in the exponential of the expression for $(G^*X)(w)$ to be monic. Recalling that the barycenters of $\mathcal{Z}$, $\mathcal{P}$ and $\mathcal{E}$ are $\mathtt{Z} = -a_1/s$, $\mathtt{P} = -b_1/r$ and $\mathtt{E} = -c_1/d$ respectively, we shall choose $\mathtt{b}$ such that one of the polynomials appearing in the expression for $(G^*X)(w)$ is centered. This provides us with the following three explicit *global* sections:

(1) $s+r+d \geq 2$ and $d \geq 2$: In this case, given $[X] \in \mathscr{E}(s, r, d)_{\mathrm{id}}/\mathrm{Aut}(\mathbb{C})$, choose $\mathtt{b} = -\frac{c_1}{d} = \mathtt{E}$ (so $G^{-1}(\mathtt{E}) = 0$); we then obtain the global section

$$\begin{aligned} \sigma : \mathscr{E}(s, r, d)_{\mathrm{id}}/\mathrm{Aut}(\mathbb{C}) &\longrightarrow \mathscr{E}(s, r, d)_{\mathrm{id}} \\ [X] &\longmapsto (G^*X)(w) = \frac{w^s + \widetilde{a}_1 w^{s-1} + \cdots + \widetilde{a}_s}{w^r + \widetilde{b}_1 w^{r-1} + \cdots + \widetilde{b}_r} \exp\left(w^d + \widetilde{c}_2 w^{d-2} + \cdots + \widetilde{c}_d\right)\tfrac{\partial}{\partial w}. \end{aligned} \tag{19}$$

That is, all three polynomials are monic and the one appearing in the exponential of the expression for $(G^*X)(w)$ is centered. A special case is when $\mathcal{Z} = \mathcal{P} = \varnothing$ and $d \geq 2$,

$$(G^*X)(w) = \exp(w^d + \widetilde{c}_2 w^{d-2} + \cdots + \widetilde{c}_d)\frac{\partial}{\partial w}.$$

Compare with § 8.6 of [3].

(2) $s+r+d \geq 2$ and $s \geq 1$: In this case, given $[X] \in \mathscr{E}(s, r, d)_{\mathrm{id}}/\mathrm{Aut}(\mathbb{C})$, choose $\mathtt{b} = -\frac{a_1}{s} = \mathtt{Z}$ (so $G^{-1}(\mathtt{Z}) = 0$); we then obtain the global section

$$\begin{aligned} \sigma : \mathscr{E}(s, r, d)_{\mathrm{id}}/\mathrm{Aut}(\mathbb{C}) &\longrightarrow \mathscr{E}(s, r, d)_{\mathrm{id}} \\ [X] &\longmapsto (G^*X)(w) = \frac{w^s + \widetilde{a}_2 w^{s-2} + \cdots + \widetilde{a}_s}{w^r + \widetilde{b}_1 w^{r-1} + \cdots + \widetilde{b}_r} \exp\left(w^d + \widetilde{c}_1 w^{d-1} + \cdots + \widetilde{c}_d\right)\tfrac{\partial}{\partial w}. \end{aligned} \tag{20}$$

That is, all three polynomials are monic and the one corresponding to the zeros of $(G^*X)(w)$ is centered.

(3) $s+r+d \geq 2$ and $r \geq 1$: In this case, given $[X] \in \mathscr{E}(s, r, d)_{\mathrm{id}}/\mathrm{Aut}(\mathbb{C})$, choose $\mathtt{b} = -\frac{b_1}{s} = \mathtt{P}$ (so $G^{-1}(\mathtt{P}) = 0$); we then obtain the global section

$$\begin{aligned} \sigma : \mathscr{E}(s, r, d)_{\mathrm{id}}/\mathrm{Aut}(\mathbb{C}) &\longrightarrow \mathscr{E}(s, r, d)_{\mathrm{id}} \\ [X] &\longmapsto (G^*X)(w) = \frac{w^s + \widetilde{a}_1 w^{s-1} + \cdots + \widetilde{a}_s}{w^r + \widetilde{b}_2 w^{r-2} + \cdots + \widetilde{b}_r} \exp\left(w^d + \widetilde{c}_1 w^{d-1} + \cdots + \widetilde{c}_d\right)\tfrac{\partial}{\partial w}. \end{aligned} \tag{21}$$

That is, all three polynomials are monic and the one corresponding to the poles of $(G^*X)(w)$ is centered.

Finally, note that any $(s, r, d)$ such that $\mathscr{E}(s, r, d)_{\mathrm{id}}$ is an $\mathrm{Aut}(\mathbb{C})$-bundle falls in one of the above cases. The non-compactness of the bundle basis follows by simple inspection of (19), (20) and (21). □

**Definition 2.14.** A *normal form* of $X \in \mathscr{E}(s, r, d)$ is a representative of its class under the pullback action $\mathcal{A}$ of $\mathrm{Aut}(\mathbb{C})$.

The explicite global sections in Lemma 2.13 immediately yield the following.

**Corollary 2.15** (Normal forms for $\mathscr{E}(s, r, d)_{\mathrm{id}}$). *For $s + r + d \geq 2$ and $d \geq 1$, global normal forms for $X \in \mathscr{E}(s, r, d)_{\mathrm{id}}$ are given by $(G^*X)(w)$ as in* (19), (20) *and* (21). □

**Remark 2.16.** The term *global* refers to the fact that the expressions for $(G^*X)(w)$ given by (19), (20) and (21) are valid for every $X \in \mathscr{E}(s, r, d)_{\text{id}}$ and also on the whole Riemann sphere $\widehat{\mathbb{C}}$.

**Remark 2.17** (Singular flat metrics in normal form). Recalling the $\text{Aut}(\mathbb{C}) \times \mathbb{S}^1$-action (5), in particular $e^{i\theta} : X \to e^{i\theta}X$ preserves the singular flat metric $g_X$ but changes the slope of the geodesic vector field $\mathfrak{Re}\,(X)$.

The normal forms given by (19), (20) and (21) can be extended to consider the action of $\text{Aut}(\mathbb{C}) \times \mathbb{S}^1$ by choosing $e^{i\theta} \in \mathbb{S}^1$ such that $e^{i\theta}e^{\widetilde{c}_d} \in \mathbb{R}^+$. This then produces the desired singular flat metric $(\widehat{\mathbb{C}}, g_X)$ in normal form. The respective third statement of Theorem 1 is now proved.

## 2.3 Obstructions for the existence of non-trivial symmetries

In this section we characterize the vector fields $X \in \mathscr{E}(s, r, d)$ that have non-trivial isotropy group $\text{Aut}(\mathbb{C})_X \cong \mathbb{Z}_{\mathtt{k}}$, with $\mathtt{k} \in \mathscr{D}$; recall (14). From Corollary 2.7 we see that there are two obstructions for the existence of $X \in \mathscr{E}(s, r, d)$ with $\text{Aut}(\mathbb{C})_X \neq \{\text{id}\}$. In particular in this section we are interested in examining Condition (2) of Corollary 2.7, so we assume that $\mathtt{k} \in \mathscr{D} \neq \varnothing$.

With this in mind we consider the partition of $\mathcal{Z}$, $\mathcal{P}$ and $\mathcal{E}$ into orbits under the action of $\text{Aut}(\mathbb{C})_X$. Recall that a set $\mathcal{B}$ is $\mathtt{k}$-*evenly distributed on circles centered about* $C$ if in addition to the elements of $\mathcal{B}$ being evenly distributed on the circles, the number of elements of $\mathcal{B}$ on each circle of positive radius is a multiple of $\mathtt{k}$.

**Remark 2.18** (Orbit structure). Recalling that $C$ is the fixed point of the discrete rotation group $\Gamma$ of order $\mathtt{k} \in \mathscr{D} \neq \varnothing$, it is evident that *the configurations* $\mathcal{Z}$, $\mathcal{P}$ *and* $\mathcal{E}$ *are* $\text{Aut}(\mathbb{C})_X$*-invariant if and only if each configuration* $\mathcal{Z}$, $\mathcal{P}$ *and* $\mathcal{E}$ *is* $\mathtt{k}$*-evenly distributed on circles* (*of any given radii* $R \geq 0$) *centered about the fixed point* $C$*, generically on more than one circle*. Moreover, as will be shown, $C \in \mathcal{Z} \cup \mathcal{P}$.

From (10), it is clear that the set of poles and zeros of $X \in \mathscr{E}(s, r, d) \setminus \mathscr{E}(s, r, d)_{\text{id}}$ do not intersect, that is $\mathcal{Z} \cap \mathcal{P} = \varnothing$; however $\mathcal{E}$ is unrelated to $\mathcal{Z}$ and $\mathcal{P}$, in the sense that $\mathcal{E} \cap \mathcal{Z}$ and $\mathcal{E} \cap \mathcal{P}$ may be non-empty.

The search of a constructive alternative for Lemma 2.6 is expressed as (A), (B) and (C) below.

(A) Choose $\mathtt{k} \in \mathscr{D}$ and let it remain fixed.

(B) For the $d$ roots $\mathcal{E}$ of the polynomial $E(z)$, recall the orbit structure of Remark 2.18 and proceed as follows:

i) Consider the partitions of $d$ as a sum of positive integers, say

$$\text{Part}(d) = \{\{d_{\iota,\kappa}\}_{\iota=1}^{\ell_\kappa} \mid d = \textstyle\sum_{\iota=1}^{\ell_\kappa} d_{\iota,\kappa},\, \kappa = 1, \ldots, p(d)\},$$

where $p(d)$ is the partition function of $d$ (the number of possible integer partitions of $d$).

ii) Let $\{d_{j,\kappa}\}_{j=1}^{\ell_\kappa}$ be a partition such that $d_{j,\kappa} = \mathtt{k}\nu_j$ for some $\nu_j \in \mathbb{N}$, say

$$d = d_{1,\kappa} + d_{2,\kappa} + \cdots + d_{j,\kappa} + \cdots + d_{\ell_\kappa,\kappa}.$$

Choose this partition and place $\mathtt{k}$ equally spaced roots on a circle $L_j$ centered about $C$ of a chosen radius $R_j > 0$, all with the same multiplicity $\nu_j$.

iii) If there are still some $d_{\iota,\kappa} = \mathtt{k}\nu_\iota$, for $\nu_\iota \in \mathbb{N}$ in the same partition, place $\mathtt{k}$ equally spaced roots on a circle $L_i$ centered about $C$ (possibly the same circle as before but the roots are to be placed on different positions), once again each root with multiplicity $\nu_\iota$. Repeat (Biii) if possible or proceed to (Biv) below.

iv) Finally, place the rest of the roots at $C$; hence $C$ will be a root of $E(z)$ of multiplicity equal to $d$ minus the number of roots (counted with multiplicity) already placed on circles $\{L_i\}$ of positive radius.

