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Topological and analytical classification of vector fields with only isochronous centres

Martín-Eduardo Frías-Armenta[a] and Jesús Muciño-Raymundo[b]*

[a]*Departamento de Matemáticas, Universidad de Sonora, Hermosillo, Mexico;*
[b]*Centro de Ciencias Matemáticas, UNAM, Campus Morelia, Morelia, Mexico*

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We study vector fields on the plane having only isochronous centres. The most familiar examples are *isochronous* vector fields, they are the real parts of complex polynomial vector fields on $\mathbb{C}$ having all their zeroes of centre type. We describe the number $N(s)$ of topologically inequivalent isochronous (singular) foliations that can appear for degree s, up to orientation preserving homeomorphisms. For each s, there exists a real analytic variety $\mathcal{I}(s)$ parametrizing the isochronous vector fields of degree s, the group of complex automorphisms of the plane $\mathrm{Aut}(\mathbb{C})$ acts on it. Furthermore, if $2 \leq s \leq 7$, then $\mathcal{I}(s)$ is a non-singular real analytic variety of dimension $s+3$, and their number of connected components is bounded by $2N(s)$. An explicit formula for the residues of the rational 1-form, canonically associated with a complex polynomial vector field with simple zeroes, is given. A collection of residues (i.e. periods) does not characterize an isochronous vector field, even up to complex automorphisms of $\mathbb{C}$. An exact bound for the number of isochronous vector fields, up to $\mathrm{Aut}(\mathbb{C})$, having the same collection of residues (periods) is given. We develop several descriptions of the quotient space $\mathcal{I}(s)/\mathrm{Aut}(\mathbb{C})$ using residues, weighted s-trees and singular flat Riemannian metrics associated with isochronous vector fields.

Keywords: ordinary differential equations; isochronous centres; residues; complex polynomials

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1. Introduction

A real vector field on $\mathbb{R}^2$ has an isochronous centre when the periods of trajectories surrounding the singular point are constant. The simplest vector fields having isochronous centres are the real parts $\mathfrak{Re}(X)$ of complex polynomial vector fields X, with non-zero pure imaginary derivative at some zero.

Real and complex isochronous centres appear in the following situations:

In the problem of linearization of centres for real vector fields of class C^{ω}, see [7,17]. They are good prospects to study the birth of limit cycles under perturbation of centres, this is the infinitesimal Hilbert's 16 problem [2,14]. In the topological classification of plane polynomial vector fields, the phase portraits of polynomial vector fields with only isochronous centres are between the simplest (see [1,4] or [23]) for the quadratic case. They are the most simple examples of Jenkins–Strebel quadratic differentials on the Riemann sphere [26, 27].

In all that follows, an *isochronous vector field*, in short, an *isochronous X* is a complex polynomial vector field on $\mathbb{C}$, having all their zeroes of centre type.

*Corresponding author. Email: muciray@matmor.unam.mx

An isochronous X has associated a weighted s-tree $\Lambda(X)$ as follows, the s vertices correspond to the zeroes, the edges are determined by adjacent centre basins and the weights are the periods. In particular, $\Lambda(X)$ determines an embedded s-tree in $\mathbb{C}$.

Following [18], we know the following three facts. Each embedded s-tree (without weights) in $\mathbb{C}$ is realized by an isochronous X. If two isochronous X_1, X_2 determine topologically equivalent isochronous foliations (up to preserving orientation homeomorphisms $\mathrm{Homeo}(\mathbb{C})^+$), then they have associated the same embedded s-tree. Furthermore, X_1, X_2 are biholomorphically equivalent (up to complex automorphisms $\mathrm{Aut}(\mathbb{C})$) if and only if they have associated isomorphic weighted s-trees.

Let $\mathbb{C}^{s+1}$ be the space of complex polynomial vector fields on $\mathbb{C}$ having degree at most s. The set of isochronous vector fields of degree s, denoted by $\mathcal{I}(s)$, forms a real analytic family. The complex Lie group $\mathrm{Aut}(\mathbb{C})$ acts holomorphically on $\mathbb{C}^{s+1}$ as changes of coordinates. We have the diagram

$$\mathbb{C}^{s+1} \supset \mathcal{I}(s) \longrightarrow \frac{\mathcal{I}(s)}{\mathrm{Aut}(\mathbb{C})} \longrightarrow \frac{\mathcal{I}(s)}{\mathrm{Homeo}(\mathbb{C})^+}. \tag{1}$$

Our goal is the computation of the above quotients from the point of view of combinatorics, topology, geometry and dynamics.

