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THERE IS A +-RAMSEY MAD FAMILY

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ABSTRACT

We answer an old question of Michael Hrušák by constructing a +-Ramsey MAD family without the need of any additional axioms beyond ZFC. We also prove that every Miller-indestructible MAD family is +-Ramsey; this improves a result of Michael Hrušák.

1. Introduction

A family $\mathcal{A} \subseteq [\omega]^\omega$ is **almost disjoint** (**AD**) if the intersection of any two different elements of $\mathcal{A}$ is finite, a MAD **family** is a maximal almost disjoint family. Almost disjoint families and MAD families have become very important in set theory, topology and functional analysis (see [7]). It is very easy to prove that the Axiom of Choice implies the existence of MAD families. However, constructing MAD families with special combinatorial or topological properties is a very difficult task without an additional hypothesis beyond ZFC. Constructing models of set theory for which certain kinds of MAD families do not exist is very difficult. We would like to mention some important examples regarding the existence or non-existence of special MAD families:

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(1) (Simon [21]) There is a MAD family which can be partitioned into two nowhere MAD families.
(2) (Mrówka [16]) There is a MAD family for which its Ψ-space has a unique compactification.
(3) (Raghavan [17]) There is a van Douwen MAD family.
(4) (Raghavan [18]) There is a model with no strongly separable MAD families.

If $\mathcal{I}$ is an ideal on ω, then by $\mathcal{I}^+$ we denote the set $\wp(\omega)\setminus\mathcal{I}$ and its elements are called $\mathcal{I}$-positive sets. If $\mathcal{A}$ is an AD family, by $\mathcal{I}(\mathcal{A})$ we denote the ideal generated by $\mathcal{A}$. In [12] Adrian Mathias proved that if $\mathcal{A}$ is a MAD family then $\mathcal{I}(\mathcal{A})^+$ is a happy family, which is a kind of Ramsey-like property. In [6] Michael Hrušák introduced a stronger Ramsey property:

Definition 1: (1) By $\mathcal{A}^\perp$ we denote the set of all $X \subseteq \omega$ such that $\mathcal{A} \cup \{X\}$ is almost disjoint.
(2) If $\mathcal{A}$ is an AD family, by $\mathcal{I}(\mathcal{A})^{++}$ we denote the set of all $X \subseteq \omega$ such that there is $\mathcal{B} \in [\mathcal{A}]^\omega$ such that $|X \cap B| = \omega$ for every $B \in \mathcal{B}$.
(3) Let $\mathcal{X} \subseteq [\omega]^\omega$, we say a tree $T \subseteq \omega^{<\omega}$ is a $\mathcal{X}$-**branching tree** if $\mathrm{suc}_T(s) \in \mathcal{X}$ for every $s \in T$ (where $\mathrm{suc}_T(s) = \{n \in \omega \mid s^\frown\langle n\rangle \in T\}$).
(4) An AD family $\mathcal{A}$ is $+$-**Ramsey** if for every $\mathcal{I}(\mathcal{A})^+$-branching tree T, there is $f \in [T]$ such that $\mathrm{im}(f) \in \mathcal{I}(\mathcal{A})^+$.

In [6] it is proved that there is a MAD family that is not $+$-Ramsey. On the other hand, $+$-Ramsey MAD families can be constructed under $\mathfrak{b} = \mathfrak{c}$, $\mathsf{cov}(\mathcal{M}) = \mathfrak{c}$, $\mathfrak{a} < \mathsf{cov}(\mathcal{M})$ or $\diamondsuit(\mathfrak{b})$ (see [6] and [8]). Michael Hrušák asked the following:

Problem 2 (Hrušák [6]): Is there a $+$-Ramsey MAD family in ZFC?

In this note we will provide a positive answer to this question. In [20] (see also [7] and [15]) Saharon Shelah developed a novel and powerful method to construct MAD families. He used it to prove that there is a completely separable MAD family if $\mathfrak{s} \leq \mathfrak{a}$ or $\mathfrak{a} < \mathfrak{s}$ and a certain PCF-hypothesis holds. Our technique for constructing a $+$-Ramsey MAD is based on the technique of Shelah (however, in this case we were able to avoid the PCF-hypothesis). It is worth mentioning that the method of Shelah has been further developed in [19] and [15] where it is proved that weakly tight MAD families exist under $\mathfrak{s} \leq \mathfrak{b}$. Our notation is mostly standard; the definition and basic properties of the cardinal invariants of the continuum used in this note can be found in [2].

2. Preliminaries

A MAD family $\mathcal{A}$ is **completely separable** if for every $X \in \mathcal{I}(\mathcal{A})^+$ there is $A \in \mathcal{A}$ such that $A \subseteq X$. This type of MAD families was introduced by Hechler in [5]. A year later, Shelah and Erdős asked the following question:

Problem 3 (Erdős–Shelah): Is there a completely separable MAD family?

It is easy to construct models where the previous question has a positive answer. It was shown by Balcar and Simon (see [1]) that such families exist assuming one of the following axioms: $\mathfrak{a} = \mathfrak{c}$, $\mathfrak{b} = \mathfrak{d}$, $\mathfrak{d} \leq \mathfrak{a}$ and $\mathfrak{s} = \omega_1$. In [20] (see also [7] and [15]) Shelah developed a novel and powerful method to construct completely separable MAD families. He used it to prove that there are such families if either $\mathfrak{s} \leq \mathfrak{a}$ or $\mathfrak{a} < \mathfrak{s}$ and a certain (so-called) PCF-hypothesis holds (which holds, for example, if the continuum is less than $\aleph_\omega$). Since the construction of Shelah of a completely separable MAD family under $\mathfrak{s} \leq \mathfrak{a}$ is key for our construction of a +-Ramsey MAD family, we will recall it on this section. This exposition is based on [15] and [7].

Definition 4:

(1) We say that S **splits** X if $S \cap X$ and $X \setminus S$ are both infinite.
(2) $\mathcal{S} \subseteq [\omega]^\omega$ is a **splitting family** if for every $X \in [\omega]^\omega$ there is $S \in \mathcal{S}$ such that S splits X.
(3) Let $S \in [\omega]^\omega$ and $\mathcal{P} = \{P_n \mid n \in \omega\}$ be an interval partition. We say S **block-splits** $\mathcal{P}$ if both of the sets
$$\{n \mid P_n \subseteq S\} \quad \text{and} \quad \{n \mid P_n \cap S = \emptyset\}$$
are infinite.
(4) A family $\mathcal{S} \subseteq [\omega]^\omega$ is called a **block-splitting family** if every interval partition is block-split by some element of $\mathcal{S}$.

Recall that the **splitting number** $\mathfrak{s}$ is the smallest size of a splitting family. It is well known that $\mathfrak{s}$ has uncountable cofinality; it is below the dominating number $\mathfrak{d}$ and independent from the unbounding number $\mathfrak{b}$ (see [2]). Regarding the smallest size of a block splitting family we have the following result of Kamburelis and Węglorz:

PROPOSITION 5 ([10]): *The smallest size of a block-splitting family is* $\max\{\mathfrak{b}, \mathfrak{s}\}$.

Some other notions of splitting are the following:

Definition 6: Let $S \in [\omega]^\omega$ and $\overline{X} = \{X_n \mid n \in \omega\} \subseteq [\omega]^\omega$.

(1) We say that S ω-**splits** $\overline{X}$ if S splits every X_n.
(2) We say that S (ω, ω)-**splits** $\overline{X}$ if both the sets $\{n \mid |X_n \cap S| = \omega\}$ and $\{n \mid |X_n \cap (\omega \setminus S)| = \omega\}$ are infinite.
(3) We say that $\mathcal{S} \subseteq [\omega]^\omega$ is an ω-**splitting family** if every countable collection of infinite subsets of ω is ω-split by some element of S.
(4) We say that $\mathcal{S} \subseteq [\omega]^\omega$ is an (ω, ω)-**splitting family** if every countable collection of infinite subsets of ω is (ω, ω)-split by some element of S.

It is easy to see that every block splitting family is an ω-splitting family. The following is a fundamental result of Mildenberger, Raghavan and Steprāns:

Proposition 7 ([15]): *There is an (ω, ω)-splitting family of size $\mathfrak{s}$.*

The key combinatorial feature of (ω, ω)-splitting families is the following result of Raghavan and Steprāns:

Proposition 8 ([19]): *If $\mathcal{S}$ is an (ω, ω)-splitting family, $\mathcal{A}$ an AD family and $X \in \mathcal{I}(\mathcal{A})^+$, then there is $S \in \mathcal{S}$ such that $X \cap S$, $X \cap (\omega \setminus S) \in \mathcal{I}(\mathcal{A})^+$.*

Given $X \subseteq \omega$ we denote $X^0 = X$ and $X^1 = \omega \setminus X$. By the previous result, if $\mathcal{A}$ is an AD family, $X \in \mathcal{I}(\mathcal{A})^+$ and $\mathcal{S} = \{S_\alpha \mid \alpha < \mathfrak{s}\}$ is an (ω, ω)-splitting family, then there are $\alpha < \mathfrak{s}$ and $\tau_X^{\mathcal{A}} \in 2^\alpha$ such that:

(1) If $\beta < \alpha$ then $X \cap S_\beta^{1-\tau_X^{\mathcal{A}}(\beta)} \in \mathcal{I}(\mathcal{A})$.
(2) $X \cap S_\alpha$, $X \setminus S_\alpha \in \mathcal{I}(\mathcal{A})^+$.

Clearly $\tau_X^{\mathcal{A}} \in 2^{<\mathfrak{s}}$ is unique, and if $Y \in [X]^\omega \cap \mathcal{I}(\mathcal{A})^+$ then $\tau_Y^{\mathcal{A}}$ extends $\tau_X^{\mathcal{A}}$. We can now prove the main result of this section:

Theorem 9 (Shelah [20]): *If $\mathfrak{s} \leq \mathfrak{a}$, then there is a completely separable MAD family.*

Proof. Let $[\omega]^\omega = \{X_\alpha \mid \alpha < \mathfrak{c}\}$. We will recursively construct $\mathcal{A} = \{A_\alpha \mid \alpha < \mathfrak{c}\}$ and $\{\sigma_\alpha \mid \alpha < \mathfrak{c}\} \subseteq 2^{<\mathfrak{s}}$ such that for every $\alpha < \mathfrak{c}$ the following holds (where $\mathcal{A}_\alpha = \{A_\xi \mid \xi < \alpha\}$):

(1) $\mathcal{A}_\alpha$ is an AD family.
(2) If $X_\alpha \in \mathcal{I}(\mathcal{A}_\alpha)^+$ then $A_\alpha \subseteq X_\alpha$.
(3) If $\alpha \neq \beta$ then $\sigma_\alpha \neq \sigma_\beta$.
(4) If $\xi < \mathrm{dom}(\sigma_\alpha)$ then $A_\alpha \subseteq^* S_\xi^{\sigma_\alpha(\xi)}$.

It is clear that if we manage to do this, then we will have achieved to construct a completely separable MAD family. Assume $\mathcal{A}_\delta = \{A_\xi \mid \xi < \delta\}$ has already been constructed. Let $X = X_\delta$ if $X_\delta \in \mathcal{I}(\mathcal{A}_\delta)^+$, and if $X_\delta \in \mathcal{I}(\mathcal{A}_\delta)$ let X be any other element of $\mathcal{I}(\mathcal{A}_\delta)^+$. We recursively find $\{X_s \mid s \in 2^{<\omega}\} \subseteq \mathcal{I}(\mathcal{A}_\delta)^+$, $\{\eta_s \mid s \in 2^{<\omega}\} \subseteq 2^{<\mathfrak{s}}$ and $\{\alpha_s \mid s \in 2^{<\omega}\}$ as follows:

(1) $X_\emptyset = X$.
(2) $\eta_s = \tau^{\mathcal{A}_\delta}_{X_s}$ and $\alpha_s = \mathrm{dom}(\eta_s)$.
(3) $X_{s^\frown 0} = X_s \cap S_{\alpha_s}$ and $X_{s^\frown 1} = X_s \cap (\omega \setminus S_{\alpha_s})$.

Note that if $t \subseteq s$, then $X_s \subseteq X_t$ and $\eta_t \subseteq \eta_s$. On the other hand, if s is incompatible with t, then η_s and η_t are incompatible. For every $f \in 2^\omega$ let

$$\eta_f = \bigcup_{n\in\omega} \eta_{f\restriction n}.$$

Since $\mathfrak{s}$ has uncountable cofinality each η_f is an element of $2^{<\mathfrak{s}}$, and if $f \neq g$ then η_f and η_g are incompatible nodes of $2^{<\mathfrak{s}}$. Since δ is smaller than $\mathfrak{c}$, there is $f \in 2^\omega$ such that there is no $\alpha < \delta$ such that σ_α extends η_f. Since $\{X_{f\restriction n} \mid n \in \omega\}$ is a decreasing sequence of elements in $\mathcal{I}(\mathcal{A}_\delta)^+$, there is $Y \in \mathcal{I}(\mathcal{A}_\delta)^+$ such that $Y \subseteq^* X_{f\restriction n}$ for every $n \in \omega$ (see [12] proposition 0.7 or [7] proposition 2).

Letting $\beta = \mathrm{dom}(\eta_f)$, we claim that if $\xi < \beta$ then $Y \cap S_\xi^{1-\eta_f(\xi)} \in \mathcal{I}(\mathcal{A})$. To prove this, let n be the first natural number such that $\xi < \mathrm{dom}(\eta_{f\restriction n})$. By our construction we know that $X_{f\restriction n} \cap S_\xi^{1-\eta_f(\xi)} \in \mathcal{I}(\mathcal{A})$, and since $Y \subseteq^* X_{f\restriction n}$ the result follows.

For every $\xi < \beta$ let $F_\xi \in [\mathcal{A}]^{<\omega}$ be such that $Y \cap S_\xi^{1-\eta_f(\xi)} \subseteq^* \bigcup F_\xi$ and let $W = \{A_\alpha \mid \sigma_\alpha \subseteq \eta_f\}$. Let

$$\mathcal{D} = W \cup \bigcup_{\xi<\beta} F_\xi$$

and note that $\mathcal{D}$ has size less than $\mathfrak{s}$, hence it has size less than $\mathfrak{a}$. In this way we conclude that $Y \restriction \mathcal{D}$ is not a MAD family, so there is $A_\delta \in [Y]^\omega$ that is almost disjoint with every element of $\mathcal{D}$ and define $\sigma_\delta = \eta_f$. We claim that A_δ is almost disjoint with $\mathcal{A}_\delta$. To prove this, let $\alpha < \delta$; in case $A_\alpha \in W$ we already know $A_\alpha \cap A_\delta$ is finite so assume $A_\alpha \notin W$. Letting $\xi = \Delta(\sigma_\delta, \sigma_\alpha)$ we know that $A_\alpha \subseteq^* S_\xi^{1-\sigma_\delta(\xi)}$ so $A_\alpha \cap A_\delta \subseteq^* \bigcup F_\xi$, but since $F_\xi \subseteq \mathcal{D}$ we conclude that A_δ is almost disjoint with $\bigcup F_\xi$ and then $A_\alpha \cap A_\delta$ must be finite. ∎

Recall that an AD family $\mathcal{A}$ is **nowhere** MAD if for every $X \in \mathcal{I}(\mathcal{A})^+$ there is $Y \in [X]^\omega$ such that Y is almost disjoint with $\mathcal{A}$. A key feature in the previous proof is that each $\mathcal{A}_\delta = \{A_\xi \mid \xi < \delta\}$ is nowhere MAD.

The first step to construct a +-Ramsey MAD family is to prove that every Miller-indestructible MAD family has this property. If $\mathcal{A}$ is a MAD family and $\mathbb{P}$ is a partial order, then we say $\mathcal{A}$ is $\mathbb{P}$-**indestructible** if $\mathcal{A}$ is still a MAD family after forcing with $\mathbb{P}$. The destructibility of MAD families has become a very important area of research with many fundamental questions still open (the reader may consult [8], [9], or [4] to learn more about the indestructibility of MAD families and ideals). The following property of MAD families plays a fundamental role in the study of destructibility:

Definition 10: A MAD family $\mathcal{A}$ is **tight** if for every

$$\{X_n \mid n \in \omega\} \subseteq \mathcal{I}(\mathcal{A})^+$$

there is $B \in \mathcal{I}(\mathcal{A})$ such that $B \cap X_n$ is infinite for every $n \in \omega$.

In [8] it is proved that every tight MAD family is Cohen-indestructible and that every tight MAD family is +-Ramsey. We will prove that every Miller-indestructible MAD family is +-Ramsey; this improves the previous result since Miller-indestructibility follows from Cohen-indestructibility (see [4]). First we need the following lemma:

LEMMA 11: *Let $\mathcal{A}$ be a MAD family and T an $\mathcal{I}(\mathcal{A})^+$-branching tree. Then there is a subtree $S \subseteq T$ with the following properties:*

(1) *If $s \in S$, there is $A_s \in \mathcal{A}$ such that $\mathrm{suc}_S(s) \in [A_s]^\omega$.*
(2) *If s and t are two different nodes of S, then $A_s \neq A_t$ and*

$$\mathrm{suc}_S(s) \cap \mathrm{suc}_S(t) = \emptyset.$$

Proof. Since T is an $\mathcal{I}(\mathcal{A})^+$-branching tree and $\mathcal{A}$ is MAD, $\mathrm{suc}_T(t)$ infinitely intersects many infinite elements of $\mathcal{A}$ for every $t \in T$. Recursively, for every $t \in T$ we choose $A_t \in \mathcal{A}$ and $B_t \in [A_t \cap \mathrm{suc}_T(t)]^\omega$ such that $B_t \cap B_s = \emptyset$ and $A_s \neq A_t$ whenever $t \neq s$. We then recursively construct $S \subseteq T$ such that, if $s \in S$, then $\mathrm{suc}_S(s) = B_s$. ∎

With the previous lemma we can now prove the following:

PROPOSITION 12: *If $\mathcal{A}$ is Miller-indestructible then $\mathcal{A}$ is +-Ramsey.*

Proof. Let $\mathcal{A}$ be a Miller-indestructible MAD family and T an $\mathcal{I}(\mathcal{A})^+$-branching tree. Let S be an $\mathcal{I}(\mathcal{A})$-branching subtree of T as in the previous lemma. We can then view S as a Miller tree. Let $\dot{r}_{\mathrm{gen}}$ be the name of the generic real and $\dot{X}$ the name of the image of $\dot{r}_{\mathrm{gen}}$.

We will first argue that $S \Vdash$ "$\dot{X} \notin \mathcal{I}(\mathcal{A})$". Assume this is not true, so there is $S_1 \leq S$ and $B \in \mathcal{I}(\mathcal{A})$ (B is an element of V) such that $S_1 \Vdash$ "$\dot{X} \subseteq B$". In this way, if t is a splitting node of S_1 then $\mathrm{suc}_{S_1}(t) \subseteq B$, but note that if $t_1 \neq t_2$ are two different splitting nodes of S_2 then $\mathrm{suc}_{S_1}(t_1)$ and $\mathrm{suc}_{S_1}(t_2)$ are two infinite sets contained in different elements of $\mathcal{A}$, so then $B \in \mathcal{I}(\mathcal{A})^+$ which is a contradiction.

In this way, $\dot{X}$ is forced by S to be an element of $\mathcal{I}(\mathcal{A})^+$, but since $\mathcal{A}$ is still MAD after performing a forcing extension of Miller forcing, we then conclude there are names $\{\dot{A}_n \mid n \in \omega\}$ for different elements of $\mathcal{A}$ such that S forces that $\dot{X} \cap \dot{A}_n$ is infinite. We then recursively build two sequences $\{S_n \mid n \in \omega\}$ and $\{B_n \mid n \in \omega\}$ such that for every $n \in \omega$ the following holds:

(1) S_n is a Miller tree and $B_n \in \mathcal{A}$.
(2) $S_0 \leq S$, and if $n < m$ then $S_m \leq S_n$.
(3) $S_n \Vdash$ "$\dot{A}_n = B_n$" (it then follows that $B_n \neq B_m$ if $n \neq m$).
(4) If $i \leq n$ then $\mathrm{stem}(S_n) \cap B_i$ has size at least n.

We then define $r = \bigcup_{n\in\omega} \mathrm{stem}(S_n)$, so clearly $r \in [S]$ and $\mathrm{im}(r) \in \mathcal{I}(\mathcal{A})^+$. ∎

The converse of the previous result is not true in general; this will be shown in Corollary 27. It is known that every MAD family of size less than $\mathfrak{d}$ is Miller-indestructible (see [4]). We can then conclude the following unpublished result of Michael Hrušák, which he proved by completely different means.

Corollary 13 (Hrušák): *Every MAD family of size less than $\mathfrak{d}$ is +-Ramsey. In particular, if $\mathfrak{a} < \mathfrak{d}$ then there is a +-Ramsey MAD family.*

3. The construction of a +-Ramsey MAD family

In this section we will construct a +-Ramsey MAD family without any extra hypothesis beyond ZFC. We will use the construction of Shelah of a completely separable MAD family; however, the previous result will help us avoid the need of a PCF-hypothesis for our construction. From now on, we will always assume that all Miller trees are formed by increasing sequences. If p is a Miller tree, we denote $\mathrm{Split}(p)$ the set of all splitting nodes of p.

Definition 14: Let p be a Miller tree. Given $f \in [p]$ we define

$$Sp(p, f) = \{f(n) \mid f \restriction n \in \mathrm{Split}(p)\}$$

and

$$[p]_{\mathrm{split}} = \{Sp(p, f) \mid f \in [p]\}.$$

We will need the following definitions:

Definition 15: Let p be a Miller tree and $H : \mathrm{Split}(p) \longrightarrow \omega$. We then define:

(1) $\mathrm{Catch}_\exists(H)$ is the set

$$\{Sp(f,p) \mid f \in [p] \wedge \exists^\infty n(f \restriction n \in \mathrm{Split}(p) \wedge f(n) < H(f \restriction n))\}.$$

(2) $\mathrm{Catch}_\forall(H)$ is the set

$$\{Sp(f,p) \mid f \in [p] \wedge \forall^\infty n(f \restriction n \in \mathrm{Split}(p) \wedge f(n) < H(f \restriction n))\}.$$

(3) Define $\mathcal{K}(p)$ as the collection of all $A \subseteq [p]_{\mathrm{split}}$ for which there is $G : \mathrm{Split}(p) \longrightarrow \omega$ such that

$$A \subseteq \mathrm{Catch}_\exists(G).$$

Note that if $\mathcal{B} = \{f_\alpha \mid \alpha < \mathfrak{b}\} \subseteq \omega^\omega$ is an unbounded family of increasing functions, then for every infinite partial function $g \subseteq \omega \times \omega$ there is $\alpha < \mathfrak{b}$ such that the set $\{n \in \mathrm{dom}(g) \mid g(n) < f_\alpha(n)\}$ is infinite. With this simple observation we can prove the following lemma.

Lemma 16: *$\mathcal{K}(p)$ is a σ-ideal in $[p]_{\mathrm{split}}$ that contains all singletons and* $\mathfrak{b} = \mathsf{add}(\mathcal{K}(p)) = \mathsf{cov}(\mathcal{K}(p))$.

Proof. In order to prove that $\mathfrak{b} \leq \mathsf{add}(\mathcal{K}(p))$, it is enough to show that if $\kappa < \mathfrak{b}$ and $\{H_\alpha \mid \alpha < \kappa\} \subseteq \omega^{\mathrm{Split}(p)}$ then $\bigcup_{\alpha<\kappa} \mathrm{Catch}_\exists(H_\alpha) \in \mathcal{K}(p)$. Since κ is smaller than $\mathfrak{b}$, we can find $H : \mathrm{Split}(p) \longrightarrow \omega$ such that if $\alpha < \kappa$, then $H_\alpha(s) < H(s)$ for almost all $s \in \mathrm{Split}(p)$. Clearly

$$\bigcup_{\alpha<\kappa} \mathrm{Catch}_\exists(H_\alpha) \subseteq \mathrm{Catch}_\exists(H).$$

Now we must prove that $\mathsf{cov}(\mathcal{K}(p)) \leq \mathfrak{b}$. Let $\mathrm{Split}(p) = \{s_n \mid n < \omega\}$ and $\mathcal{B} = \{f_\alpha \mid \alpha < \mathfrak{b}\} \subseteq \omega^\omega$ be an unbounded family of increasing functions. Given $\alpha < \mathfrak{b}$ define $H_\alpha : \mathrm{Split}(p) \longrightarrow \omega$, where

$$H_\alpha(s_n) = f_\alpha(n).$$

We will show that $\{\mathrm{Catch}_\exists(H_\alpha) \mid \alpha < \mathfrak{b}\}$ covers $[p]_{\mathrm{split}}$. Letting $f \in [p]$, define $A = \{n \mid s_n \sqsubseteq f\}$ and construct the function $g : A \longrightarrow \omega$, where

$$g(n) = f(|s_n|) + 1$$

for every $n \in A$. By the previous remark, there is $\alpha < \mathfrak{b}$ such that $f_\alpha \restriction A$ is not dominated by $g \restriction A$. It is then clear that $S_p(p,f) \in \mathrm{Catch}_\exists(H_\alpha)$. ∎

Letting p be a Miller tree and $S \in [\omega]^\omega$, we define the game $\mathcal{G}(p, S)$ as follows:

I	s_0		s_1		$\cdots$
II		r_0		r_1	

(1) Each s_i is a splitting node of p.
(2) $r_i \in \omega$.
(3) s_{i+1} extends s_i.
(4) $s_{i+1}(|s_i|) \in S$ and is bigger than r_i.

Player I wins the game if she can continue playing for infinitely many rounds. Given $S \in [\omega]^\omega$, we denote by $\mathrm{Hit}(S)$ the set of all subsets of ω that have infinite intersection with S.

Lemma 17: *Letting p be a Miller tree and $S \in [\omega]^\omega$, for the game $\mathcal{G}(p, S)$ we have the following:*

(1) *Player I has a winning strategy if and only if there is $q \leq p$ such that $[q]_{\mathrm{split}} \subseteq [S]^\omega$.*
(2) *Player II has a winning strategy if and only if there is $H : \mathrm{Split}(p) \longrightarrow \omega$ such that if $f \in [p]$, then the set $\{f \restriction n \in \mathrm{Split}(p) \mid f(n) \in S\}$ is almost contained in $\{f \restriction n \in \mathrm{Split}(p) \mid f(n) < H(f \restriction n)\}$ (in particular $[p]_{\mathrm{split}} \cap \mathrm{Hit}(S) \in \mathcal{K}(p)$).*

Proof. The first equivalence is easy so we leave it for the reader. Now assume there is a winning strategy π for II. We define $H : \mathrm{Split}(p) \longrightarrow \omega$ such that if $s \in \mathrm{Split}(p)$ then

$$\pi(\overline{x}) < H(s)$$

where $\overline{x}$ is any partial play in which player I has build s and II has played according to π (note there are only finitely many of those $\overline{x}$ so we can define $H(s)$). We want to prove that if $f \in [p]$, then $\{f \restriction n \in \mathrm{Split}(p) \mid f(n) \in S\}$ is almost contained in the set

$$\{f \restriction n \in \mathrm{Split}(p) \mid f(n) < H(f \restriction n)\}.$$

Assume this is not the case. Let B be the set of all $n \in \omega$ such that $f \restriction n \in \mathrm{Split}(p)$ with $f(n) \in S$ but $H(f \restriction n) \leq f(n)$. By our hypothesis B is infinite and then we enumerate it as $B = \{b_n \mid n \in \omega\}$ in increasing order. Consider the run of the game where I plays $f \restriction b_n$ at the n-th stage. This is possible since $f(b_n) \in S$ and $H(f \restriction b_n) \leq f(b_n)$ so I will win the game, which is a contradiction. The other implication is easy. ∎

Since $\mathcal{G}(p, S)$ is an open game for II by the Gale–Stewart theorem (see [11]) it is determined, so we conclude the following dichotomy:

Corollary 18: *If p is a Miller tree and $S \in [\omega]^\omega$ then one and only one of the following holds:*

(1) *There is $q \leq p$ such that $[q]_{\text{split}} \subseteq [S]^\omega$.*
(2) *There is $H : \text{Split}(p) \longrightarrow \omega$ such that if $f \in [p]$, then the set defined as $\{f \restriction n \in \text{Split}(p) \mid f(n) \in S\}$ is almost contained in the following set: $\{f \restriction n \in \text{Split}(p) \mid f(n) < H(f \restriction n)\}$ (and $[p]_{\text{split}} \cap \text{Hit}(S) \in \mathcal{K}(p)$).*

In particular, for every Miller tree p and $S \in [\omega]^\omega$ there is $q \leq p$ such that either $[q]_{\text{split}} \subseteq [S]^\omega$ or $[q]_{\text{split}} \subseteq [\omega \setminus S]^\omega$ (although this fact can be proved easier without the game).

Definition 19: Let p be a Miller tree and $S \in [\omega]^\omega$. We say S **tree-splits** p if there are Miller trees $q_0, q_1 \leq p$ such that

$$[q_0]_{\text{split}} \subseteq [S]^\omega \quad \text{and} \quad [q_1]_{\text{split}} \subseteq [\omega \setminus S]^\omega;$$

$\mathcal{S}$ is a **Miller tree-splitting family** if every Miller tree is tree-split by some element of $\mathcal{S}$.

It is easy to see that every Miller-tree splitting family is a splitting family and it is also easy to see that every ω-splitting family is a Miller-tree splitting family. We will now prove there is a Miller-tree splitting family of size $\mathfrak{s}$. I want to thank the referee for supplying the following argument which is simpler than the original one.

Proposition 20: *The smallest size of a Miller-tree splitting family is $\mathfrak{s}$.*

Proof. We will construct a Miller-tree splitting family of size $\mathfrak{s}$. In case $\mathfrak{b} \leq \mathfrak{s}$ there is an ω-splitting family of size $\mathfrak{s}$ (see Proposition 5 and the remark after Definition 6) and this is a Miller-tree splitting family as remarked above.

Now assume $\mathfrak{s} < \mathfrak{b}$. We will show that any splitting family of size $\mathfrak{s}$ is a Miller-tree splitting family. We argue by contradiction. Let

$$\mathcal{S} = \{S_\alpha \mid \alpha < \mathfrak{s}\}$$

be a splitting family which does not tree-split the Miller tree p. In this way, for every $\alpha < \mathfrak{s}$ there is $i(\alpha) < 2$ such that there is no $q \leq p$ for which $[q]_{\text{split}} \subseteq [S_\alpha^{i(\alpha)}]^\omega$. By Corollary 18, there is $H_\alpha : \text{Split}(p) \longrightarrow \omega$ such that for every $f \in [p]$ the following holds:

$$\{f \restriction n \in \mathrm{Split}(p) \mid f(n) \in S\} \subseteq^* \{f \restriction n \in \mathrm{Split}(p) \mid f(n) < H(f \restriction n)\}$$

or equivalently

$$\forall^\infty n(f \restriction n \in \mathrm{Split}(p) \wedge H_\alpha(f \restriction n) \leq f(n) \longrightarrow f(n) \in S_\alpha^{1-i(\alpha)}).$$

Since $\mathfrak{s} < \mathfrak{b}$ there exists $H : \mathrm{Split}(p) \longrightarrow \omega$ dominating each H_α. Take $f \in [p]$ such that for every $n \in \omega$, if $f \restriction n \in \mathrm{Split}(p)$ then $f(n) > H(f \restriction n)$. Then

$$\forall \alpha < \mathfrak{s} \forall^\infty n(f \restriction n \in \mathrm{Split}(p) \longrightarrow f(n) \in S_\alpha^{1-i(\alpha)}).$$

Let $X = \{f(n) \mid f \restriction n \in \mathrm{Split}(p)\}$; note that $X \subseteq^* S_\alpha^{1-i(\alpha)}$ for every $\alpha < \mathfrak{s}$. But this contradicts that $\mathcal{S}$ was a splitting family. ∎

The following lemma is probably well known:

Lemma 21: *Assume $\kappa < \mathfrak{d}$, and for every $\alpha < \kappa$ let $\mathcal{F}_\alpha \subseteq [\omega]^{<\omega}$ be an infinite set of disjoint finite subsets of ω and $g_\alpha : \bigcup \mathcal{F}_\alpha \longrightarrow \omega$. Then there is $f : \omega \longrightarrow \omega$ such that for every $\alpha < \kappa$ there are infinitely many $X \in \mathcal{F}_\alpha$ such that*

$$g_\alpha \restriction X < f \restriction X.$$

Proof. Given $\alpha < \kappa$, find an interval partition $\mathcal{P}_\alpha = \{P_\alpha(n) \mid n \in \omega\}$ such that for every $n \in \omega$ there is $X \in \mathcal{F}_\alpha$ such that $X \subseteq P_\alpha(n)$ (this is possible since $\mathcal{F}_\alpha$ is infinite and its elements are pairwise disjoint). Then define the function $\overline{g}_\alpha : \omega \longrightarrow \omega$ such that $\overline{g}_\alpha \restriction P_\alpha(n)$ is the constant function $\max\{g_\alpha[P_\alpha(n+1)]\}$. Since κ is smaller than $\mathfrak{d}$, we can then find an increasing function $f : \omega \longrightarrow \omega$ that is not dominated by any of the $\overline{g}_\alpha$. It is easy to prove that f has the desired property. ∎

Now we can prove the following lemma that will be useful:

Lemma 22: *Let q be a Miller tree and $\kappa < \mathfrak{d}$. If $\{H_\alpha \mid \alpha < \kappa\} \subseteq \omega^{\mathrm{Split}(q)}$ then there is $r \leq q$ such that $\mathrm{Split}(r) = \mathrm{Split}(q) \cap r$ and*

$$[r]_{\mathrm{split}} \cap \bigcup_{\alpha<\kappa} \mathrm{Catch}_\forall(H_\alpha) = \emptyset.$$

Proof. We will first prove there is $G : \mathrm{Split}(q) \longrightarrow \omega$ such that $\bigcup_{\alpha<\kappa} \mathrm{Catch}_\forall(H_\alpha)$ is a subset of $\mathrm{Catch}_\exists(G)$. Given $t \in \mathrm{Split}(q)$, let $T(t, \alpha)$ be the subtree of q such that if $f \in [T(t, \alpha)]$ then $t \sqsubseteq f$, and if $t \sqsubseteq f \restriction n$ and $f \restriction n \in \mathrm{Split}(q)$ then $f(n) \in H_\alpha(f \restriction n)$. Clearly $T(t, \alpha)$ is a finitely branching subtree of q. Then define $\mathcal{F}(t,\alpha) = \{\text{Split}_n(q) \cap T(t,\alpha) \mid n < \omega\}$ which is an infinite collection of pairwise disjoint finite sets, and let $g_{(t,\alpha)} : \bigcup \mathcal{F}(t,\alpha) \longrightarrow \omega$ given by

$$g_{(t,\alpha)}(s) = H_\alpha(s).$$

Since $\kappa < \mathfrak{d}$ by the previous lemma, we can find $G : \text{Split}(q) \longrightarrow \omega$ such that if $\alpha < \kappa$ and $t \in \text{Split}(q)$, then there are infinitely many $Y \in \mathcal{F}(t,\alpha)$ such that $g_{(t,\alpha)} \upharpoonright Y < G \upharpoonright Y$. We will now prove that $\bigcup_{\alpha<\kappa} \text{Catch}_\forall(H_\alpha) \subseteq \text{Catch}_\exists(G)$. Let $\alpha < \kappa$ and $f \in \text{Catch}_\forall(H_\alpha)$. Find $t \in \text{Split}(q)$ such that $t \sqsubseteq f$, and if $t \sqsubseteq f \upharpoonright m$ and $f \upharpoonright m \in \text{Split}(q)$ then $f(m) \in H_\alpha(f \upharpoonright m)$. Note that f is a branch through $T(t,\alpha)$. Let $Y \in \mathcal{F}(t,\alpha)$ such that $g_{(t,\alpha)} \upharpoonright Y < G \upharpoonright Y$ and, since $f \in [T(t,\alpha)]$, there is $n \in \omega$ such that $f \upharpoonright n \in Y$ so $f(n) < H_\alpha(f \upharpoonright n) < G(f \upharpoonright n)$.

Define $r \leq q$ such that $\text{Split}(r) = \text{Split}(q) \cap r$ and $\text{suc}_r(s) = \text{suc}_q(s) \setminus G(s)$. Clearly $[r]_{\text{split}}$ is disjoint from $\text{Catch}_\exists(G)$. ∎

We can then finally prove our main theorem.

Theorem 23: *There is a +-Ramsey* MAD *family.*

Proof. If $\mathfrak{a} < \mathfrak{s}$ then $\mathfrak{a}$ is smaller than $\mathfrak{d}$ so then there is a +-Ramsey MAD family (in fact, there is a Miller-indestructible MAD family, see Corollary 13). So we assume $\mathfrak{s} \leq \mathfrak{a}$ for the rest of the proof. Fix an (ω,ω)-splitting family $\mathcal{S} = \{S_\alpha \mid \alpha < \mathfrak{s}\}$ that is also a Miller-tree splitting family. Let $\{L, R\}$ be a partition of the limit ordinals smaller than $\mathfrak{c}$ such that both L and R have size continuum. Enumerate by $\{X_\alpha \mid \alpha \in L\}$ all infinite subsets of ω and by $\{T_\alpha \mid \alpha \in R\}$ all subtrees of $\omega^{<\omega}$. We will recursively construct $\mathcal{A} = \{A_\xi \mid \xi < \mathfrak{c}\}$ and $\{\sigma_\xi \mid \xi < \mathfrak{c}\}$ as follows:

(1) $\mathcal{A}$ is an AD family and $\sigma_\alpha \in 2^{<\mathfrak{s}}$ for every $\alpha < \mathfrak{c}$.
(2) If $\sigma_\alpha \in 2^\beta$ and $\xi < \beta$, then $A_\alpha \subseteq^* S_\xi^{\sigma_\alpha(\xi)}$.
(3) If $\alpha \neq \beta$, then $\sigma_\alpha \neq \sigma_\beta$.
(4) If $\delta \in L$ and $X_\delta \in \mathcal{I}(\mathcal{A}_\delta)^+$, then $A_{\delta+n} \subseteq X_\delta$ for every $n \in \omega$ (where $\mathcal{A}_\delta = \{A_\xi \mid \xi < \delta\}$).
(5) If $\delta \in R$ and T_δ is an $\mathcal{I}(\mathcal{A}_\delta)^+$-branching tree, then there is $f \in [T_\delta]$ such that $A_{\delta+n} \subseteq \text{im}(f)$ for every $n \in \omega$.

It is clear that if we manage to perform the construction, then $\mathcal{A}$ will be a +-Ramsey MAD family (and it will be completely separable too). Let δ be a limit ordinal and assume we have constructed A_ξ for every $\xi < \delta$. In case $\delta \in L$ we just proceed as in the case of the completely separable MAD family, so assume $\delta \in R$. Since $\mathcal{A}_\delta = \{A_\xi \mid \xi < \delta\}$ is nowhere-MAD (recall that $\mathcal{A}_\delta$ is nowhere-MAD by the proof of Theorem 9) we can find p, an $\mathcal{A}_\delta^\perp$-branching subtree of T_δ (recall that $\mathcal{A}_\delta^\perp$ is the set of all infinite sets that are almost disjoint with every element of $\mathcal{A}_\delta^\perp$).

First note that since $\mathcal{S}$ is a Miller-tree splitting family, for every Miller tree q there is $\alpha < \mathfrak{s}$ and $\tau_q \in 2^\alpha$ such that:

(1) S_α tree-splits q.
(2) If $\xi < \alpha$, then there is no $q' \leq q$ such that $[q']_{\text{split}} \subseteq [S_\xi^{1-\tau_q(\xi)}]^\omega$.

Note that if $q' \leq q$, then $\tau_{q'}$ extends τ_q. If $q \leq p$ and $\tau_q \in 2^\alpha$ we fix the following items:

(1) $W_0(q) = \{\xi < \alpha \mid \exists \beta < \delta(\sigma_\beta = \tau_q \restriction \xi)\}$

and

$W_1(q) = \{\xi < \alpha \mid \exists \beta < \delta(\Delta(\sigma_\beta, \tau_q) = \xi)\}$.[1]

(2) Let $\xi \in W_0(q)$. We then find β such that $\sigma_\beta = \tau_q \restriction \xi$ and define $G_{q,\xi} : \text{Split}(q) \longrightarrow \omega$ such that if $s \in \text{Split}(q)$ then

$$A_\beta \cap \text{suc}_q(s) \subseteq G_{q,\xi}(s)$$

(this is possible since q is $\mathcal{A}_\delta^\perp$-branching).

(3) Given $\xi \in W_1(q)$ we know there is no $q' \leq q$ such that

$$[q']_{\text{split}} \subseteq [S_\xi^{1-\tau_q(\xi)}]^\omega.$$

We know that there is $H_{q,\xi} : \text{Split}(q) \longrightarrow \omega$ such that if $f \in [q]$, the set defined as $\{f \restriction n \in \text{Split}(q) \mid f(n) \in S_\xi^{1-\tau_q(\xi)}\}$ is almost contained in the set

$$\{f \restriction n \in \text{Split}(q) \mid f(n) < H_{q,\xi}(f \restriction n)\}.$$

(4) If $U \in [W_0(q)]^{<\omega}$ and $V \in [W_1(q)]^{<\omega}$ choose any $J_{q,U,V} : \text{Split}(q) \longrightarrow \omega$ such that if $s \in \text{Split}(q)$, then

$$J_{q,U,V}(s) > \max\{G_{q,\xi}(s) \mid \xi \in U\}, \max\{H_{q,\xi}(s) \mid \xi \in V\}.$$

(5) $\mathcal{A}(q) = \{A_\xi \in \mathcal{A}_\delta \mid \tau_q \nsubseteq \sigma_\xi\}$.

Note that if $\xi \in W_0(q)$, then there is a unique $\beta < \delta$ such that $\sigma_\beta = \tau_q \restriction \xi$ (although the analogous remark is not true for the elements of $W_1(q)$). The following claim will play a fundamental role in the proof:

Claim 24: *If $q \leq p$, then there is $r \leq q$ such that $[r]_{\text{split}} \subseteq \mathcal{I}(\mathcal{A}(q))^+$.*

Proof of Claim 24. Let $\alpha < \mathfrak{s}$ such that $\tau_q \in 2^\alpha$. Since $\mathfrak{s} \leq \mathfrak{d}$, we know there is $r \leq q$ such that $[r]_{\text{split}}$ is disjoint from

$$\bigcup\{\text{Catch}_\forall(J_{q,U,V}) \mid U \in [W_0(q)]^{<\omega}, V \in [W_1(q)]^{<\omega}\}$$

and $\text{Split}(r) = \text{Split}(q) \cap r$. We will now prove $[r]_{\text{split}} \subseteq \mathcal{I}(\mathcal{A}(q))^+$ but assume this is not the case. Therefore, there is $f \in [r]$ and $F \in [\mathcal{A}(q)]^{<\omega}$ such that

$$X = Sp(r, f) \subseteq^* \bigcup F.$$

Let $F = F_1 \cup F_2$ and $U \in [W_0(q)]^{<\omega}$, $V \in [W_1(q)]^{<\omega}$ such that for every $A_\beta \in F_1$ there is $\xi_\beta \in U$ such that $\sigma_\beta = \tau_q \restriction \xi_\beta$, and for every $A_\gamma \in F_2$ there is $\eta_\gamma \in V$ such that $\Delta(\tau_q, \sigma_\gamma) = \eta_\gamma$. Let $D \subseteq \{n \mid f \restriction n \in \text{Split}(r)\}$ be the (infinite) set of all $n < \omega$ such that the following holds:

(1) $f \restriction n \in \text{Split}(r)$ and $f(n) \in \bigcup F$.
(2) If $\eta_\gamma \in V$ then $A_\gamma \setminus n \subseteq S_{\eta_\gamma}^{1-\tau_q(\eta_\gamma)}$.
(3) $f(n) > J_{q,U,V}(f \restriction n)$.
(4) If $\eta \in V$ and $f(n) \in S_\eta^{1-\tau_q(\eta)}$, then $f(n) < H_{q,\eta}(f \restriction n) < J_{q,U,V}(f \restriction n)$ (recall that $\{f \restriction m \in \text{Split}(q) \mid f(m) \in S_\eta^{1-\tau_q(\eta)}\}$ is almost contained in $\{f \restriction m \in \text{Split}(q) \mid f(m) < H_{q,\eta}(f \restriction m)\}$).

We first claim that if $n \in D$, $\xi_\beta \in U$ and $\eta_\gamma \in V$, then $f(n) \notin A_\beta \cup S_{\eta_\gamma}^{1-\tau_q(\eta_\gamma)}$. On one hand, since $A_\beta \cap \text{suc}_q(f \restriction n) \subseteq G_{q,\xi_\beta}(f \restriction n) < J_{q,U,V}(f \restriction n)$ and $f(n) > J_{q,U,V}(f \restriction n)$ then $f(n) \notin A_\beta$. On the other hand, if it was the case that $f(n) \in S_{\eta_\gamma}^{1-\tau_q(\eta_\gamma)}$ so $f(n) < H_{q,\eta}(f \restriction n) < J_{q,U,V}(f \restriction n)$, but we already know that $f(n) > J_{q,U,V}(f \restriction n)$. Since $n \leq f(n)$ (recall every branch through p is increasing) $f(n) \notin A_\gamma$ for every $\eta_\gamma \in V$ because $A_\gamma \setminus n \subseteq S_{\eta_\gamma}^{1-\tau_q(\eta_\gamma)}$. This implies $f(n) \notin \bigcup F$ which is a contradiction and finishes the proof of the claim. ∎

Back to the proof of the theorem, we recursively build a tree of Miller trees $\{p(s) \mid s \in 2^{<\omega}\}$ with the following properties:

(1) $p(\emptyset) = p$.
(2) $p(s^\frown i) \leq p(s)$ and the stem of $p(s^\frown i)$ has length at least $|s|$.
(3) $\tau_{p(s^\frown 0)}$ and $\tau_{p(s^\frown 1)}$ are incompatible.
(4) $[p(s^\frown i)]_{\text{split}} \subseteq \mathcal{I}(\mathcal{A}(p(s)))^+$.

This is easy to do with the aid of the previous claim. For every $g \in 2^\omega$ let $\tau_g = \bigcup \tau_{p(g \restriction m)}$. Note that if $g_1 \neq g_2$, then τ_{g_1} and τ_{g_2} are two incompatible nodes of $2^{<\mathfrak{s}}$. Since $\mathcal{A}_\delta$ has size less than the continuum, there is $g \in 2^\omega$ such that there is no $\beta < \delta$ such that σ_β extends τ_g and then $\mathcal{A}_\delta = \bigcup_{m \in \omega} \mathcal{A}(p(g \restriction m))$.

Let f be the only element of $\bigcap_{m\in\omega}[p(g \restriction m)]$. Obviously, f is a branch through p and we claim that $\mathrm{im}(f) \in \mathcal{I}(\mathcal{A}_\delta)^+$. This is easy since if $A_{\xi_1}, \ldots, A_{\xi_n} \in \mathcal{A}_\delta$, then we can find $m < \omega$ such that $A_{\xi_1}, \ldots, A_{\xi_n} \in \mathcal{A}(p(g \restriction m))$ and then we know that $Sp(p(g \restriction m+1), f) \nsubseteq^* A_{\xi_1} \cup \cdots \cup A_{\xi_n}$, and since $Sp(p(g \restriction m+1), f)$ is contained in $\mathrm{im}(f)$ we conclude that $\mathrm{im}(f) \in \mathcal{I}(\mathcal{A}_\delta)^+$.

Finally, find a partition $\{Z_n \mid n \in \omega\} \subseteq \mathcal{I}(\mathcal{A}_\delta)^+$ of $\mathrm{im}(f)$ and using the method of Shelah, construct $A_{\delta+n}$ such that $A_{\delta+n} \subseteq Z_n$. This finishes the proof. ∎

4. More constructions

In this last section, we will show the relationship between +-Ramsey and other properties of MAD families. Recall that an ideal $\mathcal{I}$ in ω is **tall** if for every $X \in [\omega]^\omega$ there is $Y \in \mathcal{I}$ such that $X \cap Y$ is infinite. Note that if $\mathcal{A}$ is an AD family, then $\mathcal{I}(\mathcal{A})$ is tall if and only if $\mathcal{A}$ is MAD. Note that the proof of Theorem 23 in fact gives the following result:

Corollary 25 ($\mathfrak{s} \leq \mathfrak{a}$): *If $\mathcal{I}$ is a tall ideal, then there is a +-Ramsey MAD family $\mathcal{A}$ such that $\mathcal{A} \subseteq \mathcal{I}$.*

The following are properties of MAD families that have been studied in the literature:

Definition 26: Let $\mathcal{A}$ be a MAD family.

(1) $\mathcal{A}$ is $\mathbb{P}$-**indestructible** if $\mathcal{A}$ remains MAD after forcing with $\mathbb{P}$ (we are mainly interested where $\mathbb{P}$ is Cohen, random, Sacks or Miller forcing).
(2) $\mathcal{A}$ is **weakly tight** if for every $\{X_n \mid n \in \omega\} \subseteq \mathcal{I}(\mathcal{A})^+$ there is $B \in \mathcal{I}(\mathcal{A})$ such that $|B \cap X_n| = \omega$ for infinitely many $n \in \omega$.
(3) $\mathcal{A}$ is **tight** if for every $\{X_n \mid n \in \omega\} \subseteq \mathcal{I}(\mathcal{A})^+$ there is $B \in \mathcal{I}(\mathcal{A})$ such that $B \cap X_n$ is infinite for every $n \in \omega$.
(4) $\mathcal{A}$ is **Laflamme** if $\mathcal{A}$ can not be extended to an F_σ-ideal.
(5) $\mathcal{A}$ is +-**Ramsey** if for every $\mathcal{I}(\mathcal{A})^+$-branching tree T, there is $f \in [T]$ such that $\mathrm{im}(f) \in \mathcal{I}(\mathcal{A})^+$.

It is known that tightness implies both weak tightness and Cohen indestructibility (see [8]). It is also easy to see that Cohen indestructibility implies Miller indestructibility and Sacks indestructibility is weaker than both Miller indestructibility and random indestructibility (see [4]).

Corollary 27 ($\mathfrak{s} \leq \mathfrak{a}$): *There is a* $+$*-Ramsey* MAD *family that is not Sacks indestructible, Laflamme or weakly tight.*

Proof. The corollary follows by the previous result. In [9] it was proved that there is a tall ideal $\mathcal{I}$ such that every MAD family contained in $\mathcal{I}$ is Sacks destructible. A similar result for weak tightness was proved in [3]. ∎

The following is a very important definition:

Definition 28: We say $\varphi : \wp(\omega) \longrightarrow \omega \cup \{\omega\}$ is a **lower semicontinuous submeasure** if the following hold:

(1) $\varphi(\omega) = \omega$.
(2) $\varphi(A) = 0$ if and only if $A = \emptyset$.
(3) $\varphi(A) \leq \varphi(B)$ whenever $A \subseteq B$.
(4) $\varphi(A \cup B) \leq \varphi(A) + \varphi(B)$ for every $A, B \subseteq X$.
(5) (lower semicontinuity) if $A \subseteq \omega$ then $\varphi(A) = \sup\{\varphi(A \cap n) \mid n \in \omega\}$.

Given a lower semicontinuous submeasure φ we define $\mathsf{Fin}(\varphi)$ as the family of those subsets of ω with finite submeasure. The following is a very interesting result of Mazur:

Proposition 29 (Mazur [13]): *$\mathcal{I}$ is an F_σ-ideal if and only if there is a lower semicontinuous submeasure such that $\mathcal{I} = \mathsf{Fin}(\varphi)$.*

If $a \subseteq \omega^{<\omega}$ we define

$$\pi(a) = \{f \in \omega^\omega \mid \exists^\infty n (f \restriction n \in a)\}.$$

Given $f \in \omega^\omega$, define

$$\widehat{f} = \{f \restriction n \mid n \in \omega\}$$

and let $\mathcal{BR} = \{\widehat{f} \mid f \in \omega^\omega\}$. By $\mathcal{J}$ we denote the ideal on $\omega^{<\omega}$ consisting of all sets $a \subseteq \omega^{<\omega}$ such that $\pi(a)$ is finite. Clearly $\mathcal{BR} \subseteq \mathcal{J}$. The next result follows easily from the results in [14], but we include a proof for the convenience of the reader:

Lemma 30: *$\mathcal{J}$ cannot be extended to an F_σ-ideal.*

Proof. Let $\varphi : \wp(\omega) \longrightarrow \omega \cup \{\omega\}$ be a lower semicontinous submeasure. We will prove that $\mathcal{J}$ is not a subset of $\mathsf{Fin}(\varphi)$. Given $s \in \omega^{<\omega}$, we denote

$$B_0(s) = \{t \in \omega^{<\omega} \mid s \subseteq t\} \quad \text{and} \quad B_1(s) = \{t \in \omega^{<\omega} \mid s \perp t\}$$

(where $s \perp t$ denotes that s and t are incompatible). Let $\omega^{<\omega} = \{s_n \mid n \in \omega\}$. We recursively construct two sequences $\langle i_n \mid n \in \omega\rangle$ and $\langle F_n \mid n \in \omega\rangle$ such that for every $n \in \omega$ the following holds:

(1) $i_n \in \{0, 1\}$.
(2) F_n is a finite subset of ω and $\varphi(F_n) \geq n + 1$.
(3) $\bigcap_{j \leq n} B_{i_j}(s_j) \in \mathsf{Fin}(\varphi)^+$.
(4) $F_n \subseteq \bigcap_{j \leq n} B_{i_j}(s_j)$.

The construction is very easy to perform. Let $G = \bigcup_{n\in\omega} F_n$. Note that $G \in \mathsf{Fin}(\varphi)^+$. Furthermore, for every $s \in \omega^{<\omega}$ either G is almost contained in $B_0(s)$ or is almost disjoint from it. It is easy to see that $G \in \mathcal{J}$ so $\mathcal{J}$ is not contained in $\mathsf{Fin}(\varphi)$. ∎

We can now prove the following:

Proposition 31 (CH): *There is a Laflamme* MAD *family that is not +-Ramsey.*

Proof. Let $\{\mathcal{I}_\alpha \mid \alpha \in \omega_1\}$ be the set of all F_σ-ideals in $\omega^{<\omega}$. We construct $\mathcal{A} = \{A_\alpha \mid \alpha < \omega_1\}$ such that the following holds:

(1) $\mathcal{A} \cup \mathcal{BR}$ is an AD family.
(2) If $s \in \omega^{<\omega}$, then $\mathcal{A}$ contains an infinite partition of $\{s^\frown n \mid n \in \omega\}$ into infinite sets.
(3) If $\mathcal{A}_\alpha \cup \mathcal{BR} \subseteq \mathcal{I}_\alpha$, then $A_\alpha \notin \mathcal{I}_\alpha$ (where $\mathcal{A}_\alpha = \{A_\xi \mid \xi < \alpha\}$).
(4) $\mathcal{A}_\alpha$ is countable.

At step α assume that $\mathcal{BR} \cup \mathcal{A}_\alpha \subseteq \mathcal{I}_\alpha$. Since $\mathcal{I}_\alpha$ is an F_σ-ideal and it contains all branches, there is $a \in \mathcal{I}_\alpha^+ \cap \mathcal{J}$. Let $\pi(a) = \{f_1, \dots, f_n\}$ and we now define $b = a \setminus (\widehat{f_1} \cup \dots \cup \widehat{f_n})$. Note that $\pi(b) = \emptyset$ and $b \in \mathcal{I}_\alpha^+$. Let φ be a lower semicontinuous submeasure such that $\mathcal{I}_\alpha = \mathsf{Fin}(\varphi)$ and let $\mathcal{A}_\alpha = \{B_n \mid n \in \omega\}$. We recursively find $s_n \subseteq b \setminus (B_0 \cup \dots \cup B_n)$ such that $\varphi(s_n) \geq n$ (this is possible since $b \in \mathcal{I}_\alpha^+$). Then $A_\alpha = \bigcup_{n\in\omega} s_n$ is the set we were looking for. It is easy to see that $\mathcal{A} \cup \mathcal{BR}$ is a Laflamme MAD family that is not +-Ramsey. ∎

We will now prove that weak tightness does not imply being +-Ramsey. Given $s \in \omega^{<\omega}$ we define

$$[s] = \{t \in \omega^{<\omega} \mid s \subseteq t\}.$$

Lemma 32: *If $A \subseteq \omega^{<\omega}$ does not have infinite antichains, then A can be covered with finitely many chains.*

Proof. Define S as the set of all unsplitting nodes of A, i.e., $s \in A$ if and only if every two extensions of s in A are compatible. Note that $S \subseteq A$ and every element of A can be extended to an element of S (otherwise A would contain a Sacks tree and hence an infinite antichain). Let $B \subseteq S$ be a maximal (finite) antichain. For every $s \in B$ let $b_s \in \omega^\omega$ be the unique branch such that $A \cap [s] \subseteq \widehat{b}_s$. Then (by the maximality of B) we conclude $A \subseteq \bigcup_{s \in B} \widehat{b}_s$. ∎

We need the following lemma:

Lemma 33: *If $A = \{A_n \mid n \in \omega\} \subseteq \wp(\omega^{<\omega})$ is a collection of infinite antichains, then there is an antichain B such that $B \cap A_n$ is infinite for infinitely many $n \in \omega$.*

Proof. We say $s \in \omega^{<\omega}$ **watches**A_n if s has infinitely many extensions in A_n. Define $T \subseteq \omega^{<\omega}$ such that $s \in T$ if and only if there are infinitely many $n \in \omega$ such that s watches A_n. Note that T is a tree. First assume there is $s \in T$ that is a maximal node. By shrinking A if needed, we may assume s watches every element of A. We now define the set $C = \{A_n \mid \exists^\infty m (A_n \cap [s^\frown m] \neq \emptyset)\}$. In case C is infinite, we can find an antichain B that has infinite intersection with every element of C. Now assume that C is finite; by shrinking A we may assume C is the empty set. In this way, for every A_n there is m_n such that $s^\frown m_n$ watches A_n. We can then find an infinite set $X \in [\omega]^\omega$ such that $m_n \neq m_r$ whenever $n \neq r$ and $n, r \in X$ (recall that s is maximal). Then $B = \bigcup_{n \in X} [s^\frown m_n] \cap A_n$ is the set we were looking for.

Now we may assume T does not have maximal nodes. If T contains a Sacks tree, then we can find an infinite antichain $Y \subseteq T$. For every $s \in Y$ we choose n_s such that s watches A_{n_s}, and if $s \neq t$ then $A_{n_s} \neq A_{n_t}$. Then $B = \bigcup_{s \in Y} [s] \cap A_{n_s}$ is the set we were looking for.

The only case left is that there is $s \in T$ that does not split in T and is not maximal. Let $f \in [T]$ be the only branch that extends s. We may assum+e s watches every element of A and every A_n is disjoint from $\widehat{f}$ (this is because A_n is an antichain and f is a branch). We say A_n is a **comb** with f if $\Delta(A_n \cap [s], \widehat{f})$ is infinite. We may assume that either every element of A is a comb with f or none is. In case all of them are combs we can easily find the desired antichain. So assume none of them are combs. In this way, for every $n \in \omega$ we can find t_n extending s but incompatible with f of minimal length such that t_n watches A_n. Since $t_n \notin T$ we can find $W \in [\omega]^\omega$ such that $t_n \neq t_m$ for all $n, m \in W$ where $n \neq m$. Then we recursively construct the desired antichain. ∎

We can then conclude the following:

PROPOSITION 34 (CH): *There is a weakly tight* MAD *family that is not +-Ramsey.*

Proof. Let $\{\overline{X}_\alpha \mid \omega \leq \alpha < \omega_1\}$ enumerate all countable sequences of infinite subsets of $\omega^{<\omega}$. Let $\mathcal{BR} = \{\widehat{f} \mid f \in \omega^\omega\}$. We construct $\mathcal{A} = \{A_\alpha \mid \alpha < \omega_1\}$ such that the following holds:

(1) Every A_α is an antichain.
(2) $\mathcal{A} \cup \mathcal{BR}$ is an AD family.
(3) If $s \in \omega^{<\omega}$, then $\mathcal{A}$ contains a partition of $\mathrm{suc}(s) = \{s^\frown n \mid n \in \omega\}$.
(4) For every $\omega \leq \alpha < \omega_1$, if $\overline{X}_\alpha = \{X_n \mid n \in \omega\} \subseteq \mathcal{I}(\mathcal{A}_\alpha \cup \mathcal{BR})^+$ then $A_\alpha \cap X_n$ is infinite for infinitely many $n \in \omega$ (where $\mathcal{A}_\alpha = \{A_\xi \mid \xi < \alpha\}$).

At step $\alpha = \{\alpha_n \mid n \in \omega\}$ assume $\overline{X}_\alpha = \{X_n \mid n \in \omega\} \subseteq (\mathcal{A}_\alpha \cup \mathcal{BR})^+$. We first claim that there is an infinite antichain $X_n' \subseteq X_n$ such that $X_n \in \mathcal{A}_\alpha^\perp$. Let $\Sigma = \{A \in \mathcal{A}_\alpha \mid |A \cap X_n| = \omega\}$. In case Σ is finite, by Lemma 32 we can find an infinite antichain $X_n' \subseteq X_n \setminus \bigcup \Sigma$. If Σ is infinite, then by Lemma 33 we can find an infinite $\Sigma' \subseteq \Sigma$ and $B_A \in [A \cap X_n]^\omega$ for $A \in \Sigma'$ such that $\bigcup\{B_A \mid A \in \Sigma'\}$ is an antichain. It is then easy to choose distinct $\{s_A \in B_A \mid A \in \Sigma'\}$ so that $X_n' = \{s_A \in B_A \mid A \in \Sigma'\} \in \mathcal{A}_\alpha^\perp$.

Let $Y_n = X_n' \setminus (A_{\alpha_0} \cup \cdots \cup A_{\alpha_n})$ which is an infinite antichain. By Lemma 33 we can find an antichain

$$A_\alpha \subseteq \bigcup_{n \in \omega} Y_n$$

such that $A_\alpha \cap Y_n$ is infinite for infinitely many $n \in \omega$.

Clearly $\mathcal{A} \cup \mathcal{BR}$ is not +-Ramsey (recall that weakly tight families are maximal). ∎

Recall that Miller indestructibility implies being +-Ramsey. We will now prove that (in particular) Sacks or random indestructibility are not enough to get +-Ramseyness. We will say a family $\mathcal{A}$ on $\omega^{<\omega}$ is a **standard $\mathcal{K}_\sigma$ family** if the following holds:

(1) $\mathcal{A}$ is an AD family.
(2) If $A \in \mathcal{A}$, either $\pi(A) = \emptyset$ or A is a finitely branching tree on $\omega^{<\omega}$.
(3) If $s \in \omega^{<\omega}$, then $\{s^\frown n \mid n \in \omega\} \in \mathcal{I}(\mathcal{A})^{++}$.

Recall that if $a \subseteq \omega^{<\omega}$, we denoted $\pi(a) = \{f \in \omega^\omega \mid \exists^\infty n (f \restriction n \in a)\}$. We now need the following lemma:

Lemma 35: *Let $\mathbb{P}$ be an ω^ω-bounding forcing and $\mathcal{A}$ a countable standard $\mathcal{K}_\sigma$ family. If $p \in \mathbb{P}$ and $\dot{b}$ is a $\mathbb{P}$-name for an infinite subset of $\omega^{<\omega}$ such that $p \Vdash$ "$\dot{b} \in \mathcal{A}^\perp$", then there are $q \leq p$ and $\mathcal{B}$ a countable standard $\mathcal{K}_\sigma$ family such that $\mathcal{A} \subseteq \mathcal{B}$ and $q \Vdash$ "$\dot{b} \notin \mathcal{B}^\perp$".*

Proof. Let $\mathcal{A} = \{T_n \mid n \in \omega\} \cup \{a_n \mid n \in \omega\}$ where T_n is a finitely branching subtree of $\omega^{<\omega}$ and $\pi(a_n) = \emptyset$ for every $n \in \omega$. We may assume that p forces that $\pi(\dot{b})$ is either empty or a singleton. We first assume there is $\dot{r}$ such that $p \Vdash$ "$\pi(\dot{b}) = \{\dot{r}\}$". Since $\mathbb{P}$ is ω^ω-bounding, we may find $p_1 \leq p$ and $T \in V$ a finitely branching well pruned subtree of $\omega^{<\omega}$ such that $p_1 \Vdash$ "$\dot{r} \in [T]$". Once again, since $\mathbb{P}$ is ω^ω-bounding we may find $p_2 \leq p_1$ and $f \in \omega^\omega$ such that the following holds:

(1) f is an increasing function.
(2) $p_2 \Vdash$ "$(T_n \cup a_n) \cap \widehat{r} \subseteq \omega^{<f(n)}$".

For each $n \in \omega$, define

$$\widetilde{T}_n = \{s \in T_n \mid f(n) \leq |s|\}$$

and define the set $\widetilde{a}_n$ as $\{t \mid \exists s \in a_n(s \in a_n \wedge f(n) \leq f(n))\}$. Let

$$K = T \setminus \bigcup_{n\in\omega}(\widetilde{T}_n \cup \widetilde{a}_n).$$

It is easy to see that K is a finitely branching tree, $p_2 \Vdash$ "$\dot{r} \in [K]$" and $K \in \mathcal{A}^\perp$. We now simply define $\mathcal{B} = \mathcal{A} \cup \{K\}$.

Now we consider the case where $\pi(\dot{b})$ is forced to be empty. Let $\dot{S}$ be the tree of all $s \in \omega^{<\omega}$ such that s has infinitely many extensions in $\dot{b}$. We will first assume there are $p_1 \leq p$ and s such that p_1 forces that s is a maximal node of $\dot{S}$. Since $\mathbb{P}$ is ω^ω-bounding, we can find a ground model interval partition $\mathcal{P} = \{P_n \mid n \in \omega\}$ and $p_2 \leq p_1$ such that if $n \in \omega$, then p_2 forces that there is $\dot{m}_n \in P_n$ such that $([s^\frown \dot{m}_n] \cap \dot{b}) \setminus (T_0 \cup \cdots \cup T_n \cup a_0 \cup \cdots \cup a_n) \neq \emptyset$. Given $n, m \in \omega$ we define $K_{n,m} = \{s^\frown i^\frown t \mid i \in P_n \wedge t \in m^m\}$. Using once again that $\mathbb{P}$ is ω^ω-bounding, we may find $p_3 \leq p_2$ and an increasing function $f : \omega \longrightarrow \omega$ such that if $n \in \omega$ then p_3 forces $(K_{n,f(n)} \cap \dot{b}) \setminus (T_0 \cup \cdots \cup T_n \cup a_0 \cup \cdots \cup a_n)$ is non-empty for every $n \in \omega$. We now define

$$a = \bigcup_{n\in\omega} K_{n,f(n)} \setminus (T_0 \cup \cdots \cup T_n \cup a_0 \cup \cdots \cup a_n).$$

It is easy to see that $\pi(a) = \emptyset$, $a \in \mathcal{A}^\perp$, and p_3 forces that a and $\dot{b}$ have infinite intersection.

Now we assume that p forces that $\dot{S}$ does not have maximal nodes. Let $\dot{r}$ be a name for a branch of $\dot{S}$. First assume that $\dot{r}$ is forced to be a branch through some element of $\mathcal{A}$. We may assume that $p \Vdash$ "$\dot{r} \in [T_0]$". Since $\mathbb{P}$ is ω^ω-bounding, we may find $p_1 \leq p$ and an increasing ground model function $f : \omega \longrightarrow \omega$ such that if $n \in \omega$, then p_1 forces that all extentions of $\dot{r} \restriction f(n)$ to $\dot{b}$ are not in $T_0 \cup \cdots \cup T_n \cup a_0 \cup \cdots \cup a_n$. Once again, we may find $p_2 \leq p_1$ and $g : \omega \longrightarrow \omega$ such that if $n \in \omega$, then $\dot{b}$ has a non-empty intersection with the set $\{\dot{r} \restriction f(n)^\frown t \mid t \in g(n)^{g(n)}\} \setminus (T_0 \cup \cdots \cup T_n \cup a_0 \cup \cdots \cup a_n)$. We now define

$$a = \bigcup_{s \in (T_0)_{f(n)}} (\{s^\frown t \mid t \in g(n)^{g(n)}\} \setminus (T_0 \cup \cdots \cup T_n \cup a_0 \cup \cdots \cup a_n)).$$

It is easy to see that a has the desired properties.

Finally, in case that $\dot{r}$ is not forced to be a branch through some element of $\mathcal{A}$, we find a finitely branching tree $T \in \mathcal{A}^\perp$ such that $p \Vdash$ "$\dot{r} \in [T]$" as we did at the beginning of the proof. If T has infinite intersection with $\dot{b}$ we are done, and if not then we apply the previous case. ∎

With a standard bookkeeping argument we can then conclude the following:

Proposition 36 (CH): *If $\mathbb{P}$ is a proper ω^ω-bounding forcing of size ω_1, then there is a* MAD *family $\mathcal{A}$ that is $\mathbb{P}$ indestructible but is not +-Ramsey.*

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THERE ARE NO P-POINTS IN SILVER EXTENSIONS

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ABSTRACT. We prove that after adding a Silver real no ultrafilter from the ground model can be extended to a P-point, and this remains to be the case in any further extension which has the Sacks property. We conclude that there are no P-points in the Silver model. In particular, it is possible to construct a model without P-points by iterating Borel partial orders. This answers a question of Michael Hrušák. We also show that the same argument can be used for the side-by-side product of Silver forcing. This provides a model without P-points with the continuum arbitrary large, answering a question of Wolfgang Wohofsky.

The first author dedicates this work to his teacher, mentor and dear friend Bohuslav Balcar. The crucial result was proved on the day of his passing.

INTRODUCTION

Ultrafilters on countable sets have become of great importance in infinite combinatorics. A non-principal ultrafilter $\mathscr{U}$ is called a *P-point* if every countable subset of $\mathscr{U}$ has a pseudointersection in $\mathscr{U}$. Recall that a set $X \subseteq \omega$ is called a *pseudointersection* of a family $\mathscr{B} \subseteq [\omega]^\omega$ if $X \setminus B$ is finite for every $B \in \mathscr{B}$. Ultrafilters of this special type have been extensively studied in set theory and topology. Walter Rudin in 1956 (see [Rud56]) proved that the topological space $\omega^* = \beta\omega \setminus \omega$ is not homogeneous assuming the continuum hypothesis CH. It is well known that the non-principal ultrafilters correspond in a natural way to points of ω^* and P-points are exactly points with neighborhoods closed under countable intersections. Rudin proved the non-homogeneity of ω^* using the following argument: CH implies that P-points exist, ultrafilters that are not P-points always exist, and a P-point and a non-P-point have different topological types. Frolík established in 1967 (see [Fro67]) that ω^* is not homogeneous without the need of the continuum hypothesis. Although Frolík's proof does not provide any specific types of ultrafilters, various distinct topological types

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of ultrafilters were later identified even without using any additional set-theoretic assumptions, see [Kun80, vM82, Ver13].

Nevertheless, P-points remain one of the central objects of research of set theorists and topologists. P-points are fundamental in forcing theory; most of the methods of preserving an ultrafilter in generic extensions require preserving some kind of a P-point, the reader may consult e.g. [BJ95, Zap09] for more details. They also appear in the study of the Tukey order [Dob15], partition calculus [BT78], model theory [Bla73], and other topics. The study of P-points is still a rich and active area of study, the reader may consult e.g. [Boo71, BHV13, RS17] for some more results and applications of P-points.

A well known result of Ketonen states that it is possible to construct a P-point if the dominating number is equal to the size of the continuum; $\mathfrak{d} = \mathfrak{c}$[1] [Ket76]. It is also possible to construct P-points if the parametrized diamond principle $\Diamond(\mathfrak{r})$[2] holds, see [MHD04] for more information on parametrized diamond principles. On the other hand a remarkable theorem of Shelah states that the existence of P-points cannot be proved using just the axioms of ZFC alone. This result was proved in 1977 and first published in [Wim82]. The reader may find the proof in [She98]. The model of Shelah is obtained by iterating the Grigorieff forcing with parameters ranging over non-meager P-filters.

Independence results are often demonstrated in models obtained by employing forcing iterations of definable posets. One possible formalization of such canonical models is treated in [MHD04]. We say that a partial order $(P, \leq)$ is Borel if there is a Polish space X such that P is a Borel subset of X, and $\leq$ is a Borel subset of $X \times X$. A *canonical model* is a model obtained by performing a countable support forcing iteration of Borel proper partial orders of length ω_2. At the *Forcing and its applications retrospective workshop* held at the Fields Institute in 2015 Michael Hrušák posed the following problem.

Problem. Do P-points exist in every canonical model?

A canonical model will contain a P-point if the steps of the iteration add unbounded reals or if no splitting reals are added—in the resulting model either $\mathfrak{d} = \mathfrak{c}$ or $\Diamond(\mathfrak{r})$ does hold. Consequently, one only needs to consider Borel ω^ω-bounding forcing notions which do add splitting reals. The best

[1] The dominating number $\mathfrak{d}$ is the least cardinality of a set of functions in ω^ω such that every function is eventually dominated by a member of that set. $\mathfrak{c}$ is the cardinality of the continuum.

[2] The *reaping number* $\mathfrak{r}$ is the smallest size of a family $\mathscr{R} \subseteq [\omega]^\omega$ such that for every $X \in [\omega]^\omega$ there is $R \in \mathscr{R}$ such that either $R \subseteq X$ or $R \subseteq \omega \setminus X$. For more information on the reaping number and cardinal characteristics of the continuum in general, the reader may consult [Bla10]. The formulation of the associated diamond principle $\Diamond(\mathfrak{r})$ is somewhat involved and since it is not used in the present paper, it is omitted.

known examples of this type of forcing are the random poset and the Silver poset. We answer the question of Hrušák in negative; Theorem 6 states that there are no P-points in the Silver model.

In [Coh79] it was claimed that there is a P-point in the random model. Unfortunately, the presented proof is incorrect and the existence of P-points in the random model is presumably unknown. We will address this issue in the Appendix section.

Problem. Are there P-points in the random model?

The existence of a model without P-points with the continuum larger that ω_2 was an open question [Woh08]. Theorem 7 states that forcing with the side-by-side product of Silver forcing produces such a model.

Our notation and terminology is mostly standard, including some folklore abuse of notation. When p is a partial function from ω to 2, we denote this by $p; \omega \to 2$ and we write $p^{-1}(1)$ instead of $p^{-1}[\{1\}]$. We say that $\mathscr{I} = \{ I_n \mid n \in \omega \}$ is an *interval partition* if there is an increasing sequence of natural numbers $\langle m_n \rangle_{n \in \omega}$ such that $m_0 = 0$ and $I_n = [m_n, m_{n+1})$.

We say that a forcing notion $\mathbf{P}$ has the *Sacks property* if for every $p \in \mathbf{P}$ and for every f such that $p \Vdash \dot{f} \in \omega^\omega$ there is $q \leq p$ and $\{ X_n \mid n \in \omega \}$ such that $X_n \in [\omega]^{n+1}$ for every $n \in \omega$, and $q \Vdash \dot{f}(n) \in X_n$ for each $n \in \omega$. It is a common practise to require in the definition of the Sacks property that $X_n \in [\omega]^{2^n}$, instead of $X_n \in [\omega]^{n+1}$ as we demanded. Nevertheless, both resulting notions are equivalent, see e.g. [GQ04, section 3].

If $p; \omega \to 2$ is a partial function we denote by by $[p]$ the set of all total function extending p, i.e. $[p] = \{ f \in 2^\omega \mid p \subseteq f \}$.

Destroying P-points with Silver reals

For a partial function $p; \omega \to 2$ we denote $\operatorname{dom} p$ the domain of p and $\operatorname{cod} p = \omega \setminus \operatorname{dom} p$. We denote the *Silver forcing* (after Jack Howard Silver, see [Mat79]) by $\mathbf{PS}$. Some authors also call this poset the Prikry–Silver forcing. It consists of all partial functions $p; \omega \to 2$ such that $\operatorname{cod} p$ is infinite, and relation $p \leq q$ is defined as $q \subseteq p$. We will always assume that $p^{-1}(1)$ is infinite for each $p \in \mathbf{PS}$, such conditions form a dense subset of the poset. If G is a generic filter for the Silver forcing, the *Silver generic real* is defined as $r = \bigcap \{ [p] \mid p \in G \}$. It is well known and easy to see that G and r can be defined from each other. A typical application of the Silver forcing is to demonstrate that the inequality $\operatorname{cof} \mathscr{N} < \mathfrak{r}$ is consistent.[3] The reader may consult [Hal17] for an introduction and more information regarding the Silver forcing. It is well known that the Silver forcing is proper and has the Sacks property.

For a partial (or total) function $p; \omega \to 2$ define an interval partition of ω by letting $I_n(p) = \left\{ k \in \omega \mid \left| k \cap p^{-1}(1) \right| = n \right\}$ for $n \in \omega$ and $\mathscr{I}(p) =$

[3] $\mathscr{N}$ is the ideal of Lebesgue null subsets of the real line.

$\{ I_n(p) \mid n \in \omega \}$. Note that if q extends p, then $\mathscr{I}(q)$ refines $\mathscr{I}(p)$, i.e. every interval of $\mathscr{I}(p)$ is the union of intervals of $\mathscr{I}(q)$. Moreover, if r is the generic real, then $\mathscr{I}(r)$ refines $\mathscr{I}(p)$ for every p in the generic filter. The proofs of the following simple observations are left for the reader.

Lemma 1. *Let $p \in \mathbf{PS}$ and $k \in \omega$ be such that $I_k(p) \subseteq \operatorname{dom} p$.*

(1) *If $q \leq p$, then $I_k(p) \in \mathscr{I}(q)$.*
(2) *$p \Vdash I_k(p) \in \mathscr{I}(\dot{r})$ (where $\dot{r}$ is the name for the generic real).*

Lemma 2. *Assume that $p, q \in \mathbf{PS}$ and $k, n \in \omega$ are such that*

(1) *$q \leq p$,*
(2) *$I_k(p) \in \mathscr{I}(q)$, and*
(3) *$\left|q^{-1}(1) \cap \min I_k(p)\right| = \left|p^{-1}(1) \cap \min I_k(p)\right| + n$.*

Then $I_k(p) = I_{k+n}(q)$.

As a consequence of these observations we conclude the following.

Corollary 3. *If $p \in \mathbf{PS}$ and $k \in \omega$ are such that $I_k(p) \subseteq \operatorname{dom} p$, then p forces that: There is $n \in \omega$, $n \leq |\operatorname{cod} p \cap \min I_k(p)|$ such that $I_k(p) = I_{k+n}(\dot{r})$ (where $\dot{r}$ is the name for the generic real).*

By $-_n$ and $=_n$ we denote the subtraction operation and congruence relation modulo n. The notation $k \in_n X$ is interpreted as 'there is $x \in X$ such that $k =_n x$.' For $X, Y \subset n$ we write $X -_n Y = \{ x -_n y \mid x \in X, y \in Y \}$.

Lemma 4. *For each $n \in \omega$ there exists $k(n) \in \omega$ such that for each set $C \in [k(n)]^n$ there exists $s \in k(n)$ such that $C \cap \left(C -_{k(n)} \{ s \}\right) = \emptyset$.*

Proof. If s does not satisfy the conclusion of the lemma, then $s \in C -_{k(n)} C$. As $\left|C -_{k(n)} C\right| \leq n^2$, any choice of $k(n) > n^2$ works as desired. □

The following proposition contains the main technical argument central for the results of this paper.

Proposition 5. *Let $\mathscr{U}$ be a non-principal ultrafilter and $\dot{\mathbb{Q}}$ be a $\mathbf{PS}$-name for a forcing such that $\mathbf{PS} * \dot{\mathbb{Q}}$ has the Sacks property. If $G \subset \mathbf{PS} * \dot{\mathbb{Q}}$ is a generic filter over V, then $\mathscr{U}$ cannot be extended to a P-point in $V[G]$.*

Before giving the formal the proof of the Proposition, let us sketch the core idea of the argument. The basic approach is the same as in the no-P-points proof of Shelah from [She98]. We will show that in order to extend $\mathscr{U}$ to an ultrafilter in the generic extension, one would need to add to $\mathscr{U}$ a particular countable set $\mathscr{D}$ of newly introduced subsets of ω, and at the same time there is no way to add to $\mathscr{U}$ also the pseudointersection of $\mathscr{D}$; for each pseudointersection Z of $\mathscr{D}$ there is $U \in \mathscr{U}$ such that $Z \cap U = \emptyset$.

The sets in $\mathscr{D}$ will be chosen as the typical independent reals added by the Silver forcing. Let r be the generic real, define d^n_i as the union of intervals $I_j(r)$ such that $j =_n i$. Although it is easy to see that each such

d^n_i is an independent real, this fact will not be explicitly needed in our argument and is therefore left for the interested reader to observe. For a fixed n the sets d^n_i form a partition of ω into n pieces, and consequently each ultrafilter extending $\mathscr{U}$ needs to contain one element of this partition, denote this set $d^n_{y(n)}$. We will show that the set $\mathscr{D} = \left\{ d^n_{y(n)} \mid n \in \omega \right\}$ works as desired.

The argument for non-existence of pseudointersections will go along the lines of the following simple claim.

Claim. *Suppose $\mathscr{D} = \{ d^n \mid n \in \omega \}$ is a subset of an ultrafilter $\mathscr{U}$ with the following property. For every function $f : \omega \to \omega$ there is an interval partition $\{ a_n \mid n \in \omega \}$ such that*

- *$f(n) < \min a_{n+1}$ for each $n \in \omega$, and*
- *$\bigcup \{ a_n \cap d^n \mid n \in \omega \} \notin \mathscr{U}$.*

Then $\mathscr{D}$ does not have a pseudointersection in $\mathscr{U}$, and consequently $\mathscr{U}$ is not a P-point.

Although the details of the sketched idea will be for technical reasons somewhat adjusted, e.g. we will use only a subset of the above defined set $\mathscr{D}$, the formal proof of Proposition 5 will roughly follow the described argument.

Proof. First we use the function k from Lemma 4 to inductively construct two increasing sequences of integers. Put $v(0) = 0$ and $m(0) = k(2)$. Assume $v(n-1)$, $m(n-1)$ are defined, put $v(n) = \sum \{ m(i) \mid i \in n \}$ and $m(n) = k\big((n+1)(v(n)+2)\big)$. Let r be the **PS** generic real in $V[G]$ added by the first stage of the iteration. For $n \in \omega$ and $i \in m(n)$ let

$$D^n_i = \bigcup \left\{ I_j(r) \mid j \in \omega, j =_{m(n)} i \right\}.$$

For a fixed n the set $\left\{ D^n_i \mid i < m(n) \right\}$ is a finite partition of ω. We will show that in $V[G]$, for every function $y \colon \omega \to \omega$ which satisfies $y(n) < m(n)$ for every $n \in \omega$, and every pseudointersection Z of $\left\{ D^n_{y(n)} \mid n \in \omega \right\}$ there is a set $U \in \mathscr{U}$ such that $U \cap Z = \emptyset$. This implies that $\mathscr{U}$ cannot be extended to a P-point in $V[G]$.

Let $(p, \dot q)$ be any condition in $\mathbf{PS} * \dot Q$, and let $\dot Z$, $\dot y$ be the corresponding names for Z and y. Utilizing the Sacks property we can assume that there are $f : \omega \to \omega$ and $\left\{ X_n \in [m(n)]^{n+1} \mid n \in \omega \right\}$ in V such that

$$(p, \dot q) \Vdash \big(\dot Z \setminus f(n) \big) \subseteq D^n_{\dot y(n)} \text{ and } \dot y(n) \in X_n.$$

Choose an interval partition $\mathscr{A} = \{ A_n \mid n \in \{-1\} \cup \omega \}$ of ω ordered in the natural way such that

(1) $f(n) < \min A_{2n}$ for each $n \in \omega$,
(2) $m(n) < \left| A_{2n+j} \cap \operatorname{cod} p \right|$ for each $n \in \omega$, $j \in 2$, and
(3) $\mathscr{I}(p)$ refines $\mathscr{A}$.

We will assume that $U_0 = \bigcup\{A_{2n+1} \mid n \in \omega\} \in \mathscr{U}$, otherwise take the interval partition $\mathscr{A}' = \langle A_{-1} \cup A_0, A_1, A_2, \dots\rangle$ instead.[4] The plan is to use the trace of extensions of p on the interval A_{2n} to control the possible behavior of the set $D^n_{y(n)} \cap A_{2n+1}$ for all $n \in \omega$ simultaneously.

Let $p_1 \in \mathbf{PS}$ be any extension of p such that $A_{2n-1} \subseteq \operatorname{dom} p_1$ and $|\operatorname{cod} p_1 \cap A_{2n}| = m(n)$ for each $n \in \omega$. Note that for any $j \in \omega$ if $I_j(p_1) \subseteq A_{2n-1}$, then $p_1 \Vdash I_j(p_1) \in \mathscr{I}(\dot{r})$. Also note that $|\operatorname{cod} p_1 \cap \min A_{2n}| = \nu(n)$ for each $n \in \omega$. Let

$$C_n = X_n -_{m(n)} \{\, i \mid i \in \nu(n)+2\,\}$$

and notice that $|C_n| \le (n+1)(\nu(n)+2)$. For $n \in \omega$ put

$$H_n = A_{2n+1} \cap \bigcup\Big\{\, I_j(p_1) \mid j \in \omega, j \in_{m(n)} C_n \,\Big\}.$$

We will now distinguish two cases. Case 1; $\bigcup\{H_n \mid n \in \omega\} \notin \mathscr{U}$, hence $U = \bigcup\{A_{2n+1} \setminus H_n \mid n \in \omega\} \in \mathscr{U}$. Pick any $p_2 < p_1$, $p_2 \in \mathbf{PS}$ such that $p_2^{-1}(1) = p_1^{-1}(1)$ and $|\operatorname{cod} p_2 \cap A_{2n}| = 1$ for each $n \in \omega$. Notice that $\mathscr{I}(p_1) = \mathscr{I}(p_2)$, $|\operatorname{cod} p_2 \cap \min(A_{2n+1})| = n+1$, and if $j \in \omega$ is such that $I_j(p_2) \subseteq A_{2n+1}$, then $I_j(p_2) \subseteq \operatorname{dom} p_2$. For each $n \in \omega$ Corollary 3 states that p_2 forces: There is $i \le n+1$ such that for each $j \in \omega$ if $I_j(p_1) \subseteq A_{2n+1}$, then $I_j(p_1) = I_{j+i}(\dot{r})$. As $(p_2, \dot{q})$ forces $\dot{y}(n) \in X_n$, it follows that if $I_j(p_1) = I_{j+i}(r) \subseteq D^n_{y(n)} \cap A_{2n+1}$, then $j \in_{m(n)} \big(X_n -_{m(n)} \{\, i \mid i \in n+2\,\}\big) \subseteq C_n$. We can conclude that:

$$(p_2, \dot{q}) \Vdash D^n_{\dot{y}(n)} \cap A_{2n+1} \subset H_n.$$

This together with

$$(p, \dot{q}) \Vdash (\dot{Z} \setminus \min A_{2n}) \subseteq D^n_{\dot{y}(n)}$$

implies that $(p_2, \dot{q}) \Vdash \dot{Z} \cap U = \emptyset$.

Case 2; $U = \bigcup\{H_n \mid n \in \omega\} \in \mathscr{U}$. Applying Lemma 4, for each $n \in \omega$ there exists $s_n \in m(n)$ such that $C_n \cap \big(C_n -_{m(n)} \{s_n\}\big) = \emptyset$. Put $t(n) = \sum\{s_i \mid i \in n\} \le \nu(n) - n$. Pick a condition $p_2 < p_1$, $p_2 \in \mathbf{PS}$ such that $|\operatorname{cod} p_2 \cap A_{2n}| = 1$ and

$$\left|p_2^{-1}(1) \cap A_{2n}\right| = \left|p_1^{-1}(1) \cap A_{2n}\right| + s_n$$

for each $n \in \omega$. Such p_2 exists as $|\operatorname{cod} p_1 \cap A_{2n}| = m(n)$. Note that in this case $|\operatorname{cod} p_2 \cap \min(A_{2n+1})| = n+1$, and if $j \in \omega$ is such that $I_j(p_2) \subseteq A_{2n+1}$, then $I_j(p_2) \subseteq \operatorname{dom} p_2$ and $I_j(p_2) = I_{j-t(n+1)}(p_1)$. For each $n \in \omega$ Corollary 3 implies that p_2 forces: There is $i \le n+1$ such that for each $j \in \omega$ if

[4] In the following proof, we will use the second assumption on the interval partition $\mathscr{A}$ only for $j = 0$. Notice however, that assuming it only for $j = 0$ at the moment of choosing $\mathscr{A}$ would not have been sufficient as if it were the case that $U_0 \notin \mathscr{U}$, we would be working with the partition $\mathscr{A}'$ instead, and $\mathscr{A}'$ would not be fulfilling the necessary requirement. The observant reader may also notice that the last assumption on $\mathscr{A}$ will in fact not be necessary in the proof.

$I_j(p_1) \subseteq A_{2n+1}$, then $I_j(p_1) = I_{j+t(n+1)+i}(\dot{r})$. As $(p_2, \dot{q})$ forces $\dot{y}(n) \in X_n$, it follows that if $I_j(p_1) = I_{j+t(n+1)+i}(r) \subseteq D^n_{y(n)} \cap A_{2n+1}$, then

$$\begin{aligned} j \in_{m(n)} \left(X_n -_{m(n)} \{\, t(n+1) \,\}\right) -_{m(n)} \{\, i \mid i \in n+2 \,\} = \\ = \left(\left(X_n -_{m(n)} \{\, t(n) \,\}\right) -_{m(n)} \{\, i \mid i \in n+2 \,\}\right) -_{m(n)} \{\, s_n \,\} \subseteq \\ \subseteq \left(X_n -_{m(n)} \{\, i \mid i \in v(n)+2 \,\}\right) -_{m(n)} \{\, s_n \,\} = C_n -_{m(n)} \{\, s_n \,\}. \end{aligned}$$

For $n \in \omega$ put

$$\bar{H}_n = A_{2n+1} \cap \bigcup \left\{\, I_j(p_1) \mid j \in \omega, j \in_{m(n)} \left(C_n -_{m(n)} \{\, s_n \,\}\right) \,\right\},$$

$H_n \cap \bar{H}_n = \emptyset$, because if $j \in_{m(n)} C_n$, then $j \notin_{m(n)} C_n -_{m(n)} \{\, s_n \,\}$.

Now

$$(p_2, \dot{q}) \Vdash D^n_{\dot{y}(n)} \cap A_{2n+1} \subset \bar{H}_n.$$

Again, together with

$$(p, \dot{q}) \Vdash (\dot{Z} \setminus \min A_{2n}) \subseteq D^n_{\dot{y}(n)}$$

we get $(p_2, \dot{q}) \Vdash \dot{Z} \cap U = \emptyset$. □

The Silver model is the result of a countable support iteration of Silver forcing of length ω_2.

Theorem 6. *There are no P-points in the Silver model.*

Proof. Denote by $\mathbf{PS}_\alpha$ the countable support iteration of Silver forcing of length α for $\alpha \leq \omega_2$. Assume V is a model of CH and let $G \subset \mathbf{PS}_{\omega_2}$ be a generic filter. Let $\mathscr{U} \in V[G]$ be a non-principal ultrafilter. For $\alpha < \omega_2$ let $\mathscr{U}_\alpha = \mathscr{U} \cap V[G_\alpha]$, where G_α is the restriction of G to $\mathbf{PS}_\alpha$. By the standard reflection argument, there is $\alpha < \omega_2$ such that $\mathscr{U}_\alpha \in V[G_\alpha]$ and it is an ultrafilter in that model. Since the next step of the iteration adds a Silver real and the tail of the iteration has the Sacks property, Proposition 5 states that $\mathscr{U}_\alpha$ cannot be extended to a P-point in $V[G]$, in particular, $\mathscr{U}$ is not a P-point. □

We show that forcing with the side-by-side product of Silver forcing also produces a model without P-points.

Theorem 7. *Assume* GCH*, let $\kappa > \omega_1$ be a cardinal with uncountable cofinality. If $\bigotimes_\kappa \mathbf{PS}$ is the countable support product of κ many Silver posets and $G \subset \bigotimes_\kappa \mathbf{PS}$ is a generic filter, then*

$$V[G] \models \text{ there are no P-points and } \mathfrak{c} = \kappa.$$

Proof. It is well known that under GCH the poset $\bigotimes_\kappa \mathbf{PS}$ is an ω_2-c.c. proper forcing notion, has the Sacks property (see e.g. [Kos92]), and $V[G] \models \mathfrak{c} = \kappa$. Assume $\mathscr{U}$ is an ultrafilter in $V[G]$. Since $\bigotimes_\kappa \mathbf{PS}$ is ω_2-c.c., there is $J \subset \kappa$ of size ω_1 such for every $A \in \mathscr{P}(\omega) \cap V$ and $q \in \bigotimes_\kappa \mathbf{PS}$ the statement $A \in \mathscr{U}$ is decided by a condition with support contained in

J and compatible with q. Choose $\alpha \in \kappa \setminus J$ and let r be the **PS** generic real added by the α-th coordinate of the product.

The theorem is now proved in the same way as Proposition 5; as the proof follows most parts of the proof of Proposition 5 in verbatim, we will focus in detail only on the points where adjustments are necessary.

Start with defining the functions v and m, consider sets D_i^n defined from the generic real r, and pick any $\bigotimes_\kappa \mathbf{PS}$ names $\dot{Z}, \dot{y}$. Let (p,q) be any condition in $\mathbf{PS} \times \bigotimes_{\kappa\setminus\{\alpha\}} \mathbf{PS} = \bigotimes_\kappa \mathbf{PS}$ which forces that $\mathscr{U}$ is a non-principal ultrafilter; we interpret $p \in \mathbf{PS}$ as the α-th coordinate and q as the other coordinates of a condition in the full product poset. We invoke the Sacks property of $\bigotimes_\kappa \mathbf{PS}$ to assume the existence of an appropriate function f and a sequence $\{X_n \mid n \in \omega\}$. Choose the interval partition $\mathscr{A}$ satisfying properties (1–3) with respect to p and consider $U_0 = \bigcup\{A_{2n+1} \mid n \in \omega\}$. As $U_0 \in V$, there is a condition $(p,q_1) < (p,q)$ deciding whether U_0 is an element of $\mathscr{U}$, because of the choice of coordinate α. We will assume that $(p,q_1) \Vdash U_0 \in \mathscr{U}$, otherwise take the interval partition $\mathscr{A}'$ instead. Follow with choosing the condition p_1 extending p, define the sets C_n and H_n for $n \in \omega$.

Now consider the set $H = \bigcup\{H_n \mid n \in \omega\}$. As $H \in V$, there is $(p_1,q_2) < (p_1,q_1)$ deciding whether $H \in \mathscr{U}$. Case 1; $(p_1,q_2) \Vdash H \notin \mathscr{U}$. Now proceed again in verbatim as in case 1 of the proof of Proposition 5; define U, choose $p_2 < p_1$, and conclude that $(p_2,q_2) \Vdash \dot{Z} \cap U = \emptyset$.

