Applications of Namba forcing to weak partition properties on trees

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Abstract

We answer several questions of Hrušák, Simon and Zindulka on weak partition relations on trees. In particular, we show that the Namba forcing on $\mathsf{add}(\mathcal{N})$ and $\mathsf{cof}(\mathcal{N})$ does not have the Sacks property. We also construct a model where there is a singular cardinal κ such that $\mathsf{cf}(\kappa)$ has the boundedness property but κ does not.

1 Introduction

In [6] Hrušák, Simon and Zindulka studied several partition relations on trees. Using a different notation, they introduced the following concepts: Given a cardinal κ, and $g : \omega \to \omega$, we say that κ is a *Zindulka cardinal* if for every coloring $\chi : \kappa^{<\omega} \to \omega$ there is a $T \in \mathbb{N}(\kappa)$ with stem $\emptyset$ such that T takes only finitely many colors on each level (where $\mathbb{N}(\kappa)$ denotes the Namba forcing on κ),[1] and κ is a *g-Zindulka cardinal* if for every coloring $\chi : \kappa^{<\omega} \to \omega$ there is a $T \in \mathbb{N}(\kappa)$ with stem $\emptyset$ such that $|\chi[T_n]| \leq g(n)$ for every $n \in \omega$ (i.e. T has at most $g(n)$ colors at the n-level). In [6] the following questions were asked:

Problem 1 (Hrušák, Simon, Zindulka) *Let $g : \omega \to \omega$ be an increasing function.*

1. *Is $\mathfrak{b}$ the first regular uncountable cardinal that is not Zindulka?*
2. *Can $\mathsf{cof}(\mathcal{N})$ be a g-Zindulka cardinal?*
3. *Can $\mathsf{add}(\mathcal{N})$ be a g-Zindulka cardinal?*

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[1]The definitions of undefined terms can be consulted in the next section.

4. *Can $\mathfrak{m}_{\sigma-linked}$ be a g-Zindulka cardinal?*

It turns out that κ is a Zindulka cardinal if and only the Namba forcing on κ does not add unbounded reals. In [4] the first question was answered positively. Here we will prove that $\mathbb{N}(\kappa)$ has the Sacks property if and only if κ is a g-Zindulka cardinal for some (any) increasing function g. We will prove that $\mathsf{cof}(\mathcal{N})$ and $\mathsf{add}(\mathcal{N})$ can not be g-Zindulka cardinals while $\mathfrak{m}_{\sigma-linked}$ may consistently be.

According to [6] a cardinal κ has the *Boundedness Property* if for every sequence $\mathcal{A} = \langle f_\alpha \mid \alpha \in \kappa \rangle$ where $f_\alpha : \omega \to \omega$, there is $g : \omega \to \omega$ such that the set $\{\alpha \mid f_\alpha < g\}$ has size κ. In [6] it was proved that if κ has the Boundedness property, then $\mathsf{cf}(\kappa)$ also has the Boundedness property. Answering a question from [6], we shall show that the converse consistently fails.

2 Preliminaries and notation

In this section we fix some notation and recall the relevant results from [4]. Let κ be a cardinal, a tree $T \subseteq \kappa^{<\omega}$ is called a κ-*Namba tree* (or just Namba tree if the cardinal κ is clear by context) if there is $s \in T$ (called the *stem of* T) such that every $t \in T$ is comparable with s; furthermore if $t \sqsubset s$ then t has just one immediate successor and if $s \sqsubseteq t$ then t has κ many immediate successors. By $\mathbb{N}(\kappa)$ we will denote the set of all κ-Namba trees ordered by inclusion; in this way, $\mathbb{N}(\omega)$ is the Laver forcing. A generic filter for $\mathbb{N}(\kappa)$ may be coded as a sequence which we will denote by $\mathfrak{n}_{gen} : \omega \to \kappa$. It is easy to see that $\mathbb{N}(\kappa)$ forces κ to have countable cofinality. Given S and T two κ-Namba trees, $S \leq_0 T$ will mean that $S \leq T$ and both S and T have the same stem. By $[T]$ we denote the set of branches of T and if $s \in T$ then we define T_s as the set of all $t \in T$ such that either $t \sqsubseteq s$ or $s \sqsubseteq t$ and $suc_T(s) = \{\alpha \in \kappa \mid s^\frown \alpha \in T\}$. By $B(T)$ we denote the set of nodes of T that extend the stem. By $stem(T)$ we denote the stem of T and $\mathbb{N}_0(\kappa)$ will denote the set of all κ-Namba trees with empty stem.

A key property of Namba forcing is the following:

Proposition 2 (Continous reading of names) *If (Y, d) is a complete metric space and $\dot{y}$ is an $\mathbb{N}(\kappa)$-name for an element of Y, then there is $S \in \mathbb{N}(\kappa)$ and a continous function $F : [S] \to Y$ such that $S \Vdash$ "$F(\dot{x}_{gen}) = \dot{y}$".*

Furthermore, it is often the case that names can be read with a *Lipschitz function.* Let T be a tree, given $F : T \to \omega$ define the function $\overline{F} : [T] \to \omega^\omega$ such that if $x \in \kappa^\omega$ and $n \in \omega$ then $\overline{F}(x) \restriction n = F(x)$. A function $H : [T] \to \omega^\omega$ is called *Lipschitz* if there is a function $F : T \to \omega$ such that $H = \overline{F}$. Clearly every Lipschitz function is continuous. If $G : \kappa^\omega \to \omega^\omega$ is a continuous function, define $G^* : \kappa^{<\omega} \to \omega^{<\omega}$ where $G^*(s) = \left(\bigcup \{t \mid G[\langle s \rangle] \subseteq \langle t \rangle\}\right) \restriction |s|$. If κ is a cardinal of uncountable cofinality, $T \in \mathbb{N}(\kappa)$ and $\dot{y}$ a $\mathbb{N}(\kappa)$-name such that $T \Vdash$ "$\dot{y} \in \omega^\omega$", then there is $S \leq_0 T$ such that:

1. If $s \in S$ then S_s decides $\dot{y} \restriction (|s| + 1)$.

2. There is $F : S \to \omega$ such that $S \Vdash$ "$\overline{F}(\mathfrak{n}_{gen}) = \dot{y}$".

The reader may consult [4] for a proof of the above fact, and the following:

Proposition 3 *Let κ be a cardinal $\mu < \mathsf{cf}(\kappa)$ and let $\{A_\alpha \mid \alpha \in \mu\}$ be a family of Borel sets of κ^ω such that $\kappa^\omega = \bigcup\limits_{\alpha<\mu} A_\alpha$. There is $T \in \mathbb{N}_0(\kappa)$ and $\alpha < \mu$ such that $[T] \subseteq A_\alpha$.*