The first main result Theorem 5.3 answers a question in [2].

The number of classes of topological isochronous foliations in $\mathcal{I}(s)/\mathrm{Homeo}(\mathbb{C})^+$ *(i.e. without bearing in mind the orientation of the trajectories) is*

$$N(s) = \frac{1}{2(s-1)} \sum_{d|(s-1)} \phi\left(\frac{s-1}{d}\right)\binom{2d}{d} - \frac{1}{2}c_{s-1} + \frac{1}{2}\chi_{\mathrm{even}}(s)c_{(s/2)-1}, \tag{2}$$

where $c_s = (1/(s+1))\binom{2s}{s}$ *denotes the sth Catalan number,* ϕ *denotes the Euler's function and* χ_{even} *is the characteristic function of even integers.*

Furthermore, for $1 \leq s \leq 15$ *the respective numbers are*

$$1, 1, 1, 2, 3, 6, 14, 34, 95, 280, 854, 2694, 8714, 28640 \text{ and } 95640. \tag{3}$$

The idea of enumeration (2) using s-trees was suggested Muciño-Raymundo in [18] Corollary 8.3, compared with the recent result of Rong [24] and see our diagram (16) for an accurate description. The analytic families are more subtle, we describe them in Theorem 6.1.

Assume $3 \leq s \leq 7$. *The set of isochronous vector fields* $\mathcal{I}(s) \subset \mathbb{C}^{s+1}$ *is a non-singular real analytic space of dimension* $s+3$.

The quotient space $\mathcal{I}(s)/\mathrm{Aut}(\mathbb{C})$ *is of dimension* $s-1$, *Hausdorff and admits a stratification by orbit types.*

In particular for $s \leq 7$,

$$\text{The number of connected components of } \mathcal{I}(s) \text{ is } \begin{cases} = 2N(s) & \text{odd } s, \\ \leq 2N(s)-1 & \text{even } s. \end{cases} \tag{4}$$

The low-dimensional quotients $\mathcal{I}(s)/\mathrm{Aut}(\mathbb{C})$, $s = 1, 2, 3$, are described in Proposition 6.3. The second assertion says (very roughly speaking) that $\mathcal{I}(s)/\mathrm{Aut}(\mathbb{C})$ is a locally finite union of manifolds, pieced together in a nice way. Equation (4) says that for odd s, the preimage of a point under $\mathcal{I}(s) \to \mathcal{I}(s)/\mathrm{Homeo}(\mathbb{C})^+$ in (1) has precisely two connected components of $\mathcal{I}(s)$, due to the change of sign $\pm X$. For even s, a more complex behaviour appears, some precise values for (4) are given in 6.1.3.

An isochronous vector field X is characterized by their associated rational 1-form ω, such that $\omega(X) \equiv 1$, having only one zero at infinity and simple poles with non-zero pure imaginary residues. Our main analytic tool is an explicit formula for these residues (see (8)). The hypothesis $s \leq 7$ in the above result depends on explicit computations with the residues. We conjecture that it remains valid for all s (see Remark 2 in Section 7). The residues of complex polynomial vector fields are the most natural complex analytic invariants under the Aut($\mathbb{C}$)-action (see Proposition 2.3). A third result is as follows (see Theorem 7.8).

Aut($\mathbb{C}$)-*orbits fail to be separated by the residues. For* $s \geq 4$, *the number of different classes of isochronous vector fields in* $\mathcal{I}(s)/\mathrm{Aut}(\mathbb{C})$ *having the same collection of residues varies between 1 and* $(s-2)!$

The determination of enough invariant quantities for the Aut($\mathbb{C}$)-action on $\mathcal{I}(s)$ is equivalent to the construction of the quotient space $\mathcal{I}(s)/\mathrm{Aut}(\mathbb{C})$. This classical idea emerged from the work of Cayley and Hilbert (see [9], p. xiii for an illuminated explanation). Even more, we deal with real analytic spaces $\mathcal{I}(s)$, non-complex analytic. Hence, we require techniques from Lie group actions (see Chapter 2 in [10]).

Very roughly speaking, we describe $\mathcal{I}(s)/\mathrm{Aut}(\mathbb{C})$, using the following points of view: realizable weighted s-trees, singular flat Riemannian metrics up to isometries, collections of residues and configurations of zeroes (see diagram (27)).