Case 2; $(p_1,q_2) \Vdash H \in \mathscr{U}$. Proceed again as in case 2 of the proof of Proposition 5; define U, find s_n for each $n \in \omega$, and choose $p_2 < p_1$. And finally conclude $(p_2,q_2) \Vdash \dot{Z} \cap U = \emptyset$. □

Concluding remarks

Theorem 6 can be stated in an axiomatic manner. Recall that $\mathscr{N}$ denotes the ideal of Lebesgue null sets and let v_0 be the ideal associated with the Silver forcing;

$$v_0 = \{A \subset 2^\omega \mid \forall p \in \mathbf{PS}\ \exists q \in \mathbf{PS}, q < p, [q] \cap A = \emptyset\}.$$

This ideal was introduced in [CRSW93] and studied in [Bre95, DPH00]. The proof of Proposition 5 can be reformulated to yield the following theorem, the detailed proof is provided in [Guz17].

Theorem 8. *The inequality* $\operatorname{cof}\mathscr{N} < \operatorname{cov} v_0$ *implies that there are no P-points.*

An alternative version of results of this paper was suggested by Jonathan Verner. The side-by-side product $\bigotimes_\omega \mathbf{PS}$ adds a Silver generic real r_α for each coordinate $\alpha \in \omega$. Consider the pair of complementary splitting reals $X_\alpha^i = \bigcup\{I_{2n+i}(r_\alpha) \mid n \in \omega\}$, $i \in 2$; an argument similar to (and less technical than) the proof of Proposition 5 demonstrates the following.

Claim. *Let $\mathcal{U}$ be a non-principal ultrafilter. The product $\bigotimes_\omega$ **PS** forces that no pseudo-intersection of $\left\{ X_\alpha^{i(\alpha)} \mid \alpha \in \omega \right\}$ is $\mathcal{U}$-positive, and this remains to be the case in each further Sacks property extension.*

Furthermore, it is possible to reason along the lines of the proof of Theorem 7 to obtain a stronger version of the theorem. These results are to be included in forthcoming publications.

Announcement 9. *Assume* GCH*, let $\kappa > \omega_1$ be a cardinal with uncountable cofinality. If $\bigotimes_\kappa$ **PS** is the countable support product of κ many Silver posets and $G \subset \bigotimes_\kappa$ **PS** is a generic filter, then*

$$\begin{aligned} V[G] \models\ & \textit{For every non-principal ultrafilter } \mathcal{U} \textit{ there exists} \\ & \{X_\alpha \mid \alpha \in \mathfrak{c}\} \subset \mathcal{U} \textit{ such that for each } y \in [\mathfrak{c}]^\omega \cap V \\ & \textit{no pseudointersection of } \{X_\alpha \mid \alpha \in y\} \textit{ is an element of } \mathcal{U}. \end{aligned}$$

The motivation for stating this theorem comes from the problem of Isbell [Isb65] which asks for the existence of two Tukey non-equivalent ultrafilters on ω. The problem can be equivalently formulated as a statement resembling the conclusion of Announcement 9, see [DT11].

Problem (Isbell)**.** Is it consistent that for each non-principal ultrafilter $\mathcal{U}$ on ω there exists $\mathcal{X} \in [\mathcal{U}]^\mathfrak{c}$ such that for each $\mathcal{Y} \in [\mathcal{X}]^\omega$ is $\bigcap \mathcal{Y} \notin \mathcal{U}$?

Appendix

At the request of the referee, we address here the situation concerning the random model. We point out the issue in the argument in [Coh79] used to reason for the existence of P-points in the random model. The reader may consult [FBH17, FB] for more information.

It is an unpublished result of K. Kunen that if ω_1 many Cohen reals are added to the ground model followed by adding ω_2 many random reals, the resulting random model will contain a P-point. Recently A. Dow proved that P-points exist in the random model provided CH and $\square_{\omega_1}$ does hold in the ground model [Dow18].

The construction in [Coh79] uses the notion of a pathway. For a recent development and general treatment of pathways see [FB].

Definition 10. A sequence $\{A_\alpha \mid \alpha \in \kappa\}$ is a *pathway* if the following conditions hold.

(1) $\omega^\omega = \bigcup \{A_\alpha \mid \alpha \in \kappa\}$,
(2) $A_\alpha \subseteq A_\beta$ for $\alpha < \beta$,
(3) A_α does not dominate $A_{\alpha+1}$,[5]
(4) if $f, g \in A_\alpha$, then $(f \operatorname{join} g) \in A_\alpha$ (where $(f_0 \operatorname{join} f_1) \in \omega^\omega$ is defined by $(f_0 \operatorname{join} f_1)(2n+i) = f_i(n)$),

[5] I.e. there is a function in $A_{\alpha+1}$ not eventually dominated by any element of A_α.

(5) if g is Turing reducible to f and $f \in A_\alpha$, then $g \in A_\alpha$.

The following is [Coh79, Theorem 1.1].

Theorem 11. *The existence of a pathway implies the existence of a P-point.*

This result is a useful tool for proving the existence of P-points in certain models. In order to prove that there is a P-point in the random model (i.e. the model obtained by adding ω_2 random reals to a model of CH), the author of [Coh79] aims to construct a pathway in the generic extension. We do not know whether there are pathways in this model. The construction from [Coh79] does not work, as we will demonstrate.

We denote $\mathbf{B}$ the random forcing and $\mathbf{B}(\omega_2)$ the poset for adding ω_2 many random reals. It is well known that if M is a countable elementary submodel of $H(\theta)$ (for some sufficiently large cardinal θ), $r\colon \omega_2 \to 2$ is a $\mathbf{B}(\omega_2)$-generic function over V, and $\pi\colon \omega \to \omega_2$ is an injective function in V (but not necessarily M), then $M[r \circ \pi]$ is a $\mathbf{B}$-generic extension of M (see [Coh79] for more details).

We outline the construction in [Coh79]. Using CH in the ground model V find $\{M_\alpha \mid \alpha \in \omega_1\}$, an increasing chain of countable elementary submodels of $H(\theta)$ such that $\omega^\omega \subset \bigcup\{M_\alpha \mid \alpha \in \omega_1\}$. Let $r\colon \omega_2 \to 2$ be a $\mathbf{B}(\omega_2)$-generic function over V. Work in $V[r]$; let Π be the set of all injective functions from ω to ω_2 in V. For every $\alpha < \omega_1$ define $A_\alpha = \bigcup\{\omega^\omega \cap M_\alpha[r \circ \pi] \mid \pi \in \Pi\}$. The argument in [Coh79] relies on $\{A_\alpha \mid \alpha \in \omega_1\}$ being a pathway. We show that this is not the case.

Fix $\mathscr{P} = \{P_n \mid n \in \omega\} \subseteq [\omega]^\omega$ a partition of ω and let $\mathrm{Q} = \{q_n \mid n \in \omega\}$ be an enumeration of the rational numbers. Furthermore, we take both $\mathscr{P}$ and the enumeration of Q to be definable. For $f, g\colon \omega \to 2$ we define $f \star g\colon \mathrm{Q} \to 2$ by declaring $f \star g(q_n) = 1$ if and only if $f \restriction P_n = g \restriction P_n$. The following proposition implies that no A_α is closed under the join operation.

Proposition 12. *Let $r\colon \omega_2 \to 2$ be a $\mathbf{B}(\omega_2)$-generic function over V, and let M be a countable elementary submodel of $H(\theta)$. There are $\pi_0, \pi_1 \in \Pi$ such that there is no $\sigma \in \Pi$ for which $M[r \circ \pi_0] \cup M[r \circ \pi_1] \subseteq M[r \circ \sigma]$.*

Proof. Let $\delta = M \cap \omega_1$. Since δ is countable ordinal, there is $S \subseteq \mathrm{Q}$ order isomorphic to δ. Now choose the functions $\pi_0, \pi_1 \in \Pi$ such that the following holds:

- If $q_n \in S$, then $\pi_0 \restriction P_n = \pi_1 \restriction P_n$.
- If $q_n \notin S$, then $\pi_0[P_n] \cap \pi_1[P_n] = \emptyset$.

Recall that both $M[r \circ \pi_0]$ and $M[r \circ \pi_1]$ are $\mathbf{B}$-generic extensions of M. Assume that $\{r \circ \pi_0, r \circ \pi_1\} \subset M[r \circ \sigma]$ for some $\sigma \in \Pi$. Then also $(r \circ \pi_0) \star (r \circ \pi_1) \in M[r \circ \sigma]$, and a simple genericity argument implies $((r \circ \pi_0) \star (r \circ \pi_1))^{-1}(1) = S \in M[r \circ \sigma]$. Now $\delta \in M[r \circ \sigma]$ is a contradiction with $M[r \circ \sigma]$ being a generic extension of M. $\square$

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The ultrafilter and almost disjointness numbers

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Abstract

We prove that every MAD family can be destroyed by a proper forcing that preserves P-points. With this result, we prove that it is consistent that $\omega_1 = \mathfrak{u} < \mathfrak{a}$, solving a nearly 20 year old problem of Shelah and a problem of Brendle. We will also present a simple proof of a result of Blass and Shelah that the inequality $\mathfrak{u} < \mathfrak{s}$ is consistent.

Introduction

Ultrafilters and MAD families[1] play a fundamental role on infinite combinatorics, set theoretic topology and other branches of mathematics. For this reason, it is interesting to study the relationship between this two objects. In this note, we will focus on the cardinal invariants associated to each of them. The *ultrafilter number* $\mathfrak{u}$ is defined as the smallest size of a base of an ultrafilter, and the The *almost disjointness number* $\mathfrak{a}$ is the smallest size of a MAD family. The consistency of the inequality $\mathfrak{a} < \mathfrak{u}$ is well known and easy to prove, in fact, it holds in the Cohen, random and Silver models, among many others. Proving the consistency of the inequality $\mathfrak{u} < \mathfrak{a}$ is much harder and used to be an open problem for a long time. In fact, it follows by the theorems of Hrušák, Moore and Džamonja that the inequality $\mathfrak{u} < \mathfrak{a}$ can not be obtained by using countable support iteration of proper Borel partial orders (see theorem 6.6 and theorem 7.2 of [51]). The consistency of $\mathfrak{u} < \mathfrak{a}$ was finally established by Shelah in [58] (see also [10]) where he proved the following theorem:

Theorem 1 (Shelah) *Let V be a model of GCH, κ a measurable cardinal and μ, λ two regular cardinals such that $\kappa < \mu < \lambda$. There is a c.c.c. forcing extension of V that satisfies $\mu = \mathfrak{b} = \mathfrak{d} = \mathfrak{u}$ and $\lambda = \mathfrak{a} = \mathfrak{c}$. In particular,* CON*(ZFC + "there is a measurable cardinal") implies* CON*(ZFC + "$\mathfrak{u} < \mathfrak{a}$").*

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[1]The reader may consult the next section for the definitions of the undefined concepts used in the introduction.

This theorem was one of the first results proved using "template iterations", which is a very powerful method that has been very useful and has been successfully applied to this day (see for example, [8], [10], [9], [26], [44]). In spite of the beauty of this result, it leaves open the following questions:

Problem 2 (Shelah [57]) *Does* CON*(ZFC) imply* CON*(ZFC +* "$\mathfrak{u} < \mathfrak{a}$"*)?*

Problem 3 (Brendle [10]) *Is it consistent that* $\omega_1 = \mathfrak{u} < \mathfrak{a}$*?*

In this note, we will provide a positive answer to both questions, by proving (without appealing to large cardinals) that every MAD family can be destroyed by a proper forcing that preserves P-points. We will also present an alternative proof of the consistency of $\mathfrak{u} < \mathfrak{s}$, which was proved first by Blass and Shelah in [6] (see also [2]).

Our motivation comes from the theorems of Shelah that establishes that the statements "$\omega_1 = \mathfrak{b} = \mathfrak{a} < \mathfrak{s} = \omega_2$" and "$\omega_1 = \mathfrak{b} < \mathfrak{a} = \mathfrak{s} = \omega_2$" are consistent (see [56]). After this impressive results, different models of $\omega_1 = \mathfrak{b} < \mathfrak{a} = \omega_2$ have been constructed (see for example [19], [11], [25] and [7]). In every case, the forcings used add Cohen reals, so no ultrafilter is preserved.

In order to construct the models of $\mathfrak{b} < \mathfrak{s}$ and $\mathfrak{b} < \mathfrak{a}$, Shelah used a creature forcing (see [56] and [1]). In [14] Brendle and Raghavan found a simpler representation of Shelah forcing as a two step iteration, which we will briefly describe (more details of the forcing will be studied in a further section).

The most natural way to increase the splitting number is to *diagonalize* an ultrafilter. In order to build a model of $\mathfrak{b} < \mathfrak{s}$, it is enough to construct (or force) an ultrafilter that can be diagonalized without adding dominating reals (even in the iteration). Denote by $\mathbb{F}_\sigma$ the set of all F_σ-filters[2] on ω. If $\mathcal{F}, \mathcal{G} \in \mathbb{F}_\sigma$ we define $\mathcal{F} \leq \mathcal{G}$ if $\mathcal{G} \subseteq \mathcal{F}$. It is not hard to see that $\mathbb{F}_\sigma$ naturally adds an ultrafilter $\mathcal{U}_{gen}$. The reader wishing to learn more about $\mathbb{F}_\sigma$ may consult [40] and [48]. It turns out that the forcing of Shelah is equivalent to the two step iteration $\mathbb{F}_\sigma * \mathbb{M}(\dot{\mathcal{U}}_{gen})$, where $\mathbb{M}(\mathcal{U}_{gen})$ is the Mathias forcing relative to $\mathcal{U}_{gen}$ (see [14]). It can be proved that $\mathbb{M}(\mathcal{U}_{gen})$ does not add dominating reals, even when iterated (see [14] or [27]).

The method to build a model of $\mathfrak{b} < \mathfrak{a}$ is similar: Given a MAD family $\mathcal{A}$, denote by $\mathbb{F}_\sigma(\mathcal{A})$ the set of all F_σ-filters $\mathcal{F}$ such that $\mathcal{F} \cap \mathcal{I}(\mathcal{A}) = \emptyset$ (where $\mathcal{I}(\mathcal{A})$ is the ideal generated by $\mathcal{A}$). Once again, we order $\mathbb{F}_\sigma(\mathcal{A})$ with inclusion. It is easy to see that $\mathbb{F}_\sigma(\mathcal{A})$ naturally adds an ultrafilter $\mathcal{U}_{gen}(\mathcal{A})$, furthermore, diagonalizing $\dot{\mathcal{U}}_{gen}(\mathcal{A})$ destroys the maximality of $\mathcal{A}$. In this case, it can be proved that $\mathbb{C}_{\omega_1} * \mathbb{F}_\sigma * \mathbb{M}(\dot{\mathcal{U}}_{gen}(\mathcal{A}))$ does not add dominating reals, even when

[2] We view filters as subspaces of 2^ω, the notion of Borel or F_σ is taken using the usual topology on 2^ω.

iterated ($\mathbb{C}_{\omega_1}$ denotes the forcing for adding ω_1-Cohen reals), see [14] or [27]. We want to point out that in both the forcing of Shelah ([56]) and the forcing used by Brendle in [7] require to add Cohen reals in an explicit way. Our work shows that adding the Cohen reals was in fact not needed.

We take a similar approach in order to build our model of $\mathfrak{u} < \mathfrak{s}$. We will first force with $\mathbb{F}_\sigma$ and then we will diagonalize $\dot{\mathcal{U}}_{gen}$. The difference, is that instead of using Mathias forcing, we will use a variant of Miller forcing. The same technique will be used to build the model of $\omega_1 = \mathfrak{u} < \mathfrak{a}$. Given a MAD family $\mathcal{A}$, we will force with $\mathbb{F}_\sigma(\mathcal{A})$ and then diagonalize in the same way as before. In both cases, we will prove that the forcings preserve all P-points of the ground model.

There is a huge work regarding the cardinal invariants $\mathfrak{b}, \mathfrak{s}, \mathfrak{d}, \mathfrak{u}$ and $\mathfrak{a}$. We would like to make some historic comments in here. As was mentioned before, the story began when Shelah ([56]) constructed models of $\omega_1 = \mathfrak{b} < \mathfrak{s}$ and $\omega_1 = \mathfrak{b} < \mathfrak{a} = \mathfrak{s}$. Shelah used a countable support iteration of creature forcings. In [7] Brendle used $\mathsf{c.c.c.}$ forcings for constructing models of $\kappa = \mathfrak{b} < \mathfrak{a} = \kappa^+$ where κ is any uncountable regular cardinal. In [25] Fischer and Steprāns constructed models of $\kappa = \mathfrak{b} < \mathfrak{s} = \kappa^+$ where κ is any uncountable regular cardinal. In [11] Brendle and Fischer used matrix iterations to prove that for any regular cardinals $\kappa < \lambda$, it is consistent that $\kappa = \mathfrak{b} = \mathfrak{a} < \mathfrak{s} = \lambda$ and if κ is bigger than a measurable cardinal, then it is consistent that $\kappa = \mathfrak{b} < \mathfrak{a} = \mathfrak{s} = \lambda$. In [14] Brendle and Raghavan find the decomposition of the original forcing of Shelah. The consistency of $\omega_1 < \mathfrak{d} < \mathfrak{a}$ and $\omega_1 < \mathfrak{u} < \mathfrak{a}$ was obtained by Shelah in [58] where he developed the technique of forcing along a template (see also [8] and [10]). In [24] Fischer and Mejía proved that it is consistent that $\omega_1 < \mathfrak{s} < \mathfrak{b} < \mathfrak{a}$ (see also [45] and [22]).

There are still many interesting open questions remaining:

Problem 4 (Roitman) *Does* $\mathfrak{d} = \omega_1$ *imply that* $\mathfrak{a} = \omega_1$*?*

Problem 5 (Brendle and Raghavan) *Does* $\mathfrak{b} = \mathfrak{s} = \omega_1$ *imply that* $\mathfrak{a} = \omega_1$*?*

Note that a positive solution to the question of Brendle and Raghavan would provide a positive solution to the problem of Roitman.

Preliminaries and notation

Let $f, g \in \omega^\omega$, define $f \leq g$ if and only if $f(n) \leq g(n)$ for every $n \in \omega$ and $f \leq^* g$ if and only if $f(n) \leq g(n)$ holds for all $n \in \omega$ except finitely many. We say a family $\mathcal{B} \subseteq \omega^\omega$ is *unbounded* if $\mathcal{B}$ is unbounded with respect to $\leq^*$. A family $\mathcal{D} \subseteq \omega^\omega$ is a *dominating family* if for every $f \in \omega^\omega$, there is $g \in \mathcal{D}$ such that $f \leq^* g$. The *bounding number* $\mathfrak{b}$ is the size of the smallest unbounded family and the *dominating number* $\mathfrak{d}$ is the smallest size of a dominating family.

We say that S *splits* X if $S \cap X$ and $X \setminus S$ are both infinite. A family $\mathcal{S} \subseteq [\omega]^{\omega}$ is a *splitting family* if for every $X \in [\omega]^{\omega}$ there is $S \in \mathcal{S}$ such that S splits X. The *splitting number* $\mathfrak{s}$ is the smallest size of a splitting family. A family $\mathcal{A} \subseteq [\omega]^{\omega}$ is *almost disjoint (AD)* if the intersection of any two different elements of $\mathcal{A}$ is finite, a MAD *family* is a maximal almost disjoint family. By $\mathsf{cov}(\mathcal{M})$ we denote the smallest size of a family of meager sets that covers the Baire space. The reader may consult the surveys [4] for the main properties of the cardinal invariants used in this paper, [32] to learn more about almost disjoint families and [31] for a survey on filters and ideals.

In this paper, a *tree* is a set of sequences closed under taking restrictions (i.e. p is a tree if whenever $s \in p$ and $n < |s|$ then $s \restriction n \in p$). If $s, t \in \omega^{<\omega}$, by $s^\frown t$ we denote the concatenation of s and t, If $n \in \omega$, we will often write $s^\frown n$ instead of $s^\frown \langle n \rangle$. In this paper, we will say that $s \in p$ is a *splitting node* if $suc_p(s) = \{n \mid s^\frown n \in p\}$ is infinite. We say that a splitting node $s \in p$ is the *stem of* p (denoted by $stem(p)$ in case it exists) if every predecessor of s has exactly one immediate successor. If p is a tree, the *set of branches of* p is defined as $[p] = \{x \mid \forall n (x \restriction n \in p)\}$. For every $s \in p$, we define the tree $p_s = \{t \in p \mid s \subseteq t \vee t \subseteq s\}$. Given $s \in \omega^{<\omega}$ define *the cone of* s as $\langle s \rangle = \{f \in \omega^{\omega} \mid s \subseteq f\}$.

Let $\mathcal{I}$ be an ideal on ω, $\mathcal{F}$ a filter on ω and $\mathcal{A}$ a MAD family. Define[3] $\mathcal{I}^{+} = \wp(\omega) \setminus \mathcal{I}$ i.e. the subsets of ω that are not in $\mathcal{I}$. We say that a forcing notion $\mathbb{P}$ *destroys* $\mathcal{I}$ if $\mathbb{P}$ adds an infinite subset of ω that is almost disjoint with every element of $\mathcal{I}$. We say that $\mathbb{P}$ *diagonalizes* $\mathcal{F}$ if $\mathbb{P}$ adds an infinite set almost contained in every element of $\mathcal{F}$. It is easy to see that $\mathbb{P}$ destroys $\mathcal{I}$ if and only if $\mathbb{P}$ diagonalizes the filter $\mathcal{I}^{*} = \{\omega \setminus A \mid A \in \mathcal{I}\}$. By $\mathcal{I}(\mathcal{A})$ we denote the ideal generated by $\mathcal{A}$ (and the finite sets). We say that $\mathbb{P}$ *destroys* $\mathcal{A}$ if $\mathcal{A}$ is no longer maximal after forcing with $\mathbb{P}$. Note that $\mathbb{P}$ destroys $\mathcal{A}$ if and only if $\mathbb{P}$ destroys the ideal $\mathcal{I}(\mathcal{A})$. The following result is well known (as well as easy to prove) and will be frequently used:

Lemma 6 *Let $\mathcal{A}$ be a MAD family. If $B \subseteq \omega$, the following are equivalent:*

1. *$B \in \mathcal{I}(\mathcal{A})^{+}$.*
2. *There is $\mathcal{A}_1 \in [\mathcal{A}]^{\omega}$ such that $|B \cap A| = \omega$ for every $A \in \mathcal{A}_1$.*

We will need to recall the definition of the Katětov order:

Definition 7 *Let A and B be two countable sets, $\mathcal{I}, \mathcal{J}$ be ideals on X and Y respectively and $f : Y \longrightarrow X$.*

1. *We say that f is a* Katětov morphism from $(Y, \mathcal{J})$ to $(X, \mathcal{I})$ *if $f^{-1}(A) \in \mathcal{J}$ for every $A \in \mathcal{I}$.*

[3] By $\wp(a)$ we denote the powerset of a.

2. *We define $\mathcal{I} \leq_{\mathsf{K}} \mathcal{J}$ ($\mathcal{I}$ is Katětov smaller that $\mathcal{J}$ or $\mathcal{J}$ is Katětov above $\mathcal{I}$) if there is a Katětov morphism from $(Y, \mathcal{J})$ to $(X, \mathcal{I})$.*

The reader may consult [33] for an interesting survey of the Katětov order on Borel ideals. The *nowhere dense ideal* nwd is the ideal on $2^{<\omega}$ where $A \in \mathsf{nwd}$ if and only if for every $s \in 2^{<\omega}$ there is $t \in 2^{<\omega}$ extending s such that no further extension of t is in A. It is easy to see that nwd is an ideal. For every $n \in \omega$ we define $C_n = \{(n, m) \mid m \in \omega\}$ and if $f : \omega \longrightarrow \omega$ let $D(f) = \{(n, m) \mid m \leq n\}$. The ideal $\mathsf{fin}\times\mathsf{fin}$ is the ideal on $\omega \times \omega$ generated by $\{C_n \mid n \in \omega\} \cup \{D(f) \mid f \in \omega^\omega\}$.

If $\mathcal{F}$ is a filter on ω (or on any countable set) we define the *Mathias forcing* $\mathbb{M}(\mathcal{F})$ *with respect to* $\mathcal{F}$ as the set of all pairs (s, A) where $s \in [\omega]^{<\omega}$ and $A \in \mathcal{F}$. If $(s, A), (t, B) \in \mathbb{M}(\mathcal{F})$ then $(s, A) \leq (t, B)$ if the following conditions hold:

1. t is an initial segment of s.
2. $A \subseteq B$.
3. $(s \setminus t) \subseteq B$.

The *Laver forcing* $\mathbb{L}(\mathcal{F})$ *with respect to* $\mathcal{F}$ is the set of all trees p such that $suc_p(s) \in \mathcal{F}$ for every $s \in p$ extending the stem of p. We say $p \leq q$ if $p \subseteq q$.

While $\mathbb{L}(\mathcal{F})$ always adds a dominating real, this may not be the case for $\mathbb{M}(\mathcal{F})$. A trivial example is taking $\mathcal{F}$ to be the cofinite filter in ω, since in this case $\mathbb{M}(\mathcal{F})$ is forcing equivalent to Cohen forcing. A more interesting example was found by Canjar in [17], where an ultrafilter whose Mathias forcing does not add dominating reals was constructed under $\mathfrak{d} = \mathfrak{c}$ (see also [28]). We say that a filter $\mathcal{F}$ is *Canjar* if $\mathbb{M}(\mathcal{F})$ does not add dominating reals. Let $\mathcal{F}$ be a filter on ω. We define the filter $\mathcal{F}^{<\omega}$ on $[\omega]^{<\omega} \setminus \{\emptyset\}$ generated by $\left\{[A]^{<\omega} \setminus \{\emptyset\} \mid A \in \mathcal{F}\right\}$. Note that $X \in (\mathcal{F}^{<\omega})^+$ if and only if for every $A \in \mathcal{F}$, there is $s \in X$ such that $s \subseteq A$. It is important to emphasize that if $X \in (\mathcal{F}^{<\omega})^+$ then by convention $\emptyset \notin X$ (recall that $\mathcal{F}^{<\omega}$ is a filter on $[\omega]^{<\omega} \setminus \{\emptyset\}$). In [35] Hrušák and Minami showed that the forcing properties of $\mathbb{M}(\mathcal{F})$ are closely related to the combinatorics of $\mathcal{F}^{<\omega}$. They proved the following result:

Proposition 8 ([35]) *Let $\mathcal{F}$ be a filter on ω. The following are equivalent:*

1. *$\mathcal{F}$ is Canjar.*
2. *For every $\{X_n \mid n \in \omega\} \subseteq (\mathcal{F}^{<\omega})^+$ there are $Y_n \in [X_n]^{<\omega}$ such that $\bigcup\limits_{n \in \omega} Y_n \in (\mathcal{F}^{<\omega})^+$.*

In [18] it was proved that a filter is Canjar if and only if is has the Menger property (as a subspace of $\wp(\omega)$). Canjar filters have been further studied in [5], [28], [27], [23] and [30].

Miller forcing based on filters

The theory of destructibility of ideals is very important in forcing theory, since many important forcing properties may be stated in these terms. For example, it is well known that a forcing $\mathbb{P}$ adds a dominating real if and only if $\mathbb{P}$ destroys $\mathsf{fin}\times\mathsf{fin}$. The reader may consult [34], [15], [39], [36] or [41] for more on destructibility of ideals.

In order to build models where $\mathfrak{a}$ is big and some other cardinal invariant is small, we need to be able to destroy a MAD family by dealing the "less damage as possible" to the ground model. The most well known forcings to destroy an ideal (or to diagonalize a filter) are the Mathias or Laver forcings relative to the ideal (filter). The following result is well known:

Proposition 9 *Let $\mathcal{F}$ be a filter on ω.*

1. *$\mathbb{L}(\mathcal{F})$ adds a dominating real.*
2. *$\mathbb{M}(\mathcal{F})$ adds a Cohen real if and only if $\mathcal{F}$ is not a Ramsey ultrafilter.*
3. *If $\mathcal{F}$ is a Ramsey ultrafilter, then $\mathbb{M}(\mathcal{F})$ adds a dominating real.*

In particular, it follows that $\mathbb{M}(\mathcal{F})$ adds either a Cohen or a dominating real. In this section, we will introduce a forcing relative a filter $\mathcal{F}$ that in some cases, it might destroy $\mathcal{F}$ without adding Cohen or dominating reals.

We say that a tree $p \subseteq \omega^{<\omega}$ is a *Miller tree* if the following conditions hold:

1. p consists of increasing sequences.
2. p has a stem.
3. For every $s \in p$, there is $t \in p$ such that $s \subseteq t$ and t is a splitting node.[4]

Usually, Miller trees are require to satisfy the following extra condition:

4. Every node of p either is an splitting node or it has exactly one immediate successor.

However, we will not assume this extra requirement. The *Miller forcing* $\mathbb{PT}$ consists of all Miller trees ordered by inclusion.[5] Miller forcing (also called "super perfect forcing") was introduced by Miller in [49], this is one of the most useful and studied forcings for adding new reals (see [2], [49] or [29]). By *split* (p)

[4]Recall that s is a splitting node of p if $suc_p(s)$ is infinite.

[5]Obviously, the trees satisfying property 4 are dense in Miller forcing. However, this does not seem to be the case for our forcings.

we denote the collection of all splitting nodes and by $split_n(p)$ we denote the collection of n-splitting nodes (i.e. $s \in split_n(p)$ if $s \in split(p)$ and s has exactly n-restrictions that are splitting nodes). Note that $split_0(p) = \{stem(p)\}$.

In [54], Sabok and Zapletal introduced the following parametrized version of Miller forcing[6]:

Definition 10 *Let $\mathcal{F}$ be a filter. By $\mathbb{Q}(\mathcal{F})$ we denote the set of all Miller trees $p \in \mathbb{PT}$ such that $suc_p(s) \in \mathcal{F}^+$ for every splitting node s. We order $\mathbb{Q}(\mathcal{F})$ by inclusion.*

Sabok and Zapletal proved some very interesting results, like the following: (the reader may consult [54] and [46] for the definitions of spl and the *Solecki ideal* $\mathcal{S}$).

Proposition 11 ([54]) *Let $\mathcal{F}$ be a filter.*

1. *$\mathbb{Q}(\mathcal{F})$ does not add Cohen reals if and only if $\mathsf{nwd} \nleq_{\mathsf{K}} \mathcal{F}^* \restriction A$ for every $A \in \mathcal{F}^+$.*
2. *If $\mathcal{F}$ is a Borel filter, then $\mathbb{Q}(\mathcal{F})$ preserves outer Lebesgue measure if and only if $\mathcal{S} \nleq_{\mathsf{K}} \mathcal{F}^* \restriction A$ for every $A \in \mathcal{F}^+$.*
3. *If $\mathcal{F}$ is a Borel filter, then $\mathbb{Q}(\mathcal{F})$ does not add splitting reals if and only if $\mathsf{spl} \nleq_{\mathsf{K}} \mathcal{F}^* \restriction A$ for every $A \in \mathcal{F}^+$.*

In some cases, $\mathbb{Q}(\mathcal{F})$ may diagonalize $\mathcal{F}$ while in some others cases not, as can be seen with the following:

Lemma 12

1. *$\mathbb{Q}(\mathsf{fin}\times\mathsf{fin}^*)$ does not destroy $\mathsf{fin}\times\mathsf{fin}$.*
2. *$\mathbb{Q}(\mathsf{nwd}^*)$ destroys nwd.*

Proof. With an easy fusion argument, it is possible to prove that $\mathbb{Q}(\mathcal{F})$ does not add dominating reals (for every $\mathcal{F}$). Alternatively, this can be proved as follows: In [54] it was proved that there is a σ-ideal $\mathcal{J}_{\mathcal{F}}$ generated by closed sets such that $\mathbb{Q}(\mathcal{F})$ is forcing equivalent to $\mathsf{Borel}(\omega^\omega)/\mathcal{J}_{\mathcal{F}}$, which does not add dominating reals by Theorem 4.1.2 of [60]. It is well known that if a forcing destroys $\mathsf{fin}\times\mathsf{fin}$, then it must add a dominating real. From this two facts, it follows that $\mathbb{Q}(\mathsf{fin}\times\mathsf{fin}^*)$ does not destroy $\mathsf{fin}\times\mathsf{fin}$. On the other hand, note that $\mathbb{Q}(\mathsf{nwd}^*)$ adds a Cohen real and it is easy to see that any forcing adding a Cohen real must destroy nwd. $\blacksquare$

We will now introduce a version of Miller forcing parametrized with a filter $\mathcal{F}$ that will always diagonalizes $\mathcal{F}$. Given $p \in \mathbb{PT}$ for every $s \in split_n(p)$ we define $spsuc_p(s) = \{t \setminus s \mid t \in split_{n+1}(p) \land s \subseteq t\}$.

[6] In [54] the authors use ideals instead of filters. Evidently, this choice is superflous.

Definition 13 *Let $\mathcal{F}$ be a filter. We say $p \in \mathbb{PT}(\mathcal{F})$ if the following holds:*

1. *$p \in \mathbb{PT}$.*
2. *If $s \in split(p)$ then $spsuc_p(s) \in (\mathcal{F}^{<\omega})^+$.*

We order $\mathbb{PT}(\mathcal{F})$ by inclusion. The following lemma contains some basic results about $(\mathcal{F}^{<\omega})^+$, which will be used implicitly.

Lemma 14 *Let $\mathcal{F}$ be a filter on ω and $X \in (\mathcal{F}^{<\omega})^+$.*

1. *If $F \in \mathcal{F}$, then $X \cap [F]^{<\omega} \in (\mathcal{F}^{<\omega})^+$.*
2. *If $X = A \cup B$, then either $A \in (\mathcal{F}^{<\omega})^+$ or $B \in (\mathcal{F}^{<\omega})^+$.*
3. *If $X = \{s_n \mid n \in \omega\}$ and $Y = \{y_n \mid n \in \omega\}$ is such that every y_n is a non-empty subset of s_n. then $Y \in (\mathcal{F}^{<\omega})^+$.*
4. *If r_{gen} is an $(\mathbb{M}(\mathcal{F}), V)$-generic real, then r_{gen} contains an element of X.*

By point 1 above we get the following:

Lemma 15 $\mathbb{PT}(\mathcal{F})$ *diagonalizes $\mathcal{F}$.*

Given $A \subseteq [\omega]^{<\omega}$ we define $minimal(A) \subseteq A$ as the set of all minimal elements of A with respect to the initial segment relation. Note that every element of A contains an element of $minimal(A)$. We conclude that if $A \in (\mathcal{F}^{<\omega})^+$ then $minimal(A) \in (\mathcal{F}^{<\omega})^+$.

Given $p \in \mathbb{PT}(\mathcal{F})$, $s \in split(p)$ and $D \subseteq \mathbb{PT}(\mathcal{F})$ an open dense subset, we define $E(D, p, s) = minimal(\{t \setminus s \mid \exists q \leq p_s (stem(q) = t \wedge q \in D)\})$. We now have the following:

Lemma 16 *If $p \in \mathbb{PT}(\mathcal{F})$, $s \in split(p)$ and $D \subseteq \mathbb{PT}(\mathcal{F})$ is an open dense subset, then $E(D, p, s) \in (\mathcal{F}^{<\omega})^+$.*

Proof. It would be enough to prove that $\{t \setminus s \mid \exists q \leq p_s (stem(q) = t \wedge q \in D)\}$ is in $(\mathcal{F}^{<\omega})^+$, which is immediate. ■

We can then prove the following:

Proposition 17 *Let $\mathcal{F}$ be a filter.*

1. *Let M be a countable elementary submodel, $p \in \mathbb{PT}(\mathcal{F})$, $D \subseteq \mathbb{PT}(\mathcal{F})$ an open dense subset with $p, D \in M$ and $s \in split(p)$. There is $q \leq p$ with $stem(q) = s$ such that $q \Vdash$ "$\dot{G} \cap D \cap M \neq \emptyset$".*

2. $\mathbb{PT}(\mathcal{F})$ *is proper.*

Proof. Let M be a countable elementary submodel, $p \in \mathbb{PT}(\mathcal{F})$, $D \subseteq \mathbb{PT}(\mathcal{F})$ an open dense subset with $p, D \in M$ and $s \in split(p)$. Since $p, D, s \in M$, it follows that $E(D, p, s) \in M$. For every t such that $t \setminus s \in E(D, p, s)$, we choose $q(t) \in M \cap D$ such that $q(t) \leq p_s$ and the stem of $q(t)$ is t. Define $q = \bigcup \{q(t) \mid t \setminus s \in E(D, p, s)\}$. We will show that q has the desired properties. Define $L = \{t \mid t \setminus s \in E(D, p, s)\}$, we will first prove that $q \in \mathbb{PT}(\mathcal{F})$. Let $z \in q$ be a splitting node. If z extends a $t \in L$, then $suc_q(z) = suc_{q(t)}(z)$, so $suc_q(z) \in (\mathcal{F}^{<\omega})^+$. Now, assume that z does not extend an element of L, it is enough to prove that $suc_q(z) = suc_p(z)$. Let n such that $z \in split_n(q)$ and $w \in split_{n+1}(q)$ such that $z \subseteq w$. Since z does not extend an element of L, we know that w can be extended to a $t \in L$, so $w \in q(t)$. Note that this argument shows that $stem(q) = s$.

We will now prove that $q \Vdash$ "$\dot{G} \cap D \cap M \neq \emptyset$". Let $q_1 \leq q$, we may assume that $q_1 \in D$. Let $w = stem(q_1)$. By construction, (recall that we took the minimal elements) we know there is $t \in L$ such that $t \subseteq w$, in this way $q_1 \leq q(t)$, so $q_1 \Vdash$ "$q(t) \in \dot{G} \cap D \cap M$" and we are done.

The second part of the lemma follows by the first part and a fusion argument. ■

We will say that a family of functions $\mathcal{B} \subseteq \omega^\omega$ is a $\mathfrak{b}$-*family* if the following holds:

1. Every element of $\mathcal{B}$ is an increasing function.
2. Given $\{f_n \mid n \in \omega\} \subseteq \mathcal{B}$ there is $g \in \mathcal{B}$ such that $f_n \leq^* g$ for every $n \in \omega$.
3. $\mathcal{B}$ is unbounded.

An example of a $\mathfrak{b}$-family would be a well-ordered unbounded family, another example is the set of all increasing functions. If $\mathcal{B}$ is a $\mathfrak{b}$-family and $\mathbb{P}$ is a partial order, we say that $\mathbb{P}$ *preserves* $\mathcal{B}$ if $\mathcal{B}$ is still unbounded after forcing with $\mathbb{P}$. Note that if $\mathbb{P}$ is a proper forcing that preserves $\mathcal{B}$, then $\mathcal{B}$ is still a $\mathfrak{b}$-family in the extension. We will need the following easy lemma:

Lemma 18 *Let $\mathcal{B}$ be a $\mathfrak{b}$-family. If $\mathcal{B} = \bigcup\limits_{n \in \omega} \mathcal{B}_n$, then there is $n \in \omega$ such that $\mathcal{B}_n$ is cofinal in $\mathcal{B}$ (i.e. for every $f \in \mathcal{B}$ there is $g \in \mathcal{B}_n$ such that $f \leq^* g$).*

Proof. We argue by contradiction. Assume this is not the case, so for every $n \in \omega$ there is $f_n \in \mathcal{B}$ such that f_n is not bounded by any element of $\mathcal{B}_n$. Since $\mathcal{B}$ is a $\mathfrak{b}$-family, we can find $g \in \mathcal{B}$ such that $f_n \leq^* g$ for every $n \in \omega$. Since $\mathcal{B} = \bigcup\limits_{n \in \omega} \mathcal{B}_n$, there must be $m \in \omega$ such that $g \in \mathcal{B}_m$, hence $f_m \leq^* g$ which is a contradiction. ■

Given a sequence $\overline{X} = \{X_n \mid n \in \omega\} \subseteq [\omega]^{<\omega} \setminus \{\emptyset\}$ and $f \in \omega^\omega$, we define the set $\overline{X}_f = \bigcup\limits_{n\in\omega} (X_n \cap \wp(f(n)))$. Note that $\mathcal{F}$ is *Canjar* if if for every sequence $\overline{X} = \{X_n \mid n \in \omega\} \subseteq (\mathcal{F}^{<\omega})^+$ there is $f \in \omega^\omega$ such that $\overline{X}_f \in (\mathcal{F}^{<\omega})^+$. Recall that by a theorem of Hrušák and Minami (see [35]), a filter $\mathcal{F}$ is Canjar if and only if $\mathbb{M}(\mathcal{F})$ does not add dominating reals. If $\mathcal{B}$ is a $\mathfrak{b}$-family, we say that $\mathcal{F}$ is *$\mathcal{B}$-Canjar* if for every sequence $\overline{X} = \{X_n \mid n \in \omega\} \subseteq (\mathcal{F}^{<\omega})^+$ there is $f \in \mathcal{B}$ such that $\overline{X}_f \in (\mathcal{F}^{<\omega})^+$ (such $\overline{X}_f$ is called a *pseudointersection according to* $\mathcal{B}$). Note that if $\mathcal{F}$ is $\mathcal{B}$-Canjar (for some $\mathfrak{b}$-family $\mathcal{B}$), then $\mathcal{F}$ is Canjar. As expected, $\mathcal{B}$-Canjar filters have a similar characterization as the one of Canjar. The following is a slight strenghtening of proposition 1 of [27], which is a generalization of the theorem of Hrušák and Minami:

Proposition 19 *Let $\mathcal{B}$ be a $\mathfrak{b}$-family. A filter $\mathcal{F}$ is a $\mathcal{B}$-Canjar filter if and only if $\mathbb{M}(\mathcal{F})$ preserves $\mathcal{B}$.*

Proof. Assume that $\mathcal{F}$ does not preserve $\mathcal{B}$, in other words, there is a name $\dot{g}$ for an increasing function such that $1_{\mathbb{M}(\mathcal{F})} \Vdash$ "$\dot{g}$ is an upper bound for $\mathcal{B}$". For every function $f \in \mathcal{B}$ let $s_f \in [\omega]^{<\omega}, n_f \in \omega$ and $F_f \in \mathcal{F}$ such that $(s_f, F_f) \Vdash$ "$\forall i \geq n_f (f(i) <_{n_f} \dot{g}(i))$". Since $\mathcal{B}$ is a $\mathfrak{b}$-family, there are $s \in [\omega]^{<\omega}, n \in \omega$ and a cofinal family $\mathcal{B}' \subseteq \mathcal{B}$ such that $s_f = s$ and $n_f = n$ for every $f \in \mathcal{B}'$.

For every $m \in \omega$ let X_m be the set of all $t \in [\omega \setminus \bigcup s]^{<\omega}$ such that there is $F \in \mathcal{F}$ with the property that $(s \cup t, F)$ decides $\langle \dot{g}(0), \dots, \dot{g}(m)\rangle$ and $(s \cup t, F) \Vdash$ "$\dot{g}(m) < max(t)$". It is easy to see that $\overline{X} = \{X_m \mid m \in \omega\}$ is a sequence of sets in $(\mathcal{F}^{<\omega})^+$. We will see that it has no pseudointersection according to $\mathcal{B}$. Since $\mathcal{B}'$ is cofinal in $\mathcal{B}$, it is enough to show that $\overline{X}$ has no pseudointersection according to $\mathcal{B}'$.

Aiming for a contradiction, assume that there is $f \in \mathcal{B}'$ such that $\overline{X}_f$ is positive. Since $\overline{X}_f \cap [F_f]^{<\omega}$ is infinite, pick $t \in \overline{X}_f \cap [F_f]^{<\omega}$ such that $t \in X_k \cap \wp(f(k))$ with $k > n$. Since $t \in X_k$ there is $F \in \mathcal{F}$ such that $(s \cup t, F) \Vdash$ "$\dot{g}(k) \leq max(t)$" and note that $(s \cup t, F) \Vdash$ "$\dot{g}(k) \leq f(k)$". In this way, $(s \cup t, F_h \cap F)$ forces both "$f(k) < \dot{g}(k)$" and "$\dot{g}(k) \leq f(k)$", which is a contradiction.

Now assume that $\mathbb{M}(\mathcal{F})$ preserves $\mathcal{B}$. Let $\overline{X} = \langle X_n \mid n \in \omega\rangle$ be a sequence of sets in $(\mathcal{F}^{<\omega})^+$. Let r_{gen} be a $(V, \mathbb{M}(\mathcal{F}))$-generic real, observe that $[r_{gen}]^{<\omega}$ intersect infinitely every member of $(\mathcal{F}^{<\omega})^+$. In this way, in $V[r_{gen}]$ we may define an increasing function $g : \omega \longrightarrow \omega$ such that $(r_{gen} \setminus n) \cap g(n)$ contains a member of X_n. Since $\mathcal{F}$ preserves $\mathcal{B}$, then there is $f \in \mathcal{B}$ such that $f \not\leq^* g$, we will see that $\overline{X}_f$ is positive. Let $F \in \mathcal{F}$ we must prove that $\overline{X}_f \cap [F]^{<\omega}$ is not empty. Since $F \in \mathcal{F}$, we know that $r_{gen} \subseteq^* F$ so there is $k \in \omega$ such that $g(k) < f(k)$ and $r_{gen} \setminus k \subseteq F$ and hence $\overline{X}_f \cap [F]^{<\omega} \neq \emptyset$. ■

Moreover, Canjar filters satisfy the following stronger property:

Lemma 20 *Let $\mathcal{B}$ be a $\mathfrak{b}$-family and $\mathcal{F}$ a $\mathcal{B}$-Canjar filter. For every family $\overline{X} = \{X_n \mid n \in \omega\} \subseteq (\mathcal{F}^{<\omega})^+$ there is $f \in \mathcal{B}$ such that for every $n \in \omega$, if $Y_n = \{s \in X_n \mid s \subseteq [f(n-1), f(n))\}$ (where $f(-1) = 0$) then $\bigcup_{n\in\omega} Y_n \in (\mathcal{F}^{<\omega})^+$.*

Proof. The idea of the proof is similar to the previous one. Let r_{gen} be a $(\mathbb{M}(\mathcal{F}), V)$-generic real. In $V[r_{gen}]$ we find an increasing function $g \in \omega^\omega$ such that for every $n \in \omega$, the following holds: $r_{gen} \cap [g(n-1), g(n))$ contains an element of X_n (where $g(-1) = 0$). Furthermore, we may assume that g is unbounded over V (this is possible since $\mathbb{M}(\mathcal{F})$ adds an unbounded real[7]). Since $\mathcal{F}$ is $\mathcal{B}$-Canjar, we can find $(s, A) \in \mathbb{M}(\mathcal{F})$ and $f \in \mathcal{B}$ such that $(s, A) \Vdash$ "$f \nleq^* \dot{g}$". We claim that f is the function we are looking for. Define $Y_n = \{s \in X_n \mid s \subseteq [f(n-1), f(n))\}$ (for every $n \in \omega$) and $Y = \bigcup_{n\in\omega} Y_n$, we must prove that $Y \in (\mathcal{F}^{<\omega})^+$. Let $B \in \mathcal{F}$, we must show that Y contains an element of B. Without lose of generality, we may assume that $B \subseteq A$. We can now find $m \in \omega$, $t \in [\omega]^{<\omega}$ and $C \in \mathcal{F}$ such that the following holds:

1. $max(s) < m$, $C \subseteq B$, $t \subseteq B$.
2. $max(s) < min(t)$.
3. $(s \cup t, C) \leq (s, B)$.
4. There is $w \in X_m$ such that $(s \cup t, C) \Vdash$ "$w \subseteq [\dot{g}(m-1), \dot{g}(m))$".
5. $(s \cup t, C) \Vdash$ "$(f(m-1) \leq \dot{g}(m-1))$" and $(s \cup t, C) \Vdash$ "$(\dot{g}(m) < f(m))$".

Such m, t and C can be obtained since $(s, B) \leq (s, A)$, so (s, B) forces that $\dot{g}$ is an unbounded real that does not dominate f. Note that $w \subseteq t$, so $w \subseteq B$ and $(s \cup t, C) \Vdash$ "$[\dot{g}(m-1), \dot{g}(m)) \subseteq [f(m-1), f(m))$", which implies that $w \in Y_m$. This finishes the proof. ■

We will need the following lemma:

Lemma 21 *Let $\mathcal{F}$ be a Canjar filter, $\{D_n \mid n \in \omega\}$ open dense subsets of $\mathbb{PT}(\mathcal{F})$ and $p \in \mathbb{PT}(\mathcal{F})$ with $stem(p) = s$. There are $q \in \mathbb{PT}(\mathcal{F})$ and $\langle F_n \rangle_{n\in\omega}$ such that the following holds:*

1. *$q \leq p$ and $stem(q) = stem(p)$.*
2. *F_n is a finite subset of $[\omega]^{<\omega} \setminus \{\emptyset\}$ for every $n \in \omega$.*
3. *For every $n \in \omega$, if $t_0 \in F_n$ and $t_1 \in F_{n+1}$, then $max(t_0) < min(t_1)$.*
4. *$spsuc_q(s) = \bigcup_{n\in\omega} F_n$.*

[7] It is well known that every σ-centered forcing adds an unbounded real.

5. *If $t \in F_n$, then $q_{s^\frown t} \in D_n$.*

Proof. For every $n \in \omega$, define $X_n = E(D_n, p, s)$ (recall that $E(D, p, s) = minimal(\{t \setminus s \mid \exists q \leq p_s (stem(q) = t \wedge q \in D)\})$), we know that $X_n \in (\mathcal{F}^{<\omega})^+$. By the previous result, we can find an increasing function $f \in \omega^\omega$ such that if $Y_n = \{s \in X_n \mid s \subseteq [f(n-1), f(n))\}$ (for every $n \in \omega$) then $\bigcup_{n\in\omega} Y_n \in (\mathcal{F}^{<\omega})^+$.

For every $t \in Y_n$, choose $q(t) \leq p_{s^\frown t}$ such that $q(t) \in D_n$ and the stem of $q(t)$ is $s \cup t$. Define $q = \bigcup_{t\in Y} q(t)$. ■

By taking all the open dense sets to be the same, we obtain the following:

Corollary 22 *Let $\mathcal{F}$ be a Canjar filter, D an open dense subset of $\mathbb{PT}(\mathcal{F})$, $p \in \mathbb{PT}(\mathcal{F})$ with $stem(p) = s$. There is $q \in \mathbb{PT}(\mathcal{F})$ such that the following holds:*

1. *$q \leq p$ and $stem(q) = stem(p)$.*
2. *If $t \in spsuc_q(s)$ then $q_{s^\frown t} \in D$.*

We can now prove the following result:

Proposition 23 *Let $\mathcal{B}$ be a $\mathfrak{b}$-family and $\mathcal{F}$ a $\mathcal{B}$–Canjar filter.*

1. *Let $p \in \mathbb{PT}(\mathcal{F})$, $s \in split_m(p)$ and $\dot{g}$ such that $p \Vdash$ "$\dot{g} \in \omega^\omega$". There are $q \in \mathbb{PT}(\mathcal{F})$, $f \in \mathcal{B}$ and $\langle F_n \rangle_{n\in\omega}$ such that the following holds:*
   - (a) *$q \leq p$, $s \in q$ and if $t \in p$ is incomparable with s, then $t \in q$ and $q_t = p_t$.*
   - (b) *For every $n \in \omega$, if $t_0 \in F_n$ and $t_1 \in F_{n+1}$, then $max(t_0) < min(t_1)$.*
   - (c) *$spsuc_q(s) = \bigcup_{n\in\omega} F_n$.*
   - (d) *If $t \in F_n$ then $q_t \Vdash$ "$\dot{g}(n) < f(n)$".*
2. *$\mathbb{PT}(\mathcal{F})$ preserves $\mathcal{B}$.*

Proof. We first prove point 1. Let $p \in \mathbb{PT}(\mathcal{F})$, $s \in split_m(p)$ and $\dot{g}$ such that $p \Vdash$ "$\dot{g} \in \omega^\omega$". Define $D_n \subseteq \mathbb{PT}(\mathcal{F})$ as the set of all q such that $q \Vdash$ "$\dot{g}(n) < max(stem(q))$". It is easy to see that each D_n is an open dense subset of $\mathbb{PT}(\mathcal{F})$. We now apply the previous lemma.

We will now prove that $\mathbb{PT}(\mathcal{F})$ preserves $\mathcal{B}$ as an unbounded family. Let $p \in \mathbb{PT}(\mathcal{F})$ and $\dot{g}$ such that $p \Vdash$ "$\dot{g} \in \omega^\omega$". By the previous point and a fusion argument, we may assume that for every $s \in split_m(p)$ there are $f_s \in \mathcal{B}$ and $\langle F^s_n \rangle_{n\in\omega}$ such that the following holds:

1. Each F^s_n is a finite subset of p.

2. $spsuc_p(s) = \bigcup_{n\in\omega} F_n^s$.

3. if $t \in F_n^s$ then $p_t \Vdash$ "$\dot{g}(n) < f_s(n)$".

Since $\mathcal{B}$ is a $\mathfrak{b}$-family, we can find $f \in \mathcal{B}$ such that $f_s \leq^* f$ for every $s \in split(p)$. For every $s \in split(p)$, let m_s such that if $i > m_s$ then $f_s(i) \leq f(i)$. We can recursively build $q \leq p$ such that $split(q) = split(p) \cap q$ and $suc_q(s) = \bigcup_{n>m_s} F_n^s$. It is easy to see that q forces that $\dot{g}$ does not dominate f. ■

It is worth mentioning that $\mathbb{PT}(\mathcal{F})$ may add dominating reals for certain filters $\mathcal{F}$. The simplest example is taking $\mathcal{F}$ to be $(\mathsf{fin}\times\mathsf{fin})^*$.

The following is a very useful fact about our forcings:

Lemma 24 (Pure decision property) *Let $\mathcal{F}$ be a Canjar filter, $p \in \mathbb{PT}(\mathcal{F})$ and A a finite set. If $\dot{x}$ is a $\mathbb{PT}(\mathcal{F})$-name such that $p \Vdash$ "$\dot{x} \in A$" then there are $q \leq p$ and $a \in A$ such that $stem(q) = stem(p)$ and $q \Vdash$ "$\dot{x} = a$".*

Proof. Let D be the set of all $q \in \mathbb{PT}(\mathcal{F})$ for which there is $a_q \in A$ such that $q \Vdash$ "$\dot{x} = a_q$". Since D is an open dense set, by the previous results, we can find $\overline{p} \leq p$ with the following properties:

1. $stem(\overline{p}) = stem(p)$.

2. If $t \in split_1(\overline{p})$ then $\overline{p}_t \in D$.

Finally, since A is finite, we can find $q \leq p$ with $stem(q) = stem(p)$ and $a \in A$ such that $q_t \Vdash$ "$\dot{x} = a$" for every $t \in split_1(q)$. It follows that $q \Vdash$ "$\dot{x} = a$". ■

By $\mathcal{K}_\sigma$ we denote the ideal generated by all σ-compact sets on the Baire space. For every function $L : \omega^{<\omega} \longrightarrow \omega$, let $K(L)$ be the set defined as $\{x \in \omega^\omega \mid \forall^\infty n\,(x(n) \leq L(x \upharpoonright n))\}$. It is easy to see that $K(L) \in \mathcal{K}_\sigma$. It is well known that for every $K \in \mathcal{K}_\sigma$ there is $f \in \omega^\omega$ such that $\mathcal{K} \subseteq \{g \in \omega^\omega \mid g \leq^* f\}$ (see [2] page 6). The following well known result follows from this remark:

Lemma 25 *Let $\mathcal{B} \subseteq \omega^\omega$ be an unbounded family. If $\mathbb{P}$ is a forcing that preserves $\mathcal{B}$, then $\mathbb{P} \Vdash$"$\mathcal{B} \notin \mathcal{K}_\sigma$".*

If T is a finite tree, we will denote its set of its maximal nodes as $[T]$. We will need the following notion:

Definition 26 *Let $p \in \mathbb{PT}(\mathcal{F})$ and $T \subseteq p$ a finite tree such that $[T] \subseteq split(p)$. We say that $q \leq_T p$ if the following conditions hold:*

1. $q \leq p$.

2. $T \subseteq q$.

3. $T \cap split\,(q) = T \cap split\,(p)\,.$

We fix a canonical bijection $d : \omega \longrightarrow \omega^{<\omega}$ such that if $d\,(m) \subseteq d\,(n)$ then $m \leq n$. Given $p \in \mathbb{PT}\,(\mathcal{F})$ and $n \in \omega$, we define the set $\widetilde{T}\,(p,n) = \left\{ s \in split\,(p) \mid d^{-1}\,(s) \leq n \right\}$. Let $T\,(p,n) \subseteq \omega^{<\omega}$ be the smallest tree such that $\widetilde{T}\,(p,n) \subseteq T\,(p,n)\,.$ It is clear that $T\,(p,n)$ is a finite subtree of p such that $[T\,(p,n)] \subseteq split\,(p)$. It is easy to see that if $q \leq_{T(p,n)} p$ and $n \leq m$, then $T\,(p,n) \subseteq T\,(q,m)\,.$

Let $p \in \mathbb{PT}\,(\mathcal{F})$ and $\mathcal{B}$ a $\mathfrak{b}$-family. We define the game $\mathcal{G}\,(\mathcal{F}, p, \mathcal{B})$ as follows:

I	p_0		p_1		p_2		...
II		n_0		n_1		n_2	

Where the following conditions hold:

1. $p_i \in \mathbb{PT}\,(\mathcal{F})$ and $n_i \in \omega$ for every $i \in \omega$.

2. $p_0 = p$.

3. $\langle n_i \rangle_{i \in \omega}$ is increasing.

4. $p_{m+1} \leq_{T_m} p_m$ where $T_m = T\,(p_m, n_m)\,.$

The player II will *win the game* $\mathcal{G}\,(\mathcal{F}, p, \mathcal{B})$ if $\bigcup T_m \in \mathbb{PT}\,(\mathcal{F})$ and $f \in \mathcal{B}$ where f is the function given by $f\,(i) = n_i$. We now have the following result:

Proposition 27 *Let $\mathcal{F}$ be a filter, $p \in \mathbb{PT}\,(\mathcal{F})$ and $\mathcal{B}$ a $\mathfrak{b}$-family. If $\mathcal{F}$ is $\mathcal{B}$-Canjar, then* I *does not have a winning strategy in $\mathcal{G}\,(\mathcal{F}, p, \mathcal{B})\,.$*

Proof. Let σ be a strategy for player I, we must prove that player II can defeat σ. Define $\{p\,(s) \mid s \in \omega^{<\omega}\} \subseteq \mathbb{PT}\,(\mathcal{F})$ as follows:

1. $p\,(\emptyset) = p$.

2. If $s = \langle n_0, ..., n_m \rangle$ then $p\,(s)$ is the tree played by player I at the m-step if he is playing according to σ and II plays n_i at the step i for $i \leq m$.

Let r_{gen} be an $(\mathbb{M}\,(\mathcal{F})\,, V)$-generic real (note that r_{gen} is generic for $\mathbb{M}\,(\mathcal{F})\,,$ not for $\mathbb{PT}\,(\mathcal{F})$). In $V\,[r_{gen}]$ we define a function $L : \omega^{<\omega} \longrightarrow \omega$ as follows: Let $s = \langle n_i \rangle_{i<m} \in \omega^{<\omega}$, we look at $p\,(s)\,,$ assume that player II plays n_i at the step i for $i < m$ and player I is following σ. Let T_m be the tree defined so far (as in the definition in the game). Let $t \in T_m \cap split\,(p\,(s))$ and since $spsuc_{p(s)}\,(t) \in (\mathcal{F}^{<\omega})^+$, it follows by genericity that there is $m_s\,(t) \in \omega$ such that there is $u_s\,(t) \in spsuc_{p(s)}\,(t)$ for which $u_s\,(t) \subseteq (r_{gen} \setminus m) \cap m_s\,(t)$. Let $L\,(s)$ such that $d^{-1}\,(t \cup u_s\,(t)) < L\,(s)$ for all $t \in T_m \cap split\,(p\,(s))$.

Since $\mathcal{F}$ is $\mathcal{B}$-Canjar, we can find $f \in \mathcal{B}$ and $(z, F) \in \mathbb{M}\,(\mathcal{F})$ such that $(z, F) \Vdash$ "$f \notin K(\dot{L})$". We claim that if II plays $f\,(n)$ at the n-step of the game, then she will win the match. Let $q = \bigcup T_m$, we must show that $q \in \mathbb{PT}\,(\mathcal{F})$.

Let $t \in q$ be a splitting node, find $n \in \omega$ such that $t \in T_n$ (where T_n is defined as in the game). We must prove that $spsuc_q\,(t) \in (\mathcal{F}^{<\omega})^+$. Let $H \in \mathcal{F}$ and note that $(z, F \cap H) \Vdash$ "$f \notin K(\dot{L})$". We know we can find $(z \cup z_0, G) \in \mathbb{M}\,(\mathcal{F})$ and $m \in \omega$ such that the following conditions hold:

1. $m > n, max\,(z)$.
2. $max\,(z) < min\,(z_0)$.
3. $(z \cup z_0, G) \leq (z, F \cap H)$ (in particular, $z_0 \subseteq H$).
4. $(z \cup z_0, G) \Vdash$ "$\dot{L}\,(f \restriction m) < f\,(m)$".

It follows by all the definitions that T_{m+1} will contain an element of the set $spsuc_{p(f \restriction m)}\,(t)$. Note that such element must be a subset of z_0, so in particular is a subset of H. This finishes the proof. ■

The previous argument was motivated by the fact that Canjar is equivalent to its "game version". This interesting result is a corollary of the theorems proved by Chodounský, Repovš and Zdomskyy in [18], we will comment more about this in a later section.

Definition 28 *We say that $D \subseteq \mathbb{PT}\,(\mathcal{F})$ is* purely dense *if the following conditions hold:*

1. *If $p \in D$ and $q \leq p$ then $q \in D$ (D is open).*
2. *For every $p \in \mathbb{PT}\,(\mathcal{F})$ and for every finite tree $T \subseteq p$ such that $[T] \subseteq split\,(p)$, there is $q \leq_T p$ such that $q \in D$.*

Intuitively, the purely dense sets are the open sets we can get in by only using the pure decision property.

We will now prove that if $\mathcal{F}$ is a Canjar filter, then $\mathbb{PT}\,(\mathcal{F})$ does not add Cohen or random reals. Recall the following notion:

Definition 29 *We say that* $c \in \omega^\omega$ *is a* half-Cohen real over V *if for every* $f \in \omega^\omega \cap V$ *the set* $\{n \mid c(n) = f(n)\}$ *is infinite.*

Obviously every Cohen real over V is half-Cohen over V. It can be proved that if one adds an unbounded real and then a half-Cohen, a Cohen real is added (see [4]). It was a long standing question of Fremlin if it was possible to add a half-Cohen real without adding a Cohen real. This problem was finally solved positively by Zapletal in [61]. We will prove that if $\mathcal{F}$ is Canjar, then $\mathbb{PT}(\mathcal{F})$ does not add a half-Cohen real. We start with the following lemma:

Lemma 30 *Let* $\mathcal{F}$ *be a Canjar filter and* $\dot{a}$ *a* $\mathbb{PT}(\mathcal{F})$*-name for a natural number. The set* $D = \{p \in \mathbb{PT}(\mathcal{F}) \mid \exists k \in \omega\, (p \Vdash \text{“}\dot{a} \neq k\text{”})\}$ *is purely dense.*

Proof. Let $p \in \mathbb{PT}(\mathcal{F})$ and $T \subseteq p$ a finite tree such that every maximal node of T is an element of $split(p)$. We need to find $q \leq_T p$ such that $q \in D$. Let $n = |T \cap split(p)|$ and for every $s \in T \cap split(p)$ let $p(s) \leq p_s$ be the biggest subtree of p_s such that every $t \in T \cap p(s)$ is a restriction of s. Using the pure decision property, for every $s \in T \cap split(p)$ we can find $q(s) \in \mathbb{PT}(\mathcal{F})$ and $k_s \in \omega$ with the following properties:

1. $q(s) \leq p(s)$ and the stem of $q(s)$ is s.
2. $q(s) \Vdash \text{“}\dot{a} \leq n+1\text{”}$ or $q(s) \Vdash \text{“}\dot{a} > n+1\text{”}$.
3. If $q(s) \Vdash \text{“}\dot{a} \leq n+1\text{”}$ then $q(s) \Vdash \text{“}\dot{a} = k_s\text{”}$.

Since $n = |T \cap split(p)|$, we can find $k \leq n+1$ such that $k \neq k_s$ for every $s \in T \cap split(p)$. Let $q = \bigcup \{q(s) \mid s \in T \cap split(p)\}$. Note that $q \leq_T p$ and $q \Vdash \text{“}\dot{a} \neq k\text{”}$, this finishes the proof. ■

We can now prove the following:

Proposition 31 *If* $\mathcal{F}$ *is a Canjar filter, then* $\mathbb{PT}(\mathcal{F})$ *does not add half-Cohen reals.*

Proof. Let $p \in \mathbb{PT}(\mathcal{F})$ and $\dot{f}$ a $\mathbb{PT}(\mathcal{F})$-name for an element of ω^ω. We must prove that there is $q \leq p$ and $g \in \omega^\omega$ such that $q \Vdash \text{“}\dot{f} \cap g = \emptyset\text{”}$. For every $n \in \omega$, define $D_n = \{q \in \mathbb{PT}(\mathcal{F}) \mid \exists k \in \omega (q \Vdash \text{“}\dot{f}(n) \neq k\text{”})\}$, note that each D_n is purely dense by the previous lemma.

Let $\preceq$ be any well order of $\mathbb{PT}(\mathcal{F})$. We will recursively define a strategy σ for player I on the game $\mathcal{G}(\mathcal{F}, p, \omega^\omega)$ as follows:

1. I starts by playing p.

2. Assume we are at round $m+1$ and I has played $p_0, ..., p_m$ while player II has played $n_0, ..., n_{m-1}$. If player II plays n_m, then player I plays p_{m+1} where p_{m+1} is the $\preceq$-least element of $\mathbb{PT}(\mathcal{F})$ such that $p_{m+1} \in D_m$ and $p_{m+1} \leq_{T_m} p_m$ (where $T_m = T(p_m, n_m)$).

Since $\mathcal{F}$ is Canjar, we know that σ is not a winning strategy for player I, which means that there is a function $d \in \omega^\omega$ such that if player II plays $d(n)$ at the n-step then she will win. Let q be the condition constructed at the end of the game. Note that $q \in \bigcap\limits_{\in\omega} D_n$ in this way, we can define a function $g : \omega \longrightarrow \omega$ such that $q \Vdash$ "$\dot{f}(n) \neq g(n)$" for every $n \in \omega$. This finishes the proof. ■

We will now prove that if $\mathcal{F}$ is Canjar, then $\mathbb{PT}(\mathcal{F})$ does not add bounded eventually different reals. By $Fn(\omega)$ we will denote the set of all functions $z : a \longrightarrow \omega$ such that $a \in [\omega]^{<\omega}$. We will need the following lemma:

Lemma 32 *Let $m \in \omega, g \in \omega^\omega$ and $\dot{f}$ be a $\mathbb{PT}(\mathcal{F})$-name for a function bounded by g. The set $D = \{p \in \mathbb{PT}(\mathcal{F}) \mid \exists z \in Fn(\omega)(m \cap dom(z) = \emptyset \wedge p \Vdash$ "$\dot{f} \cap z \neq \emptyset$")$\}$ is purely dense.*

Proof. Let $p \in \mathbb{PT}(\mathcal{F})$ and $T \subseteq p$ a finite tree such that every maximal node of T is an element of $split(p)$. We need to find $q \leq_T p$ such that $q \in D$. Let $n = |T \cap split(p)|$ and for every $s \in T \cap split(p)$ let $p(s) \leq p_s$ be the biggest subtree of p_s such that every $t \in T \cap p(s)$ is a restriction of s. Let $T \cap split(p) = \{s_i \mid i < n\}$. Using the pure decision property, for every $s_i \in T \cap split(p)$ we can find $q(s_i) \in \mathbb{PT}(\mathcal{F})$ and $k_i \in \omega$ with the following properties:

1. $q(s_i) \leq p(s_i)$ and the stem of $q(s_i)$ is s_i.
2. $q(s_i) \Vdash$ "$\dot{f}(m+i) = k_i$".

Note that $q(s_i)$ can be found (with the pure decision property) since there are only finitely many possibilities for $\dot{f}(m+i)$. We now define a function $z : [m, m+n) \longrightarrow \omega$ given by $z(m+i) = k_i$. Let $q = \bigcup \{q(s) \mid s \in T \cap split(p)\}$. Note that $q \leq_T p$ and $q \Vdash$ "$\dot{f} \cap z \neq \emptyset$", this finishes the proof. ■

We now can prove the following:

Proposition 33 *If $\mathcal{F}$ is a Canjar filter, then $\mathbb{PT}(\mathcal{F})$ does not add bounded eventually different reals.*

Proof. Let $p \in \mathbb{PT}(\mathcal{F})$, $g \in \omega^\omega$ and $\dot{f}$ an $\mathbb{PT}(\mathcal{F})$-name for an element of ω^ω bounded by g. We must prove that there is $q \leq p$ and $h \in \omega^\omega$ such that $q \Vdash$ "$\left|\dot{f} \cap h\right| = \omega$". For every $m \in \omega$, define $D_m = \{p \in \mathbb{PT}(\mathcal{F}) \mid \exists z \in Fn(\omega)(m \cap dom(z) = \emptyset \wedge p \Vdash$ "$\dot{f} \cap z \neq \emptyset$")$\}$, which we already know that is a purely dense set.

Let $\preceq$ be any well order of $\mathbb{PT}(\mathcal{F})$. We will recursively define a strategy σ for player I on the game $\mathcal{G}(\mathcal{F}, p, \omega^{\omega})$ as follows:

1. I starts by playing p.
2. Assume we are at round $m+1$ and I has played $p_0, ..., p_m$ while player II has played $n_0, ..., n_{m-1}$. If player II plays n_m, then player I plays p_{m+1} where p_{m+1} is the $\preceq$-least element of $\mathbb{PT}(\mathcal{F})$ such that $p_{m+1} \in D_m$ and $p_{m+1} \leq_{T_m} p_m$ (where $T_m = T(p_m, n_m)$).

Since $\mathcal{F}$ is Canjar, we know that σ is not a winning strategy for player I, hence there is a function $d \in \omega^{\omega}$ such that if player II plays $d(n)$ at the n-step then, she will win. Let q be the condition constructed at the end of the game. Note that $q \in \bigcap_{\in\omega} D_n$, this means that for every $n \in \omega$, we can find a finite function z such that $dom(z) = \emptyset$ and $q \Vdash$ "$z \cap \dot{f} \neq \emptyset$". We can now easily find a function $h : \omega \longrightarrow \omega$ such that $q \Vdash$ "$\left|\dot{f} \cap h\right| = \omega$". $\blacksquare$

It is well known that adding a random real adds a bounded eventually different real. In this way we conclude the following:

Corollary 34 *If $\mathcal{F}$ is a Canjar filter, then $\mathbb{PT}(\mathcal{F})$ does not add random reals.*

We also have the following lemma:

Lemma 35 *Let $T \subseteq \omega^{<\omega}$ be a finite tree, $p \in \mathbb{PT}(\mathcal{F})$ such that $[T] \subseteq split(p)$, let $n = |split(p) \cap T|$ and let A be a finite set. If $\dot{x}$ is a $\mathbb{PT}(\mathcal{F})$-name such that $p \Vdash$ "$\dot{x} \in A$", there is $q \leq_T p$ and $B \in [A]^n$ such that $q \Vdash$ "$\dot{x} \in B$".*

Proof. The lemma easily follows by applying the pure decision property n-many times. $\blacksquare$

Preservation of P-points

Let $\mathcal{U}$ be an ultrafilter and $\mathbb{P}$ a partial order. We say that $\mathbb{P}$ *preserves* $\mathcal{U}$ if $\mathcal{U}$ is the base of an ultrafilter after forcing with $\mathbb{P}$. It is well known that no ultrafilter is preserved by Cohen, random, Silver or forcings adding a dominating real. Also there is an ultrafilter that is destroyed by any forcing that adds a new real (see [2]). On the other hand, certain forcings may preserve some ultrafilters, this is the case for Sacks and Miller forcings. The preservation of P-points is particularly interesting in light of the following theorem of Shelah (see [56] or [2]):

Proposition 36 (Shelah) *Let δ be a limit ordinal, $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha < \delta \rangle$ a countable support iteration of proper forcings and let $\mathcal{U}$ be a P-point. If $\mathbb{P}_\alpha \Vdash$ "$\dot{\mathbb{Q}}_\alpha$ preserves $\mathcal{U}$" for every $\alpha < \delta$, then $\mathbb{P}_\delta$ preserves $\mathcal{U}$.*

In particular, forcings that preserve P-points do not add Cohen or random reals, even in the iteration. It is well known that Sacks forcing preserves P-points and Miller forcing preserves an ultrafilter $\mathcal{U}$ if and only if $\mathcal{U}$ is a P-point (see [49]). Note that if $\mathbb{P}$ diagonalizes $\mathcal{U}$, then $\mathbb{P}$ does not preserves $\mathcal{U}$, however, it is possible to not preserve an ultrafilter without diagonalizing it. For more on preservation of ultrafilters, the reader may consult [42], [49], [2], [47], [60] and [62].

In light of the results of the previous section, it might be tempting to conjecture that if $\mathcal{F}$ is a Canjar filter, then $\mathbb{PT}(\mathcal{F})$ preserves P-points. However, this is not the case: the simplest example is to take $\mathcal{U}$ a Canjar P-point, since $\mathbb{PT}(\mathcal{U})$ diagonalizes $\mathcal{U}$, it follows that $\mathbb{PT}(\mathcal{U})$ does not preserve $\mathcal{U}$. In this section, we will find a condition on a filter $\mathcal{F}$ that guarantees preserving a certain P-point.

We will need the following result, which is a particular case of lemma 20:

Lemma 37 *Let $\mathcal{F}$ be a Canjar filter and $X \in (\mathcal{F}^{<\omega})^+$. There is $Y \subseteq X$ such that $Y \in (\mathcal{F}^{<\omega})^+$ and for every $n \in \omega$, the set $\{s \in Y \mid s \cap n \neq \emptyset\}$ is finite.*

We can now prove the following:

Lemma 38 *Let $\mathcal{F}$ be a Canjar filter, $p \in \mathbb{PT}(\mathcal{F})$ and $c : split(p) \longrightarrow 2$. There is $q \leq p$ such that $split(q)$ is c-monochromatic.*

Proof. Assume there is no $q \leq p$ such that $split(q)$ is 0-monochromatic. We will prove that there is $q \leq p$ such that $split(q)$ is 1-monochromatic. Given $s \in split(p)$, we define $X(s)$ as the set of all $t \setminus s$ for which $t \in split(p)$, $s \subseteq t$ and $c(t) = 1$. We claim that $X(s) \in (\mathcal{F}^{<\omega})^+$. Assume this is not the case, so there is $A \in \mathcal{F}$ such that A does not contain any element of $X(s)$. Let $q \leq p_s$ such that if $t \in split(q)$, then $t \setminus s \subseteq A$. It follows that if $t \in split(q)$, then $c(t) = 0$, so $split(q)$ is 0-monochromatic, which is a contradiction. We conclude that $X(s) \in (\mathcal{F}^{<\omega})^+$ for every $s \in split(p)$. By the previous result, we can find $Y(s) \subseteq X(s)$ such that $Y(s) \in (\mathcal{F}^{<\omega})^+$ and for every $n \in \omega$, the set $\{z \in Y \mid z \cap n \neq \emptyset\}$ is finite. The proof follows by a simple fusion argument. ■

Let $\mathcal{F}$ be a filter. The *Canjar game* $\mathcal{G}_{Canjar}(\mathcal{F})$ is defined as follows:

I	X_0		X_1		X_2		...
II		Y_0		Y_1		Y_2	

Where $X_i \in (\mathcal{F}^{<\omega})^+$ and $Y_i \in [X_i]^{<\omega}$ for every $i \in \omega$. The player II *wins the game* $\mathcal{G}_{Canjar}(\mathcal{F})$ if $\bigcup_{n\in\omega} Y_n \in (\mathcal{F}^{<\omega})^+$. In [18], Chodounský, Repovš and

Zdomskyy showed that sets in $(\mathcal{F}^{<\omega})^{+}$ naturally correspond to open covers of $\mathcal{F}$ (viewed as a subspace of $\wp(\omega)$). Moreover, they proved that a filter $\mathcal{F}$ is Canjar if and only if $\mathcal{F}$ has the Menger property (see [55] for the definition of Menger property). In this way, the Canjar game is just a particular case of the Menger game that has been extensively studied in topology.

Proposition 39 ([18]) *Let $\mathcal{F}$ be a filter. The following are equivalent:*

1. *$\mathcal{F}$ is Canjar.*
2. *$\mathcal{F}$ is Menger.*
3. *Player I does not have a winning strategy in $\mathcal{G}_{Canjar}(\mathcal{F})$.*

We will now prove the following lemma:

Lemma 40 *Let $\mathcal{F}$ be a Canjar filter, $p \in \mathbb{PT}(\mathcal{F})$, $\dot{B}$ a $\mathbb{PT}(\mathcal{F})$-name such that $p \Vdash$ "$\dot{B} \in [\omega]^{\omega}$" and $s \in split(p)$. There are $q \in \mathbb{PT}(\mathcal{F})$, $B_s \subseteq \omega$ and $\langle F_n \rangle_{n\in\omega}$ such that the following holds:*

1. *$q \leq p$ and $stem(q) = s$.*
2. *F_n is a finite subset of $[\omega]^{<\omega} \setminus \{\emptyset\}$ for every $n \in \omega$.*
3. *For every $n \in \omega$, if $t_0 \in F_n$ and $t_1 \in F_{n+1}$, then $max(t_0) < min(t_1)$.*
4. *$spsuc_q(s) = \bigcup\limits_{n\in\omega} F_n$.*
5. *If $t \in F_n$, then $q_{s^\frown t} \Vdash$ "$\dot{B} \cap (n+1) = B_s \cap (n+1)$".*

Proof. We define σ a strategy for player I in $\mathcal{G}_{Canjar}(\mathcal{F})$ as follows:

1. Using the pure decision property, player I finds $q_0 \leq p$ with $stem(q) = s$ and w_0 such that $q^0 \Vdash$ "$\dot{B} \cap 1 = w_0$" and plays $X_0 = spsuc_{q^0}(s)$.
2. Assume that player II plays $Y_0 \in [X_0]^{<\omega}$. Let $l_0 \in \omega$ be the least such that $\bigcup Y_0 \subseteq l_0$ and $Z_0 = \{t \in X_0 \mid t \cap l_0 = \emptyset\} \in (\mathcal{F}^{<\omega})^{+}$. Let $\overline{q}^1 \leq q^0$ be the condition defined as $\overline{q}^1 = \bigcup\limits_{t\in Z_0} (q^0)_{s^\frown t}$. Using the pure decision property, player I finds $q^1 \leq \overline{q}^1$ with $stem(q) = s$ and w_1 such that $q^1 \Vdash$ "$\dot{B} \cap 2 = w_1$" and plays $X_1 = spsuc_{q^1}(s)$.
3. In general, at step n, the player I has constructed a decreasing sequence $\langle q^i \rangle_{i\leq n}$ where $stem(q^n) = s$, an increasing sequence $\langle w_i \rangle_{i\leq n}$ such that $q^n \Vdash$ "$\dot{B} \cap (n+1) = w_n$" has played the sequence $\langle X_i \rangle_{i\leq n}$ where $X_i = spsuc_{q^i}(s)$, he has also constructed an increasing sequence $\langle l_i \rangle_{i\leq n}$ such that $Y_i \subseteq [l_{i-1}, l_i)$ (where $l_{-1} = 0$ and Y_i is the response of player II at round i). Assume that player II plays $Y_n \in [X_n]^{<\omega}$. Let $l_n \in \omega$ be the

least such that $\bigcup Y_n \subseteq l_n$ and $Z_n = \{t \in X_n \mid t \cap l_n = \emptyset\} \in (\mathcal{F}^{<\omega})^+$. Let $\overline{q}^{n+1} \leq q^n$ be the condition defined as $\overline{q}^{n+1} = \bigcup\limits_{t \in Z_n} (q^n)_{s^\frown t}$. Using the pure decision property, player I finds $q^{n+1} \leq \overline{q}^n$ with $stem(q) = s$ and w_{n+1} such that $q^{n+1} \Vdash$ "$\dot{B} \cap (n+2) = w_{n+1}$" and plays $X_{n+1} = spsuc_{q^{n+1}}(s)$.

Since $\mathcal{F}$ is Canjar, we know that σ is not a winning strategy for the Canjar game. Consider a run in which player II defeated the strategy. Let $q = \bigcup\limits_{t \setminus s \in Y_i} (q^i)_{s^\frown t}$ and note that $q \in \mathbb{PT}(\mathcal{F})$ since player II won the game. Define $F_i = Y_i$ and $B_s = \bigcup\limits_{i \in \omega} w_i$, it is clear that this are the items we were looking for. ■

Let $\mathcal{U}$ be an ultrafilter. Recall that the *P-point game* $\mathcal{G}_{P\text{-}point}(\mathcal{U})$ is defined as follows:

I	W_0		W_1		...
II		z_0		z_1	

Where $W_i \in \mathcal{U}$ and $z_i \in [W_i]^{<\omega}$. The player II will *win the game* $\mathcal{G}_{P\text{-}point}(\mathcal{U})$ if $\bigcup\limits_{m \in \omega} z_m \in \mathcal{U}$. It is well known that player II can not have a winning strategy for this game. The following is a well known result of Galvin and Shelah (see [2] for a proof):

Proposition 41 (Galvin, Shelah) *Let $\mathcal{U}$ be an ultrafilter. The following are equivalent:*

1. *$\mathcal{U}$ is a P-point.*
2. *Player I does not have a winning strategy in $\mathcal{G}_{P\text{-}point}(\mathcal{U})$.*

Let $\mathcal{U}$ be an ultrafilter and $\mathcal{F}$ a filter. We will now define the game $\mathcal{H}(\mathcal{U}, \mathcal{F})$, which is a fusion between the P-point game and the game for $\mathbb{PT}(\mathcal{F})$. The game is defined as follows:

I	W_0		p_0		W_1		p_1		...
II		z_0		n_0		z_1		n_1	

Where the following conditions hold for every $i \in \omega$:

1. $W_i \in \mathcal{U}$.
2. $z_i \in [W_i]^{<\omega}$.

3. $p_i \in \mathbb{PT}(\mathcal{F})$.
4. $\langle n_i \rangle_{i\in\omega}$ is an increasing sequence of natural numbers.
5. $p_{m+1} \leq_{T_m} p_m$ where $T_m = T(p_m, n_m)$.

The player II will *win the game* $\mathcal{H}(\mathcal{U}, \mathcal{F})$ if $\bigcup\limits_{m\in\omega} T_m \in \mathbb{PT}(\mathcal{F})$ and $\bigcup\limits_{m\in\omega} z_m \in \mathcal{U}$.

Definition 42 *Let $\mathcal{F}$ be a filter and $\mathcal{U}$ an ultrafilter. We will say that $\mathcal{F}$ is an* $\mathcal{U}$-Canjar filter *if player I has no winning strategy in $\mathcal{H}(\mathcal{U}, \mathcal{F})$.*

Note that the previous notion is only of interest when $\mathcal{F}$ is Canjar and $\mathcal{U}$ is a P-point. It is easy to see that if $\mathcal{U}$ is an ultrafilter, then $\mathcal{U}$ is not $\mathcal{U}$-Canjar (and will also follow by the next result). Our interest in $\mathcal{U}$-Canjar filters is that (as we are about to prove), its Miller forcing preserves $\mathcal{U}$. Our proof is based on the argument of Miller that the superperfect forcing preserves P-points (see [49]).

Proposition 43 *If $\mathcal{U}$ is a P-point and $\mathcal{F}$ is an $\mathcal{U}$-Canjar filter, then $\mathbb{PT}(\mathcal{F})$ preserves $\mathcal{U}$.*

Proof. Let $p \in \mathbb{PT}(\mathcal{F})$ and $\dot{B}$ a $\mathbb{PT}(\mathcal{F})$-name such that $p \Vdash$ "$\dot{B} \in [\omega]^{\omega}$". By lemma 40, we may assume that for every $s \in split(p)$, there is $B_s \subseteq \omega$ and $\langle F_n^s \rangle_{n\in\omega}$ with the following properties:

1. F_n^s is a finite subset of $[\omega]^{<\omega} \setminus \{\emptyset\}$ for every $n \in \omega$.
2. For every $n \in \omega$, if $t_0 \in F_n^s$ and $t_1 \in F_{n+1}^s$, then $max(t_0) < min(t_1)$.
3. $spsuc_p(s) = \bigcup\limits_{n\in\omega} F_n^s$.
4. If $t \in F_n^s$, then $p_{s^\frown t} \Vdash$ "$\dot{B} \cap (n+1) = B_s \cap (n+1)$".

Furthermore, by the lemma 38 we may assume that either $B_s \in \mathcal{U}$ for all $s \in split(p)$ or $B_s \in \mathcal{U}^*$ for all $s \in split(p)$. We will assume that $B_s \in \mathcal{U}$ for all $s \in split(p)$ (in the other case we work with $\omega \setminus \dot{B}$)). Let s_0 be the stem of p. We will define a strategy σ for player I in $\mathcal{H}(\mathcal{U}, \mathcal{F})$ as follows:

1. I starts by playing $W_0 = B_{s_0}$.
2. Assume that player II plays $z_0 \in [W_0]^{<\omega}$. Let $l_0 = max(z_0)$, player I will play $p_0 = \bigcup \left\{ p_{s_0^\frown t} \mid t \in F_i^{s_0} \wedge i > l_0 \right\}$. Note that $p_0 \Vdash$ "$z_0 \subseteq \dot{B}$".
3. Assume that player II plays $n_0 \in \omega$. Now, the player I will play the set $W_1 = \bigcap \{B_s \setminus l_0 \mid s \in split(p_0) \cap T_0\}$ (where $T_0 = T(p_0, n_0)$).

4. In general, lets assume in the game it has already been played the sequence $\langle W_0, z_0, p_0, n_0, W_1...W_m, z_m, p_m\rangle$. At the same time, player I has been making sure that the sequence $\langle p_i\rangle_{i\leq m}$ has the following properties:

   (a) $p_{i+1} \leq_{T_i} p_i$ (where $T_i = T(p_i, n_i)$) for all $i < m$.

   (b) $p_i \Vdash$ "$z_i \subseteq \dot{B}$" for all $i \leq m$.

   Now, assume that player II plays $n_m \in \omega$. Player I proceeds to play $W_{m+1} = \bigcap \{B_s \setminus l_m \mid s \in split(p_m) \cap T_m\}$ where $l_m = max(z_m)$. Assume player II responds with $z_{m+1} \in [W_{m+1}]^{<\omega}$. For every $t \in T_m \cap split(p_m)$, let $p^t_m \leq p_m$ be the biggest subtree such that $stem(p^t_m) = t$ and every node in $p^t_m \cap T_m$ is contained in t. Let $q^t = \bigcup \{(p^t_m)_x \mid x \in F^t_i \wedge i > l_{m+1}\}$ (where $l_{m+1} = max(z_{m+1})$) and now player I counterattacks with $p_{m+1} = \bigcup \{q^t \mid t \in T_m \cap split(p_m)\}$. It is easy to see that $p_{m+1} \leq_{T_m} p_m$ and $p_{m+1} \Vdash$ "$z_{m+1} \subseteq \dot{B}$".

Since $\mathcal{F}$ is $\mathcal{U}$-Canjar, we know that σ is not a winning strategy. Consider a run of the game where player I followed the strategy σ, but player II was the winner. In this way, we know that $U = \bigcup_{i\in\omega} z_i \in \mathcal{U}$ and $q = \bigcup_{i\in\omega} T_i$ is a condition of $\mathbb{PT}(\mathcal{F})$. By construction, it follows that $q \Vdash$ "$U \subseteq \dot{B}$" and we are done. ■

A model of $\omega_1 = \mathfrak{u} < \mathfrak{s}$

In order to increase the splitting number, it is enough to diagonalize an ultrafilter, and to preserve the ultrafilter number, it is enough preserve a P-point. In this way, in order to construct a model of $\mathfrak{u} < \mathfrak{s}$ it is enough to find a P-point $\mathcal{W}$ and an ultrafilter $\mathcal{U}$ that is $\mathcal{W}$-Canjar. In this situation, we will have that $\mathbb{PT}(\mathcal{W})$ adds an unsplit real while preserving $\mathcal{W}$. In this section, we will use our results to build a model of $\mathfrak{u} < \mathfrak{s}$. This result is not new, it already holds in the Blass-Shelah model (see [6] or [2]). At least in the opinion of the authors, the combinatorics involved in our forcing are simpler than the ones from the Blass-Shelah forcing.

We will first focus on constructing a $\mathcal{B}$-Canjar ultrafilter for some $\mathfrak{b}$-family $\mathcal{B}$. Such ultrafilters can either be constructed under the Continuum Hypothesis or forced with a σ-closed forcing (see [7], [14], [16] or [27]). As was mentioned before, this follows by the result of Shelah and the decomposition representation of Brendle and Raghavan. In [27] the first author, Michael Hrušák and Arturo Antonio Martinez Celis-Rodriguez published a proof of the consistency of $\mathfrak{b} < \mathfrak{s}$ and $\mathfrak{b} < \mathfrak{a}$ using directly the representation of Brendle and Raghavan. This section and the following borrows some of the arguments from [27].

Given X a collection of finite non empty-subsets of ω, we define $\mathcal{C}(X) = \{A \subseteq \omega \mid \forall s \in X\, (s \cap A \neq \emptyset)\}$. The following lemma contains some of the combinatorial properties of compact sets that we will need:

Lemma 44 *Let $\mathcal{F}$ be a filter, $\mathcal{D} \subseteq \mathcal{F}$ be a compact set and $X \in (\mathcal{F}^{<\omega})^+$.*

1. *$\mathcal{C}(X)$ is a compact set.*
2. *There is $Y \in [X]^{<\omega}$ such that for every $A \in \mathcal{D}$ there is $s \in Y$ such that $s \subseteq A$.*
3. *If $\mathcal{C}(X) \subseteq \mathcal{F}$ then for every $n \in \omega$ there is $S \in [X]^{<\omega}$ such that if $A_0, ..., A_n \in \mathcal{C}(S)$ and $F \in \mathcal{D}$ then $A_0 \cap ... \cap A_n \cap F \neq \emptyset$.*
4. *If $\mathcal{U}$ is an ultrafilter and $Y \subseteq [\omega]^{<\omega}$ then $Y \in (\mathcal{U}^{<\omega})^+$ if and only if $\mathcal{C}(Y) \subseteq \mathcal{U}$.*

Proof. For item 1, it is easy to see that $\mathcal{C}(X)$ is a closed subset of $\wp(\omega)$. We now prove item 2, let $X \in (\mathcal{F}^{<\omega})^+$ and $\mathcal{D} \subseteq \mathcal{F}$ a compact set. For every $s \in X$, we define $U(s) = \{A \mid s \subseteq A\}$. It is easy to see that $U(s)$ is an open set and $\{U(s) \mid s \in X\}$ is an open cover for $\mathcal{D}$ (because $X \in (\mathcal{F}^{<\omega})^+$). Since $\mathcal{D}$ is compact, there is $Y \in [X]^{<\omega}$ such that $\{U(s) \mid s \in Y\}$ is an open cover for $\mathcal{D}$. Clearly Y is the set we were looking for.

We now prove 3, let $\mathcal{C}(X) \subseteq \mathcal{F}$ and $n \in \omega$. Given $s \in X$ define $K(s)$ as the set of all $(A_0, \ldots, A_n) \in \mathcal{C}(s)^{n+1}$ with the property that there is $F \in \mathcal{D}$ such that $A_0 \cap \ldots \cap A_n \cap F = \emptyset$. It is easy to see that $K(s)$ is a compact. Note that if $(A_0, \ldots, A_n) \in \bigcap\limits_{s \in X} K(s)$ then $A_0, \ldots, A_n \in \mathcal{C}(X) \subseteq \mathcal{F}$ and there would be $F \in \mathcal{D} \subseteq \mathcal{F}$ such that $A_0 \cap \ldots \cap A_n \cap F = \emptyset$ which is clearly a contradiction. Since the $K(s)$ are compact, then there must be $S \in [F]^{<\omega}$ such that $\bigcap\limits_{s \in S} K(s) = \emptyset$. It is easy to see that this is the S we are looking for.

We now prove 4. Let $\mathcal{U}$ be an ultrafilter and $Y \subseteq [\omega]^{<\omega}$. We will prove that $Y \notin (\mathcal{U}^{<\omega})^+$ if and only if $\mathcal{C}(Y) \nsubseteq \mathcal{U}$. First assume that $Y \notin (\mathcal{U}^{<\omega})^+$, this means that there is $A \in \mathcal{U}$ that does not contain any element of Y, so $B = \omega \setminus A$ intersects every element of Y, hence $B \in \mathcal{C}(Y)$ which implies that $\mathcal{C}(Y) \nsubseteq \mathcal{U}$. Now assume that $\mathcal{C}(Y) \nsubseteq \mathcal{U}$, so there is $B \in \mathcal{C}(Y)$ such that $B \notin \mathcal{U}$, hence $A = \omega \setminus B \in \mathcal{U}$. Since $B \in \mathcal{C}(Y)$, this implies that A does not contain any element of Y, so $Y \notin (\mathcal{U}^{<\omega})^+$. ■

We will need the following notion:

Definition 45 *Let $\mathcal{I}$ be an ideal on ω. We define $\mathbb{F}_\sigma(\mathcal{I})$ as the collection of all F_σ-filters $\mathcal{F}$ such that $\mathcal{F} \cap \mathcal{I} = \emptyset$. We order $\mathbb{F}_\sigma(\mathcal{I})$ by inclusion.*

Note that an F_σ-filter $\mathcal{F}$ is in $\mathbb{F}_\sigma(\mathcal{I})$ if and only if $\mathcal{F} \cup \mathcal{I}^*$ generates a filter. The following are some properties of this types of forcings:

Lemma 46 *Let $\mathcal{I}$ be an ideal on ω.*

1. *$\mathbb{F}_\sigma(\mathcal{I})$ is a σ-closed forcing.*
2. *$\mathbb{F}_\sigma(\mathcal{I})$ adds an ultrafilter (which we will denote by $\mathcal{U}_{gen}(\mathcal{I})$) disjoint from $\mathcal{I}$.*
3. *$\mathbb{F}_\sigma(\mathcal{I}) * \mathbb{PT}(\dot{\mathcal{U}}_{gen}(\mathcal{I}))$ and $\mathbb{F}_\sigma(\mathcal{I}) * \mathbb{M}(\dot{\mathcal{U}}_{gen}(\mathcal{I}))$ are proper forcings that destroy $\mathcal{I}$.*

If $\mathcal{A}$ is a MAD family, we will denote $\mathbb{F}_\sigma(\mathcal{A})$ instead of $\mathbb{F}_\sigma(\mathcal{I}(\mathcal{A}))$ and $\mathcal{U}_{gen}(\mathcal{A})$ instead of $\mathcal{U}_{gen}(\mathcal{I}(\mathcal{A}))$. Note that $\mathbb{F}_\sigma\left([\omega]^{<\omega}\right)$ is the collection of all F_σ-filters. In this case, we will only denote it by $\mathbb{F}_\sigma$ and by $\mathcal{U}_{gen}$ we will denote the generic ultrafilter added by $\mathbb{F}_\sigma$. The following lemma is easy and left to the reader:

Lemma 47 *If $\mathcal{I}$ is an ideal, $X \subseteq [\omega]^{<\omega}$ and $\mathcal{F} \in \mathbb{F}_\sigma(\mathcal{I})$, then $\mathcal{F} \Vdash$"$X \in (\dot{\mathcal{U}}_{gen}(\mathcal{I})^{<\omega})^+$" if and only if $\mathcal{C}(X) \subseteq \langle \mathcal{F} \cup \mathcal{I}^* \rangle$ (where $\langle \mathcal{F} \cup \mathcal{I}^* \rangle$ is the filter generated by $\mathcal{F} \cup \mathcal{I}^*$).*

In particular, if $\mathcal{F} \in \mathbb{F}_\sigma$ and $X \subseteq [\omega]^{<\omega}$, then $\mathcal{F} \Vdash$"$X \in (\dot{\mathcal{U}}_{gen}{}^{<\omega})^+$" if and only if $\mathcal{C}(X) \subseteq \mathcal{F}$. We will now prove the following:

Proposition 48 *If $\mathcal{B} \in V$ is a $\mathfrak{b}$-family, then $\mathbb{F}_\sigma$ forces that $\dot{\mathcal{U}}_{gen}$ is $\mathcal{B}$-Canjar.*

Proof. By the previous observation and since $\mathbb{F}_\sigma$ is σ-closed, it is enough to show that if $\mathcal{F} \Vdash$ "$\overline{X} = \langle X_n \rangle_{n\in\omega} \subseteq (\dot{\mathcal{U}}_{gen}^{<\omega})^+$" then there is $\mathcal{G} \leq \mathcal{F}$ and $f \in \mathcal{B}$ such that $\mathcal{C}\left(\overline{X}_f\right) \subseteq \mathcal{G}$.