Let $f, g \in \omega^\omega$, define $f \leq g$ if and only if $f(n) \leq g(n)$ for every $n \in \omega$ and $f \leq^* g$ if and only if $f(n) \leq g(n)$ holds for all $n \in \omega$ except finitely many. We say a family $\mathcal{B} \subseteq \omega^\omega$ is *unbounded* if $\mathcal{B}$ is unbounded with respect to $\leq^*$. A family $\mathcal{D} \subseteq \omega^\omega$ is a *dominating family* if for every $f \in \omega^\omega$, there is $g \in \mathcal{D}$ such that $f \leq^* g$. The *bounding number* $\mathfrak{b}$ is the size of the smallest unbounded family and the *dominating number* $\mathfrak{d}$ is the smallest size of a dominating family. Given $A, B \subseteq \omega$, by $A \subseteq^* B$ we denote that $A \setminus B$ is finite. A set $A \in [\omega]^\omega$ is a *pseudointersection* of a family $\mathcal{B}$ if $A \subseteq^* B$ for every $B \in \mathcal{B}$. The size of the continuum is denoted by $\mathfrak{c}$. The *pseudointersection number* $\mathfrak{p}$ is defined as the smallest size of a base of a filter on ω without a an infinite pseudointersection. If $\mathcal{I}$ is an ideal on a Polish space, by $\mathsf{add}(\mathcal{I})$ we denote the smallest size of a family $\mathcal{B} \subseteq \mathcal{I}$ such that $\bigcup \mathcal{N} \notin \mathcal{I}$. By $\mathsf{cof}(\mathcal{I})$ we denote the smallest size of a family $\mathcal{D} \subseteq \mathcal{I}$ such that for every $B \in \mathcal{I}$ there is $D \in \mathcal{D}$ such that $B \subseteq D$. Note that $\mathsf{add}(\mathcal{I}) \leq \mathsf{cof}(\mathcal{I})$. $\mathcal{N}$ will denote the σ-ideal of all Lebesgue null subsets of 2^ω. We will be mostly interested in $\mathsf{add}(\mathcal{N})$ and $\mathsf{cof}(\mathcal{N})$. Let $\mathbb{P}$ be a forcing notion. A set $\mathcal{B} \subseteq \mathbb{P}$ is called *linked* if every two elements of $\mathcal{B}$ are compatible. We say that $\mathbb{P}$ is *σ-linked* if there is a family $\{\mathcal{B}_n \mid n \in \omega\}$ of linked subsets of $\mathbb{P}$ such that $\mathbb{P} = \bigcup\limits_{n \in \omega} \mathcal{B}_n$. The cardinal invariant $\mathfrak{m}_{\sigma-linked}$ is the smallest κ such that there is a σ-linked forcing $\mathbb{P}$ and a family $\{D_\alpha \mid \alpha \in \kappa\}$ of dense subsets of $\mathbb{P}$ such that there is no filter $\mathcal{G} \subseteq \mathbb{P}$ for which $\mathcal{G} \cap D_\alpha \neq \emptyset$ for every $\alpha < \kappa$. The reader may consult [2] for the basic properties of the cardinal invariants used on this paper.

3 g-Zindulka cardinals

In this section we will prove that $\mathsf{cof}(\mathcal{N})$ and $\mathsf{add}(\mathcal{N})$ are not g-Zindulka cardinals while $\mathfrak{m}_{\sigma-linked}$ may consistently be (for g an increasing function).

Let κ be a cardinal and $g : \omega \to \omega$. We say that κ is a *weak Zindulka cardinal* if for every coloring $F : \kappa^{<\omega} \to \omega$ there is a $T \in \mathbb{N}(\kappa)$ such that $F[T_n]$ is finite for every $n \in \omega$, and, analogously, κ is a *weak g-Zindulka cardinal* if for every coloring $F : \kappa^{<\omega} \to \omega$ there is a $T \in \mathbb{N}(\kappa)$ such that $|F[T_n]| \leq g(n)$ is finite for every $n \in \omega$.

In [6] it was proved that $\mathfrak{b}$ is not a Zindulka cardinal and that κ is a Zindulka cardinal if and only if κ is a weak Zindulka cardinal. By the Lipschitz reading of names of κ-Namba forcing, it is easy to prove that $\mathbb{N}(\kappa)$ does not add unbounded reals if and only if κ is a weak Zindulka cardinal.

Proposition 4 ([4]) *1. κ is a Zindulka cardinal if and only if $\mathbb{N}(\kappa)$ does not add an unbounded real.*

2. $\mathfrak{b}$ is the first uncountable regular cardinal that is not a Zindulka cardinal.

Let $\mathcal{C} = \{g \in \omega^{\omega} \mid \lim(g(n)) = \infty \wedge \forall n (g(n) > 0)\}$. For any $g \in \mathcal{C}$ we define the *g-slaloms* as the set of all $S : \omega \to [\omega]^{<\omega}$ such that $|S(n)| \leq g(n)$ for every $n \in \omega$. Denote by $\mathcal{SL}_g$ the set of all g-slaloms. If $f \in \omega^{\omega}$ and $S \in \mathcal{SL}_g$ then $f \sqsubseteq^* S$ means that $f(n) \in S(n)$ holds for almost every $n \in \omega$. Given $F : \kappa^{<\omega} \to \omega$, $S : \omega \to [\omega]^{<\omega}$ and $T \in \mathbb{N}(\kappa)$ we will say that S *captures* (F, T) if $F[T_n] \subseteq S(n)$ for every $n \in \omega$. In this way, κ is g-Zindulka if and only if for every $F : \kappa^{<\omega} \to \omega$ there is a $T \in \mathbb{N}(\kappa)$ with empty stem and $S \in \mathcal{SL}_g$ such that S captures (F, T). In a similar way, we say S *almost captures* (F, T) if for every $x \in [T]$ it is the case that $\overline{F}(x) \sqsubseteq^* S$.