(C) For the placement of the poles and zeros of $X$, we proceed as in (B) replacing "$d$" and "roots of $E(z)$" with "$r$" and "roots of $P(z)$", and "$s$" and "roots of $Q(z)$", respectively. We have $\mathtt{k} \mid (s - r - 1)$, hence $\mathtt{k} \mid s$ and $\mathtt{k} \mid r$ cannot occur simultaneously, which leaves the following cases:

(a) $\mathtt{k} \nmid s$ and $\mathtt{k} \nmid r$.
(b) $\mathtt{k} \mid s$ and $\mathtt{k} \nmid r$.
(c) $\mathtt{k} \nmid s$ and $\mathtt{k} \mid r$.

Case (Ca) cannot occur: if $\mathtt{k} \nmid s$ then we must place a zero of $X$ at the fixed point $C$ of the rotation (by considering the partitions of $s$ as a sum of positive integers as in (B); it follows from the orbit structure, i.e. Remark 2.18, that at least one zero of $X$ must be placed at $C$). Similarly if $\mathtt{k} \nmid r$ then we must place a pole of $X$ at the fixed point $C$ of the rotation; but $\mathcal{Z} \cap \mathcal{P} = \varnothing$.
Case (Cb) requires a pole of $X$ at the fixed point $C$ and Case (Cc) requires a zero of $X$ at the fixed point $C$. Thus either (Cb) or (Cc) occurs, but not both.

The arithmetic conditions stated as Cases (Cb) and (Cc) above can be interpreted geometrically as *$C$ has to be either a pole or a zero of $X$*, respectively. However, since $X$ has non-trivial isotropy group, there are local restrictions on the allowed multiplicity $\nu \in \mathbb{Z} \setminus \{0\}$ of $C$.

Consider the phase portrait of $\mathfrak{Re}(X)$ in a neighborhood of the center of rotation $C \in \mathbb{C}$. This, together with the fact that the non-trivial isotropy groups are the discrete rotation groups $\mathbb{Z}_{\mathtt{k}}$ with $\mathtt{k} \in \mathscr{D}$, implies the following arithmetic conditions:

a) When $C$ is a pole of $X$ of multiplicity $-\nu \leq -1$, the phase portrait of $\mathfrak{Re}(X)$ in a neighborhood of $C$ consists of $2(\nu+1)$ hyperbolic sectors, see for instance [3]. Since hyperbolic sectors come in pairs, $\mathtt{k} \mid (\nu+1)$ is required. Furthermore the rest of the poles and all the zeros are $\mathtt{k}$-evenly distributed on circles centered about $C$, thus $r = \mathtt{k}k_r + \nu$ with $\mathtt{k} \mid (\nu+1)$ and $s = \mathtt{k}k_s$, so in fact $\mathtt{k} \mid (r+1)$.
b) When $C$ is a zero of $X$, the phase portrait of $\mathfrak{Re}(X)$ has two geometrically different cases: $\nu = 1$ and $\nu \geq 2$, see [3]. However, as it turns out, it will be enough to consider only the case $\nu \geq 2$: the phase portrait of $\mathfrak{Re}(X)$ in a neighborhood of $C$ consists of $2(\nu-1)$ elliptic sectors. Since elliptic sectors come in pairs, $\mathtt{k} \mid (\nu-1)$ is required. Furthermore the rest of the zeros and all the poles are $\mathtt{k}$-evenly distributed on circles centered about $C$, thus $s = \mathtt{k}k_s + \nu$ with $\mathtt{k} \mid (\nu-1)$ and $r = \mathtt{k}k_r$, so in fact $\mathtt{k} \mid (s-1)$ (note that the case $\nu = 1$ also satisfies this arithmetic condition).

With this in mind we can now restate Lemma 2.6.

**Proposition 2.19.** *Let $X \in \mathscr{E}(s, r, d)$. The discrete rotation group $\Gamma$ as in* (13) *with $\mathtt{k} \geq 2$ leaves invariant $X$ if and only if the following three conditions are satisfied:*

(1) *$\mathtt{k}$ is a common divisor of $(s-r-1)$ and $d$,*
(2) *either $\mathtt{k} \mid s$ and $\mathtt{k} \nmid r$, or $\mathtt{k} \nmid s$ and $\mathtt{k} \mid r$,*
(3) *$\mathcal{E}$ is $\mathtt{k}$-evenly distributed on circles centered about $C$ (with the possibility that there is a root of $E$ at $C$ with multiplicity divisible by $\mathtt{k}$).*

*Otherwise* $\mathrm{Aut}(\mathbb{C})_X = \{\mathrm{id}\}$.

*Proof.* Condition (1) is a restatement of (1) of Lemma 2.6. The discussion previous to the statement of Proposition 2.19 together with (12) are enough to show that Conditions (2) and (3) of Proposition 2.19 are equivalent to Condition (2) of Lemma 2.6. □

**Remark 2.20.** 1. Proposition 2.19 will provide an explicit realization of those $X \in \mathscr{E}(s, r, d)$ that are $\Gamma$-invariant for $\Gamma \cong \mathbb{Z}_{\mathtt{k}}$ with $\mathtt{k} \in \mathscr{D}$; see § 3.1.

2. Note that the divisibility conditions on the multiplicity $\nu \in \mathbb{Z} \setminus \{0\}$ of the pole or zero at the fixed point $C$ are automatically satisfied.
That is, if (1), (3) and ($\mathtt{k} \mid s$ and $\mathtt{k} \nmid r$) are satisfied, then $r = \mathtt{k}k_r + \nu$ for some $\nu \geq 1$ with $\mathtt{k} \nmid \nu$ and $\mathtt{k} \mid (\nu+1)$. Thus $\nu = \mathtt{k}m - 1$ for $1 \leq m \leq \frac{r+1}{\mathtt{k}}$.
Similarly, if (1), (3) and ($\mathtt{k} \nmid s$ and $\mathtt{k} \mid r$) are satisfied, then $s = \mathtt{k}k_s + \nu$ for some $\nu \geq 1$ with $\mathtt{k} \nmid \nu$ and $\mathtt{k} \mid (\nu-1)$. Thus $\nu = \mathtt{k}m + 1$ for $0 \leq m \leq \frac{s-1}{\mathtt{k}}$.
Both statements follow from (12).

3. Conditions (1), (2) and (3) of Proposition 2.19 can be restated as:
   (1) The order $\mathtt{k}$ of $\Gamma$ is either a common divisor of $s$, $r+1$ and $d$, in which case "$X$ has a pole as a fixed point of $\Gamma$ of multiplicity $-\nu = -(\mathtt{k}m-1)$ with $1 \leq m \leq \frac{r+1}{\mathtt{k}}$", or $\mathtt{k}$ is a common divisor of $s-1$, $r$ and $d$, in which case "$X$ has a zero as a fixed point of $\Gamma$ of multiplicity $\nu = \mathtt{k}m+1$ for $0 \leq m \leq \frac{s-1}{\mathtt{k}}$".

(2) The rest of the poles and zeros of $X$ are $\mathtt{k}$-evenly distributed on circles about $C$.
(3) The roots $\mathcal{E}$ of $E(z)$ are $\mathtt{k}$-evenly distributed on circles about $C$, possibly with a root at $C$ with multiplicity divisible by $\mathtt{k}$.

As an immediate consequence of Proposition 2.19 and Remark 2.20.3 we have:

**Corollary 2.21.** *The following are equivalent.*

1) $\mathscr{E}(s,r,d) = \mathscr{E}(s,r,d)_{\mathrm{id}}$.
2) (a) $\gcd(s-r-1, d) = 1$, *or* (b) $\mathtt{k} \nmid s$ *and* $\mathtt{k} \nmid r$ *for all non-trivial common divisors* $\mathtt{k}$ *of* $(s-r-1)$ *and* $d$.
3) $\gcd(s, r+1, d) = \gcd(s-1, r, d) = 1$. □

This completes the proof of assertion (2) of Theorem 1. Note that Condition (2.b) of Corollary 2.21 is required to guarantee that $\mathscr{E}(s,r,d) = \mathscr{E}(s,r,d)_{\mathrm{id}}$, as the next example shows.

**Example 2.22** (The existence of $X$ with non-trivial symmetry is not guaranteed by $\gcd(s-r-1,d) \neq 1$). Let $s = 11$, $r = 7$ and $d = 6$; then $\gcd(d, s-r-1) = \gcd(6,3) = 3 \neq 1$. However $3 \nmid 11$ and $3 \nmid 7$. Thus by Corollary 2.21, $\mathscr{E}(11,7,6) = \mathscr{E}(11,7,6)_{\mathrm{id}}$.

**Example 2.23** (Not all common divisors of $s-r-1$ and $d$ give rise to symmetry). If $s = 35$, $r = 4$ and $d = 30$, then $\gcd(s-r-1, d) = \gcd(30,30) = 30 \neq 1$. Moreover $\mathscr{D} = \{2,3,5,6,10,15,30\}$ and we see that

$$
\begin{array}{cccc}
2 \mid 35 \text{ and } 2 \nmid 4 & 3 \nmid 35 \text{ and } 3 \nmid 4 & 5 \nmid 35 \text{ and } 5 \mid 4 & 6 \nmid 35 \text{ and } 6 \nmid 4 \\
10 \nmid 35 \text{ and } 10 \nmid 4 & 15 \nmid 35 \text{ and } 15 \nmid 4 & 30 \nmid 35 \text{ and } 30 \nmid 4. &
\end{array}
$$

Hence Condition (2.b) of Corollary 2.21 is not satisfied for $\mathtt{k} = 2$ and $5$ (but we had to check for all of $\mathscr{D}$). However, as this same example clearly shows, Condition (3) of Corollary 2.21 is simpler to check because $\gcd(s, r+1, d) = \gcd(35,5,30) = 5$ and $\gcd(s-1, r, d) = \gcd(34,4,30) = 2$. It follows from Proposition 2.19 that only the groups $\mathbb{Z}_{\mathtt{k}}$ with $\mathtt{k} = 2, 5$ can be non-trivial symmetry groups for $X \in \mathscr{E}(35,4,30)$. In fact

$$
X_2(z) = \frac{z^{35}}{z^4 - 1} e^{z^{30}} \frac{\partial}{\partial z}, \quad X_5(z) = \frac{(z^5-1)^7}{z^4} e^{z^{30}} \frac{\partial}{\partial z} \in \mathscr{E}(35,4,30)
$$

are $\mathbb{Z}_2$-invariant and $\mathbb{Z}_5$-invariant, respectively. So $\mathscr{E}(35,4,30) \neq \mathscr{E}(35,4,30)_{\mathrm{id}}$.