Let $\mathcal{F} = \bigcup \mathcal{C}_n$ where each $\mathcal{C}_n$ is compact and they form an increasing chain. By a previous lemma, there is $g : \omega \longrightarrow \omega$ such that if $n \in \omega$, $F \in \mathcal{C}_n$ and $A_0, \ldots, A_n \in \mathcal{C}(X_n \cap \wp(g(n)))$ then $A_0 \cap \ldots. \cap A_n \cap F \neq \emptyset$. Since $\mathcal{B}$ is unbounded, then there is $f \in \mathcal{B}$ such that $f \nleq^* g$. We claim that $\mathcal{F} \cup \mathcal{C}\left(\overline{X}_f\right)$ generates a filter. Let $F \in \mathcal{C}_n$ and $A_0, \ldots, A_m \in \mathcal{C}\left(\overline{X}_f\right)$. We must show that $A_0 \cap \ldots. \cap A_m \cap F \neq \emptyset$. Since f is not bounded by g, we may find $r > n, m$ such that $f(r) > g(r)$. In this way, $A_0, \ldots, A_n \in \mathcal{C}(X_n \cap \wp(g(n)))$ and then $A_0 \cap \ldots. \cap A_m \cap F \neq \emptyset$. Finally, we can define $\mathcal{G}$ as the filter generated by $\mathcal{F} \cup \mathcal{C}\left(\overline{X}_f\right)$. ■

In this way, we conclude the following:

Corollary 49 *The forcing $\mathbb{F}_\sigma * \mathbb{PT}(\dot{\mathcal{U}}_{gen})$ is proper, adds an unsplit real, preserves all $\mathfrak{b}$-scales from the ground model and does not add Cohen reals.*

A forcing notion if called *weakly ω^ω-bounding* if it does not add dominating reals. Unlike the ω^ω-bounding property, the weakly ω^ω-bounding property is not preserved under two step iteration (see [1]). However, Shelah proved the following preservation result.

Proposition 50 (Shelah, see [1]) *If $\gamma \leq \omega_2$ is limit and $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha \leq \gamma \rangle$ is a countable support iteration of proper forcings and each $\mathbb{P}_\alpha$ is weakly ω^ω-bounding (over V) then $\mathbb{P}_\gamma$ is weakly ω^ω-bounding.*

Note that $\mathbb{P}$ is weakly ω^ω-bounding if and only if it preserves the unboundedness of all (one) dominating families. By applying the result of Shelah we can easily conclude the following result.

Corollary 51 *If V satisfies CH and $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha \leq \omega_2 \rangle$ is a countable support iteration of proper forcings such that $\mathbb{P}_\alpha$ forces that $\dot{\mathbb{Q}}_\alpha$ preserves the unboundedness of all well-ordered unbounded families, then $\mathbb{P}_{\omega_2}$ is weakly ω^ω-bounding.*

With this results we can conclude the following result of Shelah:

Proposition 52 (Shelah) *There is a model of $\omega_1 = \mathfrak{b} < \mathfrak{s} = \mathsf{cov}(\mathcal{M}) = \mathfrak{c} = \omega_2$.*

Proof. We perform a countable support iteration $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha \leq \omega_2 \rangle$ where $\mathbb{P}_\alpha \Vdash$ "$\dot{\mathbb{Q}}_\alpha = \mathbb{F}_\sigma * \mathbb{M}(\dot{\mathcal{U}}_{gen})$". The result follows by the previous results. ■

As was mentioned before, by the result of Brendle and Raghavan, $\mathbb{F}_\sigma * \mathbb{M}(\dot{\mathcal{U}}_{gen})$ is forcing equivalent to the original creature forcing of Shelah for $\mathfrak{b} < \mathfrak{s}$. We will now prove that if we iterate $\mathbb{F}_\sigma * \mathbb{PT}(\dot{\mathcal{U}}_{gen})$, we will get a model of $\mathfrak{u} < \mathfrak{s}$. Although we will not need the following result, it is illustrative to prove first the following:

Proposition 53 *If $\mathcal{U}$ is a P-point and $\mathcal{F}$ is an F_σ-filter, then $\mathcal{F}$ is $\mathcal{U}$-Canjar.*

Proof. Let $\mathcal{U}$ be a P-point and $\mathcal{F} = \bigcup\limits_{n\in\omega} \mathcal{C}_n$ be an F_σ-filter, where $\langle \mathcal{C}_n \rangle_{n\in\omega}$ is an increasing sequence of compact sets. We will argue by contradiction, so assume that $\mathcal{F}$ is not $\mathcal{U}$-Canjar, i.e. player I has a winning strategy for the game $\mathcal{H}(\mathcal{U}, \mathcal{F})$, call σ such strategy. We will use σ to construct a winning strategy for I in the P-point game, which will obviously entail a contradiction.

Given $X \in (\mathcal{F}^{<\omega})^+$ and $n \in \omega$, choose $Y(X, n) \in [X]^{<\omega}$ such that every element of $\mathcal{C}_n$ contains an element of $Y(X, n)$ (which is possible by lemma 44). We now define π a strategy for player I in $\mathcal{G}_{P\text{-}point}(\mathcal{U})$ as follows:

1. Player I starts by playing $W_0 = \sigma(\emptyset)$ (i.e. W_0 is the first play in the game $\mathcal{H}(\mathcal{U}, \mathcal{F})$).

2. Assume player II plays $z_0 \in [W_0]^{<\omega}$ as her response in $\mathcal{H}(\mathcal{U}, \mathcal{F})$. Let $p_0 = \sigma(W_0, z_0)$ and s_0 be the stem of p_0. Define $n_0 > d^{-1}(s_0)$ to be the least integer such that $d^{-1}(s_0{}^\frown t) < n_0$ for all $t \in Y(spsuc_{p_0}(s_0), 0)$. Player I will play (in $\mathcal{G}_{P\text{-}point}(\mathcal{U})$) $W_1 = \sigma(W_0, z_0, p_0, n_0)$ (i.e. his response in $\mathcal{H}(\mathcal{U}, \mathcal{F})$ if player II had played n_0).

3. In general assume that it has been played the sequence $\langle W_0, z_0, ..., W_m \rangle$. At the same time, in secret the player I has been constructed a partial play $\langle W_0, z_0, p_0, n_0, W_1, z_1, p_1, n_1 ..., W_m \rangle$ in the game $\mathcal{H}(\mathcal{U}, \mathcal{F})$ following σ such that for every $i < m$, the integer n_i has the following property: for every $u \in T(p_i, n_{i-1})$ (where $n_{-1} = d^{-1}(s_0)$) and for every $t \in Y(spsuc_{p_i}(u), i)$, we have that $d^{-1}(u^\frown t) < n_i$. Assume that player II plays z_m as her next response in $\mathcal{G}_{P\text{-}point}(\mathcal{U})$. Let p_m be the tree defined as $\sigma(W_0, z_0, n_0, W_1, ..., W_m, z_m)$ and let $n_m > n_{m-1}$ the least integer with the following property: for every $u \in T(p_m, n_{m-1})$ and for every $t \in Y(spsuc_{p_m}(u), m)$, we have that $d^{-1}(u^\frown t) < n_m$. Player I will play the set W_{m+1} that is defined as $\sigma(W_0, z_0, n_0, W_1, ..., W_m, z_m, p_m, n_m)$.

The game $\mathcal{G}_{P\text{-}point}(\mathcal{U})$:

I	W_0		W_1		...
II		z_0		z_1	

The game $\mathcal{H}(\mathcal{U}, \mathcal{F})$:

I	W_0		p_0		W_1		p_1		...
II		z_0		n_0		z_1		n_1	

We claim that π is a winning strategy for player I in $\mathcal{G}_{P\text{-}point}(\mathcal{U})$. Consider a run of the game in which player I played according to π. Let $Z = \bigcup\limits_{n \in \omega} z_n$, we will prove that $Z \notin \mathcal{U}$. Let $q = \bigcup\limits_{i \in \omega} T(p_i, n_i)$ be the tree that was constructed during the play. It is easy to see that $q \in \mathbb{PT}(\mathcal{F})$, but since player I was following his winning strategy σ in the side game, we know that he won, so it must be the case that $Z \notin \mathcal{U}$. This shows that π is a winning strategy for player I in $\mathcal{G}_{P\text{-}point}(\mathcal{U})$. Since player I can not have a winning strategy in the P-point game, we get a contradiction. ■

We will now prove that the forcing $\mathbb{F}_\sigma * \mathbb{PT}(\dot{\mathcal{U}}_{gen})$ preserves all ground model P-points. First we will need the following lemma, which is a slight generalization of part of lemma 44:

Lemma 54 *Let $\mathcal{F}$ be a filter, $\mathcal{D} \subseteq \mathcal{F}$ a compact set and $X_1, ..., X_n \subseteq [\omega]^{<\omega}$ such that $\mathcal{C}(X_1), ..., \mathcal{C}(X_n) \subseteq \mathcal{F}$. There are $Y_1 \in [X_1]^{<\omega}, ..., Y_n \in [X_n]^{<\omega}$ such that for every $F \in \mathcal{D}$ and for every $A_i^1, ..., A_i^n \in \mathcal{C}(Y_i)$ (with $i \leq n$), we have that $F \cap \bigcap\limits_{i,j \leq n} A_i^j \neq \emptyset$.*

Proof. Consider the space $Z = (\prod\limits_{i=1}^{n} \wp(\omega)^n) \times \mathcal{D}$, which we know is compact. Given $l \in \omega$, let $K(l)$ be the set of all $(\langle A_i^1, ..., A_i^n \rangle_{i \leq n}, F) \in Z$ such that $A_i^1, ..., A_i^n \in \mathcal{C}(X_i \cap \wp(l))$ (for every $i \leq n$) and $F \cap \bigcap\limits_{i,j \leq n} A_i^j = \emptyset$. Clearly $K(l)$ is a closed subspace. Since $\mathcal{C}(X_1), ..., \mathcal{C}(X_n), \mathcal{D} \subseteq \mathcal{F}$, we conclude that $\bigcap\limits_{l \in \omega} K(l) = \emptyset$, hence by the compactness of Z, we conclude that there is $l \in \omega$ such that $K(l) = \emptyset$. Let $Y_i = X_i \cap \wp(l)$. It is clear that these are the sets we were looking for. ■

We can now prove the following result, which is a fusion of the proofs of proposition 53 and proposition 48:

Proposition 55 *If $\mathcal{W}$ is a P-point, then $\mathbb{F}_\sigma$ forces that $\dot{\mathcal{U}}_{gen}$ is $\mathcal{W}$-Canjar.*

Proof. We will prove the proposition by contradiction. Assume there is $\mathcal{F}$ an F_σ-filter and σ such that $\mathcal{F}$ forces that σ is a winning strategy for player I in $\mathcal{H}(\mathcal{W}, \dot{\mathcal{U}}_{gen})$. Note that strategies for I are countable objects and since $\mathbb{F}_\sigma$ is σ-closed, it is enough to consider ground model strategies. Let $\mathcal{F} = \bigcup\limits_{n \in \omega} \mathcal{C}_n$ where $\langle \mathcal{C}_n \rangle_{n \in \omega}$ is an increasing sequence of compact sets. We will use σ to construct a winning strategy for I in the game $\mathcal{G}_{P\text{-}point}(\mathcal{W})$, which will be a contradiction.

Note that if p is a Miller tree such that p is a possible response of player I according to σ and $s \in split(p)$, then $\mathcal{F} \Vdash$ "$spsuc_p(s) \in (\dot{\mathcal{U}}_{gen}^{<\omega})^+$" (this is because $\mathcal{F}$ is forcing that σ is a strategy for player I, which implies that p must be a legal move), in particular $\mathcal{C}(spsuc_p(s)) \subseteq \mathcal{F}$.

For every $\mathcal{X} = \{X_1, ..., X_n\}$ such that $X_i \subseteq [\omega]^{<\omega} \setminus \{\emptyset\}$ and $\mathcal{C}(X_i) \subseteq \mathcal{F}$ for every $i \leq n$ and for every $k \in \omega$, fix a function $F_{(\mathcal{X},k)} : \mathcal{X} \longrightarrow [[\omega^{<\omega}]]^{<\omega}$ with the following properties:

1. $Y_i = F_{(\mathcal{X},k)}(X_i) \in [X_i]^{<\omega}$ for every $i \leq n$.
2. For every $B \in \mathcal{C}_k$ and for every $A_i^1, ..., A_i^n \in \mathcal{C}(Y_i)$ (with $i \leq n$), we have that $B \cap \bigcap\limits_{i,j \leq n} A_i^j \neq \emptyset$.

We know such $F_{(\mathcal{X},k)}$ exists by lemma 54. The proof now proceeds in a very similar way as the proof of proposition 53. We define π a strategy for player I in $\mathcal{G}_{P\text{-}point}(\mathcal{W})$ as follows:

1. Player I starts by playing $W_0 = \sigma(\emptyset)$.
2. Assume player II plays $z_0 \in [W_0]^{<\omega}$. Let $p_0 = \sigma(W_0, z_0)$, s_0 be the stem of p_0 and $\mathcal{X}_0 = \{spsuc_{p_0}(s_0)\}$. Define $n_0 > d^{-1}(s_0)$ to be the least integer such that $d^{-1}(s_0{}^\frown t) < n_0$ for all $t \in F_{(\mathcal{X}_0,0)}(spsuc_{p_0}(s_0))$. Player I will play (in $\mathcal{G}_{P\text{-}point}(\mathcal{W})$) $W_1 = \sigma(W_0, z_0, p_0, n_0)$.
3. In general assume that it has been played the sequence $\langle W_0, z_0, ..., W_m\rangle$. At the same time, secretly the player I has been constructing a sequence $\langle W_0, z_0, p_0, n_0, W_1, z_1, p_1, n_1 ..., W_m\rangle$ that is being forced to be a partial play of the game $\mathcal{H}(\mathcal{W}, \dot{\mathcal{U}}_{gen})$ following σ, such that for every $i < m$, the integer n_i has the following property: letting $\mathcal{X}_i$ to be the set defined as $\{spsuc_{p_i}(u) \mid u \in T(p_i, n_{i-1})\}$ (where $n_{i-1} = d^{-1}(s_0)$), for every $t \in F_{(\mathcal{X}_i,i)}(spsuc_{p_i}(u))$, we have that $d^{-1}(u^\frown t) < n_i$. Assume that player II plays z_m as her next response in $\mathcal{H}\left(\mathcal{W}, \dot{\mathcal{U}}_{gen}\right)$. Let p_m be the tree given by $\sigma(W_0, z_0, n_0, W_1, ..., W_m, z_m)$ and let $n_m > n_{m-1}$ be the least integer with the following property: letting $\mathcal{X}_m = \{spsuc_{p_m}(u) \mid u \in T(p_m, n_{m-1})\}$, for every $t \in F_{(\mathcal{X}_m,m)}(spsuc_{p_m}(u))$, we have that $d^{-1}(u^\frown t) < n_m$. Player I will play $W_{m+1} = \sigma(W_0, z_0, n_0, W_1, ..., W_m, z_m, p_m, n_m)$.

The game $\mathcal{G}_{P\text{-}point}(\mathcal{W})$:

I	W_0		W_1		...
II		z_0		z_1	

The game $\mathcal{H}(\mathcal{W}, \dot{\mathcal{U}}_{gen})$:

I	W_0		p_0		W_1		p_1		...
II		z_0		n_0		z_1		n_1	

We claim that π is a winning strategy for player I in $\mathcal{G}_{P\text{-}point}(\mathcal{W})$. Consider a run of the game in which player I played according to π. Let $Z = \bigcup_{n\in\omega} z_n$, we will prove that $Z \notin \mathcal{U}$. Let $q = \bigcup_{i\in\omega} T(p_i, n_i)$ be te tree that was constructed by player I during the play. It is easy to see that $\mathcal{F} \cup \{\mathcal{C}(spsuc_q(s)) \mid s \in split(q)\}$ generates an F_σ-filter, call it $\mathcal{K}$. Note that $\mathcal{K} \leq \mathcal{F}$ hence $\mathcal{K}$ forces that σ is a winning strategy for player I in $\mathcal{H}(\mathcal{W}, \dot{\mathcal{U}}_{gen})$. Moreover, $\mathcal{K}$ forces that $q \in \mathbb{PT}(\dot{\mathcal{U}}_{gen})$. Since player I is forced to win in $\mathcal{H}(\mathcal{W}, \dot{\mathcal{U}}_{gen})$, it must be the case that $Z \notin \mathcal{W}$. This shows that π is a winning strategy for player I in $\mathcal{G}_{P\text{-}point}(\mathcal{W})$. Since player I can not have a winning strategy in the P-point game, we get a contradiction. ■

In this way, we conclude that $\mathbb{F}_\sigma * \mathbb{PT}(\dot{\mathcal{U}}_{gen})$ preserves all ground model P-points. Note that that after forcing with $\mathbb{F}_\sigma * \mathbb{PT}(\dot{\mathcal{U}}_{gen})$, there are intermediate extensions with P-points that are not preserved ($\mathcal{U}_{gen}$ for example), however, all ground model P-points are preserved. By iterating $\mathbb{F}_\sigma * \mathbb{PT}(\dot{\mathcal{U}}_{gen})$ with countable support, we get the following result:

Corollary 56 (Blass-Shelah) *The inequality $\mathfrak{u} < \mathfrak{s}$ is consistent with ZFC.*

In [47] Mildenberger proved the following interesting result:

Proposition 57 (Mildenberger) *It is consistent that there is a proper forcing that diagonalizes an ultrafilter and preserves a P-point.*

Note that our work provides an alternative proof of the theorem of Mildenberger.

We would like to mention that in original model of Shelah of $\mathfrak{b} < \mathfrak{s}$, in the Blass-Shelah model and in our model, the almost disjointness number is equal to ω_1. In [56] (using also the results from [14]) it is proved that $\mathfrak{a} = \omega_1$ after iterating (with countable support) the forcing $\mathbb{F}_\sigma * \mathbb{M}(\dot{\mathcal{U}}_{gen})$. A similar approach works when iterating $\mathbb{F}_\sigma * \mathbb{PT}(\dot{\mathcal{U}}_{gen})$. It is also possible to use the technique of theorem 6.6 in [51] to show that $\diamondsuit(\mathfrak{b})$ holds in that model, hence $\mathfrak{a} = \omega_1$ (see [51] for the definition of $\diamondsuit(\mathfrak{b})$ and the proof that $\diamondsuit(\mathfrak{b})$ implies $\mathfrak{a} = \omega_1$). Since this result will not be used in the paper, we omit the details.

A model of $\omega_1 = \mathfrak{u} < \mathfrak{a}$

In this section, we will prove that every MAD family can be destroyed with a proper forcing that preserves P-points, answering the questions of Brendle and Shelah. First, we will prove that if $\mathcal{A}$ is a MAD family, then $\dot{\mathcal{U}}_{gen}(\mathcal{A})$ is forced to be $\mathcal{B}$-Canjar for every $\mathfrak{b}$-family $\mathcal{B}$ in the ground model. In [14] and [27] it is proved that after adding ω_1-Cohen reals, $\mathbb{F}_\sigma(\mathcal{A})$ forces that $\mathcal{U}_{gen}(\mathcal{A})$ has these properties. Obviously, we can not use this results since we do not want to add Cohen reals. In this section, we will prove that the Cohen reals were really not needed in the first place. This proof takes inspiration in the proof of $\mathfrak{b} < \mathfrak{a}$ by Brendle in [7]. When the authors were preparing the paper, they learned from Zdomskyy that he has found a different proof that the preliminary Cohen reals are not needed.

Definition 58 *We say a MAD family $\mathcal{A}$ is a* Laflamme family *if $\mathcal{I}(\mathcal{A})$ can not be extended to an F_σ ideal (or equivalently, $\mathcal{I}(\mathcal{A})^*$ can not be extended to an F_σ-filter).*

The reader may consult [41] and [50] for more information on Laflamme MAD families. Note that if $\mathcal{A}$ is not Laflamme (i.e. $\mathcal{A}$ can be extended to an

F_σ-ideal), then $\mathbb{F}_\sigma * \mathbb{PT}(\dot{\mathcal{U}}_{gen})$ destroys $\mathcal{A}$ below some condition, in this way, we only need to take care of Laflamme families. The following is a simple lemma that will be needed later:

Lemma 59 *Let $\mathcal{A}$ be a* MAD *family and $\mathcal{F} \in \mathbb{F}_\sigma(\mathcal{A})$. If there is a proper forcing $\mathbb{P}$ such that $\mathbb{P}$ forces the following statement: "There is $\mathcal{D} \in [\mathcal{A}]^\omega$ such that $\mathcal{I}(\mathcal{A})^* \subseteq \langle \mathcal{F} \cup \{\omega \setminus A \mid A \in \mathcal{D}\}\rangle$" then $\mathcal{A}$ is not Laflamme.*

Proof. Since $\mathbb{P}$ is a proper forcing, we can find a condition $p \in \mathbb{P}$ and $\mathcal{D}_1 \in [\mathcal{A}]^\omega$ in V such that $p \Vdash$ "$\dot{\mathcal{D}} \subseteq \mathcal{D}_1$". It is then easy to see that $\mathcal{I}(\mathcal{A})^* \subseteq \langle \mathcal{F} \cup \{\omega \setminus A \mid A \in \mathcal{D}_1\}\rangle$ so $\mathcal{A}$ is not Laflamme. ■

Given $X \subseteq [\omega]^{<\omega}$ and $A \in [\omega]^\omega$, we define $Catch(X, A) = \{s \in X \mid s \subseteq A\}$. We will need the following definition:

Definition 60 *Let $\mathcal{F}$ be an F_σ-filter, $X \subseteq [\omega]^{<\omega}$ and $A \in [\omega]^\omega$. We will say that $\bigstar(\mathcal{F}, X, A)$ holds, if the following conditions are satisfied:*

1. *$A \in \mathcal{F}^+$.*
2. *If $B \in [A]^\omega \cap \mathcal{F}^+$ then $Catch(X, B) \in (\mathcal{F}^{<\omega})^+$ (i.e. for every $F \in \mathcal{F}$ there is $s \in X$ such that $s \subseteq F \cap B$).*

Let $\mathcal{A}$ be a MAD family, $\mathcal{F} \in \mathbb{F}_\sigma(\mathcal{A})$ and $X \subseteq [\omega]^{<\omega}$ such that $\mathcal{C}(X) \subseteq \left\langle \mathcal{F} \cup \mathcal{I}(\mathcal{A})^* \right\rangle$. Fix $\langle \mathcal{C}_n \rangle_{n\in\omega}$ an increasing family of compact sets such that $\mathcal{F} = \bigcup \mathcal{C}_n$. The *Brendle game*[8] $\mathcal{BR}(\mathcal{A}, \mathcal{F}, X)$ is defined as follows,

I	Y_0		Y_1		Y_2		$\cdots$
II		s_0		s_1		s_2	$\cdots$

Where $Y_m \in \mathcal{I}(\mathcal{A})^*$, $s_m \in [Y_m]^{<\omega}$ intersects all the elements of $\mathcal{C}_m$ and $\max(s_m) < \min(s_{m+1})$.[9] Player I wins the game if $\bigcup_{n\in\omega} s_n$ contains an element of X. Note that this is an open game for I, i.e., if she wins, then she wins already in a finite number of steps. By the Gale-Stewart theorem (see [38]), the Brendle game is determined. We will now prove the following:

Proposition 61 *Let $\mathcal{A}$ be a Laflamme* MAD *family and $\mathcal{F} \in \mathbb{F}_\sigma(\mathcal{A})$. For every family $\{X_n \mid n \in \omega\}$ such that $\mathcal{C}(X_n) \subseteq \left\langle \mathcal{F} \cup \mathcal{I}(\mathcal{A})^* \right\rangle$, there is a countable family $\mathcal{D} \in [\mathcal{A}]^\omega$ such that $\bigstar(\mathcal{F}, A, X_n)$ holds for every $n \in \omega$ and $A \in \mathcal{D}$.*

[8] This game was based on the rank arguments used by Brendle in [7]. A similar (but different) approach using games was used by Brendle and Taylor in [16].

[9] Note that the game $\mathcal{BR}(\mathcal{A}, \mathcal{F}, X)$ does not only depend on $\mathcal{F}$, but on its representation as an increasing union of compact sets. A more formal notation would be $\mathcal{BR}(\mathcal{A}, \langle \mathcal{C}_n \rangle_{n\in\omega}, X)$.

Proof. By $V[C_\alpha]$ we denote an extension of V by adding α-Cohen reals (the reader should not be worried by the use of Cohen reals in the proof, see the paragraph after this result for more information). We first claim the following:

Claim 62 *If $X \subseteq [\omega]^{<\omega}$ is such that $\mathcal{C}(X) \subseteq \left\langle \mathcal{F} \cup \mathcal{I}(\mathcal{A})^* \right\rangle$, then in $V[C_{\omega_1}]$ the player I has a winning strategy for $\mathcal{BR}(\mathcal{A}, \mathcal{F}, X)$.*

We will prove the claim by contradiction, since $\mathcal{BR}(\mathcal{A}, \mathcal{F}, X)$ is determined, we assume that II has a winning strategy, call it π. We will choose a tree $T \subseteq \left([\omega]^{<\omega}\right)^{<\omega}$ and a family $\{B_t \mid t \in T\} \subseteq \mathcal{I}(\mathcal{A})^*$ with the following properties:

1. $\emptyset \in T$ and $B_\emptyset = \omega$ (this is just a technical step).
2. If $t = \langle s_0, s_1, ..., s_n \rangle \in T$ then $\left\langle B_{\langle s_0 \rangle}, s_0, B_{\langle s_0, s_1 \rangle}, s_1, ..., B_{\langle s_0, s_1, ..., s_n \rangle}, s_n \right\rangle$ is a legal partial play of $\mathcal{BR}(\mathcal{A}, \mathcal{F}, X)$ in which Player II is using her strategy π.

An important remark is in order here: Note that for example, for every $s \in [\omega]^{<\omega}$ there may be infinitely many $B \in \mathcal{I}(\mathcal{A})^*$ such that $\langle B, s \rangle$ is a legal partial play of $\mathcal{BR}(\mathcal{A}, \mathcal{F}, X)$ in which Player II is using her strategy π. For $B_{\langle s \rangle}$ we just choose and fix one of them. The tree T and $\{B_t \mid t \in T\}$ are recursively constructed as follows:

1. $\emptyset \in T$ and $B_\emptyset = \omega$.
2. T_1 is the set of all $\langle s \rangle$ such that $s \in [\omega]^{<\omega}$ and there is $B \in \mathcal{I}(\mathcal{A})^*$ for which $\langle B, s \rangle$ is a legal partial play of $\mathcal{BR}(\mathcal{A}, \mathcal{F}, X)$ in which Player II is using her strategy π.
3. For every s such that $\langle s \rangle \in T_1$, we choose $B_s \in \mathcal{I}(\mathcal{A})^*$ for which $\langle B_s, s \rangle$ is a legal partial play.
4. Given a node $t = \langle s_0, s_1, ..., s_n \rangle \in T$ (and we know that the sequence $\left\langle B_{\langle s_0 \rangle}, s_0, B_{\langle s_0, s_1 \rangle}, s_1, ..., B_{\langle s_0, s_1, ..., s_n \rangle}, s_n \right\rangle$ is a legal partial play) let $suc_T(t)$ be the set of all $z \in [\omega]^{<\omega}$ for which there is $B \in \mathcal{I}(\mathcal{A})^*$ such that $\left\langle B_{\langle s_0 \rangle}, s_0, B_{\langle s_0, s_1 \rangle}, s_1, ..., B_{\langle s_0, s_1, ..., s_n \rangle}, s_n, B, z \right\rangle$ is a legal partial play (in which Player II is using her strategy π). We fix $B_{t^\frown \langle z \rangle} \in \mathcal{I}(\mathcal{A})^*$ with this property.

Note that if $t = \langle s_0, s_1, ..., s_n \rangle \in T$, then $\bigcup\limits_{i \leq n} s_i$ does not contain an element of X, this is because π is a winning strategy for player II. Clearly T is a countable tree with no isolated branches, so it is equivalent to Cohen forcing when viewed as a forcing notion. Since T is countable, it appears in an intermediate extension of $V[C_{\omega_1}]$. Let $\beta < \omega_1$ such that $T \in V[C_\beta]$.

Given $Y \in \mathcal{I}(\mathcal{A})^*$ define the set D_Y of all $t = \langle s_0, s_1, ..., s_n \rangle \in T$ such that there is $i \leq n$ for which $s_i \subseteq Y$. It is easy to see that each D_Y is an open dense subset of T. Let $G \in V[C_{\omega_1}]$ be a $(T, V[C_\beta])$-generic branch through T. It is easy to see that G induces a legal play of the game in which II followed her strategy. Let $D = \bigcup G$, and since π is a winning strategy for II, we conclude that D does not contain an element of X. By genericity $D \in \left\langle \mathcal{I}(\mathcal{A})^* \cup \mathcal{F} \right\rangle^+$ however, $\omega \setminus D \in \mathcal{C}(X) \subseteq \left\langle \mathcal{I}(\mathcal{A})^* \cup \mathcal{F} \right\rangle$ which is obviously a contradiction. This finishes the proof of the claim.

We work in $V[C_{\omega_1}]$, where player I has winning strategies for all of the games $\mathcal{BR}(\mathcal{A}, \mathcal{F}, X_n)$ with $n \in \omega$. Let π_n be the winning strategy for the game $\mathcal{BR}(\mathcal{A}, \mathcal{F}, X_n)$. Let $\mathcal{W}$ be set of elements of $\mathcal{I}(\mathcal{A})^*$ that may be played by I following her winning strategy in any of these games. It is not hard to see that $\mathcal{W}$ is countable. Note that if $W \in \mathcal{W}$ then W almost contains every element of $\mathcal{A}$ except for finitely many (this is because $W \in \mathcal{I}(\mathcal{A})^*$). Let $\mathcal{A}' \subseteq \mathcal{A}$ be the set of all $A \in \mathcal{A}$ for which there is $W \in \mathcal{W}$ such that $A \not\subseteq^* W$. Note that $\mathcal{A}'$ is countable. Since $\mathcal{A}$ is Laflamme in V, by a previous lemma, $\mathcal{I}(\mathcal{A})^*$ it is not contained in $\langle \mathcal{F} \cup \{\omega \setminus B \mid B \in \mathcal{A}'\} \rangle$, so there is $A_0 \in \mathcal{A}$ such that $\omega \setminus A_0 \notin \langle \mathcal{F} \cup \{\omega \setminus B \mid B \in \mathcal{A}'\} \rangle$. This implies that $A_0 \in \mathcal{F}^+$ and A_0 is almost contained in every member of $\mathcal{W}$. We claim that $\bigstar(\mathcal{F}, A_0, X_n)$ holds for each $n \in \omega$. Let $B \in \wp(A_0) \cap \mathcal{F}^+$ we will now show that $Catch(X_n, B)$ is positive for each $n \in \omega$. Let $F \in \mathcal{F}$ and consider the following play in $\mathcal{BR}(\mathcal{A}, \mathcal{F}, X_n)$,

I	W_0		W_1		W_2		$\cdots$
II		s_0		s_1		s_2	$\cdots$

Where the W_i are played by I according to π_n, $s_i \in [B \cap F]^{<\omega}$ and intersects every element of $\mathcal{C}_i$. This is possible since $B \cap F$ is positive and is almost contained in every W_n. Since π_n is a winning strategy, this means that I wins the game,which entails that $\bigcup s_n \subseteq B \cap F$ contains an element of X_n. We can then obtain each A_{n+1} by repeating the same argument and using that $\mathcal{I}(\mathcal{A})^*$ it is not contained in $\langle \mathcal{F} \cup \{\omega \setminus B \mid B \in \mathcal{A}'\} \cup \{\omega \setminus A_0, .., \omega \setminus A_n\} \rangle$. Let $\mathcal{D}_1 = \{A_n \mid n \in \omega\}$.

We know that $V[C_{\omega_1}] \models \bigstar(\mathcal{F}, A_n, X_m)$ for every $n, m \in \omega$. However, it is easy to see that the statement $\bigstar(\mathcal{F}, A_n, X_m)$ is absolute between models of ZFC (in fact, we only need that it is downwards absolute, which is easy). So $V \models \bigstar(\mathcal{F}, A_n, X_m)$ for every $n, m \in \omega$. Since $\mathbb{C}_{\omega_1}$ has the countable chain condition, there is $\mathcal{D} \in [\mathcal{A}]^\omega$ such that $\mathbb{C}_{\omega_1} \Vdash$ "$\mathcal{D}_1 \subseteq \mathcal{D}$". By the previous remark, we may assume that that $\bigstar(\mathcal{F}, A, X_n)$ holds for every $n \in \omega$ and $A \in \mathcal{D}$. $\blacksquare$

The reader might feel that we cheated in the previous proof by adding the Cohen reals, and sincerely, we have, but it was a "legal cheating". We only used the Cohen reals to find ground model objects, and after finding them, we came back to the ground model as if nothing happened.

Given $A \in [\omega]^{\omega}$ and $l \in \omega$ define $Part_l(A)$ as the set of all sequences $\langle B_1, ..., B_l \rangle$ such that $A = \bigcup_{i \leq l} B_i$ and $B_i \cap B_j = \emptyset$ whenever $i \neq j$. Note that $Part_l(A_i)$ is a compact space with the natural topology. Also it is clear that if $A \in \mathcal{F}^+$ and $\langle B_1, ..., B_l \rangle \in Part_l(A)$ then there is $j \leq l$ such that $B_j \in \mathcal{F}^+$.

Lemma 63 *Let $\mathcal{F}$ be a filter, $\mathcal{C} \subseteq \mathcal{F}$ a compact set and $X \in (\mathcal{F}^{<\omega})^+$. Let A such that $\bigstar(A, \mathcal{F}, X)$ holds and let $l \in \omega$. There is $n \in \omega$ with the property that for all $\langle B_1, ..., B_l \rangle \in Part_l(A)$ there is $i \leq l$ such that if $F \in \mathcal{C}$ then $X \cap \wp(B_i \cap n)$ contains a subset of F.*

Proof. Let U_n be the set of all $\langle B_1, ..., B_l \rangle \in Part_l(A)$ such that there is $i \leq l$ with the property that if $F \in \mathcal{C}$ then $X \cap \wp(B_i \cap n)$ contains a subset of F. Note that $\{U_n \mid n \in \omega\}$ is an open cover (recall that $\bigstar(A, \mathcal{F}, X)$ holds and if we split A into finitely many pieces, then one of the pieces must be in $\mathcal{F}^+$) and the result follows since $Part_l(A)$ is compact. ■

We will now prove the following:

Proposition 64 *Let $\mathcal{F}$ be a filter, $\mathcal{C} \subseteq \mathcal{F}$ a compact set, $X \in (\mathcal{F}^{<\omega})^+$, $A \in [\omega]^{\omega}$ such that $\bigstar(A, \mathcal{F}, X)$ holds and $l \in \omega$. There is $Y \in [X]^{<\omega}$ such that if $C_1, ..., C_l \in \mathcal{C}(Y)$ and $F \in \mathcal{C}$ then there is $s \in Y \cap [A]^{<\omega}$ such that $s \subseteq C_1 \cap ... \cap C_l \cap F$.*

Proof. Let n such that for every $\langle B_1, ..., B_{2^l} \rangle \in Part_{2^l}(A)$ and for every $F \in \mathcal{C}$ there is $j \leq 2^l$ for which $X \cap \wp(B_j \cap n)$ contains a subset of F. Let $Y = X \cap \wp(l)$, we will see that Y has the desired properties. Let $C_1, ..., C_l \in \mathcal{C}(Y)$ and $F \in \mathcal{C}$. For every $s : l \longrightarrow 2$ define B_s as the set of all $a \in A$ such that $a \in C_i$ if and only if $s(i) = 1$. Clearly $\langle B_s \rangle_{s \in 2^l} \in Part_{2^l}(A_l)$ and we may conclude that there is s such that $Y \cap \wp(B_s \cap n)$ contains an element of F. Since $C_1, ..., C_l \in \mathcal{C}(Y)$ we conclude that s must be the constant 1 function and this entails the desired conclusion. ■

We can then finally conclude the following:

Proposition 65 *If $\mathcal{A}$ is a Laflamme MAD family, then $\mathbb{F}_{\sigma}(\mathcal{A})$ forces that $\dot{\mathcal{U}}_{gen}(\mathcal{A})$ is $\mathcal{B}$-Canjar for every $\mathfrak{b}$-family $\mathcal{B}$ in the ground model.*

Proof. It is enough to show that if $\mathcal{F} \Vdash$ "$\overline{X} = \langle X_n \rangle_{n \in \omega} \subseteq (\dot{\mathcal{U}}_{gen}(\mathcal{A})^{<\omega})^+$" then there is $\mathcal{G} \leq \mathcal{F}$ and $f \in \mathcal{B}$ such that $\mathcal{C}(\overline{X}_f) \subseteq \mathcal{G}$. Let $\mathcal{F} = \bigcup \mathcal{C}_n$ where each $\mathcal{C}_n$ is compact and they form an increasing chain. By the previous results, we may find $\{A_n \mid n \in \omega\} \subseteq \mathcal{A}$ such that $\bigstar(A_n, \mathcal{F}, X_m)$ holds for every $n, m \in \omega$. We can then find an increasing function $g : \omega \longrightarrow \omega$ such that the following holds:

***)** For every $n \in \omega$ and for every $i \leq n$, if $Y = X \cap \wp(g(n))$ then for every $C_0, ..., C_n \in \mathcal{C}(Y)$ and $F \in \mathcal{C}_n$ there is $s \in Y \cap [A_i]^{<\omega}$ such that $s \subseteq C_0 \cap ... \cap C_l \cap F$.

Since $\mathcal{B}$ is unbounded, we can find $f \in \mathcal{B}$ that is not dominated by g. It is easy to see that $\mathcal{G} = \left\langle \mathcal{F} \cup \mathcal{C}\left(\overline{X}_f\right)\right\rangle$ is a condition in $\mathbb{F}_\sigma(\mathcal{A})$ and has the desired properties. $\blacksquare$

We can conclude:

Corollary 66 *Every MAD family can be destroyed with a forcing that is proper, adds an unsplit real, preserves all $\mathfrak{b}$-families from the ground model and does not add Cohen reals.*

Proof. If $\mathcal{A}$ is not Laflamme, then it can be destroyed by $\mathbb{F}_\sigma * \mathbb{PT}(\dot{\mathcal{U}}_{gen})$ (below some condition). If $\mathcal{A}$ is Laflamme, then we can destroy it with $\mathbb{F}_\sigma(\mathcal{A}) * \mathbb{PT}(\dot{\mathcal{U}}_{gen}(\mathcal{A}))$. $\blacksquare$

By iterating the forcings in the previous corollary, we get the following:

Proposition 67 (Shelah) *There is a model of $\omega_1 = \mathfrak{b} < \mathfrak{a} = \mathfrak{s} = \mathfrak{c} = \omega_2$.*

Note that if we iterate with countable support forcings of the type $\mathbb{F}_\sigma(\mathcal{A}) * \mathbb{M}(\dot{\mathcal{U}}_{gen}(\mathcal{A}))$ and $\mathbb{F}_\sigma * \mathbb{M}(\dot{\mathcal{U}}_{gen})$, we will get a model of $\omega_1 = \mathfrak{b} < \mathfrak{a} = \mathfrak{s} = \mathsf{cov}(\mathcal{M}) = \mathfrak{c} = \omega_2$. In order to preserve P-points, we must use the Miller forcing instead of the Mathias forcing, as we are going to show now. We will prove that for every MAD family $\mathcal{A}$, the forcing $\mathbb{F}_\sigma(\mathcal{A}) * \mathbb{PT}(\dot{\mathcal{U}}_{gen}(\mathcal{A}))$ preserves all ground model P-points. We will need the following generalization of lemma 63:

Lemma 68 *Let $l \in \omega$, $\mathcal{F}$ a filter, $\mathcal{D} \subseteq \mathcal{F}$ a compact set, $X_1, ..., X_n \subseteq [\omega]^{<\omega} \setminus \{\emptyset\}$ such that $\mathcal{C}(X_1), ..., \mathcal{C}(X_n) \subseteq \mathcal{F}$ and $A \in [\omega]^\omega$ such that $\bigstar(A, \mathcal{F}, X_i)$ holds for every $i \leq n$. There is $m \in \omega$ such that for every $\langle B_1, ..., B_l\rangle \in Part_l(A)$, there is $i \leq l$ such that for every $F \in \mathcal{D}$ and for every $k \leq n$, the set $(B_i \cap F) \cap m$ contains an element of X_k.*

Proof. Let U_m be the set of all $\langle B_1, ..., B_l\rangle \in Part_l(A)$ such that there is $i \leq l$ with the property that if $F \in \mathcal{C}$ and $k \leq n$ then, $X_k \cap \wp(B_i \cap m)$ contains a subset of F. Note that $\{U_n \mid n \in \omega\}$ is an open cover and the result follows since $Part_l(A)$ is compact. $\blacksquare$

With this result, we can prove the following generalization of lemma 64:

Lemma 69 *Let $\mathcal{F}$ be a filter, $\mathcal{D} \subseteq \mathcal{F}$ a compact set, $X_1, ..., X_n \subseteq [\omega]^{<\omega}$ such that $\mathcal{C}(X_1), ..., \mathcal{C}(X_n) \subseteq \mathcal{F}$ and $A \in [\omega]^\omega$ such that $\bigstar(A, \mathcal{F}, X_i)$ holds for every $i \leq n$. There are $Y_1 \in [X_1]^{<\omega}, ..., Y_n \in [X_n]^{<\omega}$ such that for every $F \in \mathcal{D}$ and for every $C_i^1, ..., C_i^n \in \mathcal{C}(Y_i)$ (with $i \leq n$), for every $k \leq n$ there is $s \in Y_k \cap [A]^{<\omega}$ such that $s \subseteq F \cap \bigcap\limits_{i,j \leq n} C_i^j$.*

Proof. Let $l = n^2$ and by lemma 68, we know there is $m \in \omega$ such that for every $\langle B_1, ..., B_{2^l} \rangle \in Part_{2^l}(A)$ there is $i \leq 2^l$ such that for every $F \in \mathcal{D}$ and for every $k \leq n$, the set $(B_i \cap F) \cap m$ contains an element of X_k. Let $Y_i = X_i \cap \wp(m)$ for every $i \leq n$. We will see that the sets $Y_1, ..., Y_n$ have the desired properties. Let $C_i^1, ..., C_i^n \in \mathcal{C}(Y_i)$ (with $i \leq n$) and $F \in \mathcal{C}$. For every $s : n \times n \longrightarrow 2$ define B_s as the set of all $a \in A$ such that $a \in C_i^j$ if and only if $s(i,j) = 1$. Clearly $\langle B_s \rangle_{s \in 2^l} \in Part_{2^l}(A_l)$ and we may conclude that there is $s : n \times n \longrightarrow 2$ such that $Y_i \cap \wp(B_s \cap m)$ contains an element of F for every $i \leq n$. Since $C_i^1, ..., C_i^n \in \mathcal{C}(Y_i)$ we conclude that s must be the constant 1 function and this entails the desired conclusion. ■

We can now prove the following result, which is a combination of proposition 55 and proposition 65:

Proposition 70 *If $\mathcal{W}$ is a P-point and $\mathcal{A}$ is a Laflamme* MAD *family, then $\mathbb{F}_\sigma(\mathcal{A})$ forces that $\dot{\mathcal{U}}_{gen}(\mathcal{A})$ is $\mathcal{W}$-Canjar.*

Proof. We will prove the proposition by contradiction. Assume there is $\mathcal{F} \in \mathbb{F}_\sigma(\mathcal{A})$ and σ such that $\mathcal{F}$ forces that σ is a winning strategy for player I in $\mathcal{H}(\mathcal{W}, \dot{\mathcal{U}}_{gen}(\mathcal{A}))$. Since strategies for player I are countable objects and since $\mathbb{F}_\sigma(\mathcal{A})$ is σ-closed, it is enough to consider ground model strategies. Let $\mathcal{F} = \bigcup_{n \in \omega} \mathcal{C}_n$ where $\langle \mathcal{C}_n \rangle_{n \in \omega}$ is an increasing sequence of compact sets. We will use σ to construct a winning strategy for player I in the game $\mathcal{G}_{P\text{-}point}(\mathcal{W})$, which will be a contradiction.

Let L be the collection of all p_s such that p is a possible response of player I according to σ and $s \in split(p)$. In the same way as in the proof of proposition 55, if $p_s \in L$, then $\mathcal{F} \Vdash$ "$spsuc_p(s) \in (\dot{\mathcal{U}}_{gen}^{<\omega})^+$", hence in particular $\mathcal{C}(spsuc_p(s)) \subseteq \langle \mathcal{F} \cup \mathcal{I}(\mathcal{A})^* \rangle$. Since L is countable we may assume (by extending $\mathcal{F}$ if necessary) that $\mathcal{C}(spsuc_p(s)) \subseteq \mathcal{F}$ for every $p_s \in L$. By 61, we can find a family $\{A_n \mid n \in \omega\} \subseteq \mathcal{A}$ such that $\bigstar(A_n, \mathcal{F}, \mathcal{C}(spsuc_p(s)))$ holds for every $n \in \omega$ and $p_s \in L$.

For every $\mathcal{X} = \{X_1, ..., X_n\} \in [\{spsuc_p(s) \mid p \in L \wedge s \in split(p)\}]^{<\omega}$ and for every $k \in \omega$, fix a function $F_{(\mathcal{X},k)} : \mathcal{X} \longrightarrow [[\omega^{<\omega}]]^{<\omega}$ with the following properties:

1. $Y_i = F_{(\mathcal{X},k)}(X_i) \in [X_i]^{<\omega}$ for every $i \leq n$.
2. For every $B \in \mathcal{C}_k$, for every $C_i^1, ..., C_i^n \in \mathcal{C}(Y_i)$ (with $i \leq n$) and for every $k_1, k_2 \leq n$, we have that $B \cap \bigcap_{i,j \leq n} C_i^j$ contains an element of $Y_{k_1} \cap [A_{k_2}]^{<\omega}$.

We know such $F_{(\mathcal{X},k)}$ exists by the previous lemma. The proof now proceeds in a very similar way as the proof of 55. We define π a strategy for player I in $\mathcal{G}_{P\text{-}point}(\mathcal{W})$ as follows:

1. Player I starts by playing $W_0 = \sigma(\emptyset)$.

2. Assume player II plays $z_0 \in [W_0]^{<\omega}$. Let $p_0 = \sigma(W_0, z_0)$, s_0 be the stem of p_0 and $\mathcal{X}_0 = \{spsuc_{p_0}(s_0)\}$. Define $n_0 > d^{-1}(s_0)$ to be the least integer such that $d^{-1}(s_0{}^\frown t) < n_0$ for all $t \in F_{(\mathcal{X}_0,0)}(spsuc_{p_0}(s_0))$. Player I will play (in $\mathcal{G}_{P\text{-}point}(\mathcal{W})$) $W_1 = \sigma(W_0, z_0, p_0, n_0)$.

3. In general, assume that it has been played the sequence $\langle W_0, z_0, ..., W_m\rangle$. At the same time, the player I has secretly been constructing a sequence $\langle W_0, z_0, p_0, n_0, W_1, z_1, p_1, n_1 ..., W_m\rangle$ that is being forced to be a partial play of the game $\mathcal{H}(\mathcal{W}, \dot{\mathcal{U}}_{gen}(\mathcal{A}))$ following σ such that for every $i < m$, the integer n_i has the following important property: letting $\mathcal{X}_i = \{spsuc_{p_i}(u) \mid u \in T(p_i, n_{i-1})\}$ (we define $n_{-1} = d^{-1}(s_0)$), for every $t \in F_{(\mathcal{X}_i,i)}(spsuc_{p_i}(u))$, we have that $d^{-1}(u^\frown t) < n_i$. Assume that player II plays z_m as her next response in $\mathcal{H}(\mathcal{W}, \dot{\mathcal{U}}_{gen}(\mathcal{A}))$. Let p_m be the tree $\sigma(W_0, z_0, n_0, W_1, ..., W_m, z_m)$ and let $n_m > n_{m-1}$ the least integer with the following property: letting $\mathcal{X}_m = \{spsuc_{p_m}(u) \mid u \in T(p_m, n_{m-1})\}$, for every $t \in F_{(\mathcal{X}_m,m)}(spsuc_{p_m}(u))$, we have that $d^{-1}(u^\frown t) < n_m$. Player I will play $W_{m+1} = \sigma(W_0, z_0, n_0, W_1, ..., W_m, z_m, p_m, n_m)$.

The game $\mathcal{G}_{P\text{-}point}(\mathcal{W})$:

I	W_0		W_1		...
II		z_0		z_1	

The game $\mathcal{H}(\mathcal{W}, \dot{\mathcal{U}}_{gen}(\mathcal{A}))$:

I	W_0		p_0		W_1		p_1		...
II		z_0		n_0		z_1		n_1	

We claim that π is a winning strategy for player I in $\mathcal{G}_{P\text{-}point}(\mathcal{W})$. Consider a run of the game in which player I played according to π. Let $Z = \bigcup_{n\in\omega} z_n$, we will prove that $Z \notin \mathcal{U}$. Let $q = \bigcup_{i\in\omega} T(p_i, n_i)$ be the tree that was constructed by player I during the play. It is easy to see that $\mathcal{F} \cup \{\mathcal{C}(spsuc_q(s)) \mid s \in split(q)\}$ generates a condition of $\mathbb{F}_\sigma(\mathcal{A})$, call if $\mathcal{K}$. Note that $\mathcal{K} \leq \mathcal{F}$ hence $\mathcal{K}$ forces that σ is a winning strategy for player I in $\mathcal{H}(\mathcal{W}, \dot{\mathcal{U}}_{gen})$. Moreover, $\mathcal{K}$ forces that $q \in \mathbb{PT}(\dot{\mathcal{U}}_{gen}(\mathcal{A}))$. Since player I is forced to win in $\mathcal{H}(\mathcal{W}, \dot{\mathcal{U}}_{gen}(\mathcal{A}))$, it must be the case that $Z \notin \mathcal{W}$. This shows that π is a winning strategy for player I in $\mathcal{G}_{P\text{-}point}(\mathcal{W})$. Since player I can not have a winning strategy in the P-point game, we get a contradiction. ■

We can now answer the questions of Brendle and Shelah:

Theorem 71 *Every MAD family can be destroyed with a proper forcing that preserves all P-points from the ground model. In particular, it is consistent that $\omega_1 = \mathfrak{u} < \mathfrak{a} = \omega_2$.*

MAD families build up from closed sets

The *closed almost disjointness number* was introduced by Brendle and Khomskii in [13]. The invariant $\mathfrak{a}_{closed}$ is defined as the smallest number of closed sets of $[\omega]^{\omega}$ such that its union is a MAD family. Since singletons are closed, it follows that $\mathfrak{a}_{closed} \leq \mathfrak{a}$ and it is uncountable by a result of Mathias (see [43] or [59]). The following are some known results regarding this cardinal invariant:

Proposition 72

1. *(Raghavan, Törnquist independently) $\mathfrak{p} \leq \mathfrak{a}_{closed}$ (see [59]).*
2. *(Brendle and Khomskii) It is consistent that $\mathfrak{a}_{closed} < \mathfrak{b}$ (see [13] and [14]).*
3. *(Brendle and Raghavan) It is consistent that $\mathfrak{b} < \mathfrak{a}_{closed}$ (see [14]).*
4. *(Raghavan and Shelah) $\mathfrak{d} = \omega_1$ implies $\mathfrak{a}_{closed} = \omega_1$ (see [53] and [14]).*

There are still many interesting open questions regarding $\mathfrak{a}_{closed}$. The following problems are still open:

Problem 73 (Raghavan) *Does $\mathfrak{h} \leq \mathfrak{a}_{closed}$?*

Problem 74 (Brendle, Khomskii) *Does $\mathfrak{s} = \omega_1$ imply $\mathfrak{a}_{closed} = \omega_1$?*

The reader may consult [13] or [14] for more information and open problems regarding $\mathfrak{a}_{closed}$. If $\mathcal{D} \subseteq [\omega]^{\omega}$ is an AD family, we denote its *ortogonal* $\mathcal{D}^{\perp}$ as the set of all $B \subseteq \omega$ that are almost disjoint with every element of $\mathcal{D}$.

As mentioned before, Brendle and Raghavan proved that it is consistent that $\mathfrak{b} < \mathfrak{a}_{closed}$. In fact, they build two models in which this inequality holds. One is using the creature forcing of Shelah and another was using a c.c.c. forcing, similar to the model constructed in [7]. In both cases, their forcings add Cohen reals. We will combine their results with ours to build a model of $\mathfrak{b} < \mathfrak{a}_{closed}$ without adding Cohen reals, moreover; this inequality holds in the model constructed in the previous section. The key result is the following proposition, which was implicitly proved in the lemma 7 of [14]:

Proposition 75 (Brendle, Raghavan) *Let $\mathcal{D} \subseteq [\omega]^{\omega}$ be a closed AD family. If $\mathcal{U}$ is a P-point such that $\mathcal{D} \cap \mathcal{U} = \emptyset$, then there is $U \in \mathcal{U} \cap \mathcal{D}^{\perp}$.*

Proof. Note that if $\mathcal{D} \cap \mathcal{U} = \emptyset$ then $\mathcal{I}(\mathcal{D}) \cap \mathcal{U} = \emptyset$ since $\mathcal{U}$ is an ultrafilter. We argue by contradiction, assume that $\mathcal{U} \cap \mathcal{D}^{\perp} = \emptyset$. Note that this implies that for every $B \in \mathcal{U}$ there are $\{A_n \mid n \in \omega\} \subseteq \mathcal{D}$ such that $|B \cap A_n| = \omega$ for every $n \in \omega$. Consider the Laver forcing $\mathbb{L}(\mathcal{U})$. Since $\mathcal{I}(\mathcal{D})$ is an analytic set, by a result of Blass (see [3]), there is $p \in \mathbb{L}(\mathcal{U})$ such that either $[p] \subseteq \mathcal{I}(\mathcal{D})$ or $[p] \cap \mathcal{I}(\mathcal{D}) = \emptyset$. Since $\mathcal{U}$ is a P-point, we can find $q \leq p$ and $U \in \mathcal{U}$ such that $suc_q(s) =^* U$ for every $s \in q$ such that s extends the stem of q.

Let $\{A_n \mid n \in \omega\} \subseteq \mathcal{D}$ such that $A_n \cap U$ is infinite for every $n \in \omega$. On one hand, we can find a branch $f_0 \in [q]$ such that $im(f_0) \subseteq A_0$, so $im(f_0) \in \mathcal{I}(\mathcal{D})$, but on the other hand, we can find $f_1 \in [q]$ such that $im(f_1) \cap A_n$ is infinite for every $n \in \omega$, so $im(f_1) \notin \mathcal{I}(\mathcal{D})$. This two statement are in contradiction since we know that either $[q] \subseteq \mathcal{I}(\mathcal{D})$ or $[q] \cap \mathcal{I}(\mathcal{D}) = \emptyset$. ■

We can now conclude the following:

Corollary 76 *Let $\mathcal{A} = \bigcup_{\alpha \in \kappa} \mathcal{C}_\alpha$ be a MAD family such that each $\mathcal{C}_\alpha$ is a closed subset of $[\omega]^\omega$. Let $\mathcal{U}$ be a P-point such that $\mathcal{A} \cap \mathcal{U} = \emptyset$. If $\mathbb{P}$ is a forcing that diagonalizes $\mathcal{U}$, and $G \subseteq \mathbb{P}$ is a generic filter, then $V[G] \models$ "$\mathcal{A}^{V[G]} = \bigcup_{\alpha \in \kappa} \mathcal{C}_\alpha^{V[G]}$ is not a MAD family" (where $\mathcal{C}_\alpha^{V[G]}$ is the reinterpretation of $\mathcal{C}_\alpha$ in the model $V[G]$).*

Proof. Let $G \subseteq \mathbb{P}$ be a generic filter. We argue in $V[G]$. Let $B \in [\omega]^\omega$ be a pseudointersection of $\mathcal{U}$. We claim that B is almost disjoint with $\mathcal{A}^{V[G]}$. By the previous result, for every $\alpha < \kappa$ there is $U_\alpha \in \mathcal{U}$ such that the following statement holds in V : "U_α is almost disjoint with every element of $\mathcal{C}_\alpha$". Since $\mathcal{C}_\alpha$ is a closed set, this is an absolute statement, so U_α is almost disjoint with every element of $\mathcal{C}_\alpha^{V[G]}$. Furthermore, since $B \subseteq^* U_\alpha$, then B is also almost disjoint with every element of $\mathcal{C}_\alpha^{V[G]}$ for every $\alpha < \kappa$, hence B is almost disjoint with $\mathcal{A}^{V[G]}$. ■

In [17] it was proved that Canjar ultrafilters are P-points (moreover, Canjar ultrafilters are precisely the "strong P-points", see [5] for the definition of strong P-point). In this way, we can conclude the following:

Corollary 77 *The following statements are consistent with the axioms of ZFC:*

1. *(Brendle, Raghavan)* $\omega_1 = \mathfrak{b} < \mathfrak{a}_{closed} = \mathsf{cov}(\mathcal{M}) = \mathfrak{c} = \omega_2$.
2. $\omega_1 = \mathfrak{u} < \mathfrak{a}_{closed} = \omega_2$.

Open Problems

We will list some open problems the authors do not know how to answer. Regarding the forcings $\mathbb{Q}(\mathcal{F})$ of Sabok and Zapletal, we know that they might or might not diagonalize $\mathcal{F}$. It would be interesting to know the answer of the following:

Problem 78 *Is there a nice combinatorial characterization for the filters $\mathcal{F}$ for which $\mathbb{Q}(\mathcal{F})$ diagonalizes $\mathcal{F}$?*

The forcings $\mathbb{M}(\mathcal{F})$ and $\mathbb{L}(\mathcal{F})$ have been proven to be very useful (see for example [37], [52], [20], [35], [21] or [12] for some applications of this forcings). We expect the forcings $\mathbb{PT}(\mathcal{F})$ to have interesting applications as well. It would then be useful to have a deeper understanding of such forcings. For example we have the following:

Problem 79 *Let $\mathcal{F}$ be a filter.*

1. *Is there a combinatorial characterization of when $\mathbb{PT}(\mathcal{F})$ does not add Cohen reals?*
2. *Is there a combinatorial characterization of when $\mathbb{PT}(\mathcal{F})$ does not add dominating reals?*
3. *Is there a filter $\mathcal{F}$ such that $\mathbb{PT}(\mathcal{F})$ does not add dominating reals but $\mathbb{M}(\mathcal{F})$ adds dominating reals?*

It would be interesting if the previous properties have characterizations in terms of the Katětov order, similar to the results for $\mathbb{Q}(\mathcal{F})$ obtained in [54]. Regarding the preservation of P-points, we have the following:

Problem 80

1. *Let $\mathcal{U}$ be a P-point and $\mathcal{F}$ a Canjar filter such that $\mathbb{PT}(\mathcal{F})$ forces that $\mathcal{U}$ generates a non-meager filter. Does $\mathbb{PT}(\mathcal{F})$ preserves $\mathcal{U}$?*
2. *Assuming CH, is there a Canjar filter $\mathcal{F}$ such that $\mathbb{PT}(\mathcal{F})$ destroys all P-points?*

Regarding half-Cohen reals, we do not know the answer of the following questions:

Problem 81 *If $\mathbb{P}$ does not add half-Cohen reals and $\mathbb{P} \Vdash$ "$\dot{\mathbb{Q}}$ does not add half-Cohen reals", is it true that $\mathbb{P} * \dot{\mathbb{Q}}$ does not add half-Cohen reals?*

Problem 82 *If δ is a limit ordinal, $\langle \mathbb{P}_\alpha, \dot{\mathbb{R}}_\alpha \mid \alpha < \delta \rangle$ is a countable support iteration of proper forcings such that each $\mathbb{P}_\alpha$ does not add half-Cohen reals, is it true that $\mathbb{P}_\delta$ does not add half-Cohen reals? What if each $\mathbb{P}_\alpha$ does not add dominating reals?*

Recall that not adding Cohen reals is not preserved under two step iteration by the result of Zapletal in [61].

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ON $(1,\omega_1)$-WEAKLY UNIVERSAL FUNCTIONS

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ABSTRACT. A function $U : [\omega_1]^2 \longrightarrow \omega$ is called $(1,\omega_1)$-*weakly universal* if for every function $F : [\omega_1]^2 \longrightarrow \omega$ there is an injective function $h : \omega_1 \longrightarrow \omega_1$ and a function $e : \omega \longrightarrow \omega$ such that $F(\alpha,\beta) = e(U(h(\alpha), h(\beta)))$ for every $\alpha, \beta \in \omega_1$. We will prove that it is consistent that there are no $(1,\omega_1)$-weakly universal functions, this answers a question of Shelah and Steprāns. In fact, we will prove that there are no $(1,\omega_1)$-weakly universal functions in the Cohen model and after adding ω_2 Sacks reals side-by-side. However, we show that there are $(1,\omega_1)$-weakly universal functions in the Sacks model. In particular, the existence of such graphs is consistent with ♣ and the negation of the Continuum Hypothesis.

Introduction and Preliminaries. A graph $U : [\omega_1]^2 \longrightarrow 2$ is called *universal* if for every graph $F : [\omega_1]^2 \longrightarrow 2$ there is an injective function $h : \omega_1 \longrightarrow \omega_1$ such that $F(\alpha,\beta) = U(h(\alpha), h(\beta))$ for each $\alpha, \beta \in \omega_1$. It is easy to see that universal graphs exist assuming the Continuum Hypothesis, and in [14] and [15] Shelah showed that the existence of universal functions is consistent with the failure of CH. In [10] Mekler showed that the existence of universal functions $U : [\omega_1]^2 \longrightarrow \omega$ is also consistent with the failure of the Continuum Hypothesis. Universal graphs and functions were recently studied by Shelah and Steprāns in [13], where they showed that the existence of universal graphs is consistent with several values of $\mathfrak{b}$ and $\mathfrak{d}$. They also considered several variations of universal functions, in particular, the following notion was studied:

Definition 1. A function $U : [\omega_1]^2 \longrightarrow \omega$ is $(1,\omega_1)$-weakly universal if for every $F : [\omega_1]^2 \longrightarrow \omega$ there is an injective function $h : \omega_1 \longrightarrow \omega_1$ and a function $e : \omega \longrightarrow \omega$ such that $F(\alpha,\beta) = e(U(h(\alpha), h(\beta)))$ for every $\alpha, \beta \in \omega_1$.

Evidently, every universal function is $(1,\omega_1)$-weakly universal. In [13] it was proved that a function $U : [\omega_1]^2 \longrightarrow \omega$ is $(1,\omega_1)$-weakly universal if and only if for every $F : [\omega_1]^2 \longrightarrow \omega$ there is an injective function $h : \omega_1 \longrightarrow \omega_1$

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such that if $F(\alpha,\beta) \neq F(\alpha_1,\beta_1)$ then $U(h(\alpha), h(\beta)) \neq U(h(\alpha_1), h(\beta_1))$ for every $\alpha, \beta, \alpha_1, \beta_1 \in \omega_1$.

In an unpublished note of Tanmay Inamdar, it was proved that $(1,\omega_1)$-weakly universal functions exist assuming Martin's axiom for Knaster forcings (see [13]). In [13] Shelah and Steprāns asked the following:

Problem 2 ([13])**.** *Is there (in ZFC) a $(1,\omega_1)$-weakly universal function?*

In this note, we answer the previous question in the negative. For more on universal graphs and functions, the reader may consult [9] and [13].

Recall that $\clubsuit$ is the following statement:

- **$\clubsuit$):** There is a family $\{S_\alpha \mid \alpha \in LIM(\omega_1)\}$ such that each S_α is an unbounded subset of α and for every $X \in [\omega_1]^{\omega_1}$ the set $\{\alpha \mid S_\alpha \subseteq X\}$ is stationary.

The principle $\clubsuit$ is a weakening of the $\diamondsuit$ principle. It is well known that $\clubsuit$ is consistent with the failure of the Continuum Hypothesis (see [16], [5], [8] or [1]). The *stick principle* (introduced in [2]) is a weakening of $\clubsuit$:

- **$\stackrel{\bullet}{|}$):** There is a family $\{S_\alpha \mid \alpha \in \omega_1\} \subseteq [\omega_1]^\omega$ such that for every $X \in [\omega_1]^{\omega_1}$ there is an $\alpha \in \omega_1$ such that $S_\alpha \subseteq X$.

It is easy to see that the stick principle is a consequence of both $\clubsuit$ and CH. For more on $\clubsuit$ and $\stackrel{\bullet}{|}$ the reader may consult [1], [5], [7] and [4].

We say that a tree $p \subseteq 2^{<\omega}$ is a *Sacks tree* if for every $s \in p$ there is $t \in p$ extending s such that $t^\frown 0, t^\frown 1 \subseteq p$. The set of all Sacks trees is denoted by $\mathbb{S}$ and we order it by extension. Given an ordinal α, by $\mathbb{S}^\alpha$ we will denote the countable support product of α copies of Sacks forcing and by $\mathbb{S}_\alpha$ we denote α-iteration of $\mathbb{S}$ with countable support. By the *Sacks model* we mean the model obtained after forcing with $\mathbb{S}_{\omega_2}$ and by the *side-by-side Sacks model* we mean the model obtained after forcing with $\mathbb{S}^{\omega_2}$ to a model of GCH. Although the partial orders $\mathbb{S}^{\omega_2}$ and $\mathbb{S}_{\omega_2}$ are not forcing equivalent, they share very similar features. It is then interesting to point out differences between this two forcing notions. Some of the main differences between them are the following:

(1) In the Sacks model every subset of reals of size ω_2 can be mapped continuously onto the reals, while in the side-by-side Sacks model this is not the case (see [12]).
(2) The cardinal invariant $\mathfrak{hm}$[1] is evaluated differently on the Sacks model and in the side-by-side Sacks model (see [6]).
(3) The CPA axioms hold in the Sacks model but not in the side-by-side Sacks model (see [3]).

In this note, we will point out another difference: There are $(1,\omega_1)$-weakly universal functions in the Sacks model, while there are no such graphs in the side-by-side Sacks model. In [9] it was proved that $\stackrel{\bullet}{|} + \mathfrak{c} > \omega_1$ implies that there is no universal function $U : [\omega_1]^2 \longrightarrow 2$. However, our results show that the existence of $(1,\omega_1)$-weakly universal functions is even consistent with $\clubsuit + \mathfrak{c} > \omega_1$.

The countable support product of Sacks forcing. The *Sacks side-by-side model* is the model obtained by forcing with $\mathbb{S}^{\omega_2}$ over a model of the Generalized Continuum Hypothesis. We will prove that there are no $(1,\omega_1)$-weakly universal graphs in the Sacks side-by-side model.

We will need the following lemma:

Lemma 3. *There is a function $\pi : 2^\omega \longrightarrow \omega^\omega$ such that for every $r \in 2^\omega$, if f is an infinite partial function such that $f \subseteq \pi(r)$, then r is definable from f.*

Proof. Let $h : \omega \longrightarrow 2^{<\omega}$ be a definable bijection. We define $\pi : 2^\omega \longrightarrow \omega^\omega$ as follows: if $r \in 2^\omega$ and $n \in \omega$ then $\pi(r)(n) = m$ if m is the least natural number such that $h(m)$ is an initial segment of r and $h(m)$ has length at least n. It is easy to see that π has the desired property. $\square$

Note that if M is a transitive model of ZFC and $r \notin M$ then $\pi(r)$ does not contain infinite partial functions from M. We will use the following unpublished result of Baumgartner (the reader may consult [8] for a proof):

Proposition 4 (Baumgartner)**.** *The principle $\clubsuit$ holds in the Sacks side-by-side model.*

[1]The cardinal invariant $\mathfrak{hm}$ is the smallest size of a family of c_{min}-monochromatic sets required to cover the Cantor space (where $c_{min}(x,y)$ is the parity of the largest initial segment common to both x and y). It is known that $\mathfrak{c}^-, cof(\mathcal{N}) \leq \mathfrak{hm}$ (see [6]). It is an open question of Geschke if the inequality $\mathfrak{hm} < \mathfrak{r}$ is consistent. In a yet unpublished work, the author proved that the inequality $\mathfrak{hm} < \mathfrak{u}$ is consistent.

In fact, we will only use that every uncountable subset of ω_1 in the Sacks side-by-side model contains a countable ground model set. Given a function $F : [\omega_1]^2 \longrightarrow \omega$ and $U : [\omega_1]^2 \longrightarrow \omega$, we say that (h, e) is an $(1, \omega_1)$-*weakly universal embedding from F to U* if $h : \omega_1 \longrightarrow \omega_1$ is an injective function, $e : \omega \longrightarrow \omega$ and $F(\alpha, \beta) = e(U(h(\alpha), h(\beta)))$ for every $\alpha, \beta \in \omega_1$. We can now prove the following result, answering the problem of Shelah and Steprāns:

Proposition 5. *There are no $(1, \omega_1)$-weakly universal graphs in the Sacks side-by-side model.*

Proof. Let $p_0 \in \mathbb{S}^{\omega_2}$ and $\dot{U}$ such that $p_0 \Vdash$ "$\dot{U} : [\omega_1]^2 \longrightarrow \omega$". Since the product of Sacks forcing has the ω_2-chain condition, we may find $\omega_1 \leq \beta < \omega_2$ such that $p_0 \in \mathbb{S}^\beta$ and $\dot{U}$ is a $\mathbb{S}^\beta$-name. Given $\alpha < \omega_1$, let $\dot{d}_\alpha$ be name for $\pi(\dot{r}_{\beta+\alpha})$ where $\dot{r}_{\beta+\alpha}$ is the name for the $(\beta+\alpha)$-generic real. For every infinite $\alpha < \omega_1$, we fix an enumeration $\alpha = \{\alpha_n \mid n \in \omega\}$.

If $G \subseteq \mathbb{S}^{\omega_2}$ is a generic filter, in $V[G]$ we define a function $F : [\omega_1]^2 \longrightarrow \omega$ as follows: given $\omega \leq \alpha < \omega_1$ we define $F(\alpha_n, \alpha) = d_\alpha(n)$. Let $\dot{F}$ be a name for F and let $\dot{h}$ be a $\mathbb{S}^{\omega_2}$-name for an injective function from ω_1 to ω_1 and $\dot{e}$ be a $\mathbb{S}^{\omega_2}$-name for a function from ω to ω. We will see that we can find an extension q of p_0 that forces that $(\dot{h}, \dot{e})$ is not a $(1, \omega_1)$-embedding of $\dot{F}$ in $\dot{U}$.

We can first find $p_1 \leq p_0$ and a ground model injective function $g : S \longrightarrow \omega_1$ such that $p_1 \Vdash$ "$g \subseteq \dot{h}$" where $S \in [\omega_1]^\omega$ (this is possible since the stick principle holds in the Sacks side-by-side model, witnessed by the ground model countable sets). Let M be a countable elementary submodel such that $p_1, \dot{U}, \beta, \dot{F}, g, \dot{h}, \dot{e} \in M$. Let $q \leq p_1$ be a $(M, \mathbb{S}^{\omega_2})$-generic condition. We claim that q forces that $(\dot{h}, \dot{e})$ is not an $(1, \omega_1)$-embedding of $\dot{F}$ in $\dot{U}$. Assume this is not the case, so there is $q_1 \leq q$ that forces that $(\dot{h}, \dot{e})$ is an $(1, \omega_1)$-embedding of $\dot{F}$ in $\dot{U}$.

Let $G \subseteq \mathbb{S}^{\omega_2}$ be a generic filter such that $q_1 \in G$. Let $X = \beta \cup (M \cap \omega_2)$ and define G_X to be the restriction of G to $\mathbb{S}^X$. Since q_1 is a $(M, \mathbb{S}^{\omega_2})$-generic condition, it follows that $\dot{U}[G], \dot{e}[G] \in V[G_X]$. Fix $\delta \in \omega_1$ such that $S \subseteq \delta$ and $\beta + \delta \notin X$, let $A = \{n \in \omega \mid \delta_n \in S\}$. For every $\alpha \in \omega_1$, we define $f_\alpha : A \longrightarrow \omega$ the function given by $f_\alpha(n) = \dot{e}[G]\left(\dot{U}[G](g(\delta_n), \alpha)\right)$ and note that $f_\alpha \in V[G_X]$ for every $\alpha \in \omega_1$. Let $\alpha \in \omega_1$ such that $\dot{h}[G](\delta) = \alpha$.

Since $(\dot{h}[G], \dot{e}[G])$ is forced to be an $(1,\omega_1)$-embedding, if $n \in A$ then we have the following:

$$\begin{aligned} d_\delta(n) &= \dot{F}[G](\delta_n, \delta) \\ &= \dot{e}[G]\left(\dot{U}[G]\left(\dot{h}(\delta_n), \dot{h}(\delta)\right)\right) \\ &= \dot{e}[G]\left(\dot{U}[G](g(\delta_n), \alpha)\right) \\ &= f_\alpha(n) \end{aligned}$$

Hence $f_\alpha \subseteq d_\delta$, but this is a contradiction since $r_{\beta+\delta} \notin V[G_X]$. $\square$

The Cohen model. The Cohen model is the model obtained after adding ω_2-Cohen reals with finite support to a model of the Generalized Continuum Hypothesis. We will show that there are no $(1,\omega_1)$-weakly universal graphs in the Cohen model.

Lemma 6. *If $U : [\omega_1]^2 \longrightarrow \omega$ then U is not $(1,\omega_1)$-weakly universal after adding ω_2 Cohen reals.*

Proof. Let $U : [\omega_1]^2 \longrightarrow \omega$. We define the function $H : [\omega^\omega]^2 \longrightarrow \omega$ given by $H(x,y) = |x \wedge y|$ (where $x \wedge y$ denotes the largest initial segment which both x and y have in common). Let $\dot{c}_\alpha$ be the name for the α-Cohen real. Let $\dot{F}$ be a name of a function from $[\omega_1]^2$ to ω such that $\mathbb{C}_{\omega_2} \Vdash$ "$\dot{F}(\alpha,\beta) = H(\dot{c}_\alpha, \dot{c}_\beta)$".

Let $p \in \mathbb{C}_{\omega_2}$, $\dot{h}$ a name for an injective function from ω_1 to ω_1 and $\dot{e}$ a name for a function from ω to ω. We must find $q \leq p$ and $\alpha, \beta \in \omega_1$ such that $q \Vdash$ "$\dot{F}(\alpha,\beta) \neq \dot{e}U(\dot{h}(\alpha), \dot{h}(\beta))$". For every $\alpha < \omega_1$ we find $p_\alpha \leq p$ and δ_α such that $p_\alpha \Vdash$ "$\dot{h}(\alpha) = \delta_\alpha$" and $\alpha \in dom(p_\alpha)$. By the usual pruning arguments, we may find $X \in [\omega_1]^{\omega_1}$, $R \in [\omega_2]^{<\omega}$, $\overline{p} \in \mathbb{C}_{\omega_2}$ and $s \in \omega^{<\omega}$ such that the following holds:

(1) $\{dom(p_\alpha) \mid \alpha \in X\}$ forms a Δ-system with root R.
(2) $p_\alpha \restriction R = \overline{p}$ for every $\alpha \in X$.
(3) $\alpha \notin R$ for every $\alpha \in X$.
(4) $p(\alpha) = s$ for every $\alpha \in X$.

It is clear that $\{p_\alpha \mid \alpha \in \omega_1\}$ is a centered set (any finite set of conditions are compatible). Let M be a countable elementary submodel such that $\overline{p}, \{p_\alpha \mid \alpha \in \omega_1\}, \dot{e}, R \in M$. Since $M \cap \omega_2$ is countable, we may find $\alpha, \beta \in X \setminus M$ such that $\alpha \neq \beta$ and $dom(p_\alpha) \cap M = dom(p_\beta) \cap M = R$. Let

$m = U(\delta_\alpha, \delta_\beta)$. We may now find $\overline{q}$ and i such that the following conditions hold:

(1) $\overline{q} \in M$ and $\overline{q} \leq \overline{p}$.
(2) $\overline{q} \Vdash$ "$\dot{e}(m) = i$".

This is possible since $\overline{p}, \dot{e} \in M$. Let $t \in \omega^{<\omega}$ be such that $|t| > i$ and $s \subseteq t$. We now define a condition q as follows:

$$q(\xi) = \left\langle \begin{array}{cl} \overline{q}(\xi) & \text{if } \xi \in dom(\overline{q}) \\ p_\alpha(\xi) & \text{if } \xi \in dom(p_\alpha) \setminus dom(\overline{q}) \text{ and } \xi \neq \alpha \\ p_\beta(\xi) & \text{if } \xi \in dom(p_\beta) \setminus dom(\overline{q}) \text{ and } \xi \neq \beta \\ t & \text{if } \xi = \alpha \text{ or } \xi = \beta \end{array} \right.$$

Note that this is possible since $dom(\overline{q}) \subseteq M$. Clearly $q \Vdash$ "$\dot{F}(\alpha, \beta) > i$" and $q \Vdash$ "$\dot{e}(U(\dot{h}(\alpha), \dot{h}(\beta))) = i$" so $q \Vdash$ "$\dot{F}(\alpha, \beta) \neq \dot{e}(U(\dot{h}(\alpha), \dot{h}(\beta)))$". $\square$

Since Cohen forcing has the countable chain condition, we conclude the following:

Proposition 7. *There are no $(1, \omega_1)$-weakly universal graphs in the Cohen model.*

The Sacks model. The proof that there are no $(1, \omega_1)$-weakly universal graph in the Side by Side Sacks model uses that the stick principle holds in such model. It is then natural to wonder if the stick principle is enough to get the non-existence of such graphs (under the failure of the Continuum Hypothesis). Moreover, the stick principle already forbids the existence of some universal graphs, as the following result of Shelah and Steprāns shows:

Proposition 8 ([13])**.** *$\stackrel{\bullet}{\shortmid} + \mathfrak{c} > \omega_1$ implies that there is no universal function $U : [\omega_1]^2 \longrightarrow 2$.*

By the *Sacks model* we mean a model obtained by forcing with $\mathbb{S}_{\omega_2}$ over a model of the Generalized Continuum Hypothesis. In this section, we will prove that there is a $(1, \omega_1)$-weakly universal graph in the Sacks model. The following is a result of Mildenberger:

Proposition 9 ([11])**.** *$\clubsuit$ holds in the Sacks model.*

In particular, we will be able to conclude that the existence of a $(1,\omega_1)$-weakly universal graph is consistent with $\clubsuit$. As usual, if $T \subseteq 2^{<\omega}$ is a tree, we denote by $[T]$ the set of all branches (i.e. maximal linearly order sets) through T. Given $f \in 2^{\omega}$ and $T \subseteq 2^{<\omega}$ a finite tree, we say that $f \in^* [T]$ if there is $n \in \omega$ such that $f \restriction n \in [T]$. If $f \in^* [T]$, we define by $f \restriction T$ to be the unique $t \in 2^{<\omega}$ such that there is n for which $t = f \restriction n \in [T]$. For this section, we fix W as the set of all (T,f) such that $T \subseteq 2^{<\omega}$ is a finite tree and $f : [T] \longrightarrow \omega$. It is easy to see that W is a countable set.

We will need some definition and lemmas regarding iterated Sacks forcing. The following is based on [12] and [8]. If $p \in \mathbb{S}$ and $s \in 2^{<\omega}$ we define $p_s = \{t \in p \mid t \subseteq s \vee s \subseteq t\}$. Note that p_s is a Sacks tree if and only if $s \in p$. By *supp* (p) we will denote the support of p.

Definition 10. Let $p \in \mathbb{S}_\alpha$, $F \in [supp\,(p)]^{<\omega}$ and $\sigma : F \longrightarrow 2^n$. We define p_σ as follows:

(1) $supp\,(p_\sigma) = supp\,(p)$.
(2) Letting $\beta < \alpha$ the following holds:
  (a) $p_\sigma(\beta) = p(\beta)$ if $\beta \notin F$.
  (b) $p_\sigma(\beta) = p(\beta)_{\sigma(\beta)}$ if $\beta \in F$.

Similar to previous situation, p_σ is not necessarily a condition of $\mathbb{S}_\alpha$. We will say that $\sigma : F \longrightarrow 2^n$ is *consistent with* p if $p_\sigma \in \mathbb{S}_\alpha$. A condition p is (F,n)-*determined* if for every $\sigma : F \longrightarrow 2^n$ either σ is consistent with p or there is $\beta \in F$ such that $\sigma \restriction (F \cap \beta)$ is consistent with p and $(p \restriction \beta)_{\sigma \restriction (F\cap\beta)} \Vdash$ "$\sigma(\beta) \notin p(\beta)$".

We say that $p \in \mathbb{S}_\alpha$ is *determined* if for every $F \in [supp\,(p)]^{<\omega}$ and for every $n \in \omega$ there are G and m such that the following holds:

(1) $G \in [supp\,(p)]^{<\omega}$.
(2) $F \subseteq G$.
(3) $n < m$.
(4) p is (G,m)-determined.

The following result is well known:

Lemma 11 ([12])**.** *For every $p \in \mathbb{S}_\alpha$ there is a determined $q \leq p$.*

Let p be a determined condition. We say that $\langle (F_i, n_i, \Sigma_i) \mid i \in \omega \rangle$ is *a representation of* p if the following holds:

(1) $F_i \in [supp\,(p)]^{<\omega}$, $n_i \in \omega$.
(2) $F_i \subseteq F_{i+1}$ and $n_i < n_{i+1}$.
(3) $supp\,(p) = \bigcup\limits_{i \in \omega} F_i$.
(4) p is (F_i, n_i)-determined for every $i \in \omega$.
(5) Σ_i is the set of all $\sigma : F_i \longrightarrow 2^{n_i}$ such that σ is consistent with p.

We will also need the following definition:

Definition 12. Let $p \in \mathbb{S}_\alpha$ be a determined condition and $\dot{r}$ an $\mathbb{S}_\alpha$-name for an element of 2^ω. We say that p is *$\dot{r}$-canonical* if there are two sequences $\langle (F_i, n_i, \Sigma_i) \mid i \in \omega \rangle$ and $\langle \mathcal{C}_i \mid i \in \omega \rangle$ with the following properties:

(1) $\{(F_i, n_i, \Sigma_i) \mid i \in \omega\}$ is a representation of p.
(2) $\mathcal{C}_i = \{C_\sigma \mid \sigma \in \Sigma_i\}$ is a collection of disjoint clopen subsets of 2^ω.
(3) For every $\sigma \in \Sigma_i$ there is $s_\sigma \in 2^{n_i}$ such that $C_\sigma \subseteq \langle s_\sigma \rangle$.[2]
(4) If $i \in \omega$ and $\sigma \in \Sigma_i$, then $p_\sigma \Vdash$ "$\dot{r} \in C_\sigma$" (in particular, p_σ determines $\dot{r} \restriction n_i$).

In the above situation, we say that $\langle (F_i, n_i, \Sigma_i, \mathcal{C}_i) \mid i \in \omega \rangle$ is an *$\dot{r}$-canonical representation for* p. The following is lemma 6 of [12]:

Lemma 13 ([12])**.** *Let $\alpha \leq \omega_2$, $p \in \mathbb{S}_\alpha$ and $\dot{r}$ an $\mathbb{S}_\alpha$-name for an element of 2^ω such that $p \Vdash$ "$\dot{r} \notin \bigcup\limits_{\beta<\alpha} V[G_\beta]$". There is $q \leq p$ such that q is $\dot{r}$-canonical.*

With the same proof of the previous lemma, it is possible to prove the following:

Lemma 14. *Let $\alpha \leq \omega_2$, $p \in \mathbb{S}_\alpha$, $\dot{r}$ an $\mathbb{S}_\alpha$-name for an element of 2^ω such that $p \Vdash$ "$\dot{r} \notin \bigcup\limits_{\beta<\alpha} V[G_\beta]$" and $\dot{g}$ an $\mathbb{S}_\alpha$-name for an element of ω^ω. There is $q \leq p$ such that q is $\dot{r}$-canonical with $\dot{r}$-canonical representation $\langle (F_i, n_i, \Sigma_i, \mathcal{C}_i) \mid i \in \omega \rangle$ and there is $\langle h_i \mid i \in \omega \rangle$ such that the following conditions:*

(1) *$h_i : \Sigma_i \longrightarrow \omega$ for every $i \in \omega$.*
(2) *If $i \in \omega$ and $\sigma \in \Sigma_i$ then $q_\sigma \Vdash$ "$\dot{g}(i) = h_i(\sigma)$".*

[2]If $t \in 2^{<\omega}$ we define $\langle t \rangle = \{x \in 2^\omega \mid t \subseteq x\}$.

The lemma 6 of [12] is proved using a fusion argument. To prove the previous lemma we use the same fusion argument, with the extra step of deciding the respective value of $\dot{g}$ at each step. We leave the details for the reader. As before, in the above situation we say that $\langle (F_i, n_i, \Sigma_i, \mathcal{C}_i, h_i) \mid i \in \omega \rangle$ is an $(\dot{r}, \dot{g})$*-canonical representation for* q.

We can now prove the following:

Proposition 15. *Let $\eta < \omega_2, \dot{g}$ and $p \in \mathbb{S}_{\eta+1}$ such that $p \Vdash$ "$\dot{g} : \omega \longrightarrow \omega$". There is a determined $q \in \mathbb{S}_{\eta+1}$ and $\{(n, T_n, f_n) \mid n \in \omega\}$ with the property that $\{(T_n, f_n) \mid n \in \omega\} \subseteq W$ such that the following holds:*

(1) *$q \leq p$.*
(2) *$q \Vdash$ "$\dot{r}_\eta \in^* [T_n]$" for each $n \in \omega$.*
(3) *$q \Vdash$ "$\dot{g}(n) = f_n (\dot{r}_\eta \restriction T_n)$" for every $n \in \omega$.*

Proof. By the previous lemma, we can find $p_1 \leq p$ that has an $(\dot{r}_\eta, \dot{g})$-canonical representation $\{(F_i, n_i, \Sigma_i, \mathcal{C}_i, h_i) \mid i \in \omega\}$. We now have the following interesting property: If $G \subseteq \mathbb{S}_{\eta+1}$ is a generic filter with $p_1 \in G$ and $\langle r_\alpha \rangle_{\alpha \leq \eta}$ is the generic sequence, then the following holds in $V[G]$:

***):** For every $i \in \omega$ and $\sigma \in \Sigma_i$, if $r_\eta \in C_\sigma$ then $\sigma(\beta) \subseteq r_\beta$ for every $\beta \in F_i$.

This property holds because $\mathcal{C}_i = \{C_\sigma \mid \sigma \in \Sigma_i\}$ is a collection of disjoint sets. In this way, r_η is able to "code" each of the previews generic reals. Let Y be the set of all maximal $z \in 2^{<\omega}$ with the property that $\langle z \rangle \subseteq \bigcup \mathcal{C}_i$. Note that since $\mathcal{C}_i$ is a finite set of clopen sets, Y is a finite set. Let T_i be the smallest finite tree such that $Y \subseteq T_i$. Note that T_i has the following properties:

(1) $\bigcup\limits_{s \in [T_i]} \langle s \rangle = \bigcup \mathcal{C}_i$.
(2) For every $s \in [T_i]$ there is exactly one $\sigma \in \Sigma_i$ for which $\langle s \rangle \subseteq C_\sigma$ (where $\mathcal{C}_i = \{C_\sigma \mid \sigma \in \Sigma_i\}$).

For every i we have the following properties:

(1) $p_1 \Vdash$ "$\dot{r}_\eta \in [T_i]$".
(2) Let $G \subseteq \mathbb{S}_{\eta+1}$ be a generic filter with $p_1 \in G$ and $\sigma \in \Sigma_i$. If $r_\eta \in C_\sigma$ then $(p_1)_\sigma \in G$.

We now have the following claim:

Claim 16. *If $i \in \omega$, $s \in [T_i]$ and q_0, q_1 are two conditions extending p_1 such that $q_i \Vdash$ "$s \subseteq \dot{r}_\eta$" for $j \in \{0,1\}$ then there is $k \in \omega$ such that $q_0 \Vdash$ "$\dot{g}(i) = k$" and $q_1 \Vdash$ "$\dot{g}(i) = k$".*

We will prove the claim. Let $\sigma \in \Sigma_i$ such that $\langle s \rangle \subseteq C_\sigma$ and let $j < 2$. Note that since $q_j \leq p_1$ and $q_j \Vdash$ "$\dot{r}_\eta \in C_\sigma$", it follows that $q_j \Vdash$ "$(p_1)_\sigma \in \dot{G}$" (where $\dot{G}$ is a name for the generic filter), hence $q_j \Vdash$ "$\dot{g}(i) = h_i(\sigma)$", the claim follows.

For every $n \in \omega$, we define a function $f_n : [T_n] \longrightarrow \omega$ as follows: for every $s \in [T_n]$, let $f_n(s)$ such that for every $q \leq p_1$ if $q \Vdash$ "$s \subseteq \dot{r}_\eta$" then $q \Vdash$ "$\dot{g}(n) = f_n(s)$". Note that f_n is well defined by the previous claim. It is easy to see that $\{(n, T_n, f_n) \mid n \in \omega\}$ has the desired properties. $\square$

We will say that a graph $U : [\omega_1]^2 \longrightarrow W$ is $(1, \omega_1)$-*weakly universal* if for every $F : [\omega_1]^2 \longrightarrow \omega$ there is an injective $h : \omega_1 \longrightarrow \omega_1$ and a function $e : W \longrightarrow \omega$ such that $F(\alpha, \beta) = e(U(h(\alpha), h(\beta)))$. As expected, we have the following result:

Lemma 17. *If there is a $U : [\omega_1]^2 \longrightarrow W$ which is $(1, \omega_1)$-weakly universal, then there is $U_1 : [\omega_1]^2 \longrightarrow \omega$ that is $(1, \omega_1)$-weakly universal.*

Proof. Let $U : [\omega_1]^2 \longrightarrow W$ be a $(1, \omega_1)$-weakly universal graph. Fix $g : W \longrightarrow \omega$ a bijective function. We define $U_1 : [\omega_1]^2 \longrightarrow \omega$ where $U_1(\alpha, \beta) = g(U(\alpha, \beta))$. It is easy to see that U_1 is $(1, \omega_1)$-weakly universal. $\square$

For the rest of this section, we will assume the Continuum Hypothesis. Fix a large enough regular cardinal $\theta > (2^{\omega_2})^+$. We will now fix $\overline{M} = \{(M_\alpha, \in, \Vdash_{\mathbb{S}_{\eta_\alpha+1}}, p_\alpha, \eta_\alpha, \xi_\alpha, \dot{g}_\alpha) \mid \alpha \in \omega_1\}$ with the following properties:

(1) M_α is a countable elementary submodel of $H(\theta)$ with the property that $p_\alpha, \eta_\alpha, \xi_\alpha, \dot{g}_\alpha \in M_\alpha$.
(2) $\eta_\alpha < \omega_2$ and $p_\alpha \in \mathbb{S}_{\eta_\alpha+1}$.
(3) $\xi_\alpha < \omega_1$ and $p_\alpha \Vdash$ "$\dot{g}_\alpha : \xi_\alpha \longrightarrow \omega$".
(4) For every $(N, \in, \Vdash_{\mathbb{S}_{\eta+1}}, p, \eta, \xi, \dot{g})$ if the following properties hold:
   (a) N is a countable elementary submodel of $H(\theta)$ with the property that $p, \eta, \xi, \dot{g} \in N$.
   (b) $\eta < \omega_2$ and $p \in \mathbb{S}_{\eta+1}$.
   (c) $\xi < \omega_1$ and $p \Vdash$ "$\dot{g} : \xi \longrightarrow \omega$".

Then, there is $\alpha<\omega_1$ such that $(M_\alpha,\in,\Vdash_{\mathbb{S}_{\eta_\alpha+1}},p_\alpha,\eta_\alpha,\xi_\alpha,\dot{g}_\alpha)$ and $(N,\in,\Vdash_{\mathbb{S}_{\eta+1}},p,\eta,\xi,\dot{g})$ are isomorphic.

This is possible since $\mathbb{S}_{\eta+1}$ is proper and we are assuming the Continuum Hypothesis. For every $\alpha<\omega_1$, let $\delta_\alpha=M_\alpha\cap\omega_1$. We now choose $\{\beta_\alpha\mid\alpha\in\omega_1\}\subseteq\omega_1$ such that $\delta_\alpha<\beta_\alpha$ and if $\alpha_1\neq\alpha_2$ then $\beta_{\alpha_1}\neq\beta_{\alpha_2}$. For every $\alpha<\omega_1$, we also fix an enumeration $\xi_\alpha=\{\xi_\alpha(n)\mid n\in\omega\}$. By the previous lemmas, for every $\alpha<\omega_1$, we can find q_α, $\{(n,T_n^\alpha,f_n^\alpha)\mid n\in\omega\}$ such that the following holds:

(1) $q_\alpha\in\mathbb{S}_{\eta_\alpha+1}\cap M_\alpha$ and $q_\alpha\leq p_\alpha$.
(2) $\{(T_n^\alpha,f_n^\alpha)\mid n\in\omega\}\subseteq W$.
(3) $q_\alpha\Vdash$ "$\dot{r}_{\eta_\alpha}\in^*[T_n^\alpha]$" for each $n\in\omega$.
(4) $q_\alpha\Vdash$ "$\dot{g}_\alpha(\xi_\alpha(n))=f_n^\alpha(\dot{r}_{\eta_\alpha}\restriction T_n^\alpha)$" for every $n\in\omega$.