The similar problems (realization of analytic families, moduli spaces, use of weighted graphs, number of connected components, etc.) for holomorphic 1-forms on compact Riemann surfaces of genus $g \geq 1$ have received much attention (see [16,27]). However, the case of rational 1-forms having arbitrary zeroes and poles on the Riemann sphere is less studied. Our results for 'isochronous' rational 1-forms, describe new families on the Riemann sphere.

In Section 2, we construct complex manifolds which are parameter spaces for the complex polynomial vector fields with simple zeroes and their associated 1-forms. Also we recall the residues as Aut($\mathbb{C}$) invariants. In Section 3, we deal with real vector fields. A complete characterization of weighted s-trees coming from isochronous X is given in Corollary 3.6. The residue theorem imposes a restriction, but also there are other restrictions (see Example 7.3). In Section 4, the flat metric associated with an isochronous X is described as a gluing of flat cylinders. The main results are the topological classification and the construction of non-singular analytic families for $\mathcal{I}(s)$ (both are presented in Sections 5 and 6, respectively). The characterization of realizable residues by isochronous vector fields is studied in Section 7. In Section 8, explicit families of isochronous centres are described in terms of configurations of zeroes, inequalities for the residues and weighted graphs. Their dimensions and codimensions in $\mathcal{I}(s)$ are provided. As an application, Section 9 is devoted to results of bifurcations under rotation and the Hamiltonian nature of isochronous vector fields (see Sections 2 and 5). Conclusions and future directions are in Section 10.

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2. Different facets of complex polynomial vector fields

In all that follows we use the one-to-one correspondence between

(i) complex polynomial vector fields X of degree s,
(ii) rational differential forms ω having only a zero at $\infty \in \hat{\mathbb{C}}$ of multiplicity $s-2$,
(iii) orientable rational quadratic differentials $\omega \otimes \omega$, having poles of even multiplicity in $\mathbb{C}$ and a zero at $\infty \in \hat{\mathbb{C}}$ of multiplicity $2s-4$,

10. Conclusions and future directions

We have enlarged the information of diagrams and equations (1), (12), (14), (15) and (17). Our new diagram is

$$
\begin{tikzcd}
\mathcal{I}(s) \arrow[r] & \frac{\mathcal{I}(s)}{\mathrm{Aut}(\mathbb{C})} \arrow[r] \arrow[d, leftrightarrow] \arrow[dr] & \frac{\mathcal{I}(s)}{\mathrm{Homeo}\,(\mathbb{C})^+} \arrow[dr] & \\
& \left\{\begin{array}{c}\text{realizable}\\ \text{weighted}\\ s\text{-trees } \Lambda(X)\end{array}\right\} \arrow[d] \arrow[dr] & \frac{\mathcal{I}(s)}{\mathrm{Aut}(\mathbb{C})\times\mathbb{Z}_2} \arrow[r] \arrow[d, leftrightarrow] & \frac{\mathcal{I}(s)}{\mathrm{Homeo}\,(\mathbb{C})^+\times\mathbb{Z}_2} \arrow[d, leftrightarrow] \\
& \left\{\begin{array}{c}\text{isochronous}\\ \text{residues;}\\ [r_1,\ldots,r_s]\end{array}\right\} & \left\{\begin{array}{c}\text{flatmetrics of}\\ \text{isochronous}\, X,\\ \text{uptoisometries}\end{array}\right\} \arrow[r] & \left\{\begin{array}{c}\text{plane}\\ s\text{-trees}\\ \Lambda\subset\mathbb{C}\end{array}\right\}
\end{tikzcd}
\quad . \qquad (27)
$$

The $\mathbb{Z}_2$-action is induced by $X \mapsto \pm X$ and $\updownarrow$ means bijection. Having in mind the above results, we can state the following questions.

Problem. Characterize geometrically the isochronous configurations $[p_1, \ldots, p_s]$, for $s \geq 5$.

Problem. Given an isochronous configuration $[p_1, \ldots, p_s]$, construct an algorithm to determine the associated plane s-tree Λ.

In particular:

Problem. If for some isochronous configuration $[p_1, \ldots, p_s]$, with $s \geq 5$, only one point, say p_1, is in the open convex hull of the other $s-1$ points; is it true that the associated s-tree has a vertex at p_1 with degree $s-1$ (as in 8.9)?

Conjecture. For every isochronous residues $[r_1, \ldots, r_s]$, with $s \geq 5$:

the $\min_j\{|r_j|\}$ is attained at p_β in the boundary of the convex hull of the zeroes,
the $\max_j\{|r_j|\}$ is attained at p_ι in the interior of the convex hull of the zeroes.