We point out some relevant particular cases.

**Remark 2.24.** 1. For the special case $s = r = 0$, $d \geq 1$, note that $\mathscr{E}(0,0,d) = \mathscr{E}(0,0,d)_{\mathrm{id}}$ since $\gcd(-1,d) = \gcd(1,d) = 1$. See also Theorem 8.16 in [3].
2. For each $d \geq 2$ there are $X \in \mathscr{E}(0, d-1, d)$ such that $\mathrm{Aut}(\mathbb{C})_X = \mathbb{Z}_d$. Thus in fact all the cyclic groups appear as isotropy groups of $X \in \mathscr{E}(0,r,d)$ for appropriate pairs $(r,d)$.
3. For each sufficiently large pair $(r,d)$ with $\gcd(r+1,d) \neq 1$, there are infinitely many non-conformally equivalent configurations of the roots $\mathcal{E}$ of $E(z)$ and $\mathcal{P}$ of $P(z)$ which are invariant by the non-trivial $T \in \mathrm{Aut}(\mathbb{C})_X \neq \{\mathrm{id}\}$. This follows from Remark 2.18 and the fact that the quotient of the radii of an annulus is a conformal invariant; thus there are infinitely many possible configurations of the roots $\mathcal{E}$ and $\mathcal{P}$.
4. The same is also true for each sufficiently large pair $(s,d)$ with $\gcd(s-1,d) \neq 1$, of course taking into account configurations of the roots $\mathcal{E}$ and $\mathcal{Z}$.

These last two special cases can be re-stated as:

**Corollary 2.25.** *For each pair* $(r,d)$ *of sufficiently large integers with* $\mathtt{k} = \gcd(r+1,d) \neq 1$ *(or* $(s,d)$ *with* $\mathtt{k} = \gcd(s-1,d) \neq 1$*), there are infinitely many non-conformally equivalent vector fields* $X \in \mathscr{E}(0,r,d)$ *(respectively* $X \in \mathscr{E}(s,0,d)$*) having isotropy group* $\mathrm{Aut}(\mathbb{C})_X \cong \mathbb{Z}_{\mathtt{k}}$.

**Example 2.26.** Simple explicit examples of the above corollary are obtained by considering

$$
\mathscr{E}(0,5,2)_{\mathrm{pole}} \doteq \left\{ X_a(z) = \frac{e^{z^2}}{z(z^2-1)(z^2-a^2)} \frac{\partial}{\partial z} \;\middle|\; |a| \in \mathbb{R}^+ \setminus \{1\} \right\} \subset \mathscr{E}(0,5,2);
$$

then $X_a \in \mathscr{E}(0,5,2)_{\text{pole}}$ is not conformally equivalent to $X_b \in \mathscr{E}(0,5,2)_{\text{pole}}$ whenever $|a| \neq |b|$. Similarly considering

$$\mathscr{E}(5,0,2)_{\text{zero}} \doteq \left\{\widehat{X}_a(z) = z(z^2-1)(z^2-a^2)\, e^{z^2}\frac{\partial}{\partial z} \;\Big|\; |a| \in \mathbb{R}^+ \setminus \{1\}\right\} \subset \mathscr{E}(5,0,2),$$

we find that $\widehat{X}_a \in \mathscr{E}(5,0,2)_{\text{zero}}$ is not conformally equivalent to $\widehat{X}_b \in \mathscr{E}(5,0,2)_{\text{zero}}$ whenever $|a| \neq |b|$. In all cases the isotropy group are isomorphic to $\mathbb{Z}_2$ since $\mathtt{k} = 2 = \gcd(5+1,2) = \gcd(5-1,2)$.

# 3 Normal forms for $X \in \mathscr{E}(s,r,d)$ with non-trivial isotropy group

## 3.1 Realizing $X \in \mathscr{E}(s,r,d)$ with non-trivial isotropy group

Application of Theorem 2.19, with the Conditions (1), (2) and (3) as stated in Remark 2.20.3, immediately provides a way of explicitly realizing those $X \in \mathscr{E}(s,r,d)$ with non-trivial isotropy group.

**3.1.1 Zeros and poles with arbitrary multiplicity.** With the above in mind, let

$$\mathscr{D}_{\text{pole}} = \{k \in \mathbb{N} \mid k \text{ is a non-trivial common divisor of } s,\, r+1 \text{ and } d\},$$
$$\mathscr{D}_{\text{zero}} = \{k \in \mathbb{N} \mid k \text{ is a non-trivial common divisor of } s-1,\, r \text{ and } d\}.$$

**Remark 3.1.** Note that $\mathscr{D}_{\text{pole}} \cap \mathscr{D}_{\text{zero}} = \varnothing$.

We now have a way of realizing those $X \in \mathscr{E}(s,r,d)$ with non-trivial isotropy.

**Theorem 3.2** (Realizing vector fields with non-trivial symmetry). *Let $s$, $r$ and $d$ be such that $\mathscr{D}_{\text{pole}} \cup \mathscr{D}_{\text{zero}} \neq \varnothing$. Then $X \in \mathscr{E}(s,r,d)$ is a vector field with non-trivial isotropy if and only if $X(z)$ can be expressed in one of the following two forms:*

1)

$$X(z) = \frac{\prod_{j=1}^{k_s}\prod_{\ell=1}^{\mathtt{k}}\left[z - C - r_j e^{i(\theta_j + 2\pi\ell/\mathtt{k})}\right]}{(z-C)^{\nu}\prod_{j=1}^{k_r}\prod_{\ell=1}^{\mathtt{k}}\left[z - C - R_j e^{i(\alpha_j + 2\pi\ell/\mathtt{k})}\right]} \exp\left\{\mu(z-C)^{\mathtt{k}\widetilde{\nu}}\prod_{j=1}^{k_d}\prod_{\ell=1}^{\mathtt{k}}\left[z - C - \rho_j e^{i(\beta_j + 2\pi\ell/\mathtt{k})}\right]\right\}\frac{\partial}{\partial z},$$

*for choices of $\mu \in \mathbb{C}^*$, $\{r_j\}, \{R_j\}, \{\rho_j\} \subset \mathbb{R}^+$, $\{\theta_j\}, \{\alpha_j\}, \{\beta_j\} \subset \mathbb{R}$, $\mathtt{k}$, $k_s$, $k_r$, $k_d$ non-negative integers such that $s = \mathtt{k}k_s$, $d = \mathtt{k}(k_d + \widetilde{\nu})$ with $0 \le \widetilde{\nu} \le \frac{d}{\mathtt{k}}$ and $r = \mathtt{k}k_r + \nu$, $\nu = \mathtt{k}m - 1$ with $1 \le m \le \frac{r+1}{\mathtt{k}}$.*

2)

$$X(z) = \frac{(z-C)^{\nu}\prod_{j=1}^{k_s}\prod_{\ell=1}^{\mathtt{k}}\left[z - C - r_j e^{i(\theta_j + 2\pi\ell/\mathtt{k})}\right]}{\prod_{j=1}^{k_r}\prod_{\ell=1}^{\mathtt{k}}\left[z - C - R_j e^{i(\alpha_j + 2\pi\ell/\mathtt{k})}\right]} \exp\left\{\mu(z-C)^{\mathtt{k}\widetilde{\nu}}\prod_{j=1}^{k_d}\prod_{\ell=1}^{\mathtt{k}}\left[z - C - \rho_j e^{i(\beta_j + 2\pi\ell/\mathtt{k})}\right]\right\}\frac{\partial}{\partial z},$$

*for choices of $\mu \in \mathbb{C}^*$, $\{r_j\}, \{R_j\}, \{\rho_j\} \subset \mathbb{R}^+$, $\{\theta_j\}, \{\alpha_j\}, \{\beta_j\} \subset \mathbb{R}$, $\mathtt{k}$, $k_s$, $k_r$, $k_d$ non-negative integers such that $r = \mathtt{k}k_r$, $d = \mathtt{k}(k_d + \widetilde{\nu})$ with $0 \le \widetilde{\nu} \le \frac{d}{\mathtt{k}}$ and $s = \mathtt{k}k_s + \nu$, $\nu = \mathtt{k}m + 1$ with $0 \le m \le \frac{s-1}{\mathtt{k}}$.*

*Furthermore, the symmetry groups of $X$ are discrete rotation groups of order $\mathtt{k}$ given by (13) for $\mathtt{k} \in \mathscr{D}_{\text{pole}}$ if and only if $X$ has a pole at $C$ (Condition (1)), or $\mathtt{k} \in \mathscr{D}_{\text{zero}}$ if and only if $X$ has a zero at $C$ (Condition (2)).* □

**Remark 3.3.** 1. Statement (1) of Theorem 1 is a reinterpretation of Theorem 3.2.

2. For $\mathtt{k} \in \mathscr{D}_{\text{pole}} \cup \mathscr{D}_{\text{zero}}$, the group $\Gamma$ of order $\mathtt{k}$ generated by a rotation of angle $2\pi/\mathtt{k}$ around $C \in \mathbb{C}$ leaves invariant vector fields of the following form:
   (a) If $\mathtt{k} \mid s$

$$X(z) = \frac{Q((z-C)^{\mathtt{k}})}{(z-C)^{\mathtt{k}+1}P((z-C)^{\mathtt{k}})}\exp(E((z-C)^{\mathtt{k}}))\frac{\partial}{\partial z},$$

   where $Q, P, E \in \mathbb{C}[z]$, $Q$ and $P$ monic, of degree $s/\mathtt{k}$, $(r - \mathtt{k} + 1)/\mathtt{k}$, $d/\mathtt{k}$ respectively.