We now define the graph $U:[\omega_1]^2\longrightarrow W$ as follows: given $\alpha<\omega_1$ and $n\in\omega$ we define $U(\xi_\alpha(n),\beta_\alpha)=(T_n^\alpha,f_n^\alpha)$ (the value of U is not important in any other case, so if a pair (ν_1,ν_2) is not of the form $(\xi_\alpha(n),\beta_\alpha)$, we can let $U(\nu_1,\nu_2)$ be any element of W otherwise). We will show that U is forced to be $(1,\omega_1)$-weakly universal. Given $\eta<\omega_2$, in the forcing extension, we define the function $e_\eta:W\longrightarrow\omega$ given by $e_\eta(T,f)=f(\dot{r}_\eta\restriction T)$ if $\dot{r}_\eta\in^*[T]$ and $e_\eta(T,f)=0$ in other case. We need the following lemma:

Lemma 18. *Let $G\subseteq\mathbb{S}_{\omega_2}$ be a generic filter. Let $\eta<\omega_2,\xi<\omega_1$ and $g:\xi\longrightarrow\omega$ such that $g\in V[G_{\eta+1}]$. There is $\alpha\in\omega_1$ such that the following holds:*

(1) $\xi_\alpha=\xi$.
(2) $g(\xi_\alpha(n))=e_\eta(U(\xi_\alpha(n),\beta_\alpha))$ *for every* $n\in\omega$.

Proof. It is enough to show that the conditions that force the above properties are dense, in this way, there will be such condition in the generic filter. Let $p\in\mathbb{S}_{\eta+1}$, we will see we can extend p to get the desired conclusion. Let N be a countable elementary submodel such that $p,\eta,\xi,\dot{g}\in N$. We first find $\alpha<\omega_1$ such that $(M_\alpha,\in,\Vdash_{\mathbb{S}_{\eta_\alpha+1}},p_\alpha,\eta_\alpha,\xi_\alpha,\dot{g}_\alpha)$ and $(N,\in,\Vdash_{\mathbb{S}_{\eta+1}},p,\eta,\xi,\dot{g})$ are isomorphic. Let $\pi:M_\alpha\longrightarrow N$ be the isomorphism and let $q=\pi(q_\alpha)$. Note that the isomorphism fixes every ordinal smaller than δ_α (in particular each $\xi_\alpha(n)$) as well as each element in W. By the isomorphism, the following conditions hold:

(1) $q\in\mathbb{S}_{\eta+1}\cap N$ and $q\leq p$.
(2) $q\Vdash$ "$\dot{r}_\eta\in^*[T_n^\alpha]$" for each $n\in\omega$.

(3) $q \Vdash$ "$\dot{g}(\xi_\alpha(n)) = f_n^\alpha(\dot{r}_\eta \restriction T_n^\alpha)$" for every $n \in \omega$.

By the last clause, it follows that $q \Vdash$ "$\dot{g}(\xi_\alpha(n)) = e_\eta(U(\xi_\alpha(n), \beta_\alpha))$". □

We can then prove the following:

Proposition 19. *There is a $(1, \omega_1)$-weakly universal graph in the Sacks model.*

Proof. We will show that U is forced to be a $(1, \omega_1)$-weakly universal graph (note that this is enough by lemma 17). Let $p \in \mathbb{S}_{\omega_2}$ and $\dot{F}$ such that $p \Vdash$ "$\dot{F} : [\omega_1]^2 \longrightarrow \omega$". Since Sacks forcing has the ω_2-chain condition, we may assume that there is $\eta < \omega_2$ such that $p \in \mathbb{S}_\eta$ and $\dot{F}$ is an $\mathbb{S}_\eta$-name.

Given $\gamma \leq \omega_1$, we will say that an injective function $h : \gamma \longrightarrow \omega_1$ is a *partial e_η-embedding* if $F(\alpha, \beta) = e_\eta(U(h(\alpha), h(\beta)))$ for every $\alpha, \beta < \gamma$. Let G be a generic filter such that $p \in G$. We claim that in $V[G_{\eta+1}]$ the following holds:

***):** If $h : \gamma \longrightarrow \omega_1$ is a partial e_η-embedding with $\gamma < \omega_1$, then there is a partial e_η-embedding $\overline{h} : \gamma + 1 \longrightarrow \omega_1$ extending h.

We argue in $V[G_{\eta+1}]$. Let $\xi = \bigcup h[\gamma] + 1$ and note we can find $g \in V[G_{\eta+1}]$ such that $g : \xi \longrightarrow \omega$ and $g(h(\delta)) = F(\delta, \gamma)$ for all $\delta < \gamma$. By the previous lemma, there is $\alpha \in \omega_1$ such that $\xi_\alpha = \xi$ and $g(\xi_\alpha(n)) = e_\eta(U(\xi_\alpha(n), \beta_\alpha))$. We now define $\overline{h} = h \cup \{(\gamma, \beta_\alpha)\}$. Note that $\beta_\alpha \notin h[\gamma]$ since $h[\gamma] \subseteq \xi = \xi_\alpha < \delta_\alpha < \beta_\alpha$. We only need to prove that $\overline{h}$ is a partial e_η-embedding. Let $\delta < \gamma$, we can find $n \in \omega$ such that $h(\delta) = \xi_\alpha(n)$. It then follows that:

$$\begin{aligned} e_\eta(U(\overline{h}(\delta), \overline{h}(\gamma))) &= e_\eta(U(\xi_\alpha(n), \beta_\alpha)) \\ &= g(\xi_\alpha(n)) \\ &= g(h(\delta)) \\ &= F(\delta, \gamma) \end{aligned}$$

This finishes the claim. It is clear that any maximal e_η-embedding will embed F into U. □

Open questions. In general, a function $U : [\omega_1]^2 \longrightarrow \omega$ is $(1, \kappa)$-*weakly universal* if for every $F : [\omega_1]^2 \longrightarrow \omega$ there is an injective function h :

$\omega_1 \longrightarrow \omega_1$ and a function $e : \omega \longrightarrow \omega$ such that $|e^{-1}(n)| < \kappa$ for every $n \in \omega$ and $F(\alpha, \beta) = e(U(h(\alpha), h(\beta)))$ for every $\alpha, \beta \in \omega_1$. It would be interesting to know the answer of the following question:

Problem 20. *Are there $(1, \omega)$-weakly universal functions (or even $(1, 2)$-weakly universal functions) in the Sacks model?*

In fact, we conjecture that $\text{¡} + \mathfrak{c} > \omega_1$ implies that there are no $(1, \omega)$-weakly universal functions.

We would like to mention that there are no $(1, \omega_1)$-weakly universal functions after performing a pseudo-iteration of Cohen forcing, as described in [5]. It would be interesting to know what kind of universal graphs exist on the "canonical models" of set theory.

Problem 21. *Are there $(1, \omega_1)$-weakly universal functions in the random, Hechler, Laver, Miller and Mathias models?*

The purpose of the CPA axioms introduced in [3] is to provide an axiomatization of the Sacks model. In light of this work, it is then natural to ask the following:

Problem 22. *Does the existence of $(1, \omega_1)$-weakly universal functions follow from one of the CPA axioms?*

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THE ONTO MAPPING OF SIERPINSKI AND NONMEAGER SETS

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Abstract. The principle $(*)$ of Sierpinski is the assertion that there is a family of functions $\{\varphi_n : \omega_1 \longrightarrow \omega_1 \mid n \in \omega\}$ such that for every $I \in [\omega_1]^{\omega_1}$ there is $n \in \omega$ such that $\varphi_n[I] = \omega_1$. We prove that this principle holds if there is a nonmeager set of size ω_1, answering question of Arnold W. Miller. Combining our result with a theorem of Miller it then follows that $(*)$ is equivalent to *non* $(\mathcal{M}) = \omega_1$. Miller also proved that the principle of Sierpinki is equivalent to the existence of a weak version of a Luzin set, we will construct a model where all of these sets are meager yet *non* $(\mathcal{M}) = \omega_1$.

§1. Introduction.

The *principle* $(*)$ *of Sierpinski* is the following statement: There is a family of functions $\{\varphi_n : \omega_1 \longrightarrow \omega_1 \mid n \in \omega\}$ such that for every $I \in [\omega_1]^{\omega_1}$ there is $n \in \omega$ for which $\varphi_n[I] = \omega_1$. It was introduced by Sierpinski and he proved that it is a consequence of the Continuum Hypothesis. It was recently studied by Arnold W. Miller in [6], which was the motivation for this work. This principle is related to the following type of sets:

DEFINITION 1.1. Let $\mathcal{I}$ be a σ-ideal on ω^ω. We say $X = \{f_\alpha \mid \alpha < \omega_1\} \subseteq \omega^\omega$ is an *$\mathcal{I}$-Luzin set* if $X \cap A$ is at most countable for every $A \in \mathcal{I}$.

In this terminology, the Luzin sets are $\mathcal{M}$-Luzin sets (where $\mathcal{M}$ denotes the ideal of all meager sets) and the Sierpinski sets are the $\mathcal{N}$-Luzin sets (where $\mathcal{N}$ denotes the ideal of all sets with Lebesgue measure zero). Given a σ-ideal $\mathcal{I}$, its *uniformity number non* $(\mathcal{I})$ is the smallest size of a set that is not an element of $\mathcal{I}$. Clearly the existence of an $\mathcal{I}$-Luzin set implies *non* $(\mathcal{I}) = \omega_1$, but the converse is usually not true. For example, it was shown by Shelah and Judah in [4] that there are no Luzin or Sierpinski sets in the Miller model while *non* $(\mathcal{M}) =$ *non* $(\mathcal{N}) = \omega_1$ holds.

DEFINITION 1.2.

1. Given $f \in \omega^\omega$ we define $ED(f) = \{g \in \omega^\omega \mid |f \cap g| < \omega\}$.
2. $\mathcal{IE}$ is the σ-ideal generated by $\{ED(f) \mid f \in \omega^\omega\}$.

It is easy to see that each $ED(f)$ is a meager set so $\mathcal{IE} \subseteq \mathcal{M}$. It is well known that *non* $(\mathcal{IE}) =$ *non* $(\mathcal{M})$ (see [3]). In [6], Miller proved the following result:

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PROPOSITION 1.3 (Miller [6]). *The following are equivalent*:

1. *The principle* $(*)$ *of Sierpinski*.
2. *There is a family* $\{g_\alpha : \omega \longrightarrow \omega_1 \mid \alpha < \omega_1\}$ *with the property that for every* $g : \omega \longrightarrow \omega_1$ *there is* $\alpha < \omega_1$ *such that if* $\beta > \alpha$ *then* $g_\beta \cap g$ *is infinite.*
3. *There is an* $\mathcal{IE}$*-Luzin set*.

The implication from 3 to 1 is not explicit in [6] (it is implicitly proved in the lemma 6 of [6]). The referee found a very elegant and short proof of this result which we reproduce here. We are grateful with the referee for allowing us to include his proof.

PROPOSITION 1.4. *The existence of an* $\mathcal{IE}$*-Luzin set implies the principle* $(*)$ *of Sierpinski*.

PROOF. Let $\mathcal{A} = \{A_\alpha \mid \omega \leq \alpha < \omega_1\}$ be an almost disjoint family. Since there is an $\mathcal{IE}$-Luzin, for each α, we can find a family $\mathcal{F}_\alpha = \left\{f_{\alpha\beta} : A_\alpha \longrightarrow \alpha \mid \beta < \omega_1\right\}$ such that for every $g : A_\alpha \longrightarrow \alpha$ there is δ such that if $\beta > \delta$ then $f_{\alpha\beta} \cap g$ is infinite. Since $\mathcal{A}$ is an almost disjoint family, we can then construct a family $\mathcal{G} = \left\{g_\beta : \omega \longrightarrow \omega_1 \mid \omega \leq \beta < \omega_1\right\}$ such that $f_{\alpha\beta} =^* g_\beta \restriction A_\alpha$ for every $\alpha < \beta < \omega_1$.

By the previous proposition, we need to prove that for every $g : \omega \longrightarrow \omega_1$ there is $\alpha < \omega_1$ such that if $\beta > \alpha$ then $g_\beta \cap g$ is infinite. First we find δ such that $g : \omega \longrightarrow \delta$ and then we know there is γ such that if $\beta > \gamma$ then $f_{\delta\beta} \cap (g \restriction A_\delta)$ is infinite. It then follows that if $\beta > max\,\{\delta, \gamma\}$ then $g_\beta \restriction A_\delta =^* f_{\delta\beta}$ so $\left|g_\beta \cap g\right| = \omega$. ⊣

It then follows that the existence of a Luzin set implies the principle $(*)$ of Sierpinski while it implies $non\,(\mathcal{M}) = \omega_1$. Miller then asked if the principle $(*)$ of Sierpinski is a consequence of $non\,(\mathcal{M}) = \omega_1$ and we will show that this is indeed the case. In the second part of the paper, we will prove (with the aid of an inaccessible cardinal) that while $non\,(\mathcal{M}) = \omega_1$ implies the existence of a $\mathcal{IE}$-Luzin set, it does not imply the existence of a nonmeager $\mathcal{IE}$-Luzin set.

§2. $non\,(\mathcal{M}) = \omega_1$ implies the existence of an $\mathcal{IE}$-Luzin set.

We will now show that the principle $(*)$ of Sierpinski follows by $non\,(\mathcal{M}) = \omega_1$, answering the question of Miller. By *Partial* (ω^ω), we shall denote the set of all infinite partial functions from ω to ω. We start with the following lemma:

LEMMA 2.1. *If* $non\,(\mathcal{M}) = \omega_1$ *then there is a family* $X = \{f_\alpha \mid \alpha < \omega_1\}$ *with the following properties*:

1. *Each* f_α *is an infinite partial function from* ω *to* ω.
2. *The set* $\{dom\,(f_\alpha) \mid \alpha < \omega_1\}$ *is an almost disjoint family.*
3. *For every* $g : \omega \longrightarrow \omega$, *there is* $\alpha < \omega_1$ *such that* $f_\alpha \cap g$ *is infinite.*

PROOF. Let $\omega^{<\omega} = \{s_n \mid n \in \omega\}$ and we define $H : \omega^\omega \longrightarrow Partial\,(\omega^\omega)$ where the domain of $H\,(f)$ is $\{n \mid s_n \sqsubseteq f\}$ and if $n \in dom\,(H\,(f))$ then $H\,(f)\,(n) = f\,(|s_n|)$. It is easy to see that if $f \neq g$ then $dom\,(H\,(f))$ and $dom\,(H\,(g))$ are almost disjoint.

Given $g : \omega \longrightarrow \omega$, we define $N\,(g) = \{f \in \omega^\omega \mid |H\,(f) \cap g| < \omega\}$. It then follows that $N\,(g)$ is a meager set since $N\,(g) = \bigcup\limits_{k\in\omega} N_k\,(g)$, where $N_k\,(g) = \{f \in \omega^\omega \mid |H\,(f) \cap g| < k\}$, and it is easy to see that each $N_k\,(g)$ is a nowhere dense set. Finally, if $X = \{h_\alpha \mid \alpha < \omega_1\}$ is a nonmeager set then $H\,[X]$ is the family we were looking for. ⊣

With the previous lemma we can prove the following:

PROPOSITION 2.2. *If* $non(\mathcal{M}) = \omega_1$ *then the principle* $(*)$ *of Sierpinski is true.*

PROOF. Let $X = \{f_\alpha \mid \alpha < \omega_1\}$ be a family as in the previous lemma. We will build a $\mathcal{IE}$-Luzin set $Y = \{h_\alpha \mid \alpha < \omega_1\}$. For simplicity, we may assume $\{dom(f_n) \mid n \in \omega\}$ is a partition of ω.

For each $n \in \omega$, let h_n be any constant function. Given $\alpha \geq \omega$, enumerate it as $\alpha = \{\alpha_n \mid n \in \omega\}$ and then we recursively define $B_0 = dom(f_{\alpha_0})$ and $B_{n+1} = dom(f_{\alpha_n}) \setminus (B_0 \cup \cdots \cup B_n)$. Clearly $\{B_n \mid n \in \omega\}$ is a partition of ω. Let $h_\alpha = \bigcup\limits_{n\in\omega} f_{\alpha_n} \restriction B_n$, it then follows that $Y = \{h_\alpha \mid \alpha < \omega_1\}$ is an $\mathcal{IE}$-Luzin set. ⊣

§3. $non(\mathcal{M}) = \omega_1$ **does not imply the existence of a nonmeager** $\mathcal{IE}$**-Luzin set.** It is not hard to see that the $\mathcal{IE}$-Luzin set constructed in the previous proof is meager. One may then wonder if it is possible to construct a nonmeager $\mathcal{IE}$-Luzin set from $non(\mathcal{M}) = \omega_1$. We will prove that this is not the case. This will be achieved by using Todorcevic's method of forcing with models as side conditions (see [8] for more on this very useful technique).

DEFINITION 3.1. We define the forcing $\mathbb{P}_{cat}$ as the set of all $p = (s_p, \overline{M}_p, F_p)$ with the following properties:

1. $s_p \in \omega^{<\omega}$ (this is usually referred as *the stem* of p).
2. $\overline{M}_p = \{M_0, \ldots, M_n\}$ is an $\in$-chain of countable elementary submodels of $\mathsf{H}((2^{\mathfrak{c}})^{++})$.
3. $F_p : \overline{M}_p \longrightarrow \omega^\omega$.
4. $s_p \cap F_p(M_i) = \emptyset$, for every $i \leq n$.
5. $F_p(M_i) \notin M_i$ and if $i < n$ then $F_p(M_i) \in M_{i+1}$.
6. $F_p(M_i)$ is a Cohen real over M_i (i.e., if $Y \in M_i$ is a meager set then $F_p(M_i) \notin Y$).

Finally, if $p, q \in \mathbb{P}_{cat}$ then $p \leq q$ if $s_q \subseteq s_p$, $\overline{M}_q \subseteq \overline{M}_p$ and $F_q \subseteq F_p$.

The following lemma is easy and it is left to the reader:

LEMMA 3.2.

1. *If* $M \preceq \mathsf{H}((2^{\mathfrak{c}})^{+++})$ *is countable and* $p \in M \cap \mathbb{P}_{cat}$ *then there is* $f \in \omega^\omega$ *such that if* $N = M \cap \mathsf{H}((2^{\mathfrak{c}})^{++})$ *then* $\overline{p} = (s_p, \overline{M}_p \cup \{N\}, F_p \cup \{(N, f)\})$ *is a condition of* $\mathbb{P}_{cat}$ *and it extends* p.
2. *If* $n \in \omega$ *then* $D_n = \{p \in \mathbb{P}_{cat} \mid n \subseteq dom(s_p)\}$ *is an open dense subset of* $\mathbb{P}_{cat}$.

We will now prove that $\mathbb{P}_{cat}$ is a proper forcing by applying the usual "side conditions trick".

LEMMA 3.3. $\mathbb{P}_{cat}$ *is a proper forcing.*

PROOF. Let $p \in \mathbb{P}_{cat}$ and M a countable elementary submodel of $\mathsf{H}((2^{\mathfrak{c}})^{+++})$ such that $p \in M$. By the previous lemma, we know there is $f \in \omega^\omega$ such that $\overline{p} = (s_p, \overline{M}_p \cup \{N\}, F_p \cup \{(N, f)\}) \in \mathbb{P}_{cat}$ (where $N = M \cap \mathsf{H}((2^{\mathfrak{c}})^{++})$). We will now prove that $\overline{p}$ is an $(M, \mathbb{P}_{cat})$-generic condition.

Let $D \in M$ be an open dense subset of $\mathbb{P}_{cat}$ and $q \leq \overline{p}$ (we may even assume $q \in D$). We must prove that q is compatible with an element of $M \cap D$. In order to achieve this, let $q_M = (s_q, \overline{M}_q \cap M, F_q \cap M)$ it is easy to see it is a condition as well as an element of M. By elementarity, we can find $r \in M \cap D$ such that $r \leq q_M$ and $s_r = s_q$. It is then easy to see that r and q are compatible (this is easy since r and q share the same stem). $\dashv$

The next lemma shows that $\mathbb{P}_{cat}$ destroys all the ground model nonmeager $\mathcal{IE}$-Luzin families.

Lemma 3.4. *If $X = \{f_\alpha \mid \alpha < \omega_1\} \subseteq \omega^\omega$ is a nonmeager set then $\mathbb{P}_{cat}$ adds a function that is almost disjoint with uncountably many elements of X.*

Proof. Given a generic filter $G \subseteq \mathbb{P}_{cat}$, we denote the *generic real* by f_{gen} i.e., f_{gen} is the union of all the stems of the elements in G. We will show that f_{gen} is forced to be almost disjoint with uncountably many elements of X. Let $p \in \mathbb{P}_{cat}$ with stem s_p and $\alpha < \omega_1$. Choose $t \in \omega^{<\omega}$ with the same length as s_p but disjoint with it. Let $Y = \left\{g_\beta \mid \alpha < \beta < \omega_1\right\}$, where $g_\beta = t \cup \left(f_\beta \restriction [|t|, \omega)\right)$. It is easy to see that Y is a nonmeager set and then we can find $\beta > \alpha$ and $q \leq p$ such that g_β is in the image of F_q. In this way, f_{gen} is forced by q to be disjoint from g_β, so it will be almost disjoint with f_β. $\dashv$

We say a forcing notion $\mathbb{P}$ *destroys category* if there is $p \in \mathbb{P}$ such that $p \Vdash$ "$\omega^\omega \cap V \in \mathcal{M}$". It is a well known fact that a partial order $\mathbb{P}$ does not destroy category if and only if $\mathbb{P}$ does not add an eventually different real (under any condition). Given a polish space X, we denote by $NWD(X)$ the ideal of all nowhere dense subsets of X. We will need the following result of Kuratowski and Ulam (see [5]):

Proposition 3.5 (Kuratowski–Ulam). *Let X and Y be two polish spaces. If $N \subseteq X \times Y$ is a nowhere dense set, then $\{x \in X \mid N_x \in nwd(Y)\}$ is comeager (where $N_x = \{y \mid (x, y) \in N\}$).*

As a consequence of the Kuratowski–Ulam, we get the following result:

Lemma 3.6. *Let $p \in \mathbb{P}_{cat}$, $\overline{M}_p = \{M_0, \dots, M_n\}$ and $i \leq n$. Let $g_j = F_p(M_{i+j})$ and $m = n - i$. If $D \in M_i$ and $D \subseteq (\omega^\omega)^{m+1}$ is a nowhere dense set, then $(g_0, \dots, g_m) \notin D$.*

Proof. We prove it by induction over m. If $m = 0$, this is true just by the definition of $\mathbb{P}_{cat}$. Assume this is true for m and we will show it is also true for $m + 1$. Since $D \subseteq (\omega^\omega)^{m+2}$ is a nowhere dense set, then by the Kuratowski–Ulam we conclude that $A = \{h \in \omega^\omega \mid D_h \in nwd((\omega^\omega)^{m+1})\}$ is comeager and note it is an element of M_i. In this way, $g_0 \in A$ so $D_{g_0} \in nwd((\omega^\omega)^{m+1})$ and it is an element of M_{i+1}. By the inductive hypothesis, we know $(g_1, \dots, g_{m+1}) \notin D_{g_0}$ which implies $(g_0, \dots, g_m) \notin D$. $\dashv$

We will prove that $\mathbb{P}_{cat}$ does not destroy category and it is a consequence of the following result:

Lemma 3.7. *Let $p \in \mathbb{P}_{cat}$ and $\dot{g}$ a $\mathbb{P}_{cat}$-name for an element of ω^ω. Let $\langle M_n \mid n \in \omega\rangle$ be an $\in$-chain of elementary submodels of $\mathsf{H}((2^{\mathfrak{c}})^{+++})$, $h : \omega \longrightarrow \omega$ and $\{A_n \mid n \in \omega\} \subseteq [\omega]^\omega$ a family of pairwise infinite disjoint sets with the following properties:*

1. $p, \dot{g} \in M_0$.
2. $h \restriction A_n \in M_{n+1}$.
3. *If $f \in M_n \cap \omega^\omega$ then $f \cap (h \restriction A_n)$ is infinite.*

Then there is a condition $q \leq p$ such that $q \Vdash$ "$|h \cap \dot{g}| = \omega$".

Proof. Let $M = \bigcup_{n\in\omega} M_n$ and define $h_n = h \restriction A_n \in M_n$. We know that there is some $f \in \omega^\omega$ such that $\overline{p} = \left(s_p, \overline{M}_p \cup \{N\}, F_p \cup \{(N, f)\}\right) \in \mathbb{P}_{cat}$ (where $N = M \cap \mathsf{H}((2^{\mathfrak{c}})^{++})$). We will now prove that $\overline{p}$ forces that $\dot{g}$ and h will have infinite intersection. We may assume $A_n \cap n = \emptyset$ for every $n \in \omega$.

Pick any $q \leq \overline{p}$ and $k \in \omega$, we must find an extension of q that forces that $\dot{g}$ and h share a common value bigger than k. We first find $n > k$ such that $q' = \left(s_q, \overline{M}_q \cap M, F_q \cap M\right) \in M_n$. Let $m = \left|\overline{M}_q \setminus \overline{M}_{q'}\right|$ and now we define D as the set of all $t \in \omega^{<\omega}$ such that there are $l \in A_n$ and $r \in \mathbb{P}_{cat}$ with the following properties:

1. $r \leq q'$.
2. $r \in M_n$.
3. $s_q \subseteq t$ and the stem of r is t.
4. $r \Vdash$ "$\dot{g}(l) = h_n(l)$".

It is easy to see that D is an element of M_{n+1}. We now define $N(D) \subseteq (\omega^\omega)^m$ as the set of all $(f_1, \ldots, f_m) \in (\omega^\omega)^m$ such that $(f_1 \cup \cdots \cup f_m) \cap t \nsubseteq s_q$ for every $t \in D$. We claim that $N(D)$ is a nowhere dense set.

Let $z_1, \ldots, z_m \in \omega^{<\omega}$, and we may assume all of them have the same length and it is bigger than the length of s_q. We know $q' = \left(s_q, \overline{M}_{q'}, F_{q'}\right)$ and let $im(F_{q'}) = \{f_1, \ldots, f_k\}$ (where *im* denotes the image of the function). Let t_0 be any extension of s_q such that $t_0 \cap (f_{\alpha_1} \cup \cdots \cup f_{\alpha_k} \cup z_1 \cup \cdots z_m) \subseteq s_q$ and $|t_0| = |z_1|$. In this way, $q_0 = \left(t_0, \overline{M}_{q'}, F_{q'}\right)$ is a condition and is an element of M_n. Inside M_n, we build a decreasing sequence $\langle q_i \rangle_{i\in\omega}$ (starting from the q_0 we just constructed) in such a way that q_i determines $\dot{g} \restriction i$. In this way, there is a function $u : \omega \longrightarrow \omega \in M_n$ such that $q_i \Vdash$ "$\dot{g} \restriction i = u \restriction i$". Since $u \in M_n$, we may then find $l \in A_n$ such that $u(l) = h_n(l)$. Let $t = t_{l+1}$ and $r = q_{l+1}$, we may then find $z_i' \supseteq z_i$ such that $t \cap (z_1' \cup \cdots \cup z_m') \subseteq s_q$ and $|z_i'| = |t|$. In this way, we conclude that $\langle z_1', \ldots, z_m' \rangle \cap N(D) = \emptyset$ (where $\langle z_1', \ldots, z_m' \rangle = \{(g_1, \ldots, g_m) \mid \forall i \leq m\,(z_i' \subseteq g_i)\}$), so we conclude $N(D)$ is a nowhere dense set.

Let $g_1, \ldots, g_m$ be the elements of $im(F_q)$ that are not in M. Since $D \in N$ then by the previous lemma, we know that $(g_1, \ldots, g_m) \notin N(D)$. This means there are $l \in A_n$, $t \in \omega^{<\omega}$ and $r \in M_n$ such that $r \leq q'$, whose stem is t and $r \Vdash$ "$\dot{g}(l) = h_n(l)$" with the property that $t \cap (g_1 \cup \cdots \cup g_m) \subseteq s_q$, but since q is a condition, it follows that $t \cap (g_1 \cup \cdots \cup g_m) = \emptyset$. In this way, r and q are compatible, which finishes the proof. ⊣

As a corollary we get the following:

Corollary 3.8. *$\mathbb{P}_{cat}$ does not destroy category.*

Unfortunately, the iteration of forcings that does not destroy category may destroy category (this may even occur at a two step iteration, see [1]). Luckily for us, the iteration of the $\mathbb{P}_{cat}$ forcing does not destroy category as we will prove soon. First we need a couple of lemmas,

Lemma 3.9. *Let $\mathbb{P}$ be a proper forcing that does not destroy category and $p \in \mathbb{P}$. If $\dot{S}$ is a $\mathbb{P}$-name for a countable set of reals, then there is $q \leq p$ and $h : \omega \longrightarrow \omega$ such that $q \Vdash$ "$\forall f \in \dot{S}\,(|f \cap h| = \omega)$".*

Proof. First note that if $\dot{f}_0, \ldots, \dot{f}_n$ are $\mathbb{P}$-names for reals, then there is $q \leq p$ and $h : \omega \longrightarrow \omega$ such that q forces $\dot{f}_i$ and h have infinite intersection for every

$i \leq n$. To prove this, we choose a partition $\{A_{0,}, \ldots, A_n\}$ of ω in infinite sets and let $\dot{g}_i$ be the $\mathbb{P}$-name of $\dot{f}_i \upharpoonright A_i$. Since $\mathbb{P}$ does not destroy category, there are $q \leq p$ and $h_i : A_i \longrightarrow \omega$ such that q forces that h_i and $\dot{f}_i$ have infinite intersection. Clearly q and $h = \bigcup h_i$ have the desired properties.

To prove the lemma, let $\dot{S} = \{\dot{g}_n \mid n \in \omega\}$ and fix $\{A_n \mid n \in \omega\}$ a partition of ω in infinite sets. By the previous remark, we know there is a $\mathbb{P}$-name $\dot{F}$ such that $p \Vdash$ "$\dot{F} : \omega \longrightarrow Partial\ (\omega^\omega) \cap V$" such that every $\dot{F}(n)$ is forced to be a function with domain A_n and intersects infinitely $\dot{g}_0 \upharpoonright A_0, \ldots, \dot{g}_n \upharpoonright A_n$. Since $\mathbb{P}$ is a proper forcing, we can find $q \leq p$ and $M \in V$ a countable subset of $Partial\ (\omega^\omega)$ such that $q \Vdash$ "$\dot{F} : \omega \longrightarrow M$". We know that $\mathbb{P}$ does not destroy category and M is countable, so there must be $r \leq q$ and $H : \omega \longrightarrow M$ such that $r \Vdash$ "$\exists^\infty n(\dot{F}(n) = H(n))$". We may assume that the domain of $H(n)$ is A_n for every $n \in \omega$. Finally, we define $h = \bigcup\limits_{n\in\omega} H(n)$ and it is easy to see that r forces that h has infinite intersection with every element of $\dot{S}$. ⊣

We will also need the following lemma,

Lemma 3.10. *Let $\mathbb{P}$ be a proper forcing that does not destroy category, $G \subseteq \mathbb{P}$ a generic filter and X any set. Then there are $\overline{M} = \{M_n \mid n \in \omega\} \subseteq V$, $P = \{A_n \mid n \in \omega\} \subseteq V$ and $h : \omega \longrightarrow \omega \in V$ with the following properties*:

1. *Each M_n is a countable elementary submodel of $H(\kappa)$ for some big enough κ (in V).*
2. *$X \in M_0$ and $M_n \in M_{n+1}$, for every $n \in \omega$.*
3. *P is a family of pairwise infinite disjoint sets of ω.*
4. *$P, \overline{M} \in V[G]$ (while $\overline{M}$ is a subset of V, in general it will not be a ground model set, the same is true for P).*
5. *$G \cap M_n$ is a $(M_n, \mathbb{P})$-generic filter, for every $n \in \omega$.*
6. *$h \upharpoonright A_n \in M_{n+1}$ and if $f \in M_n[G]$ then $h \upharpoonright A_n \cap f$ is infinite.*

Proof. Let r be any condition of $\mathbb{P}$, we will prove that there is an extension of r that forces the existence of the desired objects. Let $\{B_n \mid n \in \omega\}$ be any definable partition of ω into infinite sets.

Claim 3.11. *If $G \subseteq \mathbb{P}$ is a generic filter with $r \in G$ then (in $V[G]$) there a sequence $\langle (N_i, p_i, h_i) \mid i \in \omega \rangle$ such that for every $i \in \omega$ the following holds*:

1. *$N_i \in V$ is a countable elementary submodel of $H(\kappa)$ (the $H(\kappa)$ of the ground model).*
2. *$r, X \in N_0$ and $N_i \in N_{i+1}$.*
3. *$p_0 \leq r$ and $\langle p_k \rangle_{k\in\omega}$ is a decreasing sequence contained in G.*
4. *p_i is $(N_i, \mathbb{P})$-generic.*
5. *$h_i : B_i \longrightarrow \omega \in N_{i+1}$.*
6. *$p_i \Vdash$ "$\forall f \in N_i[\dot{G}] \cap \omega^\omega\ (|f \cap h_i| = \omega)$".*

Assume the claim is false, so we can find $n \in \omega$ and a sequence $R = \langle (N_i, p_i, h_i) \mid i \leq n \rangle$ that is maximal with the previous properties (the point 5 is only demanded for $i < n$). Let $p \in G$ be a condition forcing R has all this features (including the maximality). Back in V, let M be a countable elementary submodel such that $\mathbb{P}, p, R \in M$. By the previous lemma, there is an $(M, \mathbb{P})$-generic condition $q \leq p$ and $g : B_{n+1} \longrightarrow \omega$ such that g is forced by q to intersect infinitely every real of $M[G]$. In this way, q forces that R could be extended by adding (M, q, g)

but this is a contradiction since $q \leq p$ so it forces R was maximal. This finishes the proof of the claim.

Fix $\langle (\dot{N}_i, \dot{p}_i, \dot{h}_i) \mid i \in \omega \rangle$ to be the name of a sequence as in the claim. We can now define a name for a function $\dot{F}$ from ω to $Partial\,(\omega^\omega) \cap V$ such that $r \Vdash$ "$\forall n(\dot{F}\,(n) = \dot{h}_n)$". As in the previous lemma, we can find a condition $p \leq r$ and $H : \omega \longrightarrow Partial\,(\omega^\omega)$ such that $p \Vdash$ "$\exists^\infty n(\dot{F}\,(n) = H\,(n))$". We may assume the domain of $H\,(n)$ is B_n and let $h = \bigcup_{n \in \omega} H\,(n)$. Let $\dot{Z} = \{\dot{z}_n \mid n \in \omega\}$ be a name for a subset of ω such that $p \Vdash$ "$\forall n\,(F\,(\dot{z}_n) = H\,(\dot{z}_n))$". If $G \subseteq \mathbb{P}$ is a generic filter such that $p \in G$ then we define $M_n = N_{\dot{z}_n[G]}$ and $A_n = B_{\dot{z}_n[G]}$, it is clear that this sets have the desired properties. ⊣

From this we can conclude the following,

Corollary 3.12. *If $\mathbb{P}$ is a proper forcing that does not destroy category then $\mathbb{P} * \mathbb{P}_{cat}$ does not destroy category.*

Proof. Let $\dot{p}$ be a $\mathbb{P}$-name for a condition of $\mathbb{P}_{cat}$ and $\dot{f}$ a $\mathbb{P}$-name for a $\mathbb{P}_{cat}$-name for a real. Let $G \subseteq \mathbb{P}$ be a generic filter. By the previous lemma, there are $h : \omega \longrightarrow \omega \in V$, an $\in$-chain of elementary submodels $\{M_n\,[G] \mid n \in \omega\}$ and a pairwise disjoint family $\{A_n \mid n \in \omega\}$ of infinite subsets of ω such that $\dot{p}\,[G], \dot{f}\,[G] \in M_0\,[G]$ and $h \restriction A_n \in M_{n+1}\,[G]$ has infinite intersection with every real in $M_n\,[G]$. Then by lemma 13, we can extend $\dot{p}\,[G]$ to a condition forcing that $\dot{f}\,[G]$ and h will have infinite intersection. ⊣

As commented before, the iteration of forcings that does not destroy category may destroy category, but the following preservation result of Dilip Raghavan shows this can only happen at the successor steps of the iteration:

Proposition 3.13 (Raghavan [7]). *Let δ be a limit ordinal and $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha < \delta \rangle$ a countable support iteration of proper forcings. If $\mathbb{P}_\alpha$ does not destroy category for every $\alpha < \delta$ then $\mathbb{P}_\delta$ does not destroy category.*

With the aid of the previous preservation theorem, we conclude the following:

Corollary 3.14. *The countable support iteration of $\mathbb{P}_{cat}$ does not destroy category.*

Putting all the pieces together, we can finally prove our theorem:

Proposition 3.15. *If the existence of an inaccessible cardinal is consistent, then so it is the following statement: non $(\mathcal{M}) = \omega_1$ and every $\mathcal{IE}$-Luzin set is meager.*

Proof. Let μ be an inaccessible cardinal, we perform a countable support iteration $\{\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha < \mu\}$ in which $\dot{\mathbb{Q}}_\alpha$ is forced by $\mathbb{P}_\alpha$ to be the $\mathbb{P}_{cat}$ forcing. It is easy to see that if $\alpha < \mu$ then $\mathbb{P}_\alpha$ has size less than μ so it has the μ-chain condition and then $\mathbb{P}_\mu$ has the μ-chain condition (see [2]). The result then follows by the previous results. ⊣

We would like to finish with some questions:

Problem 3.16. *Does $\mathbb{P}_{cat}$ preserve $\sqsubseteq^{Cohen}$? (see* [1] *chapter* 6).

Problem 3.17. *Does $\mathbb{P}_{cat}$ preserve every nonmeager set as a nonmeager set? (we only know that it preserves the ground model as a nonmeager set).*

Problem 3.18. *Is the inaccessible cardinal really needed for the last result?*

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Mathematical Logic

Mathias–Prikry and Laver type forcing; summable ideals, coideals, and +-selective filters

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Abstract We study the Mathias–Prikry and the Laver type forcings associated with filters and coideals. We isolate a crucial combinatorial property of Mathias reals, and prove that Mathias–Prikry forcings with summable ideals are all mutually bi-embeddable. We show that Mathias forcing associated with the complement of an analytic ideal always adds a dominating real. We also characterize filters for which the associated Mathias–Prikry forcing does not add eventually different reals, and show that they are countably generated provided they are Borel. We give a characterization of ω-hitting and ω-splitting families which retain their property in the extension by a Laver type forcing associated with a coideal.

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Introduction

The Mathias–Prikry and the Laver type forcings were introduced in [17] and [11] respectively. Recently, properties of these forcings were characterized in terms of properties of associated filters, see [1,5,10,12]. We continue this line of research, and investigate forcings associated with coideals.

1 Preliminaries

Our notation and terminology is fairly standard. We give here an overview of basic notions used in this paper. We sometimes neglect the formal difference between integer singletons and integers, if no confusion is likely to occur. We are mostly concerned with filters and ideals on ω and on the set of finite sets of integers $\mathrm{fin} = [\omega]^{<\omega}$. If a domain of a filter or ideal is not specified or obvious, it is assumed that the domain is ω. All filters and ideals are assumed to be proper and to extend the Fréchet filter.

For $a, b \subseteq \omega$ we write $a \subset^* b$ if $a \backslash b \in \mathrm{fin}$, $a =^* b$ if $a \subset^* b$ and $b \subset^* a$, $a < b$ if $n < m$ for each $n \in a$ and $m \in b$, and $a \sqsubseteq b$ if there is $n \in \omega$ such that $a = b \cap n$.

A *tree* T will usually be an initial subtree of the tree of finite sequences of integers $(\omega^{<\omega}, \subseteq)$ with no leaves. The space of maximal branches of T is denoted $[T]$. For $t \in T$ we denote by $T[t]$ the subtree consisting of all nodes of T compatible with t. An element $r \in T$ is called the *stem* of T if r is the maximal node of T such that $T = T[r]$. For $a \subseteq \omega$ we denote by $T^{[a]}$ the set of all nodes $t \in T$ such that $|t| \in a$ (i.e. the nodes from levels in a). A node $t \in T$ is a *branching node* of T if t has at least two immediate successors in T. For $\mathcal{X} \subset \mathcal{P}(\omega)$ we call t an $\mathcal{X}$-*branching* node if $\left\{ i \in \omega \mid t^\frown i \in T \right\} \in \mathcal{X}$. A tree is an $\mathcal{X}$-*tree* if every node of T is $\mathcal{X}$-branching.

For $\mathcal{X} \subset \mathcal{P}(\omega)$ and $A \subseteq \omega$ we write $\mathcal{X} \restriction A$ for the set $\{ X \cap A \mid X \in \mathcal{X} \}$. For a filter $\mathcal{F}$ we denote by $\mathcal{F}^*$ the dual ideal, and by $\mathcal{F}^+$ the complement of $\mathcal{F}^*$ (i.e. the $\mathcal{F}$ positive sets). For an ideal $\mathcal{I}$ we denote $\mathcal{I}^*$ the dual filter, $\mathcal{I}^+ = (\mathcal{I}^*)^+$. A complement of an ideal is called a *coideal*. We will generally not distinguish between terminology for properties of a filter and of the dual ideal, i.e. statements "$\mathcal{F}$ is φ" and "$\mathcal{F}^*$ is φ" are often regarded as synonymous. We will sometimes speak of filters on general countable sets as if they were filters on ω. In these cases statements about these filters are understood as statements about filters on ω isomorphic with them.

We call an ideal $\mathcal{I}$ *summable* if there is a function $\mu\colon \omega \to \mathbb{R}$ such that $\mathcal{I} = \left\{ I \subseteq \omega \mid \sum \{ \mu(i) \mid i \in I \} < \infty \right\}$. We say that $\mathcal{I}$ is tall if $\mathcal{I} \cap [A]^\omega \neq \emptyset$ for each $A \in [\omega]^\omega$. An ideal $\mathcal{I}$ is below an ideal $\mathcal{J}$ in the *Rudin–Keisler order*, $\mathcal{I} \leq_{\mathrm{RK}} \mathcal{J}$ if there is a function $f : \omega \to \omega$ such that $I \in \mathcal{I}$ iff $f^{-1}[I] \in \mathcal{J}$ for each $I \subseteq \omega$. We say that $\mathcal{I}$ is *Rudin–Blass* below $\mathcal{J}$, $\mathcal{I} \leq_{\mathrm{RB}} \mathcal{J}$ if the witnessing function f is finite-to-1. The Rudin–Keisler and Rudin–Blass ordering on filters is defined in the same way as on ideals. Note that for ideals is $\mathcal{I} \leq_{\mathrm{RK}} \mathcal{J}$ iff $\mathcal{I}^* \leq_{\mathrm{RK}} \mathcal{J}^*$, and similarly for $\leq_{\mathrm{RB}}$.

For a filter $\mathcal{F}$ we will consider the filter $\mathcal{F}^{<\omega}$ generated by sets $[F]^{<\omega}$ for $F \in \mathcal{F}$. If $\mathcal{F}$ is a filter on ω, then $\mathcal{F}^{<\omega}$ is a filter on fin. Notice that for $X \subset \text{fin}$ is $X \in \mathcal{F}^{<\omega+}$ iff for each $F \in \mathcal{F}$ there is $a \in X$ such that $a \subset F$, and iff for each $F \in \mathcal{F}$ there are infinitely many $a \in X$ such that $a \subset F$. The elements of $\mathcal{F}^{<\omega+}$ are sometimes called the $\mathcal{F}$-*universal sets*.

A filter $\mathcal{F}$ is a P^+-*filter* if for every sequence $\{ X_n \mid n \in \omega \} \subseteq \mathcal{F}^+$ there is a sequence $Y = \{ y_n \in [X_n]^{<\omega} \mid n \in \omega \}$ such that $\bigcup Y \in \mathcal{F}^+$.

The ideal of all meager sets of reals is denoted by $\mathscr{M}$. For $f, g \in \omega^\omega$ write $f =^\infty g$ if $\{ n \in \omega \mid f(n) = g(n) \}$ is infinite. Recall that $\text{cov}(\mathscr{M}) = \mathfrak{b}(\omega^\omega, =^\infty)$ and $\text{non}(\mathscr{M}) = \mathfrak{d}(\omega^\omega, =^\infty)$. Let V be a model of set theory. We say that $e \in \omega^\omega$ is an *eventually different* real (over V) if $e \neq^\infty f$ for each $f \in \omega^\omega \cap V$. We say that $d \in \omega^\omega$ is a *dominating* real (over V) if $f <^* d$ for each $f \in \omega^\omega \cap V$. Every dominating real is an eventually different real. For every $f \in \omega^\omega$ the set $\{ g \in \omega^\omega \mid g =^\infty f \}$ is a dense G_δ subset of ω^ω. The following proposition is well known, the proof is analogous to the proof of [2, Lemma 2.4.8].

Proposition 1 *Let V be a model of set theory. The set $V \cap \omega^\omega$ is meager in ω^ω if and only if there exists an eventually different real over V.*

The Cohen forcing for adding a subset of a set $X \subseteq \omega$ will be denoted $\mathbb{C}_X$, and $\mathbb{C}$ denotes $\mathbb{C}_\omega$. The conditions of $\mathbb{C}_X$ are finite subsets of X ordered by $\sqsupseteq$. A Cohen generic real is the union of a generic filter on $\mathbb{C}_X$.

Let $\mathcal{X}$ be a family of subsets of ω, typically a filter or a coideal. The *Mathias–Prikry forcing* $\mathbb{M}(\mathcal{X})$ associated with $\mathcal{X}$ consists of conditions of the form (s, A) where $s \in \text{fin}$ and $A \in \mathcal{X}$. Although we usually assume $s < A$, we do not require it. The ordering is given by $(s, A) \leq (t, B)$ if $t \sqsubseteq s$, $A \subseteq B$, and $s \setminus t \subset B$. Given a generic filter on $\mathbb{M}(\mathcal{X})$, we call the union of the first coordinates of conditions in the generic filter the $\mathbb{M}(\mathcal{X})$ *generic real*. Given $\mathcal{X}$ and $r \subset \omega$, we denote $G_r(\mathcal{X}) = \{ (s, A) \in \mathbb{M}(\mathcal{X}) \mid s \sqsubseteq r, r \subseteq A \}$. It is easy to see that r is an $\mathbb{M}(\mathcal{X})$ generic real iff $G_r(\mathcal{X})$ is a generic filter on $\mathbb{M}(\mathcal{X})$. Properties of $\mathbb{M}(\mathcal{F})$ when $\mathcal{F}$ is an ultrafilter were studied in [4] and for $\mathcal{F}$ a general filter in [5,12]. Since $\mathbb{M}(\mathcal{F})$ is σ-centered, it always adds an unbounded real. On the other hand, it was shown that $\mathbb{M}(\mathcal{F})$ can be weakly ω^ω-bounding and even almost ω^ω-bounding. Filters for which $\mathbb{M}(\mathcal{F})$ is weakly ω^ω-bounding are called Canjar, and these are exactly those filters for which $\mathcal{F}^{<\omega}$ is a P^+-filter.

The *Laver type forcing* associated with $\mathcal{X}$ is denoted by $\mathbb{L}(\mathcal{X})$. Conditions in this forcing is trees $T \subseteq \omega^{<\omega}$ with stem t such that every node $s \in T$, $t \leq s$, is $\mathcal{X}$-branching. The ordering of $\mathbb{L}(\mathcal{X})$ is inclusion. Given a generic filter on $\mathbb{L}(\mathcal{X})$, the generic real is the union of stems of conditions in the generic filter. The generic real is a function dominating $\omega^\omega \cap V$, unless $\mathcal{X} \cap \text{fin} \neq \emptyset$. Properties of $\mathbb{L}(\mathcal{F})$ for $\mathcal{F}$ filter were studied in [1,12].

For an ideal $\mathcal{I}$ on ω, the forcing $(\mathcal{P}(\omega)/\mathcal{I}, \subset)$ adds a generic V-ultrafilter on ω containing $\mathcal{I}^*$, which will be denoted $\mathcal{G}^{\mathcal{I}}_{\text{gen}}$. The superscript will be omitted when $\mathcal{I}$ is apparent from the context.

A family $\mathcal{X}$ is ω-*hitting* (also called ω-*tall*) if for each countable sequence $\{ A_n \in [\omega]^\omega \mid n \in \omega \}$ exists $X \in \mathcal{X}$ such that $A_n \cap X$ is infinite for each $n \in \omega$.

A family $\mathcal{X}$ is *ω-splitting* if for each countable sequence $\{ A_n \in [\omega]^\omega \mid n \in \omega \}$ exists $X \in \mathcal{X}$ such that both $A_n \cap X$ and $A_n \setminus X$ are infinite for each $n \in \omega$.

To conclude the preliminaries let us recall a useful characterization of F_σ ideals. A *lower semicontinuous submeasure* is a function $\varphi\colon \mathcal{P}(\omega) \to [0, \infty]$ such that $\varphi(\emptyset) = 0$; if $A \subseteq B$, then $\varphi(A) \leq \varphi(B)$ (monotonicity); $\varphi(A \cup B) \leq \varphi(A) + \varphi(B)$ (subadditivity); and $\varphi(A) = \sup\{ \varphi(A \cap n) \mid n \in \omega\}$ for every $A \subseteq \omega$ (lower semicontinuity).

Proposition 2 (Mazur) *Let $\mathcal{I}$ be an F_σ ideal on ω. There is a lower semicontinuous submeasure φ such that $\varphi(\{n\}) = 1$ for every $n \in \omega$, and $\mathcal{I} = \mathrm{fin}(\varphi)$.*

2 Mathias like reals and summable ideals

The original motivation for this section comes from a question of Ilijas Farah about the number of ZFC-provably distinct Boolean algebras of the form $\mathcal{P}(\omega)/\mathcal{I}$ where $\mathcal{I}$ is a 'definable' ideal [8]. Note that CH implies that all such Boolean algebras are isomorphic for F_σ ideals $\mathcal{I}$ [15]. The interpretation of 'definability' interesting in this context might be '$F_{\sigma\delta}$,' 'Borel,' or 'analytic.' The basic question was answered by Oliver [18] by showing that there are 2^ω many $F_{\sigma\delta}$ ideals for which the Boolean algebras $\mathcal{P}(\omega)/\mathcal{I}$ are provably nonisomorphic. However, these constructions are not interesting from the forcing point of view, the constructed examples are locally isomorphic to $\mathcal{P}(\omega)/\mathrm{fin}$. On the other hand, Steprāns [21] showed that there are continuum many coanalytic ideals whose quotients are pairwise forcing not equivalent.

We are interested in (anti-)classification results about forcings of this form. The first result in this direction is due to Farah ad Solecki. They showed that the Boolean algebras $\mathcal{P}(\mathbb{Q})/\mathrm{nwd}_\mathbb{Q}$ and $\mathcal{P}(\mathbb{Q})/\mathrm{null}_\mathbb{Q}$ are nonisomorphic and homogeneous, see [9]. A systematic study of such forcing notions was done by Hrušák and Zapletal [14]. They provided several examples of forcings of this form. Their results imply that for each tall summable ideal $\mathcal{I}$ there is an $F_{\sigma\delta}$ ideal denoted here $\mathrm{tr}_\mathcal{I}$ such that $\mathcal{P}(\omega)/\mathrm{tr}_\mathcal{I} = \mathbb{M}(\mathcal{I}^*) * \dot{\mathbb{Q}}$ for some $\dot{\mathbb{Q}}$, a name for a proper ω-distributive forcing notion. Therefore showing that the Mathias forcings $\mathbb{M}(\mathcal{I}^*)$ are different for various choices of summable ideals $\mathcal{I}$ seems to be a viable attempt to provide a spectrum of different forcings $\mathcal{P}(\omega)/\mathrm{tr}_\mathcal{I}$. However, the results of this section show that this approach is likely to fail, the Mathias forcings for tall summable ideals all mutually bi-embeddable.

Let us start with a general combinatorial characterization of Mathias generic reals.

Definition 3 Let $V \subseteq U$ be models of the set theory, $\mathcal{F} \subset \mathcal{P}(\omega)$ be a filter in V, and $x \in \mathcal{P}(\omega) \cap U$. We say that x is a *Mathias like real for $\mathcal{F}$ over V* if the following two conditions hold;

(1) $x \subset^* F$ for each $F \in \mathcal{F} \cap V$,
(2) $[x]^{<\omega} \cap H \neq \emptyset$ for each $H \in \mathcal{F}^{<\omega+} \cap V$.

Notice that an $\mathbb{M}(\mathcal{F})$ generic real is a Mathias like for $\mathcal{F}$. It was implicitly shown in [12] that Mathias like reals are already almost Mathias generic—it is sufficient to add a Cohen real to get the genericity. This explains why most results concerning the

Mathias forcing rely just on the fact that the generic reals are Mathias like. We provide the proof of this fact for reader's convenience.

Proposition 4 *Let $V \subseteq U$ be models of the set theory, $\mathcal{F} \subset \mathcal{P}(\omega)$ be a filter in V, and $x \in \mathcal{P}(\omega) \cap U$ be a Mathias like real for $\mathcal{F}$ over V. Let c be a $\mathbb{C}_x$ generic real over U. Then c is an $\mathbb{M}(\mathcal{F})$ generic real over V.*

Proof We need to prove that $G_c(\mathcal{F}) \cap \mathscr{D} \neq \emptyset$ for each dense subset $\mathscr{D} \in V$ of $\mathbb{M}(\mathcal{F})$, i.e. to show that the set of conditions forcing this fact is dense in $\mathbb{C}_x$. Choose any condition $s \in \mathbb{C}_x$. Denote

$$H = \{\, t \setminus s \mid (\exists F \in \mathcal{F} \mid (t, F) \in \mathscr{D}), s \sqsubseteq t \,\}.$$

Note that $H \in \mathcal{F}^{<\omega+}$, otherwise there exists $F \in \mathcal{F}$ such that $[F]^{<\omega} \cap H = \emptyset$, and the condition (s, F) has no extension in $\mathscr{D}$. Condition (2) of Definition 3 now implies that there exists $(t, F_t) \in \mathscr{D}$ such that $s \sqsubseteq t$, and $t \setminus s \subset x$. Since $x \subset^* F_t$, there is $k \in x$, $t < k$ such that $x \setminus k \subset F_t$. Hence $t \cup \{k\} \in \mathbb{C}_x$, $t \cup \{k\} < s$, and $t \cup \{k\} \Vdash (t, F_t) \in G_c(\mathcal{F}) \cap \mathscr{D}$. ◻

For a poset P we denote by $\mathrm{RO}\,(P)$ the unique (up to isomorphism) complete Boolean algebra in which P densely embeds (while preserving incompatibility), and $\mathrm{RO}\,(P)^+$ denotes the set of non-zero elements of $\mathrm{RO}\,(P)$. The relation $\lessdot$ denotes complete embedding of Boolean algebras.

Corollary 5 *Let $\mathbb{P}$ be a forcing adding a Mathias like real for a filter $\mathcal{F}$.*

(1) $\mathrm{RO}\,(\mathbb{M}(\mathcal{F})) \lessdot \mathrm{RO}\,(\mathbb{P} \times \mathbb{C})$.
(2) If $\mathbb{Q}$ is a forcing adding a Cohen real, then $\mathrm{RO}\,(\mathbb{M}(\mathcal{F})) \lessdot \mathrm{RO}\,(\mathbb{P} \times \mathbb{Q})$.

Proof Proposition 4 implies that every generic extension via $\mathbb{P} * \dot{\mathbb{C}}$ contains a generic filter on $\mathbb{M}(\mathcal{F})$ over V. Hence there is $a \in \mathrm{RO}\,(\mathbb{M}(\mathcal{F}))^+$ such that

$$\mathrm{RO}\,(\mathbb{M}(\mathcal{F})) \restriction a \lessdot \mathrm{RO}\,\left(\mathbb{P} * \dot{\mathbb{C}}\right) = \mathrm{RO}\,(\mathbb{P} \times \mathbb{C}), \qquad \circledast$$

see e.g. [22]. For each $p \in \mathbb{M}(\mathcal{F})$, the poset $\mathbb{M}(\mathcal{F}) \restriction p$ is isomorphic to $\mathbb{M}(\mathcal{F} \restriction F)$ for some $F \in \mathcal{F}$. If x is a Mathias like real for $\mathcal{F}$, then it is also Mathias like for $\mathcal{F} \restriction F$ for each $F \in \mathcal{F}$, and we can deduce from Proposition 4 that the set of elements of $\mathrm{RO}\,(\mathbb{M}(\mathcal{F}))^+$ satisfying $\circledast$ is dense. Since $\mathbb{M}(\mathcal{F})$ is c.c.c. we can find A, a countable maximal antichain of such elements. Now

$$\begin{aligned} \mathrm{RO}\,(\mathbb{M}(\mathcal{F})) &\simeq \prod_{a \in A} \mathrm{RO}\,(\mathbb{M}(\mathcal{F})) \restriction a \lessdot \prod_{\omega} \mathrm{RO}\,(\mathbb{P} \times \mathbb{C}) \\ &\simeq \mathrm{RO}\left(\mathbb{P} \times \sum_{\omega} \mathbb{C}\right) \simeq \mathrm{RO}\,(\mathbb{P} \times \mathbb{C}). \end{aligned}$$

To justify the second last isomorphism, we construct a dense embedding e of the poset $\mathbb{P} \times \sum_{\omega} \mathbb{C}$ into the complete Boolean algebra $\prod_{\omega} \mathrm{RO}\,(\mathbb{P} \times \mathbb{C})$: If t is an element of the n-th copy of $\mathbb{C}$ in $\sum_\omega \mathbb{C}$, define $e(p,t)(i) = (p,t)$ if $i = n$, and $e(p,t)(i) = \mathbf{0}$ otherwise.

If $\mathbb{Q}$ adds a Cohen generic real, then there exists some $a \in \mathrm{RO}(\mathbb{C})^+$ such that $\mathrm{RO}(\mathbb{C}) \restriction a \lessdot \mathrm{RO}(\mathbb{Q})$. Since $\mathrm{RO}(\mathbb{C}) \restriction a$ is isomorphic to $\mathrm{RO}(\mathbb{C})$, the second statement follows from the first one. □

The next lemma states that Mathias like reals behave well with respect to the Rudin–Keisler ordering on filters.

Lemma 6 *Let $\mathcal{E}, \mathcal{F}$ be filters on ω, let $f \colon \omega \to \omega$ be a function witnessing $\mathcal{F} \leq_{\mathrm{RK}} \mathcal{E}$, and x be a Mathias like real for $\mathcal{E}$. Then $f[x]$ is a Mathias like real for $\mathcal{F}$.*

Proof It is obvious that $f[x] \subset^* F$ for each $F \in \mathcal{F}$, so we need to check only condition (2) of Definition 3. Define $f^* \colon \mathrm{fin} \to \mathrm{fin}$ by

$$f^*(h) = \{a \in \mathrm{fin} \mid f[a] = h\}.$$

Claim *If $H \in \mathcal{F}^{<\omega+}$, then $\bigcup f^*[H] \in \mathcal{E}^{<\omega+}$.*

For $E \in \mathcal{E}$ is $f[E] \in \mathcal{F}$, and there is $h \in H$ such that $h \subset f[E]$. Thus $f^*(h) \cap [E]^{<\omega} \neq \emptyset$. □

Choose any $H \in \mathcal{F}^{<\omega+}$. Since x is Mathias like for $\mathcal{E}$, there exists $a \in \bigcup f^*[H]$ such that $a \subset x$. Now $f[a] \subset f[x]$ and $f[a] \in H$. □

We focus now on summable ideals. The following simple observation appears in [7].

Lemma 7 *Let $\mathcal{I}$, $\mathcal{J}$ be tall summable ideals. There exists $A \in \mathcal{P}(\omega) \setminus \mathcal{J}^*$ such that $\mathcal{I} \leq_{\mathrm{RB}} \mathcal{J} \restriction A$.*

We are now equipped to prove the bi-embeddability result.

Theorem 8 *Let $\mathcal{I}$, $\mathcal{J}$ be tall summable ideals. Then* $\mathrm{RO}(\mathbb{M}(\mathcal{I}))$ *is completely embedded in* $\mathrm{RO}(\mathbb{M}(\mathcal{J}))$.

Proof Find A as in Lemma 7 and consider the decomposition

$$\mathbb{M}(\mathcal{J}^*) = \mathbb{M}(\mathcal{J}^* \restriction A) \times \mathbb{M}(\mathcal{J}^* \restriction (\omega \setminus A)).$$

The forcing $\mathbb{M}(\mathcal{J}^* \restriction A)$ adds a Mathias real for $\mathcal{J}^* \restriction A$. Lemma 6 implies that it also adds a Mathias like real for $\mathcal{I}^*$. Since $\mathcal{J}^* \restriction (\omega \setminus A)$ is not an ultrafilter, the forcing $\mathbb{M}(\mathcal{J}^* \restriction (\omega \setminus A))$ adds a Cohen real. The conclusion now follows from Corollary 5. □

This shows that the original plan of creating many essentially different forcings by using different summable ideals is likely to fail. However, we still do not know whether the Mathias forcing is the same for every tall summable ideal.

Question 9 Are $\mathbb{M}(\mathcal{J}^*)$ and $\mathbb{M}(\mathcal{I}^*)$ equivalent forcing notions if $\mathcal{I}$ and $\mathcal{J}$ are tall summable ideals?

To conclude this section let us mention a related result of Farah [7, Proposition 3.7.1].

Proposition 10 *Assume* $\mathsf{OCA} + \mathsf{MA}$. *If $\mathcal{I}$ is a summable ideal, then $\mathcal{P}(\omega)/\mathcal{I}$ is weakly homogeneous iff $\mathcal{I} = \mathrm{fin}$.*

3 Mathias forcing with coideals

This section deals with the forcing $\mathbb{M}(\mathcal{F}^+)$ for $\mathcal{F}$ a filter on ω. We are mainly interested in the following question.

Question 11 When does $\mathbb{M}\left(\mathcal{F}^+\right)$ add dominating reals?

The following fact is well known.

Fact 12 *Let $\mathcal{I}$ be an ideal on ω. Then $\mathbb{M}\left(\mathcal{I}^+\right) = \mathcal{P}(\omega)/\mathcal{I} * \mathbb{M}(\dot{\mathcal{G}}^{\mathcal{I}}_{\text{gen}})$.*

Proposition 13 *If $\mathcal{I}$ is a Borel ideal and $\mathcal{P}(\omega)/\mathcal{I}$ does not add reals, then $\mathbb{M}\left(\mathcal{I}^+\right)$ adds a dominating real.*

Proof First assume that $\mathcal{I}$ is an F_σ ideal. Let φ be a submeasure as in Proposition 2.

Let r be a $\mathbb{M}\left(\mathcal{I}^+\right)$ generic real and notice that $r \notin \text{fin}(\varphi)$. In $V[r]$ define an increasing function $g \colon \omega \to \omega$ by letting

$$g(n) = \min\left\{k \in \omega \mid 2^n \leq \varphi(r \cap k)\right\}.$$

We will show g is a dominating real. Let $(s, A) \in \mathbb{M}\left(\mathcal{I}^+\right)$ be a condition and $f : \omega \to \omega$ a function in V. We will extend (s, A) to a condition that forces that g dominates f. Pick $m \in \omega$ such that $\varphi(s) < 2^m$ and for every $i > m$ choose $t_i \subseteq A \setminus f(i)$ such that $\max(t_i) < \min(t_{i+1})$ and $2^i \leq \varphi(t_i) < 2^{i+1}$. This is possible since $\varphi(A) = \infty$ and the φ-mass of singletons is 1. Put $B = \bigcup_{m<i} t_i$, thus $\varphi(B) = \infty$ and $(s, B) \in \mathbb{M}\left(\mathcal{I}^+\right)$. Moreover $(s, B) \leq (s, A)$, and since $(s, B) \Vdash \dot{r} \subset s \cup B$ we have that $(s, B) \Vdash f(i) < g(i)$ for $i > m$.

For the general case let $\mathcal{I}$ be an analytic ideal such that $\mathcal{P}(\omega)/\mathcal{I}$ does not add reals. If $\mathcal{G}^{\mathcal{I}}_{\text{gen}}$ is not a P-point, then it is not a Canjar filter (see e.g. [4]), and $\mathbb{M}\left(\mathcal{I}^+\right) = \mathcal{P}(\omega)/\mathcal{I} * \mathbb{M}\left(\mathcal{G}^{\mathcal{I}}_{\text{gen}}\right)$ will add a dominating real. In case $\mathcal{G}^{\mathcal{I}}_{\text{gen}}$ is a P-point, then by [13, Theorem 2.5] $\mathcal{I}$ is locally F_σ and $\mathbb{M}\left(\mathcal{I}^+\right)$ adds a dominating real as demonstrated in the first part of the proof. □

Question 14 Is there a Borel ideal $\mathcal{I}$ such that $\mathbb{M}\left(\mathcal{I}^+\right)$ does not add a dominating real?

It is easy to see that in every generic extension by $\mathbb{M}(\mathcal{F}^+)$ the ground model set of reals is meager, and thus $\mathbb{M}(\mathcal{F}^+)$ always adds an eventually different real. In [13] Michael Hrušák and Jonathan Verner asked the following question.

Question 15 Is there a Borel ideal $\mathcal{I}$ on ω such that $\mathcal{P}(\omega)/\mathcal{I}$ adds a Canjar ultrafilter?

We answer this question in negative.

Lemma 16 *If $\mathcal{I}$ is an ideal on ω such that $\mathcal{G}^{\mathcal{I}}_{\text{gen}}$ is a P-point, then $\mathcal{P}(\omega)/\mathcal{I}$ does not add reals.*

Proof Let $A \in \mathcal{I}^+$ and r a name such that $A \Vdash \dot{r} \in \omega^\omega$. Let $\mathcal{G}_{\text{gen}}$ be a $\mathcal{P}(\omega)/\mathcal{I}$ generic filter such that $A \in \mathcal{G}_{\text{gen}}$ and for every $n \in \omega$ we can find $A_n \in \mathcal{G}_{\text{gen}}$ such that $A_n \leq A$ and A_n decides $\dot{r}(n)$. Since $\mathcal{G}_{\text{gen}}$ is a P-point, there is $B \in \mathcal{G}_{\text{gen}}$ such that $B \subseteq^* A_n$ for every $n \in \omega$ (note that we can assume B is a ground model set since $\mathcal{G}_{\text{gen}}$ is generated by ground model sets). Clearly $B \leq A$ and forces $\dot{r}$ to be a ground model real. □

Corollary 17 *If $\mathcal{I}$ is an analytic ideal then $\mathcal{G}_{\text{gen}}^{\mathcal{I}}$ is not a Canjar ultrafilter.*

Proof By the previous proposition if $\mathcal{P}(\omega)/\mathcal{I}$ adds new reals then the generic filter is not a Canjar ultrafilter. Assume no new reals are added. By Proposition 13, $\mathbb{M}(\mathcal{I}^+)$ adds a dominating real and $\mathcal{G}_{\text{gen}}$ is not Canjar. □

4 Mathias–Prikry forcing and eventually different reals

We turn our attention towards the forcing $\mathbb{M}(\mathcal{F})$ for a filter $\mathcal{F}$. Our goal is the characterization of filters for which this forcing does not add eventually different reals.

A filter $\mathcal{F}$ is $+$-*Ramsey* [16] if for each $\mathcal{F}^+$-tree T there is a branch $b \in [T]$ such that $b[\omega] \in \mathcal{F}^+$.

Definition 18 Let $\mathcal{F}$ be a filter on ω. We say that $\mathcal{F}$ is $+$-*selective* if for every sequence $\{X_n \mid n \in \omega\} \subseteq \mathcal{F}^+$ there is a selector

$$S = \{a_n \in X_n \mid n \in \omega\} \in \mathcal{F}^+.$$

Every $+$-Ramsey filter is $+$-selective and every $+$-selective filter is a P^+-filter.

Let M be an extension of the universe of sets V. We say that $r \in \omega^\omega \cap M$ is an eventually different real over V if the set $\{n \in \omega \mid r(n) = f(n)\}$ is finite for each $f \in \omega^\omega \cap V$. We say that a forcing $\mathbb{P}$ does not add an eventually different real iff there is no eventually different real over V in any generic extension by forcing $\mathbb{P}$.

Theorem 19 *Let $\mathcal{F}$ be a filter. The following are equivalent;*

(1) Forcing $\mathbb{M}(\mathcal{F})$ does not add an eventually different real,
(2) $\mathcal{F}^{<\omega}$ is $+$-selective,
(3) $\mathcal{F}^{<\omega}$ is $+$-Ramsey.

Proof The implication (3) ⇒ (2) is clear. We start with (2) ⇒ (1).

Let $\mathcal{F}^{<\omega}$ be $+$-selective and x be an $\mathbb{M}(\mathcal{F})$ name for a function in ω^ω. Enumerate $\text{fin} = \langle s_i \mid i \in \omega\rangle$ such that $\max s_i \le i$ for each $i \in \omega$. Let $\{a_i \mid i \in \omega\}$ be a partition of ω into infinite sets, and denote by $a_i(k)$ the k-th element of a_i. For $k \in \omega$ let

$$X_k = \Big\{ t \in \text{fin} \mid k < \min t \text{ and } \forall i < k \colon \exists h_i^t(k) \in \omega \colon \exists F \\ \in \mathcal{F} \colon (s_i \cup t, F) \Vdash \dot{x}(a_i(k)) = h_i^t(k) \Big\}.$$

Claim $X_k \in \mathcal{F}^{<\omega+}$ *for each $k \in \omega$.*

Let $k \in \omega$. We need to show that for each $G \in \mathcal{F}$ there exists $t \in X_k$ such that $t \subset G$. Put $t_0 = \emptyset$, $F_0 = G \setminus (k+1)$, and for $i < k$ proceed with an inductive construction as follows.

Suppose t_i, F_i were defined, we will define t_{i+1}, F_{i+1}, $h_i^t(k)$. Find a condition $p = (s_i \cup t_{i+1}, F_{i+1}) < (s_i \cup t_i, F_i)$ and $h_i^t(k) \in \omega$ such that $p \Vdash \dot{x}(a_i(k)) = h_i^t(k)$. Finally put $t = t_k$, and notice that $t \in X_k$, $t \subset G$.

Let $S \in \mathcal{F}^{<\omega+}$ be a selector for $\langle X_k \mid k \in \omega \rangle$ guaranteed by the $+$selectivity of $\mathcal{F}^{<\omega}$. Define $g;\ \omega \to \omega$ by $g(a_i(k)) = h_i^t(k)$ if $t \in S$ and $i < k$. We claim that $\Vdash |\{ g(n) = \dot{x}(n) \mid n \in \omega \}| = \omega$.

Let (s_i, G) be a condition and n be an integer. There exists $k > n$, i and $t \in X_k \cap S$ such that $t \subset G$. Thus there is $F \in \mathcal{F}$ such that

$$(s_i \cup t, F) \Vdash \dot{x}(a_i(k)) = h_i^t(k) = g(a_i(k)).$$

Put $p = (s_i \cup t, F \cap G) < (s_i, G)$. Now $n < k \leq a_i(k)$ and $p \Vdash \dot{x}(a_i(k)) = g(a_i(k))$.

To prove (1) $\Rightarrow$ (3) assume $\mathbb{M}(\mathcal{F})$ does not add an eventually different real. Let T be an $\mathcal{F}^{<\omega+}$-tree and r be an $\mathbb{M}(\mathcal{F})$ generic real. For $n \in \omega$ let $O_n = \{ a \in [T] \mid \exists m > n : a(m) \subset r \setminus n \}$. Note that each such O_n is an open dense subset of $[T]$. Now $G = \bigcap \{ O_n \mid n \in \omega \}$ is a dense G_δ set, and Proposition 1 implies that there exists some $b \in G \cap V$. We claim that b is the desired branch for which $b[\omega] \in \mathcal{F}^{<\omega+}$. Otherwise there is $F \in \mathcal{F}$ such that $b[\omega] \cap F^{<\omega} = \emptyset$, which contradicts $r \subset^* F$ and $|[r]^{<\omega} \cap b[\omega]| = \omega$. $\square$

The last part of the proof in fact demonstrated the following.

Theorem 20 *Let $V \subseteq U$ be models of the set theory, $\mathcal{F} \subset \mathcal{P}(\omega)$ be a filter in V. If U contains a Mathias like real for $\mathcal{F}$ but no eventually different real over V, then $\mathcal{F}$ is $+$-Ramsey.*

The implication (2) $\Rightarrow$ (3) of Theorem 19 can be proved directly with the same proof as is used in [19, Lemma 2]. Although this implication holds true for filters of the form $\mathcal{F}^{<\omega}$, this is not the case for filters in general. The filter on $2^{<\omega}$ generated by complements of $\subseteq$-chains and $\subseteq$-antichains is an F_σ $+$-selective filter which is not $+$-Ramsey.

The following proposition is a direct consequence of [16, Theorem 2.9].

Proposition 21 *Let $\mathcal{F}$ be a Borel filter. $\mathcal{F}$ is $+$-Ramsey if and only if $\mathcal{F}$ is countably generated.*

Corollary 22 *If $\mathcal{F}$ is a Borel filter on ω and $\mathbb{M}(\mathcal{F})$ does not add an eventually different real, then $\mathbb{M}(\mathcal{F})$ is forcing equivalent to the Cohen forcing.*

Proof If $\mathbb{M}(\mathcal{F})$ does not add an eventually different, then the Borel $\mathcal{F}^{<\omega}$ is $+$-Ramsey and hence countably generated. Thus $\mathcal{F}$ is also countably generated and $\mathbb{M}(\mathcal{F})$ has a countable dense subset. $\square$

It is not hard to see that any forcing of size less than $\mathrm{cov}(\mathscr{M})$ can not add an eventually different real, so we have another proof of the following well known result,

Corollary 23 *If $\mathcal{I}$ is a Borel ideal which is not countably generated then* $\mathrm{cov}(\mathscr{M}) \leq \mathrm{cof}(\mathcal{I})$.

Corollary 22 can be derived directly from [20, Conclusion 9.16], which says that if a Suslin c.c.c. forcing adds a non-Cohen real, then it makes the set of ground model reals meager. See also [23, Corollary 3.5.7].

5 Laver type forcing

We will address the question of preserving hitting families with Laver type forcing. Since every forcing adding a real destroys some maximal almost disjoint family, it only makes sense to ask for survival of hitting families with some additional properties. Preservation of ω-hitting and ω-splitting families with Laver forcing $\mathbb{L}$ was studied in [6]. A characterization of the strong preservation of these properties with forcing $\mathbb{L}(\mathcal{F})$ for a filter $\mathcal{F}$ was given in [1]. Preservation of spliting familes with $\mathbb{L}(\mathcal{F})$ was also studied in [3]. We utilize methods used in [6] to characterize ω-hitting and ω-splitting families for which the Laver forcing $\mathbb{L}(\mathcal{F}^+)$ preserves the ω-hitting and the ω-splitting property.

Definition 24 Let $\mathcal{X} \subset \mathcal{P}(\omega)$ be a family of sets and let $\mathcal{F}$ be a filter on ω. We say that $\mathcal{X}$ is $\mathcal{F}^+$-*ω-hitting* if for every countable set of functions $\{ f_n : \omega \to \omega \mid n \in \omega \}$ such that $f_n[\omega] \in \mathcal{F}^+$ for each $n \in \omega$, there exists $X \in \mathcal{X}$ such that $f_n[X] \in \mathcal{F}^+$ for each $n \in \omega$.

Obviously, every $\mathcal{F}^+$-ω-hitting family must be ω-hitting.

Proposition 25 *Let $\mathcal{F}$ be a filter on ω and let $\mathcal{X} \subset \mathcal{P}(\omega)$. The following are equivalent;*

(1) $\mathcal{X}$ is $\mathcal{F}^+$-ω-hitting,
(2) $\mathbb{L}(\mathcal{F}^+)$ preserves "$\check{\mathcal{X}}$ is ω-hitting."

Proof Start with (1) implies (2). For conditions $S, T \in \mathbb{L}(\mathcal{F}^+)$, where the stem of T is $r \in \omega^k$, we write $S <^n T$ if $S < T$ and $S \cap \omega^{k+n} = T \cap \omega^{k+n}$.

Let θ be a large enough cardinal and let $M \prec H_\theta$ be a countable elementary submodel containing $\mathcal{F}$. Let $X \in \mathcal{X}$ be such that $f[X] \in \mathcal{F}^+$ for each $f : \omega \to \omega$, $f \in M$ such that $f[\omega] \in \mathcal{F}^+$.

Claim A *Let $A \in M$ be an $\mathbb{L}(\mathcal{F}^+)$-name, and $S \in \mathbb{L}(\mathcal{F}^+) \cap M$ be a condition such that $S \Vdash \dot{A} \in [\omega]^\omega$. There exists $S' <^0 S$ such that for each $T' < S'$ there is $t \in T'$ such that $S'[t] \in M$ and $S'[t] \Vdash \check{X} \cap \dot{A} \neq \emptyset$.*

Since S is countable and A is a name for an infinite set, we can inductively build a sequence $\{ \langle t_n, k_n, R_n \rangle \mid n \in \omega \} \in M$ such that

- $t_n \in S$, $k_n \in \omega$, $R_n \in \mathbb{L}(\mathcal{F}^+)$,
- $R_n <^0 S[t_n]$,
- $R_n \Vdash k_n \in \dot{A}$,
- $k_n \neq k_m$ for $n \neq m$,
- $I = \{t_n \mid n \in \omega\}$ is a maximal antichain in S.

Put $S' = \bigcup \{ R_n \mid k_n \in X \}$. Let r be the stem of S. We only need to show that for each $s \in S$ such that $r \leq s < t$ for some $t \in I$, the set $\left\{ i \in \omega \mid s^\frown i \in S' \right\}$ is in $\mathcal{F}^+$. Define a function $f : \omega \to \omega$ in M by $f : k_n \mapsto i$ if $t_n \geq s^\frown i$ for $n \in \omega$, and $f : k \mapsto 0$ otherwise. Note that $\left\{ i \in \omega \mid s^\frown i \in S \right\} \subseteq f[\omega] \in \mathcal{F}^+$ since I is maximal bellow s. Thus $f[X] = \left\{ i \in \omega \mid s^\frown i \in S' \right\} \in \mathcal{F}^+$. $\square$

Let $T \in \mathbb{L}(\mathcal{F}^+) \cap M$ be a condition with stem r. Enumerate $\{ A_n \mid n \in \omega \}$ all $\mathbb{L}(\mathcal{F}^+)$-names belonging to M such that $\Vdash \dot{A}_n \in [\omega]^\omega$ for each $n \in \omega$. We will inductively construct a fusion sequence of conditions $\{ T_n \mid n \in \omega \}$ starting with $T_0 = T$ such that

- $T_{n+1} <^n T_n$ for each $n \in \omega$,
- for each $T' < T_n$ there is $t \in T'$ such that $T_n[t] \in M$ and $T_n[t] \Vdash \dot{A}_n \cap \check{X} \neq \emptyset$.

Suppose that T_n is constructed and use the inductive hypothesis to find a maximal antichain $J \subset \{ t \in T_n \mid n + |r| < |t|, T_n[t] \in M \}$ in T_n. For each $t \in J$ use Claim A for $S = T_n[t]$ and $A = A_{n+1}$ to get $T_n'[t] <^0 T_n[t]$ as in the statement of the claim. Now $T_{n+1} = \bigcup \left\{ T_n'[t] \mid t \in J \right\}$ is as required.

Once this sequence is constructed put $R = \bigcap \{ T_n \mid n \in \omega \} \in \mathbb{L}(\mathcal{F}^+)$. Now $R \Vdash \dot{A}_n \cap \check{X} \neq \emptyset$ for each $n \in \omega$, and the implication is proved.

For the other direction, assume there are functions $\{ f_n : \omega \to \omega \mid n \in n \}$ such that $f_n[\omega] \in \mathcal{F}^+$, and for each $X \in \mathcal{X}$ there is $n \in \omega$ such that $f_n[X] \in \mathcal{F}^*$. Fix $\{ b_n \in [\omega]^\omega \mid n \in \omega \}$, a partition of ω into infinite sets. Let $\dot{\ell}$ be a name for the $\mathbb{L}(\mathcal{F}^+)$ generic real, and define a name for $\dot{A}_n^k \subset \omega$ by declaring $\dot{A}_n^k = f_n^{-1} \left[\dot{\ell}[b_n \backslash k] \right]$ for each $k, n \in \omega$. Inductively define $T \in \mathbb{L}(\mathcal{F}^+)$ such that $t^\frown i \in T$ iff $i \in f_n[\omega]$ for $t \in T^{[b_n]}$. Notice that T forces that $\dot{A}_n^k$ is infinite for each $k, n \in \omega$.

Take any $X \in \mathcal{X}$ and let $S < T$ be a condition with stem r. There is $n \in \omega$ such that $f_n[X] \in \mathcal{F}^*$. Put

$$S' = S \backslash \left\{ s \in T \mid \exists t \in T^{[b_n]}, r < t : \exists i \in f[X] : t^\frown i \subseteq s \in S \right\}.$$

Note that $S' \in \mathbb{L}(\mathcal{F}^+)$ since we removed only $\mathcal{F}^*$ many immediate successors of each splitting node of S. Also notice that $S' \Vdash X \cap \dot{A}_n^{|r|} = \emptyset$. Thus for each $X \in \mathcal{X}$ the condition T forces that X does not have infinite intersection with all sets $\dot{A}_n^k$, and $\mathcal{X}$ is not ω-hitting in the extension. $\square$

We can formulate the "splitting" version of the previous result. A similar result for $\mathbb{L}(\mathcal{F})$, where $\mathcal{F}$ is a filter, is contained in [3, Section 6].

Definition 26 Let $\mathcal{X} \subset \mathcal{P}(\omega)$ be a family of sets and let $\mathcal{F}$ be a filter on ω. We say that $\mathcal{X}$ is *$\mathcal{F}^+$-ω-splitting* if for every countable set of functions $\{ f_n : \omega \to \omega \mid n \in \omega \}$ such that $f_n[\omega] \in \mathcal{F}^+$ for each $n \in \omega$, there exists $X \in \mathcal{X}$ such that $f_n[X], f_n[\omega \backslash X] \in \mathcal{F}^+$ for each $n \in \omega$.

Again, every $\mathcal{F}^+$-ω-splitting family is ω-splitting. The same proof as before with the obvious adjustments gives us the following.

Proposition 27 *Let $\mathcal{F}$ be a filter on ω and let $\mathcal{X} \subset \mathcal{P}(\omega)$. The following are equivalent;*

(1) $\mathcal{X}$ is $\mathcal{F}^+$-ω-splitting,
(2) $\mathbb{L}(\mathcal{F}^+)$ preserves "$\check{\mathcal{X}}$ is ω-splitting."

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GENERIC EXISTENCE OF MAD FAMILIES

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Abstract. In this note we study generic existence of maximal almost disjoint (MAD) families. Among other results we prove that Cohen-indestructible families exist generically if and only if $\mathfrak{b} = \mathfrak{c}$. We obtain analogous results for other combinatorial properties of MAD families, including Sacks-indestructibility and being $+$-Ramsey.

§**1. Introduction.** An infinite family $\mathscr{A}$ of infinite subsets of ω is *almost disjoint* (*AD*) if the intersection of any two distinct elements of $\mathscr{A}$ is finite. It is *maximal almost disjoint* (*MAD*) if it is not properly included in any larger AD family or, equivalently, if given an infinite $X \subseteq \omega$ there is an $A \in \mathscr{A}$ such that $|A \cap X| = \omega$.

Many MAD families with special combinatorial or topological properties can be constructed using set-theoretic assumptions like CH, MA, or $\mathfrak{b} = \mathfrak{c}$. However, special MAD families are notoriously difficult to construct in ZFC alone. The reason being the lack of a device ensuring that a recursive construction of a MAD family would not prematurely terminate, an object that would serve a similar purpose as independent linked families do for the construction of special ultrafilters (see [16]). There is also a definite lack of negative (i.e., consistency) results. The following problem due to J. Steprāns presents one the basic open test problems for understanding the behaviour of MAD families in forcing extensions.

PROBLEM 1.1 ([25]). *Is there a Cohen-indestructible MAD family in* ZFC?

As we mentioned before, the main difficulty lies in ensuring that a recursive construction of a MAD family does not terminate prematurely. This can be done typically either by means of cardinality considerations alone or by using an ad hoc construction for the problem at hand. In this paper we focus on the former.

The following is one of the most important definitions in this note.

DEFINITION 1.2. Let P be a property of MAD families. We say *MAD families with property P exist generically* if every AD family of size less than $\mathfrak{c}$ can be extended to a MAD family with property P.

We begin with a simple example. Recall that a MAD family $\mathcal{A}$ is *completely separable* if every subset of ω which can not be almost covered by finitely many

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elements of $\mathcal{A}$ contains an element of $\mathcal{A}$. It is not known, whether completely separable MAD families exist in ZFC [24]. However, it is easy to see that completely separable MAD families exist generically if and only if $\mathfrak{a} = \mathfrak{c}$. On the one hand, assuming $\mathfrak{a} = \mathfrak{c}$, a straightforward and well-known recursive construction permits to extend any AD family of size less than $\mathfrak{c}$ to a completely separable MAD family, while if $\mathfrak{a} < \mathfrak{c}$, then there is a MAD family of size less than $\mathfrak{c}$, which is not completely separable[1] and since it is already maximal can not be extended to a completely separable MAD family.

One of the main results in this paper is the following theorem which gives a partial answer to the Problem 1.1.

THEOREM 1.3. *Cohen-indestructible families exist generically if and only if* $\mathfrak{b} = \mathfrak{c}$.

Extensions of AD families to maximal ones have been previously investigated by Leathrum in [19] and by Fuchino, Geschke, and Soukup in [8]. Generic existence of ultrafilters has been introduced by Canjar in [7] and was recently investigated by Brendle and Flašková in [5].

Given a forcing notion $\mathbb{P}$, a MAD family $\mathscr{A}$ is $\mathbb{P}$-*indestructible* if $\mathscr{A}$ remains maximal after forcing with $\mathbb{P}$. It follows from the proof of $\mathfrak{b} \leq \mathfrak{a}$ (see [2]) that if $\mathbb{P}$ adds a dominating real then it destroys every MAD family from the ground model, so the definition is only interesting when $\mathbb{P}$ does not add dominating reals. Our main focus is on Sacks and Cohen indestructible MAD families.

If $\mathscr{A}$ is an AD family on ω (or any countable set) we denote by $\mathscr{A}^{\perp}$ the set of all infinite $X \subseteq \omega$ that are almost disjoint with every element of $\mathscr{A}$. If $\mathscr{I}$ is an ideal on ω, we denote by $\mathscr{I}^{+}$ as those subsets of ω that are not in $\mathscr{I}$. We shall only consider ideals which extend the ideal of finite sets. If $X \in \mathscr{I}^{+}$ then by $\mathscr{I} \restriction X$ we will denote the restriction of $\mathscr{I}$ to X, that is, $\mathscr{I} \restriction X = \{I \cap X : I \in \mathscr{I}\}$ which is an ideal on X. We say $\mathscr{I}$ is *tall* if for every infinite $X \subseteq \omega$ there is an infinite $A \in \mathscr{I}$ such that $A \subseteq X$. There is a close relationship between MAD families and definable ideals (typically Borel of low complexity). We shall also investigate the connection here.

DEFINITION 1.4. Let $\mathscr{I}$ be an ideal (on a countable set). Then:

(1) We define $\mathsf{cov}^{*}(\mathscr{I})$ as the least size of a family $\mathscr{B} \subseteq \mathscr{I}$ such that for every infinite $X \in \mathscr{I}$ there is $B \in \mathscr{B}$ for which $B \cap X$ is infinite.
(2) If $\mathscr{I}$ is tall, we define $\mathsf{cov}^{+}(\mathscr{I})$ as the least size of a family $\mathscr{B} \subseteq \mathscr{I}$ such that for every $Y \in \mathscr{I}^{+}$ there is $B \in \mathscr{B}$ for which $B \cap Y$ is infinite.
(3) We say an AD family $\mathscr{A} \subseteq \mathscr{I}$ is a *MAD family restricted to* $\mathscr{I}$ if for every infinite $X \in \mathscr{I}$ there is $A \in \mathscr{A}$ such that $|X \cap A| = \omega$.
(4) $\mathfrak{a}(\mathscr{I})$ is the least size of a MAD family restricted to $\mathscr{I}$.
(5) For tall ideals, we define $\mathfrak{a}^{+}(\mathscr{I})$ as the least size of an AD family $\mathscr{A}$ such that $\mathscr{A} \cup \mathscr{A}^{\perp} \subseteq \mathscr{I}$ (or in other words, if $Y \in \mathscr{I}^{+}$ then there is $A \in \mathscr{A}$ such that $|Y \cap A| = \omega$).

Note that if $\mathscr{I}$ is tall, then $\mathsf{cov}^{*}(\mathscr{I})$ is just the least size of a family $\mathscr{B} \subseteq \mathscr{I}$ such that for every infinite $X \subseteq \omega$ there is $B \in \mathscr{B}$ for which $B \cap X$ is infinite, also $\mathsf{cov}^{+}(\mathscr{I}) \leq \mathsf{cov}^{*}(\mathscr{I})$. In general, $\mathsf{cov}^{*}(\mathscr{I}) \leq \mathfrak{a}(\mathscr{I})$ and for tall ideals $\mathsf{cov}^{+}(\mathscr{I}) \leq \mathfrak{a}^{+}(\mathscr{I})$ and $\mathfrak{a}^{+}(\mathscr{I}) \leq \mathfrak{a}(\mathscr{I})$. For the definitions of the classical invariants of the continuum see [2].

[1] Every completely separable MAD family has size $\mathfrak{c}$.

§2. Destructibility of MAD families. Let $\mathscr{I}$ be a tall ideal on ω and $\mathbb{P}$ be a forcing notion. Recall that $\mathscr{I}$ is $\mathbb{P}$-*indestructible* if $\mathscr{I}$ remains tall after forcing with $\mathbb{P}$ and otherwise it is $\mathbb{P}$-*destructible*. It is easy to see that a MAD family $\mathscr{A}$ is $\mathbb{P}$-indestructible if and only if $\mathscr{I}(\mathscr{A})$ is $\mathbb{P}$-indestructible. For the sake of the reader, we will now quote some results about destructibility. The key is to associate to every σ-ideal on the Baire space (Cantor space) an ideal on $\omega^{<\omega}$ $(2^{<\omega})$ and this is done by the notion of trace ideal.

Definition 2.1 ([6]). Given $a \subseteq \omega^{<\omega}$ we define $\pi(a) = \{b \in \omega^\omega \colon \exists^\infty n (b \restriction n \in a)\}$. If I is a σ-ideal on ω^ω define its *trace ideal* $tr(I) = \{a \subseteq \omega^{<\omega} \colon \pi(a) \in I\}$.

Clearly $\pi(a)$ is a G_δ set for every $a \subseteq \omega^{<\omega}$. The Katětov order plays a crucial role when studying destructibility of ideals and MAD families.

Definition 2.2. Let A, B be two countable sets and $\mathscr{I}$, $\mathscr{J}$ be two ideals on A and B, respectively.

(1) We say that $\mathscr{I}$ is *Katětov below* $\mathscr{J}$ (denoted by $\mathscr{I} \leq_K \mathscr{J}$) if there is a function $f\colon B \longrightarrow A$ such that $f^{-1}[I] \in \mathscr{J}$, for all $I \in \mathscr{I}$. The function f is called a *Katětov function*. We say that $\mathscr{I}$ is a *Katětov-Blass* below $\mathscr{J}$ ($\mathscr{I} \leq_{KB} \mathscr{J}$) if the function f may be taken finite-to-one (in this case f is called a Katětov-Blass function).
(2) We say that $\mathscr{I}$ is *Katětov equivalent to* $\mathscr{J}$ if $\mathscr{I} \leq_K \mathscr{J}$ and $\mathscr{J} \leq_K \mathscr{I}$ and we denote it by $\mathscr{I} \simeq_K \mathscr{J}$, the analogous definition holds for the Katětov-Blass order.

Observe that if a forcing notion $\mathbb{P}$ destroys an ideal $\mathscr{J}$ and $\mathscr{I} \leq_K \mathscr{J}$, then $\mathbb{P}$ also destroys $\mathscr{I}$. In fact, suppose that $\mathbb{P}$ destroys $\mathscr{J}$ and that $f\colon B \longrightarrow A$ witness $\mathscr{I} \leq_K \mathscr{J}$. Find a $\mathbb{P}$-name $\dot{X}$ for an infinite subset of B such that $\Vdash$ "$|\dot{X} \cap J| < \omega$ for all $J \in \mathscr{J}$." Note that $\Vdash$ "$f''\dot{X} \notin \mathscr{I}$," so in particular is infinite and it also witness that $\mathscr{I}$ is not tall in the extension. It is also immediate to see that if $X \in \mathscr{I}^+$ then $\mathscr{I} \leq_{KB} \mathscr{I} \restriction X$. An ideal $\mathscr{I}$ is called *Katětov-Blass uniform* if $\mathscr{I}$ is Katětov-Blass equivalent to all its restrictions (equivalently, if $X \in \mathscr{I}^+$, then $\mathscr{I} \restriction X \leq_{KB} \mathscr{I}$).

Given a σ-ideal I on ω^ω, $\mathbb{P}_I$ denotes the collection of all Borel sets in I^+ ordered by the I-almost inclusion. The ideal I has the *continuous reading of names* [26] if for all $B \in \mathbb{P}_I$ and each Borel function $f : B \longrightarrow \omega^\omega$, there is a Borel set $C \in I^+$ such that $C \subseteq B$ and $f \restriction C$ is continuous. We shall need the following result of Hrušák and Zapletal.

Proposition 2.3 ([12]). *Let I be a σ-ideal on ω^ω such that $\mathbb{P}_I$ is proper with continuous reading of names and $\mathscr{J}$ be an ideal on ω. Then the following are equivalent*:

(1) *There is a $B \in \mathbb{P}_I$ such that $B \Vdash$ "$\mathscr{J}$ is not tall."*
(2) *There is an $a \in tr(I)^+$ such that $\mathscr{J} \leq_K tr(I) \restriction a$.*

In many cases occurring in practice, particularly for the forcing notions discussed in this article, the previous items are also equivalent to there is an $a \in tr(I)^+$ such that $\mathscr{J} \leq_{KB} tr(I) \restriction a$.

Let I be a σ-ideal on ω^ω, recall that I is *continuously homogeneous* if for every $B \in \mathbb{P}_I$ there is a continuous function $f\colon \omega^\omega \longrightarrow B$ such that $f^{-1}[A] \in I$ for every $A \in I \restriction B$.

COROLLARY 2.4 ([12]). *Let I be continuously homogeneous σ-ideal on ω^ω such that $\mathbb{P}_I$ is proper with continuous reading of names such that $tr(I)$ is Katětov-Blass uniform and $\mathscr{J}$ be an ideal on ω. Then the following are equivalent:*

(1) $\mathscr{J}$ *is $\mathbb{P}_I$-destructible,*
(2) $\mathscr{J} \leq_K tr(I)$,
(3) $\mathscr{J} \leq_{KB} tr(I)$.

Let nwd be the ideal of all nowhere dense sets of the rational numbers, ctbl be the σ-ideal of all countable sets in the Baire space, and K_σ be the ideal generated by all σ-compact sets of the Baire space. It can be shown that $tr(\mathcal{M})$ is Katětov equivalent to nwd and it is easy to see that both $\mathcal{M}$ and ctbl are continuously homogeneous. Therefore, we can conclude the following.