Lemma 5 *Let κ be a cardinal of uncountable cofinality, $g \in \mathcal{C}$, $F : \kappa^{<\omega} \to \omega$ and $T \in \mathbb{N}(\kappa)$. Then the following are equivalent:*

1. *There is $S \in \mathcal{SL}_g$ and $T' \leq_0 T$ such that S captures (F, T').*
2. *There is $S \in \mathcal{SL}_g$ and $T' \leq_0 T$ such that S almost captures (F, T').*

Proof. Clearly 1 implies 2 , we will show that 2 implies 1, let S and T' as in 2. Given $x \in [T']$ define $a_x = \{n \mid F(x \upharpoonright n) \notin S(n)\}$ and we also define $b_x = \{(n, F(x \upharpoonright n)) \mid n \in a_x\}$. Note that both a_x and b_x are finite sets. Given $a \in [\omega]^{<\omega}$ and $b \in [\omega \times \omega]^{<\omega}$ let $B(a, b) = \{x \in [T] \mid a_x = a \wedge b_x = b\}$. Clearly each $B(a, b)$ is a Borel set and $[T'] = \bigcup\limits_{a,b} B(a, b)$ and since every $B(a, b)$ is Borel and κ has uncountable cofinality, there are $T'' \leq_0 T'$ and a, b such that $[T''] \subseteq B(a, b)$. Let $a = \{n_i \mid i < l\}$ and $b = \{(n_i, m_i) \mid i \in l\}$ define $S' : \omega \to [\omega]^{<\omega}$ such that $S'(k) = S(k)$ if $k \notin a$ and $S'(n_i) = \{m_i\}$ for every $i < l$. Then S' captures (F, T''). ■

In this way, if κ has uncountable cofinality, then κ is a g-Zindulka cardinal if and only if for every $F : \kappa^{<\omega} \to \omega$ there is a $T \in \mathbb{N}(\kappa)$ with empty stem and $S \in \mathcal{SL}_g$ such that S almost captures (F, T).

Next we recall the notation concerning Borel Tukey order on cardinal invariants.

We say $(A, B, \to)$ is a *Borel invariant* if A, B are Borel subsets of some polish space, and $\to \subseteq A \times B$ is also Borel such that for every $a \in A$ there is a $b \in B$ such that $a \to b$. The *evaluation of* $(A, B, \to)$ (denoted by $\|A, B, \to\|$) is defined as the minimum size a family $D \subseteq B$ such that for every $a \in A$ there is a $d \in D$ such that $a \to d$.

If $(A^-, A^+, \to_A)$ and $(B^-, B^+, \to_B)$ are two Borel invariants then $(A^-, A^+, \to_A) \leq_{\mathsf{BT}} (B^-, B^+, \to_B)$ if there are Borel functions $F^- : A^- \to B^-$ and $F^+ : B^+ \to A^+$ such that for every $a \in A^-$ and $b \in B^+$, if $F^-(a) \to_B b$ then $a \to_A F^+(b)$.

We write $(A^-, A^+, \to_A) \simeq_{\mathsf{BT}} (B^-, B^+, \to_B)$ if both $(A^-, A^+, \to_A) \leq_{\mathsf{BT}} (B^-, B^+, \to_B)$ and $(B^-, B^+, \to_B) \leq_{\mathsf{BT}} (A^-, A^+, \to_A)$.

It is easy to see that if $(A^-, A^+, \to_A) \simeq_{\mathsf{BT}} (B^-, B^+, \to_B)$ then $(A^+, A^-, {}_A\not\leftarrow) \simeq_{\mathsf{BT}} (B^+, B^-, {}_B\not\leftarrow)$.

Given a Borel invariant $(A, B, \to)$ and a forcing notion $\mathbb{P}$, we say that $\mathbb{P}$ *destroys* $(A, B, \to)$ if there is a $\mathbb{P}$-name $\dot{r}$ such that $\mathbb{P} \Vdash$ "$\dot{r} \in A$" and if $b \in B$ (with $b \in V$) then $\mathbb{P} \Vdash$ "$\dot{r} \nrightarrow b$".

Given two Borel invariants $(A^-, A^+, \to_A) \leq_{\mathsf{BT}} (B^-, B^+, \to_B)$ and a forcing notion $\mathbb{P}$. If $\mathbb{P}$ destroys $(A^-, A^+, \to_A)$ then $\mathbb{P}$ destroys $(B^-, B^+, \to_B)$.

The following proposition is well known, but we include a proof for the convenience of the reader:

Proposition 6 *Let $f, g \in \mathcal{C}$ then $(\omega^\omega, \mathcal{SL}_f, \sqsubseteq^*) \simeq_{BT} (\omega^\omega, \mathcal{SL}_g, \sqsubseteq^*)$. Moreover, there are continuous $R : \omega^\omega \to \omega^\omega$, $H : \mathcal{SL}_f \to \mathcal{SL}_g$, and $k \in \omega$ such that:*

1. *$\forall x \in \omega^\omega$ $\forall S \in \mathcal{SL}_f$ if $R(x) \sqsubseteq^* S$ then $x \sqsubseteq^* H(S)$.*
2. *$\forall x \in \omega^\omega$ and $\forall S \in \mathcal{SL}_f$ if $R(x) \sqsubseteq S$ then $(\forall m > k)(x(m) \in H(S)(m))$.*

Proof. We define an interval partition $\mathcal{P} = \{P_n \mid n \in \omega\}$ such that for every $n, m \in \omega$ if $P_n \subseteq m$ then $f(n) \leq g(m)$. Let $\{t_n \mid n \in \omega\}$ be an enumeration of all functions $p : s \to \omega$ where $s \in [\omega]^{<\omega}$ and define $R : \omega^\omega \to \omega^\omega$ such that if $m \in \omega$ and $x \in \omega^\omega$ then $x \upharpoonright P_{m+1} = t_{R(x)(m)}$. Now define $H : \mathcal{SL}_f \to \mathcal{SL}_g$ such that if $S \in \mathcal{SL}_f$ then the following holds:

If $m \in P_0$, let $H(S)(m) = \{0\}$.

If $m \in P_{i+1}$, let $S(i) = \{l_1, ..., l_{f(n)}\}$ and $H(S)(m) = \{t_{l_1}(m), ..., t_{l_{f(n)}}(m)\}$.

It is easy to see that both R and H are continuous. Let $k = \max(P_0)$, we will prove 3 and 2 will follow by the proof of 3. Let $x \in \omega^\omega$, $S \in \mathcal{SL}_f$ and $m > k$ such that $R(x) \sqsubseteq S$. Let $i \in \omega$ such that $m \in P_{i+1}$, since $R(x)(i) \in S(i)$ then $x(m) = (x \upharpoonright P_{i+1})(m) = t_{R(x)(i)}(m) \in H(S)(m)$. ■

In particular, a forcing notion $\mathbb{P}$ has the Sacks property if and only if it does not destroy $(\omega^\omega, \mathcal{SL}_f, \sqsubseteq^*)$ for some (every) $f \in \mathcal{C}$. It is easy to see that if κ has uncountable cofinality, then $\mathbb{N}(\kappa)$ has the Sacks property if and only if κ is a weak f-Zindulka cardinal for some (every) $f \in \mathcal{C}$.

