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DYNAMICS OF SINGULAR COMPLEX ANALYTIC VECTOR FIELDS WITH ESSENTIAL SINGULARITIES I

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Dedicated to Luz Ximena; she loves the pictures.

ABSTRACT. We tackle the problem of understanding the geometry and dynamics of singular complex analytic vector fields X with essential singularities on a Riemann surface M (compact or not). Two basic techniques are used. (a) In the complex analytic category on M, we exploit the correspondence between singular vector fields X, differential forms ω_X (with $\omega_X(X) \equiv 1$), orientable quadratic differentials $\omega_X \otimes \omega_X$, global distinguished parameters $\Psi_X(z) = \int^z \omega_X$, and the Riemann surfaces $\mathcal{R}_X$ of the above parameters. (b) We use the fact that all singular complex analytic vector fields can be expressed as the *global* pullback via certain maps of the holomorphic vector fields on the Riemann sphere, in particular, via their respective Ψ_X.

We show that under certain analytical conditions on Ψ_X, the germ of a singular complex analytic vector field determines a decomposition in angular sectors; center C, hyperbolic H, elliptic E, parabolic P sectors but with the addition of suitable copies of a new type of *entire angular sector* $\mathscr{E}$, stemming from $X(z) = e^z \frac{\partial}{\partial z}$. This extends the classical theorems of A. A. Andronov *et al.* on the decomposition in angular sectors of real analytic vector field germs.

The Poincaré–Hopf index theory for $\mathfrak{Re}(X)$ local and global on compact Riemann surfaces, is extended so as to include the case of suitable isolated essential singularities.

The inverse problem: determining which cyclic words $\mathcal{W}_X$, comprised of hyperbolic, elliptic, parabolic and entire angular sectors, it is possible to obtain from germs of singular analytic vector fields, is also answered in terms of local analytical invariants.

We also study the problem of when and how a germ of a singular complex analytic vector field having an essential singularity (not necessarily isolated) can be extended to a suitable compact Riemann surface.

Considering the family of entire vector fields $\mathcal{E}(d) = \{X(z) = \lambda e^{P(z)} \frac{\partial}{\partial z}\}$ on the Riemann sphere, where $P(z)$ is a polynomial of degree d and $\lambda \in \mathbb{C}^*$, we completely characterize the local and global dynamics of this class of vector fields, compute analytic normal forms for $d = 1, 2, 3$, and show that for $d \geq 3$ there are an infinite number of topological (phase portrait) classes of $\mathfrak{Re}(X)$, for $X \in \mathcal{E}(d)$. These results are based on the work of R. Nevanlinna, A. Speisser and M. Taniguchi on entire functions Ψ_X.

Finally, on the topological decomposition of real vector fields into canonical regions, we extend the results of L. Markus and H. E. Benzinger to meromorphic on $\mathbb{C}$ vector fields X, with an essential singularity at $\infty \in \widehat{\mathbb{C}}$, whose Ψ_X^{-1} have d logarithmic branch points over d finite asymptotic values and d logarithmic branch points over ∞.

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1. Introduction

We consider singular complex analytic vector fields X, having local expressions $\{f_j(z)\frac{\partial}{\partial z}\}$ on a Riemann surface M (compact or not), admitting a singular set $\mathcal{S}_X$. The singular set $\mathcal{S}_X$ consists of:

- zeros, poles, *isolated essential singularities* and
- accumulation points of the above kind of points on M, for the sake of simplicity we will call these *essential singularities.*

As brief terminology, analytic means complex analytic.

In order to describe the global structure of the real trajectory solutions of X, consider the following naive question:
How explicitly can we globally describe a singular analytic vector field X on a Riemann surface M?

We use two main techniques. The first technique is the one-to-one correspondence on M (respectively on $(\mathbb{C}, 0)$ in the case of germs) between:

1) Singular analytic vector fields X.
2) Singular analytic differential forms ω_X, where $\omega_X(X) \equiv 1$.
3) Singular analytic orientable quadratic differentials $\omega_X \otimes \omega_X$.
4) Flat metrics g_X associated to the quadratic differentials $\omega_X \otimes \omega_X$, with suitable singularities, provided with a real geodesic vector field.
5) Global singular analytic (additively automorphic, probably multivalued) distinguished parameters

$$\Psi_X(z) = \int^z \omega_X.$$

6) The Riemann surfaces $\mathcal{R}_X$ associated to the maps Ψ_X.

Which enables us to transfer results from one area to the others; see Lemma 2.6 for the detailed correspondence statement.