COROLLARY 2.5 ([10] and [6]). *Let $\mathscr{I}$ be an ideal on ω, then the following holds:*

(1) *$\mathscr{I}$ is Cohen-destructible if and only if $\mathscr{I} \leq_{KB}$ nwd,*
(2) *$\mathscr{I}$ is Sacks-destructible if and only if $\mathscr{I} \leq_{KB} tr(\mathsf{ctbl})$,*
(3) *$\mathscr{I}$ is Miller-destructible if and only if $\mathscr{I} \leq_{KB} tr(K_\sigma)$.*

Sacks destructibility is particularly interesting due to the following result.

PROPOSITION 2.6 ([10]). *If $\mathbb{P}$ adds a new real, then $\mathbb{P}$ destroys $tr(\mathsf{ctble})$. Therefore, if $\mathscr{I}$ is Sacks-destructible, then it is $\mathbb{Q}$-destructible by any forcing $\mathbb{Q}$ that adds a new real.*

PROOF. Let $r \in V[G]$ be a new real and set $\widehat{r} = \{r \restriction n : n \in \omega\}$. Note that if $a \in (tr(\mathsf{ctbl})) \cap V$, then $r \notin \pi(a)$. This implies that $\widehat{r} \cap a =^* \emptyset$, hence $tr(\mathsf{ctbl})^V$ was destroyed. $\dashv$

By a similar argument, we can show the following:

PROPOSITION 2.7 ([6]). *If $\mathbb{P}$ adds an unbounded real, then $\mathbb{P}$ destroys $tr(K_\sigma)$. Therefore, if $\mathscr{I}$ is Miller-destructible, then it is $\mathbb{Q}$-destructible by any forcing $\mathbb{Q}$ that adds an unbounded real.*

It is easy to show that if $\mathscr{I} \leq_K \mathscr{J}$, then $\mathsf{cov}^*(\mathscr{J}) \leq \mathsf{cov}^*(\mathscr{I})$. It is a result of Keremedis [15] (see also [1]) that $\mathsf{cov}^*(\mathsf{nwd}) = \mathsf{cov}(\mathcal{M})$, it is not hard to see that $\mathsf{cov}^*(tr(\mathsf{ctbl})) = \mathfrak{c}$ and $\mathsf{cov}^*(tr(K_\sigma)) = \mathfrak{d}$. If $\mathscr{A}$ is a MAD family, then it is straight forward to check that $\mathsf{cov}^*(\mathscr{I}(\mathscr{A})) = |\mathscr{A}|$.

COROLLARY 2.8. *Let $\mathscr{A}$ be a MAD family. Then:*

(1) *If $\mathscr{A}$ is Cohen-destructible, then $\mathsf{cov}(\mathcal{M}) \leq |\mathscr{A}|$,*
(2) *If $\mathscr{A}$ is Sacks-destructible, then $|\mathscr{A}| = \mathfrak{c}$,*
(3) *If $\mathscr{A}$ is Miller-destructible, then $\mathfrak{d} \leq |\mathscr{A}|$.*

It follows.

COROLLARY 2.9 ([10] and [6]). (1) *If $\mathfrak{a} < \mathsf{cov}(\mathcal{M})$, then there is a Cohen-indestructible MAD family,*
(2) *If $\mathfrak{a} < \mathfrak{c}$, then there is a Sacks-indestructible MAD family,*
(3) *If $\mathfrak{a} < \mathfrak{d}$, then there is a Miller-indestructible MAD family.*

Given an AD family $\mathscr{A}$ we say that it is $\mathscr{I}$-MAD if $\mathscr{I}(\mathscr{A}) \not\leq_{KB} \mathscr{I}$.

PROPOSITION 2.10. *$\mathscr{I}$-MAD families exist generically if and only if $\mathfrak{a}^+(\mathscr{I}) = \mathfrak{c}$.*

PROOF. First assume $\mathfrak{a}^+(\mathscr{I}) = \mathfrak{c}$. We will show that $\mathscr{I}$-MAD families exist generically. Let $\mathscr{A}$ be an AD family of size less than $\mathfrak{c}$ we will show how to extend it to a MAD family $\mathscr{B}$ so that $\mathscr{I}(\mathscr{B}) \nleq_{KB} \mathscr{I}$. Let $\langle f_\alpha : \alpha \in \mathfrak{c}\rangle$ be an enumeration of the set of all finite to one functions from $\omega^{<\omega}$ to ω. We will construct recursively an increasing sequence $\langle \mathscr{B}_\alpha : \alpha \in \mathfrak{c}\rangle$ of AD families such that:

(1) $\mathscr{B}_0 = \mathscr{A}$,
(2) $|\mathscr{B}_\alpha| < \mathfrak{c}$,
(3) $f_\alpha : \omega^{<\omega} \longrightarrow \omega$ is not a Katětov function from $(\omega^{<\omega}, \mathscr{I})$ into $(\omega, \mathscr{I}(\mathscr{B}_{\alpha+1}))$.

Assume we are at step α and $f_\alpha : \omega^{<\omega} \longrightarrow \omega$ is a Katětov function from $(\omega^{<\omega}, \mathscr{I})$ into $(\omega, \mathscr{I}(\mathscr{B}_\alpha))$. Let $\mathscr{D} = f_\alpha^{-1}[\mathscr{B}_\alpha]$ and note that, since f_α is finite to one, $\mathscr{D}$ is an almost disjoint family and it is contained in $\mathscr{I}$. Since $\mathscr{D}$ has size less than $\mathfrak{c}$, it follows that $\mathscr{D}^\perp$ is not contained in $\mathscr{I}$ so there is $X \notin \mathscr{I}$ that is almost disjoint with $\mathscr{D}$. Let $Y = f_\alpha[X]$ and note that $\mathscr{B}_\alpha = \mathscr{B} \cup \{Y\}$ is almost disjoint and $f_\alpha : \omega^{<\omega} \longrightarrow \omega$ is no longer a Katětov function.

For the forward implication, note that if $\mathfrak{a}^+(\mathscr{I}) < \mathfrak{c}$, then there is an AD family $\mathscr{A}$ of size strictly less than continuum such that $\mathscr{A} \cup \mathscr{A}^\perp \subseteq \mathscr{I}$, and obviously $\mathscr{A}$ can not be extended to a MAD family not K-below $\mathscr{I}$. ⊣

We now address the question of when $\mathbb{P}_I$-indestructible MAD families exist generically.

Corollary 2.11. *Let I be a σ-ideal with continuous reading of names such that $tr(I)$ is KB-uniform. Then $\mathbb{P}_I$-indestructible MAD families exist generically if and only if $\mathfrak{a}^+(tr(I)) = \mathfrak{c}$.*

§3. Cohen-indestructibility. We will now show that $\mathfrak{b} = \mathfrak{a}^+(\mathsf{nwd})$. First we give several formulations of $\mathfrak{b}$ that might be of independent interest.

Proposition 3.1. *Let κ be an infinite cardinal, then the following are equivalent*:

(1) $\kappa < \mathfrak{b}$.
(2) *If $\mathscr{A}$ is an AD family of size κ and $\{C_n : n \in \omega\} \subseteq \mathscr{A}^\perp$, then there is $X \in \mathscr{A}^\perp$ such that $C_n \subseteq^* X$ for every $n \in \omega$, i.e., $\mathscr{A}^\perp$ is a P-ideal.*
(3) *If $\mathscr{A}$ is an AD family of size κ and $\{C_n : n \in \omega\} \subseteq \mathscr{A}^\perp$, then there is $X \in \mathscr{A}^\perp$ such that $|X \cap C_n| = \omega$ for every $n \in \omega$.*
(4) *If $\mathscr{A}$ is an AD family of size κ and $\{C_n : n \in \omega\} \subseteq \mathscr{A}^\perp$, then there is $X \in \mathscr{A}^\perp$ such that $X \cap C_n \neq \emptyset$ for every $n \in \omega$.*
(5) *If $\mathscr{A}$ is an AD family of size κ and $\{C_n : n \in \omega\} \subseteq \mathscr{A}^\perp$ is a pairwise disjoint family, then there is $X \in \mathscr{A}^\perp$ such that $|X \cap C_n| = \omega$ for every $n \in \omega$.*
(6) *If $\mathscr{A}$ is an AD family of size κ and for every pairwise disjoint family $\{C_n : n \in \omega\} \subseteq \mathscr{A}^\perp$, then there is $X \in \mathscr{A}^\perp$ such that $X \cap C_n \neq \varnothing$ for every $n \in \omega$.*

Proof. Obviously 2 implies 3, and 3 implies 5. It is easy to see that 4 implies 3 by splitting each C_n in countably many disjoint parts. By a similar reasoning it follows that 5 and 6 are equivalent. We now show that 1 implies 2. Let $\kappa < \mathfrak{b}$, $\mathscr{A} = \{A_\alpha : \alpha \in \kappa\}$ be an AD family in ω and $\{C_n : n \in \omega\} \subseteq \mathscr{A}^\perp$. Enumerate each $C_n = \{c_n^i : i \in \omega\}$ and for every $\alpha < \kappa$, define $f_\alpha : \omega \longrightarrow \omega$ in such a way that $A_\alpha \cap C_n \subseteq \{c_n^i : i < f_\alpha(n)\}$, for every $n \in \omega$. Since $\kappa < \mathfrak{b}$, we can find an increasing function $g : \omega \longrightarrow \omega$ that dominates each f_α, and define $X = \{c_n^{g(n)} : n \in \omega\}$. It is clear that X has the desired properties.

We now show that 5 implies 1 by contrapositive. Let $\mathscr{B} \subseteq \omega^\omega$ be an unbounded family of size κ, define the function $H : \omega^\omega \longrightarrow 2^\omega$ by $H\left(\langle x_n\rangle_{n\in\omega}\right) = 0^{x_0}{}^\frown 1^\frown 0^{x_1}{}^\frown 1^\frown 0^{x_2}{}^\frown \cdots$ (where 0^{x_n} is the sequence of x_n consecutive 0's). Let $\mathbb{Q} \subseteq 2^\omega$ be the set of all sequences that are eventually zero, it is not hard to see that H is an homeomorphism between ω^ω and $2^\omega \setminus \mathbb{Q}$. Given $b \in 2^\omega$ define $\widehat{b} = \{b \upharpoonright n : n \in \omega\}$ and let $\mathscr{A} = \{\widehat{H(f)} : f \in \mathscr{B}\}$ which clearly is an AD family and note that $\{\widehat{q} : q \in \mathbb{Q}\} \in \mathscr{A}^\perp$. We must now show there is no $X \in \mathscr{A}^\perp$ such that $|X \cap \widehat{q}| = \omega$ for every $q \in \mathbb{Q}$.

Suppose this is not the case. For each $n \in \omega$, define $U_n = \{b \in 2^\omega : |\widehat{b} \cap X| \geq n\}$ and note that each U_n is an open set, hence $G = \bigcap U_n$ is a G_δ set and $\mathbb{Q} \subseteq G$ while $G \cap H(\mathscr{B}) = \emptyset$. Let $K = 2^\omega \setminus G$. It is clear that $H[\mathscr{B}] \subseteq K$ and K is σ-compact. It follows that $H^{-1}[K]$ is σ-compact and contains $\mathscr{B}$, which is a contradiction. ⊣

We are now ready to prove the main theorem of the section.

THEOREM 3.2. *Cohen-indestructible MAD families exist generically if and only if* $\mathfrak{b} = \mathfrak{c}$.

PROOF. It suffices to show that $\mathfrak{b} = \mathfrak{a}^+(\mathsf{nwd})$. First, let $\kappa < \mathfrak{b}$ be given. Let us show that $\kappa < \mathfrak{a}^+(\mathsf{nwd})$. Let $\mathscr{A}$ be an AD family in $\mathbb{Q}$ such that $\mathscr{A} \subseteq \mathsf{nwd}$, we must prove that $\mathscr{A}^\perp$ is not contained in nwd. Let $\langle U_n : n \in \omega\rangle$ be a base for the topology of the rational numbers. Since $\kappa < \mathfrak{b}$, then it is also smaller than $\mathfrak{a}$, so $\mathscr{A}_n = \mathscr{A} \upharpoonright U_n$ is not a MAD family in U_n (note that $U_n \notin \mathscr{I}(\mathscr{A})$ as the elements of $\mathscr{A}$ are nowhere dense) so we can find an infinite $C_n \subseteq U_n$ that is almost disjoint from every element of $\mathscr{A}_n$. Using $\kappa < \mathfrak{b}$ and the previous proposition, we can find an $X \in \mathscr{A}^\perp$ that intersects every C_n, and hence it is dense.

In order to show that $\mathfrak{a}^+(\mathsf{nwd}) \leq \mathfrak{b}$, we will construct an AD family $\mathscr{A}$ of size $\mathfrak{b}$ such that both $\mathscr{A}$ and $\mathscr{A}^\perp$ are contained in nwd. Recursively, we construct families $\{\mathscr{A}_s : s \in \omega^{<\omega}\}$ and $\{\overline{C}_s : s \in \omega^{<\omega}\}$ such that:

(1) $\mathscr{A}_\emptyset$ is an AD family on ω of size $\mathfrak{b}$ which is not maximal,
(2) $\overline{C}_s = \{C_s(n) : n \in \omega\}$ are pairwise disjoint infinite sets,
(3) $\mathscr{A}_{s^\frown\langle n\rangle}$ is an AD family on $C_s(n)$ of size $\mathfrak{b}$ which is not maximal,
(4) $\overline{C}_\emptyset$ is a partition of ω and $\overline{C}_{s^\frown\langle n\rangle}$ is a partition of $C_s(n)$,
(5) $\overline{C}_s \subseteq \mathscr{A}_s^\perp$ and if $Y \in [\omega]^\omega$ intersects infinitely every $C_s(n)$ then $Y \notin \mathscr{A}_s^\perp$,
(6) For every $a, b \in \omega$ there are s and n such that $|\{a, b\} \cap C_s(n)| = 1$.

In order to do this, fix an enumeration $\{\{a_k, b_k\} : k \in \omega\}$ of $[\omega]^2$. Using Proposition 3.1(5), there exists an AD family $\mathscr{A}$ of size $\mathfrak{b}$ on ω and a pairwise disjoint family $\{C(n) : n \in \omega\} \subseteq \mathscr{A}^\perp$ such that if $Y \in [\omega]^\omega$ intersects infinitely every $C(n)$ then $Y \notin \mathscr{A}^\perp$. Put $\mathscr{A}_\emptyset = \mathscr{A}$ and by a adding finitely many points to each $C(n)$ we may assume $\{C(n) : n \in \omega\}$ forms a partition of ω. Moreover, by making finite changes we can assume that $|\{a_0, b_0\} \cap C(0)| = 1$. Now set $\overline{C}_\emptyset = \{C(n) : n \in \omega\}$.

Suppose that we have constructed $\mathscr{A}_s$ and $\overline{C}_s$ for $s \in \omega^{\leq m}$. Again using Proposition 3.1(5), for every $n \in \omega$ and $s \in \omega^m$, there exists an AD family $\mathscr{A}$ of size $\mathfrak{b}$ on $C_s(n)$ and a pairwise disjoint family $\{C(k) : k \in \omega\} \subseteq \mathscr{A}^\perp$ such that if $Y \in [\omega]^\omega$ intersects infinitely every $C(k)$ then $Y \notin \mathscr{A}^\perp$. Put $\mathscr{A}_{s^\frown\langle n\rangle} = \mathscr{A}$ and by a adding finitely many points to each $C(k)$ we may assume $\{C(k) : k \in \omega\}$ forms a partition of $C_s(n)$. Moreover, by making finite changes, we can assume that if

$a_m, b_m \in C_s(n)$ then $|\{a_m, b_m\} \cap C(0)| = 1$. Now set $\overline{C}_{s^\frown\langle n\rangle} = \{C(k) : k \in \omega\}$. This concludes the construction.

Let τ be the topology on ω generated by declaring each $C_s(n)$ clopen. It follows from a result of Sierpiński (see [17] or [14]) that (ω, τ) is homeomorphic to the rational numbers with the usual topology. Let $\mathscr{A} = \bigcup_{s\in\omega^{<\omega}} \mathscr{A}_s$, it suffices to show that $\mathscr{A} \cup \mathscr{A}^\perp \subseteq \mathsf{nwd}$. Let $A \in \mathscr{A}_t$ and $\overline{C}_s(n)$ first note that if $A \cap \overline{C}_s(n) \neq \varnothing$ then t and s are incompatible, by further extending s if necessary we may assume that s extends t. By (5) we extend s even further to s' so that $A \cap \overline{C}(n) \neq \varnothing$ is finite and then using (6) we can find an open subset of $\overline{C}_s(n)$ disjoint from A. Thus, $\mathscr{A} \subseteq \mathsf{nwd}$. The argument for $\mathscr{A}^\perp$ is analogous. ⊣

A closely related notion is that of a tight MAD family [20].

Definition 3.3. We say a MAD family $\mathscr{A}$ is *tight* if for every $\langle X_n : n \in \omega\rangle \subseteq \mathscr{I}(\mathscr{A})^+$ there is $B \in \mathscr{I}(\mathscr{A})$ such that $|B \cap X_n| = \omega$ for every $n \in \omega$.

Every tight family is Cohen indestructible and every Cohen indestructible family has a restriction that is tight (see [11, 18]). In particular, the existence of a tight MAD family is equivalent to the existence of a Cohen indestructible MAD family.

Corollary 3.4. *Tight families exist generically if and only if* $\mathfrak{b} = \mathfrak{c}$.

Proof. If tight families exist generically then obviously there exist Cohen-indestructible MAD families, therefore $\mathfrak{b}$ must be equal to $\mathfrak{c}$. The other implication follows from standard recursive construction. ⊣

We will now show that there are also tight families in many models where $\mathfrak{b}$ equals to ω_1. The following guessing principle was defined in [22].

$\diamondsuit(\mathfrak{b})$: For every Borel coloring $C : 2^{<\omega_1} \longrightarrow \omega^\omega$ there is a $G : \omega_1 \longrightarrow \omega^\omega$ such that for every $R \in 2^{\omega_1}$ the set $\{\alpha : C(R \restriction \alpha) \ {}^*\!\not\geq G(\alpha)\}$ is a stationary set (such G is called a *guessing sequence* for C).

A coloring $C : 2^{<\omega_1} \longrightarrow \omega^\omega$ is *Borel* if for every α, the function $C \restriction 2^\alpha$ is Borel in the classical sense. It is easy to see that $\diamondsuit(\mathfrak{b})$ implies that $\mathfrak{b} = \omega_1$ and in [22] it was proved that it implies $\mathfrak{a} = \omega_1$. The following answers a question of Hrušák and García-Ferreira from [11].

Proposition 3.5. *Assuming* $\diamondsuit(\mathfrak{b})$, *there is a tight MAD family.*

Proof. For every $\alpha < \omega_1$, fix an enumeration $\alpha = \{\alpha_n : n \in \omega\}$. Using a suitable coding, the coloring C will be defined on pairs $t = (\mathscr{A}_t, X_t)$, where $\mathscr{A}_t = \langle A_\xi : \xi < \alpha\rangle$ and $X_t = \langle X_n : n \in \omega\rangle$. We define $C(t)$ to be the constant 0 function in case $\mathscr{A}_t$ is not an almost disjoint family or X_t is not a sequence of elements in $\mathscr{I}(\mathscr{A}_t)^+$. In the other case, define an increasing function $C(t) : \omega \longrightarrow \omega$ such that for every $n \in \omega$ and $i \leq n$ the set $X_i \cap (C(t-1), C(t)) \setminus A_{\alpha_0} \cup \cdots \cup A_{\alpha_n}$ is not empty (where $C(t)(-1) = 0$).

By $\diamondsuit(\mathfrak{b})$ there is a guessing sequence $G : \omega_1 \longrightarrow \omega^\omega$ for C, changing G if necessary, we may assume that all the $G(\alpha)$ are increasing and if $\alpha < \beta$ then $G(\alpha) <^* G(\beta)$. We will now construct our MAD family by recursion on ω_1: Let $\{A_n : n \in \omega\}$ be a partition of ω. Suppose we have defined A_ξ for all $\xi < \alpha$, we put $A_\alpha = \bigcup_{n\in\omega} (G(\alpha)(n) \setminus A_{\alpha_0} \cup \cdots \cup A_{\alpha_n})$ in case this is an infinite set, otherwise just take any A_α that is almost disjoint with $\mathscr{A}_\alpha$.

We will now show that $\mathscr{A}$ is a tight family. Let $X = \langle X_n : n \in \omega \rangle$ where each $X_n \in \mathscr{I}(\mathscr{A})^+$. Consider the branch $R = (\langle A_\xi : \xi < \omega_1 \rangle, X)$ and pick $\beta > \omega$ such that $C(R \restriction \beta) \; ^* \not\geq G(\beta)$. It follows from the construction that A_β intersects infinitely every X_n. ⊣

With the aid of a result of [22] we can conclude the following.

Proposition 3.6 ([22]). *Let $\langle \mathbb{Q}_\alpha : \alpha \in \omega_2 \rangle$ be a sequence of Borel proper partial orders where each $\mathbb{Q}_\alpha$ is forcing equivalent to $\mathscr{P}(2)^+ \times \mathbb{Q}_\alpha$ and let $\mathbb{P}_{\omega_2}$ be the countable support iteration of this sequence. Then there is a tight family in any forcing extension by $\mathbb{P}_{\omega_2}$.*

Proof. If $\mathbb{P}_{\omega_2}$ forces $\mathfrak{b} = \mathfrak{c}$ then tight families exist generically, otherwise, it follows from [22] that $\mathbb{P}_{\omega_2}$ forces $\diamondsuit(\mathfrak{b})$ and, hence forces the existence of tight families. ⊣

The following weakening of tightness was introduced in [11].

Definition 3.7. We say that $\mathscr{A}$ is *weakly tight* if for every $\{B_n : n \in \omega\} \subseteq \mathscr{I}(\mathscr{A})^+$ there is $A \in \mathscr{A}$ such that $|A \cap B_n| = \omega$ for infinitely many $n \in \omega$.

Obviously every tight family is weakly tight. The proof of the next proposition is virtually identical to the proof of Corollary 3.4.

Proposition 3.8. *Weakly tight families exist generically if and only if $\mathfrak{b} = \mathfrak{c}$.*

Mildenberger, Raghavan, and Steprāns (see [23] and [21]) proved that if $\mathfrak{s} \leq \mathfrak{b}$ then there is a weakly tight family. However, it is still open if it is possible to construct a weakly tight family without any additional axioms beyond ZFC.

The following invariant was introduced by Shelah in [24].

Definition 3.9. We define $\mathfrak{a}_*$ as the minimal size of an AD family $\mathscr{A}$ such that there are (disjoint) $C_0, C_1, C_2, \ldots \in \mathscr{A}^\perp$ such that for every $B \in [\omega]^\omega$ if $C_n \cap B \neq^* \emptyset$ for infinitely many C_n then there is $A \in \mathscr{A}$ for which $A \cap B \neq^* \emptyset$.

The relation of $\mathfrak{a}_*$ with the other cardinal invariants is the following.

Proposition 3.10 ([24]). $\mathfrak{b} \leq \mathfrak{a}_* \leq \mathfrak{a}$.

Proof. By the characterization of $\mathfrak{b}$, it is clear that $\mathfrak{b} \leq \mathfrak{a}_*$. In order to prove that $\mathfrak{a}_* \leq \mathfrak{a}$, let $\mathscr{A}$ be a MAD family of minimum size. Choose $C_0, C_1, C_2, \ldots \in \mathscr{A}$ and let $\mathscr{A}_1 = \mathscr{A} \setminus \{C_n : n \in \omega\}$ we will show that $\mathscr{A}_1$ is a witness for $\mathfrak{a}_*$. Assume $C_n \cap B \neq^* \emptyset$ for infinitely many C_n, then it follows that $B \setminus C_0 \cup \cdots \cup C_n$ is infinite for every $n \in \omega$, so we may find $B' \subseteq B$ that is almost disjoint from every C_n. Since $\mathscr{A}$ is MAD, it follows that there is an $A \in \mathscr{A}$ such that $A \cap B \neq^* \emptyset$. ⊣

Proposition 3.11. *$\mathfrak{a}_*$ has uncountable cofinality.*

Proof. Assume $\mathrm{cof}(\mathfrak{a}_*) = \omega$ and let $\mathcal{A}$ be an AD family of size $\mathfrak{a}_*$ and $\{C_n \mid n \in \omega\} \subseteq \mathcal{A}^\perp$ such that for every B with the property that there are infinitely many $n \in \omega$ such that $|C_n \cap B| = \omega$ then there is $A \in \mathcal{A}$ for which $|A \cap B| = \omega$. Since $\mathfrak{a}_*$ has countable cofinality then we can find an increasing chain $\{A_n \mid n \in \omega\} \subseteq \wp(\mathcal{A})$ such that $\mathcal{A} = \bigcup_{n \in \omega} \mathcal{A}_n$ and $|\mathcal{A}_n| < \mathfrak{a}_*$ for every $n \in \omega$.

(1) Since $|\mathcal{A}_0| < \mathfrak{a}_*$ then we can find $B_0 \in \mathcal{A}_0^\perp$ such that $D_0 = \{n : |C_n \cap B_0| = \omega\}$ is infinite. Let $m_0 = min(D_0)$.
(2) Let $\mathcal{A}_1' = \mathcal{A}_1 \restriction B_0$ since $|\mathcal{A}_1'| < \mathfrak{a}_*$ then we can find $B_1 \subseteq B_0$ such that $B_1 \in \mathcal{A}_1^\perp$ and $D_1 = \{n > m_0 \mid |(C_n \cap B_0) \cap B_1| = \omega\}$ is infinite. Let $m_1 = min(D_1)$.

(3) Let $\mathcal{A}_2' = \mathcal{A}_2 \restriction B_1$ since $|\mathcal{A}_2'| < \mathfrak{a}_*$ then we can find $B_2 \subseteq B_1$ such that $B_2 \in \mathcal{A}_2^{\perp}$ and $D_2 = \{n > m_1 \mid |(C_n \cap B_1) \cap B_2| = \omega\}$ is infinite. Let $m_2 = min\,(D_2)$.

$\vdots$

Finally, let $X = \bigcup_{i\in\omega} (B_i \cap C_{m_i})$ then X intersects infinitely every C_{m_i} and $X \in \mathcal{A}^{\perp}$ which is a contradiction. ⊣

With the previous proposition we can conclude the following,

Corollary 3.12. *There is a model where* $\mathfrak{a}_* < \mathfrak{a}$.

Proof. In [4] Brendle constructed a model where $\mathfrak{a}$ has countable cofinality. By the previous proposition, it is clear that $\mathfrak{a}_* < \mathfrak{a}$ holds in that model. ⊣

§4. Sacks-indestructibility. For simplicity, call $\mathfrak{a}_{Sacks} = \mathfrak{a}^+\,(tr\,(\mathsf{ctbl}))$ which is the least size of an AD family $\mathscr{A} \subseteq tr\,(\mathsf{ctbl})$ such that for every $X \in tr\,(\mathsf{ctbl})^+$ there is $A \in \mathscr{A}$ such that $A \cap X$ is infinite. Recall that Sacks-indestructible MAD families exist generically if and only if $\mathfrak{a}_{Sacks} = \mathfrak{c}$. Likewise, call $\mathfrak{a}^+_{Miller} = \mathfrak{a}\,(tr\,(K_\sigma))$ and as before, Miller indestructible MAD families exist generically if and only if $\mathfrak{a}_{Miller} = \mathfrak{c}$. Since $\mathsf{ctbl} \subseteq K_\sigma \subseteq \mathcal{M}$ then $\mathfrak{b} = \mathfrak{a}\,(\mathsf{nwd})^+ \leq \mathfrak{a}_{Miller} \leq \mathfrak{a}_{Sacks}$.

The following result is an easy one but is very important.

Corollary 4.1. *If* $\mathfrak{a} \leq \mathfrak{a}_{Sacks}$ *then there is a Sacks-indestructible family.*

Proof. Assume $\mathfrak{a} \leq \mathfrak{a}_{Sacks}$. On the one hand if $\mathfrak{a} < \mathfrak{c}$, then any MAD family of size $\mathfrak{a}$ is Sacks indestructible. On the other hand, if $\mathfrak{a} = \mathfrak{c}$, then $\mathfrak{a}_{Sacks} = \mathfrak{c}$ and so there are also Sacks indestructible MAD families. ⊣

We do not know if $\mathfrak{a}$ can consistently be bigger than $\mathfrak{a}_{Sacks}$.

Problem 4.2. *Is* $\mathfrak{a} \leq \mathfrak{a}_{Sacks}$?

Given $s \in 2^{<\omega}$ we define $\langle s\rangle_{<\omega} = \{t \in 2^{<\omega} : s \sqsubseteq t\}$, it is clear that if $X \cap \langle s\rangle_{<\omega} \neq \emptyset$ for every $s \in 2^{<\omega}$ then $X \notin tr\,(\mathsf{ctbl})$. Let $\mathscr{BR}$ be the ideal of $2^{<\omega}$ generated by branches, in this way $\mathscr{BR}^{\perp}$ is the ideal of all well-founded subsets of $2^{<\omega}$, its elements are called *off-branch,* it is clear that $\mathscr{BR}^{\perp} \subseteq tr\,(\mathsf{ctbl})$. We have the following simpler characterization of $\mathsf{cov}^+\,(tr\,(\mathsf{ctbl}))$:

Lemma 4.3. $\mathsf{cov}^+\,(tr\,(\mathsf{ctbl}))$ *is the minimum size of a family* $\mathscr{B} \subseteq \mathscr{BR}^{\perp}$ *such that for every* $A \in tr\,(\mathsf{ctbl})^+$ *there is* $B \in \mathscr{B}$ *such that* $|A \cap B| = \omega$.

Proof. Call μ the minimum size a family $\mathscr{B} \subseteq \mathscr{BR}^{\perp}$ such that for every $A \in tr\,(\mathsf{ctbl})^+$ there is $B \in \mathscr{B}$ such that $|A \cap B| = \omega$. It is clear that $\mathsf{cov}^+\,(tr\,(\mathsf{ctbl})) \leq \mu$, we shall now prove the other inequality. We split the proof in two cases: if $\mathsf{cov}^+\,(tr\,(\mathsf{ctbl})) = \mathfrak{c}$ then there is nothing to prove, so assume $\mathsf{cov}^+\,(tr\,(\mathsf{ctbl}))$ is less than size of the continuum and let $\mathscr{B} \subseteq tr\,(\mathsf{ctbl})$ witness this fact. Since $2^{\omega\times\omega} \cong 2^{\omega}$ we may find a partition $\{[T_\alpha] : \alpha < \mathfrak{c}\}$ of 2^ω where each T_α is a Sacks tree. Since $\mathscr{B} \subseteq tr\,(\mathsf{ctbl})$ and has size less than $\mathfrak{c}$, then there is T_α such that $\pi\,(B) \cap [T_\alpha] = \emptyset$ for every $B \in \mathscr{B}$. The splitting nodes of T_α is isomorphic to $2^{<\omega}$ and for every $B \in \mathscr{B}$ it is the case that $B \cap T_\alpha$ is off-branch in T_α. ⊣

Using an analogous argument, we can prove the following.

Lemma 4.4. $\mathfrak{a}_{Sacks}$ *is the smallest size of an almost disjoint family* $\mathscr{A} \subseteq \mathscr{BR}^{\perp}$ *such that* $\mathscr{A} \cup \mathscr{A}^{\perp} \subseteq tr\,(\mathsf{ctbl})$.

We call a maximal AD family restricted to $\mathscr{BR}^{\perp}$ a *MOB* family. In [19] Leathrum defined $\mathfrak{o}$ as the smallest size of a MOB family and he showed that $\mathfrak{a} \leq \mathfrak{o}$.

LEMMA 4.5. $\mathfrak{a}_{Sacks} \leq \mathfrak{o}$.

PROOF. Let $\mathscr{O}$ be a MOB family of size $\mathfrak{o}$, then $\mathscr{O} \subseteq \mathscr{BR}^{\perp}$ and any $A \in \mathscr{O}^{\perp}$ must be contained in a union of finitely many branches, therefore $A \in tr\,(\mathsf{ctbl})$. ⊣

We have the following inequalities:

LEMMA 4.6. $\mathsf{cov}^{+}\,(tr\,(\mathsf{ctbl})) \leq min\left\{\mathfrak{a}_{Sacks}, \mathsf{cov}^{*}\left(\mathscr{BR}^{\perp}\right)\right\}$.

PROOF. The inequality $\mathsf{cov}^{+}\,(tr\,(\mathsf{ctbl})) \leq \mathfrak{a}_{Sacks}$ follows by definition, and for the other it is enough to recall that any $B \in tr\,(\mathsf{ctbl})^{+}$ contains an infinite antichain. ⊣

Now we compare them with some of the category related cardinal invariants.

PROPOSITION 4.7. $\mathsf{cov}\,(\mathcal{M}) \leq \mathsf{cov}^{+}\,(tr\,(\mathsf{ctbl}))$.

PROOF. Let $\kappa < \mathsf{cov}\,(\mathcal{M})$ and $\mathscr{A} = \{A_\alpha : \alpha \in \kappa\} \subseteq \mathscr{BR}^{\perp}$, we ought to find $B \in tr\,(\mathsf{ctbl})^{+}$ that is AD with $\mathscr{A}$. Let $\mathbb{P}$ be the partial order of all finite trees contained in $2^{<\omega}$ ordered by end extension. Obviously, $\mathbb{P}$ is isomorphic to Cohen forcing. Let $\dot{T}_{gen}$ be the name for the generic tree, clearly $\dot{T}_{gen}$ is forced to be a Sacks tree. For every $\alpha < \kappa$ define the set D_α of all $T \in \mathbb{P}$ such that if $s \in T$ is a maximal node, then $\langle s\rangle_{\leq\omega} \cap A_a = \emptyset$. It is straightforward to see that D_α is dense. Since $\kappa < \mathsf{cov}\,(\mathcal{M})$ then we can find, in the ground model, a filter that intersects every D_α and the result follows. ⊣

We recursively define $S = \{t_s : s \in 2^{<\omega}\}$ as follows:

We will now compare the Miller related invariants with the unbounding number. Recall that $\mathsf{cov}^{*}\,(tr\,(K_\sigma)) = \mathfrak{d}$.

PROPOSITION 4.8. $\mathsf{cov}^{+}\,(tr\,(K_\sigma)) = \mathfrak{b}$.

PROOF. We first show that $\mathsf{cov}^{+}\,(tr\,(K_\sigma)) \leq \mathfrak{b}$. Let $\{f_\alpha : \alpha \in \mathfrak{b}\}$ be an unbounded family of strictly increasing functions. For every $s \in \omega^{<\omega}$ and $\alpha < \mathfrak{b}$, we define $T_\alpha\,(s)$ as the downward closed subtree of the set consisting of nodes of the form $s^\frown \langle n\rangle^\frown t$, where $n \in \omega$ and $t \in \omega^{f_\alpha(n)}$.

Note that each $T_\alpha\,(s)$ is in $tr\,(K_\sigma)$, as $\pi(T_\alpha\,(s))$. Now let $A \subseteq 2^{<\omega}$ be such that $\pi\,(A)$ is unbounded. Find $s \in \omega^{<\omega}$ such that for infinitely many $n \in \omega$, $s^\frown \langle n\rangle$ has a successor in A. For each $n \in \omega$, let $g(n)$ be the minimum integer k so that there is a $t \in \omega^k$ with $s \frown \langle n\rangle t \in A$. Using that $\{f_\alpha : \alpha \in \mathfrak{b}\}$ is an unbounded family, we can find $\alpha < \mathfrak{b}$ so that $f_\alpha * \nleq g$. It follows that $A \cap T_\alpha\,(s)$ is infinite.

Now, let $\kappa < \mathfrak{b}$ and $\{A_\alpha : \alpha \in \kappa\} \subseteq tr\,(K_\sigma)$, we must show it is not a covering family. Since $\kappa < \mathfrak{b}$ we can find f that bounds $\pi\,(A_\alpha)$ for every $\alpha < \kappa$. Let T be the tree such that every branch though T is bigger or equal than f, we may assume $T = \omega^{<\omega}$.

For every $s \in \omega^{<\omega}$ choose $b_s \in \langle s\rangle$ and given $\alpha < \kappa$ define $f_\alpha : \omega^{<\omega} \longrightarrow \omega$ be such that if $m \geq f_\alpha\,(s)$ then $b_s \restriction m \notin A_\alpha$ (recall A_α is off-branch in $T = \omega^{<\omega}$) since κ is less than $\mathfrak{b}$, we may find $g : \omega^{<\omega} \longrightarrow \omega$ that dominates every f_α. We define a Miller tree S in the following way:

(1) The stem of S is $b_\emptyset \restriction g\,(\emptyset)$,
(2) If $s \in S$ is a splitting node, then $\mathrm{succ}_S\,(s) = \omega$,
(3) If s is a splitting node, then the next splitting node below $s^\frown \langle n\rangle$ is $b_{s^\frown\langle n\rangle} \restriction g\,(s^\frown \langle n\rangle)$.

Let $B = \text{split}(S)$ which obviously is $tr(K_\sigma)$ positive, we now claim $B \cap A_\alpha$ is finite for every α but this is clear. ⊣

Likewise, $\mathsf{cov}^+(\mathsf{nwd})$ is smaller than $\mathsf{cov}^*(\mathsf{nwd})$.

PROPOSITION 4.9. $\mathsf{cov}^+(\mathsf{nwd}) = \mathsf{add}(\mathcal{M})$.

PROOF. On the one hand, $\mathsf{cov}^+(\mathsf{nwd}) \leq \mathsf{cov}^*(\mathsf{nwd}) = \mathsf{cov}(\mathcal{M})$ and on the other hand, $\mathsf{cov}^+(\mathsf{nwd}) \leq \mathsf{cov}^+(K_\sigma) = \mathfrak{b}$. Now we proceed to prove $\mathsf{add}(\mathcal{M}) \leq \mathsf{cov}^+(\mathsf{nwd})$. Let $\kappa < \mathsf{add}(\mathcal{M})$ and $\{N_\alpha : \alpha \in \kappa\} \subseteq \mathsf{nwd}$. Let $\{U_n : n \in \omega\}$ be a base for $\mathbb{Q}$ and note that since $\kappa < \mathsf{cov}^*(\mathsf{nwd})$, then we may find an infinite $B_n \subseteq U_n$ almost disjoint from every N_α. Define $h_\alpha : \omega \longrightarrow [\mathbb{Q}]^{<\omega}$ where $h_\alpha(n) = B_n \cap A_\alpha$. Since $\kappa < \mathfrak{b}$ then there is $g : \omega \longrightarrow [\mathbb{Q}]^{<\omega}$ such that for every α it is the case that $h_\alpha(n) \subseteq g(n)$ for almost all $n \in \omega$, we may further assume that $g(n) \subseteq B_n$. Define $B = \bigcup_{n\in\omega}(B_n \setminus g(n))$ then B is dense and almost disjoint with each A_α. ⊣

In [13] Kamburelis and Węglorz introduced the following definitions,

DEFINITION 4.10. (1) $\mathfrak{s}(\mathscr{B}_0)$ is the smallest size of a family of open sets $\mathscr{U} \subseteq \mathscr{P}(2^\omega)$ such that for every infinite antichain $\{s_n : n \in \omega\} \subseteq 2^{<\omega}$ there is $U \in \mathscr{U}$ such that both sets $\{n : \langle s_n \rangle \subseteq U\}$ and $\{n : \langle s_n \rangle \cap U = \emptyset\}$ are infinite.

(2) Given $x \in 2^\omega$ and $n \in \omega$, let $r(x, n)$ be the sequence of length $n+1$ that agrees with x in the first n places but disagrees in the last one.

(3) Let $x \in 2^\omega$, $A \in [\omega]^\omega$, and $U \subseteq 2^\omega$ an open set. We say U *separates* (x, A) if $x \notin U$ and there are infinitely many $n \in A$ such that $\langle r(x,n) \rangle \subseteq U$.

(4) $\mathfrak{sep}$ is the smallest size of a family of open sets $\mathscr{U}$ such that for every (x, A) there is $U \in \mathscr{U}$ that separates (x, A).

It was later proved by Brendle in [3] that the two previous invariants are actually equal.

PROPOSITION 4.11. $\mathsf{cov}^*\left(\mathscr{BR}^\perp\right) = \mathfrak{sep}$.

PROOF. We first show that $\mathfrak{sep} \leq \mathsf{cov}^*\left(\mathscr{BR}^\perp\right)$. Let $\mathscr{B} \subseteq \mathscr{BR}^\perp$ be a witness for $\mathsf{cov}^*\left(\mathscr{BR}^\perp\right)$, we might assume it is closed under finite changes. For every $B \in \mathscr{B}$, let $\mathscr{U}_B = \bigcup\{\langle s \rangle : s \in B\}$. We will show that $\{\mathscr{U}_B : B \in \mathscr{B}\}$ witness $\mathfrak{sep}$. Let $x \in 2^\omega$, $A \in [\omega]^\omega$ and define the off-branch family $Y = \{r(x,n) : n \in A\}$ then find $B \in \mathscr{B}$ such that $B \cap Y$ is infinite. Since B is off-branch, by taking a finite subset of it we may assume no restriction of x is in B, it then follows that $\mathscr{U}_B$ separates (x, A).

We will now show $\mathsf{cov}^*\left(\mathscr{BR}^\perp\right) \leq \mathfrak{s}(\mathscr{B}_0)$. Let $\left\{U_\beta : \beta < \mathfrak{s}(\mathscr{B}_0)\right\}$ be a witness for $\mathfrak{s}(\mathscr{B}_0)$ and $\{f_\alpha : \alpha < \mathfrak{b}\}$ be an unbounded set of functions where each $f_\alpha : \omega \longrightarrow [2^{<\omega}]^{<\omega}$ and if $n < m$ then $f_\alpha(n) \subseteq f_\alpha(m)$. For every $\beta < \mathfrak{s}(\mathscr{B}_0)$, let $A_\beta = \left\{s_n^\beta : n \in \omega\right\} \subseteq 2^{<\omega}$ be the set of all minimal nodes of $\left\{s : \langle s \rangle \subseteq U_\beta\right\}$, note that they form an antichain. For every $\alpha < \mathfrak{b}$ and $\beta < \mathfrak{s}(\mathscr{B}_0)$, define $B(\alpha, \beta) = A_\beta \cup \bigcup\limits_{n\in\omega}\left(f_\alpha(n) \cap \left\langle s_n^\beta \right\rangle_{<\omega}\right)$, observe that this is an off-branch set. We will show that for every infinite off branch Y there are $\alpha < \mathfrak{b}$ and $\beta < \mathfrak{s}(\mathscr{B}_0)$ such that $B(\alpha, \beta)$ intersects Y infinitely.

Let $A \subseteq Y$ be an infinite antichain, first find $\beta < \mathfrak{s}(\mathscr{B}_0)$ such that the set $X = \left\{t \in A \colon \langle t\rangle \subseteq U_\beta\right\}$ is infinite. Define $g : \omega \longrightarrow [2^{<\omega}]^{<\omega}$ such that for every $n \in \omega$ there is $m > n$ such that there is $t \in A$ and t extends s^{β}_{m}. Let $\alpha < \mathfrak{b}$ such that $g(n) \subseteq f_\alpha(n)$ for infinitely many $n \in \omega$. It is then clear that $B(\alpha, \beta)$ intersects A infinitely. Finally since $\mathfrak{b} \leq \mathfrak{s}(\mathscr{B}_0)$ (see [3]) we get the desired result. ⊣

It then follows by a result of Kamburelis and Węglorz (see [13]) that $\mathrm{cov}^*\left(\mathscr{BR}^{\perp}\right) \leq \mathrm{cof}(\mathcal{N})$.

§5. +-Ramsey MAD families. Let $\mathscr{I}$ be an ideal, we say $T \subseteq \omega^{<\omega}$ is $\mathscr{I}^+$-*branching* if for every $s \in T$, the set $\mathrm{succ}_T(s) = \{n \colon s^\frown\langle n\rangle\} \in \mathscr{I}^+$. We say $\mathscr{I}$ is $+$-*Ramsey* if for every $\mathscr{I}^+$-branching tree T, there is $b \in [T]$ such that $\mathrm{ran}(b) \in \mathscr{I}^+$. An AD family $\mathscr{A}$ is called +-Ramsey if $\mathscr{I}(\mathscr{A})$ is +-Ramsey. The following was introduced in [9].

Definition 5.1. $\mathfrak{ra}$ is the minimum size of an AD family that is not +-Ramsey.

In respect to the generic existence of +-Ramsey families, we have the following:

Proposition 5.2. $\mathfrak{ra} = \mathfrak{c}$ *if and only if* +-*Ramsey families exist generically.*

Proof. First assume $\mathfrak{ra} = \mathfrak{c}$ and let $\mathscr{A}$ be an AD family of size less than the continuum. Enumerate $\{T_\alpha \colon \alpha < \mathfrak{c}\}$ all the trees in $\omega^{<\omega}$. Recursively, we shall construct a sequence $\langle \mathscr{A}_\alpha \colon \alpha < \mathfrak{c}\rangle$ of AD families such that:

(1) $\mathscr{A}_0 = \mathscr{A}$,
(2) If $\alpha < \beta$ then $\mathscr{A}_\alpha \subseteq \mathscr{A}_\beta$ and if γ is limit then $\mathscr{A}_\gamma = \bigcup_{\delta<\gamma} \mathscr{A}_\delta$,
(3) Every $\mathscr{A}_\alpha$ has size less than $\mathfrak{c}$,
(4) For every $\alpha < \mathfrak{c}$ either T_α is not a $\mathscr{I}^+(\mathscr{A}_{\alpha+1})$-branching tree or there is $b \in [T_\alpha]$ such that $\mathrm{ran}(b) \in \mathscr{I}(\mathscr{A}_{\alpha+1})^{++}$.

It is clear that if the construction can be carried out, we just extend $\bigcup_{\alpha<\mathfrak{c}} \mathscr{A}_\alpha$ to a MAD family and this will be a +-Ramsey MAD family. Assume $\mathscr{A}_\alpha$ has been defined, we will see how to define $\mathscr{A}_{\alpha+1}$. First consider the case where there is $s \in T_\alpha$ such that $\mathrm{succ}_{T_\alpha}(s) \notin \mathscr{I}(\mathscr{A}_\alpha)^{++}$. If $\mathrm{succ}_{T_\alpha}(s) \in \mathscr{I}(\mathscr{A}_\alpha)$ then we just define $\mathscr{A}_{\alpha+1} = \mathscr{A}_\alpha$ otherwise we can find an infinite $A \subseteq \mathrm{succ}_{T_\alpha}(s)$ that is AD with $\mathscr{A}_\alpha$, so we just define $\mathscr{A}_{\alpha+1} = \mathscr{A}_\alpha \cup \{A\}$. Now assume $\mathrm{succ}_{T_\alpha}(s) \in \mathscr{I}(\mathscr{A}_\alpha)^{++}$ for every $s \in T_\alpha$. Since $\mathscr{A}_\alpha$ has size less than $\mathfrak{ra}$ then we know there is $b \in [T_\alpha]$ such that $\mathrm{ran}(b) \in \mathscr{I}(\mathscr{A}_\alpha)^+$. In case $\mathrm{ran}(b) \in \mathscr{I}(\mathscr{A}_\alpha)^{++}$ then we can just define $\mathscr{A}_{\alpha+1} = \mathscr{A}_\alpha$, in the other case as before, we can find pairwise disjoint $\{A_n \colon n \in \omega\} \subseteq [b]^\omega \cap \mathscr{A}_\alpha^+$ and let $\mathscr{A}_{\alpha+1} = \mathscr{A}_\alpha \cup \{A_n \colon n \in \omega\}$.

Now assume $\mathfrak{ra} \leq \mathfrak{c}$, let $\mathscr{A}$ be an non +-Ramsey AD family of size less than $\mathfrak{c}$. In this way, we know there is T a $\mathscr{I}(\mathscr{A}_\alpha)^+$ branching tree such that if $b \in [T]$ then $\mathrm{ran}(b) \in \mathscr{I}(\mathscr{A}_\alpha)$. ⊣

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The ultrafilter number and $\mathfrak{hm}$

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Abstract

The cardinal invariant $\mathfrak{hm}$ is defined as the minimum size of a family of $\mathsf{c}_{\mathsf{min}}$-monochromatic sets that cover 2^ω (where $\mathsf{c}_{\mathsf{min}}(x,y)$ is the parity of the biggest initial segment both x and y have in common). We prove that $\mathfrak{hm} = \omega_1$ holds in the Shelah's model of $\mathfrak{i} < \mathfrak{u}$ so the inequality $\mathfrak{hm} < \mathfrak{u}$ is consistent with the axioms of ZFC. This answers a question of Thilo Weinert. We prove that $\mathfrak{a} = \omega_1$ also holds in that model.

Introduction

In [3] Stefan Geshke, Menachem Kojman, Wieslaw Kubiś and Rene Schipperus defined the coloring $\mathsf{c}_{\mathsf{min}} : [2^\omega]^2 \longrightarrow 2$ given by $\mathsf{c}_{\mathsf{min}}(x,y) = 0$ if $\Delta(x,y)$ is even and $\mathsf{c}_{\mathsf{min}}(x,y) = 1$ in case $\Delta(x,y)$ is odd (where $\Delta(x,y)$ is the length of the largest initial segment that x and y have in common). They defined the cardinal invariant $\mathfrak{hm}$ as the smallest size of a family of $\mathsf{c}_{\mathsf{min}}$-monochromatic sets that covers 2^ω. Since every $\mathsf{c}_{\mathsf{min}}$-monochromatic set is nowhere dense, it clearly follows that $\mathsf{cov}(\mathcal{M}) \leq \mathfrak{hm}$[1]. However, the cardinal invariant $\mathfrak{hm}$ is much larger than $\mathsf{cov}(\mathcal{M})$:

Proposition 1 ([3] and [4]) *$\mathsf{cof}(\mathcal{N}), \mathfrak{c}^- \leq \mathfrak{hm}$ (where $\mathfrak{c}^- = \mathfrak{c}$ if $\mathfrak{c}$ is a limit cardinal and if $\mathfrak{c} = \kappa^+$ then $\mathfrak{c}^- = \kappa$).*

Therefore, $\mathfrak{hm}$ is bigger than all the cardinal invariants that appear in the Cichoń diagram. On the other hand, it is known that the inequality $\mathfrak{hm} < \mathfrak{c}$ is consistent; in fact, it holds in the Sacks model ([3]) and in the Miller lite model ([4]). It is interesting to compare it with the other cardinal invariants of the continuum. The following is an interesting open problem of Stefan Geshke:

Problem 2 (Geshke) *Is the inequality $\mathfrak{hm} < \mathfrak{r}$ consistent?*

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[1] By $\mathsf{cov}(\mathcal{M})$ we denote the smallest family of meager sets that are needed to cover 2^ω and by $\mathsf{cof}(\mathcal{N})$ we denote the cofinality of the ideal of Lebesgue-null subsets of 2^ω.

Where $\mathfrak{r}$ denotes the smallest size of a *reaping family*, i.e. the smallest size of a family $\mathcal{R} \subseteq [\omega]^{\omega}$ such that for every $A \in [\omega]^{\omega}$, there is $R \in \mathcal{R}$ such that either $R \subseteq^* A$ or $R \subseteq^* \omega \setminus A$. This problem is still open. Thilo Weinert made some interesting advances with respect to the question in his thesis (and it was after reading the first chapters of his thesis that the author became interested on the topic). In [10], Weinert asked the following:

Problem 3 (Weinert) *Is the inequality* $\mathfrak{hm} < \mathfrak{u}$ *consistent?*

In here, $\mathfrak{u}$ denotes the *ultrafilter number*, the smallest size of a base of an (non-principal) ultrafilter in ω. In this note, we will provide a positive answer to the question. Since $\mathfrak{r} \leq \mathfrak{u}$ our solution to the problem of Weinert, may be viewed as a partial solution to the problem of Geshke. The reader may consult [3], [5] and [4] to learn more about $\mathfrak{hm}$.

In [8] Saharon Shelah built a model of $\mathfrak{i} < \mathfrak{u}$ and we will prove that $\mathfrak{hm} < \mathfrak{u}$ holds in that model too. A variant of the forcing of Shelah was recently used in [2] by David Chodounský, Vera Ficher and Jan Grebík in order to show that the inequality $\mathfrak{f} < \mathfrak{u}$ is consistent ($\mathfrak{f}$ is the *free sequence number* introduced by Donald Monk, the reader may consult the interesting paper [2] for the definition of $\mathfrak{f}$ and will not be used in here).

Recall that a family $\mathcal{A} \subseteq [\omega]^{\omega}$ is an *almost disjoint family* (AD) if the intersection of any two of its elements is finite and $\mathcal{A}$ is MAD if it is maximal with this property. The *almost disjointness number* $\mathfrak{a}$ is defined as the smallest size of a MAD family. In the last part of the paper, we will prove that $\mathfrak{a} = \omega_1$ holds in the model of Shelah. We would like to point out that the following problem of [10] is still open:

Problem 4 (Weinert) *Is the inequality* $\mathfrak{hm} < \mathfrak{a}$ *consistent?*

This problem seems very hard since the only known method to construct a models of $\mathsf{cof}(\mathcal{N}) < \mathfrak{a}$ is with the aid of templates (see [1] and [9]) but this method does not seem to help with the question.

Our notation is mostly standard. A tree $T \subseteq X^{<\omega}$ is a set closed under taking subsequences and the *set of branches of* T (denoted by $[T]$) is the set $\{f \in X^{\omega} \mid \forall n \in \omega\, (f \upharpoonright n \in T)\}$. If $s, t \in X^{<\omega}$, by $s^\frown t$ we denote the concatenation of s and t, if $x \in X$, we will often write $s^\frown x$ instead of $s^\frown \langle x \rangle$. Given a tree $T \subseteq X^{<\omega}$ and $s \in T$, define $suc_T(s) = \{x \in X \mid s^\frown x \in T\}$. Given $n \in \omega$, the *n-level of* T is defined as $T_n = \{s \in T \mid |s| = n\}$.

Shelah's forcing with respect to an ultrafilter

Given a partial order $\mathbb{P}$ and an ultrafilter $\mathcal{U}$, we say that $\mathbb{P}$ *ultradestroys* $\mathcal{U}$ if $\mathbb{P}$ forces that $\mathcal{U}$ is no longer the base for an ultrafilter. In [8] Shelah designed a forcing to ultradestroy any given ultrafilter with causing "minimal damage" to the ground model. Since this forcing is not very well-known, we will review its basic properties. The results of section are mostly due to Shelah. The partial order may seem difficult at first, so for the convenience of the reader, after defining the partial order, we will include a very informal description of it, the forcing resembles the conflicts in organizing a party for infinitely many guests that may love or hate each other.

Definition 5 *Let $\mathcal{U}$ be an ultrafilter. A set $E = \{E_n \mid n \in \omega\} \subseteq \wp(\omega)$ is called an $\mathcal{U}$-*partition *if E satisfy the following properties:*

1. *E is a pairwise disjoint family of elements of $\mathcal{U}^*$ (where $\mathcal{U}^*$ denotes the dual ideal of $\mathcal{U}$ i.e. $\mathcal{U}^* = \{\omega \setminus A \mid A \in \mathcal{U}\}$).*
2. *$dom(E) = \bigcup_{n\in\omega} E_n \in \mathcal{U}$.*

In other words, E is an $\mathcal{U}$-partition if E is a partition of some element in $\mathcal{U}$ such that all of its classes are in $\mathcal{U}^*$. Given $a \in dom(E)$ we define $[a]_E$ as the unique $E_n \in E$ such that $a \in E_n$. We define $A_E = \{a_E(n) \mid n \in \omega\}$ where $a_E(n) = min(E_n)$ and this set will be called the *leaders of* E.

Definition 6 *Let E and E' be two $\mathcal{U}$-partitions, we say that $E' <_{part} E$ if every E'-class is the union of E-classes.*

Equivalently, E' can be constructed by throwing away entire classes of E and merging (i.e. taking unions) of some classes of E (obviously, making sure that the new classes are elements of $\mathcal{U}^*$). Note that if $E' <_{part} E$ then $dom(E') \subseteq dom(E)$ and $A_{E'} \subseteq A_E$.

Definition 7 *Let $\mathcal{U}$ be an ultrafilter. The* Shelah Party forcing with respect to *$\mathcal{U}$ (denoted by $\mathbb{S}(\mathcal{U})$) is the set of all $p = (E_p, H_p, Love_p, Hate_p)$ with the following properties:*

1. *E_p is a $\mathcal{U}$-partition (let $A_p = A_{E_p}$, $a_p(n) = a_{E_p}(n)$ and $[a_p(n)]_p = [a_p(n)]_{E_p}$).*
2. *$Love_p = \{Love_p(a_p(n)) \mid n \in \omega\}$ and $Hate_p = \{Hate_p(a_p(n)) \mid n \in \omega\}$.*
3. *If $n \in \omega$ then $\{Love_p(a_p(n)), Hate_p(a_p(n))\}$ is a partition of $[a_p(n)]_p$ and $a_p(n) \in Love_p(a_p(n))$.*
4. *$H_p = \left\{H_p^m \mid m \notin dom(E_p)\right\}$.*

5. *If* $m \notin dom(E_p)$ *then* $H_p^m : 2^{A_p \cap m} \longrightarrow 2.$

Given $p, q \in \mathbb{S}(\mathcal{U})$, *define* $p \leq q$ *if the following holds:*

1. $E_p <_{part} E_q$ *(so* $A_p \subseteq A_q$*).*
2. *If* $a \in A_p$ *(so* $a \in A_q$ *and its class was not thrown away) then* $Love_q(a) \subseteq Love_p(a)$ *and* $Hate_q(a) \subseteq Hate_p(a)$.
3. *If* $a \in A_q$ *and there is* $b \in A_p$ *such that* $b < a$ *and* $a \in [b]_p$ *(so the class of* a *was merged with a previous class) then we have the following:*
   (a) *If* $a \in Love_p(b)$ *then* $Love_q(a) \subseteq Love_p(b)$ *and* $Hate_q(a) \subseteq Hate_p(b)$.
   (b) *If* $a \in Hate_p(b)$ *then* $Love_q(a) \subseteq Hate_p(b)$ *and* $Hate_q(a) \subseteq Love_p(b)$.
4. *If* $a \in A_q \setminus dom(E_p)$ *and* $n \in [a]_q$ *then* $H_p^n : 2^{A_p \cap n} \longrightarrow 2$ *is defined as follows:*
   (a) H_p^a *may be any function (with domain* $2^{A_p \cap n}$ *and codomain* 2*).*
   (b) *If* $f \in 2^{A_p \cap n}$ *and* $n \in Love_q(a)$ *then* $H_p^n(f) = H_p^a(f \upharpoonright a)$ *and if* $n \in Hate_q(a)$ *then* $H_p^n(f) = 1 - H_p^a(f \upharpoonright a)$.
5. *If* $n \notin dom(E_q)$ *and* $f \in 2^{A_p \cap n}$ *then*
   $H_p^n(f) = H_q^n\left(f \cup \left\{\left(a, H_p^a(f \upharpoonright a)\right) \mid a \in A_q \wedge a < n\right\}\right).$

It is hard to understand all of this conditions at first, so we offer an intuitive and informal meaning of the forcing. Imagine we are throwing a big party at the Hilbet Hotel and every natural number is invited (which of course does not mean all of them will show up). At the night of the party, each number will decide in order if they will go to the party or not (each number is aware of the decision of the previous numbers before her). So when asked n if she is going to the party or not, she will see which numbers before her are going and then she will decide either to assist or not. A condition $p = (E_p, H_p, Love_p, Hate_p) \in \mathbb{S}(\mathcal{U})$ offer us the following information about who will be assisting to the party:

1. If $a \in A_p$ and $m \in [a]_p$, then we have the following:
   (a) If $m \in Love_p(a)$ then m is saying "I will go to the party if and only if a is going".
   (b) If $m \in Hate_p(a)$ then m is saying "I will go to the party if and only if a is not going".
2. If $n \notin dom(E_p)$ and $f \in 2^{A_p \cap n}$ then:

(a) In case $H_p^n(f) = 1$ then the number n is saying "if the set of leaders before me that are going to the party is $f^{-1}(\{1\})$ then I will go to the party".

(b) In case $H_p^n(f) = 0$ then the number n is saying "if the set of leaders before me that are going to the party is $f^{-1}(\{1\})$ then I will not go to the party".

Note that the condition p offers no information of whether the leaders will go to the party. Note that if $n \notin dom(E_p)$ and H_p^n is the constant 1 function, then n has decided to assist the party while if it is the constant 0 function, then she has decided to skip it. If q is also a condition on $\mathbb{S}(\mathcal{U})$ then $q \leq p$ just means $E_q <_{part} E_p$ and that the previous promises are kept.

Definition 8 *Let $G \subseteq \mathbb{S}(\mathcal{U})$ be a generic filter. In $V[G]$ we define* the generic real x_{gen} *as the set of all $n \in \omega$ such that there is $p \in G$ for which $n \notin dom(E_p)$ and H_p^n is the constant 1 function.*

In the informal description, the generic real is the set of the numbers that assisted to the party. Recall that if $p \in \mathbb{S}(\mathcal{U})$ and $n \notin dom(E_p)$ then the domain of H_p^n is $2^{A_p \cap n}$. Since we can always throw away a given class, for every $p \in \mathbb{S}(\mathcal{U})$ and for every $n \in \omega$, there is $q \leq p$ such that $n \notin dom(E_q)$ and H_q^n is a constant function (this is the case if $A_q \cap n$ is empty). The following lemma follows from the definitions:

Lemma 9 ([8]) *Let $p \in \mathbb{S}(\mathcal{U})$, $a \in A_p$ and $n \notin dom(E_p)$.*

1. *$p \Vdash$ "$(Love_p(a) \subseteq \dot{x}_{gen}) \vee (Love_p(a) \cap \dot{x}_{gen} = \emptyset)$".*
2. *$p \Vdash$ "$(Hate_p(a) \subseteq \dot{x}_{gen}) \vee (Hate_p(a) \cap \dot{x}_{gen} = \emptyset)$".*
3. *There are $q_0, q_1 \leq p$ such that the following holds: $q_0 \Vdash$ "$Love_p(a) \subseteq \dot{x}_{gen}$" and $q_1 \Vdash$ "$Love_p(a) \cap \dot{x}_{gen} = \emptyset$".*
4. *There are $q_0, q_1 \leq p$ such that the following holds: $q_0 \Vdash$ "$Hate_p(a) \subseteq \dot{x}_{gen}$" and $q_1 \Vdash$ "$Hate_p(a) \cap \dot{x}_{gen} = \emptyset$".*
5. *H_p^n is the constant 1 function if and only if $p \Vdash$ "$n \in \dot{x}_{gen}$".*
6. *H_p^n is the constant 0 function if and only if $p \Vdash$ "$n \notin \dot{x}_{gen}$".*

Now it is possible to conclude the following:

Lemma 10 ([8]) $\mathbb{S}(\mathcal{U})$ *ultradestroys $\mathcal{U}$, in fact, both $\dot{x}_{gen}$ and $\omega \setminus \dot{x}_{gen}$ are forced to intersect infinitely every element of $\mathcal{U}$.*

Proof. We will prove that $\dot{x}_{gen}$ is forced to intersect every element of $\mathcal{U}$, the proof for $\omega \setminus \dot{x}_{gen}$ is similar. Let $B \in \mathcal{U}$, $p \in \mathbb{S}(\mathcal{U})$ and $n \in \omega$. It is enough to prove that there is $q \leq p$ such that $q \Vdash$ "$B \cap \dot{x}_{gen} \nsubseteq n$". Since $dom(E_p) \in \mathcal{U}$, there is $m > n$ such that $m \in B \cap dom(E_p)$. Since $m \in dom(E_p)$, we know there is a (unique) $a \in A_p$ such that $m \in [a]_p$. Define a condition q with the following properties:

1. $q \leq p$.
2. $dom(E_q) = dom(E_p) \setminus [a]_p$.
3. If $b \in dom(E_p) \setminus [a]_p$ then $[b]_p = [b]_q$.
4. If $m \in Love_p(a)$ then H_q^a is the constant 1 function, and if $m \in Hate_p(a)$, then H_q^a is the constant 0 function.

It is easy to see that $q \Vdash$ "$m \in B \cap \dot{x}_{gen}$", so we are done. ■

The following definitions will be frequently used:

Definition 11 *Let $p, q \in \mathbb{S}(\mathcal{U})$ and $n \in \omega$. We define the following orders on $\mathbb{S}(\mathcal{U})$:*

1. *$p \leq_n q$ if $p \leq q$ and $[a_i(p)]_p = [a_i(q)]_q$ for every $i \leq n$.*
2. *$p \leq_n^* q$ if $p \leq_{n-1} q$ and $a_n(p) = a_n(q)$.*[2]
3. *$p \leq_n^{**} q$ if $p \leq_n^* q$ and $dom(E_p) = dom(E_q)$.*

In other words, $p \leq_n q$ if p extends q and the first n-classes of E_q do not get thrown away nor they merge with other classes. Meanwhile, $p \leq_n^* q$ if $p \leq_{n-1} q$ and the n-class of E_q was not thrown away (but it could have been merge with other later classes). In a similar fashion, $p \leq_n^{**} q$ means that $p \leq_n^* q$ and no class was thrown away. Note that $p \leq_0^* q$ means that p extends q and the 0-class was not thrown away.

Definition 12 *Let $q, r, q', r' \in \mathbb{S}(\mathcal{U})$ and $n \in \omega$. We say $\langle q, r, q', r' \rangle$ is an n-*nice sequence* if the following holds:*

1. *$q \perp r$ (i.e. q and r are incompatible).*
2. *If $n < i$ then $[a_q(i)]_q = [a_r(i)]_r$.*
3. *$q' \leq_n^* q$ and $r' \leq_n^* r$.*
4. *If $n + 1 < i$ then $[a_{q'}(i)]_{q'} = [a_{r'}(i)]_{r'}$.*

[2]By $p \leq_{-1} q$ we simply mean $p \leq q$.

5. $dom\left(E_q\right) \setminus dom\left(E_{q'}\right) \subseteq \left[a_{r'}\left(n\right)\right]_{r'} \cup \left[a_{r'}\left(n+1\right)\right]_{r'}$ *and* $dom\left(E_r\right) \setminus dom\left(E_{r'}\right) \subseteq \left[a_{q'}\left(n\right)\right]_{q'} \cup \left[a_{q'}\left(n+1\right)\right]_{q'}.$

We will also need the following notions:

Definition 13 *Let $p \in \mathbb{S}\left(\mathcal{U}\right)$, $D \subseteq \mathbb{S}\left(\mathcal{U}\right)$, $n \in \omega$ and $D \subseteq \mathbb{S}\left(\mathcal{U}\right)$.*

1. *D is $\leq_n$*-dense below *p if for every $q \leq_n p$ there is $r \leq_n q$ such that $r \in D$.*
2. *D is $\leq_n$*-open below *p if for every r such that $r \leq_n p$ and $r \in D$ then $q \in D$ whenever $q \leq_n r$.*
3. *D is $\leq_n$*-open dense below *p if it is both $\leq_n$-open and $\leq_n$-dense below p.*
4. *The same definitions apply for $\leq_n^*$.*

The following lemma is the base for several constructions in the paper:

Lemma 14 *Let $n \in \omega$ and q, r be two incompatible conditions such that if $n < i$ then $\left[a_q\left(i\right)\right]_q = \left[a_r\left(i\right)\right]_r$. Let D_q be an $\leq_{n+1}^*$-open, $\leq_n^*$-dense set below q and D_r be an $\leq_{n+1}^*$-open, $\leq_n^*$-dense set below r. Then there are $q' \in D_q$ and $r' \in D_r$ such that $\langle q, r, q', r' \rangle$ is an n-nice sequence.*

Proof. We first find $q_1 \leq_n^* q$ such that $q_1 \in D_q$. Let $S = \left(dom\left(q\right) \setminus dom\left(q_1\right)\right) \cup \left(\left[a_{q_1}\left(n\right)\right]_{q_1} \setminus \left[a_q\left(n\right)\right]_q\right)$ and note it is an element of $\mathcal{U}^*$. Let r_1 be any condition such that:

1. $r_1 \leq_n^* r$.
2. $\left[a_{r_1}\left(n\right)\right]_{r_1} = \left[a_r\left(n\right)\right]_r \cup S$.
3. $dom\left(E_{r_1}\right) = dom\left(E_r\right).$
4. E_{r_1} and E_{q_1} are equal on $\bigcup\{\left[a_{q_1}\left(i\right)\right]_{q_1} \mid n < i\}$ (recall that if $n < i$ then $\left[a_q\left(i\right)\right]_q = \left[a_r\left(i\right)\right]_r$).

Now we find $r_2 \leq_n^* r_1$ (so $r_2 \leq_n^* r$) such that $r_2 \in D_r$. Let $Z = (dom\left(r_1\right) \setminus dom\left(r_2\right)) \cup \left(\left[a_{r_2}\left(n\right)\right]_{r_2} \setminus \left[a_{r_1}\left(n\right)\right]_{r_1}\right)$ which is also an element of $\mathcal{U}^*$. Let q_2 be any condition such that:

1. $q_2 \leq_{n+1}^* q_1$.
2. $\left[a_{q_2}\left(n+1\right)\right]_{q_2} = \left[a_{q_1}\left(n+1\right)\right]_{q_1} \cup Z \cup \left[a_{r_2}\left(n+1\right)\right]_{r_2}.$
3. $dom\left(E_{q_2}\right) = dom\left(E_{q_1}\right).$
4. E_{q_2} and E_{r_2} are equal on $\bigcup\{\left[a_{r_2}\left(i\right)\right]_{r_2} \mid n+1 < i\}$.

Since $q_2 \leq^*_{n+1} q_1$ and D_q is $\leq^*_{n+1}$-open, then $q_2 \in D_q$. It is clear that $\langle q, r, q_2, r_2 \rangle$ has the desired properties. ■

Recall the following notion,

Definition 15 *Let $\mathbb{P}$ be a partial order. We say $\left(\mathbb{P}, \langle \leq_n \rangle_{n \in \omega}\right)$ is* axiom A *if the following holds:*

1. *If $n \in \omega$ then $\leq_n$ is a partial order on $\mathbb{P}$.*
2. *If $p \leq_0 q$ then $p \leq q$.*
3. *If $p \leq_{n+1} q$ then $p \leq_n q$.*
4. *(Fusion property) if $\langle p_n \rangle_{n \in \omega}$ is a sequence such that $p_{n+1} \leq_{n+1} p_n$ then there is $p_\omega \in \mathbb{P}$ such that $p_\omega \leq_n p_n$ for every $n \in \omega$.*
5. *(Freezing property) if $p \in \mathbb{P}$, $n \in \omega$ and $\mathcal{A}$ is a maximal antichain below p, then there is $q \leq_n p$ such that $\{r \in \mathcal{A} \mid r \text{ is compatible with } p\}$ is countable.*

Obviously if $\left(\mathbb{P}, \langle \leq_n \rangle_{n \in \omega}\right)$ satisfy axiom A then $\mathbb{P}$ is proper. The axiom A structure is often very useful. Unfortunately, it does not seem that $\mathbb{S}(\mathcal{U})$ has an axiom A structure (note however, that $\mathbb{S}(\mathcal{U})$ is $< \omega_1$-proper by the results in [8], so by a theorem of Tetsuya Ishiu, ([7]) $\mathbb{S}(\mathcal{U})$ is forcing equivalent to a partial order with an axiom A structure, unfortunately, this does not seem to help us). The purpose of the following definitions and results, are to obtain a similar structure to the one of an axiom A forcing.

Definition 16 *We say $T = \langle q_i, r_i \rangle_{i < \omega}$ is a* 1-fusion sequence *if the following holds:*

1. *q_0 and r_0 are incomparable but $E_{q_0} = E_{r_0}$.*
2. *If $i < \omega$ then $\langle q_i, r_i, q_{i+1}, r_{i+1} \rangle$ is an i-nice sequence.*

We will say $T = \langle q_i, r_i \rangle_{i \leq n+1}$ is a 1-*finite fusion sequence* if it satisfy the previous points for every $i < n$.

Definition 17 *Let $\langle p_i \rangle_{i \in \omega} \subseteq \mathbb{S}(\mathcal{U})$ such that $p_{i+1} \leq^*_i p_i$ for every $i \in \omega$. We define* the limit of $\langle p_i \rangle_{i \in \omega}$ *as $Lim\left(\langle p_i \rangle_{i \in \omega}\right) = p = (E_p, H_p, Love_p, Hate_p)$ as follows:*

1. *$dom(E_p) = \bigcup\limits_{i < \omega} \left[a_{p_{i+1}}(i)\right]_{p_{i+1}}$.*
2. *$\left[a_p(i)\right]_{E_p} = \left[a_{p_{i+1}}(i)\right]_{p_{i+1}}$.*

3. $Love_p\left(a_p\left(i\right)\right) = Love_{p_{i+1}}\left(a_{p_{i+1}}\left(i\right)\right)$
4. $Hate_p\left(a_p\left(i\right)\right) = Hate_{p_{i+1}}\left(a_{p_{i+1}}\left(i\right)\right).$
5. If $m \notin dom\left(E_p\right)$ then $H_p^m = H_{p_m}^m.$

Note however, that $p = Lim\left(\langle p_i\rangle_{i\in\omega}\right)$ may not be a condition of $\mathbb{S}\left(\mathcal{U}\right)$ since it might be the case that $dom\left(E_p\right)$ is not an element of $\mathcal{U}$. However, if p is indeed a condition, then $p \leq_i^* p_i$ for every $i \in \omega$. The following result plays the role of a fusion sequence in an Axiom A forcing:

Lemma 18 *If $T = \langle q_i, r_i\rangle_{i<\omega}$ is a 1-fusion sequence in $\mathbb{S}\left(\mathcal{U}\right)$ then there is $\overline{p}$ that either is a lower bound of $\langle\, q_i\rangle_{i<\omega}$ or it is a lower bound of $\langle\, r_i\rangle_{i<\omega}$. In fact, either $Lim\left(\langle q_i\rangle_{i\in\omega}\right) \in \mathbb{S}\left(\mathcal{U}\right)$ or $Lim\left(\langle r_i\rangle_{i\in\omega}\right) \in \mathbb{S}\left(\mathcal{U}\right).$*

Proof. Assume $q = Lim\left(\langle q_i\rangle_{i\in\omega}\right) \notin \mathbb{S}\left(\mathcal{U}\right)$, we will that $r = Lim\left(\langle r_i\rangle_{i\in\omega}\right)$ is a condition in $\mathbb{S}\left(\mathcal{U}\right)$. In order to show this, we must first argue that $dom\left(E_{q_0}\right) = dom\left(E_q\right) \cup dom\left(E_r\right)$. If $n \in dom\left(E_{q_0}\right) \setminus dom\left(E_q\right)$ we then may find $i \in \omega$ such that $n \in dom\left(q_i\right) \setminus dom\left(q_{i+1}\right)$. Since $\langle q_i, q_{i+1}, r_i, r_{i+1}\rangle$ is i-nice then $n \in \left[a_{r_{i+1}}\left(i\right)\right]_{r_{i+1}} \cup \left[a_{r_{i+1}}\left(i+1\right)\right]_{r_{i+1}}$ so $n \in dom\left(E_r\right)$. Since $dom\left(E_{q_0}\right) \in \mathcal{U}$ and $\mathcal{U}$ is an ultrafilter, it must be the case that $dom\left(E_r\right) \in \mathcal{U}$. ■

In the above case, we would say that $Lim\left(\langle q_i\rangle_{i\in\omega}\right)$ (or $Lim\left(\langle r_i\rangle_{i\in\omega}\right)$ in case $Lim\left(\langle q_i\rangle_{i\in\omega}\right) \notin \mathbb{S}\left(\mathcal{U}\right)$) is a fusion for T. For the case of $\leq^{**}$the situation is simpler:

Lemma 19 *If $\langle p_i\rangle_{i\in\omega} \subseteq \mathbb{S}\left(\mathcal{U}\right)$ is a sequence such that $p_{i+1} \leq_i^{**} p_i$ for every $i \in \omega$ then there is $q \in \mathbb{S}\left(\mathcal{U}\right)$ such that $q \leq p_i$ for every $i \in \omega$.*

We will need the following,

Definition 20 *Let $p \in \mathbb{S}\left(\mathcal{U}\right)$, $n > 0$ and $h : n \longrightarrow 2$. We define $p\left[h\right]$ as the condition extending p with the following properties:*

1. $dom\left(E_{p[h]}\right) = dom\left(E_p\right) \setminus \bigcup\limits_{i<n} \left[a_p\left(i\right)\right]_p.$
2. If $m \in dom\left(E_{p[h]}\right)$ then $\left[m\right]_{p[h]} = \left[m\right]_p.$
3. If $a \in A_{p[h]}$ then $Love_{p[h]}\left(a\right) = Love_p\left(a\right)$ and $Hate_{p[h]}\left(a\right) = Hate_p\left(a\right).$
4. If $i < n$ then $H_{p[h]}^{a_p(i)}$ is the constant function with value $h\left(i\right).$

In other words, $p\left[h\right]$ is obtained by throwing out the first n-classes and tell their leaders to "follow h". It is not hard to see that for a fix n, the set $\{p\left[h\right] \mid h \in 2^n\}$ is a maximal finite antichain below p.

Definition 21 *Let $D \subseteq \mathbb{S}(\mathcal{U})$ be an open dense set below p. Define $\widetilde{D}(n) = \left\{q \leq_n p \mid \forall h \in 2^{n+1} (q[h] \in D)\right\}$.*

We now have the following:

Lemma 22 *If $D \subseteq \mathbb{S}(\mathcal{U})$ is an open dense set below p, then $\widetilde{D}(n)$ is $\leq_n$-open dense below p.*

Proof. We first show that $\widetilde{D}(n)$ is $\leq_n$-dense below p. Let $q \leq_n p$ and enumerate $2^{n+1} = \{h_i \mid i < k\}$, we can then recursively find a sequence $\langle q_i \rangle_{i<k+1}$ with the following properties:

1. $q_0 = q$.
2. $\langle q_i \rangle_{i<k+1}$ is $\leq_n$-decreasing.
3. $q_{i+1}[h_i] \in D$.

It is then easy to see that $q_{k+1} \in \widetilde{D}(n)$. Finally, $\widetilde{D}(n)$ is open since whenever $r \leq_n q$ then $r[h] \leq q[h]$ for every $h : n+1 \longrightarrow 2$. ■

Armed with the previous results, we can finally prove the following:

Corollary 23 ([8]) *If $\mathcal{U}$ is an ultrafilter, then $\mathbb{S}(\mathcal{U})$ is proper and has the Sacks property.*

Proof. We will prove that $\mathbb{S}(\mathcal{U})$ is proper. Let M be a countable elementary submodel of some $\mathsf{H}(\kappa)$ and $p \in M$. Let $\{D_n \mid n \in \omega\}$ enumerate all open dense subsets of $\mathbb{S}(\mathcal{U})$ that belong to M. In a similar way as before, it is easy to see that $\widetilde{D}(n)$ is $\leq^*_{n+1}$-open and $\leq^*_n$-dense. It is also clear that each $\widetilde{D}(n)$ is an element of M, so we can construct $T = \langle q_i, r_i \rangle_{i<\omega}$ a 1-fusion sequence with the following properties:

1. $T \subseteq M$.
2. $q_0, r_0 \leq p$.
3. $q_{i+1}, r_{i+1} \in \widetilde{D}(i)$.

It is easy to see that if q is a fusion of T, then q will be an $(M, \mathbb{S}(\mathcal{U}))$-generic condition extending p. The proof of the Sacks property is nearly identical. ■

Many of the arguments in this sections were based on the ones of [8]. Instead of our fusion sequences, Shelah uses games to prove the properness and the Sacks property. We decided to use the fusion sequences instead of games because it will be easier to work with sequences rather than with games when dealing with the iteration.

Forcing with $\mathbb{S}(\mathcal{U})$ preserves $\mathsf{c}_{\mathsf{min}}$-covering

It is easy to see that if $X \subseteq 2^{\omega}$ is $\mathsf{c}_{\mathsf{min}}$-monochromatic, then the closure of $\overline{X}$ is also $\mathsf{c}_{\mathsf{min}}$-monochromatic. In this way, $\mathfrak{hm}$ is the smallest size of a family of $\mathsf{c}_{\mathsf{min}}$-monochromatic closed sets that covers 2^{ω}. We will say that a tree $T \subseteq 2^{<\omega}$ is $\mathsf{c}_{\mathsf{min}}$*-monochromatic* if $[T]$ is $\mathsf{c}_{\mathsf{min}}$-monochromatic. Note that this means that either T only splits at odd levels or at even levels. We say that a forcing notion $\mathbb{P}$ *preserves* $\mathsf{c}_{\mathsf{min}}$*-covering* if for every $p \in \mathbb{P}$ and for every $\mathbb{P}$-name $\dot{x}$ for a real in 2^{ω}, there are $q \leq p$ and $T \in V$ a $\mathsf{c}_{\mathsf{min}}$-monochromatic tree such that $q \Vdash$ "$\dot{x} \in [T]$".

Definition 24 *Let $\mathbb{P}$ be a partial order, $p \in \mathbb{P}$ and $\dot{x}$ a $\mathbb{P}$-name for a new element of 2^{ω}. We define $\dot{x}[p] = \bigcup \{t \in 2^{<\omega} \mid p \Vdash \text{“}t \subseteq \dot{x}\text{”}\}$.*

Since $\dot{x}$ is forced to be a new real, then $\dot{x}[p] \in 2^{<\omega}$. Note that for every $p \in \mathbb{P}$ there are $q, r \leq p$ such that $\dot{x}[q]$ and $\dot{x}[r]$ are incompatible elements of $2^{<\omega}$. In this section, we will prove that Shelah's party forcing $\mathbb{S}(\mathcal{U})$ preserves $\mathsf{c}_{\mathsf{min}}$-covering. We will deal with the iteration in later sections. For the rest of this section, fix $\dot{x}$ a $\mathbb{S}(\mathcal{U})$-name for a new real.

Definition 25 *Let $p \in \mathbb{S}(\mathcal{U})$.*

1. *We say p is $\dot{x}$-*nice *if $p[h]$ determines $\dot{x} \restriction n$ for every $h : n \longrightarrow 2$.*
2. *Let $S_p(\dot{x}) = \bigcup \{\dot{x}[p[h]] \mid h \in 2^{<\omega}\}$.*

It is easy to see that if p is $\dot{x}$-nice, then $S_p(\dot{x})$ is a tree and $p \Vdash$ "$\dot{x} \in [S_p(\dot{x})]$". Recall that a tree $p \subseteq 2^{<\omega}$ is a *Sacks tree* if for for every $s \in p$ there is $t \in p$ such that $s \subseteq t$ and t is a *splitting node of* p (i.e. $t^\frown 0,\ t^\frown 1 \in p$). Note that since $\dot{x}$ is a new real, then $S_p(\dot{x})$ must be a Sacks tree.

Lemma 26 ([8]) *The set of all $\dot{x}$-nice conditions is a dense set.*

Proof. Let $p \in \mathbb{S}(\mathcal{U})$ and for each $n \in \omega$ let D_n be the set of all $q \leq_n^* p$ such that if $h : n+1 \longrightarrow 2$ then $q[h]$ determines $\dot{x} \restriction n$. It is easy to see that each D_n is both $\leq_{n+1}^*$-open and $\leq_n^*$-dense. We can obtain an $\dot{x}$-nice condition extending p using a fusion argument. ■

We now have the following lemma:

Lemma 27 *Let p be $\dot{x}$-nice and $n > 0$.*

1. *If $h : n \longrightarrow 2$ then there is $q \leq_n p$ such that $\dot{x}[q[h^\frown 0]]$ and $\dot{x}[q[h^\frown 1]]$ are incompatible (as nodes in the tree $2^{<\omega}$).*

2. *There is $r \leq_n p$ such that for that $\dot{x}[r[h^\frown 0]]$ and $\dot{x}[r[h^\frown 1]]$ are incompatible every $h : n \longrightarrow 2$*

Proof. Let $h : n \longrightarrow 2$ and since $S_p(\dot{x})$ is a Sacks tree, there must be $h^\frown 0 \subseteq g_0^0$, g_0^1 and $h^\frown 1 \subseteq g_1^0$, g_1^1 such that both $\dot{x}[p[g_0^0]]$ and $\dot{x}[p[g_0^1]]$ are incompatible and both $\dot{x}[p[g_1^0]]$ and $\dot{x}[p[g_1^1]]$ are incompatible. We may also assume there is m such that all g_0^0, g_0^1, g_1^0 and g_1^1 have m as its domain. There must be i, j such that $\dot{x}[p[g_0^i]]$ and $\dot{x}[p[g_1^j]]$ are incompatible, with out loosing generality, we may assume $i = j = 0$. We then define the condition $q \leq p$ with the following properties:

1. $E_q = E_p \setminus \bigcup\{[a_p(l)]_p \mid l \in (n, m)\}$.
2. If $l \in (n, m)$ and $f : 2^{A_p \cap a_p(l)} \longrightarrow 2$ then the following holds:
   (a) If $f(a_p(n)) = 0$ then $H_q^{a_p(l)}(f) = g_0^0(l)$.
   (b) If $f(a_p(n)) = 1$ then $H_q^{a_p(l)}(f) = g_1^0(l)$.

Note that $q[h^\frown 0] = p[g_0^0]$ and $q[h^\frown 1] = p[g_1^0]$, so the result immediately follows. The second part of the lemma is proved by applying iteratively the first part. ■

We need the following notion:

Definition 28 *We say an $\dot{x}$-nice condition p is* n-separating *if for all $h, g : n+1 \longrightarrow 2$ then $\dot{x}[p[h]]$ and $\dot{x}[p[g]]$ are incomparable whenever $h \neq g$. We say p is* ω-separating *if it is n-separating for every $n \in \omega$.*

Note that if p is n-separating and $q \leq_n^* p$ then q is n-separating too. This is because if $h : n+1 \longrightarrow 2$ then $q[h] \leq p[h]$.

Lemma 29 *If p is n-separating then $D = \{q \leq_n^* p \mid q$ is $(n+1)$-separating$\}$ is $\leq_{n+1}^*$-open and $\leq_n^*$-dense below p.*

Proof. Note that D is $\leq_{n+1}^*$-open by the previous remark, we will see it is also $\leq_n^*$-dense below p. Let $q \leq_n^* p$ and by the previous lemma, we can find $r \leq_{n+1} q$ such that for that $\dot{x}[r[h^\frown 0]]$ and $\dot{x}[r[h^\frown 1]]$ are incomparable for every $h : n+1 \longrightarrow 2$. Since p is n-separating then r is $(n+1)$-separating. ■

We can now conclude the following:

Corollary 30 *The set of all ω-separating conditions is open dense.*

Proof. Clearly this set is open, we will now show it is dense. Let $p \in \mathbb{S}(\mathcal{U})$ we will prove p has a ω-separating extension, note we may assume that p is $\dot{x}$-nice and 0-separating (by a previous lemma). We then recursively construct a 1-fusion sequence $T = \langle q_i, r_i \rangle_{i<\omega}$ with the following properties:

1. q_0 and r_0 are two incompatible extensions of p with $E_{q_0} = E_{r_0}$.
2. Both q_i, r_i are i-separating.

This can be done by the previous lemmas and is easy to see that any fusion of T will have the desired properties. ■

If $s, t \in 2^{<\omega}$ are two incomparable finite sequences, define $\mathsf{c}_{\mathsf{min}}(s,t)$ to be the parity of the length of the biggest initial segment shared both by s and t. Note that this is an abuse of notation since the domain of $\mathsf{c}_{\mathsf{min}}$ is $[2^\omega]^2$. We will now define the following:

Definition 31 *Let $p \in \mathbb{S}(\mathcal{U})$, $n \in \omega$ and $i < 2$. Then p is called (n,i)-faithful if $\mathsf{c}_{\mathsf{min}}(\dot{x}[p[h]], \dot{x}[p[g]]) = i$ for every $h, g \in 2^{n+1}$ such that $h \neq g$.*

Note that if p is (n,i)-faithful and $q \leq_n^* p$ then q is also (n,i)-faithful. We can finally prove the main result on this section:

Proposition 32 *If $\mathcal{U}$ is an ultrafilter, then $\mathbb{S}(\mathcal{U})$ preserves $\mathsf{c}_{\mathsf{min}}$-covering.*

Proof. Fix a condition p that is ω-separating. Given $n \in \omega$, $q \leq p$ and $h : n \longrightarrow 2$, let $I_q(h)$ be the union of all the classes $[a_q(m)]_q$ with the following properties:

1. $n < m$.
2. For every $g \in 2^m$ such that $h \subseteq g$ we have that $\mathsf{c}_{\mathsf{min}}(\dot{x}[q[g^\frown 0]], \dot{x}[q[g^\frown 1]]) = 1$.

We will now proceed by cases, first assume there are $q \leq p$ and $h \in 2^{<\omega}$ such that $I_q(h) \in \mathcal{U}$. Let $r \leq q[h]$ such that $dom(E_r) = I_q(h)$ and every H_r^l is a constant function for $l \in A_q \setminus A_r$. It is then easy to see that $S_r(\dot{x})$ is a 1-monochromatic tree and we are done.

Now we assume that $I_q(h) \notin \mathcal{U}$ for every $q \leq p$ and $h \in 2^{<\omega}$. Given $q \leq p$ and $n < \omega$ let $F_{n+1}(q) = \{r \leq_n^* q \mid r \text{ is } (n+1,0)\text{-faithful}\}$.

Claim 33 *If q is $(n,0)$-faithful then $F_{n+1}(q)$ is $\leq_{n+1}^*$-open and $\leq_n^*$-dense below q.*

It is clear that $F_{n+1}(q)$ is $\leq_{n+1}^*$-open below q, we will now prove it is also $\leq_n^*$-dense. Let $r \leq_n^* q$ we will extend r to a $(n+1,0)$-faithful condition, let $2^{n+1} = \{h_i \mid i < k\}$. We know $B = I_r(h_1) \cup ... \cup I_r(h_k) \in \mathcal{U}^*$ (since each $I_r(h_i) \in \mathcal{U}^*$ by hypothesis) and since B is the union of E_r-classes, then there is $m > n+1$ such that $[a_r(m)]_r \cap B = \emptyset$. Since $a_r(m) \notin I_r(h_i)$ then for every $i < k$ there is $g_i : m \longrightarrow 2$ extending h_i such that $\mathsf{c}_{\mathsf{min}}(\dot{x}[r[g_i^\frown 0]], \dot{x}[r[g_i^\frown 1]]) = 0$. We now define a condition r_1 with the following properties:

1. $r_1 \leq_n r$.
2. $dom\left(E_{r_1}\right) = dom\left(E_r\right) \setminus \bigcup\left\{\left[a_r\left(l\right)\right]_r \mid n < l < m\right\}$.
3. If $m \leq l$ then $\left[a_{r_1}\left(l\right)\right]_{r_1} = \left[a_r\left(l\right)\right]_r$.
4. If $n < l < m$ and $f \in 2^{A_{r_1} \cap a_r(l)}$ then $H_{r_1}^{a_r(l)}\left(f\right) = g_i\left(l\right)$ where $i < k$ is (the unique) such that $f\left(a_r\left(j\right)\right) = h_i\left(j\right)$ for every $j < n$.

It is now easy to see that r_1 is $(n+1, 0)$-faithful, this finishes the proof of the claim.

Continuing with the proof, pick any $m \in I_p\left(\emptyset\right)$, we know there is $g \in 2^m$ such that $\mathsf{c}_{\mathsf{min}}\left(\dot{x}\left[p\left[g^\frown 0\right]\right], \dot{x}\left[p\left[g^\frown 1\right]\right]\right) = 0$. With the aid of this g, we can obtain (in the same way as before) a $p_1 \leq p$ such that p_1 is $(0,0)$-faithful. We now recursively construct a 1-fusion sequence $T = \langle q_i, r_i \rangle_{i<\omega}$ with the following properties:

1. q_0 and r_0 are two incompatible extensions of p_1 with $E_{q_0} = E_{r_0}$.
2. Both q_i and r_i are $(i, 0)$-faithful.

Let $\overline{p}$ be a fusion of T. Then $\overline{p}$ is an extension of p and $S_{\overline{p}}\left(\dot{x}\right)$ is a 0-monochromatic tree. ■

The iteration of the Shelah forcing

Our next task is to prove that if we perform a countable support iteration $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha \leq \omega_2 \rangle$ such that for if $\alpha < \omega_2$ then $\mathbb{P}_\alpha \Vdash$ "$\dot{\mathbb{Q}}_\alpha = \mathbb{S}(\dot{\mathcal{U}}_\alpha)$" where $\dot{\mathcal{U}}_\alpha$ is a $\mathbb{P}_\alpha$-name for an ultrafilter, then $\mathbb{P}_{\omega_2}$ preserves $\mathsf{c}_{\mathsf{min}}$-covering. We do not know in general if the iteration of forcings that preserve $\mathsf{c}_{\mathsf{min}}$-covering preserves $\mathsf{c}_{\mathsf{min}}$-covering, but we will be able to prove it for our forcings. In this section, we will prove some technical lemmas regarding the iteration that will be useful in the following sections. The purpose of this section will be to extend our work with fusion sequences to every $\mathbb{P}_\alpha$. For the following sections, fix a countable support iteration $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha \leq \omega_2 \rangle$ as before.

Definition 34 *Let $\alpha \leq \omega_2$, $F \in [\alpha]^{<\omega}$, $\eta : F \longrightarrow \omega$ and $p, q \in \mathbb{P}_\alpha$.*

1. *Define $p \leq_{F,\eta} q$ if $p \leq q$ and for every $\beta \in F$ it is the case that $p \restriction \beta \Vdash$ "$p\left(\beta\right) \leq_{\eta(\beta)} q\left(\beta\right)$".*
2. *$p \leq^*_{F,\eta} q$ if $p \leq q$ and for every $\beta \in F$ it is the case that $p \restriction \beta \Vdash$ "$p\left(\beta\right) \leq^*_{\eta(\beta)} q\left(\beta\right)$".*
3. *If $\sigma \in \prod\limits_{\beta \in F} 2^{\eta(\beta)}$ we define the condition $p * \sigma \in \mathbb{P}_\alpha$ as follows:*

(a) *If* $\delta \notin F$ *then* $(p * \sigma) \upharpoonright \delta \Vdash$ "$(p * \sigma)(\delta) = p(\delta)$".

(b) *If* $\delta \in F$ *then* $(p * \sigma) \upharpoonright \delta \Vdash$ "$(p * \sigma)(\delta) = p(\delta)[\sigma(\delta)]$" *(recall that* $\sigma(\delta) \in 2^{\eta(\delta)}$*).*

In a similar way as before, we define the following:

Definition 35 *Let* $\alpha \leq \omega_2$, $F \in [\alpha]^{<\omega}$, $\eta : F \longrightarrow \omega$, $p \in \mathbb{P}_\alpha$ *and* $D \subseteq \mathbb{P}_\alpha$.