(b) If $\mathtt{k} \mid r$

$$X(z) = \frac{(z-C)Q((z-C)^{\mathtt{k}})}{P((z-C)^{\mathtt{k}})} \exp(E((z-C)^{\mathtt{k}})) \frac{\partial}{\partial z},$$

where $Q, P, E \in \mathbb{C}[z]$, $Q$ and $P$ monic, of degree $(s-1)/\mathtt{k}$, $r/\mathtt{k}$, $d/\mathtt{k}$ respectively.

3. Note that the expressions in Theorem 3.2 are not normal forms for $X \in \mathscr{E}(s, r, d) \setminus \mathscr{E}(s, r, d)_{\text{id}}$. However, by requiring (as in the proof of Lemma 2.13, specifically (18), that $\mathtt{a} = \mu^{-1/d}$ and $\mathtt{b} = C$, so that the polynomial in the exponent of $(G^* X)(w)$ is monic and $G^{-1}(C) = 0$) we immediately obtain normal forms for $X \in \mathscr{E}(s, r, d) \setminus \mathscr{E}(s, r, d)_{\text{id}}$:

$$X(z) = \frac{1}{z^{\nu}} \frac{\prod_{j=1}^{k_s} \prod_{\ell=1}^{\mathtt{k}} [z - r_j e^{i(\theta_j + 2\pi\ell/\mathtt{k})}]}{\prod_{j=1}^{k_r} \prod_{\ell=1}^{\mathtt{k}} [z - R_j e^{i(\alpha_j + 2\pi\ell/\mathtt{k})}]} \exp\left\{ z^{\mathtt{k}\tilde{\nu}} \prod_{j=1}^{k_d} \prod_{\ell=1}^{\mathtt{k}} \left[ z - \rho_j e^{i(\beta_j + 2\pi\ell/\mathtt{k})} \right] \right\} \frac{\partial}{\partial z}, \quad \text{and}$$

$$X(z) = z^{\nu} \frac{\prod_{j=1}^{k_s} \prod_{\ell=1}^{\mathtt{k}} [z - r_j e^{i(\theta_j + 2\pi\ell/\mathtt{k})}]}{\prod_{j=1}^{k_r} \prod_{\ell=1}^{\mathtt{k}} [z - R_j e^{i(\alpha_j + 2\pi\ell/\mathtt{k})}]} \exp\left\{ z^{\mathtt{k}\tilde{\nu}} \prod_{j=1}^{k_d} \prod_{\ell=1}^{\mathtt{k}} \left[ z - \rho_j e^{i(\beta_j + 2\pi\ell/\mathtt{k})} \right] \right\} \frac{\partial}{\partial z}, \quad \text{respectively.}$$

4. Once again by choosing $e^{i\theta} \in \mathbb{S}^1$ such that $e^{i\theta} e^{\tilde{c}_d} \in \mathbb{R}^+$, as in Remark 2.17, the normal form given above for $X \in \mathscr{E}(s, r, d) \setminus \mathscr{E}(s, r, d)_{\text{id}}$ can be extended to the corresponding singular flat metric $(\widehat{\mathbb{C}}, g_X)$ in normal form.

**3.1.2 Simple zeros and simple poles in $\mathbb{C}$.** Recall that if a discrete rotation group $\Gamma < \text{Aut}(\mathbb{C})$ leaves invariant a vector field $X \in \mathscr{E}(s, r, d)$ it is necessary that $X$ have a pole or zero at the center $C \in \mathbb{C}$ of rotation of $\Gamma$. Obviously the family of vector fields $X \in \mathscr{E}(s, r, d)$ with only simple poles and zeros is dense in $\mathscr{E}(s, r, d)$. Moreover, we observe the following dichotomy.

1) If there is a (simple) pole of $X$ at the fixed point $C \in \mathbb{C}$, then $\text{Aut}(\mathbb{C})_X \cong \mathbb{Z}_2$, the number of (simple) zeros of $X$ is even, the number of (simple) poles of $X$ is odd and the number of roots (counted with multiplicity) of the polynomial in the exponential is even.
2) If there is a (simple) zero of $X$ at the fixed point $C \in \mathbb{C}$, then there are no restrictions other than those given by the orbit structure (Remark 2.18).

Formally we have:

**Corollary 3.4.** *Let $X \in \mathscr{E}(s, r, d)$ have only simple poles and zeros, with a non-trivial isotropy group* $\text{Aut}(\mathbb{C})_X$.

(1) *If the center of rotation of* $\text{Aut}(\mathbb{C})_X$ *is a pole of $X$, then*
  a) $\text{Aut}(\mathbb{C})_X \cong \mathbb{Z}_2$,
  b) *$s$ and $d$ are even, $r$ is odd,*
  c) *the vector field $X$ is of the form*

$$X(z) = \frac{\prod_{j=1}^{k_s} [(z-C)^2 - q_j^2]}{(z-C) \prod_{j=1}^{k_r} [(z-C)^2 - p_j^2]} \exp\left\{ \mu (z-C)^{2\tilde{\nu}} \prod_{j=1}^{k'_d} [(z-C)^2 - e_j^2] \right\} \frac{\partial}{\partial z},$$

*where $k_r = \frac{r-1}{2} \geq 0$, $k_s = \frac{s}{2} \geq 0$, $k'_d = \frac{d-2\tilde{\nu}}{2} \geq 0$ with $\tilde{\nu} \in \mathbb{N} \cup \{0\}$, all the $\{p_j\} \subset \mathbb{C} \setminus \{0\}$ and $\{q_j\} \subset \mathbb{C} \setminus \{0\}$ are distinct, but the $\{e_j\} \subset \mathbb{C}$ need not be distinct.*

(2) *If the center of rotation of* $\text{Aut}(\mathbb{C})_X$ *is a zero of $X$, then*
  a) $\text{Aut}(\mathbb{C})_X \cong \mathbb{Z}_{\mathtt{k}}$ *occurs for each* $\mathtt{k} \geq 2$,
  b) $s = \mathtt{k}k_s + 1 \geq 1$, $r = \mathtt{k}k_r \geq 0$ *and* $d = \mathtt{k}k_d \geq 1$ *with* $k_s, k_r, k_d \in \mathbb{N} \cup \{0\}$,

c) *these vector fields $X$ are of the form*

$$X(z) = \frac{(z-C)\prod_{j=1}^{k_s}\prod_{\ell=1}^{\mathrm{k}}\left[z-C-r_j e^{i(\theta_j+2\pi\ell/\mathrm{k})}\right]}{\prod_{j=1}^{k_r}\prod_{\ell=1}^{\mathrm{k}}\left[z-C-R_j e^{i(\alpha_j+2\pi\ell/\mathrm{k})}\right]}\exp\left\{\mu(z-C)^{\mathrm{k}\widetilde{\nu}}\prod_{j=1}^{k_d}\prod_{\ell=1}^{\mathrm{k}}\left[z-C-\rho_j e^{i(\beta_j+2\pi\ell/\mathrm{k})}\right]\right\}\frac{\partial}{\partial z},$$

*for choices of* $\{r_j\}, \{R_j\}, \{\rho_j\} \subset \mathbb{R}^+$, $\{\theta_j\}, \{\alpha_j\}, \{\beta_j\} \subset \mathbb{R}$ *and* $\widetilde{\nu} \in \mathbb{N} \cup \{0\}$ *such that* $\{r_j e^{i(\theta_j+2\pi\ell/\mathrm{k})}\}$ *and* $\{R_j e^{i(\alpha_j+2\pi\ell/\mathrm{k})}\}$ *are distinct, but the* $\{\rho_j e^{i(\beta_j+2\pi\ell/\mathrm{k})}\}$ *need not necessarily be distinct.*

*Proof.* For (1) consider Theorem 3.2.1. We immediately see that if $X \in \mathscr{E}(s,r,d)$ has only simple poles and zeros with non-trivial isotropy group fixing a (simple) pole of $X$, then the multiplicity $-\nu$ of the fixed simple pole is $-1 = -(\mathrm{k}m - 1)$, hence $\mathrm{k} = 2$.

The proof of (2) follows as a direct application of Theorem 3.2.2. □

**Example 3.5.** Let $X(z) = \frac{e^{z^2}}{z(z^2+1)}\frac{\partial}{\partial z} \in \mathscr{E}(0,3,2)$. Its isotropy group is $\mathrm{Aut}(\mathbb{C})_X = \mathbb{Z}_2$, see Figure 3 (c).

# 4 Singular complex analytic dictionary and $\Gamma$-symmetry

## 4.1 A dictionary in the singular complex analytic category

The interplay between different singular complex analytic tensors, as stated below, is a general principle. A complete proof can be found in [3] § 2.2 with further discussion in [4].

**Proposition 4.1** (Singular complex analytic dictionary). *On any (not necessarily compact) Riemann surface $M$ there is a canonical one to one correspondence between:*

1) *Singular complex analytic vector fields $X$.*
2) *Singular complex analytic differential forms $\omega_X$ satisfying $\omega_X(X) \equiv 1$.*
3) *Singular complex analytic orientable quadratic differentials $\omega_X \otimes \omega_X$.*
4) *Singular flat metrics $(M, g_X)$ with suitable singularities, trivial holonomy and provided with a real geodesic vector field $\mathfrak{Re}(X)$, arising from $\omega_X \otimes \omega_X$ satisfying $g_X(\mathfrak{Re}(X), \mathfrak{Re}(X)) \equiv 1$ and $g_X(\mathfrak{Re}(X), \mathfrak{Im}(X)) \equiv 0$.*
5) *Global singular complex analytic (possibly multivalued) distinguished parameters $\Psi_X(z) = \int^z \omega_X : M \to \widehat{\mathbb{C}}_t$.*
6) *Pairs $(\mathcal{R}_X, \pi^*_{X,2}(\frac{\partial}{\partial t}))$ consisting of branched Riemann surfaces $\mathcal{R}_X$, associated to the maps $\Psi_X$, and the vector fields $\pi^*_{X,2}(\frac{\partial}{\partial t})$ under the projection $\pi_{X,2} : \mathcal{R}_X \longrightarrow \widehat{\mathbb{C}}_t$.* □

To better understand the dictionary, the adjectives "singular complex analytic" should be clear for each of the objects in Proposition 4.1. The singular set $\mathrm{Sing}(X)$ of $X$ is composed of zeros, poles, essential singularities and accumulation points of the above (and in analogous way for (2) and (3)). The *singular flat metric $g_X$ with singular set* $\mathrm{Sing}(X)$ is the flat Riemannian metric on $M\backslash\mathrm{Sing}(X)$ defined as the pullback under $\Psi_X : (M, g_X) \to (\mathbb{C}_t, \delta)$, where $\delta$ is the usual flat Riemannian metric on $\mathbb{C}_t$; recall Definition 2.4. The topology of the phase portrait of $\mathfrak{Re}(X)$ and the geometry of $g_X$ are subjects of current interest; some sources can be found in [3] in § 1, p. 133, § 5 p. 159 and Table 2. See [4] for visualization aspects. Independent applications of geometric structures associated to flat metrics $(\widehat{\mathbb{C}}, g_X)$ can be found in [18].