Proposition 7 *Let κ be a cardinal and $f, g \in \mathcal{C}$. Then κ is a f-Zindulka cardinal if and only if κ is a g-Zindulka cardinal.*

Proof. It is easy to see that if κ has countable cofinality then κ is not a Zindulka cardinal, so, in particular, it is neither f-Zindulka nor g-Zindulka. Now assume that κ is an f-Zindulka cardinal of uncountable cofinality. Fix $R : \omega^\omega \to \omega^\omega$, $H : \mathcal{SL}_f \to \mathcal{SL}_g$ and k as in the previous proposition. Let $F : \kappa^{<\omega} \to \omega$, we can then find $T \in \mathbb{N}_0(\kappa)$ and $G : B(T) \to \omega$ such that $T \Vdash$ "$\overline{G}(\mathfrak{n}_{gen}) = R\overline{F}(\mathfrak{n}_{gen})$". Since κ is f-Zindulka we can then find $S \in \mathcal{SL}_f$ and $T' \leq_0 T$ such S captures (G, T'). We claim that $H(S)$ almost captures (F, T').

Let $t \in T'$ such that $|t| = n > k$. We will show that $F(t) \in H(S)(n)$. Since S captures (G, T') we know that $G(t) \in S(n)$. Let $\mathfrak{n} : \omega \to \kappa$ be a generic branch through T' extending t. In this way, $R\overline{F}(\mathfrak{n})(n) = \overline{G}(\mathfrak{n}_{gen})(n) = G(t)$ so $R\overline{F}(\mathfrak{n})(n) \in S(n)$. In this way, $F(t) = \overline{F}(\mathfrak{n})(n) \in H(S)(n)$. ■

Let $f, g \in \mathcal{C}$ and $S_1 \in \mathcal{SL}_f$, $S_2 \in \mathcal{SL}_g$. Define $S_1 \leq S_2$ if $S_1(n) \subseteq S_2(n)$ for every $n \in \omega$ and $S_1 \leq^* S_2$ if $S_1(n) \subseteq S_2(n)$ holds for almost all $n \in \omega$. We now recursively build functions $\{f_n \mid n \in \omega\} \subseteq \omega^\omega$ as follows: $f_0(m) = m+1$ for every $m \in \omega$ and $f_{n+1}(m) = (m+1)^2 f_n(m)$. Finally, let $f_\omega : \omega \to \omega$ such that $f_n \leq^* f_\omega$ for every $n \in \omega$.

Lemma 8 *Let κ be such that $\mathbb{N}(\kappa)$ has the Sacks property and let $n \in \omega$. If $\{S_\alpha \mid \alpha \in \kappa\} \subseteq \mathcal{SL}_{f_n}$ then there is $A \in [\kappa]^\kappa$ and $S \in \mathcal{SL}_{f_{n+1}}$ such that $S_\alpha \leq S$ for every $\alpha \in A$.*

Proof. Let $\mathfrak{n} : \omega \to \kappa$ be a generic sequence for $\mathbb{N}(\kappa)$. In $V[\mathfrak{n}]$ we define $Z : \omega \to [\omega]^{<\omega}$ where $Z(m) = \bigcup\limits_{i \leq m} S_{\mathfrak{n}(i)}(m)$. Note that $|Z(m)| \leq (m+1) f_n(m)$. Let $[\omega]^{<\omega} = \{t_m \mid m \in \omega\}$ and (still in $V[\mathfrak{n}]$) we define $h : \omega \to \omega$ such that $Z(m) = t_{h(m)}$ for every $m \in \omega$. Since $\mathbb{N}(\kappa)$ has the Sacks property then there is $T \in \mathbb{N}(\kappa)$ and $W : \omega \to [\omega]^{<\omega}$ such that if $m \in \omega$ then $|W(m)| \leq m+1$ and $T \Vdash$ "$h(m) \in W(m)$". Without losing generality, we may assume that if $i \in W(m)$ then $|t_i| \leq (m+1) f_n(m)$, and let $S : \omega \to [\omega]^{<\omega}$ be defined by $S(m) = \bigcup\limits_{i \in W(m)} t_i$. Note that $|S(m)| \leq (m+1)^2 f_n(m)$ so $S \in \mathcal{SL}_{f_{n+1}}$, and $T \Vdash$ "$Z \leq S$". Let s be the stem of T and $A = suc_T(s)$. We claim that if $\alpha \in A$ and $m > |s|+1$ then $S_\alpha(m) \subseteq S(m)$. Let $T' \leq T$ such that $s^\frown\alpha \subseteq st(T')$, in this way, $T' \Vdash$ "$S_\alpha(m) \subseteq Z(m)$" and then $T' \Vdash$ "$S_\alpha(m) \subseteq S(m)$" so $S_\alpha(m) \subseteq S(m)$. Since κ has uncountable cofinality, it is then easy to find $A' \in [A]^\kappa$ and S' a finite modification of S such that $S_\alpha \leq S'$ for every $\alpha \in A'$. ■

We have the following combinatorial characterization of the Sacks property for κ-Namba forcing:

Proposition 9 *Let κ be a cardinal. Then $\mathbb{N}(\kappa)$ has the Sacks property if and only if κ is a g-Zindulka cardinal for some (every) $g \in \mathcal{C}$.*

Proof. If κ has countable cofinality then κ is not g-Zindulka for some (every) $g \in \mathcal{C}$ and since Laver forcing $\mathbb{N}(\omega)$ does not have the Sacks property, neither does $\mathbb{N}(\kappa)$. We now assume that κ has uncountable cofinality. Since every f-Zindulka cardinal is a weak f-Zindulka cardinal, it follows that if κ is f-Zindulka then $\mathbb{N}(\kappa)$ has the Sacks property.

Let κ be a cardinal of uncountable cofinality such that $\mathbb{N}(\kappa)$ has the Sacks property. We will prove that κ is an f_ω-Zindulka cardinal. Let $F : \kappa^{<\omega} \to \omega$, define a rank function $rk : \kappa^{<\omega} \to OR \cup \{\infty\}$ as follows:

1. $rk(s) = 0$ if there are $n \in \omega$, $T \in \mathbb{N}(\kappa)$ with stem s and $S \in \mathcal{SL}_{f_n}$ that captures (T, F).
2. $rk(s) \leq \alpha$ if $|\{\xi \mid rk(s^\frown\xi) < \alpha\}| = \kappa$.