1. *D is* $(F,\eta)^*$-dense below p *if for every* $q \leq^*_{F,\eta} p$ *there is* $r \leq^*_{F,\eta} q$ *such that* $r \in D$.
2. *D is* $(F,\eta)^*$-open below p *if for every* r *such that* $r \leq^*_{F,\eta} p$ *and* $r \in D$ *then* $q \in D$ *whenever* $q \leq^*_{F,\eta} r$.
3. *D is* $(F,\eta)^*$-open dense below p *if it is both* $(F,\eta)^*$*-open and* $(F,\eta)^*$*-dense below* p.
4. *D is* (F,η)-dense below p *if for every* $q \leq_{F,\eta} p$ *there is* $r \leq_{F,\eta} q$ *such that* $r \in D$.
5. *D is* (F,η)-open below p *if for every* r *such that* $r \leq_{F,\eta} p$ *and* $r \in D$ *then* $q \in D$ *whenever* $q \leq_{F,\eta} r$.
6. *D is* (F,η)-open dense below p *if it is both* (F,η)*-open and* (F,η)*-dense below* p.

Given a function $\eta : F \longrightarrow \omega$, we define the function $\eta + 1 : F \longrightarrow \omega$ given by $(\eta+1)(\alpha) = \eta(\alpha) + 1$.

The following is the generalization of a 1-fusion sequence:

Definition 36 *Let* M *be a countable elementary submodel of* $\mathsf{H}(\kappa)$ *(for some big enough regular* κ*) such that* $\mathbb{P}_\alpha \in M$, $L = \{(F_n, \eta_n) \mid n \in \omega\}$ *such that* $|F_n| \leq n$, $\bigcup\limits_{n\in\omega} F_n = M \cap \alpha$, $\eta_n : F_n \longrightarrow \omega$ *are such that* $\eta_n \leq \eta_{n+1} \upharpoonright F_n$ *and if* $\beta \in M \cap \alpha$ *then* $\langle \eta_n(\beta) \rangle_{n\in\omega} \longrightarrow \infty$. *We say* $T \subseteq (\mathbb{P}_\alpha \cap M)^{<\omega}$ *is an* α-fusion tree *(with respect to* M *and* L*) if the following holds:*

1. $|T_0| = 1$.
2. *If* $s = \langle p_0, ..., p_n \rangle \in T$ *then* $p_{m+1} \leq^*_{F_m,\eta_m} p_m$ *for every* $m < n$.
3. *If* $F_{n+1} = F_n$ *and* $s = \langle p_0, ..., p_n \rangle \in T_n$ *then* s *has only one immediate successor in* T.
4. *For every* $n \in \omega$ *such that there is a (unique)* $\gamma \in F_{n+1} \setminus F_n$ *there are two* $\mathbb{P}_\gamma$*-names* $\dot{a}_n, \dot{b}_n$ *such that if* $s = \langle p_0, ..., p_n \rangle \in T_n$ *then there are* $q_s, r_s \in \mathbb{P}_\alpha \cap M$ *with the following properties:*

(a) $suc_T(s) = \{q_s, r_s\}$.

(b) $q_s \restriction \gamma = r_s \restriction \gamma$.

(c) $q_s \restriction \gamma \Vdash$ "$q_s(\gamma) = \dot{a}_n$" and $r_s \restriction \gamma \Vdash$ "$r_s(\gamma) = \dot{b}_n$".

(d) $q_s \restriction \gamma$ $(= r_s \restriction \gamma)$ forces that $\dot{a}_n, \dot{b}_n$ are incomparable elements of $\mathbb{S}(\dot{\mathcal{U}}_\gamma)$ but $E_{\dot{a}_n} = E_{\dot{b}_n}$.

5. Assume $s = \langle p_0, ..., p_n \rangle$, $t = \langle p'_0, ..., p'_n \rangle$, $m = \Delta(s,t)$ and $\gamma \in F_{m+1} \backslash F_m$. If $p_n \restriction \gamma = p'_n \restriction \gamma$ the following holds:

   (a) In case $F_n \neq F_{n+1}$ the following holds:

      i. $q_s \restriction \gamma = q_t \restriction \gamma$ and $r_s \restriction \gamma = r_t \restriction \gamma$.

      ii. $q_s \restriction \gamma \Vdash$ "$\langle p_n(\gamma), p'_n(\gamma), q_s(\gamma), q_t(\gamma) \rangle$ is an $\eta_n(\gamma)$-nice sequence".

      iii. $r_s \restriction \gamma \Vdash$ "$\langle p_n(\gamma), p'_n(\gamma), r_s(\gamma), r_t(\gamma) \rangle$ is an $\eta_n(\gamma)$-nice sequence".

   (b) In case $F_n = F_{n+1}$ and $suc_T(s) = \{p_{n+1}\}$ and $suc_T(t) = \{p'_{n+1}\}$ then we have the following:

      i. $p_{n+1} \restriction \gamma = p'_{n+1} \restriction \gamma$.

      ii. $p_{n+1} \restriction \gamma \Vdash$ "$\langle p_n(\gamma), p'_n(\gamma), p_{n+1}(\gamma), p'_{n+1}(\gamma) \rangle$ is an $\eta_n(\gamma)$-nice sequence".

Note that the definition of 1-fusion sequence from the previous section is essentially the same as the one of a 1-fusion tree, the only difference is the presence for the elementary submodel (in fact, the role of M is simply to help us with bookkeeping arguments, but it really could be avoided if we wanted). If a tree $T \subseteq (\mathbb{P}_\alpha \cap M)^{<\omega}$ of height k satisfy all the above properties but only for $n \leq k$, we will say that T is an *α-finite fusion tree of height k*.

Given a tree $S \subseteq \mathbb{P}_\alpha^{<\omega}$ and $\beta < \alpha$ we define $S \restriction \beta$ as the tree in $\mathbb{P}_\beta^{<\omega}$ obtained by restricting every condition of S to β. Let $\beta, \alpha \in M$, $\beta < \alpha$, $L = \{(F_n, \eta_n) \mid n \in \omega\}$ and T an α-fusion tree, define $F'_n = F_n \cap \beta$, $\eta'_n = \eta_n \restriction F'_n$, it is easy to see that $T \restriction \beta$ is a β-fusion tree.

Definition 37 *Let T be an α-fusion tree and $q \in \mathbb{P}_\alpha$. We say q is* compatible with T *if q forces that q is a lower bound for some (possibly new) branch of T.*

We need the following:

Lemma 38 *Let $\alpha, \beta \in M \preceq H(\kappa)$ with $\beta < \alpha$ and T an α-fusion tree. Assume there is $q \in \mathbb{P}_\beta$ that is compatible with $T \restriction \beta$. Then there is $\dot{r}$ such that $q^\frown \dot{r} \in \mathbb{P}_\alpha$ and $q^\frown \dot{r}$ is compatible with T. Moreover, if $\dot{R}$ is a $\mathbb{P}_\beta$-name for a branch through $T \restriction \beta$ such that q forces that q is a lower bound for $\dot{R}$ then there is $\dot{R}'$ a $\mathbb{P}_\alpha$-name for a branch through T such that $q^\frown \dot{r}$ forces that $q^\frown \dot{r}$ is a lower bound for $\dot{R}'$ and $\dot{R}' \restriction \beta = \dot{R}$.*

Proof. We prove it by induction on α, the case $\alpha = 1$ follows from lemma 18. Now assume $\alpha = \delta + 1$ and the lemma holds for δ. We first tackle the case of $\beta = \delta$. Let $G \subseteq \mathbb{P}_\beta$ be a generic filter such that $q \in G$. Find n the first natural number such that $\beta \in F_n$. In $V[G]$ define $H = \{s \in T \mid s \restriction \beta \in \dot{R}[G] \wedge |s| > n\}$. It is easy to see that $|T_m \cap H| = 2$ for every $m > n$. Moreover, for every $m > n$ there must be $r_m \in \mathbb{P}_\beta$ and $\dot{c}_m, \dot{e}_m$ such that $T_m \cap H = \{r_m{}^\frown\dot{c}_m, r_m{}^\frown\dot{e}_m\}$. Since T is an α-fusion tree then $S = (\langle \dot{c}_m \rangle, \langle \dot{e}_m \rangle)_{m>n}$ is forced to be a 1-fusion tree. By lemma 18 there is $\dot{a}$ a $\mathbb{P}_\beta$-name for a condition in $\mathbb{S}(\dot{\mathcal{U}}_\alpha)$ that is forced to be a lower bound of either $\langle \dot{c}_m \rangle_{m>n}$ or $\langle \dot{e}_m \rangle_{m>n}$ (note however, that we can not know which possibility occurs without extending q). Then $q{}^\frown\dot{a}$ is the condition we were looking for. In case $\beta < \delta$ the result follows by the previous case and the inductive hypothesis.

Assume α is a limit ordinal and let $\langle \delta_n \rangle_{n\in\omega} \subseteq M$ be an increasing sequence such that $\bigcup \delta_n = \bigcup (M \cap \alpha)$, we also assume $\beta = \delta_0$ and $F_n \subseteq \delta_{n+1}$. We then apply successively the inductive hypothesis for the trees $T \restriction \delta_{n+1}$. ■

The next task is to be able to extend finite fusion trees:

Lemma 39 *Let $\alpha \in M \preceq H(\kappa)$ and T an α-finite fusion tree of height n. For every $s = \langle p_0, ..., p_n \rangle \in T$ let $D_s^0, D_s^1 \in M$ be $(F_n, \eta_n + 1)^*$-open and $(F_n, \eta_n)^*$-dense sets below p_n. Then there is $\widetilde{T}$ with the following properties:*

1. *$\widetilde{T}$ is a α-finite fusion tree of height $n+1$.*
2. *$\widetilde{T}$ is an end-extention of T.*
3. *If $s = \langle p_0, ..., p_n \rangle \in T$ and $F_n = F_{n+1}$ then $suc_{\widetilde{T}}(s) \subseteq D_s^0$.*
4. *If $s = \langle p_0, ..., p_n \rangle \in T$ and $F_n \neq F_{n+1}$ then one of the elements of $suc_{\widetilde{T}}(s)$ is in D_s^0 and the other is in D_s^1.*

Proof. We prove it by induction on α. The case $\alpha = 1$ was already done in 14. Now assume $\alpha = \beta + 1$ and the lemma holds for β, let $S = T \restriction \beta$. There are several cases to consider:

Case 40 $\beta \notin F_{n+1}$.

Let $s = \langle p_0, ..., p_n \rangle \in T$ and define E_s^i as the set of all $q \leq p_n \restriction \beta$ such that there is $\dot{r}$ for which $q{}^\frown\dot{r} \leq^*_{F_n,\eta_n} p_n$ and $q{}^\frown\dot{r} \in D_s^i$. We apply the inductive hypothesis. Note that in this case, T is "essentially" a β-fusion tree.

Case 41 $\beta \in F_{n+1} \setminus F_n$.

For every $s = \langle p_0, ..., p_n \rangle \in T$ we define E_s as the set of all $q \leq p_n \restriction \beta$ such that there are $\dot{a}$ and $\dot{b}$ such that $q{}^\frown\dot{a}$, $q{}^\frown\dot{b} \leq^*_{F_n,\eta_n} p_n$, $q{}^\frown\dot{a} \in D_s^0$, $q{}^\frown\dot{b} \in D_s^1$ and q forces that $\dot{a}, \dot{b}$ are incomparable elements of $\mathbb{S}(\dot{\mathcal{U}}_\beta)$ but $E_{\dot{a}} = E_{\dot{b}}$. It is not hard to see that E_s is an $(F_n \cap \beta, \eta_n \restriction \beta + 1)^*$-open and $(F_n \cap \beta, \eta_n \restriction \beta)^*$-dense set below $p_n \restriction \beta$ (this follows from 14). We apply the inductive hypothesis (here we are only using one dense set for each terminal node).

Case 42 $\beta \in F_n$.

Let $s = \langle p_0, ..., p_n \rangle \in T$ and note that there is a unique $t = \langle p'_0, ..., p'_n \rangle \in T$ such that $p_n \restriction \beta = p'_n \restriction \beta$ and $p_n(\beta) \neq p'_n(\beta)$ (and then s and t get identify by restricting to β). Let $z = s \restriction \beta = t \restriction \beta$ (which is a node in $T \restriction \beta$) and then for each $i < 2$ we define E^i_z as the set of all $q \leq p_n \restriction \beta$ such that there are $\dot{r}_s$ and $\dot{r}_t$ for which $q^\frown \dot{r}_s \leq^*_{F_n,\eta_n} p_n$, $q^\frown \dot{r}_t \leq^*_{F_n,\eta_n} p'_n$, $q^\frown \dot{r}_s \in D^i_s$, $q^\frown \dot{r}_t \in D^i_t$ and $q \Vdash$ "$\langle p_n(\beta), p'_n(\beta), \dot{r}_s, \dot{r}_t \rangle$ is an $\eta_n(\beta)$-nice sequence". We apply the inductive hypothesis.

Now, assume α is a limit ordinal. We first find $\beta \in M$ such that $F_{n+1} \subseteq \beta$. We apply the induction hypothesis to $T \restriction \beta$ and a similar argument as the one used in the case $\beta \notin F_{n+1}$. ■

Given an open dense set $D \subseteq \mathbb{P}_\alpha$ and F, η we define $\widetilde{D}_{F,\eta} = \{p \in \mathbb{P}_\alpha \mid \forall \sigma \in \prod\limits_{\delta \in F} 2^{\eta(\delta)} (p * \sigma \in D)\}$. The following lemma is easy and left to the reader:

Lemma 43 $\widetilde{D}_{F,\eta} = \{p \in \mathbb{P}_\alpha \mid \forall \sigma \in \prod\limits_{\delta \in F} 2^{\eta(\delta)} (p * \sigma \in D)\}$ *is* (F, η)*-open dense.*

Preserving $\mathsf{c}_{\mathsf{min}}$*-covering at successor steps*

With the tools developed in the last section, we can finally start the proof that the iteration of the Shelah's party forcing preserves $\mathsf{c}_{\mathsf{min}}$-covering. The proof will proceed by induction, the base case has already been done, we will do the successor step in this section and the limit step in the next one. This proof takes inspiration in the result by Geshke that the iteration of the Miller lite forcing preserves $\mathsf{c}_{\mathsf{min}}$-covering (such proof is also splitted in cases). For this section assume $\alpha = \beta + 1$ and $\dot{x}$ is the name of a real that was not added at the β-step.

Definition 44 *Let T and S be two finite subtrees of $2^{<\omega}$ and $i < 2$. We say that $\mathsf{c}_{\mathsf{min}}(T, S) = i$ if for every maximal node t of T and every maximal node s of S the following holds:*

1. *t and s are incompatible.*
2. *$\mathsf{c}_{\mathsf{min}}(t, s) = i$.*

Note that if $\mathsf{c}_{\mathsf{min}}(T, S) = i$ and $\overline{T}$ and $\overline{S}$ are end-extentions of T and S respectively, then $\mathsf{c}_{\mathsf{min}}(\overline{T}, \overline{S}) = i$ holds as well. Recall that $\mathbb{P}$ is a partial order, $\dot{y}$ is a $\mathbb{P}$-name for an element in 2^ω and $p \in \mathbb{P}$, we defined $\dot{y}[p] = \bigcup \{t \mid p \Vdash \text{"}t \subseteq \dot{y}\text{"}\}$. Also remember that if $p \in \mathbb{S}(\mathcal{U})$, we defined $S_p(\dot{y}) = \bigcup \{\dot{y}[p[h]] \mid h \in 2^{<\omega}\}$.

Definition 45 *Let $q \in \mathbb{P}_\beta$ and $\dot{p}$ a $\mathbb{P}_\beta$-name for a condition of $\mathbb{S}(\dot{\mathcal{U}}_\beta)$ and $n \in \omega$. We say that q* determines $\dot{p}$ to the n-level *if there is a tree $T_n^{\dot{p}}(q)$ such that $q \Vdash$ "$T_n^{\dot{p}}(q) = \bigcup \left\{ \dot{x}(p[h]) \mid h \in 2^{n+1} \right\}$".*

By the results of the previous sections, we know that if $q \in \mathbb{P}_\beta$ and $\dot{p}_1$ is a $\mathbb{P}_\beta$-name for a condition of $\mathbb{S}(\dot{\mathcal{U}}_\beta)$, then we can find $\dot{p}_2$ a $\mathbb{P}_\beta$-name for a condition of $\mathbb{S}(\dot{\mathcal{U}}_\beta)$ such that q forces that $\dot{p}_2$ is an extension of $\dot{p}_1$, $\dot{p}_2$ is ω-separative and $S_{p(\beta)}(\dot{x})$ is forced to be a monochromatic Sacks tree (although we might not know of which color without extending q first).

Definition 46 *Let $q \in \mathbb{P}_\beta$ and $\dot{p}$ is a $\mathbb{P}_\beta$-name for a condition of $\mathbb{S}(\dot{\mathcal{U}}_\beta)$ that is ω-separative and $S_{p(\beta)}(\dot{x})$ is forced to be an i-monochromatic Sacks tree. Let $F \in [\beta]^{<\omega}$ and $\eta : F \longrightarrow \omega$. We say $(q, \dot{p})$ is* (F, η, n, i)-faithful *if the following holds:*

1. *If $\sigma \in \prod\limits_{\gamma \in F} 2^{\eta(\gamma)}$ then $q * \sigma$ determines $\dot{p}$ to the n-level.*
2. *If $\sigma, \sigma' \in \prod\limits_{\gamma \in F} 2^{\eta(\gamma)}$ and $\sigma \neq \sigma'$ then $\mathsf{c}_{\mathsf{min}}(T_n^{\dot{p}}(q * \sigma), T_n^{\dot{p}}(q * \sigma')) = i$.*

The following lemma follows from the definitions:

Lemma 47 *Assume $(q, \dot{p})$ is $(F, \eta, n+1, i)$-faithful. Let $\dot{p}^0$ be the condition obtained by merging $[a_{\dot{p}}(n)]_p$ with $[a_{\dot{p}}(n+1)]_p$ and $a_{\dot{p}}(n+1) \in Love_{p(\beta)}(a_{\dot{p}}(n))$ and let $\dot{p}^1$ be the condition obtained by merging $[a_{\dot{p}}(n)]_p$ with $[a_{\dot{p}}(n+1)]_p$ and $a_{\dot{p}}(n+1) \in Hate_{p(\beta)}(a_{\dot{p}}(n))$. Then $(q, \dot{p}^0)$ and $(q, \dot{p}^1)$ are (F, η, n, i)-faithful and if $\sigma \in \prod\limits_{\gamma \in F} 2^{\eta(\gamma)}$ then $\mathsf{c}_{\mathsf{min}}(T_n^{\dot{p}^0}(q * \sigma), T_n^{\dot{p}^1}(q * \sigma)) = 1$.*

We will need the following:

Lemma 48 *Assume $(q, \dot{p})$ is (F, η, n, i)-faithful.*

1. *The set $\{r \leq^*_{F,\eta} q \mid (r, \dot{p})$ is $(F, \eta, n+1, i)$-faithful $\}$ is $(F, \eta)^*$-dense and $(F, \eta+1)^*$-open.*
2. *Let $\delta \in F$ and define $\eta' : F \longrightarrow \omega$ where $\eta'(\xi) = \eta(\xi)$ if $\xi \neq \delta$ and $\eta'(\delta) = \eta(\delta) + 1$. Then the set D consisting of all $r \leq^*_{(F,\eta)} q$ such that there is $\dot{p}_1$ such that $r \Vdash$ "$\dot{p}_1 \leq^{**}_n \dot{p}$" and $(r, \dot{p}_1)$ is (F, η', n, i)-faithful is $(F, \eta)^*$-dense and $(F, \eta+1)^*$-open.*

Proof. We will first prove 1. It is easy to see that $D = \{r \leq^*_{F,\eta} q \mid (r, \dot{p})$ is $(F, \eta, n+1, i)$-faithful $\}$ is $(F, \eta+1)^*$-open. We will now prove that D is $(F, \eta)^*$-dense. Let $q_1 \leq_{F,\eta} q$ and fix an enumeration $\prod\limits_{\gamma \in F} 2^{\eta(\gamma)} = \{\sigma_j \mid j < k\}$. We can recursively build conditions $q_1^0, ..., q_1^{k+1}$ with the following properties:

1. $\langle q_1^0, ..., q_1^{k+1}\rangle$ is a $\leq_{(F,\eta)^*}$-decreasing sequence.
2. $q_1^0 = q_1$.
3. $q_1^{i+1} * \sigma_j$ determines $\left(S_{p(\beta)}(\dot{x})\right)_{n+2}$.

We claim that q_1^{k+1} is the condition we are looking for. In order to prove this, note that if $j_1 \neq j_2$, then $\mathsf{c}_{\mathsf{min}}(T_{n+1}^{\dot{p}}\left(q_1^{k+1} * \sigma_{j_1}\right), T_{n+1}^{\dot{p}}\left(q_1^{k+1} * \sigma_{j_2}\right)) = i$ because $T_{n+1}^{\dot{p}}\left(q_1^{k+1} * \sigma_{j_1}\right)$ and $T_{n+1}^{\dot{p}}\left(q_1^{k+1} * \sigma_{j_2}\right)$ are end-extentions of trees that already have that property.

Now we will prove 2, we will show D is $(F,\eta)^*$-dense. Let $q_1 \leq_{F,\eta}^* q$ and by the first part of the lemma, we can find $r \leq_{F,\eta}^* q_1$ that is $(F,\eta,n+1,i)$-faithful. Let $\dot{p}^0$ and $\dot{p}^1$ defined as in 47, then we define a $\mathbb{S}(\mathcal{U}_\beta)$-name $\dot{p}_1$ such that if $\sigma \in \prod_{\gamma \in F} 2^{\eta'(\gamma)}$ then $r * \sigma \Vdash$ "$\dot{p}_1 = \dot{p}^{\sigma(\eta(\delta))}$". It follows that $(r, \dot{p}_1)$ is (F,η',n,i)-faithful. It is easy to see that D is $(F,\eta+1)^*$-open. ■

We will now prove the most important result on this section:

Proposition 49 *Let $\alpha = \beta + 1$ and assume $\dot{x}$ is a $\mathbb{P}_\alpha$-name for a real that was not added by $\mathbb{P}_\beta$. If p is any condition of $\mathbb{P}_\alpha$ then there are $q \leq p$ and $S \in V$ a $\mathsf{c}_{\mathsf{min}}$-monochromatic tree such that $q \Vdash$ "$\dot{x} \in [S]$".*

Proof. Let $q \in \mathbb{P}_\beta$ and $\dot{p}$ a $\mathbb{P}_\beta$-name for a condition of $\mathbb{S}(\dot{\mathcal{U}}_\beta)$ that is ω-separative and $S_{p(\beta)}(\dot{x})$ is an $i$monochromatic Sacks tree (for some $i < 2$). Let M be a countable elementary submodel of $\mathsf{H}(\kappa)$ (for a big large enough κ) such that $\alpha, q, \dot{p}, \dot{\mathcal{U}}_\beta, \dot{x} \in M$. Let $L = \{(F_n, \eta_n) \mid n \in \omega\}$ such that $|F_n| \leq n$, $\bigcup_{n\in\omega} F_n = M \cap \beta$, $\eta_n : F_n \longrightarrow \omega$ are such that $\eta_n \leq \eta_{n+1} \upharpoonright F_n$ and if $\xi \in M \cap \beta$ then $\langle \eta_n(\xi)\rangle_{n\in\omega} \longrightarrow \infty$. We build a β-fusion tree $T = \{q_s \mid s \in 2^{<\omega}\}$ and $\{\dot{p}_s \mid s \in 2^{<\omega}\}$ as follows:

1. $q_\emptyset \leq q$, $q_\emptyset \Vdash$ "$\dot{p}_\emptyset \leq \dot{p}$".
2. $(q_s, \dot{p}_s)$ is $\left(F_{|s|}, \eta_{|s|}, |s|+1, i\right)$-faithful.
3. If $s \in 2^n$ and $j < 2$ then $q_{s^\frown j} \leq_{F_n,\eta_n} q_s$ and $q_{s^\frown j} \Vdash$ "$\dot{p}_{s^\frown j} \leq_n^{**} \dot{p}_s$".
4. T_{n+1} is constructed from T_n as follows: let $s \in 2^n$ and since $(q_s, \dot{p}_s)$ is $(F_n, \eta_n, n+1, i)$-faithful we then find $\dot{p}_s^0$ and $\dot{p}_s^1$ as in 47. For each $j < 2$ let $D_s^j = \{r \leq_{F_n,\eta_n}^* q_s \mid (r, \dot{p}_s^j)$ is $(F_{n+1}, \eta_{n+1}, n+2, i)$-faithful $\}$ and we apply 2 and 4 to construct T_{n+1}.

Clearly T can be constructed. Given $s \in 2^n$ define $K(q_s) = \bigcup\{T_n^{\dot{p}_s}(q_s * \sigma) \mid \sigma \in \prod_{\gamma \in F_n} 2^{\eta_n(\gamma)}\}$ and note that (by faithfulness) $K(q_s)$ is an i-monochromatic tree. Furthermore, $K(q_{s^\frown 0})$ and $K(q_{s^\frown 1})$ are two end extensions of $K(p_s)$ and $\mathsf{c}_{\mathsf{min}}(K(q_{s^\frown 0}), K(q_{s^\frown 1})) = i$. In this way, $K = \bigcup K(p_s)$ is an i-monochromatic tree, which is an element of V. Let $\overline{q}$ be a condition compatible with T and $\dot{R}$ a name for a branch through T such that $\overline{q}$ forces $\dot{R}$ to be contained in the generic filter. In this way $\overline{p} = Lim(\{\dot{p}_{R\restriction n} \mid n \in \omega\})$ is a name for a condition of $\mathbb{S}(\dot{\mathcal{U}}_\beta)$ (since it is forced to be the limit of a $\leq^{**}$-decreasing sequence). In this way $r = \overline{q}^\frown \overline{p}$ is an extension of $q^\frown \dot{p}$ and $r \Vdash$ "$\dot{x} \in [K]$". $\blacksquare$

Preserving $\mathsf{c}_{\mathsf{min}}$-*covering at limit steps*

The last task is to prove that $\mathsf{c}_{\mathsf{min}}$-covering is preserved at limit steps. For this section let α be a limit ordinal and $\dot{x}$ a $\mathbb{P}_\alpha$-name for an element of 2^ω that was not added by any $\mathbb{P}_\beta$ for $\beta < \alpha$ (note that α must have countable cofinality). Given $i < 2$, define E_i as the set of all $p \in \mathbb{P}_\alpha$ such that for every $\beta < \alpha$ and for every $q \leq p$ there are $q' \leq q$ and q_0, q_1 with the following properties:

1. $q_0 \restriction \beta = q_1 \restriction \beta = q' \restriction \beta$.
2. $q_0, q_1 \leq q'$.
3. $\mathsf{c}_{\mathsf{min}}(\dot{x}[q_0], \dot{x}[q_1]) = i$.

The following is lemma 30 of [3] (in [3] the lemma is only stated for iteration of Sacks forcing, but it is stated that it is true for any iteration, see also lemma 6.11 of [4]).

Lemma 50 ([3]) *Both E_0 and E_1 are open and $E_0 \cup E_1$ is an open dense set.*

We will need the following concepts:

Definition 51 *Let $F \in [\alpha]^{<\omega}$, $\eta : F \longrightarrow \omega$, $i < 2$, $\delta = max(F) + 1$ and $p \in \mathbb{P}_\alpha$.*

1. *We say p is (F, η, i)-*faithful *if $p \in E_i$ and for every $\sigma, \sigma' \in \prod_{\gamma \in F} 2^{\eta(\gamma)}$ such that $\sigma \neq \sigma'$ then $\mathsf{c}_{\mathsf{min}}(\dot{x}[p * \sigma], \dot{x}[p * \sigma']) = i$.*
2. *We say p is (F, η, i)-*super faithful *if p is (F, η, i)-faithful and there are p^0, p^1 such that:*
   (a) $p^0, p^1 \leq p$.
   (b) $p^0 \restriction \delta = p^1 \restriction \delta = p \restriction \delta$.
   (c) If $\sigma \in \prod_{\gamma \in F} 2^{\eta(\gamma)}$ then $\mathsf{c}_{\mathsf{min}}(\dot{x}[p^0 * \sigma], \dot{x}[p^1 * \sigma']) = i$.

In the above case, we say that p^0 and p^1 witness that p is (F, η, i)-super faithful.

Note that if $p \in E_i$ then p is $(\emptyset, \emptyset, i)$-faithful. Also note that if $\beta \notin F$ and p is (F, η, i)-faithful then p is $(F \cup \{\beta\}, \eta \cup \{(\beta, 0)\}, i)$ -faithful.

Lemma 52 *If p is (F, η, i)-faithful then there is $q \leq_{F,\eta} p$ such that q is (F, η, i)-super faithful.*

Proof. Let $\delta = max(F) + 1$. Using that $p \in E_i$, we can recursively find $q \leq_{F,\eta} p \restriction \delta$ such that for every $\sigma \in \prod\limits_{\gamma \in F} 2^{\eta(\gamma)}$ there are q_0^σ, q_1^σ with the following properties:

1. $q_0^\sigma \restriction \delta = q_1^\sigma \restriction \delta = q \restriction \delta$.
2. $q_0^\sigma, q_1^\sigma \leq q$.
3. $\mathsf{c}_{\mathsf{min}}(\dot{x}[q_0^\sigma], \dot{x}[q_1^\sigma]) = i$.

Let $q_0, q_1 \in \mathbb{P}_\delta$ such that $q_0 \restriction \delta = q_1 \restriction \delta = q \restriction \delta$ and if $\sigma \in \prod\limits_{\gamma \in F} 2^{\eta(\gamma)}$ then $q_0 * \sigma = q_0^\sigma$ and $q_1 * \sigma = q_1^\sigma$. We now find $\dot{r}_0$ and $\dot{r}_1$ two $\mathbb{P}_\sigma$-names such that if $\sigma \in \prod\limits_{\gamma \in F} 2^{\eta(\gamma)}$ then $q_0 * \sigma \Vdash$ "$\dot{r}_0 = q_0^\sigma \restriction [\delta, \alpha)$" and $q_1 * \sigma \Vdash$ "$\dot{r}_1 = q_1^\sigma \restriction [\delta, \alpha)$". it is easy to see that $q_0{}^\frown \dot{r}_0$ and $q_1{}^\frown \dot{r}_1$ witness that q is (F, η, i)-super faithful. ■

We will now prove the following,

Lemma 53 *Let p be (F, η, i)-faithful and $\beta \in F$. Define $\eta' : F \longrightarrow \omega$ where $\eta'(\xi) = \eta(\xi)$ if $\xi \neq \beta$ and $\eta'(\beta) = \eta(\beta) + 1$. Then $D = \{q \leq^*_{F,\eta} p \mid q$ is (F, η', i)-faithful$\}$ is $(F, \eta)^*$-dense and $(F, \eta + 1)^*$-open.*

Proof. It is easy to see that it is $(F, \eta + 1)^*$-open, we will prove it is also $(F, \eta)^*$-dense. Let $\delta = max(F) + 1$ and $q \leq^*_{F,\eta} p$. By the previous lemma, we may assume q is (F, η, i)-super faithful. Let q^0 and q^1 witness that q is (F, η, i)-super faithful. For every $\sigma \in \prod\limits_{\gamma \in F} 2^{\eta(\gamma)}$ and for each $j < 2$ let $\sigma_j \in \prod\limits_{\gamma \in F} 2^{\eta'(\gamma)}$ such that $\sigma_j(\beta) = \sigma(\beta)^\frown j$ and $\sigma_j(\gamma) = \sigma(\gamma)$ for every $\gamma \neq \beta$. Let $\dot{r}$ be a $\mathbb{P}_\delta$-name such that $(q * \sigma_j) \restriction \delta \Vdash$ "$\dot{r} = q_j \restriction [\delta, \alpha)$" for every $\sigma \in \prod\limits_{\gamma \in F} 2^{\eta(\gamma)}$ and for each $j < 2$. The condition $\overline{q} = (q \restriction \delta)^\frown \dot{r}$ has the desired properties. ■

We can now prove that the $\mathsf{c}_{\mathsf{min}}$-covering is preserved at limit steps:

Proposition 54 *Let α be a limit ordinal and assume $\dot{x}$ is a $\mathbb{P}_\alpha$-name for a real that was not added by $\mathbb{P}_\beta$ for every $\beta < \alpha$. If p is any condition of $\mathbb{P}_\alpha$ then there are $q \leq p$ and $S \in V$ a $\mathsf{c}_{\mathsf{min}}$-monochromatic tree such that $q \Vdash$ "$\dot{x} \in [S]$".*

Proof. We may assume there is $i<2$ such that $p\in E_i$. Let M be a countable elementary submodel of $\mathsf{H}(\kappa)$ (for a big large enough κ) such that $\alpha,p,\dot{x}\in M$. Let $L=\{(F_n,\eta_n)\mid n\in\omega\}$ such that $|F_n|\leq n$, $\bigcup\limits_{n\in\omega}F_n=M\cap\alpha$, $\eta_n:F_n\longrightarrow\omega$ are such that $\eta_n\leq\eta_{n+1}\restriction F_n$ and if $\xi\in M\cap\alpha$ then $\langle\eta_n(\xi)\rangle_{n\in\omega}\longrightarrow\infty$. We now find an increasing sequence $\langle\delta_n\rangle_{n\in\omega}\in M^\omega$ such that $\bigcup\limits_{n\in\omega}\delta_n=\alpha$ (recall that α has countable cofinality). We build a fusion tree $T=\{p_s\mid s\in 2^{<\omega}\}$ with the following properties:

1. $p_\emptyset\leq p$.
2. p_s is $\left(F_{|s|},\eta_{|s|},i\right)$ super faithful.
3. T_{n+1} is constructed from T_n as follows: For every $s\in 2^n$ we first find p_s^0 and p_s^1 witnessing that p_s is (F_n,η_n,i) super faithful. Let $D_s^i=\{q\leq^*_{F_n,\eta_n}p_s^i\mid q$ is (F_{n+1},η_{n+1},i)-faithful$\}$ which is $(F,\eta)^*$-dense and $(F,\eta+1)^*$-open below p_s^i. We apply 4 and get a tree T'_{n+1}. We now extend every maximal node of T'_{n+1} to a (F_{n+1},η_{n+1},i) super faithful condition and this is T_{n+1}.

Given $s\in 2^n$ let $K(p_s)=\bigcup\{\dot{x}[p_s*\sigma]\mid\sigma\in\prod\limits_{\gamma\in F_n}2^{\eta_n(\gamma)}\}$ and note that (by faithfulness) $K(p_s)$ is an i-monochromatic tree. Furthermore, $K(p_{s^\frown 0})$ and $K(p_{s^\frown 1})$ are two end extensions of $K(p_s)$ such that $K(p_{s^\frown 0})\cap K(p_{s^\frown 1})=K(p_s)$. In this way, $K=\bigcup K(p_s)$ is an i-monochromatic tree and if q is compatible with T then $q\Vdash$ "$\dot{x}\in[K]$". $\blacksquare$

After all our hard work, we can finally prove the main result of the paper:

Corollary 55 *The inequality $\mathfrak{hm}<\mathfrak{u}$ is consistent with the axioms of ZFC.*

Proof. We start with a model V of the Generalized Continuum Hypothesis. We perform a countable support iteration $\langle\mathbb{P}_\alpha,\dot{\mathbb{Q}}_\alpha\mid\alpha\leq\omega_2\rangle$ such that for if $\alpha<\omega_2$ then $\mathbb{P}_\alpha\Vdash$ "$\dot{\mathbb{Q}}_\alpha=\mathbb{S}(\dot{\mathcal{U}}_\alpha)$" where $\dot{\mathcal{U}}_\alpha$ is a $\mathbb{P}_\alpha$-name for an ultrafilter. Furthermore, with a carefully chosen bookkeeping devise we can make sure that $\mathbb{P}_{\omega_2}\Vdash$ "$\mathfrak{u}=\omega_2$". By the results in the last sections, we know that $\mathbb{P}_{\omega_2}$ preserves $\mathsf{c}_{\mathsf{min}}$-covering, so $\mathbb{P}_{\omega_2}\Vdash$ "$\mathfrak{hm}=\omega_1$". $\blacksquare$

MAD families in the Shelah model

By the Shelah model we mean the iteration as in the last result. We already know that $\mathfrak{hm}=\omega_1$ holds in such model, so all the cardinal invariants found on the Cichoń diagram are small. Furthermore, in that model $\mathfrak{u}=\omega_2$ and $\mathfrak{i}=\omega_1$ (see [8]). The only one of the usual invariants that is missing to compute in such model is $\mathfrak{a}$. In this last section, we will prove that there is a MAD family of size ω_1 in the Shelah model.

Let $f, g \in \omega^\omega$, define $f \leq^* g$ if and only if $f(n) \leq g(n)$ for all $n \in \omega$ except finitely many. A family $\mathcal{D} \subseteq \omega^\omega$ is a *dominating family* if for every $f \in \omega^\omega$ there is $g \in \mathcal{D}$ such that $f \leq^* g$. The *dominating number* $\mathfrak{d}$ is defined as the least size of a dominating family. In [6] Michael Hrušák introduced a diamond principle for the dominating number:

$\diamondsuit_{\mathfrak{d}}$ There is a sequence $\langle d_\alpha \mid \alpha < \omega_1 \rangle$ where $d_\alpha : \alpha \longrightarrow \omega$ such that for every $f : \omega_1 \longrightarrow \omega$ the set $\{\alpha > \omega \mid f \restriction \alpha \leq^* d_\alpha\}$. The sequence is called a $\diamondsuit_{\mathfrak{d}}$*-sequence.*

It is easy to see that $\diamondsuit_\delta$ implies that $\mathfrak{d} = \omega_1$. It is an old problem of Roitman if $\mathfrak{d} = \omega_1$ implies $\mathfrak{a} = \omega_1$, however, Michael Hrušák proved the following:

Proposition 56 ([6]) $\diamondsuit_{\mathfrak{d}}$ *implies* $\mathfrak{a} = \omega_1$.

in order to show that $\mathfrak{a} = \omega_1$ holds in the Shelah model, we will prove that $\diamondsuit_{\mathfrak{d}}$ holds in such model. Recall that $\diamondsuit$ is the following statement:

$\diamondsuit$ There is $\mathcal{D} = \{D_\alpha \mid \alpha \in \omega_1\}$ with $D_\alpha \subseteq \alpha$ such that for every $X \subseteq \omega_1$, the set $\{\alpha \mid X \cap \alpha = D_\alpha\}$ is stationary.

We will use $\diamondsuit$ in order to construct a $\diamondsuit_{\mathfrak{d}}$-sequence for the Shelah model. By $LIM(\omega_1)$ we denote the set of all countable limit ordinals. We start with the following:

Lemma 57 *Assume V is a model of $\diamondsuit$ and let κ be a regular large enough cardinal. Then there is a sequence $\langle (M_\alpha, p_\alpha, \dot{f}_\alpha) \rangle_{\alpha \in LIM(\omega_1)}$ such that for every $\alpha < \omega_1$ the following holds:*

1. *M_α is a countable elementary submodel of $\mathsf{H}(\kappa)$ such that $\mathbb{P}_{\omega_2}, p_\alpha, \dot{f}_\alpha \in M_\alpha$ (where $\mathbb{P}_{\omega_2}$ is the iteration of the Shelah party forcing).*
2. *$p_\alpha \in \mathbb{P}_{\omega_2}$ and $p_\alpha \Vdash$ "$\dot{f}_\alpha : \omega_1 \longrightarrow \omega$".*

With the property that for every $p \in \mathbb{P}_{\omega_2}$ and $\dot{f}$ such that $p \Vdash$ "$\dot{f} : \omega_1 \longrightarrow \omega$" there is a countable $N \preceq \mathsf{H}(\kappa)$ for which $\mathbb{P}_{\omega_2}, p, \dot{f} \in N$ and $\alpha < \omega_1$ such that $M_\alpha \cap \omega_1 = \alpha$ and the structures $(N, \in, \mathbb{P}_{\omega_2}, p, \dot{f})$ and $(M_\alpha, \in, \mathbb{P}_{\omega_2}, p_\alpha, \dot{f}_\alpha)$ are isomorphic.

Proof. Using $\diamondsuit$, we can find a sequence $\langle \mathfrak{A}_\alpha = (\alpha, \rhd_\alpha, P_\alpha, r_\alpha, h_\alpha) \rangle_{\alpha \in LIM(\omega_1)}$ such that for every structure $\mathfrak{A} = (\omega_1, \rhd, P, r, h)$ there are stationary many α such that $\mathfrak{A}_\alpha$ is a substructure of $\mathfrak{A}$. Given α a limit ordinal, in case there are a countable $M \preceq \mathsf{H}(\kappa)$, $p \in \mathbb{P}_{\omega_2}$, $\dot{f}$ such that $\mathbb{P}_{\omega_2}, p, \dot{f} \in M$, $M \cap \alpha = \alpha$, $p \Vdash$ "$\dot{f} : \omega_1 \longrightarrow \omega$" and $(M, \in, \mathbb{P}_{\omega_2}, p, \dot{f})$ is isomorphic to $\mathfrak{A}_\alpha$ then we choose one of them and define $M_\alpha = M$, $p_\alpha = p$ and $\dot{f}_\alpha = \dot{f}$. If there is no M satisfying those properties, we just take any $(M_\alpha, p_\alpha, \dot{f}_\alpha)$ satisfying the properties 1 and 2. We will now prove $\mathcal{D} = \{(M_\alpha, p_\alpha, \dot{f}_\alpha) \mid \alpha \in LIM(\omega_1)\}$ has the desired properties.

Let $p \in \mathbb{P}_{\omega_2}$ and $\dot{f}$ such that $p \Vdash$ "$\dot{f} : \omega_1 \longrightarrow \omega$". Recursively, we build $\{N_\alpha \mid \alpha < \omega_1\}$ a continous $\in$-chain of countable elementary submodels of $\mathsf{H}(\kappa)$ such that $p, \dot{f}, \mathbb{P}_{\omega_2} \in N_0$. Let $N = \bigcup\limits_{\alpha \in \omega_1} N_\alpha$, since N has size ω_1 then we can define a structure $\mathfrak{A} = (\omega_1, \rhd, P, r, h)$ that is isomorphic to $(N, \in, \mathbb{P}_{\omega_2}, p, \dot{f})$. Let $F : \omega_1 \longrightarrow N$ be an isomorphism.

It is easy to see that $\{\alpha \in LIM(\omega_1) \mid N_\alpha \cap \omega_1 = \alpha \wedge F[\alpha] = N_\alpha\}$ is a club. In this way, we can find α such that $F[\alpha] = N_\alpha, N_\alpha \cap \omega_1 = \alpha$ and $\mathfrak{A}_\alpha$ is a substructure of $\mathfrak{A}$. Note that N_α, p and $\dot{f}$ satisfy the conditions of the definition at step α, so $(M_\alpha, \in, \mathbb{P}_{\omega_2}, p_\alpha, \dot{f}_\alpha)$ is isomorphic to $\mathfrak{A}_\alpha$ hence it is also isomorphic to $(N, \in, \mathbb{P}_{\omega_2}, p, \dot{f})$ (of course it might be the case $M_\alpha = N_\alpha$ but this is highly unlikely). ■

With the lemma, we can now prove the following:

Proposition 58 *$\Diamond_{\mathfrak{d}}$ holds in the Shelah model.*

Proof. It is well known that we may assume that $\Diamond$ holds in V. Fix a sequence $\langle (M_\alpha, p_\alpha, \dot{f}_\alpha) \rangle_{\alpha \in LIM(\omega_1)}$ as in the previous lemma, we will now define $\mathcal{D} = \{d_\alpha : \alpha \longrightarrow \omega \mid \alpha < \omega_1\}$. In case $M_\alpha \cap \omega_1 \neq \alpha$, let d_α be any constant function. Fix α such that $M_\alpha \cap \omega_1 = \alpha$, we will see how to define d_α.

Let $L = \{(F_n, \eta_n) \mid n \in \omega\}$ such that $|F_n| \leq n$, $\bigcup\limits_{n \in \omega} F_n = M \cap \alpha$, $\eta_n : F_n \longrightarrow \omega$ are such that $\eta_n \leq \eta_{n+1} \restriction F_n$ and if $\xi \in M \cap \alpha$ then $\langle \eta_n(\xi) \rangle_{n \in \omega} \longrightarrow \infty$. Fix an enumeration $\alpha = \{\alpha_n \mid n \in \omega\}$. For every $\beta < \alpha$ define $D^\alpha_\beta = \{q \mid \exists n (q \Vdash$ "$\dot{f}_\alpha(\beta) = n$")$\}$ and recall that the set $\widetilde{D^\alpha_\beta}(F, \eta) = \{q \mid \forall \sigma \in \prod\limits_{\delta \in F} 2^{\eta(\delta)} \exists n_\sigma (q * \sigma \Vdash$ "$\dot{f}_\alpha(\beta) = n_\sigma$")$\}$ is $(F, \eta + 1)^*$-open, $(F, \eta)^*$-dense and it is an element of M_α. We can build a fusion tree $T = \{p_s \mid s \in 2^{<\omega}\}$ with the following properties:

1. $p_\emptyset = p_\alpha$.
2. If $s \in 2^{n+1}$ then $p_s \in \widetilde{D^\alpha_{\beta_n}}(F_n, \eta_n)$.

Define $d_\alpha : \alpha \longrightarrow \omega$ such that if $n \in \omega$ then $p_s \Vdash$ "$\dot{f}_\alpha(\alpha_n) < d_\alpha(\alpha_n)$" for every $s \in 2^{n+1}$. We will prove that $\mathcal{D} = \{d_\alpha : \alpha \longrightarrow \omega \mid \alpha < \omega_1\}$ will be a $\Diamond_{\mathfrak{d}}$-sequence after forcing with $\mathbb{P}_{\omega_2}$. Let $p \in \mathbb{P}_{\omega_2}$ and $\dot{f}$ such that $p \Vdash$ "$\dot{f} : \omega_1 \longrightarrow \omega$". Applying the previous lemma, we can find a countable N for which $\mathbb{P}_{\omega_2}, p, \dot{f} \in N$ and $\alpha < \omega_1$ such that $M_\alpha \cap \omega_1 = \alpha$ and the structures $(N, \in, \mathbb{P}_{\omega_2}, p, \dot{f})$ and $(M_\alpha, \in, \mathbb{P}_{\omega_2}, p_\alpha, \dot{f}_\alpha)$ are isomorphic. Let $H : M_\alpha \longrightarrow N$ be the isomorphism.

Let $F_n' = H(F_n)$ and $\eta_n' = H(\eta_n)$. Let $L' = \{(F_n', \eta_n') \mid n \in \omega\}$. For every $s \in 2^{<\omega}$ define $p_s' = H(p_s)$. In this way, $p_\emptyset' = p$ and it is easy to see that $T' = \{p_s' \mid s \in 2^{<\omega}\}$ is a fusion tree.

Let q be compatible with T' (which is obviously an extension of p). We will now prove that $q \Vdash$ "$\dot{f} \upharpoonright \alpha \leq d_\alpha$". Let $n \in \omega$ and $s \in 2^{n+1}$, since $p_s \Vdash$ "$\dot{f}_\alpha(\alpha_n) < d_\alpha(\alpha_n)$" it follows that $p_s' \Vdash$ "$\dot{f}(\alpha_n) < d_\alpha(\alpha_n)$". Finally, since q is compatible with T', it follows that $\{p_s' \mid s \in 2^{n+1}\}$ is predense below q, so $q \Vdash$ "$\dot{f}(\beta_n) < d_\alpha(\beta_n)$". ■

Open questions

Much of the work in this paper would be simplified if $\mathbb{S}(\mathcal{U})$ was an Axiom A forcing, so we could ask the following:

Problem 59 *Let $\mathcal{U}$ be an ultrafilter. Is it possible to give $\mathbb{S}(\mathcal{U})$ an Axiom A structure?*

Unfortunately, we conjecture that this is not possible. An iteration theorem would greatly simplify the work in this paper, as well as in [3] and [4]. We could ask the following:

Problem 60 *If $\mathbb{P}$ preserves $\mathsf{c}_{\mathsf{min}}$-covering and $\mathbb{P} \Vdash$"$\dot{\mathbb{Q}}$ preserves $\mathsf{c}_{\mathsf{min}}$-covering", is it true that $\mathbb{P} * \dot{\mathbb{Q}}$ preserves $\mathsf{c}_{\mathsf{min}}$-covering?*

Problem 61 *If δ is a limit ordinal, $\left\langle \mathbb{P}_\alpha, \dot{\mathbb{R}}_\alpha \mid \alpha < \delta \right\rangle$ is a countable support iteration of proper forcings such that each $\mathbb{P}_\alpha$ preserves $\mathsf{c}_{\mathsf{min}}$-covering, is it true that $\mathbb{P}_\delta$ preserves $\mathsf{c}_{\mathsf{min}}$-covering?*

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Accepted in Topology and its applications

ON POSPÍŠIL IDEALS

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ABSTRACT. We study a class of ideals introduced by Pospíšil. We answer a question of the second author by proving that there is an $F_{\sigma\delta\sigma}$ ideal $\mathcal{I}$ such that every filter of character less than $\mathfrak{c}$ can be extended to an $\mathcal{I}$-ultrafilter. We also prove that this statement is consistently false for $F_{\sigma\delta}$-ideals.

Dedicated to the memory of William Wistar Comfort.

0.1. **Introduction.** Ultrafilters and independent families occupy one of the central places in Wis Comfort's research (see e.g. [8, 4, 5, 6, 7]). We revisit an old construction of B. Pospíšil involving both concepts. In his 1939 paper [17], Pospíšil proved that there is an ultrafilter on ω of character $\mathfrak{c}$. He did it by defining a certain filter from an independent family of size $\mathfrak{c}$, and then proved that any ultrafilter extending the filter has character $\mathfrak{c}$ (see [4, 2.6 and 2.7]). It is this filter of his, or the dual ideal, which is the main object of study here.

Recall that a family $\mathcal{X} = \{X_\alpha \mid \alpha \in \kappa\} \subseteq [\omega]^\omega$ is *independent* if for any two disjoint $F, G \in [\kappa]^{<\omega}$ the set

$$(\bigcap_{\alpha\in F} X_\alpha) \cap \bigcap_{\beta\in G} (\omega \setminus X_\beta)$$

is infinite. The fact that there are independent families of size $\mathfrak{c}$ was probably first proved by G. Fichtenholz and L. Kantorovitch [9]. In retrospect, their construction provides an independent family which is *perfect* as a subspace of the Cantor set.

Given a perfect independent family P define the *Pospíšil ideal of* P (denoted by $\mathsf{Pos}(P)$) as the ideal generated by the finite sets and

$$\{\omega \setminus x \mid x \in P\} \cup \{\bigcap C \mid C \in [P]^\omega\}.$$

Given an ideal $\mathcal{I}$ on a set X, J. Baumgartner [1] introduced the notion of an $\mathcal{I}$-ultrafilter as follows: An ultrafilter $\mathcal{U}$ on ω is an $\mathcal{I}$-*ultrafilter* if for every function $f : \omega \to X$ there is a $U \in \mathcal{U}$ such that $f[U] \in \mathcal{I}$. The notion of an $\mathcal{I}$-ultrafilter is closely tied with the *Katetov*

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order[1] as an ultrafilter $\mathcal{U}$ is an $\mathcal{I}$-ultrafilter if and only if $\mathcal{I} \nleq_{\mathsf{K}} \mathcal{U}^*$. Most of the standard properties of ultrafilters can then be characterized in this way, using Borel ideals[2] of low complexity:

(1) $\mathcal{U}$ is a Ramsey ultrafilter if and only if $\mathcal{U}$ is an $\mathcal{ED}$-ultrafilter.
(2) $\mathcal{U}$ is a P-point if and only if $\mathcal{U}$ is a $\mathsf{Fin}\times\mathsf{Fin}$-ultrafilter if and only if $\mathcal{U}$ is a conv-ultrafilter.
(3) $\mathcal{U}$ is a Q-point if and only if $\mathcal{ED}_{\mathsf{Fin}} \nleq_{\mathsf{KB}} \mathcal{U}^*$.
(4) $\mathcal{U}$ is a nowhere dense ultrafilter if and only if $\mathcal{U}$ is a nwd-ultrafilter.
(5) $\mathcal{U}$ is rapid if and only if $\mathcal{J} \nleq_{\mathsf{KB}} \mathcal{U}^*$ for any analytic P-ideal $\mathcal{J}$.

The reader may consult [1, 2, 11, 12] for more information. In other words, the Katětov order naturaly stratifies ultrafilters by "upward cones" of Borel ideals. Ultrafilters satisfying any of the above properties cannot be constructed in ZFC alone, so one has to wonder whether this stratification may consistently be vacuous. On the other hand, extending Pospíšil's argumentation slightly, one can show that $\mathsf{Pos}(P)$-ultrafilters do exist in ZFC, in fact, they *exist generically*, i.e. any filter of character $< \mathfrak{c}$ can be extended to a $\mathsf{Pos}(P)$-ultrafilter. However, $\mathsf{Pos}(P)$ is analytic, and appears not to be Borel. This led the second author to ask:

Problem 1 ([12])**.** *Is there a Borel ideal $\mathcal{I}$ such that $\mathcal{I}$-ultrafilters exist in ZFC?*

We shall answer this question in the positive by defining a Borel (in fact $F_{\sigma\delta\sigma}$) version of the Pospíšil ideal. Then we shall show that the complexity cannot be lowered, i.e. consistently $\mathcal{I}$-ultrafilters do not exist generically for any $F_{\sigma\delta}$ ideal $\mathcal{I}$.

We conclude this introduction by fixing some notation. Given $A, B \subseteq \omega$ we will say that *A is an almost subset of B* (or *B almost covers A*) if $A \setminus B$ is finite, this will be denoted by $A \subseteq^* B$. If $A \subseteq \omega$ we denote by A^* the complement of A and if $\mathcal{X} \subseteq \wp(\omega)$[3] we define $\mathcal{X}^* = \{A^* \mid A \in \mathcal{X}\}$. If $\mathcal{I}$ is an ideal, $\mathcal{I}^+$ denotes the family of all subsets of ω that are not in $\mathcal{I}$. Given $A \in \mathcal{I}^+$, the *restriction of $\mathcal{I}$ to A* is defined as $\mathcal{I} \cap \wp(A)$. We say a family $\mathcal{B} \subseteq \mathcal{I}$ is *cofinal in $\mathcal{I}$* if for every $A \in \mathcal{I}$ there is $B \in \mathcal{B}$ such that $A \subseteq B$. By $\mathsf{cof}(\mathcal{I})$ we denote the smallest size of a cofinal family of $\mathcal{I}$. In this note, a *tree* T will be a subset of

[1] Recall that given two ideals $\mathcal{I}$ and $\mathcal{J}$ on sets X and Y respectively we say that *$\mathcal{I}$ is Katětov below $\mathcal{J}$* and denote by $\mathcal{I} \leq_{\mathsf{K}} \mathcal{J}$, if there is a function $f : Y \to X$ such that $f^{-1}[I] \in \mathcal{J}$ for every $I \in \mathcal{I}$ (see [14]) We say that *$\mathcal{I}$ is Katětov-Blass below $\mathcal{J}$* and denote by $\mathcal{I} \leq_{\mathsf{KB}} \mathcal{J}$ if, moreover, the witnessing function is finite-to-one.

[2] For every $n \in \omega$ we define $C_n = \{(n, m) \mid m \in \omega\}$ and if $f : \omega \longrightarrow \omega$ let $D(f) = \{(n, m) \mid m \leq n\}$. Recall that:

(1) Fin is the ideal of all finite subsets of ω.
(2) $\mathcal{ED}$ is the ideal on $\omega \times \omega$ generated by $\{C_n \mid n \in \omega\}$ and (the graphs of) functions from ω to ω.
(3) $\mathcal{ED}_{\mathsf{Fin}}$ is the restriction of $\mathcal{ED}$ to $\Delta = \{(m, n) \mid m \leq n\}$.
(4) $\mathsf{Fin}\times\mathsf{Fin}$ is the ideal on $\omega \times \omega$ generated by $\{C_n \mid n \in \omega\} \cup \{D(f) \mid f \in \omega^\omega\}$.
(5) conv is the ideal on $[0, 1] \cap \mathbb{Q}$ generated by all sequences converging to a real number.
(6) nwd is the ideal on $\mathbb{Q}$ generated by all nowhere dense sets.

[3] If X is a set, we denote its power set by $\wp(X)$

$\omega^{<\omega}$ closed under taking initial segments. If T is a tree and $n \in \omega$, we denote $T_n = T \cap \omega^n$. The *branches of* T is defined as $[T] = \{f \mid \forall n\,(f \restriction n \in T)\}$.

0.2. **Pospíšil ideals.** We will say that a perfect tree $T \subseteq 2^{<\omega}$ is *independent* if the set of its *branches* $[T]$ is independent[4]. Abusing notation a bit, given an independent tree $T \subseteq 2^{<\omega}$ we shall denote the Pospíšil ideal $\mathsf{Pos}([T])$ simply by $\mathsf{Pos}(T)$, i.e. as the ideal generated by the finite sets and $\{x^* \mid x \in [T]\} \cup \{\bigcap C \mid C \in [[T]]^\omega\}$.

We now present the following mild extension of Pospíšil's argument crucial for our considerations.

Lemma 2. *If $T \subseteq 2^{<\omega}$ is an independent tree then $\mathsf{Pos}(T)$ is a proper ideal and $\mathsf{cof}\,(\mathcal{J}) = \mathfrak{c}$ for every ideal $\mathcal{J}$ extending $\mathsf{Pos}(T)$.*

Proof. First we will show that $\mathsf{Pos}(T)$ is a proper ideal, i.e. that $\omega \notin \mathsf{Pos}(T)$. Let $x_0, ..., x_n \in [T]$ and $C_0, ..., C_m$ be countable subsets of $[T]$. For each $i \leq m$ we choose $y_i \in C_i$ such that $y_i \notin \{x_0, ..., x_n\}$. Clearly $(\bigcap C_0) \cup ... \cup (\bigcap C_m)$ is a subset of $y_0 \cup ... \cup y_m$. Since T is independent, then $x_0 \cup ... \cup x_n$ does not almost cover $\omega \setminus (y_0 \cup ... \cup y_m)$, hence $\omega \notin \mathsf{Pos}(T)$.

Now, aiming toward a contradiction assume that there is an ideal $\mathcal{J}$ such that $\mathsf{Pos}(T) \subseteq \mathcal{J}$ and $\mathsf{cof}(\mathcal{J}) < \mathfrak{c}$. Let $\mathcal{B} \subseteq \mathcal{J}$ be a cofinal family of size less than $\mathfrak{c}$. Since $\{x^* \mid x \in [T]\} \subseteq \mathcal{J}$ then there is $C \in [[T]]^\omega$ and $B \in \mathcal{J}$ such that $x^* \subseteq B$ for every $x \in C$. On one hand $\bigcup\limits_{x \in C} x^* \subseteq B$ and in the other hand

$$\bigcap C = \left(\bigcup_{x \in C} x^* \right)^*$$

belongs to $\mathcal{J}$ so $\omega \in \mathcal{J}$ which is a contradiction. $\square$

Recall that an ideal $\mathcal{I}$ is P^- (see [13]) if for every $\{X_n \mid n \in \omega\} \subseteq \mathcal{I}^*$ there is $Y \in \mathcal{I}^+$ such that $Y \subseteq^* X_n$ for every $n \in \omega$. We shall need the following proposition in the next section.

Proposition 3. *If T is an independent tree then $\mathsf{Pos}(T)$ is not* P^-.

Proof. Let $D \subseteq [T]$ be a countable dense set. Clearly $D \subseteq \mathcal{I}^*$, we will show that every pseudo-intersection of D is in $\mathsf{Pos}(T)$. Let A be a pseudo-intersection of D, we must show that $A \in \mathsf{Pos}(T)$. Let $f : D \longrightarrow \omega$ such that $A \setminus d \subseteq f(d)$ for every $d \in D$. We may assume that there are two $x, y \in D$ such that $A \subseteq x \cap y$. Choose $\{x_n \mid n \in \omega\}$, $\{y_n \mid n \in \omega\}$ two subsets of $D \setminus \{x, y\}$ such that $x \restriction n = x_n \restriction n$ and $y \restriction n = y_n \restriction n$ for every $n \in \omega$. We recursively define two increasing sequences of natural numbers $\langle n_i \rangle_{i \in \omega}$ and $\langle m_i \rangle_{i \in \omega}$ as follows:

(1) $n_0 = 0$.

[4] We are identifing a set with its characteristic function

(2) $m_i > \mathsf{max}\{f(x_{n_i}), n_i\}$.
(3) $n_{i+1} > \mathsf{max}\{f(y_{m_i}), m_i\}$.

This is very easy to do. Let $X = \{x_{n_i} \mid i \in \omega\} \cup \{x\}$ and $Y = \{y_{m_i} \mid i \in \omega\} \cup \{y\}$. It is easy to see that $A \subseteq \bigcap X \cup \bigcap Y$ so $A \in \mathsf{Pos}(T)$. □

Recall that an ideal $\mathcal{I}$ is *ω-hitting* if for every $\{X_n \mid n \in \omega\} \subseteq [\omega]^\omega$ there is $B \in \mathcal{I}$ such that $|B \cap X_n| = \omega$ for every $n \in \omega$.

Unlike in the previous result, we will see that there are examples of Pospíšil ideals that are ω-hitting, but there are also some that are not. In order to prove this, we need the following definitions: Let $T \subseteq 2^{<\omega}$ be an independent tree.

(1) We say that T is a *hitting tree* if whenever $s \in T$, for almost all $n \in \omega$ there is $t \in T_{n+1}$ extending s such that $t(n) = 1$.
(2) We say that T has the *generic property* if for every $n \in \omega$, $X \subseteq T_n$ and $c : X \longrightarrow 2$ there are infinitely many $m > n$ such that for every $s \in T_{m+1}$ if $s \restriction n \in X$ then $s(m) = c(s \restriction n)$.

Proposition 4. *Let $T \subseteq 2^{<\omega}$ be an independent tree.*

(1) *If T has the generic property then $\mathsf{Pos}(T)$ is not ω-hitting.*
(2) *If T is a hitting tree then $\mathsf{Pos}(T^*)$ is ω-hitting.*

Proof. Let T be an independent tree with the generic property. Let $\mathcal{W}$ be the family of all pairs $p = (X, Y)$ such that there is an $n \in \omega$ such that X and Y are two disjoint non empty subsets of T_n. Given $p = (X, Y) \in \mathcal{W}$ such that $X, Y \in T_n$, let $W_p \in [\omega]^\omega$ such that for every $m \in W_p$ and for every $s \in T_{m+1}$ if $s \restriction n \in X$ then $s(m) = 1$ and if $s \restriction n \in Y$ then $s(m) = 0$. Since $\mathcal{W}$ is a countable family, then $\{W_p \mid p \in \mathcal{W}\}$ is a countable family of infinite sets.

We claim that no element of $\mathsf{Pos}(T)$ has infinite intersection with each W_p. Let $A \in \mathsf{Pos}(T)$, we may assume there are $x_0, ..., x_n \in [T]$ and $C_0, ..., C_m$ countable subsets of T such that $A = \bigcup\limits_{i \leq n} x_i^* \cup \bigcup\limits_{j \leq m} (\bigcap C_j)$. For every $j \leq m$ we choose $y_j \in C_j$ such that $y_j \notin \{x_0, ..., x_n\}$ and $y_j \neq y_k$ if $j \neq k$. We may then find $l \in \omega$ such any two different elements of $\{x_i \mid i \leq n\} \cup \{y_j \mid j \leq m\}$ differ before l. We now define $p = (X, Y)$ where $X = \{x_i \restriction l \mid i \leq n\}$ and $Y = \{y_j \restriction l \mid j \leq m\}$. Note that if $k \in W_p$ then $k \in x_0 \cap ... \cap x_n$ and $k \notin y_0 \cup ... \cup y_m$ so $k \notin A$.

Finally, it is easy to see that if T is a hitting tree then $[T] \subseteq \mathsf{Pos}(T^*)$ is already ω-hitting. □

It should be noted here that both kinds of trees actually exist:

Proposition 5. *There are independent trees $T, S \subseteq 2^{<\omega}$ such that T has the generic property and S is a hitting tree.*

Proof. By $\mathbb{T}$ we denote the set of all finite trees $p \subseteq 2^{<\omega}$ such that all maximal nodes of p have the same height, this common value will be denoted by $ht(p)$. Given $p, q \in \mathbb{T}$ we define the following:

1. $p \leq_0 q$ if $p \cap 2^{ht(q)} = q$ (hence $q \subseteq p$).
2. $p \leq_1 q$ if $p \leq_0 q$ and for every $n \in \omega$ and $s \in q$ if $ht(q) \leq n < ht(p)$ then there is $t \in p$ extending s such that $t(n) = 1$.

By $\mathsf{max}(p)$ we denote the set of maximal nodes of p. We define the following sets:

1. For every $n \in \omega$ we define $D_0(n)$ as the set of all $p \in \mathbb{T}$ such that $|s^{-1}(0)|$, $|s^{-1}(1)| \geq n$ for every $s \in \mathsf{max}(p)$.
2. For every $n \in \omega$ we define $D_1(n)$ as the set of all $p \in \mathbb{T}$ such that there is $k \in \omega$ such that $n < k < ht(p)$ and every node in p_k is an splitting node.
3. For every $n \in \omega$ we define $D_2(n)$ as the set of all $p \in \mathbb{T}$ such that there is $k \in \omega$ such that $n < k < ht(p)$ and for every $Z \in [p_k]^n$ and $c : Z \longrightarrow 2$ the set $\bigcap_{s \in Z} s^{-1}(c(s))$ has size at least n.
4. For every $n, m \in \omega$ we define $B_{n,m}$ as the set of all $p \in \mathbb{T}$ such that for every $X \subseteq p_n$ and $c : X \longrightarrow 2$ there are $j_0 < ... < j_m < ht(p)$ such that for every $i \leq m$ and for every $s \in p_{j_i+1}$ if $s \restriction n \in X$ then $s(m) = c(s \restriction n)$.

Let $\mathcal{D} = \{D_i(n) \mid i < 3 \wedge n \in \omega\}$ and $\mathcal{B} = \mathcal{D} \cup \{B_{n,m} \mid n, m \in \omega\}$. It is easy to see that each $D_i(n)$ is $\leq_1$-dense (i.e. for every $p \in \mathbb{T}$ there is $q \in D_i(n)$ such that $q \leq_1 p$) and each $B_{n,m}$ is $\leq_0$-dense. By the Rasiowa–Sikorski lemma (see [15]) there are $G_0, G_1 \subseteq \mathbb{T}$ with the following properties:

1. G_0 is a filter in $(\mathbb{T}, \leq_0)$.
2. $G_0 \cap W \neq \emptyset$ for every $W \in \mathcal{B}$.
3. G_1 is a filter in $(\mathbb{T}, \leq_1)$.
4. $G_1 \cap W \neq \emptyset$ for every $W \in \mathcal{D}$.

It is then easy to see that $T = \bigcup G_0$ has the generic property and $S = \bigcup G_1$ is a hitting tree. □

0.3. **$\mathcal{I}$-Ultrafilters.** We will now provide a positive answer to Problem 1 here.

Given an ideal $\mathcal{I}$, we say that *$\mathcal{I}$-ultrafilters exist generically* if every filter of character less than $\mathfrak{c}$ can be extended to an $\mathcal{I}$-ultrafilter. Generic existence of $\mathcal{I}$-ultrafilters can be conveniently characterized by the *generic existence number* or *exterior cofinality* $\mathsf{cof}^*(\mathcal{I})$ defined as the smallest cofinality of an ideal $\mathcal{J}$ such that $\mathcal{I} \subseteq \mathcal{J}$, introduced and studied studied by Brendle and Flašková in [2] and, independently, by Hong and Zhang in [10][5]:

[5]In [2] the cardinal invariant is denoted by $\mathfrak{ge}(\mathcal{I})$, and in [10] by $\mathsf{non}^{**}(\mathcal{I})$.

Lemma 6 ([2], [10]). *If $\mathcal{I}$ is an ideal on ω then $\mathcal{I}$-ultrafilters exist generically if and only if $\mathsf{cof}^*(\mathcal{I}) = \mathfrak{c}$.*

In particular, by Lemma 2 if T is an independent tree, then $\mathsf{Pos}(T)$-ultrafilters exist generically.

The problem is that $\mathsf{Pos}(T)$ does not seem to be Borel. However, we will now prove that every ideal $\mathsf{Pos}(T)$ can be extended to a Borel ideal, and as cof^* is increasingly monotone , $\mathcal{I}$-ultrafilters exist generically also for this new, Borel, ideal $\mathcal{I}$ as well. Now, the existence of such a Borel ideal can be deduced directly from a theorem of H. Sakai [19] who showed that every analytic ideal can be extended to a Borel one. However, Sakai's proof does not give any bound on the complexity. We shall give an explicit definition of an $F_{\sigma\delta\sigma}$ ideal extending $\mathsf{Pos}(T)$ here.

Given a set A and $m \in \omega$ we define $Z_m(A)$ as the set of all $\overline{y} = (\overline{y}(i))_{i<m} \in A^m$ such that $\overline{y}(i) \neq \overline{y}(j)$ whenever $i \neq j$. Let $T \subseteq 2^{<\omega}$ be an independent tree.

(1) For every $\overline{x} \in [T]^n$ let $C(\overline{x}) = \bigcup\limits_{i<n} (x(i)^*)$ and $D(\overline{x}) = \bigcap\limits_{i<n} x(i)$.
(2) For every $\overline{x} \in [T]^n$ and $\overline{y}_1, ..., \overline{y}_k \in Z_m([T])$ we define $H(\overline{x}, \overline{y}_1, ..., \overline{y}_m) = C(\overline{x}) \cup \bigcup\limits_{j \leq k} D(\overline{y}_j)$.
(3) For every $n > 0$ we define $\mathcal{H}(n)$ as the set of all $A \subseteq \omega$ such that for every $m > n$ there are $k \geq 1$, $\overline{x} \in [T]^n$ and $\overline{y}_1, ..., \overline{y}_k \in Z_m([T])$ such that $A \subseteq H(\overline{x}, \overline{y}_1, ..., \overline{y}_k)$.

It is easy to see that $\mathcal{H}(n) \subseteq \mathcal{H}(n+1)$ for every $n \in \omega$. We now introduce the following definition:

Definition 7. *If $T \subseteq 2^{<\omega}$ is an independent tree, we define $\mathsf{Pos_B}(T) = \bigcup\limits_{n\in\omega} \mathcal{H}(n)$.*

We will need the following lemma:

Lemma 8. *Let $n > 0$. If $A, B \in \mathcal{H}(n)$ then $A \cup B \in \mathcal{H}(2n)$.*

Proof. Let $A, B \in \mathcal{H}(n)$ and $m > 2n$. Since $A, B \in \mathcal{H}(n)$ then there are $\overline{x}, \overline{a} \in [T]^n$, $\overline{y}_1, ..., \overline{y}_{k_1} \in Z_m([T])$ and $\overline{b}_1, ..., \overline{b}_{k_2} \in Z_m([T])$ such that $A \subseteq H(\overline{x}, \overline{y}_1, ..., \overline{y}_{k_1})$ and $B \subseteq H\left(\overline{a}, \overline{b}_1, ..., \overline{b}_{k_2}\right)$. It follows that

$$A \cup B \subseteq H(\overline{x}^\frown\overline{a}, \overline{y}_1, ..., \overline{y}_{k_1}, \overline{b}_1, ..., \overline{b}_{k_2}).$$

$\square$

We now have the following result:

Proposition 9. *Let $T \subseteq 2^{<\omega}$ be an independent tree.*

(1) *$\mathsf{Pos_B}(T)$ is an $F_{\sigma\delta\sigma}$-ideal extending $\mathsf{Pos}(T)$.*
(2) *$\mathsf{Pos_B}(T)$ can not be extended to an $F_{\sigma\delta}$-ideal.*

Proof. We will first prove $\mathsf{Pos_B}(T)$ is an ideal. It is closed under unions by the previous lemma, so it is enough to prove that $H(\overline{x}, \overline{y}_1, ..., \overline{y}_k) \neq \omega$ for every $\overline{x} \in [T]^n$ and $\overline{y}_1, ..., \overline{y}_k \in Z_m([T])$ with $n < m$. Since $n < m$ then there is $i < m$ such that $z = \overline{y}_1(i) \neq \overline{x}(j)$ for every

$j < n$. Note that if $z \in im(\overline{y}_j)$ (for any $j \leq k$) then $D(\overline{y}_j) \subseteq z$. Since T is independent, we know that z^* is not almost contained in $C(\overline{x}) \cup \bigcup\{D(\overline{y}_j) \mid z \notin im(\overline{y}_j)\}$ so $H(\overline{x}, \overline{y}_1, ..., \overline{y}_m)$ does not almost contain z^*.

We will now prove that $\mathsf{Pos}(T) \subseteq \mathsf{Pos}_{\mathsf{B}}(T)$. Let $x \in [T]$ and $C = \{y_i \mid i \in \omega\} \in [[T]]^{\omega}$. Since $x^* \cup \bigcap C \subseteq x^* \cup (y_0 \cap ... \cap y_m)$ for every $m \in \omega$ then $x^* \cup \bigcap C \in \mathcal{H}(1)$, hence $\mathsf{Pos}(T) \subseteq \mathsf{Pos}_{\mathsf{B}}(T)$.

Next we shall prove that $\mathsf{Pos}_{\mathsf{B}}(T)$ is an $F_{\sigma\delta\sigma}$ ideal. Let $a_m(T)$ be the set of finite sequences $\overline{s} = (s_1, ..., s_m) \in T^m$ such that s_i is incompatible with s_j whenever $i \neq j$. For every $\overline{s} = (s_1, ..., s_m) \in a_m(T)$ we define $\langle \overline{s} \rangle = \{(y_1, ..., y_m) \mid \forall i \leq m\, (y_i \in [T_{s_i}])\}$. For every $0 < n < m$, $k > 0$ and $\overline{s}_1, ..., \overline{s}_k \in a_m(T)$ we define

$$\mathcal{H}(n, m, \overline{s}_1, ..., \overline{s}_k) = \{H(\overline{x}, \overline{y}_1, ..., \overline{y}_k) \mid \overline{x} \in [T]^n \wedge \forall i \leq k\, (\overline{y}_i \in \langle \overline{s}_i \rangle)\}.$$

Note that $\mathcal{H}(n, m, \overline{s}_1, ..., \overline{s}_k)$ is a closed set since it is a continous image of $[T]^n \times \prod_{i \leq k} \langle \overline{s}_i \rangle$. In this way, the subset closure of $\mathcal{H}(n, m, \overline{s}_1, ..., \overline{s}_k)$ (denoted by $\mathcal{H}_{\downarrow}(n, m, \overline{s}_1, ..., \overline{s}_k)$) is closed as well.

Let $\mathcal{B}(n, m) = \bigcup_{k \in \omega} \{\mathcal{H}_{\downarrow}(n, m, \overline{s}_1, ..., \overline{s}_k) \mid \overline{s}_1, ..., \overline{s}_k \in a_m(T)\}$ and note that $\mathcal{B}(n, m)$ is an F_σ-set. Clearly $\mathcal{H}(n) = \bigcap_{m > n} \mathcal{B}(n, m)$ so each $\mathcal{H}(n)$ is an $F_{\sigma\delta}$-set and then $\mathsf{Pos}_{\mathsf{B}}(T) = \bigcup_{n > 0} \mathcal{H}(n)$ is an $F_{\sigma\delta\sigma}$-set. Finally, by results of Solecki, Laczkovich and Recław (see [21] and [16]), no $F_{\sigma\delta}$ ideal is Katětov above $\mathsf{Fin} \times \mathsf{Fin}$. Since $\mathsf{Fin} \times \mathsf{Fin} \leq_{\mathsf{K}} \mathsf{Pos}_{\mathsf{B}}(T)$ by Proposition 3, and the fact (see [13]) that an ideal $\mathcal{I}$ is P^- if and only if $\mathsf{Fin} \times \mathsf{Fin} \nleq_{\mathsf{K}} \mathcal{I}$, the result follows. □

We can then conclude the desired result:

Corollary 10. *There is an $F_{\sigma\delta\sigma}$-ideal $\mathcal{I}$ for which $\mathcal{I}$-ultrafilters exist generically.*

The previous result is optimal in the sense that it is no longer true for $F_{\sigma\delta}$-ideals, as we will prove now.

Definition 11 (see [20, 18, 3])**.** *An ideal $\mathcal{I}$ is* Shelah-Steprāns *if for every sequence $\{s_n \mid n \in \omega\} \subseteq [\omega]^{<\omega} \setminus \{\emptyset\}$ one of the following holds:*

1. *There is $A \in \mathcal{I}$ such that $A \cap s_n \neq \emptyset$ for every $n \in \omega$.*
2. *There is $B \in \mathcal{I}$ such that $s_n \subseteq B$ for infinitely many $n \in \omega$.*

Let $\mathcal{I}$ be an ideal and $X \subseteq [\omega]^{<\omega} \setminus \{\emptyset\}$. We say that X witness that $\mathcal{I}$ is not Shelah-Steprāns if neither of the two possibilities above hold for X.

Lemma 12. *Let $\mathcal{I}$ be an analytic ideal, and let $X = \{s_n \mid n \in \omega\}$ witness that $\mathcal{I}$ is not Shelah-Steprāns. If $\mathbb{P}$ is a forcing notion, then X still witnesses that $\mathcal{I}$ is not Shelah-Steprāns after forcing with $\mathbb{P}$.*

Proof. Let $\mathcal{B} = \{A \mid \forall n\, (A \cap s_n \neq \emptyset)\}$ and $\mathcal{D} = \{B \mid \exists^{\infty} n\, (s_n \subseteq B)\}$. Both $\mathcal{B}$ and $\mathcal{D}$ are Borel sets, and not being Shelah-Steprāns simply means that $\mathcal{I} \cap (\mathcal{B} \cup \mathcal{D}) = \emptyset$. Since this is an analytic statement, it holds in any forcing extension. □

We say that an ideal $\mathcal{I}$ is *nowhere Shelah-Steprāns* if $\mathcal{I} \restriction A$ is not Shelah-Steprāns for every $A \in \mathcal{I}^+$. By Schoenfield's absoluteness we have the following result.

Lemma 13. *Let $\mathcal{I}$ be a nowhere Shelah-Steprāns analytic ideal. If $\mathbb{P}$ is a forcing notion, then $\mathcal{I}$ is still nowhere Shelah-Steprāns after forcing with $\mathbb{P}$.*

Recall that given an ideal $\mathcal{I}$ on ω (or on any countable set), the *Mathias forcing* $\mathbb{M}(\mathcal{I})$ associated with $\mathcal{I}$ is the set of all pairs (s, A) where $s \in [\omega]^{<\omega}$ and $A \in \mathcal{I}$. If $(s, A), (t, B) \in \mathbb{M}(\mathcal{I})$ then $(s, A) \leq (t, B)$ if the following conditions hold:

(1) t is an initial segment of s.
(2) $B \subseteq A$.
(3) $(s \setminus t) \cap B = \emptyset$.

If $G \subseteq \mathbb{M}(\mathcal{I})$ is a generic filter, we define *the generic real* as $r_{gen} = \bigcup \{s \mid \exists A ((s, A) \in G)\}$.

Lemma 14. *Let $\mathcal{I}$ be a nowhere Shelah-Steprāns analytic ideal. If $G \subseteq \mathbb{M}(\mathcal{I})$ is a generic filter then the following holds in $V[G]$:*

(1) $r_{gen} \in \mathcal{I}^+$.
(2) *If $A \in V \cap \mathcal{I}^+$ then $r_{gen} \cap A \in \mathcal{I}^+$.*
(3) *If $A \in V \cap \mathcal{I}$ then $r_{gen} \cap A$ is finite.*

Proof. Note that the first item follows from the second by taking $A = \omega$. Let $A \in V \cap \mathcal{I}^+$, we will prove that $r_{gen} \cap A \in \mathcal{I}^+$. Since $\mathcal{I}$ is nowhere Shelah-Steprāns and $A \in \mathcal{I}^+$ then there is $X = \{s_n \mid n \in \omega\} \subseteq [A]^{<\omega} \setminus \{\emptyset\}$ with the following properties:

(1) For every $B \in \mathcal{I}$ there is $n \in \omega$ such that $s_n \cap B = \emptyset$.
(2) If $W \in [\omega]^{\omega}$ then $\bigcup\limits_{n \in W} s_n \in \mathcal{I}^+$.

Furthermore, since $\mathcal{I}$ is analytic the two previous properties hold in forcing extention of V. By a a simple genericity argument and the first property, we can conclude that there are infinitely many $n \in \omega$ such that $s_n \subseteq r_{gen} \cap A$ and then $r_{gen} \cap A \in \mathcal{I}^+$ by the second property.

The third item is easy and holds for every ideal. □

The following result was proved in [3]:

Proposition 15. *If $\mathcal{I}$ is a Borel ideal, then $\mathcal{I}$ is Shelah-Steprāns if and only if $\mathsf{Fin} \times \mathsf{Fin} \leq_{\mathsf{K}} \mathcal{I}$.*

As was mentioned earlier, no $F_{\sigma\delta}$-ideal is Katětov above $\mathsf{Fin} \times \mathsf{Fin}$, so $F_{\sigma\delta}$-ideals are nowhere Shelah-Steprāns. The following result is based on the results of [2].

Proposition 16. *It is consistent that $\mathcal{I}$-ultrafilters do not exist generically for every analytic nowhere Shelah-Steprāns ideal (in particular for $F_{\sigma\delta}$-ideals).*

Proof. Given a model of set theory W, we define $\mathbb{P}(W)$ as the finite support iteration of the Mathias forcing of all analytic nowhere Shelah-Steprāns ideal. Let V be a model where $\mathfrak{c} = \omega_2$. We perform a finite support iteration $\{\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha < \omega_1\}$ where $\mathbb{P}_\alpha \Vdash$ "$\dot{\mathbb{Q}}_\alpha = \dot{\mathbb{P}}(V_\alpha)$" where V_α is the model obtained after forcing with $\mathbb{P}_\alpha$. We will argue that V_{ω_1} is the desired model. Let $\mathcal{I} \in V_{\omega_1}$ be an analytic nowhere Shelah-Steprāns ideal. Since $\mathcal{I}$ can be coded by a real, there is $\alpha < \omega_1$ such that $\mathcal{I} \in V_\alpha$, and by Shoenfield's Absoluteness Theorem $V_\alpha \models \mathcal{I}$ is Shelah-Steprāns. Given $\beta > \alpha$ let $r^{\mathcal{I}}_\beta$ be the $\mathbb{M}(\mathcal{I})$ generic real added by $\mathbb{P}_{\beta+1}$. Let $x^{\mathcal{I}}_\beta = \omega \setminus r^{\mathcal{I}}_\beta$ and define $\mathcal{J}$ as the ideal generated by $\left\{x^{\mathcal{I}}_\beta \mid \alpha < \beta < \omega_1\right\}$. By the previous result, it follows that $\mathcal{J}$ is a proper ideal and $\mathcal{I} \subseteq \mathcal{J}$ so $\mathsf{cof}^*(\mathcal{I}) \leq \mathsf{cof}(\mathcal{J}) = \omega_1$. □

In [2] Brendle and Flašková proved that if $\mathcal{I}$ is an F_σ-ideal then $\mathsf{cof}^*(\mathcal{I}) \leq \mathsf{cof}(\mathcal{N})$ (where $\mathcal{N}$ denotes the ideal of all null sets). This can actually be deduced directly using some results that can be currently found in the literature: In [19] Sakai proved that there is an analytic P-ideal $\mathcal{P}_{\mathsf{max}}$ such that $\mathcal{I} \leq_{\mathsf{KB}} \mathcal{P}_{\mathsf{max}}$ where $\mathcal{I}$ is either an F_σ-ideal or an analytic P-ideal. In particular, $\mathsf{cof}^*(\mathcal{I}) \leq \mathsf{cof}(\mathcal{P}_{\mathsf{max}})$ for every F_σ-ideal $\mathcal{I}$. In [22] Todorcevic showed that the cofinality of every analytic P-ideal is at most $\mathsf{cof}(\mathcal{N})$. Therefore, we conclude that if $\mathcal{I}$ is an F_σ-ideal then $\mathsf{cof}^*(\mathcal{I}) \leq \mathsf{cof}(\mathcal{N})$. The following questions remain open:

Problem 17. *Is there an F_σ-ideal $\mathcal{I}$ for which $\mathcal{I}$-ultrafilters exist?*

Problem 18. *Is there an $F_{\sigma\delta}$-ideal $\mathcal{I}$ for which $\mathcal{I}$-ultrafilters exist? What about the density zero ideal or $\mathcal{P}_{\mathsf{max}}$?*

Note that by the aforementioned result of Sakai [19] the non-existence of a $\mathcal{P}_{\mathsf{max}}$-ultrafilter would imply negative answer to Problem 17, i.e. consistency of $\mathcal{I} \leq_{\mathsf{K}} \mathcal{U}^*$ for every ultrafilter $\mathcal{U}$ and every F_σ ideal $\mathcal{I}$.

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Restricted MAD families

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Abstract

Let $\mathcal{I}$ be an ideal on ω. By $\mathsf{cov}^*(\mathcal{I})$ we denote the least size of a family $\mathcal{B} \subseteq \mathcal{I}$ such that for every infinite $X \in \mathcal{I}$ there is $B \in \mathcal{B}$ for which $B \cap X$ is infinite. We say that an AD family $\mathcal{A} \subseteq \mathcal{I}$ is a *MAD family restricted to* $\mathcal{I}$ if for every infinite $X \in \mathcal{I}$ there is $A \in \mathcal{A}$ such that $|X \cap A| = \omega$. Let $\mathfrak{a}(\mathcal{I})$ be the least size of an infinite MAD family restricted to $\mathcal{I}$. We prove that If $\max\{\mathfrak{a},\mathsf{cov}^*(\mathcal{I})\} = \omega_1$ then $\mathfrak{a}(\mathcal{I}) = \omega_1$. We conclude that if $\mathcal{I}$ is tall and $\mathfrak{c} \leq \omega_2$ then $\mathfrak{a}(\mathcal{I}) = \max\{\mathfrak{a},\mathsf{cov}^*(\mathcal{I})\}$. We use these results to prove that if $\mathfrak{c} \leq \omega_2$ then $\mathfrak{o} = \overline{\mathfrak{o}}$ and that $\mathfrak{a}_s = \max\{\mathfrak{a},\mathsf{non}(\mathcal{M})\}$.

1 Introduction and preliminaries

We say that $\mathcal{A} \subseteq \wp(\omega)$[1] is an *almost disjoint family* (*AD*) if the intersection of any two of its elements is finite and $\mathcal{A}$ is *MAD* if it is maximal with respect to this property. MAD families have played a very important role in set theory, functional analysis and topology (see [12]). It follows by Zorn's lemma that every AD family can be extended to a MAD family; however, we may still wonder how the extensions of an AD family might be. This has been previously studied by Leathrum in [18] and was the object of study in [21]. Understanding how AD families can be extended to MAD families is fundamental in order to study certain combinatorial aspects of MAD families. This is relevant in the study of *forcing indestructibility of MAD families.* Given a MAD family $\mathcal{A}$ and a forcing $\mathbb{P}$, we say that $\mathcal{A}$ is $\mathbb{P}$-*destructible* if $\mathcal{A}$ is no longer maximal after forcing with $\mathbb{P}$. For example, it is known that if $\mathcal{A}$ is a MAD family on $\mathbb{Q}$ such that every element of $\mathcal{A}$ is nowhere dense, then it will be destroyed by Cohen forcing. The reader that wishes to learn more about destructibility of MAD families may consult [13], [7], [15], [8], [11] or [17].

In order to state the main results of the paper, we need the following notions:

Definition 1 *Let $\mathcal{I}$ be an ideal (in a countable set).*

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[1] If X is a set, by $\wp(X)$ we denote its power set.

1. *We define* $\mathsf{cov}^*(\mathcal{I})$ *as the least size of a family* $\mathcal{B} \subseteq \mathcal{I}$ *such that for every infinite* $X \in \mathcal{I}$ *there is* $B \in \mathcal{B}$ *for which* $B \cap X$ *is infinite.*
2. *We say that an* AD *family* $\mathcal{A} \subseteq \mathcal{I}$ *is a* MAD family restricted to $\mathcal{I}$ *if for every infinite* $X \in \mathcal{I}$ *there is* $A \in \mathcal{A}$ *such that* $|X \cap A| = \omega$.
3. $\mathfrak{a}(\mathcal{I})$ *is the least size of an infinite* MAD *family restricted to* $\mathcal{I}$.

In the next section, we will prove our main combinatorial lemma: If the maximum of $\mathfrak{a}$ and $\mathsf{cov}^*(\mathcal{I})$ is ω_1 then $\mathfrak{a}(\mathcal{I}) = \omega_1$. This is a simple, yet very useful result. We will see several applications of this theorem. One of our applications deal with the *off-branch numbers* of Leathrum (see [18]):

Definition 2 1. *A set* $B \subseteq 2^{<\omega}$ *is called* off-branch *if it has finite intersection with every branch of* $2^{<\omega}$ *(i.e. if* $r \in 2^{\omega}$ *then* $B \cap \{r \upharpoonright n \mid n \in \omega\}$ *is finite).*

2. $\mathfrak{o}$ *is the smallest size of a maximal family of almost disjoint off-branch sets.*
3. $\overline{\mathfrak{o}}$ *is the smallest size of a maximal family of almost disjoint antichains of* $2^{<\omega}$.

It is easy to see that $\mathfrak{o} \leq \overline{\mathfrak{o}}$. It is an open question of Leathrum if the inequality $\mathfrak{o} < \overline{\mathfrak{o}}$ is consistent. We do not know the answer to this question, but we will prove prove that $\mathfrak{o} = \omega_1$ implies that $\overline{\mathfrak{o}} = \omega_1$. In particular, it is not possible to get the inequality if the size of the continuum is at most ω_2. We will also use our result to provide answers to some questions found in [21].

Another of our applications deals with the cardinal invariant $\mathfrak{a}_s$, which is defined as the smallest size of a maximal family of eventually different partial functions. In [6] Brendle showed that it is consistent that $\mathsf{max}\{\mathfrak{a},\mathsf{non}(\mathcal{M})\} < \mathfrak{a}_s$ (where $\mathfrak{a}$ is smallest size of a MAD family and $\mathsf{non}(\mathcal{M})$ is the smallest size of a non-meager subset of the Baire space). In the model of Brendle, the continuum has size at least ω_3. This is no coincidence, we will prove that if the continuum has size at most ω_2, then $\mathfrak{a}_s = \mathsf{max}\{\mathfrak{a},\mathsf{non}(\mathcal{M})\}$.

The cardinal invariant $\mathfrak{a}^+(\omega_1)$ (introduced in [21]) is defined as the least κ such that every AD family of size ω_1 can be extended to a MAD family of size at most κ. In [21] it was proved that it is consistent that $\omega_2 = \mathfrak{a}^+(\omega_1) < \mathfrak{c}$ (where $\mathfrak{c}$ denotes the cardinality of the continuum). Nevertheless, the following problem is still open:

Problem 3 ([21]) *Is* $\omega_1 = \mathfrak{a}^+(\omega_1) < \mathfrak{c}$ *consistent? In other words, is the statement "Every* AD *family of size* ω_1 *can be extended to a* MAD *family of size* ω_1*" consistent with the negation of the Continuum Hypothesis?*

We do not know the answer to the problem, but we will derive some consequences from the assumption that $\omega_1 = \mathfrak{a}^+(\omega_1) < \mathfrak{c}$ and show that it fails in most of the known models of set theory.

Our notation is mostly standard. If $\mathcal{X}$ is a set of subsets of ω, we denote by $\mathcal{X}^{\perp}$ the set of all infinite $A \subseteq \omega$ that are almost disjoint from every element of $\mathcal{X}$. If $\mathcal{I}$ is an ideal in ω, we denote by $\mathcal{I}^{+}$ as those subsets of ω that are not in $\mathcal{I}$. If $X \in \mathcal{I}^{+}$ then by $\mathcal{I} \restriction X$ we denote the restriction of $\mathcal{I}$ to X. We say that $\mathcal{I}$ is *tall* if for every infinite $X \subseteq \omega$ there is $A \in \mathcal{I}$ such that $A \subseteq X$. The relationship between MAD families and definable ideals (typically Borel of low complexity) has been an active area of research (see e.g. [12], [7]).

If $\mathcal{J}$ is a σ-ideal of a Polish space X, we denote by $\mathsf{cov}(\mathcal{J})$ the smallest size of a subfamily of $\mathcal{J}$ that covers X. By $\mathsf{non}(\mathcal{J})$ we denote the smallest size of a subset of X that it is not in $\mathcal{J}$. By $\mathcal{M}$ we denote the σ-ideal of all meager sets in 2^{ω}, and by $\mathcal{N}$ we denote the σ-ideal of all Lebesgue null subsets of 2^{ω}.

The size of the continuum is denoted by $\mathfrak{c}$. Let $f, g \in \omega^{\omega}$, define $f \leq g$ if $f(n) \leq g(n)$ for every $n \in \omega$, and $f \leq^{*} g$ if $f(n) \leq g(n)$ for almost all $n \in \omega$ except finitely many. We say a family $\mathcal{B} \subseteq \omega^{\omega}$ is *unbounded* if $\mathcal{B}$ is unbounded with respect to $\leq^{*}$. The *bounding number* $\mathfrak{b}$ is the size of the smallest unbounded family. We say that S *splits* X if $S \cap X$ and $X \setminus S$ are both infinite. A family $\mathcal{S} \subseteq [\omega]^{\omega}$ is a *splitting family* if for every $X \in [\omega]^{\omega}$ there is $S \in \mathcal{S}$ such that S splits X. The *splitting number* $\mathfrak{s}$ is the smallest size of a splitting family. The reader may consult the survey [4] for the main properties of the cardinal invariants used in this paper.

1.1 Main combinatorial result

Let $\mathcal{I}$ be an ideal. It is easy to see that $\mathsf{cov}^{*}(\mathcal{I}) \leq \mathfrak{a}(\mathcal{I})$. Furthermore, if $\mathcal{I}$ is a tall ideal, then every MAD family restricted to $\mathcal{I}$ is actually a MAD family in the usual sense; hence $\mathfrak{a} \leq \mathfrak{a}(\mathcal{I})$. Hence, if $\mathcal{I}$ is tall, then $\max\{\mathfrak{a}, \mathsf{cov}^{*}(\mathcal{I})\} \leq \mathfrak{a}(\mathcal{I})$. We will show that there is a deeper connection between these cardinals.