The graph of $\Psi_X$ is

$$\mathcal{R}_X = \{(z,t) \mid t = \Psi_X(z)\} \subset M \times \widehat{\mathbb{C}}_t,$$

which is a Riemann surface provided with the vector field induced by $(\widehat{\mathbb{C}}, \frac{\partial}{\partial t})$ via the projection of $\pi_{X,2}$, say $(\mathcal{R}_X, \pi^*_{X,2}(\frac{\partial}{\partial t}))$. Moreover, the singular flat metric from this pair coincides with $g_X = \Psi^*_X\delta$ since $\pi_{X,1}$ is an isometry (the isometry is to be understood on the complement of the corresponding singular set of $X$ in $M$, and hence in $\mathcal{R}_X$). We summarize all this in the diagram

$$\begin{tikzcd}
(M, X) \arrow[drr, "\Psi_X"'] & & (\mathcal{R}_X, \pi^*_{X,2}(\frac{\partial}{\partial t})) \arrow[ll, "\pi_{X,1}"'] \arrow[d, "\pi_{X,2}"] \\
& & (\widehat{\mathbb{C}}_t, \frac{\partial}{\partial t}).
\end{tikzcd} \tag{22}$$

In the presence of non-trivial symmetries we have:

**Proposition 4.2** (The dictionary under $\Gamma$-symmetry). *Let $\Gamma$ be a fixed subgroup of the complex automorphisms* $\mathrm{Aut}(M)$ *having quotient* $\mathrm{proj} : M \longrightarrow M/\Gamma$ *to a Riemann surface.*

1. *On $M$ there is a canonical one to one correspondence between:*
   1) *$\Gamma$-invariant singular complex analytic vector fields $X$.*
   2) *$\Gamma$-invariant singular complex analytic differential forms $\omega_X$ satisfying $\omega_X(X) \equiv 1$.*
   3) *$\Gamma$-invariant singular complex analytic orientable quadratic differentials $\omega_X \otimes \omega_X$.*
   4) *$\Gamma$-invariant singular flat metrics $(M, g_X)$ with suitable singularities, trivial holonomy and provided with a real geodesic vector field $\mathfrak{Re}\,(X)$, arising from $\omega_X \otimes \omega_X$ satisfying $g_X(\mathfrak{Re}\,(X), \mathfrak{Re}\,(X)) \equiv 1$ and $g_X(\mathfrak{Re}\,(X), \mathfrak{Im}\,(X)) \equiv 0$.*
   5) *$\Gamma$-invariant global singular complex analytic (possibly multivalued) distinguished parameters $\Psi_X$.*
   6) *Pairs $(\mathcal{R}_X, \pi^*_{X,2}(\frac{\partial}{\partial t}))$ consisting of branched Riemann surfaces $\mathcal{R}_X$ associated to the $\Gamma$-invariant maps $\Psi_X$.*
2. *Moreover, any $X$ (respectively $\Psi_X$) on $M$ which is invariant by a non-trivial $\Gamma <$* $\mathrm{Aut}(M)$ *can be recognized as a lifting of a suitable vector field $Y$ (respectively function $\Psi_Y$) on $M/\Gamma$, as in the following diagram:*

$$\begin{tikzcd}
& & (M, X) \arrow[ddl, "\mathrm{proj}_*"'] \arrow[drr, "\Psi_X"] & & (\mathcal{R}_X, \pi^*_{X,2}(\frac{\partial}{\partial t})) \arrow[ll, "\pi_{X,1}"'] \arrow[d, "\pi_{X,2}"] \arrow[ddl, "\widetilde{\mathrm{proj}}_*"'] \\
& & & & (\widehat{\mathbb{C}}_t, \frac{\partial}{\partial t}) \arrow[ddl, "\mathrm{id}"] \\
& (M/\Gamma, Y) \arrow[drr, "\Psi_Y"'] & & (\mathcal{R}_Y, \pi^*_{Y,2}(\frac{\partial}{\partial t})) \arrow[ll, "\pi_{Y,1}"] \arrow[d, "\pi_{Y,2}"'] & \\
& & & (\widehat{\mathbb{C}}_t, \frac{\partial}{\partial t}) &
\end{tikzcd} \tag{23}$$

*Proof.* By hypothesis $\mathrm{proj} : M \longrightarrow M/\Gamma$ determines a connected Riemann surface as a target, thus Diagram (22) holds true both for $M$ and $M/\Gamma$. We want to show that $\mathrm{proj}_* X \doteq Y$ is a well defined vector field on $M/\Gamma$.

From a local point of view, let $(\mathbb{C}, 0)$ denote local charts of $M$ where 0 corresponds to a fixed point for some $g : M \to M$ where $g \neq \mathrm{id}$ in $\Gamma$. Without loss of generality, we assume that $\mathrm{proj}^{-1}(\mathrm{proj}(\mathbb{C}, 0))$ is connected in $M$.

Note that $X$ is necessarily singular at $(\mathbb{C}, 0)$. The trouble is that the local behaviour of $X$ is unknown. The computation of $Y$ from the germ $((\mathbb{C}, 0), X)$ uses geometrical arguments. The fundamental domain of $\mathrm{proj} : (\mathbb{C}, 0) \longmapsto (\mathbb{C}, 0)/\Gamma$ is an angular sector $\{0 \leq arg(z) \leq 2\pi/\kappa\} \subset (\mathbb{C}, 0)$, $\kappa \geq 2$. Using the singular flat metric $g_X$ and the frame of geodesic vector fields $\mathfrak{Re}\,(X)$, $\mathfrak{Im}\,(X)$ on the angular sectors (recall Theorem 4.1 (4)), the values of $X$ at the borders of an angular sector coincide, hence the germ $Y$ on $((\mathbb{C}, 0)/\Gamma, Y)$ is well defined. For poles, zeros and the simplest isolated essential singularities at $(\mathbb{C}, 0)$ explicit computations are provided in Table 1, which in itself is of independent interest.

The global existence of $Y$ on $M/\Gamma$ follows by an analytic continuation argument. Diagram (23) for vector fields follows immediately, where $\mathrm{proj}_*$ and $\widetilde{\mathrm{proj}}_*$ are the maps induced by proj on $M$ and $\mathcal{R}_X$ respectively.

Finally, the use of the dictionary given by Proposition 4.1 extends Diagram (23) to singular complex analytic 1-forms $\omega_X$ and functions $\Psi_X$, where $g \in \Gamma$ acts on functions as $\Psi_X \mapsto \Psi_X \circ g$. Assertions (2) and (5) are done. ∎

As a matter of record, Table 1 summarizes the local behaviour of germs $((\mathbb{C}, 0), X)$ and the quotient $((\mathbb{C}, 0)/\Gamma, Y)$ for poles, zeros and the simplest family of isolated essential singularities. The linear vector field $\lambda z \frac{\partial}{\partial z}$ has complete isotropy group $\mathbb{C}^*$; however only discrete groups are considered for Proposition 4.2. Note that Table 1 makes sense globally; in the last row we use $(\widehat{\mathbb{C}}, \infty)$ as germ domain.

**Table 1:** Computation of $Y = \mathrm{proj}_* X$ given a germ $((\mathbb{C}, 0), X)$

| | **on $(\mathbb{C}, 0)$** | | | **on $(\mathbb{C}, 0)/\Gamma$** | |
|---|---|---|---|---|---|
| **normal form for a germ $X$** | **order $\nu \in \mathbb{Z}$, residue $\mathtt{r} \in \mathbb{C}$** | **isotropy group $\Gamma$** | **vector field $Y$** | **differential 1-form $\omega_Y$** | **quadratic differential $\omega_Y \otimes \omega_Y$** |
| $\frac{1}{z^\nu} \frac{\partial}{\partial z}$ | $-\nu \leq -1$ | $\mathbb{Z}_{\mathtt{k}}, \mathtt{k} \mid (\nu + 1)$ | $\frac{1}{w^{(\nu+1)/\mathtt{k}-1}} \frac{\partial}{\partial w}$ | $w^{(\nu+1)/\mathtt{k}-1} dw$ | $w^{2(\nu+1)/\mathtt{k}-2} dw^2$ |
| $\lambda z \frac{\partial}{\partial z}$ | $\nu = 1, \mathtt{r} = \lambda$ | $\mathbb{C}^* \rhd \mathbb{Z}_{\mathtt{k}}$ | $\frac{\lambda w}{\mathtt{k}} \frac{\partial}{\partial w}$ | $\frac{\mathtt{k}}{\lambda w} dw$ | $\frac{\mathtt{k}^2}{\lambda^2 w^2} dw^2$ |
| $z^2 \frac{\partial}{\partial z}$ | $\nu = 2$ | id | $w^2 \frac{\partial}{\partial w}$ | $\frac{1}{w^2} dw$ | $\frac{1}{w^4} dw^2$ |
| $z^\nu \frac{\partial}{\partial z}$ | $\nu \geq 3, \mathtt{r} = 0$ | $\mathbb{Z}_{\mathtt{k}}, \mathtt{k} \mid (\nu - 1)$ | $w^{(\nu-1)/\mathtt{k}+1} \frac{\partial}{\partial w}$ | $\frac{1}{w^{(\nu-1)/\mathtt{k}+1}} dw$ | $\frac{1}{w^{2(\nu-1)/\mathtt{k}+2}} dw^2$ |
| $\frac{z^\nu}{1+\lambda z^{\nu-1}} \frac{\partial}{\partial z}$ | $\nu \geq 3, \mathtt{r} = \lambda \neq 0$ | id | $\frac{w^\nu}{1+\lambda w^{\nu-1}} \frac{\partial}{\partial w}$ | $\frac{1+\lambda w^{\nu-1}}{w^\nu} dw$ | $\frac{(1+\lambda w^{\nu-1})^2}{w^{2\nu}} dw^2$ |
| $e^{z^d} \frac{\partial}{\partial z}$ | $\nu \geq 3, \mathtt{r} = 0$ | $\mathbb{Z}_{\mathtt{k}}, \mathtt{k} \mid d$ | $e^{w^{d/\mathtt{k}}} \frac{\partial}{\partial w}$ | $e^{-w^{d/\mathtt{k}}} dw$ | $e^{-w^{2d/\mathtt{k}}} dw^2$ |

## 4.2 Description of $Y = \mathrm{proj}_* X$ for $X \in \mathscr{E}(s, r, d)$

For $X \in \mathscr{E}(s, r, d)$, recall that the isotropy group $\mathrm{Aut}(\mathbb{C})_X \cong \mathbb{Z}_{\mathtt{k}}$ is given by (13) for some $\mathtt{k} \in \mathscr{D}_{\text{poles}} \cup \mathscr{D}_{\text{zeros}}$ and $\mathrm{proj} : \widehat{\mathbb{C}}_z \longrightarrow \widehat{\mathbb{C}}_z/\mathbb{Z}_{\mathtt{k}} = \widehat{\mathbb{C}}_w$.