As far as we know, unifying (1)–(5) came from the idea of (local) distinguished parameters of $\omega_X \otimes \omega_X$, see K. Strebel [66], pp. 20–21, where many of the geometric and dynamical aspects encoded in meromorphic quadratic differentials are described. The brothers Nevanlinna [53], pp. 298–303, A. Speiser [64] and M. Taniguchi [67], [68] have studied the relationship between (5) and (6) for entire and meromorphic functions. We look at the larger family of singular analytic objects, where the study of the global nature of (5) and (6) brings new insight into the description of vector fields.

The second technique is presented §3. We follow Riemann's idea: "every compact hyperelliptic Riemann surface M can be described as a ramified covering on the sphere $\widehat{\mathbb{C}}$, where the placings and orders of the ramification values determine M"; see [51], Lecture I for this synthesis in the general case. Our assertion is:

"Every singular analytic vector field X on M (respectively on $(\mathbb{C}, 0)$) can be expressed as the pullback, via certain singular analytic probably multivalued maps Ψ_X and Φ_X, of the simplest analytic vector fields $\frac{\partial}{\partial t}$ or $-w\frac{\partial}{\partial w}$ on the Riemann sphere $\widehat{\mathbb{C}}$."

As a concrete answer to the naive question, one has that the following commutative diagram holds true:

$$
\begin{tikzcd}
& (M, X) \arrow[dl, "\Psi_X"'] \arrow[dr, "\Phi_X"] & \\
\left(\widehat{\mathbb{C}}, \frac{\partial}{\partial t}\right) \arrow[rr, "\exp(-t)"'] & & \left(\widehat{\mathbb{C}}, -w\frac{\partial}{\partial w}\right),
\end{tikzcd}
\tag{1.1}
$$

where $\Phi_X = \exp \circ (-\Psi_X)$; see §3.2, in particular Remark 3.6 for the accurate statement of the diagram. In the language of differential equations:

- $X = \Psi_X^*(\frac{\partial}{\partial t})$ means that X has a *global flow–box*, *i.e.* the local rectificability can be analytically continued to M minus the singular set $\mathcal{S}_X$.
- $X = \Phi_X^*(-w\frac{\partial}{\partial w})$ states that X is the *global Newton vector field of* Φ_X. The usual interest stems from the fact that Newton vector fields X have sinks exactly at the zeros of Φ_X, and sources exactly at the poles of Φ_X.

Even in the case of univalued maps, by Picard's Theorem, in the vicinity of an isolated essential singularity of X the (a priori) *local* maps Ψ_X and Φ_X cover $\widehat{\mathbb{C}}$ minus two points. Hence, the local complexity of X at an isolated essential singularity on $(\mathbb{C}, 0)$ must be studied using the *global* maps Ψ_X and Φ_X. This can be readily seen in the example of $X(z) = \mathrm{e}^{z^d}\frac{\partial}{\partial z}$ where the essential singularity is at $\infty \in \widehat{\mathbb{C}}$.

Given a germ $\big((\mathbb{C}, 0), X\big)$ of a singular analytic vector field with an isolated essential singularity at 0, roughly speaking, we have the following *analytic invariants of* X:

- *class* p, taking values in $\mathbb{N} \cup \{0, \infty\}$,
- p*–order*, with values in $\mathbb{R} \cup \{\infty\}$,
- *residue* and *semi–residues* (of the respective ω_X), taking values in $\mathbb{C}$,
- *configurations of critical and asymptotic values* (of the respective Ψ_X),

see §4 and Corollary 4.18.

We study local topological/analytical invariants of $\big((\mathbb{C}, 0), X\big)$ in §5. We convene that the real trajectories of the associated real analytic vector field $\mathfrak{Re}\,(X)$ are to be called trajectories of the singular analytic vector field X (the phase portraits of X and $\mathfrak{Re}\,(X)$ coincide). Recall that a germ of a real analytic vector field Z on $(\mathbb{R}^2, 0)$ having one isolated zero, admits a decomposition in angular sectors: hyperbolic, elliptic, and parabolic (see §5.1), thanks to a classical theorem of A. A. Andronov *et al.* (see [8], pp. 86, [37], pp. 144).