The following lemma is well known. We prove it for the sake of completeness:

Lemma 4 *Let $\mathcal{C} = \{C_n \mid n \in \omega\} \subseteq [\omega]^{\omega}$ be a partition of ω. There is an almost disjoint family $\mathcal{D}$ such that:*

1. *$\mathcal{D} \subseteq \mathcal{C}^{\perp}$,*
2. *$|\mathcal{D}| = \mathfrak{a}$, and*
3. *for every $X \in \mathcal{C}^{\perp}$ there is $D \in \mathcal{D}$ such that $|D \cap X| = \omega$.*

Proof. Let $\mathcal{A}$ be a MAD family of size $\mathfrak{a}$. We may assume there is $\mathcal{B} = \{A_n \mid n \in \omega\} \subseteq \mathcal{A}$ that is a partition of ω. Let $f : \omega \longrightarrow \omega$ be a bijection such that $f[A_n] = C_n$. It is easy to see that $\mathcal{D} = f[\mathcal{A} \setminus \mathcal{B}]$ has the desired properties. ■

We can now prove the following:

Proposition 5 *If $\max\{\mathfrak{a}, \mathsf{cov}^{*}(\mathcal{I})\} = \omega_1$ then $\mathfrak{a}(\mathcal{I}) = \omega_1$.*

Proof. Let $\mathcal{B} = \{B_\alpha \mid \alpha \in \omega_1\} \subseteq \mathcal{I}$ such that for every $X \in [\omega]^\omega$ there is $\alpha \in \omega_1$ for which $X \cap B_\alpha$ is infinite. We may assume that $\{B_n \mid n \in \omega\} \subseteq \mathcal{B}$ is a partition of ω. We will recursively build a sequence of AD families $\langle \mathcal{A}_\alpha \rangle_{\alpha \in \omega_1}$ such that:

1. $\langle \mathcal{A}_\alpha \rangle_{\alpha \in \omega_1}$ is an increasing chain of almost disjoint families of size ω_1,
2. $\mathcal{A}_\alpha \restriction B_\alpha$ is a (possibly finite) MAD family in B_α for every $\alpha \in \omega_1$, and
3. $\overline{\mathcal{A}_\alpha} = \mathcal{A}_\alpha \setminus \bigcup_{\xi<\alpha} \mathcal{A}_\xi \subseteq \wp(B_\alpha)$.

We start by choosing $\{\mathcal{A}_n \mid n \in \omega\}$ such that $\overline{\mathcal{A}_n}$ is MAD family of subsets of B_n of size ω_1 for every $n \in \omega$. Assume $\alpha \leq \omega_1$ is an infinite ordinal, and we have already build all the $\mathcal{A}_\xi$ for $\xi < \alpha$. We shall see how to find $\mathcal{A}_\alpha$. In case $\bigcup_{\xi<\alpha} \mathcal{A}_\xi \restriction B_\alpha$ is already a MAD family in B_α, we define $\mathcal{A}_\alpha = \bigcup_{\xi<\alpha} \mathcal{A}_\xi$. So we assume that $\bigcup_{\xi<\alpha} \mathcal{A}_\xi \restriction B_\alpha$ is not maximal in B_α. Enumerate $\alpha = \{\alpha_n \mid n \in \omega\}$. Note that $B_\alpha \not\subseteq^* B_{\alpha_0} \cup ... \cup B_{\alpha_m}$ for all $m \in \omega$. If this was not the case, every infinite subset of B_α would have infinite intersection with some B_{α_i} and therefore $\mathcal{A}_{\alpha_0} \cup ... \cup \mathcal{A}_{\alpha_m}$ would be MAD in B_α.

Define $C_n = \left(B_{\alpha_n} \setminus \bigcup_{i<n} B_{\alpha_i} \right) \cap B_\alpha$. By possibly taking a subsequence and making finite changes, we may assume all the C_n are infinite and form a partition of B_α. Since $\mathfrak{a} = \omega_1$, we may find $\mathcal{D}$ an AD family in B_α of size ω_1 such that every $C_n \in \mathcal{D}^\perp$, and if $X \subseteq B_\alpha$ has finite intersection with every C_n, then there is $D \in \mathcal{D}$ such that $|X \cap D| = \omega$.

We define $\mathcal{A}_\alpha = \left(\bigcup_{\xi<\alpha} \mathcal{A}_\xi \right) \cup \mathcal{D}$, and prove that $\mathcal{A}_\alpha \restriction B_\alpha$ is a MAD family in B_α. To see this let $X \in [B_\alpha]^\omega$, and proceed by cases. In case that there is $n \in \omega$ such that $X \cap C_n$ is infinite, the result follows since $\mathcal{A}_{\alpha_n} \restriction B_{\alpha_n}$ is MAD. In case that $X \cap C_n$ is finite for every $n \in \omega$, the result follows by the way we chose $\mathcal{D}$.

Let $\mathcal{A} = \bigcup_{\alpha<\omega_1} \mathcal{A}_\alpha$. It is clear that it is an AD family contained in $\mathcal{I}$, and note that if $X \in \mathcal{I} \cap [\omega]^\omega$ then there is some B_α such that $B_\alpha \cap X$ is infinite and therefore (since $\mathcal{A} \restriction B_\alpha$ is MAD) then there is en element of $\mathcal{A}$ with infinite intersection with X, so $\mathcal{A}$ is be MAD. ■

From the result, we get the following corollary:

Corollary 6 *Assume $\mathfrak{c} \leq \omega_2$ and let $\mathcal{I}$ be an ideal.*

1. *If $\mathfrak{a} \leq \mathfrak{a}(\mathcal{I})$, then $\mathfrak{a}(\mathcal{I}) = \max\{\mathfrak{a}, \mathsf{cov}^*(\mathcal{I})\}$.*
2. *If $\mathcal{I}$ is tall, then $\mathfrak{a}(\mathcal{I}) = \max\{\mathfrak{a}, \mathsf{cov}^*(\mathcal{I})\}$.*

Proof. Assume $\mathfrak{c} \leq \omega_2$ and that $\mathfrak{a} \leq \mathfrak{a}(\mathcal{I})$. In case $\max\{\mathfrak{a}, \mathsf{cov}^*(\mathcal{I})\} = \omega_1$ we get $\mathfrak{a}(\mathcal{I}) = \omega_1$ by the last result, if $\max\{\mathfrak{a}, \mathsf{cov}^*(\mathcal{I})\} = \omega_2$ then $\mathfrak{a}(\mathcal{I}) = \omega_2$ because $\max\{\mathfrak{a}, \mathsf{cov}^*(\mathcal{I})\} \leq \mathfrak{a}(\mathcal{I})$ (recall that $\mathfrak{a} \leq \mathfrak{a}(\mathcal{I})$).

Finally, if $\mathcal{I}$ is tall, then $\mathfrak{a} \leq \mathfrak{a}(\mathcal{I})$, so the second assertion follows from the first. ∎

1.2 Some applications

In this section, we will get some applications of the results proved in the last section. Given $r \in 2^\omega$, denote $\widehat{r} = \{r \upharpoonright n \mid n \in \omega\}$. By $\mathcal{BR}$ we denote the ideal on $2^{<\omega}$ generated by $\{\widehat{r} \mid r \in 2^\omega\}$. In this way, $X \subseteq 2^{<\omega}$ belongs to the ideal $\mathcal{BR}$ if and only if X can be covered by finitely many branches. The elements of $\mathcal{BR}^\perp$ are often called the *off-branch sets*. Note that every antichain is an off-branch set, but there are off-branch sets that are not the union of finitely many antichains. The cardinal invariant $\mathfrak{a}\left(\mathcal{BR}^\perp\right)$ was introduced by Leathrum in [18] and it is denoted by $\mathfrak{o}$. Witnesses of $\mathfrak{a}\left(\mathcal{BR}^\perp\right)$ are usually called *MOB families.* Although $\mathcal{BR}^\perp$ is not a tall ideal, the following result was proved by Leathrum:

Proposition 7 (Leathrum) $\mathfrak{a} \leq \mathfrak{o}$.

In this way, we may conclude the following:

Corollary 8 *If $\mathfrak{c} \leq \omega_2$, then $\mathfrak{o} = \max\{\mathfrak{a}, \mathsf{cov}^*\left(\mathcal{BR}^\perp\right)\}$.*

Let $\mathcal{AT}$ be the ideal on $2^{<\omega}$ generated by antichains. The invariant $\mathfrak{a}(\mathcal{AT})$ was also studied by Leathrum and it is denoted as $\overline{\mathfrak{o}}$. In this way, $\overline{\mathfrak{o}}$ is the smallest size of a maximal almost disjoint family of antichains. Since $\mathcal{AT} \subseteq \mathcal{BR}^\perp$ and every infinite off-branch set contains an infinite antichain, it follows that $\mathsf{cov}^*\left(\mathcal{BR}^\perp\right) \leq \mathsf{cov}^*(\mathcal{AT})$ and $\mathfrak{o} \leq \overline{\mathfrak{o}}$. The following is the most interesting problem regarding the off-branch numbers:

Problem 9 ([18]) *Is $\mathfrak{o} = \overline{\mathfrak{o}}$?*

We do not know the answer to the problem, but we will prove that this is the case if size of the continuum is at most ω_2. We will need the following notions due to Kamburelis and Weglorz (see [16]):

Definition 10 1. *A family of open sets $\mathcal{U} \subseteq \wp(2^\omega)$ is called an* open splitting family *if for every infinite antichain $\{s_n \mid n \in \omega\} \subseteq 2^{<\omega}$ there is $U \in \mathcal{U}$ such that both sets $\{n \mid \langle s_n \rangle \subseteq U\}$[2] and $\{n \mid \langle s_n \rangle \cap U = \emptyset\}$ are infinite.*

2. *$\mathfrak{s}(\mathcal{B}_0)$ is the smallest size of an open splitting family.*

[2] If $s \in 2^{<\omega}$, define $\langle s \rangle = \{f \in 2^\omega \mid s \subseteq f\}$.

3. *Given $x \in 2^\omega$ and $n \in \omega$ let $r(x,n)$ be the sequence of length $n+1$ that agrees with x in the first n places but disagrees in the last one.*

4. *Let $x \in 2^\omega$, $A \in [\omega]^\omega$ and $U \subseteq 2^\omega$ an open set. We say that U* separates *(x,A) if $x \notin U$ and there are infinitely many $n \in A$ such that $\langle r(x,n)\rangle \subseteq U$.*

5. *$\mathfrak{sep}$ is the smallest size of a family of open sets $\mathcal{U}$ such that for every (x,A) there is $U \in \mathcal{U}$ that separates (x,A).*

Kamburelis and Weglorz proved that $\mathfrak{s}(\mathcal{B}_0) = \max\{\mathfrak{s}, \mathfrak{sep}\}$. However, in [5] Brendle proved that this two cardinal invariants are equal. In fact he proved the following:

Proposition 11 ([5]) *1. $\mathsf{non}(\mathcal{M}) \leq \mathfrak{sep}$.*

2. $\mathfrak{sep} = \mathfrak{s}(\mathcal{B}_0)$.

Note that the second assertion follows from the first since $\mathfrak{s} \leq \mathsf{non}(\mathcal{M})$ and $\mathfrak{s}(\mathcal{B}_0) = \max\{\mathfrak{s}, \mathfrak{sep}\}$. In [8] the authors proved that $\mathsf{cov}^*\left(\mathcal{BR}^\perp\right) = \mathfrak{sep}$. The same argument shows that in fact $\mathsf{cov}^*(\mathcal{AT}) = \mathfrak{sep}$. We will provide the whole argument for completeness. First we will need a definition and a lemma:

Definition 12 *Let $U \subseteq 2^\omega$ be an open set. Define A_U as the set of all minimal $s \in 2^{<\omega}$ for which $\langle s\rangle \subseteq U$.*

It is easy to see that $A_U \subseteq 2^{<\omega}$ is an antichain and if U is not clopen, then A_U is infinite. We now have the following:

Lemma 13 *Let $\mathcal{U}$ be an open splitting family and $C \subseteq 2^{<\omega}$ an infinite antichain. There is $U \in \mathcal{U}$ such that there are infinitely many $t \in A_U$ for which there is $s \in C$ such that $\langle s\rangle \subseteq \langle t\rangle$.*

Proof. By a compactness argument, we can find $C_1 = \{s_n \mid n \in \omega\}$ and $r \in 2^\omega$ with the following properties:

1. $C_1 \subseteq C$,
2. $r \restriction n \subseteq s_m$ for every $n \in \omega$ and for almost every $m \in \omega$,
3. $r \notin \bigcup_{n\in\omega} \langle s_n\rangle$.

Since $\mathcal{U}$ is an open splitting family, we know there is $U \in \mathcal{U}$ such that both sets $\{n \mid \langle s_n\rangle \subseteq U\}$ and $\{n \mid \langle s_n\rangle \cap U = \emptyset\}$ are infinite. We claim that U has the desired properties. First note that $r \notin U$, this is because if $r \in U$ then the set $\{n \mid \langle s_n\rangle \cap U = \emptyset\}$ would be finite. Since every set $\langle t\rangle$ is clopen, then for every $t \in A_U$, $\langle t\rangle$ can only contained finitely many elements of C_1. ■

We have the following strengthening of Proposition 4.11 of [8]:

Proposition 14 *cov*$^*\left(\mathcal{BR}^{\perp}\right) = $ *cov*$^*(\mathcal{AT}) = \mathfrak{sep}$.

Proof. We first show that $\mathfrak{sep} \leq \mathsf{cov}^*\left(\mathcal{BR}^{\perp}\right)$. Let $\mathcal{B} \subseteq \mathcal{BR}^{\perp}$ be a witness for *cov*$^*\left(\mathcal{BR}^{\perp}\right)$. We may even assume it is closed under finite changes. For every $B \in \mathcal{B}$ let $\mathcal{U}_B = \bigcup \{ \langle s \rangle \mid s \in B\}$ we will show that $\{\mathcal{U}_B \mid B \in \mathcal{B}\}$ witness $\mathfrak{sep}$. Let $x \in 2^{\omega}$, $A \in [\omega]^{\omega}$ and define the off-branch family $Y = \{r(x,n) \mid n \in A\}$ then find $B \in \mathcal{B}$ such that $B \cap Y$ is infinite. Since B is off-branch, by taking a finite subset of it we may assume no restriction of x is in B, it then follows that $\mathcal{U}_B$ separates (x, A).

We will now show that $\mathsf{cov}^*(\mathcal{AT}) \leq \mathfrak{s}(\mathcal{B}_0)$. This completes the proof since $\mathsf{cov}^*\left(\mathcal{BR}^{\perp}\right) \leq \mathsf{cov}^*(\mathcal{AT})$ and $\mathfrak{sep} = \mathfrak{s}(\mathcal{B}_0)$. Let $\{U_\beta \mid \beta < \mathfrak{s}(\mathcal{B}_0)\}$ be an open splitting family. By Bartoszyński's characterization of $\mathsf{non}(\mathcal{M})$ (see [3] Lemma 2.4.8) there is a family $\mathcal{F} = \{f_\alpha \mid \alpha < \mathsf{non}(\mathcal{M})\}$ with the following properties:

1. $f_\alpha : \omega \longrightarrow 2^{<\omega}$, and
2. for every $g : W \longrightarrow 2^{<\omega}$, where $W \in [\omega]^{\omega}$, there is $\alpha < \mathsf{non}(\mathcal{M})$ such that there are infinitely many $n \in W$ such that $f_\alpha(n) = g(n)$.

For every $\beta < \mathfrak{s}(\mathcal{B}_0)$ we fix an enumeration $A_{U_\beta} = \{s_n^\beta \mid n \in \omega\} \subseteq 2^{<\omega}$ (recall that A_{U_β} is the set of all minimal nodes of $\{s \mid \langle s \rangle \subseteq U_\beta\}$). For every $\alpha < \mathsf{non}(\mathcal{M})$ and $\beta < \mathfrak{s}(\mathcal{B}_0)$ we define $B(\alpha,\beta) = \{f_\alpha(n) \mid s_n^\beta \subseteq f_\alpha(n)\}$ which is an antichain since A_{U_β} is an antichain. Let $\mathcal{B}$ be the set of all $B(\alpha,\beta)$ where $\alpha < \mathsf{non}(\mathcal{M})$ and $\beta < \mathfrak{s}(\mathcal{B}_0)$. We will prove that that for every infinite antichain Y there is $B(\alpha,\beta) \in \mathcal{B}$ such that $B(\alpha,\beta) \cap Y$ is infinite.

Since $\mathcal{U}$ is an open splitting family, we know there is $\beta < \mathfrak{s}(\mathcal{B}_0)$ such that there are infinitely many $s_n^\beta \in A_{U_\beta}$ for which there is $t \in Y$ such that $s_n^\beta \subseteq t$. Let $W = \{n \mid \exists t \in Y(s_n^\beta \subseteq t)\}$ which is an infinite set. Define $g : W \longrightarrow 2^{<\omega}$ such that for every $n \in W$ the following hold:

1. $s_n^\beta \subseteq g(n)$, and
2. $g(n) \in Y$.

We can now find $\alpha < \mathsf{non}(\mathcal{M})$ such that there are infinitely many $n \in W$ for which $f_\alpha(n) = g(n)$. It is then clear that $B(\alpha,\beta) \cap Y$ is infinite. Finally since $\mathsf{non}(\mathcal{M}) \leq \mathfrak{s}(\mathcal{B}_0)$ by the theorem of Brendle, we conclude that $|\mathcal{B}| = \mathfrak{s}(\mathcal{B}_0)$ and we get the desired result. ■

From this we can conclude that both $\mathfrak{o}$ and $\overline{\mathfrak{o}}$ are equal to $\max\{\mathfrak{a}, \mathfrak{sep}\}$ in case $\mathfrak{c} \leq \omega_2$. We get the following:

Corollary 15 *1.* $\mathfrak{o} = \omega_1$ *implies* $\overline{\mathfrak{o}} = \omega_1$.

2. If $\mathfrak{c} \leq \omega_2$ *then* $\mathfrak{o} = \overline{\mathfrak{o}} = \max\{\mathfrak{a}, \mathfrak{sep}\}$.

As was mentioned before, it is still an open problem if $\mathfrak{o} = \overline{\mathfrak{o}}$. Getting the consistency of $\mathfrak{o} < \overline{\mathfrak{o}}$ will most likely be very hard. Our result shows that countable support iteration can not be used to solve this problem and long finite support iterations will not work either since $\mathsf{cov}(\mathcal{M}) \leq \mathfrak{o}$.

For every $n \in \omega$ we define $C_n = \{(n, m) \mid m \in \omega\}$. Recall that $\mathcal{ED}$ is the ideal on $\omega \times \omega$ generated by $\{C_n \mid n \in \omega\}$ and (the graphs of) functions from ω to ω. It is easy to see that $\mathcal{ED}$ is a tall ideal. The invariant $\mathfrak{a}_s$ is defined as the smallest size of a maximal family of eventually different partial functions. In other words, $\mathfrak{a}_s$ is the smallest size of a family $\mathcal{B}$ with the following properties:

1. For every $f \in \mathcal{B}$ there is $A \in [\omega]^{\omega}$ such that $f : A \longrightarrow \omega$,
2. for every $f \neq g \in \mathcal{B}$ $\{n \in dom\,(f) \cap dom\,(g) \mid f\,(n) = g\,(n)\}$ is finite, and
3. for every function $h : A \longrightarrow \omega$ with $A \in [\omega]^{\omega}$ there is $f \in \mathcal{B}$ such that there are infinitely many $n \in \omega$ for which $f\,(n) = h\,(n)$.

We now have the following result:

Lemma 16 $\mathfrak{a}_s = \mathfrak{a}\,(\mathcal{ED})$.

Proof. Let $\mathcal{B}$ be a maximal family of eventually different partial functions. Define $\mathcal{D} = \mathcal{B} \cup \{C_n \mid n \in \omega\}$. Note that $\mathcal{D} \subseteq \mathcal{ED}$ and it is easy to see that $\mathcal{D}$ is a MAD family. In this way we conclude that $\mathfrak{a}\,(\mathcal{ED}) \leq \mathfrak{a}_s$.

For the other inequality, let $\mathcal{A} \subseteq \mathcal{ED}$ be a MAD family contained in $\mathcal{ED}$. Note that for every $A \in \mathcal{A}$, there is a pair (X_A, F_A) with the following properties:

1. There is $m \in \omega$ such that $X_A \subseteq C_0 \cup ... \cup C_m$,
2. F_A is a finite set of disjoint partial functions, and
3. $A = X_A \cup \{(n, h\,(n)) \mid h \in F_A \wedge n \in dom\,(h)\}$.

Let $\mathcal{B} = \bigcup\limits_{A \in \mathcal{A}} F_A$. It is easy to see that $|\mathcal{B}| = |\mathcal{A}|$, and that $\mathcal{B}$ is a maximal family of eventually different partial functions. So $\mathfrak{a}_s \leq \mathfrak{a}\,(\mathcal{ED})$. ■

In [14] it was proved that $\mathsf{cov}^*\,(\mathcal{ED}) = \mathsf{non}(\mathcal{M})$. We conclude the following:

Corollary 17 *If* $\mathfrak{c} \leq \omega_2$, *then* $\mathfrak{a}_s = \max\{\mathfrak{a}, \mathit{non}(\mathcal{M})\}$.

We should mention that the hypothesis $\mathfrak{c} \leq \omega_2$ is needed. In [6] Brendle used the technique of forcing along a template to prove the following:

Proposition 18 (Brendle) *The following is relatively consistent with the axioms of ZFC:* $\mathfrak{a} = \omega_1$, $\mathit{non}(\mathcal{M}) = \omega_2$ *and* $\mathfrak{a}_s = \mathfrak{c} = \omega_3$.

Our result shows that the theorem of Brendle can not be improved.

Definition 19 *If $a \subseteq \omega^{<\omega}$ we define $\pi(a) = \{f \in \omega^\omega \mid \exists^\infty n (f \restriction n \in a)\}$. Let $\mathcal{I}$ be a σ-ideal on ω^ω (or 2^ω). We define $tr(\mathcal{I})$* the trace ideal of $\mathcal{I}$ *(which will be an ideal on $\omega^{<\omega}$ or $2^{<\omega}$) where $a \in tr(\mathcal{I})$ if and only if $\pi(a) \in \mathcal{I}$.*

Note that if $a \subseteq \omega^{<\omega}$ then $\pi(a)$ is a G_δ set (furthermore, every G_δ set is of this form). While both $tr(\mathcal{M})$ and $tr(\mathcal{N})$ are Borel, in general, the trace ideals are not Borel (see [15] for more information). By nwd we will denote the ideal of the nowhere dense subsets of the rational numbers. It is well known that $tr(\mathcal{M})$ is equivalent to nwd. By $\mathcal{NDN}$ we will denote the ideal $tr(\mathcal{M}) \cap tr(\mathcal{N})$. In [21] the cardinal invariants $\mathfrak{a}(tr(\mathcal{M}))$, $\mathfrak{a}(tr(\mathcal{N}))$ and $\mathfrak{a}(\mathcal{NDN})$ were studied. In that paper, the following results were proven:

Proposition 20 ([21])

1. $\mathsf{cov}(\mathcal{M}), \mathfrak{a} \leq \mathfrak{a}(\mathsf{nwd})$,
2. $\mathsf{cov}(\mathcal{N}), \mathfrak{a} \leq \mathfrak{a}(tr(\mathcal{N}))$, *and*
3. $\mathfrak{a}(\mathsf{nwd})$, $\mathfrak{a}(tr(\mathcal{N})) \leq \mathfrak{a}(\mathcal{NDN})$.

and the following question was asked:

Problem 21 ([21]) *Are the inequalities between $\mathfrak{a}(\mathsf{nwd})$, $\mathfrak{a}(tr(\mathcal{N}))$, $\mathfrak{a}(\mathcal{NDN})$ consistently strict and complete?*

We can readily provide the following:

Corollary 22 *Both $\mathfrak{a}(\mathsf{nwd}) < \mathfrak{a}(tr(\mathcal{N}))$ and $\mathfrak{a}(tr(\mathcal{N})) < \mathfrak{a}(\mathsf{nwd})$ are consistent.*

Proof. It is easy to see that both nwd and $tr(\mathcal{N})$ are tall ideals. It is a theorem of Keremedis that $\mathsf{cov}(\mathcal{M}) = \mathsf{cov}^*(\mathsf{nwd})$ (see e.g. [2] for a proof). In this way, if $\mathfrak{c} = \omega_2$ and $\mathfrak{a} = \omega_1$ then $\mathfrak{a}(\mathsf{nwd}) = \mathsf{cov}(\mathcal{M})$ and $\mathfrak{a}(tr(\mathcal{N})) = \mathsf{cov}^*(tr(\mathcal{N}))$. Furthermore, in [9] it was proved that $\mathsf{cov}(\mathcal{N}) \leq \mathsf{cov}^*(tr(\mathcal{N})) \leq \mathsf{non}(\mathcal{M})$. From these results it is clear that in the Cohen model (the model obtained after adding ω_2 Cohen reals to a model of CH) the inequality $\mathfrak{a}(tr(\mathcal{N})) < \mathfrak{a}(\mathsf{nwd})$ holds and in the random model (the model obtained after adding ω_2 random reals to a model of CH) the inequality $\mathfrak{a}(\mathsf{nwd}) < \mathfrak{a}(tr(\mathcal{N}))$ holds. $\blacksquare$

Problem 23 *Is $\mathfrak{a}(\mathcal{NDN})$ the maximum of $\mathfrak{a}(\mathsf{nwd})$ and $\mathfrak{a}(tr(\mathcal{N}))$?*

Another problem from [21] is the following:

Problem 24 *Are $\mathfrak{a}(\mathsf{nwd})$, $\mathfrak{a}(tr(\mathcal{N}))$, $\mathfrak{a}(\mathcal{NDN})$ incomparable with $\mathfrak{o}, \overline{\mathfrak{o}}$, $\mathfrak{a}_s$?*

We will provide some partial answers to the question. We start with the following:

Proposition 25 *$\mathfrak{a}(\mathsf{nwd})$ and $\mathfrak{a}_s$ are incomparable.*

Proof. We know that if $\mathfrak{c} = \omega_2$ and $\mathfrak{a} = \omega_1$ then $\mathfrak{a}(\mathsf{nwd}) = \mathsf{cov}(\mathcal{M})$ and $\mathfrak{a}_s = \mathsf{non}(\mathcal{M})$. The result follows since $\mathsf{cov}(\mathcal{M})$ and $\mathsf{non}(\mathcal{M})$ are independent (with $\mathfrak{c} = \omega_2$). ∎

Regarding $\mathfrak{a}\left(tr\left(\mathcal{N}\right)\right)$ and $\mathfrak{a}_s$ we have the following:

Proposition 26 *It is consistent that* $\mathfrak{a}\left(tr\left(\mathcal{N}\right)\right) < \mathfrak{a}_s$.

Proof. By Proposition 4.1 of [15], we know that $\mathsf{cov}(\mathcal{N}) \leq \mathsf{cov}^*\left(tr\left(\mathcal{N}\right)\right) \leq \mathsf{max}\{\mathfrak{d},\mathsf{cov}(\mathcal{N})\}$. In this way, in order to obtain a model of $\mathfrak{a}\left(tr\left(\mathcal{N}\right)\right) < \mathfrak{a}_s$, it is enough to find a model of $\mathfrak{c} = \mathsf{non}(\mathcal{M}) = \omega_2$, and $\mathfrak{a} = \mathfrak{d} = \mathsf{cov}(\mathcal{N}) = \omega_1$. The existence of such models is well known, for example, they can be obtained by iterating the Mathias forcing associated to the ideal $\mathcal{ED}$. ∎

Finally, the next proposition follows from Corollary 6 and known inequalities between cardinal invariants:

Proposition 27 *If* $\mathfrak{c} \leq \omega_2$ *then the following hold:*

1. $\mathfrak{a}_s \leq \mathfrak{o}$,
2. $\mathfrak{a}\left(tr\left(\mathcal{N}\right)\right) \leq \mathfrak{a}_s$, *and*
3. $\mathfrak{a}(\mathit{nwd}) \leq \mathfrak{o}$.

Next we consider the ideal $\mathcal{K}$, i.e. the ideal generated by the finitely branching subtrees of $\omega^{<\omega}$ Regarding the cardinal invariant $\mathfrak{a}\left(\mathcal{K}\right)$, we have the following:

Proposition 28 $\mathfrak{a} \leq \mathfrak{a}\left(\mathcal{K}\right)$.

Proof. For every $n \in \omega$, let $z_n : n \longrightarrow \omega$ be the constant 0 function. Define $X_n = \{z_n{}^\frown i \mid i \in \omega\}$ and $X = \bigcup_{n\in\omega} X_n$. Let $f : X \longrightarrow \omega$ be a bijection and define $A_n = f\left[X_n\right]$. We now find a family $\mathcal{B} = \{K_\alpha \mid \omega \leq \alpha < \mathfrak{a}\left(\mathcal{K}\right)\} \subseteq \mathcal{K}$ such that for every infinite $Y \in \mathcal{K}$ there is α such that $|K_\alpha \cap Y| = \omega$. For every $\omega \leq \alpha < \mathfrak{a}\left(\mathcal{K}\right)$ let $A_\alpha = f\left[X \cap K_\alpha\right]$ and $\mathcal{A} = \{A_\alpha \mid \alpha < \mathfrak{a}\left(\mathcal{K}\right)\} \setminus [\omega]^{<\omega}$. We claim that $\mathcal{A}$ is a MAD family.

Note that if $K_\alpha \in \mathcal{B}$ and $n \in \omega$, then $X_n \cap K_\alpha$ is finite, this implies that $\mathcal{A}$ is an almost disjoint family. In order to prove that $\mathcal{A}$ is maximal, note that for every $Z \subseteq \omega$, if Z is almost disjoint with every A_n for $n \in \omega$, then $f^{-1}\left(Z\right) \in \mathcal{K}$ and hence there is $K_\alpha \in \mathcal{B}$ such that $f^{-1}\left(Z\right) \cap K_\alpha$ is infinite, which implies that $Z \cap A_\alpha$ is infinite. ∎

It follows that if $\mathfrak{c} \leq \omega_2$ then $\mathfrak{a}\left(\mathcal{K}\right) = \mathsf{max}\left\{\mathfrak{a}, \mathfrak{d}\right\}$. The cardinal invariant $\mathfrak{a}_T$ is defined as the smallest size of a maximal AD family of finitely branching subtrees of $\omega^{<\omega}$ (or $2^{<\omega}$). This cardinal invariant has been studied by Miller ([19]) and Newelski ([20]). It is easy to see that $\mathfrak{a}_T$ is the smallest cardinality of a partition of ω^ω into disjoint compact sets. It follows that $\mathfrak{d} \leq \mathfrak{a}_T$. Spinas ([24]) proved that the inequality $\mathfrak{d} < \mathfrak{a}_T$ is consistent, answering a question on

[10]. The invariants $\mathfrak{a}_T$ and $\mathfrak{a}(\mathcal{K})$ seem very similar, It would be tempting to conjecture that in fact $\mathfrak{a}(\mathcal{K}) = \mathfrak{a}_T$. We will now prove that this is not the case.

Recall that a tree $p \subseteq 2^{<\omega}$ is a *Sacks tree* if for for every $s \in p$ there is $t \in p$ such that $s \subseteq t$ and t is a *splitting node of* p (i.e. $t^\frown 0$, $t^\frown 1 \in p$). Recall that Sacks forcing is the set of Sacks trees ordered by inclusion. The following forcing notion was introduced by Miller in [19]:

Definition 29 *Let $\mathcal{C} = \{C_\alpha \mid \alpha \in \omega_1\}$ be a partition of 2^ω into compact sets. $\mathbb{P}(\mathcal{C})$ is the collection of all p such that the following hold:*

1. *$p \subseteq 2^{<\omega}$ is a Sacks tree, and*
2. *if $\alpha < \omega_1$, then $C_\alpha \cap [p]$ is nowhere dense in $[p]$ (i.e. for every $s \in p$, there is $t \in p$ such that $s \subseteq t$ and $\langle t \rangle \cap [p] \cap C_\alpha = \emptyset$).*

If $p, q \in \mathbb{P}(\mathcal{C})$, then $p \leq q$ if $p \subseteq q$.

In [19] Miller proved that $\mathbb{P}(\mathcal{C})$ is proper, has the Laver property and forces that $\mathcal{C}$ is no longer a partition of 2^ω. In [24] Spinas showed that $\mathbb{P}(\mathcal{C})$ is ω^ω-bounding. It follows by the results of Miller and Spinas that $\mathbb{P}(\mathcal{C})$ even has the Sacks property. We will prove that $\mathbb{P}(\mathcal{C})$ does not increase $\mathfrak{a}$. We fix $\mathcal{C} = \{C_\alpha \mid \alpha \in \omega_1\}$ a partition of 2^ω into compact sets. We will need some basic results about the forcing $\mathbb{P}(\mathcal{C})$:

Definition 30 *We say that $X = \{x_s \mid s \in \omega^{<\omega}\} \subseteq 2^\omega$ is* nice *if the following conditions hold:*

1. *For every $s \in \omega^{<\omega}$, the sequence $\langle x_{s^\frown n} \rangle_{n\in\omega}$ converges to x_s; furthermore, $\Delta(x_s, x_{s^\frown n}) < \Delta(x_s, x_{s^\frown n+1})$*[3]*,*
2. *for every $s, t, z \in \omega^{<\omega}$, if $s \subseteq t \subseteq z$, then $\Delta(x_s, x_t) < \Delta(x_s, x_z)$, and*
3. *for every $s \in \omega^{<\omega}$, let $\alpha_s < \omega_1$ such that $x_s \in C_{\alpha_s}$, and*
4. *If $s \subseteq t$ then $\alpha_s \neq \alpha_t$.*

The following was proved implicitly in [24]:

Lemma 31 *Let p be a Sacks tree. If there is a nice $X = \{x_s \mid s \in \omega^{<\omega}\}$ that is dense in $[p]$, then $p \in \mathbb{P}(\mathcal{C})$.*

Proof. We need to prove that every C_β is nowhere dense in $[p]$. Let $\beta < \omega_1$ and $t \in p$. Since X is nice and dense in $[p]$, we can find $s \in \omega^{<\omega}$ such that $t \subseteq x_s$ and $\alpha_s \neq \beta$. Since $x_s \notin C_\beta$ and C_β is closed, there is $z \in p$ such that $t \subseteq z \subseteq x_s$ and $\langle z \rangle \cap p \cap C_\beta = \emptyset$. ■

Recall that an AD family. $\mathcal{A}$ is *tight* if for every $\{X_n \mid n \in \omega\} \subseteq \mathcal{I}(\mathcal{A})^+$ there is $B \in \mathcal{I}(\mathcal{A})$ such that $B \cap X_n$ is infinite for every $n \in \omega$. Note that tight

[3] If $x, y \in 2^\omega$ and $x \neq y$, denote $\Delta(x, y) = \min\{n \mid x(n) \neq y(n)\}$.

AD families are MAD familie. In [13] it was proved that tight MAD families exists assuming $\mathfrak{b} = \mathfrak{c}$ and that they are Cohen-indestructible (recall that if $\mathcal{A}$ is a MAD family and $\mathbb{P}$ is a partial order, then $\mathcal{A}$ is $\mathbb{P}$-*indestructible* if $\mathcal{A}$ is still maximal after forcing with $\mathbb{P}$). We will prove that tight MAD families are $\mathbb{P}(\mathcal{C})$-indestructible, in fact, we will prove something stronger:

Definition 32 *Let $\mathcal{A}$ be a tight MAD family. We say that a proper forcing $\mathbb{P}$* strongly preserves the tightness of $\mathcal{A}$ *if for every $p \in \mathbb{P}$, M a countable elementary submodel of $\mathsf{H}(\kappa)$ (where κ is a large enough regular cardinal) such that $\mathbb{P}, \mathcal{A}, p \in M$ and $B \in \mathcal{I}(\mathcal{A})$ for which $|B \cap Y| = \omega$ for every $Y \in \mathcal{I}(\mathcal{A})^+ \cap M$, there is $q \leq p$ an $(M, \mathbb{P})$-generic condition such that $q \Vdash$ "$\forall \dot{Z} \in \left(\mathcal{I}(\mathcal{A})^+ \cap M[\dot{G}]\right)\left(\left|\dot{Z} \cap B\right| = \omega\right)$" (where $\dot{G}$ denotes the name of the generic filter). We say that q is an $(M, \mathbb{P}, \mathcal{A}, B)$*-generic condition.

It is easy to see that if $\mathbb{P}$ strongly preserves the tightness of $\mathcal{A}$, then $\mathcal{A}$ is a tight MAD family after forcing with $\mathbb{P}$. We will need the following well known fact:

Lemma 33 *Let $\mathcal{A}$ be an AD family, $\mathbb{P}$ a partial order, $\dot{B}$ a $\mathbb{P}$-name for a subset of ω and $p \in \mathbb{P}$ such that $p \Vdash$ "$\dot{B} \in \mathcal{I}(\mathcal{A})^+$". The set*

$$C = \{n \mid \exists q \leq p \ (q \Vdash \text{“} n \in \dot{B} \text{”})\} \in \mathcal{I}(\mathcal{A})^+ .$$

Proof. Since $\dot{B}$ is forced to be a subset of C, the result follows. ■

If p is a Sacks tree and $s \in p$, let $p_s = \{t \in p \mid s \subseteq t \vee t \subseteq s\}$.

Proposition 34 *If $\mathcal{A}$ is a tight MAD family and $\mathbb{P}(\mathcal{C}) = \{C_\alpha \mid \alpha \in \omega_1\}$ is a partition of 2^ω in compact sets, then $\mathbb{P}(\mathcal{C})$ strongly preserves the tightness of $\mathcal{A}$.*

Proof. Let $p \in \mathbb{P}(\mathcal{C})$, M a countable elementary submodel of $\mathsf{H}(\kappa)$ (where κ is a large enough regular cardinal) such that $\mathcal{C}, \mathcal{A}, p \in M$ and $B \in \mathcal{I}(\mathcal{A})$ for which $|B \cap Y| = \omega$ for every $Y \in \mathcal{I}(\mathcal{A})^+ \cap M$. Let $\{D_n \mid n \in \omega\}$ be an enumeration of all open dense subsets of $\mathbb{P}(\mathcal{C})$ that are in M and fix $\left\{\dot{Z}_n \mid n \in \omega\right\}$ an enumeration of all $\mathbb{P}_\gamma$-names for elements of $\mathcal{I}(\mathcal{A})^+$ that are in M such that every name appears infinitely many times in the enumeration. We will recursively construct $\langle p_n \rangle_{n \in \omega}$ and $X = \{x_s \mid s \in \omega^{<\omega}\}$ such that the following conditions hold:

1. $p_0 = p$,
2. $\langle p_n \rangle_{n \in \omega}$ is a decreasing sequence and $p_n \in M$ for every $n \in \omega$,
3. X is nice and $X \subseteq 2^\omega \cap M$,
4. $X \subseteq [p_n]$ for every $n \in \omega$, and
5. For every $s \in \omega^n$ and $i, m \in \omega$ if $m = \Delta(x_s, x_{s^\frown i})$ and $t = (x_{s^\frown i}) \restriction m$ then $(p_{n+1})_t \in D_n$ and $(p_{n+1})_t \Vdash$ "$\left(\dot{Z}_n \cap B\right) \setminus n \neq \emptyset$".

To start let $p_0 = p$, and let $x_\emptyset$ be any element of $[p_0] \cap M$. Assume we have defined p_n and $\left\{x_s \mid s \in \omega^{\leq n}\right\}$, we will define p_{n+1} and $\left\{x_s \mid s \in \omega^{n+1}\right\}$.

Let $s \in \omega^n$ and choose $l \in \omega$ such that $l > \Delta(x_s, x_{s'})$ for all $s' \subsetneq s$. Define Y_s as the set of all $m > l$ such that $x_s \restriction m$ is a splitting node of p_n. For every $m \in Y_s$, let $t_m = (x_s \restriction m)^\frown (1 - x_s(m))$ (which is a node p_n) and let $p^s_m = (p_n)_{t_m}$, clearly $p^s_m \in M$. Let $C = \{j \mid \exists r \leq p^s_m (r \Vdash \text{“}j \in \dot{Z}_n\text{”})\}$. Since $C \in \mathcal{I}(\mathcal{A})^+$, there is $j \in C \cap B$ such that $j > n$. We choose $r^s_m \leq p^s_m$ such that $r^s_m \in M$ and $r^s_m \Vdash \text{“}j \in \dot{Z}_n\text{”}$. We may further assume that $r^s_m \in D_n$ and $[r^s_m] \cap C_{\alpha_z} = \emptyset$ for every $z \subseteq s$ (recall that if $z \in \omega^{<\omega}$, α_z denoted the unique ordinal such that $x_z \in C_{\alpha_z}$). Let $Y_s = \left\{m^i_s \mid i \in \omega\right\}$. For every $i \in \omega$, choose $x_{s^\frown i}$ be any branch in $r^s_{m^i_s}$ and let $p_{n+1} = \bigcup\{r^s_{m^i_s} \mid s \in \omega^n \wedge i \in \omega\}$.

We now let $q = \bigcap\limits_{n\in\omega} p_n$. It is easy to see that X is a dense subset of $[q]$ and q is a Sacks tree, so $q \in \mathbb{P}(\mathcal{C})$. Moreover, it is not hard to see that q is an $(M, \mathbb{P}(\mathcal{C}), \mathcal{A}, B)$*-generic condition.* ■

It follows that the forcings $\mathbb{P}(\mathcal{C})$ preserve tight MAD families. We now need to take care of the iteration. We will prove that the countable support iteration of forcings that strongly preserve $\mathcal{A}$-tightness, also strongly preserve $\mathcal{A}$-tightness. Our proof will be a variation of the preservation of properness under countable support iteration by Shelah ([22]). First we do the two step iteration:

Lemma 35 *Let $\mathcal{A}$ be a tight MAD family. If $\mathbb{P}$ is a proper forcing that strongly preserves the tightness of $\mathcal{A}$ and $\dot{\mathbb{Q}}$ is a $\mathbb{P}$-name for a proper forcing such that $\mathbb{P} \Vdash$ “$\dot{\mathbb{Q}}$ strongly preserves the tightness of $\mathcal{A}$”. Furthermore, if $B \in \mathcal{I}(\mathcal{A})$, M is a countable elementary submodel with $\mathcal{A}, \mathbb{P}, \dot{\mathbb{Q}} \in M$, $p \in \mathbb{P}$, is an $(M, \mathbb{P}, \mathcal{A}, B)$-generic condition and $\dot{q}$ is a $\mathbb{P}$-name for an element of $\dot{\mathbb{Q}}$ such that $p \Vdash$ “$\dot{q}$ is an $(M\left[\dot{G}\right], \dot{\mathbb{Q}}, \mathcal{A}, B)$-generic condition”, then $(p, \dot{q})$ is an $(M, \mathbb{P} * \dot{\mathbb{Q}}, \mathcal{A}, B)$-generic condition.*

Proof. Let $G \subseteq \mathbb{P} * \dot{\mathbb{Q}}$ be a generic filter with $(p, \dot{q}) \in G$, denote $G_{\mathbb{P}}$ the projection of G to $\mathbb{P}$. Since p is an $(M, \mathbb{P}, \mathcal{A}, B)$-generic condition, it follows that B has infinite intersection with every element of $M[G_P] \cap \mathcal{I}(\mathcal{A})^+$. Finally, since $\dot{q}$ is forced to be an $(M\left[\dot{G}\right], \dot{\mathbb{Q}}, \mathcal{A}, B)$-generic condition, then B will have infinite intersection with every element of $M[G] \cap \mathcal{I}(\mathcal{A})^+$. Finally, note that $(p, \dot{q})$ is an $(M, \mathbb{P} * \dot{\mathbb{Q}})$-condition (see [1]). ■

We will now prove the “proper iteration lemma” ([1] Lemma 2.8) for $(M, \mathbb{P}, \mathcal{A})$-generic conditions. In the following, if $\mathcal{P} = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha \leq \gamma\rangle$ is a countable support iteration of proper forcings and $\alpha \leq \gamma$, by $\Vdash_\alpha$ we will denote $\Vdash_{\mathbb{P}_\alpha}$ and by $\dot{G}_\alpha$ the canonical name for a $\mathbb{P}_\alpha$-generic filter.

Proposition 36 *Let $\mathcal{A}$ be a tight MAD family. Let $\mathcal{P} = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha \leq \gamma\rangle$ be a countable support iteration of proper forcings such that $\mathbb{P}_\alpha \Vdash_\alpha$ “$\dot{\mathbb{Q}}_\alpha$ strongly preserves the tightness of $\mathcal{A}$”. Let $B \in \mathcal{I}(\mathcal{A})$, M be a countable elementary submodel of $\mathsf{H}(\kappa)$ (where κ is a large enough regular cardinal) with $\mathcal{A}, \mathcal{P}, \gamma \in M$.*

For every $\alpha \in M \cap \gamma$ and an $(M, \mathbb{P}_\alpha, \mathcal{A}, B)$-generic condition $p \in \mathbb{P}_\alpha$ the following holds:

If $\dot{q}$ is a $\mathbb{P}_\alpha$-name such that $p \Vdash_\alpha$ "$\dot{q} \in \mathbb{P}_\gamma \cap M$" and $p \Vdash_\alpha$ "$\dot{q} \restriction \alpha \in \dot{G}_\alpha$", then there is an $(M, \mathbb{P}_\alpha, \mathcal{A}, B)$-generic condition$\overline{p} \in \mathbb{P}_\gamma$ such that $\overline{p} \restriction \alpha = p$ and $\overline{p} \Vdash_\gamma$ "$\dot{q} \in \dot{G}$".

Proof. We will prove the proposition by induction on γ. The case where γ is a successors follows easily by the last lemma, so we assume that γ is a limit ordinal and the proposition holds for every ordinal smaller that γ. Let $\langle \alpha_n \rangle_{n \in \omega}$ be an increasing sequence of ordinals in $M \cap \gamma$ such that $\alpha_0 = \alpha$ and $\bigcup \alpha_n = \bigcup M \cap \gamma$. We fix an enumeration $\{D_n \mid n \in \omega\}$ of all open dense sets of $\mathbb{P}_\gamma$ that are in M and fix $\left\{\dot{Z}_n \mid n \in \omega\right\}$ an enumeration of all $\mathbb{P}_\gamma$-names for elements of $\mathcal{I}(\mathcal{A})^+$ that are in M such that every name appears infinitely many times in the enumeration. We will recursively construct sequences $\langle \dot{q}_n \rangle_{n \in \omega}$, $\langle p_n \rangle_{n \in \omega}$ and $\langle \dot{m}_n \rangle_{n \in \omega}$ with the following properties:

1. $p_0 = p$, $\dot{q}_0 = \dot{q}$,
2. $p_n \in \mathbb{P}_{\alpha_n}$ is an $(M, \mathbb{P}_{\alpha_n}, \mathcal{A}, B)$-generic condition,
3. $p_{n+1} \restriction \alpha_n = p_n$,
4. $\dot{q}_n$ is a $\mathbb{P}_{\alpha_n}$-name such that $p_n \Vdash_{\alpha_n}$ "$\dot{q}_n \in \mathbb{P}_\gamma \cap M$" and $p_n \Vdash_{\alpha_n}$ "$\dot{q}_n \restriction \alpha_n \in \dot{G}_{\alpha_n}$",
5. $p_{n+1} \Vdash_{\alpha_{n+1}}$ "$\dot{q}_{n+1} \leq \dot{q}_n$" and $p_{n+1} \Vdash_{\alpha_{n+1}}$ "$\dot{q}_{n+1} \in D_n$", and
6. $\dot{m}_n$ is a $\mathbb{P}_\gamma$-name for a natural number such that $p_{n+1} \Vdash_{\alpha_n}$ "$\dot{q}_n \Vdash_\gamma$ "$\dot{m}_n \in \left(\dot{Z}_n \cap B\right) \setminus n$"".

Assume we have constructed $\dot{q}_n, p_n$ and $\dot{m}_n$. We will see how to construct $\dot{q}_{n+1}, p_{n+1}$ and $\dot{m}_{n+1}$. Let $G_{\alpha_n} \subseteq \mathbb{P}_{\alpha_n}$ be a generic filter with $p_n \in G_{\alpha_n}$. We know that $\dot{q}_n [G_{\alpha_n}] \in \mathbb{P}_\gamma \cap M$ and $\dot{q}_n [G_{\alpha_n}] \restriction \alpha_n \in G_{\alpha_n}$. We now argue in $V [G_{\alpha_n}]$: Since p_n is an $(M, \mathbb{P}_{\alpha_n})$-generic condition, there is $r \in D_n \cap M$ such that $r \leq \dot{q}_n [G_{\alpha_n}]$ and $r \restriction \alpha_n \in G_{\alpha_n}$. Let W be the set of all $m \in \omega$ such that there is $\overline{r} \in \mathbb{P}_\gamma$ such that the following holds:

1. $\overline{r} \leq r$,
2. $\overline{r} \restriction \alpha_n \in G_{\alpha_n}$, and
3. $\overline{r} \Vdash_\gamma$ "$m \in \dot{Z}_n \setminus n$".

Clearly $W \in M [G_{\alpha_n}] \cap \mathcal{I}(\mathcal{A})^+$. Since p_n is an $(M, \mathbb{P}_{\alpha_n}, \mathcal{A}, B)$-generic condition, there is $m_n \in B$ and $q_{n+1} \in \mathbb{P}_\gamma$ such that $q_{n+1} \leq r$, $q_{n+1} \restriction \alpha_n \in G_{\alpha_n}$ and $q_{n+1} \Vdash_\gamma$ "$m_n \in \dot{Z}_n \setminus n$". Back in V, let $\dot{q}_{n+1}$ and $\dot{m}_n$ be names for q_{n+1} and m_n that are forced by p_n to have all the properties above. We now apply the inductive hypothesis on γ_{n+1} and find p_{n+1} with the desired properties.

Let $\overline{p} = \bigcup_{n\in\omega} p_n$, it is easy to see that $\overline{p}$ is an $(M, \mathbb{P}_\gamma)$-generic condition and $\overline{p} \Vdash$ "$\dot{q}_n \in G_\gamma$" for every $n \in \omega$ (see the proof of Lemma 2.8 in [1] for more details). It is clear that $\overline{p}$ is a $(M, \mathbb{P}_\gamma, \mathcal{A}, B)$-generic condition. ■

We conclude the following:

Corollary 37 *Let $\mathcal{A}$ be a tight* MAD *family. If $\mathcal{P} = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha \leq \gamma \rangle$ is a countable support iteration of proper forcings such that $\mathbb{P}_\alpha \Vdash_\alpha$ "$\dot{\mathbb{Q}}_\alpha$ strongly preserves the tightness of $\mathcal{A}$", then $\mathbb{P}_{\omega_2} \Vdash$ "$\mathcal{A}$ is a tight* MAD *family".*

We can now prove the consistency result:

Proposition 38 *It is consistent that $\mathfrak{a}(\mathcal{K}) < \mathfrak{a}_T$.*

Proof. We start with a model of GCH and perform a countable support iteration $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha < \omega_2 \rangle$ such that $\mathbb{P}_\alpha$ forces that $\dot{\mathbb{Q}}_\alpha$ is a forcing of the type $\mathbb{P}(\mathcal{C})$. Furthermore, with a suitable bookkeeping we make sure that $\mathfrak{a}_T = \mathfrak{c} = \omega_2$ holds in the final extension. Since $\mathbb{P}_{\omega_2}$ strongly preserves the tightness of $\mathcal{A}$, it follows that there is a (tight) MAD family of size ω_1. Moreover, $\mathfrak{d} = \omega_1$ holds in the extension since each $\mathbb{P}_\alpha$ is ω^ω-bounding. In this way, $\mathfrak{a} = \mathfrak{d} = \omega_1$ hence $\mathfrak{a}(\mathcal{K}) = \omega_1$. ■

We do not, however, know the answer to the following question:

Problem 39 *Is $\mathfrak{a}(\mathcal{K}) \leq \mathfrak{a}_T$?*

It seems difficult to produce a model of $\mathfrak{a}_T < \mathfrak{a}(\mathcal{K})$. A model of $\mathfrak{a}_T < \mathfrak{a}(\mathcal{K})$ and $\mathfrak{c} = \omega_2$ would be a model of $\mathfrak{d} = \omega_1$ and $\mathfrak{a} = \omega_2$, which would answer a famous open problem of Roitman. In fact, in all known models of $\mathfrak{c} = \omega_2$ the equality $\mathfrak{d} = \mathfrak{a}(\mathcal{K})$ holds. It is possible to build models of $\mathfrak{d} < \mathfrak{a}(\mathcal{K})$ by template iterations (see [23] and [6]) but this approach does not seem to help in order to build a model of $\mathfrak{a}_T < \mathfrak{a}(\mathcal{K})$.

1.3 Remarks on $\omega_1 = \mathfrak{a}^+(\omega_1) < \mathfrak{c}$

Recall that $\mathfrak{a}^+(\omega_1)$ is defined as the least κ such that every AD family of size ω_1 can be extended to a MAD family of size at most κ. In this way, $\omega_1 = \mathfrak{a}^+(\omega_1)$ is equivalent to the assertion that every AD of size ω_1 can be extended to a MAD family of size ω_1. This is obviously true under CH, but it is unknown if it is consistent with the failure of the Continuum Hypothesis:

Problem 40 ([21]) *Is it consistent that $\omega_1 = \mathfrak{a}^+(\omega_1) < \mathfrak{c}$?*

In [21] it was proved that it is consistent that $\omega_2 = \mathfrak{a}^+(\omega_1) < \omega_3 = \mathfrak{c}$, so at least $\mathfrak{a}^+(\omega_1)$ is consistently less than $\mathfrak{c}$. One "rule of thumb" which one learns when working on cardinal invariants, is that if an invariant is consistently less than $\mathfrak{c}$, then this will already happen in the Sacks model. This intuition is formalized by the following interesting theorem of Zapletal: (see [25] chapter 6).

Proposition 41 (LC) *If $\mathfrak{j}$ is a tame invariant*[4] *such that $\mathfrak{j} < \mathfrak{c}$ is consistent, then "$\mathfrak{j} = \omega_1$" holds in the Sacks model.*

Unfortunately, the theorem of Zapletal can not be applied to $\mathfrak{a}^+(\omega_1)$. Furthermore, it follows by the results on [21] that $\mathfrak{a}^+(\omega_1) = \mathfrak{c}$ holds in the Sacks model. In this section, we will derive some consequences of $\omega_1 = \mathfrak{a}^+(\omega_1) < \mathfrak{c}$. Our main tool is the following result:

Proposition 42 *The following are equivalent:*

1. $\mathfrak{a}^+(\omega_1) = \omega_1$.
2. $\mathfrak{a} = \omega_1$ *and for every ideal $\mathcal{I}$ on ω, if $\mathsf{cov}^*(\mathcal{I}) \leq \omega_1$, then $\mathsf{cov}^*(\mathcal{I}^\perp) \leq \omega_1$.*

Proof. We first assume that $\mathfrak{a}^+(\omega_1) = \omega_1$. Let $\mathcal{I}$ be an ideal on ω such that $\mathsf{cov}^*(\mathcal{I}) \leq \omega_1$. Clearly $\mathfrak{a}^+(\omega_1) = \omega_1$ implies $\mathfrak{a} = \omega_1$, so $\max\{\mathfrak{a}, \mathsf{cov}^*(\mathcal{I})\} = \omega_1$ which we already know implies that $\mathfrak{a}(\mathcal{I}) = \omega_1$. In this way, there is $\mathcal{A} \subseteq \mathcal{I}$ a MAD family restricted to $\mathcal{I}$ such that $|\mathcal{A}| = \omega_1$. Since $\mathfrak{a}^+(\omega_1) = \omega_1$, we can find an AD family $\mathcal{B} \subseteq \mathcal{A}^\perp$ such that $\mathcal{A} \cup \mathcal{B}$ is a MAD family and $|\mathcal{B}| \leq \omega_1$. By the maximality of $\mathcal{A}$, it follows that $\mathcal{B} \subseteq \mathcal{I}^\perp$. Since $\mathcal{A} \cup \mathcal{B}$ is a MAD family, we have that $\mathcal{B}$ is a MAD family restricted to $\mathcal{I}^\perp$, so $\mathsf{cov}^*(\mathcal{I}^\perp) \leq |\mathcal{B}| \leq \omega_1$.

We now assume that $\mathfrak{a} = \omega_1$ and if $\mathcal{I}$ is a ideal on ω such that $\mathsf{cov}^*(\mathcal{I}) \leq \omega_1$, then $\mathsf{cov}^*(\mathcal{I}^\perp) \leq \omega_1$. We will prove that $\mathfrak{a}^+(\omega_1) = \omega_1$. Let $\mathcal{A}$ be an AD family of size ω_1, we must prove that $\mathcal{A}$ can be extended to a MAD family of the same size. It suffices to show that $\mathfrak{a}\left(\mathcal{I}(\mathcal{A})^\perp\right) \leq \omega_1$. If $\mathfrak{a}\left(\mathcal{I}(\mathcal{A})^\perp\right) = \omega$ there is nothing to prove, so we assume that $\omega_1 \leq \mathfrak{a}\left(\mathcal{I}(\mathcal{A})^\perp\right)$, which implies that $\mathfrak{a} \leq \mathfrak{a}\left(\mathcal{I}(\mathcal{A})^\perp\right)$. It is easy to see that $\mathsf{cov}^*(\mathcal{I}(\mathcal{A})) = \omega_1$, hence $\mathsf{cov}^*\left(\mathcal{I}(\mathcal{A})^\perp\right) \leq \omega_1$. ■

The ideal $\mathcal{WF}$ is defined as the ideal generated by well-founded subtrees of $\omega^{<\omega}$. It is easy to see that $\mathcal{K}^\perp = \mathcal{WF}$. Recall that an ideal $\mathcal{I}$ on ω is called *Fréchet* (or *nowhere tall*) if for every $A \in \mathcal{I}^+$ there is $B \in [A]^\omega$ such that $B \in \mathcal{I}^\perp$. It is not hard to see that $\mathcal{BR}$, $\mathcal{BR}^\perp$, $\mathcal{WF}$ and $\mathcal{K}$ are Fréchet ideals.

It is easy to see that $\mathcal{I} \subseteq \mathcal{I}^{\perp\perp}$ for any ideal $\mathcal{I}$, while $\mathcal{I}$ is Fréchet if and only if $\mathcal{I} = \mathcal{I}^{\perp\perp}$. It follows that $\mathcal{I}^\perp$ is a Fréchet ideal for any ideal $\mathcal{I}$.

It is easy to calculate the following:

Lemma 43

1. $\mathsf{cov}^*(\mathcal{BR}) = \mathfrak{c}$.
2. $\mathsf{cov}^*(\mathcal{K}) = \mathfrak{d}$.
3. $\mathsf{cov}^*(\mathcal{WF}) = \mathfrak{b}$.

Proof. We will first prove that $\mathsf{cov}^*(\mathcal{BR}) = \mathfrak{c}$. Let $\kappa < \mathfrak{c}$ and $\mathcal{X} = \{B_\alpha \mid \alpha < \kappa\}$ a subset of $\mathcal{BR}$. For every $\alpha < \kappa$ there is a set $F_\alpha \in [2^\omega]^{<\omega}$ such that $B_\alpha \subseteq \bigcup\limits_{\alpha<\kappa} \widehat{r}$.

[4] The reader may consult [25] for the definition of tame invariant.

Since $\kappa < \mathfrak{c}$ there is $x \notin \bigcup\limits_{\alpha<\kappa} F_{\alpha}$. Clearly $\widehat{x} \in \mathcal{BR}$ and it is almost disjoint with B_{α}. It follows that $\mathsf{cov}^*(\mathcal{BR}) = \mathfrak{c}$.

We will now prove that $\mathsf{cov}^*(\mathcal{K}) = \mathfrak{d}$. It is easy to see that for every finitely branching tree $T \subseteq \omega^{<\omega}$ there is $f \in \omega^{\omega}$ such that $[T] \subseteq \{h \in \omega^{\omega} \mid h \leq f\}$. It follows from this fact that $\mathsf{cov}^*(\mathcal{K}) \leq \mathfrak{d}$. We will now prove that $\mathfrak{d} \leq \mathsf{cov}^*(\mathcal{K})$. Let $\mathcal{D} \subseteq \mathcal{K}$ such that for every infinite $X \in \mathcal{K}$ there is $D \in \mathcal{D}$ such that $X \cap D$ is infinite. We may assume that every element of $\mathcal{D}$ is a finitely branching tree. Since $\omega^{\omega} \subseteq \mathcal{K}$, it follows that $\omega^{\omega} = \bigcup\limits_{T\in\mathcal{D}} [T]$, which implies that $\mathfrak{d} \leq \mathsf{cov}^*(\mathcal{K})$.

Finally, we will show that $\mathsf{cov}^*(\mathcal{WF}) = \mathfrak{b}$. We will first prove that $\mathsf{cov}^*(\mathcal{WF})$ is at most $\mathfrak{b}$. Let $\mathcal{B} \subseteq \omega^{\omega}$ be an unbounded family of increasing functions with $|\mathcal{B}| = \mathfrak{b}$. For every $s \in \omega^{<\omega}$, let $\langle s\rangle_{<\omega} = \{t \in \omega^{<\omega} \mid s \subseteq t\}$. Fix an enumeration $\omega^{<\omega} = \{t_n \mid n \in \omega\}$.

If $f \in \mathcal{B}$, $s \in \omega^{<\omega}$ and $n \in \omega$, define $X_n(s,f) = \left\{t_i \in \langle s^\frown n\rangle_{<\omega} \mid i \leq f(n)\right\}$ and let $X(s,f) = \bigcup\limits_{n\in\omega} X_n(s,f)$. It is easy to see that $\{X(s,f) \mid f \in \mathcal{B} \wedge s \in \omega^{<\omega}\}$ is a witness for $\mathsf{cov}^*(\mathcal{WF})$, so $\mathsf{cov}^*(\mathcal{WF}) \leq \mathfrak{b}$.

We will now prove that $\mathfrak{b} \leq \mathsf{cov}^*(\mathcal{WF})$. Let $\mathcal{D} \subseteq \mathcal{WF}$ such that for every infinite $A \in \mathcal{WF}$ there is $W \in \mathcal{D}$ such that $A \cap W$ is infinite and $|\mathcal{D}| = \mathsf{cov}^*(\mathcal{WF})$. For every $n \in \omega$, let $r_n \in \omega^{\omega}$ such that $r_n(0) = n$ and $r_n(m) = 0$ for every $m > 0$. Let $W \in \mathcal{D}$, since W is contained in a well-founded tree, we can find a function $g_W : \omega \longrightarrow \omega$ such that $\widehat{r_n} \cap W \subseteq \omega^{g_W(n)}$ for every $n \in \omega$. It is easy to see that $\{g_W \mid W \in \mathcal{D}\}$ is an unbounded family. ■

We can now prove the following:

Proposition 44 *If $\mathfrak{a}^+(\omega_1) = \omega_1 < \mathfrak{c}$ then*

1. $\omega_1 < \mathfrak{sep}$, *and*
2. $\mathfrak{d} = \omega_1$.

Proof. First assume that $\mathfrak{sep} = \omega_1$. Since $\mathfrak{sep} = \mathsf{cov}^*\left(\mathcal{BR}^{\perp}\right) = \omega_1$ and we are assuming that $\mathfrak{a}^+(\omega_1) = \omega_1$ holds, we conclude that $\mathfrak{c} = \mathsf{cov}^*\left(\mathcal{BR}^{\perp}\right) = \mathsf{cov}^*\left(\mathcal{BR}^{\perp\perp}\right) = \omega_1$ which is in contradiction with our hypothesis.

Since $\mathfrak{a}^+(\omega_1) = \omega_1$ implies $\mathfrak{a} = \omega_1$, we conclude that $\mathsf{cov}^*(\mathcal{WF}) = \mathfrak{b} = \omega_1$, which implies that $\omega_1 = \mathsf{cov}^*\left(\mathcal{WF}^{\perp}\right) = \mathsf{cov}^*(\mathcal{K}) = \mathfrak{d}$. ■

It follows from the result that $\mathfrak{a}^+(\omega_1) = \omega_1$ fails in the Sacks, Cohen, Hechler, Laver, Mathias and Miller models. We will now prove that it also fails in the random model. By μ we will denote the standard measure on 2^{ω} and μ^* denotes the exterior measure.

Lemma 45 *There is a set $A \subseteq 2^{\omega}$ such that $\mu^*(A) = 1$ and $|A| = \mathsf{non}(\mathcal{N})$.*

Proof. Let $S \subseteq [0,1]$ be the set of all $x \in [0,1]$ such that there is $B \subseteq 2^{\omega}$ with $|B| = \mathsf{non}(\mathcal{N})$ for which $x \leq \mu^*(B)$. Let z be the supremum of S (note that $S \neq \emptyset$). We claim that $z \in S$. Since z is the supremum of S, there is an increasing sequence $\langle z_n \rangle_{n\in\omega}$ of elements of S that converges to z. For every $n \in \omega$, we choose $B_n \subseteq 2^\omega$ such that $z_n \leq \mu^*(B_n)$ and $|B_n| = \mathsf{non}(\mathcal{N})$. Clearly $B = \bigcup_{n\in\omega} B_n$ has size $\mathsf{non}(\mathcal{N})$ and $z = \mu^*(B)$. We now claim that $z = 1$. We argue by contradiction, assume that $z < 1$. Let $B \subseteq 2^\omega$ such that $|B| = \mathsf{non}(\mathcal{N})$ and $\mu^*(B) = z$. Since $\mu^*(B) < 1$, there is a non-null compact set $C \subseteq 2^\omega$ such that $B \cap C = \emptyset$. Let $A \subseteq C$ such that $A \notin \mathcal{N}$ and $|A| = \mathsf{non}(\mathcal{N})$. Let $D = A \cup B$, clearly D has size $\mathsf{non}(\mathcal{N})$ and $z < \mu^*(D)$, which is a contradiction. ■

We can now prove the following:

Proposition 46 **($\mathfrak{a}^+(\omega_1) = \omega_1$)** *If $\mathsf{non}(\mathcal{N}) = \omega_1$, then $\mathsf{cov}^*(tr(\mathcal{N})) = \omega_1$.*

Proof. Assume that $\mathsf{non}(\mathcal{N}) = \omega_1$. Let $X \subseteq 2^\omega$ such that $\mu^*(X) = 1$ and $|X| = \mathsf{non}(\mathcal{N}) = \omega_1$. Define $\mathcal{A} = \{\widehat{r} \mid r \in X\}$, clearly $\mathcal{A}$ is an AD family of size ω_1 and $\mathcal{A} \subseteq tr(\mathcal{N})$. We claim that $\mathcal{A}^\perp \subseteq tr(\mathcal{N})$. Let $B \in tr(\mathcal{N})^+$, thus, $\pi(B)$ is a non-null G_δ set. Since $\mu^*(X) = 1$, there is an $r \in X \cap \pi(B)$, which implies that $\widehat{r} \cap B$ is infinite, so $B \notin \mathcal{A}^\perp$.

Since $\mathfrak{a}^+(\omega_1) = \omega_1$, there is a MAD family $\mathcal{B}$ such that $\mathcal{A} \subseteq \mathcal{B}$ and $|\mathcal{B}| = \omega_1$. By the comment above, we know that $\mathcal{B} \subseteq tr(\mathcal{N})$, Since $\mathcal{B}$ is a MAD family, it follows that $\mathsf{cov}^*(tr(\mathcal{N})) \leq |\mathcal{B}| = \omega_1$. ■

Since $\mathsf{cov}(\mathcal{N}) \leq \mathsf{cov}^*(tr(\mathcal{N}))$ and $\mathsf{non}(\mathcal{N}) = \omega_1$ holds in the random model, we can conclude the following:

Corollary 47 $\mathfrak{a}^+(\omega_1) = \omega_1$ *fails in the random model.*

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Some Structural Aspects of the Katětov Order on Borel Ideals

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Abstract We prove that the Katětov order on Borel ideals (1) contains a copy of $\mathscr{P}(\omega)/\mathbf{Fin}$, consequently it has increasing and decreasing chains of lenght $\mathfrak{b}$; (2) the sequence $\mathbf{Fin}^{\alpha}$ $(\alpha < \omega_1)$ is a strictly increasing chain; and (3) in the Cohen model, Katětov order does not contain any increasing nor decreasing chain of length $\mathfrak{c}$, answering a question of Hrušák (2011).

Keywords Borel ideals · Katětov order · Cohen model

1 Introduction

The *Katětov order* is defined on ideals on ω as follows: $\mathsf{I} \leq_K \mathsf{J}$ if there is a function $f \in \omega^{\omega}$ such that $f^{-1}(I) \in \mathsf{J}$ for all $I \in \mathsf{I}$. This order is a generalization of the better-known Rudin-Keisler order. It is a powerful tool for the study of some properties about ideals and filters, like Ramsey type properties, Fubini property, classes of ultrafilters, destructibility of ideals by forcing, among other (see [3, 7, 8]). Frequently, the combinatorial properties about ideals have definable critical ideals in the Katětov order. In this paper we study some structural aspects of the Katětov order restricted to the family of Borel ideals, as an

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order-type. Clearly, the Katětov order among the whole family of ideals is more complicated. Another fragment of it that has been studied is the family of ideals generated by maximal almost-disjoint families (see [6]). Two relevant structural properties of Katětov order on definable ideals, the named *category and measure dichotomies*, are proved by Hrušák in [5]. A more complete study about ideals on ω is available in [4].

The properties we are going to prove are described in the abstract and they correspond with the number of each section. The notation we use is standard, and mainly follows [10].

2 Summable Ideals in the Katětov Order

We now prove that the Katětov order on Borel ideals contains a copy of $\mathscr{P}(\omega)/\mathbf{Fin}$, ordered by $\subseteq^*$. More specifically, this copy is contained inside the family of summable ideals. This result is analogous to another obtained by Ilijas Farah (Theorem 1.12.1(c) in [2]) about the Rudin-Blass order. Recall that an ideal I is *summable* if there is a function f from ω to $[0, \infty)$ satisfying $\lim_{n\to\infty} f(n) = 0$, $\sum_{n\in\omega} f(n) = \infty$ and

$$\mathsf{I} = \mathsf{I}_f := \left\{ A \subseteq \omega : \sum_{n\in A} f(n) < \infty \right\}.$$

Clearly, by definition, summable ideals are tall F_σ P-ideals. Let us denote by Σ the family of summable ideals.

Theorem 1 *There is an order embedding φ from $\mathscr{P}(\omega)/\mathbf{Fin}$ into Σ.*

Proof Let us recursively construct two sequences of real numbers p_n and r_n ($n \in \omega$) as follows: $r_0 = 1$, $p_0 = 0$, $p_{n+1} \geq ((r_n + 1)p_n + 1)\, r_n^{-1}$ and $r_{n+1} = 2^{-n-1}(p_{n+1} - p_n)^{-1}$: By definning intervals $I_n = [p_n, p_{n+1} - 1]$ we have constructed a partition of ω in intervals satisfying $\min(I_{n+1}) = \max(I_n) + 1$ and

(1) $|I_n| r_n \geq |\bigcup_{j<n} I_j|$ and
(2) $|I_n| r_{n+1} \leq 2^{-n-1}$.

For each infinite subset A of ω, let us define a function $f_A : \omega \to (0, 1]$ such that for every $k < \omega$

$$f_A(k) = \begin{cases} r_n & \text{if } k \in I_n \text{ and } n \notin A \\ r_{n+1} & \text{if } k \in I_n \text{ and } n \in A \end{cases}$$

□

The Theorem follows immediately from claims below.

Claim For every infinite and coinfinite subset A of ω, I_{f_A} is a non-trivial tall ideal.

Proof (Proof of claim) Note that

$$\sum_{n<\omega} f_A(n) = \sum_{j<\omega} \sum_{n\in I_j} f_A(n) \geq \sum_{j\in\omega\setminus A} r_j |I_j| \geq \sum_{j\in\omega\setminus A} |\bigcup_{i<j} I_i| = \infty.$$

□

Claim If $A, B \in [\omega]^\omega$ and $A \subseteq^* B$ then $\mathsf{I}_{f_A} \leq_K \mathsf{I}_{f_B}$.

Proof (Proof of claim) Note that if $A \subseteq^* B$ then $f_B \leq^* f_A$ and then $\mathsf{I}_{f_A} \subseteq \mathsf{I}_{f_B}$, and so $\mathsf{I}_{f_A} \leq_K \mathsf{I}_{f_B}$. □

Claim If $A, B \in [\omega]^\omega$ and $|A \setminus B| = \aleph_0$ then $\mathsf{I}_{f_A} \not\leq_K \mathsf{I}_{f_B}$.

Proof (Proof of claim) Let φ be in ω^ω and let us prove that φ is not a witness for $\mathsf{I}_{f_A} \leq_K \mathsf{I}_{f_B}$. First note that for any $n < \omega$ there is $F_n \subseteq I_n$ such that $|F_n| \geq \frac{1}{2}|I_n|$ and, either $\varphi(x) < \min(I_n)$ for all $x \in F_n$ or $\varphi(x) \geq \min(I_n)$ for all $x \in F_n$. Then we have two cases: □

Case 1 *The family* $C = \{n \in A \setminus B : x \in F_n \to \varphi(x) < \min(I_n)\}$ *is infinite*. Note that by condition (1) and the pigeonhole principle, for any $n \in C$ there is $k_n \in \bigcup_{j<n} I_j$ such that $|\varphi^{-1}[\{k_n\}] \cap F_n| \geq \frac{1}{2r_n}$. Note that for any $n \in A \setminus B$, $\sum_{i \in \varphi^{-1}(k_n)} f_B(i) \geq r_n \cdot \frac{1}{2r_n} = \frac{1}{2}$. If $\{k_n : n \in C\}$ is finite, then it belongs I_{f_A}. In other case we can take an infinite $C' \subseteq C$ such that for every $j < \omega$, $|\{k_n : n \in C'\} \cap I_j| \leq 1$, Then, we have that $\bigcup_{n \in C'} \varphi^{-1}[\{k_n\}] \notin \mathsf{I}_{f_B}$ but $\{k_n : n \in C'\} \in \mathsf{I}_{f_A}$. Hence in this case, φ is not a witness for $\mathsf{I}_{f_A} \leq_K \mathsf{I}_{f_B}$.

Case 2 *The family* $D = \{n \in A \setminus B : x \in F_n \to \varphi(x) \geq \min(I_n)\}$ *is infinite*. Note that $Y = \bigcup_{n \in D} F_n$ is an I_{f_B} positive set and $J = \varphi'' Y \in \mathsf{I}_{f_A}$ since $\sum_{n \in J} f_A(n) \leq \sum_{y \in Y} f_A(\varphi(y)) = \sum_{n \in D} \sum_{y \in F_n} f_A(\varphi(y)) \leq \sum_{n \in D} r_{n+1}|F_n| \leq \sum_{n \in D} 2^{-n-1}$. Hence in case 2, φ is not a witness for $\mathsf{I}_{f_A} \leq_K \mathsf{I}_{f_B}$.

Corollary 1 Σ *ordered by the Katětov order contains increasing and decreasing chains of lenght* $\mathfrak{b}$*, and antichains of size* $\mathfrak{c}$.

3 The Ideals $\mathbf{Fin}^\alpha$ in the Katětov Order

We investigate an increasing chain in the Katětov order of lenght ω_1.

Definition 1 (**Katětov** [9]**, also see** [1]) A countable set X_α and an ideal $\mathbf{Fin}^\alpha$ on X_α ($\alpha < \omega_1$) are defined by recursion as follows: $X_0 = \{0\}$, $\mathbf{Fin}^0 = \{\emptyset\}$, $X_{\alpha+1} = \omega \times X_\alpha$,

$$\mathbf{Fin}^{\alpha+1} = \{A \subseteq X_{\alpha+1} : (\exists m)(\forall n \geq m)\{r \in X_\alpha : (n, r) \in A\} \in \mathbf{Fin}^\alpha\},$$

and if α is a limit ordinal, then $X_\alpha = \bigcup_{\beta<\alpha}\{\beta\} \times X_\beta$ and

$$\mathbf{Fin}^\alpha = \{A \subseteq X_\alpha : (\exists \beta < \alpha)(\forall \gamma \geq \beta)\{r \in X_\gamma : (\gamma, r) \in A\} \in \mathbf{Fin}^\gamma\}.$$

Proposition 1 *The sequence* $\{\mathbf{Fin}^\alpha : \alpha < \omega_1\}$ *is strictly* $\leq_K$*-increasing.*

Proof First note that the projection $\pi_{X_\alpha} : X_{\alpha+1} \to X_\alpha$ is a witness for $\mathbf{Fin}_\alpha \leq_K \mathbf{Fin}_{\alpha+1}$. We conclude that the sequence is $\leq_K$-increasing by showing that if α is limit and $\beta = \gamma + 1 < \alpha$ then $\mathbf{Fin}^\beta \leq_K \mathbf{Fin}^\alpha$. Let $\{\alpha_n : n < \omega\}$ be an enumeration of $\alpha \setminus \gamma$, and let φ_0 be a bijection from $\bigcup_{\delta<\gamma}\{\delta\} \times X_\delta$ onto $\{1\} \times X_1$, and for $0 < n < \omega$, let $\varphi_n : X_{\alpha_n} \to X_\gamma$ be a witness of $\mathbf{Fin}^\gamma \leq_K \mathbf{Fin}^{\alpha_n}$. Now we define the Katětov function desired by $\varphi(\delta, r) = (n, \varphi_n(r))$, if $\gamma \leq \delta = \alpha_n$, and $\varphi(\delta, r) = (1, \varphi_0(\delta, r))$ if $\delta < \gamma$. Let us prove that φ works.

Let A be in $\mathbf{Fin}^{\beta}$, and $k \in \omega$ such that $\{r \in X_{\gamma} : (m, r) \in A\} \in \mathbf{Fin}^{\gamma}$, for all $m \geq k$. Let ε be the maximum of the family $\{\alpha_1, \ldots, \alpha_k, \beta\}$. Then, for all ordinal $\varepsilon < \xi < \alpha$, $\{r \in X_{\xi} : (\xi, r) \in \varphi^{-1}(A)\} = \varphi_m^{-1}\{r \in X_{\gamma} : (m, r) \in A\}$, where $\xi = \alpha_m$. Since φ_m witnesses $\mathbf{Fin}^{\gamma} \leq_K \mathbf{Fin}^{\xi}$, we are done.

For the strictness, it will be sufficient to prove $\mathbf{Fin}^{\alpha+1} \not\leq_K \mathbf{Fin}^{\alpha}$ for all $\alpha < \omega_1$. Let us suppose not, and let α be the minimal with respect to the property $\mathbf{Fin}^{\alpha+1} \leq_K \mathbf{Fin}^{\alpha}$, and let $f : X_{\alpha} \to X_{\alpha+1}$ a witness for this. For simplicity, let us denote $\omega' = \omega$ if α is a succesor, and $\omega' = \alpha$ if not, and for $\beta \in \omega'$ let β' be equal to $\alpha - 1$ if α is a successor, and $\beta' = \beta$ if not. For $\beta < \omega'$ and $k < \omega$, let us define $Y_{(\beta,k)} = \{j \in X_{\beta'} : f(\beta, j) \in \{k\} \times X_{\alpha}\}$. □

Case 1 *The set* $B = \{\beta < \omega' : (\exists k < \omega) Y_{(\beta,k)} \in (\mathbf{Fin}^{\beta'})^{+}\}$ *is unbounded.* For some cofinal family C of B, we can find an increasing sequence $\langle k_{\beta} : \beta \in C\rangle$ of natural numbers, satisfying $Y_{(\beta,k_{\beta})} \in (\mathbf{Fin}^{\beta'})^{+}$ for all $\beta \in C$. In this case, for every $\beta \in C$, we can consider the function $f_{\beta} : Y_{(\beta,k_{\beta})} \to X_{\alpha}$ given by $f_{\beta}(r) = proj_{X_{\alpha}}(f(\beta, r))$. From $\mathbf{Fin}^{\beta'} \restriction Y_{(\beta,k_{\beta})} \geq_K \mathbf{Fin}^{\beta'}$ and the minimality of α, [1] f_{β} is not a Katětov function, and then, for all $\beta \in C$, we can find a $\mathbf{Fin}^{\beta'}$-positive set A_{β} such that $f_{\beta}(A_{\beta}) \in \mathbf{Fin}^{\alpha}$. Define $A = \bigcup_{\beta \in C}\{k_{\beta}\} \times f_{\beta}(A_{\beta})$. Clearly, $A \in \mathbf{Fin}^{\alpha+1}$ but $f^{-1}(A) \supseteq \bigcup_{\beta \in C}\{\beta\} \times A_{\beta} \in (\mathbf{Fin}^{\alpha})^{+}$. This is a contradiction.

Case 2 *B is bounded.* Let $\langle \beta_n : n < \omega\rangle$ a cofinal increasing sequence in α with $\beta_0 > \sup B$. For all $n < \omega$, the function $g_n : X_{\beta_n} \to X_{\alpha+1}$ given by $g_n(r) = f(\beta_n, r)$ is not a Katětov function, then there is a $\mathbf{Fin}^{\beta_n'}$-positive set A_n such that $g_n(A_n) \in \mathbf{Fin}^{\alpha+1}$. Define $C_n = g_n(A_n) \setminus ((n+1) \times X_{\alpha})$. Note that for all n, $g_n^{-1}(C_n) \in (\mathbf{Fin}^{\beta_n'})^{+}$, because $f^{-1}(\{k\} \times X_{\alpha}) \cap \{\beta_j\} \times X_{\beta_j'} \in \mathbf{Fin}^{\beta_j'}$ for all k and j. Hence, $C = \bigcup_n C_n \in \mathbf{Fin}^{\alpha+1}$ but $f^{-1}(C) \supseteq \bigcup_n \{\beta_n\} \times g_n^{-1}(C_n) \in (\mathbf{Fin}^{\alpha})^{+}$, a contradiction again.

4 Chains in Katětov Order on Borel Ideals and the Cohen Model

In [4], M. Hrušák asked if there are increasing or decreasing $\leq_K$-chains of Borel ideals with lenght $\mathfrak{c}$. This section is dedicated to prove that, consistently, this is not the case. Let $\mathbb{C}_{\omega_2}$ be the forcing for adding ω_2-many Cohen reals. We first prove some facts about families of $\aleph_2$-many $\mathbb{C}_{\omega_2}$-names. Let us recall that every automorphism φ of $\mathbb{C}_{\omega_2}$, induces an automorphism $\overline{\varphi}$ of $V^{\mathbb{C}_{\omega_2}}$ (the family of $\mathbb{C}_{\omega_2}$-names on V) recursively defined by $\overline{\varphi}(\dot{A}) = \{\langle \overline{\varphi}(\dot{a}), \varphi(p)\rangle : \langle \dot{a}, p\rangle \in \dot{A}\}$, satisfying that for any $\mathbb{C}_{\omega_2}$-generic filter G on V,

$$val_G(\dot{A}) = val_{\varphi(G)}(\overline{\varphi}(\dot{A})).$$

Note that $\overline{\varphi}^{-1} = \overline{\varphi^{-1}}$ for all automorphism φ of $\mathbb{C}_{\omega_2}$.

Lemma 1 *Let V a model of CH and $\{\dot{A}_{\alpha} : \alpha < \omega_2\}$ a family of $\mathbb{C}_{\omega_2}$-names for real numbers. Then, there exists an automorphism φ of $\mathbb{C}_{\omega_2}$and some $\alpha < \beta < \omega_1$ such that $\varphi^{-1} = \varphi$ and $\overline{\varphi}(\dot{A}_{\alpha}) = \dot{A}_{\beta}$.*

[1] It is an easy fact that for every ideal I and every I-positive set X, the restriction $\mathsf{I} \restriction X = \{A \subseteq X : A \in \mathsf{I}\}$ is an ideal on X which is Katětov above I.

Proof For every $\alpha < \omega_2$, let X_α be the support of $\dot{A}_\alpha$. By Fodor's lemma, there is a root $R \subseteq \omega_1$ and $Y \in [\omega_2]^{\omega_2}$ such that for any $\alpha < \beta \in Y$, $X_\alpha \cap X_\beta = R$. For each $\alpha \in Y$, let C_α be a sequence $\langle (D_n^\alpha, E_n^\alpha) : n < \omega \rangle$ satisfying:

1. $D_n^\alpha \cup E_n^\alpha$ is a maximal antichain in $\mathbb{C}_{X_\alpha}$, and
2. $p \in D_n^\alpha$ implies $\overline{p} \Vdash \dot{A}_\alpha(n) = 0$ and $p \in E_n^\alpha$ implies $\overline{p} \Vdash \dot{A}_\alpha(n) = 1$, where $\overline{p}$ is the obvious extension of p to $\mathbb{C}_{\omega_2}$.

Every X_α is a countable set and then, for every $\alpha, \beta \in Y$, $\mathbb{C}_{X_\alpha} \cong \mathbb{C}_\omega \cong \mathbb{C}_{X_\beta}$. Let us consider each C_α as a subset of $\mathbb{C}_\omega$. Since $V \models CH$ and there are $\mathfrak{c} = \omega_1$-many countable sequences of pairs of countable subsets of ω, there are some $\alpha < \beta \in Y$ such that $C_\alpha = C_\beta$, i.e. there is an isomorphism ψ from $\mathbb{C}_{X_\alpha}$ onto $\mathbb{C}_{X_\beta}$ such that for all $n < \omega$, $p \in D_n^\alpha$ iff $\psi(p) \in D_n^\beta$ and $p \in E_n^\alpha$ iff $\psi(p) \in E_n^\beta$. Let us define the requested automorphism by

$$\varphi(p)(\gamma) = \begin{cases} p(\gamma) & \text{if } \alpha \neq \gamma \neq \beta \\ \psi(p)(\beta) & \text{if } \gamma = \alpha \\ \psi^{-1}(p)(\alpha) & \text{if } \gamma = \beta. \end{cases}$$

It is clear by definition that $\varphi = \varphi^{-1}$ and $\overline{\varphi}(\dot{A}_\alpha) = \dot{A}_\beta$. □

Note that the automorphism φ also satisfies that if $\dot{f}$ is the name of a witness for $\dot{I} \leq_K \dot{\mathsf{J}}$, then $\overline{\varphi}(\dot{f})$ is a name for a witness for $\overline{\varphi}(\dot{\mathsf{I}}) \leq_K \overline{\varphi}(\dot{\mathsf{J}})$. By the classical procedure, we can consider Borel ideals as real numbers (the Borel codes in 2^ω) and the Katětov order becomes a preorder on real numbers, that satisfies the hypothesis of the next Theorem.

Theorem 2 *Let V be a model of CH, $\leq$ a preorder relation on 2^ω satisfying that for all $\dot{x}, \dot{y}$ $\mathbb{C}_{\omega_2}$-names for elements of 2^ω and all automorphism φ of $\mathbb{C}_{\omega_2}$, $V[G] \models \dot{x} \leq \dot{y}$ iff $V[\varphi(G)] \models \dot{x} \leq \dot{y}$, for all $\mathbb{C}_{\omega_2}$-generic filter G on V. Then, in $V[G]$, there are no increasing nor decreasing $\leq$-chains of lenght $\mathfrak{c}$.*

Proof Suppose that in $V[G]$ exists an increasing chain of lenght ω_2. Let $\{\dot{r}_\alpha : \alpha < \omega_2\}$ be a family of $\mathbb{C}_{\omega_2}$-names such that $\mathbb{C}_{\omega_2} \vdash \dot{r}_\alpha < \dot{r}_\beta$ for $\alpha < \beta < \omega_2$. By Lemma 1 there are an automorphism φ of $\mathbb{C}_{\omega_2}$ and $\alpha < \beta < \omega_2$ such that $\overline{\varphi}(\dot{r}_\alpha) = \dot{r}_\beta$ and $\overline{\varphi}(\dot{r}_\beta) = \dot{r}_\alpha$. Then, for any $\mathbb{C}_{\omega_2}$-generic filter G on V, $val_G(\dot{r}_\alpha) < val_G(\dot{r}_\beta)$, obviously, the same holds for the $\mathbb{C}_{\omega_2}$-generic filter $\varphi(G)$, i.e. $val_{\varphi(G)}(\dot{r}_\alpha) < val_{\varphi(G)}(\dot{r}_\beta)$. However, since $\overline{\varphi}(\dot{r}_\alpha) = \dot{r}_\beta$, $val_G(\dot{r}_\alpha) = val_{\varphi(G)}(\overline{\varphi}(\dot{r}_\alpha)) = val_{\varphi(G)}(\dot{r}_\beta)$ and $val_G(\dot{r}_\beta) = val_{\varphi(G)}(\overline{\varphi}(\dot{r}_\beta)) = val_{\varphi(G)}(\dot{r}_\alpha)$, and hence,

$$val_{\varphi(G)}(\dot{r}_\beta) = val_G(\dot{r}_\alpha) < val_G(\dot{r}_\beta) = val_{\varphi(G)}(\dot{r}_\alpha),$$

which is a contradiction. Analogously the decreasing case can be proved. □

Corollary 2 *In the Cohen model, the Katětov order does not contain increasing nor decreasing chains of Borel ideals with lenght $\mathfrak{c}$.*

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Cardinal Invariants of Strongly Porous Sets

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In this work we study cardinal invariants of the ideal **SP** of strongly porous sets on ${}^{\omega}2$. We prove that $\mathsf{add}(\mathbf{SP}) = \omega_1$, $\mathsf{cof}(\mathbf{SP}) = \mathfrak{c}$ and that it is consistent that $\mathsf{non}(\mathbf{SP}) < \mathsf{add}(\mathcal{N})$, answering questions of Hrušák and Zindulka. We also find a connection between strongly porous sets on ${}^{\omega}2$ and the Martin number for σ-linked partial orders, and we use this connection to construct a model where all the Martin numbers for σ-k-linked forcings are mutually different.

03E17, 28A75; 03E35, 03E50

1 Introduction

The study of σ-porous sets began in 1967 in [5] and since then, many applications have been found. One of these appears in [10], where the authors proved that, given a Banach space X with a separable dual and a continuous convex function f on X, the set of points in which f is not Fréchet differentiable is σ-porous. Other applications can be found in [2], [7], [8], [11], [12] and [18].

We shall study the notion of strong porosity: Given a metric space $\langle X, d\rangle$, a subset $A \subseteq X$ is *strongly porous* if there is a $p > 0$ such that for any $x \in X$ and any $0 < r < 1$, there is $y \in X$ such that $B_{p\cdot r}(y) \subseteq B_r(x) \setminus A$. In this paper we will refer to strongly porous sets as porous sets. We shall work mostly with porous sets in ${}^{\omega}2$: We will say that a set $A \subseteq {}^{\omega}2$ is n-porous if for every $s \in {}^{<\omega}2$ there is a $t \in {}^{n}2$ such that $\langle s^\frown t\rangle \cap A = \emptyset$. By $s^\frown t$ we denote the concatenation of s followed by t, and by $\langle s\rangle$ we denote the cone of s, that is $\langle s\rangle = \{f \in {}^{\omega}2 : s \subseteq f\}$. It is easy to see (see [6]) that a set $A \subseteq {}^{\omega}2$ is porous if and only if there is an $n \in \omega$ such that A is n-porous. A set A in a metric space $\langle X, d\rangle$ is σ-porous if and only if it is σ-lower porous (see [16]), where A is *lower porous* if for every $x \in X$ there exist $\rho_x > 0$ and $r_x > 0$ such that for any $0 \le r \le r_x$ there is $y \in X$ such that $B_{\rho_x\cdot r}(y) \subseteq B_r(x) \setminus A$. Another classical notion of porosity is upper porosity: A set A in a metric space $\langle X, d\rangle$ is *upper porous* if for every $x \in X$ there is $\rho_x > 0$ and a sequence $r_n \to 0$ such that for every $n \in \omega$

there is $y_n \in X$ such that $B_{\rho_x \cdot r_n}(y_n) \subseteq B_{r_n}(x) \setminus A$. It is easy to see that lower porosity implies upper porosity.

We will denote the σ-ideal generated by porous sets on ${}^{\omega}2$ by $\mathbf{SP}$, the σ-ideal generated by n-porous sets by $\mathbf{SP}_n$, and the σ-ideal generated by upper porous sets by $\mathbf{UP}$. Observe that $\mathbf{SP}_1$ is the ideal of countable sets of ${}^{\omega}2$. Further research about different types of porosity can be found in [15], [16], [17], [19] and [20].

Cardinal invariants of these σ-ideals have been studied in [3], [6], [12], [13] and [14]. Recall that, given a σ-ideal $\mathcal{I}$ over a set X, the following are the standard cardinal invariants of $\mathcal{I}$:

$$\begin{aligned}
\mathsf{add}(\mathcal{I}) &= \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \wedge \bigcup \mathcal{A} \notin \mathcal{I}\}, \\
\mathsf{cov}(\mathcal{I}) &= \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \wedge \bigcup \mathcal{A} \neq X\}, \\
\mathsf{non}(\mathcal{I}) &= \min\{|Y| : Y \subseteq X \wedge Y \notin \mathcal{I}\}, \\
\mathsf{cof}(\mathcal{I}) &= \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \wedge \forall B \in \mathcal{I}\, (\exists A \in \mathcal{A}\, (B \subseteq A))\}.
\end{aligned}$$

In [6] the authors proved that the cardinal invariants of the σ-ideal of lower porous sets in the real line are the same as the cardinal invariants of $\mathbf{SP}$. They proved that $\mathsf{non}(\mathbf{SP}) < \mathfrak{m}_{\sigma\text{-centered}}$ is consistent, that $\mathsf{cov}(\mathbf{SP}) > \mathsf{cof}(\mathcal{N})$ is consistent, and that $\mathsf{cov}(\mathbf{SP}) < \mathfrak{c}$ is consistent, where $\mathfrak{m}_{\sigma\text{-centered}}$ is the smallest cardinal where the Martin's axiom for σ-centered forcings fails and $\mathcal{N}$ is the ideal of sets of Lebesgue measure zero.

There are some analogous inequalities that hold for $\mathbf{UP}$. In [14], M. Repický proved that $\mathsf{non}(\mathbf{UP}) \geq \mathfrak{m}_{\sigma\text{-centered}}$ and $\mathsf{cov}(\mathbf{UP}) \leq \mathsf{cof}(\mathcal{N})$ holds. He also proved [12] that $\mathsf{non}(\mathbf{UP}) \geq \mathsf{add}(\mathcal{N})$ and in [3], J. Brendle proved that $\mathsf{add}(\mathbf{UP}) = \omega_1$ and $\mathsf{cof}(\mathbf{UP}) = \mathfrak{c}$ hold. In [6], the authors asked if the last three inequalities hold also for the $\mathbf{SP}$ ideal. In this work we show that $\mathsf{add}(\mathbf{SP}) = \omega_1$, $\mathsf{cof}(\mathbf{SP}) = \mathfrak{c}$ and that it is consistent that $\mathsf{non}(\mathbf{SP}) < \mathsf{add}(\mathcal{N})$.

Given $k \in \omega$ and a forcing notion $\mathbb{P}$ a subset $A \subseteq \mathbb{P}$ is *k-linked* if for every collection $\{a_i : i \in k\}$ of k elements of A, there is an $a \in \mathbb{P}$ such that for every $i \in k$, $a \leq a_i$. $\mathbb{P}$ is *σ-k-linked* if $\mathbb{P}$ is the countable union of k-linked subsets of $\mathbb{P}$. We will denote by $\mathfrak{m}_k$ the *Martin number for σ-k*-linked forcings, that is, the smallest cardinal κ such that there is a σ-k-linked forcing $\mathbb{P}$ and κ many $\mathbb{P}$-dense subsets of $\mathbb{P}$ such that no filter of $\mathbb{P}$ intersects them all.

If X, Y are sets, then ${}^{Y}X$ is the set of all functions from Y to X and ${}^{<\omega}X = \bigcup_{n\in\omega} {}^{n}X$. If $\sigma, s \in {}^{<\omega}X$, then we will denote that σ is an initial segment of s by $\sigma \sqsubseteq s$. A set $T \subseteq {}^{<\omega}X$ is a tree if it is closed under restrictions to initial segments. If $T \subseteq {}^{<\omega}X$ is a tree, then by $[T]$ we denote the set of branches of T, that is, $[T] = \{f \in {}^{\omega}X : \forall n \in$

$\omega\ (f \restriction n \in T)\}$. With $\mathsf{end}(T)$ we will denote the end nodes of T, that is the nodes of T without extensions. In our forcing notation, the stronger conditions are the smaller ones. In general, our notation follows [1].