**Proposition 4.3.** *Let $d \geq 1$ and $X \in \mathscr{E}(s, r, d)$, having the isotropy group* $\mathrm{Aut}(\mathbb{C})_X \cong \mathbb{Z}_{\mathtt{k}}$ *with* $\mathtt{k} \geq 2$. *The quotient vector field $Y = \mathrm{proj}_* X$ has the following characteristics.*

(1) *$Y \in \mathscr{E}(s', r', d')$ has $s'$ zeros, $r'$ poles and an essential singularity of 1-order $d' = d/\mathtt{k}$ at $\infty$, where*
  - *$s' = s/\mathtt{k}$ and $r' = \frac{r+1}{\mathtt{k}} - 1$ when $C$ is a pole of $X$,*
  - *$r' = r/\mathtt{k}$ and $s' = \frac{s-1}{\mathtt{k}} + 1$ when $C$ is a zero of $X$.*

(2) *The isotropy of $Y$ in* $\mathrm{Aut}(\mathbb{C})$ *is trivial.*

(3) *The phase portrait of $\mathfrak{Re}\,(X)$ is the pullback via $\{z \mapsto e^{2\pi i/\mathtt{k}} z + \mathtt{b}\}$ of the phase portrait of $\mathfrak{Re}\,(Y)$.*

Of course this proposition can be stated for other $\Gamma$-invariant subgroups, not only for the maximal one (which is $\mathrm{Aut}(\mathbb{C})_X$); the proof remains basically the same so the details are left to the interested reader.

*Proof.* Since $\Psi_X(z) = \int^z \omega_X$, Diagram (23) commutes and assertions (2) and (3) follow. Now we compute the nature of the singularities of $Y$. We have $d > 1$, hence $\infty$ is an isolated essential singularity of $X$ having $2d$ entire sectors; see [3, § 5.3.1 p. 151, Figure 3 p. 153]. By Theorem (A) p. 130 and Corollary 10.1 p. 216 in [3], it follows that, since proj is $\mathtt{k}$ to 1 around $\infty$ and since $\mathtt{k} \mid d$, the phase portrait of $\mathfrak{Re}\,(\mathrm{proj}_*(X))$ has $2d' = 2(d/\mathtt{k})$ entire sectors at $\infty \in \widehat{\mathbb{C}}_z$.

For the number $s'$ of zeros and $r'$ of poles of $\mathrm{proj}_*(X)$, recalling Theorem 3.2 we need to consider two cases: ($\mathtt{k} \mid s$ and $\mathtt{k} \nmid r$) and ($\mathtt{k} \nmid s$ and $\mathtt{k} \mid r$).

**Case ($\mathtt{k} \mid s$ and $\mathtt{k} \nmid r$): $C$ is a pole of $X$.** Note that $r = \mathtt{k} k_r + \nu$ with $k_r, \nu \in \mathbb{N} \cup \{0\}$, $\mathtt{k} \nmid \nu$, $\mathtt{k} \mid (\nu + 1)$ and $s = \mathtt{k} k_s$ with $k_s \in \mathbb{N} \cup \{0\}$. In this case the fundamental region, induced by $T_{\mathtt{k}}$, has exactly $k_r + \nu$ poles of $X$ ($C$ being a pole of multiplicity $-\nu$) and $k_s$ zeros of $X$. The phase portrait of $\mathfrak{Re}\,(X)$ has $2(\nu + 1)$ hyperbolic sectors at $C$.

On the other hand, $\mathrm{proj}_*(X)$ corresponds to a vector field $Y$ on $\widehat{\mathbb{C}}/\,\mathrm{Aut}(\mathbb{C})_X$ and a local condition at $\mathrm{proj}(C)$ must be met: $Y$ should have a pole of order $\nu'$, hence $Y$ is required to have $2(\nu'+1)$ hyperbolic sectors at $\mathrm{proj}(C)$

and hence $\frac{2(\nu+1)}{\mathrm{k}} = 2(\nu' + 1)$, so $\nu' = \frac{\nu+1}{\mathrm{k}} - 1$. In other words: the local condition is equivalent to $\mathrm{k} \mid (\nu + 1)$. Thus $\mathrm{proj}_*(X) \in \mathscr{E}(s', r', d')$ for $s' = s/\mathrm{k}$, $d' = d/\mathrm{k}$ and $r' = k_r + \nu'$ where $\nu' = \frac{\nu+1}{\mathrm{k}} - 1$, so $r' = \frac{r+1}{\mathrm{k}} - 1$.

**Case ($\mathrm{k} \nmid s$ and $\mathrm{k} \mid r$): $C$ is a zero of $X$.** In this case $r = \mathrm{k}k_r$ with $k_r \in \mathbb{N} \cup \{0\}$ and $s = \mathrm{k}k_s + \nu$ with $k_s, \nu \in \mathbb{N} \cup \{0\}$, $\mathrm{k} \nmid \nu$, $\mathrm{k} \mid (\nu - 1)$. The corresponding argument then yields that $\mathrm{proj}_*(X) \in \mathscr{E}(s', r', d')$ for $r' = r/\mathrm{k}$, $d' = d/\mathrm{k}$ and $s' = \frac{s-1}{\mathrm{k}} + 1$. □

See for instance Examples 2.9, 3.5 and Figures 1 (a), 3 (c) respectively.

**Remark 4.4.** By hypothesis in Proposition 4.2, $\Gamma$ is a fixed subgroup, hence the center of rotation $C \in \mathbb{C}$ is also fixed. The map $\mathrm{proj}_*$ is well defined on $\mathcal{U}_{\mathrm{k}} = \{X \in \mathscr{E}(s, r, d) \mid \mathrm{Aut}(\mathbb{C})_X \cong \mathbb{Z}_{\mathrm{k}}\}$. Thus Proposition 4.3 provides a certain reducibility property $\mathcal{U}_{\mathrm{k}} \to \mathscr{E}(s', r', d/\mathrm{k})_{\mathrm{id}} : X \mapsto \mathrm{proj}_* X = Y$.

## 4.3 Rational vector fields

Relaxing the condition that $d \geq 1$, i.e. considering $d = 0$, we have the family

$$\mathscr{E}(s, r, 0) = \left\{ X(z) = \lambda \frac{Q(z)}{P(z)} \frac{\partial}{\partial z} \,\middle|\, \text{monic } Q,\ P \in \mathbb{C}[z], \deg Q = s, \deg P = r, \lambda \in \mathbb{C}^* \right\}$$

of rational vector fields on the sphere with $s$ zeros and $r$ poles on $\mathbb{C}$. The main difference between the cases $d = 0$ and $d \geq 1$ is the dynamical behaviour of $\infty \in \widehat{\mathbb{C}}$. By Poincaré–Hopf theory, $X \in \mathscr{E}(s, r, 0)$ has $\infty \in \widehat{\mathbb{C}}$ as

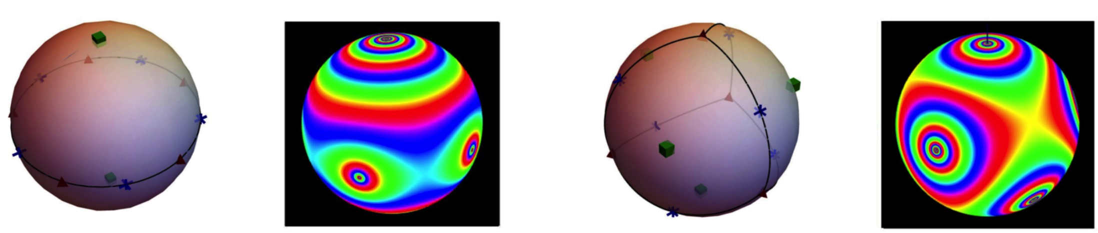
a) a *regular point* when $2 - s + r = 0$,
b) a *zero of order* $\mu$ when $\mu = 2 - s + r \geq 1$, and
c) a *pole of order* $-\nu$ when $\nu = 2 - s + r \leq -1$.

Considering $X$ on $\widehat{\mathbb{C}}$, the total number of zeros minus the total number of poles is 2. Obviously, as the following examples show, generically for $X \in \mathscr{E}(s, r, 0)$ the isotropy group $\mathrm{Aut}(\widehat{\mathbb{C}})_X$ *does not fix* $\infty \in \widehat{\mathbb{C}}$ (and hence strays from the present work). For further examples and a classification of rational vector fields with finite isotropy on the Riemann sphere, see [2].

**Example 4.5.** Consider

$$X(z) = \lambda \frac{z(z^n - 1)}{z^n + 1} \frac{\partial}{\partial z} \in \mathscr{E}(n + 1, n, 0) \text{ for } n \geq 3. \tag{24}$$

As shown in [2], the isotropy group is a dihedral group $\mathrm{Aut}(\widehat{\mathbb{C}})_X \cong \mathbb{D}_n$. In this case $\{z \mapsto -1/z\} \in \mathrm{Aut}(\widehat{\mathbb{C}})_X$, hence $\infty \in \widehat{\mathbb{C}}$ is not a fixed point of the isotropy group. See Figures 2 (A) and 2 (B).