We propose complex analytic angular sectors; center C, hyperbolic H, elliptic E, parabolic P and a new type of *entire* angular sector $\mathscr{E}$ (see Figure 3) based upon the entire vector field $X(z) = \mathrm{e}^{z}\frac{\partial}{\partial z}$ at the point $\infty \in \widehat{\mathbb{C}}$. Using these new sectors, a large family of germs of singular analytic vector fields X on $(\mathbb{C}, 0)$, with 0 an isolated essential singularity, admit a similar decomposition. Roughly speaking, a germ X determines an *admissible word* whenever it has an associated cyclic word $\mathcal{W}_X$ in the alphabet $C, H, E, P, \mathscr{E}$, arising from the topology of the trajectories of X at the singularity, diagrammatically (see (5.15)):

$$\big((\mathbb{C}, 0), X\big) \longmapsto \mathcal{W}_X = W_1 W_2 \cdots W_k, \quad W_\iota \in \{C, H, E, P, \mathscr{E}\}.$$

Essentially $\mathcal{W}_X$ is well defined modulo the relation $E\mathscr{E}H \sim \mathscr{E}$ (see Remark 5.18). The relation arises from to the ambiguity in the choice of representative for the germ of X.

As a concrete example, recall that $\big((\mathbb{C},0),X\big)$ having a pole, a simple zero with pure imaginary linear part or a multiple zero with zero residue determine the words

$$\frac{1}{z^k}\frac{\partial}{\partial z} \longmapsto \underbrace{H\cdots H}_{2k+2}, \qquad iz\frac{\partial}{\partial z} \longmapsto C, \qquad z^s\frac{\partial}{\partial z} \longmapsto \underbrace{E\cdots E}_{2s-2} \quad s\geq 2.$$

An accurate description in the meromorphic case is in Table 2, §5. An example with an isolated essential singularity on $(\widehat{\mathbb{C}},\infty)$ is

$$\frac{e^{z^d}}{d\,z^{d-1}}\frac{\partial}{\partial z} \longmapsto \underbrace{\mathscr{E}\cdots\mathscr{E}}_{2d} \qquad \text{for } d\geq 1;$$

see Example 5.15.2.

Let h, e, p, ϵ denote the number of hyperbolic H, elliptic E, parabolic P and entire $\mathscr{E}$ angular sectors, respectively, in an admissible word $\mathcal{W}_X$. In §6 the Poincaré–Hopf index theory for $\mathfrak{Re}\,(X)$ is extended, starting from the classic formula of I. Bendixson, to vector fields with isolated essential singularities.

Theorem (A) (Local and global Poincaré–Hopf theory).

1) Let $\big((\mathbb{C},0),X\big)$ be a germ of a singular analytic vector field with an isolated singularity at 0 and further suppose that X determines an admissible word $\mathcal{W}_X$. Then the Poincaré–Hopf index of X at 0 is

$$PH(X,0) = 1 + \frac{e-h+\epsilon}{2}.$$

2) Let X be a singular analytic vector field on a compact M having a discrete set of poles, zeros and isolated essential singularities determining admissible words at each singularity, then

$$\chi(M) = \sum_{q\in M} PH(X,q).$$

Exploration of the conditions under which a germ of a singular analytic vector field can be extended to a compact Riemann surface yields the following result.

Theorem (B) (Extension of vector field germs to compact Riemann surfaces). *Let $\big((\mathbb{C},0),X\big)$ be a germ of a singular analytic vector field having a nonnecessarily isolated singularity at 0.*

1) There exists an extended singular analytic vector field $\widetilde{X}$ on a compact Riemann surface $M_{\mathfrak{g}}$, for each genus $\mathfrak{g}\geq 0$, such that:
the germ of $\widetilde{X}$ at some $p\in M$ is holomorphically equivalent to the germ X, with $\widetilde{X}$ having an additional finite number of zeros and poles.

2) If, in addition, X determines an admissible word $\mathcal{W}_X$, then there exists an extended $\widetilde{X}$ on $\widehat{\mathbb{C}}$, having at most an additional pole and a finite number of simple zeros.

3) If, in addition, X determines an admissible word $\mathcal{W}_X$, having Poincaré–Hopf index $PH(X,0)=2-2\mathfrak{g}$, for $\mathfrak{g}\in\mathbb{N}\cup\{0\}$, and residue $Res(\omega_X,0)=0$, then there exists an extended $\widetilde{X}$ on a compact Riemann surface $M_{\mathfrak{g}}$, of genus $\mathfrak{g}$, having

a) no other singularities on $M_{\mathfrak{g}}=\widehat{\mathbb{C}}$, when $\mathfrak{g}=0$, *b) two new simple poles and no other singularities on $M_{\mathfrak{g}}$, when $\mathfrak{g} \geq 1$.*

Where assertions (1), (2) and (3) correspond to Theorem 7.1, Corollary 7.3 and Corollary 10.3, respectively.