3. $rk(s) = \alpha$ if $rk(s) \leq \alpha$ and there is no $\beta < \alpha$ such that $rk(s) \leq \beta$.

4. $rk(s) = \infty$ if there is no α such that $rk(s) \leq \alpha$.

We will first prove that $rk(s) \neq \infty$ for every $s \in \kappa^{<\omega}$. Assume this is not the case, we can then recursively build $T \in \mathbb{N}(\kappa)$ such that $stem(T) = s$ and $rk(t) = \infty$ for every $t \in B(t)$. We then arrive at a contradiction since κ was a weak f_0-Zindulka cardinal.

We now claim that $rk(\emptyset) = 0$. Assume this is not the case, then we can find $s \in \kappa^{<\omega}$ such that $rk(s) = 1$ and let $A = \{\alpha \mid rk(s^\frown\alpha) = 0\}$. Note that $|A| = \kappa$ since $rk(s) = 1$. For every $\alpha \in A$ choose $n_\alpha \in \omega, T_\alpha \in \mathbb{N}(\kappa)$ with stem $s^\frown\alpha$ and $S_\alpha \in \mathcal{SL}_{f_{n_\alpha}}$ such that S_α captures (T_α, F). We can then find $n \in \omega$ such that $B = \{\alpha \in A \mid n_\alpha = n\}$ has size κ. By the previous lemma, there are $C \in [B]^\kappa$ and $S \in \mathcal{SL}_{f_{n+1}}$ such that $S_\alpha \leq S$ for every $\alpha \in C$. If $T = \{s \upharpoonright i \mid i \leq |s|\} \cup \bigcup\limits_{\alpha \in C} T_\alpha$ then S captures (F, T), which contradicts $rk(s) = 1$.

Hence, there are $n \in \omega$, $T \in \mathbb{N}(\kappa)$ and $S \in \mathcal{SL}_{f_n}$ such that $stem(T) = \emptyset$ and S captures (T, F). Since $f_n \leq^* f_\omega$ we can find $S_1 \in \mathcal{SL}_{f_\omega}$ such that S_1 almost captures (T, F). $\blacksquare$

Fix a family $C = \{C^n_m \mid n, m \in \omega\}$ of clopen subsets of 2^ω of Lebesgue measure at most $\frac{1}{2^n}$ such that for every clopen $D \subseteq 2^\omega$ of measure at most $\frac{1}{2^n}$ then there is $m \in \omega$ such that $D = C^n_m$. Given $f : \omega \to \omega$ let $N(f) = \bigcap\limits_{n \in \omega} \bigcup\limits_{i > n} C^i_{f(i)}$ which clearly is a null set. Then $\{N(f) \mid f \in \omega^\omega\}$ is a cofinal family of null sets (see [1] lemma 3.2). Given $f, g \in \omega^\omega$ define $f \leq_{\mathcal{N}} g$ if $N(f) \subseteq N(g)$. Recall the following important result:

Proposition 10 (see [1]) $(\omega^\omega, \mathcal{SL}, \sqsubseteq^*) \simeq_{\mathsf{BT}} (\omega^\omega, \omega^\omega, \leq_{\mathcal{N}})$.

I.e. a forcing $\mathbb{P}$ has the Sacks property if and only if every null set in an extension by $\mathbb{P}$ is contained in a ground model null set.

Theorem 11 *Neither $\mathbb{N}(\mathsf{add}(\mathcal{N}))$ nor $\mathbb{N}(\mathsf{cof}(\mathcal{N}))$ have the Sacks property.*

Proof. We first show that $\mathbb{N}(\mathsf{cof}(\mathcal{N}))$ does not have the Sacks property. Let $D = \{N_\alpha \mid \alpha \in \mathsf{cof}(\mathcal{N})\}$ be a cofinal family of null sets. Given $\beta < \mathsf{cof}(\mathcal{N})$ define $D_\beta = \{x_\alpha \mid \alpha \leq \beta\}$. Since D_β is not cofinal, there is $M_\beta \in \mathcal{N}$ such that $M_\beta \nsubseteq N_\alpha$ for every $\alpha \leq \beta$.

Let $\mathfrak{n} : \omega \to \mathsf{cof}(\mathcal{N})$ be a generic sequence for $\mathbb{N}(\mathsf{cof}(\mathcal{N}))$. In $V[\mathfrak{n}]$ let $M = \bigcup\limits_{n \in \omega} M_{\mathfrak{n}(n)}$ which is clearly a null set. We claim that M is not contained in any element of $\mathcal{N} \cap V$, it is enough to prove that if $\alpha \in \mathsf{cof}(\mathcal{N})$ then $M \nsubseteq N_\alpha$. By genericity, there is $m \in \omega$ such that $\alpha < \mathfrak{n}(m)$, since $M_{\mathfrak{n}(m)} \subseteq M$ while $M_\beta \nsubseteq N_\alpha$ we conclude that $M \nsubseteq N_\alpha$.

Now we prove that $\mathbb{N}(\mathsf{add}(\mathcal{N}))$ does not have Sacks property. This is a dual argument to the above. Let $B = \{N_\alpha \mid \alpha \in \mathsf{add}(\mathcal{N})\} \subseteq \mathcal{N}$ such that $\bigcup B \notin \mathcal{N}$. Given $\beta < \mathsf{add}(\mathcal{N})$ define $B_\beta = \{x_\alpha \mid \alpha \leq \beta\}$. Since $\beta < \mathsf{add}(\mathcal{N})$, then $M_\beta = \bigcup\limits_{\alpha \leq \beta} N_\alpha$ is a null set. Let $\mathfrak{n} : \omega \to \mathsf{add}(\mathcal{N})$ be a generic sequence for

$\mathbb{N}(\mathsf{add}(\mathcal{N}))$. In $V[\mathfrak{n}]$ define the null set $M = \bigcup_{n\in\omega} M_{\mathfrak{n}(n)}$, we claim that M is not contained in any ground model null set. Let $A \in \mathcal{N} \cap V$ and since $\bigcup B \notin \mathcal{N}$ then there is $\alpha \in \mathsf{add}(\mathcal{N})$ such that $N_\alpha \nsubseteq A$. By genericity, there is $m \in \omega$ such that $\alpha < \mathfrak{n}(m)$. Note that $M_{\mathfrak{n}(m)} \subseteq M$ and on the other hand, $M_{\mathfrak{n}(m)} \nsubseteq A$ because $N_\alpha \nsubseteq A$ and $N_\alpha \subseteq M_{\mathfrak{n}(m)}$. ■

In [4] it was proved that if κ is a regular cardinal such that $\kappa < \mathsf{add}(\mathcal{N})$ then $\mathbb{N}(\kappa)$ has the Sacks property. We can then conclude the following:

Corollary 12 *The cardinal invariant* $\mathsf{add}(\mathcal{N})$ *is the least regular cardinal* κ *that is not a* g*-Zindulka cardinal for every (some)* $g \in \mathcal{C}$.