(A) (B) (C) (D)

**Figure 2:** Phase portraits of Example 4.5. We have set $\lambda = -i$, so the zeros of $X$ are centers. (A) and (C) represent the divisors of $X$: zeros appear as red pyramids, poles appear as blue crosses. (B) and (D) visualize the corresponding phase portraits. Borders of the strip flows correspond to streamlines of the field. (A) and (B) correspond to (24) with $n = 5$ which has isometry group isomorphic to $\mathbb{D}_5$, and (C) and (D) correspond to (25) which has isometry group isomorphic to $A_4$.

From the perspective of Proposition 4.2, $\widehat{\mathbb{C}}/\mathbb{D}_n = \widehat{\mathbb{C}}$ and

$$\mathrm{proj} : \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{C}}, \quad \mathrm{proj}_* X(w) = n\lambda \frac{w(w-1)}{w+1}\frac{\partial}{\partial w} \doteq Y(w).$$

Moreover, a quick calculation involving partial fractions shows that the distinguished parameter $\Psi_X(z) = \frac{2}{n}\log(1-z^n) - \log(z)$ is multivalued and has $\mathbb{D}_n$-symmetry.

Now consider

$$X(z) = \lambda \frac{4z^7 + 7\sqrt{2}z^4 - 4z}{4z^6 - 20\sqrt{2}z^3 - 4}\frac{\partial}{\partial z} \in \mathscr{E}(7,6,0). \tag{25}$$

In this case, as shown in [2], the isotropy group $\mathrm{Aut}(\widehat{\mathbb{C}})_X \cong \mathbb{A}_4$, the isometry group of the tetrahedron. Note that $\infty \in \widehat{\mathbb{C}}$ is a vertex of the corresponding tetrahedron and since the vertices are in the same orbit of $\mathrm{Aut}(\widehat{\mathbb{C}})_X$, it follows that $\infty \in \widehat{\mathbb{C}}$ is not a fixed point of the isotropy group. See Figures 2 (C) and 2 (D). Similarly, from the perspective of Proposition 4.2, $\widehat{\mathbb{C}}/A_4 = \widehat{\mathbb{C}}$ and

$$\mathrm{proj} : \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{C}}, \quad \mathrm{proj}_* X(w) = 4\lambda w \frac{\partial}{\partial w} \doteq Y(w).$$

Once again, the distinguished parameter

$$\Psi_X(z) = -i\Big(2\tanh^{-1}\Big(\frac{4\sqrt{2}z^3}{9} + \frac{7}{9}\Big) + \log(z)\Big)$$

is multivalued and has $A_4$-symmetry.

**Remark 4.6.** The above behaviour of $\Psi_X$ is worth noting: $\Psi_X$ is a single-valued function if and only if $\omega_X$ has zero residue on all its poles.

The cases $s = d = 0$ and $r = d = 0$ are of special interest.

**4.3.1 The families $\mathscr{E}(0, r, 0)$.** A particularly interesting case is $\mathscr{E}(0, r, 0)$; the condition that $\infty \in \widehat{\mathbb{C}}$ is a fixed point of $\mathrm{Aut}(\widehat{\mathbb{C}})_X$ is automatically satisfied. In this case, there is a zero of multiplicity $r+2$ at $\infty \in \widehat{\mathbb{C}}$, and there are multi-saddles in $\mathbb{C}$.

The family $\mathscr{E}(0, r, 0)$ appears in Kaplan [21] and Boothby [7], [8]. On the other hand, Morse and Jenkins [25] studied whether a foliation on the plane with multi-saddles as singularities can be recognized as the level curves of an harmonic function; see also Bott [9] § 8 and [26]. Using the dictionary given by Proposition 4.1, we recognize

$$X(z) = \frac{1}{P(z)}\frac{\partial}{\partial z} \longleftrightarrow \Psi(z) = \int^{z} P(\zeta)d\zeta.$$

As an immediate corollary of Theorem 1 we obtain:

**Corollary 4.7** (Analytical and metric classification of $\mathscr{E}(0, r, 0)$).

1) *The families $\mathscr{E}(0, r, 0)$ and $\mathscr{E}(0, r, 0)_{\mathrm{id}}$ coincide if and only if $r + 1$ is prime.*

*For $r \geq 2$ we have:*

2) $\pi_1 : \mathscr{E}(0, r, 0)_{\mathrm{id}} \longrightarrow \mathscr{E}(0, r, 0)_{\mathrm{id}}/\mathrm{Aut}(\mathbb{C})$ *is a holomorphic trivial principal bundle, and* $\pi_2 \circ \pi_1 : \mathscr{E}(0, r, 0)_{\mathrm{id}} \longrightarrow \mathscr{E}(0, r, 0)_{\mathrm{id}}/(\mathrm{Aut}(\mathbb{C}) \times \mathbb{S}^1)$ *is a real analytic trivial principal bundle.*
3) *If $X \in \mathscr{E}(0, r, 0) \setminus \mathscr{E}(0, r, 0)_{\mathrm{id}}$ then there exists a rotation group $\Gamma \cong \mathbb{Z}_{\mathrm{k}}$ with $\mathrm{k} \in \mathscr{D} \setminus \{1\}$ and $\mathrm{k} \nmid r$ that leaves invariant*

$$X(z) = \frac{\lambda}{(z-C)^{\nu}\prod_{j=1}^{k_r}\prod_{\ell=1}^{\mathrm{k}}[z - C - (R_j e^{i\alpha_j})^{\ell/\mathrm{k}}]}\frac{\partial}{\partial z},$$

*where $r = \mathrm{k}k_r + \nu$, $\{R_j\} \subset \mathbb{R}^+$, $\{\alpha_j\} \subset \mathbb{R}$ and $\nu \in \mathbb{N}$.*

Furthermore the corresponding normal form is given by (21) with $s = d = 0$,

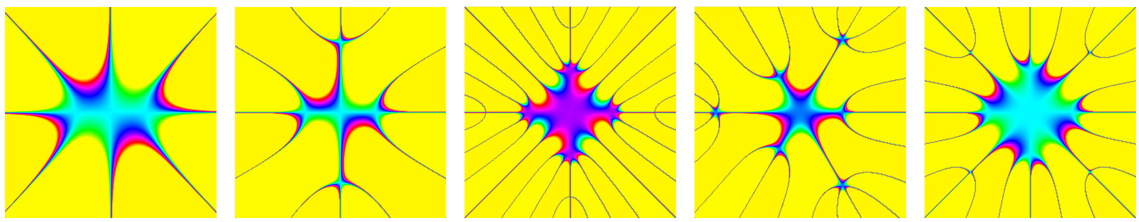
$$X(z) = \frac{1}{z^r + b_2 z^{r-2} + \cdots + b_r}\frac{\partial}{\partial z}.$$

**Example 4.8.** Consider $X_1(z) = \frac{1}{z(z^2-1)}\frac{\partial}{\partial z} \in \mathscr{E}(0,3,0)$ and $X_2(z) = \frac{1}{z(z^2-1)(z^2+4)}\frac{\partial}{\partial z} \in \mathscr{E}(0,5,0)$. Both have isotropy group isomorphic to $\mathbb{Z}_2$, in agreement with Corollary 3.4, see Figure 3 (A), (B).

Let $X(z) = \frac{\lambda}{z^3(z^4-1)^2(z^4-16)}\frac{\partial}{\partial z} \in \mathscr{E}(0,15,0)$. Considering the partition $r = 15 = 3 + (4+4) + 4$, and since $4 \mid (15+1)$, we have $\mathrm{Aut}(\mathbb{C})_X \cong \mathbb{Z}_4$ as can readily be seen by checking with (12), see Figure 3 (C).

Consider $X(z) = \frac{\lambda}{z^2(z^3-1)(z^3+8)^2}\frac{\partial}{\partial z} \in \mathscr{E}(0,11,0)$. From the partition $r = 11 = 2 + (3+3) + 3$, and since $3 \mid (11+1)$, it follows that $\mathrm{Aut}(\mathbb{C})_X \cong \mathbb{Z}_3$ as can readily be seen by checking with (12), see Figure 3 (D).

Since $r = 11 = 3+4+4$ and $4 \mid (11+1)$, also $\mathrm{Aut}(\mathbb{C})_X \cong \mathbb{Z}_4$ is possible: $X(z) = \frac{\lambda}{z^3(z^4-1)(z^4+16)}\frac{\partial}{\partial z} \in \mathscr{E}(0,11,0)$ realizes it, see Figure 3 (E).



(A) (B) (C) (D) (E)

**Figure 3:** Phase portraits of $\mathfrak{Re}(X)$ in $\mathscr{E}(0,r,0)$ having non-trivial isotropy. Borders of the strip flows correspond to streamlines of the field. (A) shows $X \in \mathscr{E}(0,3,0)$, (B) shows $X \in \mathscr{E}(0,5,0)$, both with simple poles and simple zeros and having isotropy group $\mathrm{Aut}(\mathbb{C})_X \cong \mathbb{Z}_2$, see Example 4.8.1. (C) corresponds to $X \in \mathscr{E}(0,15,0)$ with isotropy group isomorphic to $\mathbb{Z}_4$, see Example 4.8.2. (D) and (E) correspond to $X \in \mathscr{E}(0,11,0)$ with (D) having isotropy group isomorphic to $\mathbb{Z}_3$ and (E) having isotropy group isomorphic to $\mathbb{Z}_4$, see Example 4.8.3.

### 4.3.2 The families $\mathscr{E}(s,0,0)$.

For the case $\mathscr{E}(s,0,0)$, the condition that $\infty \in \widehat{\mathbb{C}}$ is a fixed point of $\mathrm{Aut}(\widehat{\mathbb{C}})_X$ is automatically satisfied for $s \geq 3$: $X$ has a pole of order $2-s$ at $\infty \in \widehat{\mathbb{C}}$. Dynamically this corresponds to the case of singularities consisting of centers, sources, sinks and flowers on $\mathbb{C}$ and a multi-saddle at $\infty$.

The polynomial vector fields $X \in \mathscr{E}(s,0,0)$ have been studied by Douady et al. [13], Branner et al. [10], Frías-Armenta et al. [17] and Rousseau [30] amongst others.