In §8, we provide a complete answer to the problem, which was originally considered by K. Hockett and S. Ramamurti [34], of describing the families of vector fields on $\widehat{\mathbb{C}}$,

$$\mathcal{E}(d) = \left\{ X(z) = \lambda e^{P(z)} \frac{\partial}{\partial z} \;\middle|\; P(z) \text{ a polynomial of degree } d \geq 1, \lambda \in \mathbb{C}^* \right\}.$$

Explicit examples of $X \in \mathcal{E}(d)$ are presented in Figures 1, 3, 15, 16 and 19.

It is natural to hope that vector fields in the family $\mathcal{E}(d)$ should have a description as admissible words at $(\widehat{\mathbb{C}}, \infty)$. In order to answer this, and to also analytically classify the family $\mathcal{E}(d)$, in §8.3, we introduce the main global combinatorial object: the *d–configuration tree* Λ_X associated to $X \in \mathcal{E}(d)$.

Very roughly speaking, d–configuration trees are weighted trees with complex parameters as their weights; compare with [65] and [54]. They provide an accurate description of the Riemann surface $\mathcal{R}_X$, as a ramified covering $\pi_{X,2} : \mathcal{R}_X \to (\widehat{\mathbb{C}}, \frac{\partial}{\partial t})$ (see diagram (2.6)), by encoding the placement and ramification index of the ramification points of $\mathcal{R}_X$. The main results of this section are incorporated, without all the technical details; see the following.

Theorem (C) (Analytical and topological classification of $\mathcal{E}(d)$).

1) Singular analytic vector fields in $\mathcal{E}(d)$ are in one-to-one correspondence with classes of d–configuration trees, i.e.

$$\mathcal{E}(d) \cong \left\{ [\Lambda_X] \mid \Lambda_X \text{ is a } d\text{–configuration tree} \right\}.$$

2) The normal forms in $\mathcal{E}(d)/Aut(\mathbb{C})$, for $d \leq 3$, can be given as follows:

For d=1, $$e^z \frac{\partial}{\partial z}.$$

For d=2, $$\mu\, e^{z^2} \frac{\partial}{\partial z}, \qquad \mu \in \mathbb{C}^*.$$

For d=3, $$\mu\, e^{\left(-\frac{1}{3}z^3 + pz\right)} \frac{\partial}{\partial z}, \qquad \mu \in \mathbb{C}^*,\ p \in \mathbb{C},$$

and in particular, the geometry of $\mathcal{E}(3)/Aut(\mathbb{C})$ is related to Airy's function.

3) For $d = 1, 2$, there are exactly d topological classes of $\Re(X)$ for $X \in \mathcal{E}(d)$.

4) For each $d \geq 3$, there are an infinite number of topological classes of $\Re(X)$ for $X \in \mathcal{E}(d)$.

5) A germ $\big((\widehat{\mathbb{C}}, \infty), X\big)$ is the restriction of $X \in \mathcal{E}(d)$ if and only if $\infty \in \widehat{\mathbb{C}}$ is an isolated essential singularity and the admissible word $\mathcal{W}_X$ satisfies that
   1) the residue of the word $Res(\mathcal{W}_X) = 0$,
   2) the Poincaré–Hopf index of the word $PH(\mathcal{W}_X) = 2$,
   3) it has exactly $\epsilon = 2d$ class 1 entire sectors $\mathscr{E}$.

The correspondence of (1) through (5) and the text being: (1) with Theorem 8.16, (2) with Theorem 8.24 and §8.6.1, (3) and (4) with Theorem 8.31, and (5) with Corollary 10.1.

From Theorem (C.5) it follows that not all admissible words arise from singular analytic vector fields in $\mathcal{E}(d)$. Roughly speaking, the relationship between a germ of a singular analytic vector field and the admissible word is that: half the number ϵ of class 1 entire sectors $\mathscr{E}$ in the admissible word $\mathcal{W}_X$ corresponds to the 1–order of the distinguished parameter Ψ_X. This is the contents of Theorems 9.1 and 9.8.

Theorem (D) (Local realization and recognition of analytical invariants from admissible words).