Since the inequality $\mathfrak{m}_{\sigma-linked} < \mathsf{add}(\mathcal{N})$ is consistent, we conclude that $\mathfrak{m}_{\sigma-linked}$ may consistently be a g-Zindulka cardinal.

4 The Boundedness property

We call a subtree $T \subseteq \kappa^{<\omega}$ a *broom tree* if there is $s \in T$ such that s has κ immediate successors and every other node has just one successor. The statement $\kappa \rightsquigarrow_{\mathsf{b}} (\kappa)^{<\omega}_{\omega}$ means that for every coloring $\chi : \kappa^{<\omega} \to \omega$ there is a finitely colored broom tree. On the other hand, $\kappa \rightsquigarrow_{\mathsf{W}} (\kappa)^{<\omega}_{\omega}$ means that for any coloring $\chi : \kappa^{<\omega} \to \omega$ there is a finitely colored tree $T \subseteq \kappa^{<\omega}$ of size κ. Obviously $\kappa \rightsquigarrow_{\mathsf{b}} (\kappa)^{<\omega}_{\omega}$ implies $\kappa \rightsquigarrow_{\mathsf{W}} (\kappa)^{<\omega}_{\omega}$. Furthermore, if κ has uncountable cofinality, then $\kappa \rightsquigarrow_{\mathsf{b}} (\kappa)^{<\omega}_{\omega}$ if and only if $\kappa \rightsquigarrow_{\mathsf{W}} (\kappa)^{<\omega}_{\omega}$. However, this relations are not equivalent as the next result shows:

Proposition 13 *If* $\kappa = \mathfrak{c}^{+\omega}$ [2] *then* $\kappa \rightsquigarrow_{\mathsf{W}} (\kappa)^{<\omega}_{\omega}$ *but* $\kappa \not\rightsquigarrow_{\mathsf{b}} (\kappa)^{<\omega}_{\omega}$.

Proof. We will first show that $\kappa \rightsquigarrow_{\mathsf{W}} (\kappa)^{<\omega}_{\omega}$. Let $\chi : \kappa^{<\omega} \to \omega$ and let $S = \kappa^{<\omega}$. Since $\mathfrak{c}^+$ is a Zindulka cardinal (see [4]) then we may find $T(0) \subseteq S_{\langle 0\rangle}$ with $T(0) \in \mathbb{N}(\mathfrak{c}^+)$ that is finitely colored. In the same way, we may find $T(1) \subseteq S_{\langle 1,0\rangle}$ finitely colored with $T(1) \in \mathbb{N}(\mathfrak{c}^{++})$ and then we find $T(2) \subseteq S_{\langle 1,1,0\rangle}$ finitely colored with $T(2) \in \mathbb{N}(\mathfrak{c}^{+++})$ ... After ω steps, we define $T = \bigcup_{n\in\omega} T(n)$ and it is clear that it is finitely colored and of size κ.

Now we will show that $\kappa \not\rightsquigarrow_{\mathsf{b}} (\kappa)^{<\omega}_{\omega}$, actually we will prove that if μ has countable cofinality then $\mu \not\rightsquigarrow_{\mathsf{b}} (\mu)^{<\omega}_{\omega}$. Let $\chi : \mu^{<\omega} \to \omega$ such that for every $s \in \mu^{<\omega}$ and every $n \in \omega$, the set $\{\alpha \mid \chi(s^\frown\alpha) = n\}$ is bounded, then clearly there can not be a finitely colored broom tree. ■

Following [6] we say that κ has the *Boundedness property* if for every sequence $\mathcal{A} = \langle f_\alpha \mid \alpha \in \kappa\rangle$ where $f_\alpha : \omega \to \omega$, there is $g : \omega \to \omega$ such that the set $\{\alpha \mid f_\alpha < g\}$ has size κ. $BP(\kappa)$ will abbreviate that κ has the boundedness property.

[2] i.e. $\kappa = \bigcup\{\mathfrak{c}, \mathfrak{c}^+, \mathfrak{c}^{++}, ...\}$

As was pointed before, $\aleph_\omega \not\to_{\mathsf{b}} (\aleph_\omega)^{<\omega}_\omega$. We shall show that the analogous statement for the weak arrow is independent of ZFC. Given S a set of ordinals, we will denote by $V[C_S]$ as the extension obtained by adding S Cohen reals.

Proposition 14 *The statement* $\aleph_\omega \rightsquigarrow_{\mathsf{W}} (\aleph_\omega)^{<\omega}_\omega$ *is independent from* ZFC.

Proof. If $\mathfrak{c} < \aleph_\omega$ then $\aleph_\omega \rightsquigarrow_{\mathsf{W}} (\aleph_\omega)^{<\omega}_\omega$ by the previous result, so we just need to build a model where the relation does not hold. Assume $V \models \mathsf{GCH}$ and consider $V[C_{\aleph_\omega}]$ the forcing extension obtained by adding $\aleph_\omega$ Cohen reals. In $V[C_{\aleph_\omega}]$ define $\chi : \aleph_\omega^{<\omega} \to \omega$ where $\chi(\langle \alpha_0, ..., \alpha_n \rangle) = \sum\limits_{i,j<n} c_{\alpha_i}(j)$ and assume there is T a finitely colored tree of size $\aleph_\omega$ and $g : \omega \to \omega$ such that $T \leq g$. Clearly, there must be $s \in T$ that has (at least) ω_1 successors. Let $n = |s|$ and $\langle \beta_\xi \mid \xi \in \omega_1 \rangle \subseteq suc_T(s)$. Define $h : \omega \to \omega$ given by $h(m) = g(n+m)$ then it follows that $c_{\beta_\xi} \leq h$ for every $\xi \in \omega_1$ which is clearly impossible since h must had appeared in an intermediate extension where only ω Cohen reals has been added. ■

We will need the following result of Hirschorn (see [5]).

Proposition 15 ([5]) *If c is a Cohen real over V, then* $\mathfrak{p}^V = \mathfrak{p}^{V[c]}$.