In other order of ideas. For $s \geq 2$, the symplectic structure making Hamiltonian an isochronous vector field in Proposition 9.3 has punctures at the zeroes.

Problem. Given an isochronous vector field, does exists a smooth symplectic structure on all $\mathbb{R}^2$ with the above property?

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References

[1] A.J. Álvarez, A. Gasull, and R. Prohens, *Configurations of critical points in complex polynomial differential equations*, Nonlinear Anal. 71 (2009), pp. 923–934.
[2] A.J. Álvarez, A. Gasull, and R. Prohens, *Topological classification of polynomial complex differential equations with all the critical points of centre type*, J. Difference Equ. Appl. 16(5–6) (2010), pp. 411–423.
[3] F. Bergeron, G. Labelle, and P. Leroux, *Combinatorial Species and Tree-like Structures*, Cambridge University Press, Cambridge, 1998.

[4] B. Branner and K. Dias, *Classification of complex polynomial vector fields in one variable*, J. Difference Equ. Appl. 16 (5–6) (2010), pp. 463–517.
[5] K. Broughan, *Corrigenda for holomorphic flows in simply connected regions have no limit cycles*, Meccanica 42 (2007), p. 213.
[6] M. Chaperon and S. López de Medrano, *Some regularities and singularities appearing in the study of polynomials and operators*, Astérisque 323 (2009), pp. 123–160.
[7] J. Chavarriga and M. Sabatini, *A survey of isochronous centers*, Qual. Theory Dyn. Syst. 1 (1999), pp. 1–70.
[8] C. Chicone, *Ordinary Differential Equations with Applications*, Springer, New York, 2006.
[9] I. Dolgachev, *Lectures on Invariant Theory*, Cambridge University Press, Cambridge, 2003.
[10] J.J. Duistermaat and J.A.C. Kolk, *Lie Groups*, Springer-Verlag, Berlin, 2000.
[11] F. Harary and E.M. Palmer, *Graphical Enumeration*, Academic Press, New York, London, 1973.
[12] F. Harary, G. Prins, and W.T. Tutte, *The number of plane trees*, Indag. Math. 26 (1964), pp. 319–329.
[13] J. Harris, *Algebraic Geometry*, Springer-Verlag, Berlin, 1992.
[14] I.D. Iliev, *The number of limit cycles due to polynomial perturbations of the harmonic oscillator*, Math. Proc. Cambridge Philos. Soc. 127 (2) (1999), pp. 317–322.
[15] F. Klein, *On Riemann's Theory of Algebraic Functions and Their Integrals*, Dover, New York, 1963.
[16] M. Konstsevich and A. Zorich, *Connected components of the moduli spaces of Abelian differentials with prescribed singularities*, Invent. Math. 153 (2003), pp. 631–678.
[17] P. Mardesić, C. Rousseau, and B. Toni, *Linearization of isochronous centers*, J. Differ. Equ. 121 (1) (1995), pp. 67–108.
[18] J. Muciño-Raymundo, *Complex structures adapted to smooth vector fields*, Math. Ann. 322 (2002), pp. 229–265.
[19] J. Muciño-Raymundo and C. Valero-Valdéz, *Bifurcations of meromorphic vector fields on the Riemann sphere*, Ergod Theory Dynam. Syst. 15 (1995), pp. 1211–1222.
[20] J. Muciño-Raymundo and C. Valero-Valdéz, *Geometry and dynamics of the residue theorem*, Morfismos 5 (1) (2001), pp. 1–16.
[21] R. Otter, *The number of trees*, Ann. Math. 49 (1948), pp. 583–599.
[22] G. Pólya, *Kombinatorische anzahlbestimmungen für gruppen, graphen und chemische verbindungen*, Acta Math. 68 (1937), pp. 145–254.
[23] J. Reyn, *Phase Portraits of Planar Quadratic Systems*, Springer, New York, 2007.
[24] F. Rong, *A note on the topology classification of complex polynomial differential equations with only cetre singularities*, J. Difference Equ. Appl. 18 (11) (2012), pp. 1947–1949.
[25] M. Sabatini, *Characterizing isochronous centers by Lie brackets*, Diff. Equ. Dyn. Syst. 5 (1) (1997), pp. 91–99.
[26] K. Strebel, *Quadratic Differentials*, Ergebnisse der mathematik Undihrer Grenzgebiete 3, Folge Band 5, Springer-Verlag, Berlin, 1984.
[27] A. Yu, *Quadratic differentials and weighted graphs on compact surfaces*, Anal. Math. Phys. (Trends Math.) (2009), pp. 473–505.