Once again by Theorem 1 we obtain:

**Corollary 4.9** (Analytical and metric classification of $\mathscr{E}(s,0,0)$).

1) *The families $\mathscr{E}(s,0,0)$ and $\mathscr{E}(s,0,0)_{\mathrm{id}}$ coincide if and only if $s-1$ is prime.*

*For $s \geq 3$ we have:*

2) $\pi_1 : \mathscr{E}(s,0,0)_{\mathrm{id}} \longrightarrow \mathscr{E}(s,0,0)_{\mathrm{id}}/\mathrm{Aut}(\mathbb{C})$ *is a holomorphic trivial principal bundle, and* $\pi_2 \circ \pi_1 : \mathscr{E}(s,0,0)_{\mathrm{id}} \longrightarrow \mathscr{E}(0,r,0)_{\mathrm{id}}/(\mathrm{Aut}(\mathbb{C}) \times \mathbb{S}^1)$ *is a real analytic trivial principal bundle.*

3) *If $X \in \mathscr{E}(s,0,0) \setminus \mathscr{E}(s,0,0)_{\mathrm{id}}$ then there exists a rotation group $\Gamma \cong \mathbb{Z}_{\mathrm{k}}$ with $\mathrm{k} \in \mathscr{D} \setminus \{1\}$ and $\mathrm{k} \nmid s$ that leaves invariant $X$. Furthermore*

$$X(z) = \lambda\,(z-C)^{\nu} \prod_{j=1}^{k_s} \prod_{\ell=1}^{\mathrm{k}} [z - C - (r_j e^{i\theta_j})^{\ell/\mathrm{k}}]$$

*where $s = \mathrm{k}k_s + \nu$, $\{r_j\} \subset \mathbb{R}^+$, $\{\theta_j\} \subset \mathbb{R}$ and $\nu \in \mathbb{N}$ such that $k \mid (\nu - 1)$.*

The corresponding normal form given by (20) with $r = d = 0$ and $s \geq 3$ is $X(z) = (z^s + a_2 z^{s-2} + \cdots + a_s)\frac{\partial}{\partial z}$.

**Example 4.10.** As an example consider $\mathscr{E}(7,0,0)$; note that $\mathscr{D} = \{1,3,6\}$. The vector field $X(z) = z^4(z^3-1)\frac{\partial}{\partial z}$ has $\mathrm{Aut}(\mathbb{C})_X \cong \mathbb{Z}_3$. In this case, there is a saddle at $\infty \in \widehat{\mathbb{C}}_z$ with 12 hyperbolic sectors (corresponding to a pole of $X$ of multiplicity $-5 = -(7-2)$). See Figure 4 for the phase portrait in the vicinity of the origin.

The distinguished parameter $\Psi_X(z) = \frac{1}{3z^3} + \frac{1}{3}\log(1-z^3) - \log(z)$ has $\mathbb{Z}_3$-symmetry and is once again multivalued.

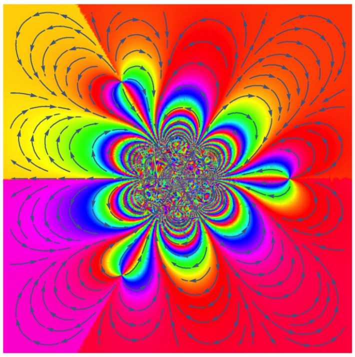


**Figure 4:** Phase portrait of $\mathfrak{Re}\,(X)$ for $X(z) = z^4(z^3 - 1)\frac{\partial}{\partial z}$ in $\mathscr{E}(7, 0, 0)$, with isotropy group $\mathrm{Aut}(\mathbb{C})_X = \mathbb{Z}_3$, see Example 4.10. Borders of the strip flows correspond to streamlines of the field.

**Acknowledgements:** The authors wish to thank Adolfo Guillot for useful comments, as well as the anonymous referee for the generous criticism that improved the present work.

# References


[1] R. D. M. Accola, Strongly branched coverings of closed Riemann surfaces. *Proc. Amer. Math. Soc.* **26** (1970), 315–322. MR262485 Zbl 0212.42501

[2] A. Alvarez-Parrilla, M. E. Frías-Armenta, C. Yee-Romero, Classification of rational differential forms on the Riemann sphere, via their isotropy group. Preprint 2018, arXiv:1811.04342 [math.GT]

[3] A. Alvarez-Parrilla, J. Muciño-Raymundo, Dynamics of singular complex analytic vector fields with essential singularities I. *Conform. Geom. Dyn.* **21** (2017), 126–224. MR3623567 Zbl 1368.32020

[4] A. Alvarez-Parrilla, J. Muciño-Raymundo, Y. Solorza, C. Yee-Romero, On the geometry, flows and visualization of singular complex analytic vector fields on Riemann surfaces. In: *Proceedings of the* 2018 *Workshop in Holomorphic Dynamics*, *Serie Papirhos, Instituto de Matemáticas, UNAM, México*, 21–109, 2019. arXiv:1811.04157 [math.GT]

[5] K. Biswas, R. Perez-Marco, Log-Riemann surfaces, Caratheodory convergence and Euler's formula. In: *Geometry, groups and dynamics*, volume 639 of *Contemp. Math.*, 197–203, Amer. Math. Soc. 2015. MR3379828 Zbl 1360.30031

[6] K. Biswas, R. Perez-Marco, Uniformization of simply connected finite type log-Riemann surfaces. In: *Geometry, groups and dynamics*, volume 639 of *Contemp. Math.*, 205–216, Amer. Math. Soc. 2015. MR3379829 Zbl 1360.30032

[7] W. M. Boothby, The topology of regular curve families with multiple saddle points. *Amer. J. Math.* **73** (1951), 405–438. MR42692 Zbl 0042.17603

[8] W. M. Boothby, The topology of the level curves of harmonic functions with critical points. *Amer. J. Math.* **73** (1951), 512–538. MR43456 Zbl 0044.31802

[9] R. Bott, Marston Morse and his mathematical works. *Bull. Amer. Math. Soc.* (*N.S.*) **3** (1980), 907–950. MR585177 Zbl 0469.01012

[10] B. Branner, K. Dias, Classification of complex polynomial vector fields in one complex variable. *J. Difference Equ. Appl.* **16** (2010), 463–517. MR2642463 Zbl 1203.37080

[11] S. A. Broughton, Superelliptic surfaces as $p$-gonal surfaces. In: *Riemann and Klein surfaces, automorphisms, symmetries and moduli spaces*, volume 629 of *Contemp. Math.*, 15–28, Amer. Math. Soc. 2014. MR3289630 Zbl 1346.14074

[12] I. Dolgachev, *Lectures on invariant theory*. Cambridge Univ. Press 2003. MR2004511 Zbl 1023.13006

[13] A. Douady, J. F. Estrada, P. Sentenac, Champs de vecteurs polynomiaux sur $\mathbb{C}$. Unpublished preprint, 2005.

[14] P. Doyle, C. McMullen, Solving the quintic by iteration. *Acta Math.* **163** (1989), 151–180. MR1032073 Zbl 0705.65036

[15] J. J. Duistermaat, J. A. C. Kolk, *Lie groups*. Springer 2000. MR1738431 Zbl 0955.22001

[16] H. M. Farkas, I. Kra, *Riemann surfaces*. Springer 1992. MR1139765 Zbl 0764.30001

[17] M. E. Frías-Armenta, J. Muciño-Raymundo, Topological and analytic classification of vector fields with only isochronous centers. *J. Difference Equ. Appl.* **19** (2013), 1694–1728. MR3173513 Zbl 1320.37010

[18] A. Guillot, Complex differential equations and geometric structures on curves. In: *Geometrical themes inspired by the N-body problem*, volume 2204 of *Lecture Notes in Math.*, 1–47, Springer 2018. MR3753115 Zbl 1409.37053

[19] J. Harris, *Algebraic geometry*. Springer 1992. MR1182558 Zbl 0779.14001

[20] B. Howard, J. Millson, A. Snowden, R. Vakil, The equations for the moduli space of $n$ points on the line. *Duke Math. J.* **146** (2009), 175–226. MR2477759 Zbl 1161.14033

[21] W. Kaplan, Topology of level curves of harmonic functions. *Trans. Amer. Math. Soc.* **63** (1948), 514–522. MR25159 Zbl 0032.07103
[22] S. Kerckhoff, H. Masur, J. Smillie, Ergodicity of billiard flows and quadratic differentials. *Ann. of Math.* (2) **124** (1986), 293–311. MR855297 Zbl 0637.58010
[23] F. Klein, *Lectures on the icosahedron and the solution of equations of the fifth degree*. Dover Publications, New York, 1956. MR0080930 Zbl 0072.25901
[24] J. C. Magaña-Cáceres, Classification of rational 1-forms on the Riemann sphere up to $PSL(2,\mathbb{C})$. *Bol. Soc. Mat. Mex.* (3) **25** (2019), 597–617. MR4022309 Zbl 1430.30021
[25] M. Morse, The existence of pseudoconjugates on Riemann surfaces. *Fund. Math.* **39** (1952), 269–287 (1953). MR57338 Zbl 0050.08501
[26] J. Muciño-Raymundo, Complex structures adapted to smooth vector fields. *Math. Ann.* **322** (2002), 229–265. MR1893915 Zbl 1047.37032
[27] J. Muciño-Raymundo, C. Valero-Valdés, Bifurcations of meromorphic vector fields on the Riemann sphere. *Ergodic Theory Dynam. Systems* **15** (1995), 1211–1222. MR1366317 Zbl 0863.58056
[28] R. Nevanlinna, Über Riemannsche Flächen mit endlich vielen Windungspunkten. *Acta Math.* **58** (1932), 295–373. MR1555350 Zbl 0004.35504
[29] R. Nevanlinna, *Analytic functions*. Springer 1970. MR0279280 Zbl 0199.12501
[30] C. Rousseau, The bifurcation diagram of cubic polynomial vector fields on $\mathbb{CP}^1$. *Canad. Math. Bull.* **60** (2017), 381–401. MR3679715 Zbl 1372.37095
[31] K. Strebel, *Quadratic differentials*. Springer 1984. MR743423 Zbl 0547.30001
[32] M. Taniguchi, Explicit representation of structurally finite entire functions. *Proc. Japan Acad. Ser. A Math. Sci.* **77** (2001), 68–70. MR1829377 Zbl 1024.32001
[33] M. Taniguchi, Synthetic deformation space of an entire function. In: *Value distribution theory and complex dynamics* (*Hong Kong,* 2000), volume 303 of *Contemp. Math.*, 107–136, Amer. Math. Soc. 2002. MR1943529 Zbl 1019.32013
[34] A. Tyurin, The classical geometry of vector bundles. In: *Algebraic geometry* (*Ankara,* 1995), 347–378, Dekker 1997. MR1483334 Zbl 0906.14009