We may define a natural two cardinal variation of the boundedness property, given κ, λ the statement $BP(\kappa, \lambda)$ will mean that for every sequence $\langle f_\alpha \mid \alpha \in \kappa \rangle$ of reals, there is $g : \omega \to \omega$ such that the set $\{\alpha < \kappa \mid f_\alpha < g\}$ has size at least λ. Obviously, $BP(\kappa)$ is the same as $BP(\kappa, \kappa)$. We know that both $BP(\mathfrak{b})$ and $BP(\mathfrak{d})$ are false, however we have the following result:

Proposition 16 *The statement* $BP(\mathfrak{d}, \mathfrak{b})$ *is independent from* ZFC.

Proof. To get a model where $BP(\mathfrak{d}, \mathfrak{b})$ fails, assume $V \models \mathsf{GCH}$ and add ω_2 Cohen reals, then it is clear that $\mathfrak{b} = \omega_1, \mathfrak{d} = \omega_2$ and $BP(\omega_2, \omega_1)$ fails because of the Cohen reals. To build a model where $BP(\mathfrak{d}, \mathfrak{b})$ holds, start with a model of $\mathfrak{p} = \mathfrak{c} = \omega_2$ and add ω_1 Cohen reals, clearly in the extension $\mathfrak{b} = \omega_1$.

We now show that in $V[C_{\omega_1}]$ we get $\mathfrak{d} = \omega_2$. Assume this is not the case, so there must be a dominating family $\mathcal{F} = \{f_\alpha \mid \alpha \in \omega_1\} \in V[C_{\omega_1}]$. For every $p \in \mathbb{C}_{\omega_1}$ and $\alpha \in \omega_1$ define $f^p_\alpha : \omega \to \omega$ given by $f^p_\alpha(n) = \min\{m \mid \exists r \leq p(r \Vdash$ "$\dot{f}_\alpha(n) = m$")$\}$ and note that $\{f^p_\alpha \mid \alpha \in \omega_1 \wedge p \in \mathbb{C}_{\omega_1}\}$ belongs to the ground model. This is a family of size ω_1, so there is $g \in V$ that is not dominated by any f^p_α. However, since $\mathcal{F}$ is dominating in the extension, there must be $\alpha \in \omega_1$ and $p \in \mathbb{C}_{\omega_1}$ such that $p \Vdash$ "$g < f_\alpha$" which would then imply $g < f^p_\alpha$ which is a contradiction.

It only remains to prove $BP(\omega_2, \omega_1)$ so (in $V[C_{\omega_1}]$) take a sequence $A = \langle f_\alpha \mid \alpha \in \omega_2 \rangle$ and since every real appears in an intermediate extension, then there is a countable α such that $A \cap V[C_\alpha]$ has size ω_1. Note that $W = V[C_\alpha]$ is equivalent to a single Cohen extension and since $\mathfrak{p}^W = \omega_2$, then ω_1 has the boundedness property in W, so we may find a function that dominates uncountable many elements of $A \cap W$. ■

In [6] it was proved that if $\kappa \rightsquigarrow_{\mathsf{b}} (\kappa)^{<\omega}_{\omega}$ then $\mathsf{cof}(\kappa) \rightsquigarrow_{\mathsf{b}} (\mathsf{cof}(\kappa)^{<\omega}_{\omega})$ and it was asked if the converse is also true. We will now answer this question negatively.

Given partial orders $\mathbb{P}$ and $\mathbb{Q}$, we say $\mathbb{P}$ is a *regular (or complete) suborder of* $\mathbb{Q}$ (which we denote by $\mathbb{P} \leq_r \mathbb{Q}$) if $\mathbb{P} \subseteq \mathbb{Q}$, the order and incomparability relation of $\mathbb{P}$ are the order and incomparability relations of $\mathbb{Q}$ restricted to $\mathbb{P}$ and every maximal antichain (dense) of $\mathbb{P}$ is also a maximal antichain (predense) of $\mathbb{Q}$. This is equivalent that for every $q \in \mathbb{Q}$ there is $p \in \mathbb{P}$ such that if $p' \leq p$ then $p' \parallel q$, such p (which in general is not unique) is called a *reduction of* q. If $\mathbb{P} \leq_r \mathbb{Q}$ and $G \subseteq \mathbb{Q}$ is generic, then $G \cap \mathbb{P}$ is generic for $\mathbb{P}$. For more details the reader may consult [7].

The key for our result is the next lemma (which we took from [3] but we proved it here for the sake of completeness).

Lemma 17 ([3]) *Assume $V \subseteq W$, $\mathbb{P} \in V$ and $\mathbb{Q} \in W$. Moreover (in W) $\mathbb{P} \leq_r \mathbb{Q}$ and there is $c \in W$ which is unbounded for V. Let $G_{\mathbb{Q}} \subseteq \mathbb{Q}$ be a generic $(W, \mathbb{Q})$-generic filter and let $G_{\mathbb{P}} = G_{\mathbb{Q}} \cap \mathbb{P}$. Then $V[G_{\mathbb{P}}] \subseteq W[G_{\mathbb{Q}}]$ and c is unbounded for $V[G_{\mathbb{P}}]$.*

Proof. Assume this is not the case, so there is $\dot{f} \in V$ and $q \in \mathbb{Q}$ such that $q \Vdash$ "$c < \dot{f}$". Let $p \in \mathbb{P}$ be a reduction of q. In V, define $h : \omega \to \omega$ where $h(n) = min\{m \mid \exists r \leq p(r \Vdash \text{“}\dot{f}(n) = m\text{”})\}$. Since $h \in V$ then there is $n \in \omega$ such that $h(n) < c(n)$. Find $r \leq p$ such that $r \Vdash$ "$\dot{f}(n) = h(n)$" and since p is a reduction of q there is $\overline{r} \in \mathbb{Q}$ such that $\overline{r} \leq r, q$. Note that $\overline{r}$ forces $\dot{f}(n) < c(n)$ and $\dot{f}(n) > c(n)$ which is a contradiction. ■

Given $\mathcal{F} = \langle f_\alpha \mid \alpha \in \omega_1 \rangle$ define $\mathbb{H}(\mathcal{F})$ (the *Hechler forcing restricted to* $\mathcal{F}$) as the set of all pairs of the form $(s, \mathcal{G})$ where $s \in \omega^{<\omega}$ and $\mathcal{G} \in [\omega_1]^{<\omega}$. If $(s_1, \mathcal{G}_1), (s_2, \mathcal{G}_2) \in \mathbb{H}(\mathcal{F})$ then define $(s_1, \mathcal{G}_1) \leq (s_2, \mathcal{G}_2)$ is $s_1 \subseteq s_2$, $\mathcal{G}_2 \subseteq \mathcal{G}_1$ and if $i \in dom(s_1) \setminus dom(s_2)$ and $\alpha \in \mathcal{G}_2$ then $s_1(i) > f_\alpha(i)$. Now we are ready for the announced consistency result:

Proposition 18 *Assume* GCH *holds in V. There is $\mathbb{P}$ such that if $G \subseteq \mathbb{P}$ is generic, then in $V[G]$ the following hold,*

1. *$\omega_1 < \mathfrak{b}$ so ω_1 has the boundedness property,*
2. *$\mathfrak{d} = \aleph_{\omega_1+1}$,*
3. *$\aleph_{\omega_1}$ does not have the boundedness property.*

Proof. Let $\mathbb{P} = \mathbb{C}_{\aleph_{\omega_1+1}} * \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha \in \aleph_{\omega_1+1} \rangle$ where $\mathbb{P}_\alpha \Vdash$ "$\exists \dot{\mathcal{F}}_\alpha \in [\omega^\omega]^{\omega_1} (\dot{\mathbb{Q}}_\alpha = \mathbb{H}(\dot{\mathcal{F}}_\alpha))$" and we iterate with finite support. Moreover, (with a suitable book-keeping device) we arrange that every sequence of reals of length ω_1 in the final model is used at some successor step. It is clear that $\omega_1 < \mathfrak{b}$ and when we prove that $\aleph_{\omega_1}$ does not have the boundedness property, it will follow that $\aleph_{\omega_1} \leq \mathfrak{d}$

but since $\mathfrak{b} \leq \mathsf{cof}(\mathfrak{d})$ then we may conclude that $\aleph_{\omega_1} < \mathfrak{d}$. We will now prove that $\aleph_{\omega_1}$ does not have the boundedness property.

Let $\langle c_\alpha \mid \alpha \in \aleph_{\omega_1} \rangle$ be the first $\aleph_{\omega_1}$ Cohen reals added by $\mathbb{P}$, we will show that if $g : \omega \to \omega \in V[G]$ then the set $\{\alpha \mid c_\alpha \leq^* g\}$ has size less than $\aleph_{\omega_1}$. Let $\dot{g}$ be a name for g and find M and elementary submodel of $H(\theta)$ (for some big enough θ) such that ${}^{\omega_1}M \subseteq M$, $\mathbb{P}, \dot{g} \in M$ and let $S = \aleph_{\omega_1} \cap M$. Obviously $V[C_S] \subseteq V[C_{\aleph_{\omega_1+1}}]$ and if $\alpha \notin S$ then c_α is unbounded for $V[C_S]$. Now we define another finite support iteration $\overline{\mathbb{P}} = \mathbb{C}_S * \langle \overline{\mathbb{P}}_\alpha, \overline{\dot{\mathbb{Q}}}_\alpha \mid \alpha \in \aleph_{\omega_{1+1}} \rangle$ where $\overline{\mathbb{P}}_\alpha \Vdash$ "$\overline{\dot{\mathbb{Q}}}_\alpha = \mathbb{H}(\dot{\mathcal{F}}_\alpha)$" if $\alpha \in M$ and $\overline{\mathbb{P}}_\alpha \Vdash$ "$\overline{\dot{\mathbb{Q}}}_\alpha = \{\emptyset\}$" in the other case. It is not evident that this is well defined, since although $\dot{\mathcal{F}}_\alpha$ is a $\mathbb{P}_\alpha$-name for a sequence of reals, at the moment it is not clear it is also a $\overline{\mathbb{P}}_\alpha$-name, however, the next claim will take care of this problem:

Claim 19 *If $\alpha < \aleph_{\omega_1+1}$ then the following holds:*

1_α) $\mathbb{C}_S * \overline{\mathbb{P}}_\alpha \leq_r \mathbb{C}_{\aleph_{\omega_1+1}} * \mathbb{P}_\alpha$,

2_α) *If $\alpha \in M$ and $a \in (\mathbb{C}_{\aleph_{\omega_1+1}} * \mathbb{P}_\alpha) \cap M$ then there is $b \in \left(\mathbb{C}_S * \overline{\mathbb{P}}_\alpha\right) \cap M$ that is* equivalent to a *(i.e. $a \leq b$ and $b \leq a$).*

3_α) *If $\alpha \in M$ and $\dot{f} \in M$ is a $\mathbb{C}_{\aleph_{\omega_1+1}} * \mathbb{P}_\alpha$ name for a real, then there is $\dot{h} \in M$ a $\mathbb{C}_S * \overline{\mathbb{P}}_\alpha$-name such that that $1 \Vdash$ "$\dot{f} = \dot{h}$".*

We first note that 1_α and 2_α imply 3_α. Given $\alpha, \dot{f} \in M$ then without loss of generality, we may assume $\dot{f} = \{\{n\} \times A_n \mid n \in \omega\}$ where $A_n \subseteq \mathbb{C}_{\aleph_{\omega_1+1}} * \mathbb{P}_\alpha$ is a maximal antichain deciding the value of n. Since ${}^{\omega}M \subseteq M$ and using 2_α we can easily construct such an h. We will now prove 1_α and 2_α by induction.

Assume they hold for α, we need to show they hold for $\alpha + 1$. We will first assume $\alpha \in M$, let $a \in (\mathbb{C}_{\aleph_{\omega_1+1}} * \mathbb{P}_\alpha) \cap M$ then we may assume $a = (s, p, z, \mathcal{G})$ where $(s, p) \in \mathbb{C}_{\aleph_{\omega_1+1}} * \mathbb{P}_\alpha$, $z \in \omega^{<\omega}$ and $\mathcal{G} \in [\omega_1]^{<\omega}$. Since $a \in M$ then $(s, p) \in (\mathbb{C}_{\aleph_{\omega_1+1}} * \mathbb{P}_\alpha) \cap M$ and by our hypothesis, there is $(s', p') \in \left(\mathbb{C}_S * \overline{\mathbb{P}}_\alpha\right)$ equivalent to (s, p) and then $b = (s', p', z, \mathcal{G})$ is equivalent to a. In case $\alpha \notin M$ then $1_{\alpha+1}$ and $2_{\alpha+1}$ are trivially true. Now assume α is limit, then 1_α follows by lemma of 10 of [3] and 2_α follows since we are taking direct limit.

In this way, we may conclude that g is in some forcing extension of $V[C_S]$ so we may conclude that if $\alpha \notin S$ then $c_\alpha \not\leq^* g$ by the previous lemma. ■

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