1) *A germ* $\big((\mathbb{C},0),X\big)$ *of a singular analytic vector field having an isolated essential singularity at* 0, *and whose distinguished parameter* Ψ_X *satisfies that* Ψ_X^{-1} *has as unique singularities* d *logarithmic branch points over* d *finite asymptotic values* $\{a_j\} \subset \mathbb{C}_t$ *and* d *logarithmic branch points over* ∞, *determines an admissible word* $\mathcal{W}_X$ *composed of sectors of type* H, E, P *and* $\mathscr{E}$. *In particular, there are* $\epsilon = 2d$ *sectors of type* $\mathscr{E}$.
2) *Every admissible word* $\mathcal{W}_X$ *in the alphabet* C, H, E, P, $\mathscr{E}$ *comes from a germ* $\big((\mathbb{C},0),X\big)$ *of an isolated essential singularity of a complex analytic vector field.*

   For $\epsilon \geq 2$, *the distinguished parameter* Ψ_X *satisfies that* Ψ_X^{-1} *has* $\epsilon/2$ *logarithmic branch points over* $\epsilon/2$ *finite asymptotic values and* $\epsilon/2$ *logarithmic branch points over* ∞.

In §11 Appendix, we recall that the topological decomposition of the phase portrait for real vector fields into *canonical regions* (spiral, annulus and strips), essentially provided by L. Markus [47] for C^1 plane vector fields, has been enhanced by H. E. Benzinger [10] to include the rational category. We extend these results to the case of meromorphic vector fields on $\mathbb{C}$ with an essential singularity at $\infty \in \widehat{\mathbb{C}}$, whose Ψ_X^{-1} have d logarithmic branch points over d finite asymptotic values and d logarithmic branch points over ∞. See Corollary 11.3. The reader is encouraged to consult Table 3 in §11; it provides a quick reference to the canonical regions used coherently throughout the work.

Naturally, the space of singular analytic vector fields, on $\widehat{\mathbb{C}}$, with an essential singularity is extremely complex. As an example of this complexity, we look at the class 2 vector field $X(z) = e^z e^{-e^z} \frac{\partial}{\partial z}$ (see Figures 2 and 5) which also has an isolated essential singularity at $\infty \in \widehat{\mathbb{C}}$. With this example in mind, a new family of angular sectors can be constructed. The theory presented here is not enough to completely understand the class 2 case, except Theorem (B.1) which does apply and Theorem (A) which applies with minor modifications. This could be the subject of a further study.

The different categories in which we work: Our approach to singular analytic vector fields uses different techniques. As an attempt to clarify where things are going, let us recall that we are working in: the complex analytic category (X, ω_X, Ψ_X); the flat singular surfaces category (g_X on M); the combinatorial and topological categories (d–configuration trees Λ_X, admissible words $\mathcal{W}_X$, topological phase portraits classes of $\mathfrak{Re}\,(X)$).

In particular, for the family $\mathcal{E}(d)$, diagram (1.2) shows some of the main relations between the combinatorial, analytical, topological and geometrical categories. Here, $\{d\text{–}configuration\ trees\ [\Lambda_X]\}$ means the space of classes of d–configuration trees; the normal forms are $\mathcal{E}(d)/Aut(\mathbb{C})$; the quotient $\mathcal{E}(d)/Aut(\mathbb{C}) \times S^1$ means the space of classes of flat metrics $\{g_X\}$ up to orientation preserving isometries; and the quotient $\mathcal{E}(d)/Homeo(\mathbb{C})^+$ means the space of phase portrait classes of $\mathfrak{Re}\,(X)$ up to orientation preserving homeomorphisms (see Definition 3.1.1).

$$
(1.2)\qquad
\begin{tikzcd}
\mathcal{E}(d) \arrow[r, "\pi_1"] \arrow[d, "\cong"] & \frac{\mathcal{E}(d)}{Aut(\mathbb{C})} \arrow[r, "\pi_3"] \arrow[d, "\cong"] \arrow[dr, "\pi_2"] & \frac{\mathcal{E}(d)}{Aut(\mathbb{C})\times S^1} \cong \left\{ \begin{array}{c} \text{flat} \\ \text{metrics} \\ g_X \end{array} \right\} \\
\left\{ \begin{array}{c} d\text{–configuration} \\ \text{trees } [\Lambda_X] \end{array} \right\} & \left\{ \begin{array}{c} \text{normal} \\ \text{forms} \end{array} \right\} & \frac{\mathcal{E}(d)}{Homeo(\mathbb{C})^+} \cong \left\{ \begin{array}{c} \text{vector} \\ \text{fields} \\ \mathfrak{Re}\,(X) \end{array} \right\}.
\end{tikzcd}
$$

The following sources provided us enlightenment on many of the topics considered in the present work. We apologize in advance for our extremely brief and necessarily incomplete list.

As far as we are aware, the roots of the geometrical study of differential forms started in 1857 with B. Riemann: he distinguished differentials of the 1st kind (holomorphic), 2nd kind (meromorphic, zero residues) and 3rd kind (meromorphic, nonzero residues); see [58] pp. 96–97. Later on F. Klein implicitly described the geometrical behaviour of trajectories of ω_X, see [43] pp. 1–9.

Modern treatises focusing on the trajectory structure of quadratic differentials: J. A. Jenkins [39], K. Strebel [66].

Transcendental singularities of meromorphic functions and Riemann surfaces: R. Nevanlinna [53], [54], M. Taniguchi [67], [68], W. Bergweiler *et al.* [12].

The point of view of differential equations (meromorphic vector fields): J. Gregor [26], [27], O. Hájek [31], [32], [33], N. A. Lukashevich [46], L. Brickman *et al.* [14], M. Sabatini [59], J. Muciño–Raymundo *et al.* [50], D. J. Needhan *et al.* [52], E. P. Volokitin *et al.* [70], A. Alvarez–Parrilla *et al.* [4], A. Garijo *et al.* [24], B. Branner *et al.* [13], E. Frías–Armenta *et al.* [23].

Partial versions of the correspondence (1)–(6): J. Muciño–Raymundo *et al.* [50], J. Muciño–Raymundo [49], A. Bustinduy *et al.* [17], [18].

Modern explicit proofs of the normal forms for zeros and poles of vector fields and quadratic differentials: J. A. Jenkins [39] ch. 3, J. Gregor [26], [27], L. V. Ahlfors [1] pp. 111, L. Brickman *et al.* [14], K. Strebel [66] ch. III, A. Garijo *et al.* [24].

The case of singularities for entire vector fields: K. Hockett *et al.* [34], A. Garijo *et al.* [25].

Relations with discrete dynamical systems: S. Smale [63], M. Shub *et al.* [61], H. E. Benzinger [10], H. T. Jonger *et al.* [40], A. Douady *et al.* [19], X. Buff *et al.* [16].

Flat metrics: K. Strebel [66], R. Peretz [57], J. Muciño–Raymundo *et al.* [50], H. Masur *et al.* [48], J. Muciño–Raymundo [49], M. Kontsevich *et al.* [44], J. P. Bowman *et al.* [15].

Graphs associated to global analytic functions: A. Speiser [65], R. Nevanlinna [54], G. Elfving [22], M. Shub *et al.* [61], H. T. Jongen *et al.* [40].

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origin. Furthermore, there are 6 entire $\mathscr{E}$ sectors in the vicinity of infinity, so the admissible word at ∞ is

$$E\mathscr{E}HH\mathscr{E}EE\mathscr{E}HH\mathscr{E}EE\mathscr{E}HH\mathscr{E}E,$$

hence the Poincaré–Hopf index of X at ∞ is 4, while the admissible word at 0 is $HHHHHH$, and its Poincaré–Hopf index is -2, and of course

$$\chi(\widehat{\mathbb{C}}) = PH(X,\infty) + PH(X,0) = 4 - 2.$$

Example 11.6 (Example 4.17 revisited). Consider the vector field $X(z) = \frac{e^{z^3}}{3z^3-1}\frac{\partial}{\partial z}$; see Figure 21 (b). Once again ∞ is an isolated essential singularity of X, but now X has 3 simple poles located at $\frac{1}{\sqrt[3]{3}}$, $\frac{e^{i2\pi/3}}{\sqrt[3]{3}}$ and $\frac{e^{-i2\pi/3}}{\sqrt[3]{3}}$. The admissible word at ∞ is (starting from the top in Figure 21 (b)):

$$\mathscr{E}HH\mathscr{E}EE\mathscr{E}HH\mathscr{E}EEE\mathscr{E}HH\mathscr{E}EEE,$$

so the Poincaré–Hopf index of X at ∞ is 5, while the admissible word at each simple pole is $HHHH$, and the corresponding Poincaré–Hopf index is -1, and of course once again

$$\chi(\widehat{\mathbb{C}}) = PH(X,\infty) + PH\left(X, \frac{1}{\sqrt[3]{3}}\right) + PH\left(X, \frac{e^{i2\pi/3}}{\sqrt[3]{3}}\right) + PH\left(X, \frac{e^{-i2\pi/3}}{\sqrt[3]{3}}\right) = 2.$$

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