2 Additivity and cofinality

The main goal of this section is to prove that $\mathsf{add}(\mathbf{SP}) = \omega_1$ and $\mathsf{cof}(\mathbf{SP}) = \mathfrak{c}$. We will use the following notion.

Definition 2.1 *Let $k \in \omega$. A tree $T \subseteq {}^{<\omega}2$ is a k-porous tree if for every $s \in {}^{<\omega}2$ there is $t \in {}^{k}2$ such that $s^\frown t \notin T$.*

Note that $A \subseteq {}^{\omega}2$ is k-porous if and only if there is a k-porous tree T such that $[T]$ contains A.

Theorem 2.2 *There is a family $\{T_f : f \in {}^{\omega}2\}$ of 2-porous trees such that for every $X \in \mathbf{SP}$, the set $\{f \in {}^{\omega}2 : [T_f] \subseteq X\}$ is countable.*

Proof We will construct the family $\{T_f : f \in {}^{\omega}2\}$ as follows: For every $a \subseteq {}^{<\omega}2$ such that $|a| = 2^n$, let $\varphi_a : a \to {}^{n}2$ be a bijective function. For every $i \in \omega$, let $\psi_i : \{a \subseteq {}^{i}2 : \exists k \in \omega\ (|a| = 2^k)\} \to \omega \setminus \{0\}$ be an injective function. If $a \subseteq {}^{i}2$ and $|a| = 2^k$, define

$$\sigma_a = \langle 0, \underbrace{1, \ldots, 1}_{2\psi_i(a) \text{ times}}, 0\rangle.$$

For each $\sigma \in {}^{<\omega}2$, we will recursively define a finite tree T_σ as follows: $T_\emptyset = \{\emptyset\}$ and if T_σ is defined, then

$$T_{\sigma^\frown i} = \{s \in {}^{<\omega}2 : \exists t \in \mathsf{end}(T_\sigma)\ (\exists j \in \omega\ (\exists a \subseteq {}^{|\sigma|+1}2$$
$$(|a| = 2^j \wedge \sigma^\frown i \in a \wedge s \sqsubseteq t^\frown \sigma_a^\frown \varphi_a(\sigma^\frown i))))\} \cup \{s \in {}^{<\omega}2 : \exists t \in \mathsf{end}(T_\sigma)\ (s \sqsubseteq t^\frown \langle 1, 1\rangle)\}.$$

It is easy to see that, for each $\sigma \in {}^{<\omega}2$, T_σ is a finite 2-porous tree and that if $\sigma \sqsubseteq \tau$, then $T_\sigma \subseteq T_\tau$. For each $f \in {}^{\omega}2$, define $T_f = \bigcup_{n\in\omega} T_{f\restriction n}$. It follows easily that each T_f is a 2-porous tree.

We will show that the family $\{T_f : f \in^{\omega} 2\}$ is the family we were looking for: Let $X \in \mathbf{SP}$. Without loss of generality we will assume that $X = \bigcup_{i\in\omega}[T_i]$, where T_i

is an $i+1$-porous tree. We must show that the set $B = \{f \in {}^{\omega}2 : [T_f] \subseteq X\}$ is countable: For each $s,t \in {}^{<\omega}2$ and each $n \in \omega$, define $B_{s,t,n} = \{f \in {}^{\omega}2 : t \sqsubseteq f, s \in T_t \wedge [T_f] \cap \langle s \rangle \subseteq [T_n]\}$. We will see that $B \subseteq \bigcup_{s,t \in {}^{<\omega}2, n \in \omega} B_{s,t,n}$: If f is such that $f \in B$, then $[T_f] \subseteq \bigcup_{n \in \omega} [T_n]$. Using the Baire Category Theorem we can find $s \in T_f$ and $n \in \omega$ such that $[T_f] \cap \langle s \rangle \subseteq [T_n]$. Find $k \in \omega$ such that $s \in T_{f \restriction k}$. It follows that $f \in B_{s, f\restriction k, n}$. To finish the proof we will see that each $B_{s,t,n}$ has at most $2^{n+1}-1$ elements: Suppose this is not the case and let $s,t \in {}^{<\omega}2$, $n \in \omega$ and $\{f_i\}_{i<2^{n+1}} \subseteq B_{s,t,n}$. Extend s to σ such that $\sigma \in \mathsf{end}(T_t)$. Let $j \in \omega$ be such that the set $a = \{f_i \restriction j : i < 2^{n+1}\}$ has 2^{n+1} elements and let

$$s_0 = \sigma^\frown \langle \underbrace{1, \ldots, 1}_{2\cdot(j-|t|-1) \text{ times}} \rangle^\frown \sigma_a .$$

The tree T_n is $n+1$-porous, so there is a $\tau \in 2^{n+1}$ such that $s_0^\frown \tau \notin T_n$. Find $k < 2^{n+1}$ such that $\varphi_a(f_k \restriction j) = \tau$ and observe that $s_0^\frown \tau = s_0^\frown \varphi_a(f_k \restriction j) \in T_{f_k}$. As a consequence, $[T_{f_k}] \cap \langle s \rangle \nsubseteq [T_n]$, but this contradicts the fact that $f_k \in B_{s,t,n}$. This implies that each $B_{s,t,n}$ is finite, and therefore B is countable. □

We can now prove the main result of this section.

Corollary 2.3 $\mathsf{add}(\mathbf{SP}) = \omega_1, \mathsf{cof}(\mathbf{SP}) = \mathfrak{c}$.

Proof Let $\{T_f : f \in {}^{\omega}2\}$ be the family given by the theorem above. If $H \subseteq {}^{\omega}2$ is an uncountable set, then $\bigcup\{[T_f] : f \in H\} \notin \mathbf{SP}$. As a consequence, $\mathsf{add}(\mathbf{SP}) = \omega_1$. On the other hand, if $\kappa < \mathfrak{c}$ and if $\{X_\alpha : \alpha < \kappa\} \subseteq \mathbf{SP}$, then there is an $f \in {}^{\omega}2$ such that, for every $\alpha < \kappa$, $[T_f] \nsubseteq X_\alpha$ and therefore $\mathsf{cof}(\mathbf{SP}) = \mathfrak{c}$. □

Observe that this last proof can be used to show that $\mathsf{add}(\mathbf{SP}_n) = \omega_1 = \mathsf{add}(\mathbf{SP})$ and $\mathsf{cof}(\mathbf{SP}_n) = \mathfrak{c} = \mathsf{cof}(\mathbf{SP})$.

3 Uniformity number

In this section we will study some properties concerning the uniformity number of porosity ideals. We will prove the consistency of $\mathsf{non}(\mathbf{SP}) < \mathsf{add}(\mathcal{N})$. We will also develop some tools that we will use later in the paper. We will need the following concept, inspired by the concept of a k-Sacks tree in ${}^{<\omega}k$.

Definition 3.1 *Let $k > 1$. A tree $T \subseteq {}^{<\omega}k$ is a k-*anti-Sacks tree *if for every $s \in T$ there is $i < k$ such that $s^\frown\langle i\rangle \notin T$. We will denote by $\mathbf{AS}_k$ the σ-ideal generated by the branches of k-anti-Sacks trees.*

This notion is the analogue of the notion of 1-porous tree in ${}^{<\omega}k$ and it is closely related to the k-Sacks forcing. Recall that a *k-Sacks tree T* is a tree on ${}^{<\omega}k$ such that for every $s \in T$, there is a $t \in T$ such that, for every $i < k$, $t^\frown i \in T$. The *k-Sacks forcing* $\mathbb{S}_k$ is the collection of all k-Sacks trees ordered by reverse inclusion. It is well-known that the k-Sacks forcing is equivalent to $\mathrm{Borel}({}^{\omega}k)/\mathbf{AS}_k$.

Using a similar argument to the one we gave in the last section, it is possible to show that $\mathsf{add}(\mathbf{AS}_k) = \omega_1$ and that $\mathsf{cof}(\mathbf{AS}_k) = \mathfrak{c}$. Alternatively, a proof of this fact can be found in [9].

The ideals $\mathbf{SP}_k$ and $\mathbf{AS}_{2^k}$ share many properties. Many of the results in this work will concern about properties of the ideals $\mathbf{AS}_k$ that are also valid for the ideal $\mathbf{SP}_k$, and the proofs for both ideals are almost the same. Whenever we state a property about one of these ideals that is also valid for the other one, we will only give the proof for the ideal $\mathbf{AS}_k$.

We shall introduce a notion that we will use to keep the uniformity number small in a forcing extension.

Definition 3.2 *Let $\mathbb{P}$ be a forcing notion and let $A \subseteq {}^{\omega}k$ ($A \subseteq {}^{\omega}2$) be such that $A \notin \mathbf{AS}_k$ ($A \notin \mathbf{SP}_k$). We say that $\mathbb{P}$* strongly preserves $\mathsf{non}(\mathbf{AS}_k)$ in A ($\mathbb{P}$ strongly preserves $\mathsf{non}(\mathbf{SP}_k)$ in A) *if for every $\mathbb{P}$-name $\dot{X}$ of a k-anti-Sacks tree (k-porous tree) there is a $Y \in \mathbf{AS}_k$ ($Y \in \mathbf{SP}_k$) such that, for every $x \in A$, if $x \notin Y$ then $\mathbb{P} \Vdash$ "$x \notin [\dot{X}]$". We will say that $\mathbb{P}$* strongly preserves $\mathsf{non}(\mathbf{AS}_k)$ ($\mathbb{P}$ strongly preserves $\mathsf{non}(\mathbf{SP}_k)$) *if $\mathbb{P}$ strongly preserves $\mathsf{non}(\mathbf{AS}_k)$ in ${}^{\omega}k$ ($\mathbb{P}$ strongly preserves $\mathsf{non}(\mathbf{SP}_k)$ in ${}^{\omega}2$).*

It is easy to see that, if $\mathbb{P}$ strongly preserves $\mathsf{non}(\mathbf{AS}_k)$ in A, then $\mathbb{P} \Vdash$ "$A \notin \mathbf{AS}_k$" and if $\mathbb{P}$ strongly preserves $\mathsf{non}(\mathbf{AS}_k)$, then $\mathbb{P}$ strongly preserves $\mathsf{non}(\mathbf{AS}_k)$ in A for every $A \subseteq {}^{\omega}k$. The following lemma is easy to prove.

Lemma 3.3 *Suppose that a forcing notion $\mathbb{P}$ strongly preserves $\mathsf{non}(\mathbf{SP}_k)$ for every $k \in \omega$, then $\mathbb{P} \Vdash$ "${}^{\omega}2 \cap V \notin \mathbf{SP}$".*

Proof The proof is straightforward from the definitions. $\square$

The next lemma shows that there is a connection between anti-Sacks trees and σ-k-linked forcings.

Lemma 3.4 *Let $\mathbb{P}$ be a forcing notion. If $\mathbb{P}$ is σ-k-linked, then $\mathbb{P}$ strongly preserves* $\mathsf{non}(\mathbf{AS}_k)$.

Proof Let $\{\mathbb{P}_i : i \in \omega\} \subseteq \mathbb{P}$ be a sequence of k-linked subsets such that $\mathbb{P} = \bigcup_{i\in\omega} \mathbb{P}_i$. Let $\dot{A}$ be a $\mathbb{P}$-name of a k-anti-Sacks tree. Define $T_n \subseteq {}^{\omega}k$ as follows:

$$T_n = \{s \in {}^{<\omega}k : \exists p \in \mathbb{P}_n(p \Vdash \text{“}s \in \dot{A}\text{"})\}$$

We claim that, for each $n \in \omega$, T_n is a k-anti-Sacks tree. Suppose this is not the case, so there is an $s \in T_n$ such that, for every $i \in k$, $s^\frown i \in T_n$. For every $i \in k$, we can pick a condition $p_i \in \mathbb{P}_n$ such that $p_i \Vdash \text{“}s^\frown i \in \dot{A}\text{"}$. Let $p \in \mathbb{P}$ be such that, for every $i \in k$, $p \leq p_i$. Then $p \Vdash \text{“}\forall i \in k(s^\frown i \in \dot{A})\text{"}$. This contradicts the fact that $\dot{A}$ is a $\mathbb{P}$-name of a k-anti-Sacks tree.

To conclude the proof, note that for every $x \in {}^{\omega}k$, if $p \Vdash \text{“}x \in [\dot{A}]\text{"}$, then $x \in [T_n]$, where n is such that $p \in \mathbb{P}_n$. □

The lemma above is optimal in the sense that, for each k, there is a σ-$(k-1)$-linked forcing $\mathbb{P}_k$ such that $\mathbb{P}_k \Vdash \text{“}{}^{\omega}k \cap V \in \mathbf{AS}_k\text{"}$ and therefore $\mathbb{P}_k$ does not strongly preserve $\mathbf{AS}_k$. This will be proved in the next section. There is also a relation between porous sets in ${}^{\omega}2$ and σ-k-linked forcings.

Lemma 3.5 *Let $\mathbb{P}$ be a forcing notion. If $\mathbb{P}$ is σ-2^k-linked, then $\mathbb{P}$ strongly preserves* $\mathsf{non}(\mathbf{SP}_k)$.

Proof The proof is similar to the previous lemma. □

We shall show that strong preservation of $\mathsf{non}(\mathbf{AS}_k)$ and $\mathsf{non}(\mathbf{SP}_k)$ is preserved by finite support iterations.

Lemma 3.6 *Let $A \subseteq {}^{\omega}k$, let $\mathcal{I} \in \{\mathbf{SP}_n, \mathbf{AS}_k : n > 0, k > 1\}$,*

(1) *if $\mathbb{P}$ is a forcing notion such that $\mathbb{P}$ strongly preserves $\mathsf{non}(\mathcal{I})$ in A and $\dot{\mathbb{Q}}$ is a $\mathbb{P}$-name for a forcing such that $\mathbb{P} \Vdash$ “$\dot{\mathbb{Q}}$ strongly preserves $\mathsf{non}(\mathcal{I})$ in A", then $\mathbb{P} * \dot{\mathbb{Q}}$ strongly preserves $\mathsf{non}(\mathcal{I})$ in A,*

(2) *if $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha \leq \gamma\rangle$ is a finite support iteration of c.c.c. forcings such that $\mathbb{P}_\alpha \Vdash$ “$\dot{\mathbb{Q}}_\alpha$ strongly preserves $\mathsf{non}(\mathcal{I})$ in A" for each $\alpha \in \gamma$, then $\mathbb{P}_\gamma$ strongly preserves $\mathsf{non}(\mathcal{I})$ in A.*

Proof We will only show the case when $\mathcal{I} = \mathbf{AS}_k$ as the other cases are similar. The part (1) is easy, we will proceed with part (2) by induction on γ. It is easy to see that the lemma holds for successor ordinals, and if γ has uncountable cofinality we can use a standard reflection argument to show that $\mathbb{P}$ strongly preserves $\mathsf{non}(\mathbf{AS}_k)$ in A, so it is enough to show that the lemma holds for $\gamma = \omega$: let $\dot{T}$ be a $\mathbb{P}$-name of a k-anti-Sacks tree. For each $n \in \omega$, let $\dot{T}_n$ be a $\mathbb{P}_n$-name for the following set.

$$\dot{T}_n = \{s \in {}^{<\omega}k : \mathbb{P}_{(n,\omega)} \Vdash \text{“}s \in \dot{T}\text{"}\}.$$

It is easy to see that each $\dot{T}_n$ is name for a k-anti-Sacks tree. Now we use that each $\mathbb{P}_n$ strongly preserves $\mathsf{non}(\mathbf{AS}_k)$ to find a family $\{T_i^j : i,j \in \omega\}$ such that, for each $n \in \omega$, if $x \in A$ and $x \notin \bigcup_{i\in\omega}[T_i^n]$, then $\mathbb{P}_n \Vdash \text{“}x \notin [\dot{T}_n]\text{"}$. It is easy to see that the set $Y = \bigcup\{[T_i^j] : i,j \in \omega\}$ is the set we are looking for. $\square$

We will now prove the consistency of $\mathsf{non}(\mathbf{SP}) < \mathsf{add}(\mathcal{N})$. For constructing the model we are looking for, we will use the amoeba forcing $\mathbb{A}$ in the following presentation:

$$\mathbb{A} = \{B \in \text{Borel}(2^\omega) : \mu(B) > \frac{1}{2}\}.$$

Here Borel(2^ω) represents the collection of Borel subsets of the Cantor space and μ is the standard Lebesgue measure over ${}^\omega 2$. The order is given by $A \leq B$ if and only if $A \subseteq B$. The following lemma is well-known (see e.g. [1]). We include the simple proof for the sake of completeness.

Lemma 3.7 *The amoeba forcing is σ-n-linked for every $n \in \omega$.*

Proof Let $n \in \omega$. For every clopen C in 2^ω, define

$$\mathbb{A}_C = \{A \in \mathbb{A} : \mu(C \setminus A) < \frac{1}{n} \cdot (\mu(C) - \frac{1}{2})\}$$

We will show that $\mathbb{A} = \bigcup\{\mathbb{A}_C : C \text{ is a clopen in } 2^\omega\}$: Let $A \in \mathbb{A}$ and let $\varepsilon > 0$ such that $\mu(A) = \frac{1}{2} + \varepsilon$. Find an open set $U \subseteq 2^\omega$ such that $A \subseteq U$ and $\mu(U \setminus A) < \frac{\varepsilon}{n}$. Now find a clopen set $C \subseteq U$ such that $\mu(C) > \frac{1}{2} + \varepsilon$. Then

$$\mu(C \setminus A) < \mu(U \setminus A) < \frac{\varepsilon}{n} = \frac{1}{n} \cdot (\frac{1}{2} + \varepsilon - \frac{1}{2}) < \frac{1}{n} \cdot (\mu(C) - \frac{1}{2}).$$

Therefore $A \in \mathbb{A}_C$. Now we must show that, for every clopen set $C \subseteq 2^\omega$, the intersection K of an arbitrary family $\{A_j : j \in n\} \subseteq \mathbb{A}_C$ is an element of $\mathbb{A}$. This is a consequence of the following calculations:

$$\mu(C) \leq \mu(K) + \sum_{j\in n} \mu(C \setminus A_j) < \mu(K) + \frac{1}{n} \cdot (\sum_{j\in n} \mu(C) - \frac{1}{2}) = \mu(K) + \mu(C) - \frac{1}{2}.$$

As a consequence, $\frac{1}{2} < \mu(K)$. Therefore $K \in \mathbb{A}$. $\square$

We are ready to prove the main result of this section. The method of the proof was suggested to us by J. Brendle.

Theorem 3.8 *If* ZFC *is consistent, then* $\mathsf{ZFC} + \mathsf{non}(\mathbf{SP}) < \mathsf{add}(\mathcal{N})$ *is consistent.*

Proof Start with a model V such that $V \models CH$. Let $\mathbb{P} = \{\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \omega_2\}$ be a finite support iteration of the amoeba forcing. It follows from the lemmas above that $\mathbb{P}$ strongly preserves $\mathsf{non}(\mathbf{SP}_k)$ for every $k \in \omega$ and therefore $\mathbb{P} \Vdash$ "$2^\omega \cap V \notin \mathbf{SP}$". As a consequence, we have that $V[G] \models \mathsf{non}(\mathbf{SP}) = \omega_1$. It is a known fact (see [1]) that $V[G] \models \mathsf{add}(\mathcal{N}) = \omega_2$, hence $V[G] \models \mathsf{non}(\mathbf{SP}) < \mathsf{add}(\mathcal{N})$. $\square$

4 Martin numbers of σ-k-linked forcings

It is easy to see that $\mathfrak{m}_2 \leq \mathfrak{m}_3 \leq \ldots$ and, for each $k > 1$, it is possible to get the consistency of $\mathfrak{m}_k < \mathfrak{m}_{k+1}$ by forcing with a finite support iteration of σ-$(k+1)$-linked forcings over a model of CH. In [4], the authors constructed a model where all the cardinals of the form $\mathfrak{m}_{2^k}$ are different. In this section we will construct a model where all the Martin numbers $\mathfrak{m}_i$ are different at the same time. In this model, the cardinals $\mathsf{non}(\mathbf{AS}_i)$ will be different all at once (so will be the cardinals $\mathsf{non}(\mathbf{SP}_i)$). We will use the following forcing notions. Given $k > 2$ let

$\mathbb{P}_k = \{\langle s, F\rangle :$ **(a)** s is a finite k-anti-Sacks tree of height $\mathsf{ht}(s)$,
(b) $F \in [{}^\omega k]^{<\omega}$, and $\lceil F \restriction \Delta_F \rceil$ is a finite k-anti-Sacks tree,
(c) $s \subseteq \lceil F \restriction \Delta_F + 1 \rceil\}$,

where $F \restriction k = \{f \restriction k : f \in F\}$, $\lceil F \rceil = \{s \in {}^{<\omega}k : \exists f \in F\ (s \subseteq F)\}$ and $\Delta_F = \min\{n \in \omega : |F \restriction n| = |F|\}$. The order is defined by $\langle s', F'\rangle \leq \langle s, F\rangle$ if and only if $s \subseteq s'$ and $F \subseteq F'$. This forcing notion will be used to work with the ideal $\mathbf{AS}_k$. For the ideal $\mathbf{SP}_k$, we will be using a similar forcing notion. Given $k > 1$:

$\mathbf{P}_k = \{\langle s, F\rangle :$ **(a)** s is a finite k-porous tree of height $\mathsf{ht}(s)$,
(b) $F \in [{}^\omega 2]^{<\omega}$, and $\lceil F \restriction \Delta_F \rceil$ is a finite k-porous tree,
(c) $s \subseteq \lceil F \restriction \Delta_F + 1 \rceil\}$,

The order is defined by $\langle s', F'\rangle \leq \langle s, F\rangle$ if and only if $s \subseteq s'$ and $F \subseteq F'$. We will be using the following proposition.

Proposition 4.1 *Given a* $k > 2$ *and an* $i > 1$, $\mathbb{P}_k \Vdash$ "${}^\omega k \cap V \in \mathbf{AS}_k$" *and* $\mathbf{P}_i \Vdash$ "${}^\omega 2 \cap V \in \mathbf{SP}_i$".

Proof We will only check that $\mathbb{P}_k \Vdash$ "${}^{\omega}k \cap V \in \mathbf{AS}_k$" as the other part is similar: It is easy to see that, for every $f \in {}^{\omega}k$ and $n \in \omega$, the following sets are dense in $\mathbb{P}$:

$$D_f = \{\langle s, F\rangle \in \mathbb{P}_k : \exists \sigma \in {}^{<\omega}k\, (\sigma^\frown f \restriction (\omega \setminus |\sigma|) \in F)\},$$
$$E_n = \{\langle s, F\rangle \in \mathbb{P}_k : \Delta_F > n \wedge s = \lceil F \restriction \Delta_F + 1\rceil\}.$$

If $G \subseteq \mathbb{P}_k$ is a filter meeting all these dense sets, then, using that the sets E_n are dense, it follows that $T = \bigcup\{s : \exists F(\langle s, F\rangle \in G)\}$ is a k-anti-Sacks tree. If $\sigma \in {}^{<\omega}k$ and if $C[\sigma] = \{\sigma^\frown x \restriction (\omega \setminus |\sigma|) : x \in [T]\}$, then, using that the D_f are dense, it follows that ${}^{\omega}k \cap V \subseteq \bigcup\{C[\sigma] : \sigma \in {}^{<\omega}k\} \in \mathbf{AS}_k$. □

The last proposition together with the Lemma 3.4 implies that $\mathbb{P}_k$ is not σ-k-linked. In contrast with this last observation, we have the following proposition.

Proposition 4.2 *For each $k > 1$, $\mathbb{P}_{k+1}$ is σ-k-linked and $\mathbf{P}_k$ is σ-$(2^k - 1)$-linked.*

Proof Again, we will only check that $\mathbb{P}_{k+1}$ is σ-k-linked, the other part is done in a similar way: For any two finite $(k+1)$-anti-Sacks trees s, t of height $\mathsf{ht}(s), \mathsf{ht}(t)$ respectively, define

$$P(s,t) = \{\langle s, F\rangle \in \mathbb{P}_{k+1} : \mathsf{ht}(t) > \Delta_F \wedge F \restriction \mathsf{ht}(t) = t\}.$$

It is easy to see that $\mathbb{P}_{k+1} = \bigcup\{P(s,t) : s, t$ are finite $(k+1)$-anti-Sacks trees $\}$. We will show that every $P(s,t)$ is k-linked: Let $\{\langle s, F_i\rangle : i < k\} \subseteq P(s,t)$ and let $F = \bigcup_{i<k} F_i$. We must show that $\langle s, F\rangle \in \mathbb{P}_{k+1}$. The properties **(a)** and **(c)** are easily verified, so the only thing left to do is to show that $\lceil F \restriction \Delta_F\rceil$ is a $(k+1)$-anti-Sacks tree: Let $s \in \lceil F \restriction \Delta_F\rceil$. If $|s| < \mathsf{ht}(t)$, then, because $F \restriction \mathsf{ht}(t) = t$, it is possible to find an $i \in k$ such that $s^\frown\langle i\rangle \notin \lceil F \restriction \Delta_F + 1\rceil$. If $|s| \geq \mathsf{ht}(t)$, then, for every $i < k$, s only has (at most) one immediate successor in F_i and therefore it is always possible to find a $j \in k$ such that $s^\frown\langle j\rangle \notin \lceil F \restriction \Delta_F + 1\rceil$. □

From these last two propositions we get the following result.

Corollary 4.3 *For each $k > 1$, $\mathfrak{m}_k \leq \mathsf{non}(\mathbf{AS}_{k+1})$ and $\mathfrak{m}_{2^k-1} \leq \mathsf{non}(\mathbf{SP}_k)$.*

Proof This follows easily from the proof of the Proposition 4.1 and the last proposition. □

For the proof of the main theorem we will need the following notion.

Definition 4.4 *Given a regular cardinal κ and $\mathcal{I} \in \{\mathbf{SP}_n, \mathbf{AS}_k : n > 0, k > 1\}$, we will say that a set L is $\langle \kappa, \mathcal{I} \rangle$-*Luzin *if $|L| = \kappa$ and $\mathcal{I} \restriction L = [L]^{<\kappa}$.*

Observe that the existence of a $\langle \kappa, \mathcal{I} \rangle$-Luzin set implies that $\mathsf{non}(\mathcal{I}) \leq \kappa$. Recall that Cohen reals are added at every limit step of countable cofinality of a finite support iteration of arbitrary length. One common application of Cohen reals is that they are used to construct Luzin sets with special properties. The following lemma is one of those applications.

Lemma 4.5 *Let κ be a regular cardinal, let $i > 2$, $k > 1$ and let $\mathbb{L} = \langle \mathbb{L}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha \in \kappa \rangle$ be a finite support iteration of length κ such that $\mathbb{L}_\alpha \Vdash$ "$\dot{\mathbb{Q}}_\alpha = \mathbb{P}_i * \mathbf{P}_k$", then $\mathbb{L} \Vdash$ "There is a $\langle \kappa, \mathbf{AS}_i \rangle$-Luzin set and there is a $\langle \kappa, \mathbf{SP}_k \rangle$-Luzin set".*

Proof Working in $V[G]$, let $L = \{f_\alpha : \alpha \in \kappa \wedge \alpha \text{ has countable cofinality }\}$ be a family of Cohen reals such that each f_α is added at the α-th stage of the iteration. Using the Proposition 4.1, it is easy to show that $V[G] \models [L]^{<\kappa} \subseteq \mathbf{AS}_i \restriction L$. On the other hand, if $T \in V[G]$ is such that $V[G] \models T$ is an i-anti-Sacks tree, then, by a standard reflection argument, there is an intermediate model such that $T \in V[G(\beta)]$. As a consequence, $V[G] \models \forall \gamma > \beta (f_\gamma \notin [T])$. This implies that $V[G] \models \mathbf{AS}_i \restriction L \subseteq [L]^{<\kappa}$. The $\langle \kappa, \mathbf{SP}_k \rangle$-Luzin set is found in a similar way. $\square$

In the lemma above, it is clear that if we replace $\mathbb{L}_\alpha \Vdash$ "$\dot{\mathbb{Q}}_\alpha = \mathbb{P}_i * \mathbf{P}_k$" for $\mathbb{L}_\alpha \Vdash$ "$\dot{\mathbb{Q}}_\alpha = \mathbb{P}_i$", then, in the extension, we still have a $\langle \kappa, \mathbf{AS}_i \rangle$-Luzin set. The following theorem is the main tool we will use to prove the main result of this section.

Theorem 4.6 *If* ZFC *is consistent, then* $\mathsf{ZFC} + \forall i > 2\,(\exists L_i\,(L_i \text{ is } \langle \aleph_i, \mathbf{AS}_i \rangle\text{-Luzin})) + \forall k > 1\,(\exists L'_i\,(L'_i \text{ is } \langle \aleph_{2^k}, \mathbf{SP}_k \rangle\text{-Luzin}))$ *is consistent.*

Proof Let $\mathbb{L} = \langle \mathbb{L}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha \in \omega_\omega \rangle$ be a finite support iteration of length ω_ω such that, for each $i > 1$ and each $\alpha \in [\omega_i, \omega_{i+1})$, $\mathbb{L}_\alpha \Vdash$ "$\dot{\mathbb{Q}}_\alpha = \mathbb{P}_{i+1} * \mathbf{Q}_{i+1}$", where $\mathbf{Q}_{i+1} = \mathbf{P}_{i+1}$ when $i+1$ is a number of the form $2^k + 1$ and $\mathbf{Q}_{i+1} = \{\emptyset\}$ in all the other cases (for $\alpha < \omega_2$, $\mathbb{L}_\alpha \Vdash$ "$\dot{\mathbb{Q}}_\alpha = \{\emptyset\}$"). We will show that the extension is the model we are looking for: As usual in this work, we will only show that there are $\langle \aleph_i, \mathbf{AS}_i \rangle$-Luzin sets for every $i > 2$ (the rest can be done in a similar way): Using the lemma above, for each $i > 2$, in $V[G_{\omega_i}]$ there is a $\langle \aleph_i, \mathbf{AS}_i \rangle$-Luzin set L_i. The only thing left to do is to show that L_i remains $\langle \aleph_i, \mathbf{AS}_i \rangle$-Luzin in $V[G]$. Using that $\mathbb{L}$ is c.c.c. it is easy to see that, in $V[G]$, $[L_i]^{<\omega_i} \subseteq \mathbf{AS}_i \restriction L_i$, so we only need to show that $\mathbf{AS}_i \restriction L_i \subseteq [L_i]^{<\omega_i}$ holds in $V[G]$: First, using Lemma 3.4 and Lemma 3.6, we observe that $\mathbb{L}_{[\omega_i,\omega_\omega]}$ strongly preserves $\mathsf{non}(\mathbf{AS}_i)$ in L_i, so if $\dot{T}$ is a $\mathbb{L}_{[\omega_i,\omega_\omega]}$-name of a i-anti-Sacks tree, then, in $V[G_{\omega_i}]$, there is a $X \in \mathbf{AS}_i \upharpoonright L_i$ such that $\mathbb{L}_{[\omega_i,\omega_\omega]} \Vdash$ "$[\dot{T}] \cap L_i \subseteq X$". Then it follows that $\mathbf{AS}_i \upharpoonright L_i \subseteq [L_i]^{<\omega_i}$ holds in $V[G]$. $\square$

The actual value of $\mathfrak{c}$ in the model above may depend on V. For example, if $V \models \mathsf{GCH}$, then it is easy to see that $V[G] \models \mathfrak{c} = \aleph_{\omega+1}$.

Lemma 4.7 *Let κ be a regular cardinal, let $\mathcal{I} \in \{\mathbf{SP}_n, \mathbf{AS}_k : n > 0, k > 1\}$ and let L be a $\langle \kappa, \mathcal{I} \rangle$-Luzin. If $\mathbb{P}$ is a forcing notion such that $|\mathbb{P}| < \kappa$, then $\mathbb{P}$ strongly preserves $\mathsf{non}(\mathcal{I})$ in L.*

Proof We will do the case when $\mathcal{I} \in \{\mathbf{AS}_i : i > 1\}$, the other cases are similar: Let $\dot{A}$ be a $\mathbb{P}$-name of an i-anti-Sacks tree and let $\mathbb{P} = \{p_\alpha : \alpha \in \mu\}$. For each $\alpha \in \mu$ define $T_\alpha = \{s \in {}^{<\omega}k : p_\alpha \Vdash \text{"}s \in \dot{A}\text{"}\}$. It follows that each T_α defines an i-anti-Sacks tree. If $Y = \bigcup\{[T_\alpha] \cap L : \alpha \in \mu\}$ then $Y \in \mathbf{AS}_k$. If $x \in L$ and $p_\alpha \Vdash$ "$x \in [\dot{A}]$", then $x \in [T_\alpha] \cap L \subseteq Y$. $\square$

We are ready to prove the main result of this section.

Theorem 4.8 *If* ZFC *is consistent, then* $\mathsf{ZFC} + \forall k > 1\ (\mathfrak{m}_k = \mathsf{non}(\mathbf{AS}_{k+1}) = \aleph_{k+1}) + \forall i > 1\ (\mathfrak{m}_{2^i-1} = \mathsf{non}(\mathbf{SP}_i) = \aleph_{2^i+1}) + \mathsf{non}(\mathbf{SP}) = \aleph_{\omega+1}$ *is consistent.*

Proof Start with a model V like the one constructed in Theorem 4.6 and $V \models \mathfrak{c} = \aleph_{\omega+1}$. Using a standard bookkeeping argument, it is possible to construct a finite support iteration $\mathbb{P}$ of length $\omega_{\omega+1}$ of σ-k-linked forcings of size smaller than $\aleph_{k+1}$ (for every $k > 1$), such that any partial order which appears in an intermediate model is listed cofinally along the iteration. Now, using the lemmas 3.4, 3.6 and 4.7, it is possible to show that, for every $k > 2$, $\mathbb{P}$ strongly preserves $\mathsf{non}(\mathbf{AS}_k)$ in L_k. If $G \subseteq \mathbb{P}$ is a generic filter over V, then $V[G] \models \mathsf{non}(\mathbf{AS}_k) \leq \aleph_k$. We note that, as each small σ-k-linked forcing appears in an intermediate model in the iteration, we have $V[G] \models \aleph_{k+1} \leq \mathfrak{m}_k$. As a consequence $V[G] \models \aleph_{k+1} = \mathfrak{m}_k = \mathsf{non}(\mathbf{AS}_{k+1})$. Using a similar argument, it is possible to show that, for each $i > 1$, $V[G] \models \aleph_{2^i+1} = \mathfrak{m}_{2^i-1} = \mathsf{non}(\mathbf{SP}_i)$. To finish the proof, use the fact that $\mathsf{non}(\mathbf{SP})$ does not have countable cofinality and that, for every $n \in \omega$, $\mathsf{non}(\mathbf{SP}_n) \leq \mathsf{non}(\mathbf{SP})$ to show that $V[G] \models \mathsf{non}(\mathbf{SP}) = \mathfrak{c} = \aleph_{\omega+1}$. $\square$

It follows from $\mathbf{SP}_1 \subseteq \mathbf{SP}_2 \subseteq \mathbf{SP}_3 \subseteq \ldots$ that $\omega_1 = \mathsf{non}(\mathbf{SP}_1) \leq \mathsf{non}(\mathbf{SP}_2) \leq \mathsf{non}(\mathbf{SP}_3) \leq \ldots \leq \mathsf{non}(\mathbf{SP})$ and we proved in the theorem above that each inequality can be consistently strict. It is important to remark that none of these numbers is comparable with $\mathfrak{m}_{\sigma\text{-centered}}$. An argument for this can be found in [6].

5 The covering number

It follows from the fact that $\mathbf{AS}_2 \subseteq \mathbf{AS}_3 \subseteq \dots$ that $\mathsf{cov}(\mathbf{SP}) \leq \dots \leq \mathsf{cov}(\mathbf{AS}_3) \leq \mathsf{cov}(\mathbf{AS}_2) = \mathfrak{c}$. We can show that every pair of these numbers can be consistently different.

Proposition 5.1 *Let $k > 1$, if* ZFC *is consistent, then* ZFC$+\mathsf{cov}(\mathbf{AS}_{k+1}) < \mathsf{cov}(\mathbf{AS}_k)$ *is consistent*[1].

Proof Let V be a model such that $V \models \mathsf{cov}(\mathbf{AS}_k) = \mathfrak{c} = \omega_2$. Let $\mathbb{P}$ be a finite support iteration of length ω_1 of the $\mathbb{P}_{k+1}$ forcing defined above and let $G \subseteq \mathbb{P}$ be a generic filter over V. It follows that $\mathbb{P}$ is an iteration of σ-k-linked forcing notions and therefore $\mathbb{P}$ strongly preserves $\mathsf{non}(\mathbf{AS}_k)$. In $V[G]$, consider the family $C = \{V[G_\alpha] \cap {}^\omega(k+1) : \alpha < \omega_1\}$. Using Proposition 4.1, it is easy to show that $V[G] \models C \subseteq \mathbf{AS}_{k+1}$ and $V[G] \models \bigcup C = {}^\omega(k+1)$. As a consequence we have that $V[G] \models \mathsf{cov}(\mathbf{AS}_{k+1}) = \omega_1$. On the other hand, if $\{\dot{T}_\alpha : \alpha \in \omega_1\}$ is a collection of $\mathbb{P}$-names for k-anti-Sacks trees, then we can use the fact that $\mathbb{P}$ strongly preserves $\mathsf{non}(\mathbf{AS}_k)$ to show that there is a collection $\{C_\alpha : \alpha \in \omega_1\} \subseteq \mathbf{AS}_k$ such that if $x \in {}^\omega k$ and $x \notin \bigcup\{C_\alpha : \alpha \in \omega_1\}$, then $\mathbb{P} \Vdash$ "$x \notin \bigcup_{\alpha \in \omega_1}[\dot{T}_\alpha]$". This, together with $V \models \mathsf{cov}(\mathbf{AS}_k) > \omega_1$, implies that $V[G] \models \mathsf{cov}(\mathbf{AS}_{k+1}) < \mathsf{cov}(\mathbf{AS}_k)$. □

An alternative proof of this proposition follows from the results proven in [9]. If $k > 1$, then a tree $T \subseteq {}^{<\omega}\omega$ is a *k-tree* if every $s \in T$ has at most k immediate successors. A forcing notion $\mathbb{P}$ has the *k-localization property* if $\mathbb{P} \Vdash$ "$\forall f \in {}^\omega\omega\ (\exists T \in V\ (T \textit{ is a } k\textit{-tree and } f \in [T]))$". It is easy to see that if $\mathbb{P}$ has the k-localization property, then $\mathbb{P} \Vdash$ "$\bigcup(\mathbf{AS}_{k+1} \cap V) = {}^\omega k + 1$". Let $\mathbb{S}_k = \{T \subseteq {}^{<\omega}k : \forall s \in T(\exists t \in T(\forall i \in k(s \sqsubseteq t \wedge t^\frown i \in T)))\}$ be the k-Sacks forcing ordered by inclusion. It turns out that $\mathbb{S}_k$ is forcing equivalent to $\mathrm{Borel}({}^\omega k)/\mathbf{AS}_k$ and that if $\mathbb{P}$ is the countable support iteration or the countable support product of length ω_2 of the forcing $\mathbb{S}_k$, then $\mathbb{P}$ has the k-localization property (see [9]). As a consequence, in the extension $\mathsf{cov}(\mathbf{AS}_{k+1}) = \omega_1$ and $\mathsf{cov}(\mathbf{AS}_k) = \omega_2$.

Obviously it is impossible to separate infinitely many of the $\mathsf{cov}(\mathbf{SP}_n)$ at the same time. This suggests the following:

Question 5.2 *How many of the $\mathsf{cov}(\mathbf{SP}_n)$ can be separated at the same time?*

[1]A similar theorem for the ideals $\mathbf{SP}_k$ can be proved using the same argument.

We do not even know how to separate three of them. Another question we have is the following:

Question 5.3 *Is it possible to get the consistency of* $\mathsf{ZFC} + \forall k \in \omega(\mathsf{cov}(\mathbf{SP}) < \mathsf{cov}(\mathbf{SP}_k))$ *?*

We are also interested in the relationship between $\mathsf{non}(\mathbf{SP})$ and $\mathsf{cov}(\mathbf{SP})$. It follows from the fact that the Cohen forcing is σ-centered that, in the Cohen model, $\mathsf{non}(\mathbf{SP}) < \mathsf{cov}(\mathbf{SP})$. However, we do not know if it is possible to construct a model where $\mathsf{non}(\mathbf{SP}) > \mathsf{cov}(\mathbf{SP})$.

Question 5.4 *Is* $\mathsf{non}(\mathbf{SP}) \leq \mathsf{cov}(\mathbf{SP})$ *?*

A related question, as to whether $\mathsf{non}(\mathbf{AS}_n) \leq \mathsf{cov}(\mathbf{AS}_n)$ was asked in [9]. Finally, we would like to discuss about the relation of the cardinal numbers of the ideals $\mathbf{SP}_k$ and $\mathbf{AS}_{2^k}$. In this work we showed that these ideals share a lot of properties, however we do not know if they share the same cardinal invariants. There is a connection between 2^k-anti-Sacks trees and k-porous sets given by the following argument: Let $\varphi_k : 2^k \to {}^k2$ be a bijective function. Let $\psi_k : {}^\omega(2^k) \to {}^\omega 2$ defined by $\psi_k(x) = \varphi_k(x(0))^\frown\varphi_k(x(1))^\frown \ldots$ Clearly, if $\psi_n(A) \in \mathbf{SP}_n$, then $A \in \mathbf{AS}_{2^n}$. We do not know if this can be used to show a relation between the cardinal invariants of the ideals $\mathbf{SP}_k$ and $\mathbf{AS}_{2^k}$.

Question 5.5 *Is* $\mathsf{non}(\mathbf{SP}_k) = \mathsf{non}(\mathbf{AS}_{2^k})$ *? Is* $\mathsf{cov}(\mathbf{SP}_k) = \mathsf{cov}(\mathbf{AS}_{2^k})$ *?*

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Canjar Filters

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Abstract If $\mathscr{F}$ is a filter on ω, we say that $\mathscr{F}$ is *Canjar* if the corresponding Mathias forcing does not add a dominating real. We prove that any Borel Canjar filter is F_σ, this solves a problem of Hrušák and Minami. We give several examples of Canjar and non-Canjar filters, in particular, we construct a MAD family such that the corresponding Mathias forcing adds a dominating real. This answers a question of Brendle. Then we prove that in all the "classical" models of ZFC there are MAD families whose Mathias forcing does not add a dominating real. We also study ideals generated by branches, and we uncover a close relation between Canjar ideals and the selection principle $S_{fin}(\Omega,\Omega)$ on subsets of the Cantor space.

1 Introduction

Given a filter $\mathscr{F}$ and a forcing notion $\mathbb{P}$, we say that $\mathbb{P}$ *diagonalizes* $\mathscr{F}$ if it adds a pseudointersection to $\mathscr{F}$. There are two classical partial orders for diagonalizing a filter $\mathscr{F}$, the *Laver forcing* relative to $\mathscr{F}$, denoted by $\mathbb{L}(\mathscr{F})$, which consists of all trees of height ω that have a stem and above it the set of successors of every node is a member of $\mathscr{F}$, and there is also the *Mathias forcing* relative to $\mathscr{F}$, which is defined as $\mathbb{M}(\mathscr{F}) = \left\{(s,A) \mid s \in [\omega]^{<\omega} \wedge A \in \mathscr{F}\right\}$, the order is given by $(s,A) \leq (z,B)$ whenever z is an initial segment of s, $s-z \subseteq B$ and $A \subseteq B$. These partial orders have many properties in common, but in general they are distinct forcing notions; for example, it is easy to see that $\mathbb{L}(\mathscr{F})$ always adds a dominating real, while this is not necessarily the case for $\mathbb{M}(\mathscr{F})$. It is folklore knowledge that if $\mathscr{U}$ is a Ramsey ultrafilter, then $\mathbb{M}(\mathscr{U})$ is equivalent to $\mathbb{L}(\mathscr{U})$, hence adds a dominating real (this has been implicitly proved in [13]). On the other hand, under $\mathfrak{d} = \mathfrak{c}$, Canjar constructed an ultrafilter whose Mathias forcing does not add a dominating real (see [5]). We call such type of filters *Canjar filters*. We say that an ideal $\mathscr{I}$ is a *Canjar ideal* if its dual filter $\mathscr{I}^* = \{\omega - X \mid X \in \mathscr{I}\}$ is a Canjar filter. Canjar filters have been investigated in [7] and [3], this paper is a continuation of that line of research.

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In [7] Hrušák and Minami found a combinatorial reformulation of being Canjar. If W is a countable set, we denote by $fin(W)$ as the set of all non empty finite subsets of W. If $\mathscr{I}$ is an ideal on W, we define the ideal $\mathscr{I}^{<\omega}$ as the set of all $A \subseteq fin(W)$ such that there is $Y \in \mathscr{I}$ with the property that $a \cap Y \neq \emptyset$ for all $a \in A$. We will write fin instead of $fin(W)$ when is clear of context. Recall that $\mathscr{I}$ is a P^+*-ideal* if every decreasing sequence of positive sets has a positive pseudointersection. The characterization of Hrušák and Minami is the following.

Proposition 1 ([7]) *$\mathscr{I}$ is a Canjar ideal if and only if $\mathscr{I}^{<\omega}$ is a P^+-ideal.*

In [4] Brendle showed that every F_σ ideal is a Canjar ideal. It was asked by Hrušák and Minami if every Borel Canjar ideal must be F_σ and one of the main results of this article is to answer this question positively. In order to achieve this, we will extend a characterization of Canjar ultrafilters by Blass, Hrušák and Verner in [3].

We say that a MAD family is *Canjar* if the ideal generated by it is Canjar. In [4] Brendle showed that under $\mathfrak{b} = \mathfrak{c}$ there is a non Canjar MAD family, and asked if it is possible to construct one in ZFC. We show that this is indeed the case. We then turn our attention to constructing a Canjar MAD family, and we show that in many of the "classical" models of ZFC there is one. We do not know if this is true in general.

We also study ideals generated by branches, and we show that there is a connection between Canjar ideals and selection principles on the Cantor space.[1]

Using the previous ideas, in [**?**] we gave alternative proofs of the consistency of $\mathfrak{b} < \mathfrak{a}$ and $\mathfrak{b} < \mathfrak{s}$ (which were proved by Shelah [19]).

Our notation is standard and follows mostly [1], by $\mathscr{I}^+$ we will denote the set of subsets of ω that are not in $\mathscr{I}$ and are called the *positive sets with respect to $\mathscr{I}$* or $\mathscr{I}$-positive sets. Whenever a, b are two sets, $a - b$ will denote the set theoretic difference of a and b. The definition of the basic cardinal invariants such as $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{d}$, $\mathfrak{r}$, non$(\mathscr{M})$ and cov$(\mathscr{M})$ may be consulted in [2].

2 Canjar Ideals

Given $A \subseteq fin$, we denote by $\mathscr{C}(A)$ as the set of all $X \subseteq \omega$ such that $a \cap X \neq \emptyset$ for all $a \in A$. We may identify $\wp(\omega)$ [2] with 2^ω, which is homeomorphic to the Cantor set endowed with the product topology. In this way, we can talk about topological properties (like compact, F_σ or Borel) of families of subsets of ω. The next lemma is easy and its proof is left to the reader.

Lemma 1
1. *If $A \subseteq fin$, then $\mathscr{C}(A)$ is compact, and if $A \in (\mathscr{I}^{<\omega})^+$ then $\mathscr{C}(A) \subseteq \mathscr{I}^+$.*
2. *If $\mathscr{C} \subseteq \wp(\omega)$ is compact and $X \subseteq \omega$ intersects every element of $\mathscr{C}$, then there is $F \in [X]^{<\omega}$ such that F intersects every element of $\mathscr{C}$.*
3. *If $\mathscr{C}_1, \ldots, \mathscr{C}_n$ are compact, then $\mathscr{D} = \{A_1 \cap \ldots \cap A_n \mid A_i \in \mathscr{C}_i\}$ is also compact.*

A slightly less trivial lemma is the following.

Lemma 2 *Let $\mathscr{F}$ be a filter , $X \subseteq fin$ be such that $\mathscr{C}(X) \subseteq \mathscr{F}$ and $\mathscr{D}$ compact with $\mathscr{D} \subseteq \mathscr{F}$. Then, for every $n \in \omega$ there is $S \in [X]^{<\omega}$ such that if $A_0, \ldots, A_n \in \mathscr{C}(S)$ and $F \in \mathscr{D}$ then $A_0 \cap \ldots \cap A_n \cap F \neq \emptyset$.*

Proof Given $s \in X$ define $K(s)$ as the set of all $(A_0, \ldots, A_n) \in \mathscr{C}(s)^{n+1}$ with the property that there is $F \in \mathscr{D}$ such that $A_0 \cap \ldots \cap A_n \cap F = \emptyset$, this is a compact set by the previous lemma. Note that if $(A_0,\ldots,A_n)\in\bigcap\limits_{s\in X}K(s)$ then $A_0,\ldots,A_n\in\mathscr{C}(X)\subseteq\mathscr{F}$ and there would be $F\in\mathscr{D}\subseteq\mathscr{F}$ such that $A_0\cap\ldots\cap A_n\cap F=\emptyset$ which is clearly a contradiction. Since the $K(s)$ are compact, there must be $S\in[F]^{<\omega}$ such that $\bigcap\limits_{s\in S}K(s)=\emptyset$. It is easy to see that this is the S we are looking for. □

Now we prove the theorem of Canjar using the characterization of Hrušák and Minami. This is an elaboration of the proof that there is a P-point under $\mathfrak{d}=\mathfrak{c}$ (see [2]).

Proposition 2 ([5]) *If $\mathfrak{d}=\mathfrak{c}$ then there is a Canjar ultrafilter.*

Proof Let $\langle\overline{X}_\alpha\mid\alpha\in\mathfrak{c}\rangle$ be an enumeration of all decreasing sequences of subsets of $[\omega]^{<\omega}$. Recursively, we will construct a continuous increasing sequence of filters $\langle\mathscr{U}_\alpha\mid\alpha\in\mathfrak{c}\rangle$ such that for all $\alpha<\mathfrak{c}$,

1. $\mathscr{U}_\alpha$ is the union of less than $\mathfrak{d}$ compact sets,
2. either $\overline{X}_\alpha$ is not a sequence of $\mathscr{U}^{<\omega}$ positive sets or it has a pseudointersection P such that $\mathscr{C}(P)\subseteq\mathscr{U}_{\alpha+1}$.

We begin by setting $\mathscr{U}_0$ to be the cofinite subsets of ω and we take the union at limit stages. Assume that we have already defined $\mathscr{U}_\alpha$, we will see how to define $\mathscr{U}_{\alpha+1}$. In case $\overline{X}_\alpha=\langle X_n\mid n\in\omega\rangle$ is not a sequence of $\mathscr{U}^{<\omega}$ positive sets we just do $\mathscr{U}_{\alpha+1}=\mathscr{U}_\alpha$. Now assume that each $X_n\in\mathscr{U}^+$, which implies that $\mathscr{C}(X_n)\subseteq\mathscr{U}^+$, we will find a compact set $\mathscr{D}$ such that $\mathscr{U}_\alpha\cup\mathscr{D}$ generates a filter, and this will be $\mathscr{U}_{\alpha+1}$, by point 3 of lemma 1, $\mathscr{U}_{\alpha+1}$ will be generated by less than $\mathfrak{c}$ compact sets.

In case there is $n\in\omega$ such that $\mathscr{C}(X_n)$ is not contained in $\mathscr{U}_\alpha$, we choose $Y\in\mathscr{C}(X_n)-\mathscr{U}_\alpha$ and define $\mathscr{D}=\{\omega-Y\}$. In this way, $\overline{X}_\alpha$ is no longer a sequence of positive sets. So assume $\mathscr{C}(X_n)\subseteq\mathscr{U}_\alpha$ for each $n\in\omega$. Let $\mathscr{U}_\alpha=\bigcup\limits_{\beta\in\kappa}\mathscr{C}_\beta$ where $\mathscr{C}_\beta$ is compact and κ is less than $\mathfrak{d}$. By the previous lemma, for every $\beta<\kappa$ we can define a function $f_\beta:\omega\longrightarrow\omega$ such that for every $n\in\omega$ there is $S\in[X_n]^{<\omega}$ with $S\subseteq\wp\big(f_\beta(n)\big)$ such that if $A_0,\ldots,A_{n+1}\in\mathscr{C}(S)$ and $F\in\mathscr{C}_\beta$ then $A_0\cap\ldots\cap A_{n+1}\cap F\neq\emptyset$. Since $\{f_\beta\mid\beta<\kappa\}$ is not dominating, there is g that is not dominated by any of the f_β. Let $P=\bigcup\limits_{n\in\omega}\wp(g(n))\cap X_n$. It is clear that P is a pseudointersection. Now we claim that $\mathscr{U}_\alpha\cup\mathscr{C}(P)$ generates a filter. For this, let $F\in\mathscr{U}_\alpha$ and $B_0,\ldots,B_n\in\mathscr{C}(P)$. We must show $B_0\cap\ldots\cap B_n\cap F\neq\emptyset$. Pick $\beta<\kappa$ such that $F\in\mathscr{C}_\beta$, and since $g\nleq^* f_\beta$, there is $m>n$ such that $g(m)>f_\beta(m)$. By the construction, then there is $S\in[X_m]^{<\omega}$ with $S\subseteq\wp\big(f_\beta(m)\big)\subseteq\wp(g(m))$ such that if $A_0,\ldots,A_{n+1}\in\mathscr{C}(S)$ then $A_0\cap\ldots\cap A_{n+1}\cap F\neq\emptyset$, but clearly $B_0,\ldots,B_n\in\mathscr{C}(S)$ so we are done.

Finally, let $\mathscr{U}=\bigcup\limits_{\alpha<\mathfrak{c}}\mathscr{U}_\alpha$. Then, by the construction, $\mathscr{U}$ is a Canjar ultrafilter. □

In [11] Laflamme introduced the following notion for ultrafilters.

Definition 1 *We say that $\mathscr{I}$ is a* strong P^+-ideal *if for every increasing sequence $\langle\mathscr{C}_n\mid n\in\omega\rangle$ of compact sets with $\mathscr{C}_n\subseteq\mathscr{I}^+$, there is an interval partition $\mathscr{P}=\langle P_n\mid n\in\omega\rangle$ such that if $\langle X_n\mid n\in\omega\rangle$ is a sequence with $X_n\in\mathscr{C}_n$ for all $n\in\omega$ then $\bigcup\limits_{n\in\omega}(X_n\cap P_n)\in\mathscr{I}^+$.*

Laflamme noted without a proof that Canjar ultrafilters were strong P^+-filters and asked if these two notions were equivalent. This was answered positively by Blass, Hrušák and Verner in [3]. We will now extend their result to the general case.

Definition 2 *We say that $\mathscr{I}$ is a* coherent strong P^+-ideal *if for every increasing sequence $\langle \mathscr{C}_n \mid n \in \omega \rangle$ of compact sets with $\mathscr{C}_n \subseteq \mathscr{I}^+$, there is an interval partition $\mathscr{P} = \langle P_n \mid n \in \omega \rangle$ such that if $\langle X_n \mid n \in \omega \rangle$ is a sequence with the following "coherence property" for $\mathscr{P}$,*

1. *$X_n \in \mathscr{C}_n$ for all $n \in \omega$,*
2. *if $n < m$ then $X_m \cap P_n \subseteq X_n \cap P_n$.*

then $\bigcup_{n\in\omega} X_n \cap P_n \in \mathscr{I}^+$.

Note that the coherence property is satisfied if the $\langle X_n \mid n \in \omega \rangle$ is decreasing, as well as, when $\mathscr{I}$ is the dual of an ultrafilter. We will now prove that an ideal is Canjar if and only if it satisfies the coherent strong P^+-ideal property.

Proposition 3 ([3] for ultrafilters) *An ideal $\mathscr{I}$ is Canjar if and only if $\mathscr{I}$ is a coherent strong P^+-ideal.*

Proof First assume that $\mathscr{I}$ is a Canjar ideal. Let $\langle \mathscr{C}_n \mid n \in \omega \rangle$ be an increasing sequence of compact sets with $\mathscr{C}_n \subseteq \mathscr{I}^+$. For every $n \in \omega$ define A_n as the set of all $a \in [\omega]^{<\omega}$ such that if $X \in \mathscr{C}_n$ then $a \cap X \neq \emptyset$. We will see that $A_n \in (\mathscr{I}^{<\omega})^+$. Let $B \in \mathscr{I}$. We must find an element of A_n that is disjoint from B. For every $y \notin B$ define $V_y = \{X \in \mathscr{C}_n \mid y \in X\}$. Since $\mathscr{C}_n \subseteq \mathscr{I}^+$, we conclude that $\langle V_y \mid y \notin B \rangle$ is an open cover of $\mathscr{C}_n$ so there is a finite $a \subseteq \omega - B$ such that $\mathscr{C}_n = \bigcup_{y\in a} V_y$. Therefore $a \in A_n$ and $a \cap B = \emptyset$.

In this way $\langle A_n \mid n \in \omega \rangle$ is a decreasing sequence of positive sets and since $\mathscr{I}$ is Canjar, there is $A \subseteq^* A_n$ with $A \in (\mathscr{I}^{<\omega})^+$. We may as well assume that $A \subseteq A_0$. Define an interval partition $\mathscr{P} = \langle P_n \mid n \in \omega \rangle$ in such a way that for all $n \in \omega$ if $a \in A - A_n$ then $a \subseteq \bigcup_{i<n} P_i$. We will see that this is the partition we are looking for. Let $\langle X_n \mid n \in \omega \rangle$ be a sequence with the coherence property for $\mathscr{P}$. We will show that $X = \bigcup_{n\in\omega} X_n \cap P_n \in \mathscr{I}^+$. It is enough to show that X intersects every element of A (because if $X \in \mathscr{I}$ then A will be in $\mathscr{I}^{<\omega}$ which is a contradiction). Let $a \in A$ and define $n = \max\{m \mid a \cap \bigcup_{i\leq m} P_i \neq \emptyset\}$. Since $a \nsubseteq \bigcup_{i<n} P_i$, a must be in A_n, hence $a \cap X_n \neq \emptyset$. By the coherence property, we know that $\bigcup_{i\leq n} X_n \cap P_i \subseteq \bigcup_{i\leq n} X_i \cap P_i \subseteq X$ so $a \cap X \neq \emptyset$.

Now assume that $\mathscr{I}$ is a coherent strong P^+-ideal. We shall show that $\mathscr{I}^{<\omega}$ is a P^+-ideal. Let $\langle A_n \mid n \in \omega \rangle \subseteq (\mathscr{I}^{<\omega})^+$ be a decreasing sequence. We must find a positive pseudointersection. For every $n \in \omega$ define $\mathscr{C}_n = \{X \subseteq \omega \mid \forall a \in A_n (a \cap X \neq \emptyset)\}$. Since $\mathscr{C}_n$ is an intersection of compact sets, it is compact and it is easy to see that $\mathscr{C}_n \subseteq \mathscr{I}^+$. Let $\mathscr{P} = \langle P_n \mid n \in \omega \rangle$ be an interval partition witnessing that $\mathscr{I}$ is a coherent strong P^+-ideal. Call $E_n = \bigcup_{i\leq n} P_i$ and define $A = \bigcup_{n\in\omega} (A_n \cap \wp(E_n))$. Clearly $A \subseteq^* A_n$ for every $n \in \omega$ so it remains to show that A is positive. Assume this is not the case, so there is $B \in \mathscr{I}$ that intersects every element of A. Define $X_n = (B \cap E_n) \cup (\omega - E_n)$ and note that $X_n \in \mathscr{C}_n$ and $\langle X_n \mid n \in \omega \rangle$ satisfies the coherence property for $\mathscr{P}$. In this way $B = \bigcup_{n\in\omega} (X_n \cap P_n) \in \mathscr{I}^+$ which is a contradiction. $\square$

As an application, we will show that all F_σ ideals are Canjar.

Proposition 4 ([4]) *Every F_σ ideal is a Canjar ideal.*

Proof Let $\mathscr{I}$ be an F_σ ideal. We will show that it is a coherent strong P^+-ideal. By a theorem of Mazur (see [15]) there is a lower semicontinuous submeasure $\varphi : \wp(\omega) \longrightarrow [0,\infty]$ [3] such that $\mathscr{I} = \{A \mid \varphi(A) < \omega\}$.

Let $\langle \mathscr{C}_n \mid n \in \omega \rangle$ be an increasing sequence of compact positive sets. Since each $\mathscr{C}_n$ is compact, it is easy to recursively construct an interval partition $\langle P_n \mid n \in \omega \rangle$ such that $\varphi(P_n \cap Y) > n$ for each $Y \in \mathscr{C}_n$. In this way, it is clear that $\bigcup_{n\in\omega} X_n \cap P_n \in \mathscr{I}^+$, whenever $X_n \in \mathscr{C}_n$. □

Actually, in [4] Brendle showed that if $\mathscr{I}$ is the union of less than $\mathfrak{d}$ compact sets, then $\mathscr{I}$ is Canjar. In [7] it was asked if every Borel Canjar ideal is F_σ, in the next section we will prove that this is indeed the case.

3 Borel Canjar Ideals

Recall another notion introduced by Laflamme and Leary in [12]. We say that a tree $T \subseteq ([\omega]^{<\omega})^{<\omega}$ is an *$\mathscr{I}^+$-tree of finite sets* if for every $t \in T$, there is $X_t \in \mathscr{I}^+$ such that $suc_T(t) = [X_t]^{<\omega}$.

Definition 3 *We say that $\mathscr{I}$ is a* P^+(tree)-ideal *if for every $\mathscr{I}^+$-tree of finite sets T, there is $b \in [T]$ such that $\bigcup_{n\in\omega} b(n) \in \mathscr{I}^+$.*

We will show that Canjar ideals are $P^+(tree)$,

Proposition 5 *If $\mathscr{I}$ is Canjar, then $\mathscr{I}$ is $P^+(tree)$.*

Proof Let $T \subseteq ([\omega]^{<\omega})^{<\omega}$ be an $\mathscr{I}^+$-tree of finite sets. For convenience, denote by $\omega^{\nearrow\omega}$ the set of all increasing finite sequences of natural numbers. We define a subtree $T' = \{t_s \mid s \in \omega^{\nearrow\omega}\} \subseteq T$ in the following way,

1. $t_\emptyset = \emptyset$,
2. $t_{\langle n \rangle} = X_\emptyset \cap [0,n)$ for every $n \in \omega$,
3. $t_{\langle n_0,\dots,n_{m+1}\rangle} = X_{t_{\langle n_0 \dots n_m\rangle}} \cap [n_m, n_{m+1})$.

Let $Y_\emptyset = X_\emptyset$. If $s^\frown\langle n \rangle \in \omega^{\nearrow\omega}$ define $Y_{s^\frown\langle n\rangle} = (Y_s \cap n) \cup (X_{s^\frown\langle n\rangle} - n)$. Call $\mathscr{C}_n = \{Y_s \mid s \in \omega^{\nearrow\omega} \wedge |s| \leq n\}$. It is easy to see that $\langle \mathscr{C}_n \mid n \in \omega\rangle$ is an increasing sequence of compact positive sets (for example, one may note that if $Y \in \overline{\mathscr{C}_{n+1}}$ then either $Y \in \{Y_s \mid |s| = n+1\}$ or it is in the closure of $\mathscr{C}_n$). Find $\mathscr{P} = \langle P_n \mid n \in \omega\rangle$ an interval partition that witnesses that $\mathscr{I}$ is Canjar. Define the function $l : \omega \longrightarrow \omega$ where $l(n)$ is the right end-point of P_n and consider the branch $b = \langle t_{l\restriction n}\rangle$. We will see that $\bigcup_{n\in\omega} t_{l\restriction n} \in \mathscr{I}^+$. Note that $Y_{l\restriction n} \in \mathscr{C}_n$ and $\langle Y_{l\restriction n} \mid n \in \omega\rangle$ satisfie the coherence property for $\mathscr{P}$ so $\bigcup_{n\in\omega} Y_{l\restriction n} \cap P_n \in \mathscr{I}^+$ but $\bigcup_{n\in\omega} Y_{l\restriction n} \cap P_n = \bigcup_{n\in\omega} t_{l\restriction n}$ which is what we were looking for. □

However, being Canjar is a stronger notion than being $P^+(tree)$, we will later see an example of an ideal that is $P^+(tree)$ but not Canjar.

Theorem 1 *If $\mathscr{I}$ is a Borel ideal, then the following are equivalent,*

1. *$\mathscr{I}$ is Canjar,*

2. *$\mathscr{I}$ is F_σ,*
3. *$\mathscr{I}$ is $P^+(tree)$.*

Proof The equivalence between 2 and 3 was proved by Hrušák and Meza in [8] and the other equivalence follows from the previous results. □

In [5] Canjar proved that if a forcing notion adds a dominating real, then it must have size at least $\mathfrak{d}$. It follows that every ideal generated by less than $\mathfrak{d}$ sets is Canjar, since its Mathias forcing has a dense set of size less than $\mathfrak{d}$. With this observation and the previous theorem, we can conclude the following result of Veličković and Louveau,

Corollary 1 (Veličković, Louveau see [14]) *If $\mathscr{I}$ is a Borel non F_σ-ideal then $cof(\mathscr{I}) \geq \mathfrak{d}$.*

Note that there are Borel (non F_σ) ideals of cofinality $\mathfrak{d}$, one example is $FIN \times FIN$ which is the ideal in $\omega \times \omega$ generated by all columns $C_n = \{(n,m) \mid m \in \omega\}$ and all $A \subseteq \omega \times \omega$ such that A intersects every C_n in a finite set.

4 Canjar MAD Families

Given an almost disjoint family $\mathscr{A}$, we denote by $\mathscr{I}(\mathscr{A})$ the ideal generated by $\mathscr{A}$. We say $\mathscr{A}$ is *Canjar* if $\mathscr{I}(\mathscr{A})$ is Canjar. In [4] Brendle constructed a non Canjar MAD family under $\mathfrak{b} = \mathfrak{c}$ and asked if it is possible to construct one without additional axioms. We now answer his question in the affirmative.

Proposition 6 *There is a non Canjar MAD family.*

Proof Let $\mathscr{P} = \{A_n \mid n \in \omega\}$ be a partition of ω. For every $n \in \omega$ choose $\mathscr{B}_n$ an almost disjoint family of subsets of A_n. Construct a tree $T \subseteq ([\omega]^{<\omega})^{<\omega}$ such that for every $t \in T$ there is $n_t \in \omega$ with the property that $suc(t) = [A_{n_t}]^{<\omega}$ and make sure that if $t \neq s$ then $n_t \neq n_s$, and for every m there is a t such that $n_t = m$. For every branch $b \in [T]$ let $A_b = \bigcup\limits_{n\in\omega} b(n)$ and note that $\mathscr{A} = \{A_b \mid b \in [T]\} \cup \bigcup\{\mathscr{B}_n \mid n \in \omega\}$ is an almost disjoint family and $\mathscr{P} \subseteq \mathscr{I}(\mathscr{A})^{++}$. Let $\mathscr{A}'$ be any MAD family extending $\mathscr{A}$. Note that $\mathscr{P} \subseteq \mathscr{I}(\mathscr{A}')^+$ so T is an $\mathscr{I}(\mathscr{A}')^+$-tree of finite sets but it has no positive branch. □

Interestingly, we do not know if there is a Canjar MAD family in ZFC. Obviously they exist under $\mathfrak{a} < \mathfrak{d}$. We will now give some sufficient conditions for the existence of a Canjar MAD family. Usually, we will construct a MAD family $\mathscr{A} = \{A_\alpha \mid \alpha \in \kappa\}$ recursively and in such case we will denote by $\mathscr{A}_\alpha = \{A_\xi \mid \xi < \alpha\}$. Call *Part* the set of all interval partitions (partitions in finite sets) of ω. We may define an order on *Part* as follows: given $\mathscr{P}, \mathscr{Q} \in Part$ we say that $\mathscr{P} \leq^* \mathscr{Q}$ if for almost all $Q \in \mathscr{Q}$ there is $P \in \mathscr{P}$ such that $P \subseteq Q$. In [2] it is proved that the smallest size of a dominating family of interval partitions is $\mathfrak{d}$.

First we will give a combinatorial reformulation of $\min\{\mathfrak{d},\mathfrak{r}\}$.

Proposition 7 *If κ is an infinite cardinal, then $\kappa < min\{\mathfrak{d},\mathfrak{r}\}$ if and only if for every $\langle \mathscr{P}_\alpha \mid \alpha \in \kappa \rangle$ family of interval partitions of ω, there is an interval partition $\mathscr{Q} = \{Q_n \mid n \in \omega\}$ with the property that there are disjoint $A, B \in [\omega]^\omega$ such that for all $\alpha < \kappa$, both $\bigcup\limits_{n\in A} Q_n$ and $\bigcup\limits_{n\in B} Q_n$ contain infinitely many intervals of $\mathscr{P}_\alpha$.*

Proof Let $\kappa < \min\{\mathfrak{d}, \mathfrak{r}\}$ and $\langle \mathscr{P}_\alpha \mid \alpha \in \kappa \rangle$ be a family of interval partitions. We may assume that for every $\mathscr{P}_\alpha$ and $n \in \omega$ there is a $\mathscr{P}_\beta$ such that every interval of $\mathscr{P}_\beta$ contains n intervals of $\mathscr{P}_\alpha$. Define $f_\alpha : \omega \longrightarrow \omega$ such that $f_\alpha(n)$ is the left point of $\mathscr{P}_\alpha$ (so $f_\alpha(0) = 0$). Since $\kappa < \mathfrak{d}$, there is $g : \omega \longrightarrow \omega$ such that g is not dominated by any f_α, we may as well assume that g is increasing and $g(0) = 0$. Define the interval partition $\mathscr{Q} = \{Q_n \mid n \in \omega\}$ where $Q_n = [g(n), g(n+1))$. Let M_α be the set of all $n \in \omega$ such that Q_n contains an interval of $\mathscr{P}_\alpha$.

Claim 1 *M_α is infinite for every $\alpha < \kappa$.*

By the assumption on our family, it is enough to show that each M_α is not empty. Since $g \nleq^* f_\alpha$, there is $n \in \omega$ such that $f_\alpha(n) < g(n)$. But then it follows that some interval of $\mathscr{P}_\alpha$ must be contained in one Q_m with $m < n$.

Since $\kappa < \mathfrak{r}$, we know that $\{M_\alpha \mid \alpha < \kappa\}$ is not a reaping family, so there are disjoint $A, B \in [\omega]^\omega$ such that $\omega = A \cup B$ and for every α, both $M_\alpha \cap A$ and $M_\beta \cap B$ are infinite. It is clear that A and B are the sets we were looking for.

Now we must show that the conclusion of the proposition fails for $\kappa = \mathfrak{d}$ and $\kappa = \mathfrak{r}$. Let $\mathscr{R} = \{M_\alpha \mid \alpha \in \mathfrak{r}\}$ be a reaping family. Define $\mathscr{P}_\alpha$ such that every interval of $\mathscr{P}_\alpha$ contains one point of M_α. Assume there is an interval partition $\mathscr{Q} = \{Q_n \mid n \in \omega\}$ and $A, B \in [\omega]^\omega$ as in the proposition. Let $X = \bigcup_{n \in A} Q_n$. Then no M_α reaps X, which is a contradiction since $\mathscr{R}$ was a reaping family.

Finally, let $\langle \mathscr{P}_\alpha \mid \alpha \in \mathfrak{d} \rangle$ be a dominating family of partitions and let $\mathscr{Q}$ be any other partition. Then there is a P_α such that every interval of P_α contains two intervals of $\mathscr{Q}$, so obviously there can not be any A and B as required. $\square$

Using the proposition, we may prove the following result.

Proposition 8 *If $\mathfrak{d} = \mathfrak{r} = \mathfrak{c}$ then there is a Canjar MAD family of size continuum (In particular, there is one if $\mathfrak{b} = \mathfrak{c}$ or $cov(\mathscr{M}) = \mathfrak{c}$).*

Proof Let $\mathscr{B}$ be a MAD family of size $\mathfrak{c}$. Enumerate $\langle \overline{X}_\alpha \mid \omega \leq \alpha < \mathfrak{c} \rangle$ the set of decreasing sequences of chains of finite subsets of ω and let $[\omega]^\omega = \{Y_\alpha \mid \omega \leq \alpha < \mathfrak{c}\}$. We will recursively construct a MAD family $\mathscr{A} = \{A_\alpha \mid \alpha \in \mathfrak{c}\}$ and $\mathscr{P} = \{P_\alpha \mid \alpha \in \mathfrak{c}\}$ such that,

1. for every $A_\xi \in \mathscr{A}_\alpha$ there is $B_\xi \in \mathscr{B}$ such that $A_\xi \subseteq B_\xi$. In this way, $\mathscr{A}_\alpha$ is almost disjoint but it is not MAD,
2. if $\overline{X}_\alpha$ is a decreasing sequence of positive sets of $(\mathscr{I}(\mathscr{A}_\alpha)^{<\omega})^+$ then P_α is a pseudointersection,
3. if $\beta \leq \alpha$ then $P_\alpha \in (\mathscr{I}(\mathscr{A}_\alpha)^{<\omega})^+$,
4. if Y_α is almost disjoint with $\mathscr{A}_\alpha$ then $A_\alpha \subseteq Y_\alpha$.

It should be obvious that if we manage to do the construction, then we would have built a Canjar MAD family. We start by taking any partition $\{A_n \mid n \in \omega\}$ of ω in infinite sets. Assume that we have already defined $\mathscr{A}_\alpha$, we will see how to find A_α. If $\overline{X}_\alpha$ is not a sequence of elements in $(\mathscr{I}(\mathscr{A}_\alpha)^{<\omega})^+$ then we define $P_\alpha = fin$. Otherwise, (since $\mathfrak{d} = \mathfrak{c}$) we may find P_α a positive pseudointersection.

Now assume that Y_α is almost disjoint with $\mathscr{A}_\alpha$ (if not, take as Y_α any other set almost disjoint from $\mathscr{A}_\alpha$, note there is always one since $\mathscr{A}_\alpha$ is not MAD). Call $\mathscr{D}$ the set of all finite unions of elements of $\mathscr{A}_\alpha$ and for every $\xi \leq \alpha$ and $B \in \mathscr{D}$ define an interval partition $\mathscr{P}_{\xi B} = \{P_{\xi B}(n) \mid n \in \omega\}$ with the following properties:

1. for every $n \in \omega$ there is $s \subseteq P_{\xi B}(n)$ such that $s \in P_{\xi}$ and $s \cap B = \emptyset$,
2. every $P_{\xi B}(n)$ contains an element of Y_{α}.

Since $\left\langle \mathscr{P}_{\xi B} \mid \xi \leq \alpha \wedge B \in \mathscr{B} \right\rangle$ has size less than $\max\{\mathfrak{d}, \mathfrak{r}\}$, by the previous result, there is an interval partition $\mathscr{Q} = \{Q_n \mid n \in \omega\}$ and C, D disjoint such that both $\bigcup_{n \in C} Q_n$ and $\bigcup_{n \in D} Q_n$ contains infinitely many intervals of each $\mathscr{P}_{\xi B}$. Define $A'_{\alpha} = \bigcup_{n \in C} (Q_n \cap Y_{\alpha})$, then A'_{α} satisfies all the requirements except that it may not be contained in some element of $\mathscr{B}$. However, since $\mathscr{B}$ is MAD we may find $B_{\alpha} \in \mathscr{B}$ such that $A'_{\alpha} \cap B_{\alpha}$ is infinite and then we just define $A_{\alpha} = A'_{\alpha} \cap B_{\alpha}$. $\square$

Given an almost disjoint family $\mathscr{A}$, we will denote by $(\mathscr{I}(\mathscr{A})^{<\omega})^{++}$ the set of all $X \in (\mathscr{I}(\mathscr{A})^{<\omega})^{+}$ such that there is $\{A_n \mid n \in \omega\} \subseteq \mathscr{A}$ with the property that each A_n contains infinitely many elements of X. Note that if $\mathscr{A}'$ is an almost disjoint family with $\mathscr{A} \subseteq \mathscr{A}'$ and $X \in (\mathscr{I}(\mathscr{A})^{<\omega})^{++}$ then $X \in (\mathscr{I}(\mathscr{A}')^{<\omega})^{+}$. The purpose of this definition is the following: assume that we want to construct (recursively) $\mathscr{A} = \{A_{\alpha} \mid \alpha \in \kappa\}$ a Canjar MAD family, at some stage α of the construction, we may look at some decreasing sequence $\langle X_n \mid n \in \omega \rangle \subseteq (\mathscr{I}(\mathscr{A}_{\alpha})^{<\omega})^{+}$ and somehow we manage to find P_{α} a pseudointersection with $P_{\alpha} \in (\mathscr{I}(\mathscr{A}_{\alpha})^{<\omega})^{+}$, we must make sure that P remains positive in the future extensions of $\mathscr{A}_{\alpha}$. In the previous proof, we made sure that at each step of the construction, we preserved the positiveness of all the P_{α}. Another approach would be to make sure that $P_{\alpha} \in (\mathscr{I}(\mathscr{A}_{\alpha})^{<\omega})^{++}$.

Lemma 3 *If $\mathscr{A}$ is an almost disjoint family such that for every decreasing sequence $\langle X_n \mid n \in \omega \rangle$ of $(\mathscr{I}(\mathscr{A})^{<\omega})^{+}$ then there is a pseudointersection $P \in (\mathscr{I}(\mathscr{A})^{<\omega})^{++}$, then $\mathscr{A}$ is a Canjar MAD family.*

Proof The proof is left to the reader. $\square$

Lemma 4 *Let $\mathscr{A} = \{A_n \mid n \in \omega\}$ be an almost disjoint family and let $\langle X_n \mid n \in \omega \rangle$ in $(\mathscr{I}(\mathscr{A})^{<\omega})^{+}$ be a decreasing sequence. Then there is an increasing $f : \omega \longrightarrow \omega$ such that for every $n \in \omega$ there is $s_n \in \wp(f(n) - f(n-1)) \cap X_n$ and $s_n \cap (A_0 \cup \ldots \cup A_n) = \emptyset$ (for ease of writing, assume that $f(-1) = 0$).*

Proof Easy. $\square$

Moreover, note that f can be obtained in a completely definable way. We must also remark that if we define $P = \bigcup_{n \in \omega} X_n \cap \wp(f(n))$ and $B = \bigcup_{n \in \omega} (f(n) - A_0 \cup \ldots \cup A_n)$ then P will be a positive pseudointersection of $\{X_n : n \in \omega\}$, B will contain infinitely many elements of P and $\mathscr{A} \cup \{B\}$ will be an AD family.

The following guessing principle was defined in [16].

- $\diamondsuit(\mathfrak{b})$: For every Borel coloring $C : 2^{<\omega_1} \longrightarrow \omega^{\omega}$ there is a $G : \omega_1 \longrightarrow \omega^{\omega}$ such that for every $R \in 2^{\omega_1}$ the set $\{\alpha \mid C(R \upharpoonright \alpha) \not\geq^{*} G(\alpha)\}$ is stationary (such G is called a *guessing sequence for C*).

Recall that a coloring $C : 2^{<\omega_1} \longrightarrow \omega^{\omega}$ is *Borel* if for every α, the function $C \upharpoonright 2^{\alpha}$ is Borel. It is easy to see that $\diamondsuit(\mathfrak{b})$ implies that $\mathfrak{b} = \omega_1$ and in [16] it is proved that it also implies $\mathfrak{a} = \omega_1$.

Proposition 9 *Assuming $\diamondsuit(\mathfrak{b})$, there is a Canjar MAD family.*

Proof For every $\alpha < \omega_1$ fix an enumeration $\alpha = \{\alpha_n \mid n \in \omega\}$. With a suitable coding, the coloring C will be defined on pairs $t = (\mathscr{A}_t, X_t)$ where $\mathscr{A}_t = \langle A_\xi \mid \xi < \alpha \rangle$ and $X_t = \langle X_n \mid n \in \omega \rangle$. We define $C(t)$ to be the constant 0 function in case $\mathscr{A}_t$ is not an almost disjoint family or if X_t is not a decreasing sequence of $(\mathscr{I}(\mathscr{A})^{<\omega})^+$. In the other case, let $C(t)$ be the function obtained by the previous lemma with $\mathscr{A} = \{A_{\alpha_n} \mid n \in \omega\}$ and X_t. Using $\diamondsuit(\mathfrak{b})$, let $G : \omega_1 \longrightarrow \omega^\omega$ be a guessing sequence for C. By changing G if necessary, we may assume that all the $G(\alpha)$ are increasing and if $\alpha < \beta$ then $G(\alpha) <^* G(\beta)$.

We will now define our MAD family: start by taking $\{A_n \mid n \in \omega\}$ a partition of ω. Having defined A_ξ for all $\xi < \alpha$, we proceed to define

$$A_\alpha = \bigcup_{n \in \omega} \left(G(\alpha)(n) - A_{\alpha_0} \cup \ldots \cup A_{\alpha_n} \right)$$

in case this is an infinite set, otherwise take any A_α that is almost disjoint from $\mathscr{A}_\alpha$. We will see that $\mathscr{A}$ is a Canjar MAD family. Let $X = \langle X_n \mid n \in \omega \rangle$ be a decreasing sequence in $(\mathscr{I}(\mathscr{A})^{<\omega})^+$. Consider the branch $R = \left(\langle A_\xi \mid \xi < \omega_1 \rangle, X \right)$ and pick $\beta^0, \beta^1, \beta^2, \ldots$ such that $C(R \restriction \beta^n) \; {}^* \!\ngeq G(\beta^n)$. Choose α bigger than all the β^n and define $h = G(\alpha)$ and $P = \bigcup_{n \in \omega} \wp(h(n)) \cap X_n$. It is clear that P is a pseudointersection of X. We will now just show that $P \in (\mathscr{I}(\mathscr{A}_\alpha)^{<\omega})^{++}$ and we will do this by proving that each A_{β^n} contains infinitely many elements of P.

Fix $n \in \omega$ and Let $t = R \restriction \beta^n$. Since $C(t) \; {}^* \!\ngeq G(\beta^n)$ we may find m such that $C(t)(m) < G(\beta^n)(m) < h(m)$. In such case (by the property of $C(t)$) there is $s \in \wp(C(t)(m)) \cap X_m$ disjoint from $A_{\beta_0^n}, \ldots A_{\beta_m^n}$ and then $s \subseteq A_{\beta^n}$ and $s \in P$. $\square$

We quote an instance of a very general theorem from [16].

Proposition 10 ([16]) *Let $\langle \mathbb{Q}_\alpha \mid \alpha \in \omega_2 \rangle$ be a sequence of Borel proper partial orders where each $\mathbb{Q}_\alpha$ is forcing equivalent to $\wp(2)^+ \times \mathbb{Q}_\alpha$ and let $\mathbb{P}_{\omega_2}$ be the countable support iteration of this sequence. If $\mathbb{P}_{\omega_2} \Vdash$ "$\mathfrak{b} = \omega_1$" then $\mathbb{P}_{\omega_2} \Vdash$ "$\diamondsuit(\mathfrak{b})$".*

With the aid of the previous result, we can prove that there are Canjar MAD families in many of the models obtained by countable support iteration.

Corollary 2 *Let $\langle \mathbb{Q}_\alpha \mid \alpha \in \omega_2 \rangle$ be a sequence of Borel proper partial orders where each $\mathbb{Q}_\alpha$ is forcing equivalent to $\wp(2)^+ \times \mathbb{Q}_\alpha$ and let $\mathbb{P}_{\omega_2}$ be the countable support iteration of this sequence. Let $G \subseteq \mathbb{P}_{\omega_2}$ be generic, then there is a Canjar MAD family in $V[G]$.*

Proof If in $V[G]$ happens that $\mathfrak{b}$ is ω_2 then we already know there is a Canjar MAD family. Otherwise $\mathfrak{b} = \omega_1$ and then $\diamondsuit(\mathfrak{b})$ holds in $V[G]$ so there is a Canjar MAD family. $\square$

Recall that a forcing is *ω^ω-bounding* if it does not add unbounded reals (or, equivalently, the ground model reals still form a dominating family). Given a forcing $\mathbb{P}$ and a Canjar MAD family $\mathscr{A}$, we say that $\mathscr{A}$ is $\mathbb{P}$ MAD-*Canjar indestructible* if it remains Canjar MAD after forcing with $\mathbb{P}$. We will see that under CH, no proper ω^ω-bounding forcing of size ω_1 can destroy all Canjar MAD families. If $\mathbb{P}$ is a partial order, $\dot{a}$ is a $\mathbb{P}$ name and $G \subseteq \mathbb{P}$ is a generic filter, we will denote by $\dot{a}[G]$ the evaluation of $\dot{a}$ according to the generic filter G.

Proposition 11 *Assume* CH *and let $\mathbb{P}$ be a proper ω^ω-bounding forcing of size ω_1. Then there is a $\mathbb{P}$ MAD-Canjar indestructible family.*

Proof Using the Continuum Hypothesis and the properness of $\mathbb{P}$, we may find a set $H = \{(p_\alpha, \dot{W}_\alpha) \mid \alpha \in \omega_1\}$ such that for all p and $\dot{X}$, if p forces that $\dot{X}$ is a decreasing sequence, then there is α such that $p \leq p_\alpha$ and $p_\alpha \Vdash$ "$\dot{W}_\alpha = \dot{X}$".

We will construct a MAD family $\mathscr{A} = \{A_\alpha \mid \alpha \in \omega_1\}$ such that if p_α forces that $\dot{W}_\alpha$ is a decreasing sequence of positive sets in $(\mathscr{I}(\mathscr{A}_\alpha)^{<\omega})^+$, then there is $q \leq p_\alpha$ with the property that there is $\dot{P}_\alpha$ such that q forces that $\dot{P}_\alpha$ is a pseudointersection of $\dot{W}_\alpha$ and that $\dot{P}_\alpha$ is in $(\mathscr{I}(\mathscr{A}_\alpha)^{<\omega})^{++}$ (hence q will force that $\dot{P}_\alpha$ is in $(\mathscr{I}(\mathscr{A})^{<\omega})^+$).

First take $\{A_n \mid n \in \omega\}$ a partition of ω. Assume that we have defined $\mathscr{A}_\alpha$. We will see how to define $\mathscr{A}_{\alpha+\omega}$. In case p_α does not force that $\dot{W}_\alpha$ is a decreasing sequence of positive sets in $(\mathscr{I}(\mathscr{A}_\alpha)^{<\omega})^+$ then take $\mathscr{A}_{\alpha+\omega}$ be any almost disjoint family extending $\mathscr{A}_\alpha$. Now assume otherwise, write $\alpha = \{\alpha_n \mid n \in \omega\}$ and let $G \subseteq \mathbb{P}$ be a generic filter with $p_\alpha \in G$. Since $\mathscr{A}_\alpha$ is countable and $\dot{W}_\alpha[G] = \langle \dot{W}_\alpha(n)[G] \mid n \in \omega \rangle \in V[G]$ is a sequence of positive sets in $V[G]$, there is an interval partition $\mathscr{P} = \{P_n \mid n \in \omega\} \in V[G]$ such that for all $n \in \omega$, there is $s_n \subseteq P_n$ such that $s_n \in \dot{W}_\alpha(n)[G]$ and s_n is disjoint from $A_{\alpha_0} \cup \ldots \cup A_{\alpha_n}$. Define $P_\alpha = \bigcup (P_n \cap \dot{W}_\alpha(n)[G])$. Let $q' \leq p_\alpha$ force that $\dot{\mathscr{P}}$ is an interval partition and every $\dot{P}_n$ contains an element in $\dot{W}_\alpha(n)$ disjoint from $A_{\alpha_0} \cup \ldots \cup A_{\alpha_n}$. Since $\mathbb{P}$ is ω^ω-bounding, there is $q \leq q'$ and $\mathscr{Q} = \{Q_n \mid b \in \omega\}$ a ground model partition such that $q \Vdash$ "$\dot{\mathscr{P}} \leq \mathscr{Q}$". Let $\{D_n \mid n \in \omega\}$ be a partition of ω with $D_n = \{d_n^i \mid i \in \omega\}$. Define $A_{\alpha+n} = \bigcup_{n \in \omega} \left(P_{d_n^i} - A_{\alpha_0} \cup \ldots A_{\alpha_n}\right)$, then $\mathscr{A}_{\alpha+\omega}$ is an AD family and q forces that each $A_{\alpha+n}$ contains infinitely many elements of $\dot{P}_\alpha$. □

Corollary 3 *There are Canjar MAD families in the Cohen, Random, Hechler, Sacks, Laver, Miller and Mathias model.*

Proof We have already proved it for the models obtained by countable support iteration and in the Cohen and Hechler models since $\mathrm{cov}(\mathscr{M})$ is equal to $\mathfrak{c}$. It only remains to check it for the Random real model. Assume CH and denote by $\mathbb{B}(\kappa)$ the forcing notion for adding κ random reals. Let $G \subseteq \mathbb{B}(\omega_2)$ be a generic filter, we want to see that there is a Canjar MAD family in $V[G]$. By the previous proposition, we know there is $\mathscr{A}$ a $\mathbb{B}(\omega_1)$ MAD-Canjar indestructible family. It is easy to see that $\mathscr{A}$ is $\mathbb{B}(\omega_2)$ MAD-Canjar indestructible (since every new real in $V[G]$ appears in an intermediate extension after adding only ω_1 random reals). □

Although there still may be models without Canjar MAD families, it is easy to show that there are always uncountable Canjar almost disjoint families. Let $C_n = \{n\} \times \omega$ and given a family of increasing functions $\mathscr{B} = \{f_\alpha \mid \alpha \in \omega_1\} \subseteq \omega^\omega$ such that if $\alpha < \beta$ then $f_\alpha <^* f_\beta$ define $\mathscr{A}_\mathscr{B} = \mathscr{B} \cup \{C_n \mid n \in \omega\}$ and note that it is an almost disjoint family.

Proposition 12 *There is a family $\mathscr{B} = \{f_\alpha \mid \alpha \in \omega_1\}$ such that $\mathscr{A}_\mathscr{B}$ is Canjar, so there is an uncountable Canjar almost disjoint family.*

Proof If $\omega_1 < \mathfrak{d}$ then any $\mathscr{B}$ will do, so assume that $\mathfrak{d} = \omega_1$. Let $\mathscr{B} = \{f_\alpha \mid \alpha \in \omega_1\}$ be a well-ordered dominating family. For every $\alpha < \omega_1$ define $L_\alpha = \{(n,m) \mid m < f_\alpha(n)\}$ and for a given X define $X(\alpha) = X \cap [L_\alpha]^{<\omega}$. We will show that

$\mathscr{I}(\mathscr{A}_{\mathscr{B}})^{<\omega}$ is a P^+-ideal and to show that, we will need the following "reflection property" due to Nyikos (see [17]),

Claim 2 *If $X \in (\mathscr{I}(\mathscr{A})^{<\omega})^+$ then $X(\alpha) \in (\mathscr{I}(\mathscr{A})^{<\omega})^+$ for some $\alpha < \omega_1$.*

Assume this is not the case, so for every $\alpha < \omega_1$ the set $X(\alpha) \in \mathscr{I}(\mathscr{A}_{\mathscr{B}})^{<\omega}$, which means there is $F_\alpha \in [\alpha]^{<\omega}$ and $n_\alpha \in \omega$ such that $Z_\alpha = \bigcup_{\xi \in F_\alpha} f_\xi \cup \bigcup_{i \leq n_\alpha} C_i$ intersects every element of $X(\alpha)$. By a trivial application of elementary submodels, there are $S \subseteq \omega_1$ a stationary set, F a finite subset of ω_1 and $n \in \omega$ such that $F = F_\alpha$ and $n_\alpha = n$ for every $\alpha \in S$, call $Z = \bigcup_{\xi \in F} f_\xi \cup \bigcup_{i \leq n} C_i \in \mathscr{I}(\mathscr{A}_{\mathscr{B}})$.

Given $s \subseteq \omega \times \omega$, define $\pi(s) = \{n \mid \exists m((n,m) \in s)\}$. As $X \in (\mathscr{I}(\mathscr{A}_{\mathscr{B}})^{<\omega})^+$ we may find a sequence $Y = \{x_n \mid n \in \omega\} \subseteq X$ such that $x_n \cap Z = \emptyset$ and $\max(\pi(x_n)) < \min(\pi(x_{n+1}))$ for all $n \in \omega$. Since $\mathscr{B}$ is a well-ordered dominating family of increasing functions, there is $\alpha \in S$ such that the set $Y \cap L_\alpha$ is infinite. Note that $Z_\alpha = Z$ so $x_n \cap Z_\alpha = \emptyset$ for all $x_n \in Y \cap L_\alpha$ which contradicts the choice of F_α and n_α.

We are ready to show that $\mathscr{I}(\mathscr{A}_{\mathscr{B}})^{<\omega}$ is a P^+-ideal. Let $\langle X_n \mid n \in \omega \rangle$ be a decreasing sequence of positive sets. Find α such that $X_n(\alpha) \in (\mathscr{I}(\mathscr{A})^{<\omega})^+$ for all $n \in \omega$ (this is possible because if $\beta < \gamma$ and $X_n(\beta)$ is positive $X_n(\gamma)$ is positive). Let $\alpha = \{\alpha_n \mid n \in \omega\}$. For every $n \in \omega$ choose $x_n \in X_n$ such that x_n is disjoint from $\bigcup_{i \leq n} f_{\alpha_i} \cup \bigcup_{i \leq n} C_i$ then it is easy to see that $X = \{x_n \mid n \in \omega\}$ is a positive pseudointersection. □

In particular,

Corollary 4 *There is a non Borel Canjar ideal generated by ω_1 sets.*

Proof By the previous result, we know there is $\mathscr{B} = \{f_\alpha \mid \alpha \in \omega_1\}$ such that $\mathscr{I}(\mathscr{A}_{\mathscr{B}})$ is Canjar, it is enough to show it is not F_σ. Assume otherwise, so it must be F_σ. Let $\mathscr{I}(\mathscr{A}_{\mathscr{B}}) = \bigcup_{n \in \omega} C_n$ where each C_n is a compact set. Clearly, there is $n \in \omega$ such that C_n contains uncountably many elements of $\mathscr{B}$. Note that $C_n \cap \mathscr{B} = C_n \cap \omega^\omega$ so $A = C_n \cap \mathscr{B}$ is a Borel set. For a given Z subset of a Polish space, recall the following definition (see [20])

$\mathsf{OCA}(Z)$**:** If $c : Z^2 \longrightarrow 2$ is a coloring such that $c^{-1}(0)$ is open, then either Z has an uncountable 0-monochromatic set, or Z is the union of countable many 1-monochromatic sets.

In [20] it is proved that $\mathsf{OCA}(Z)$ is true for every analytic set, so in particular $\mathsf{OCA}(A)$ is true. However, we will arrive to a contradiction using the same argument that OCA implies that $\mathfrak{b} = \omega_2$ (see [20]). □

5 Ideals Generated by Branches

If $b \in 2^\omega$ we denote by $\widehat{b} = \{b \restriction n \mid n \in \omega\}$. Let A be a dense, co-dense subset of 2^ω. We define $\mathscr{I}_A$ the *branching ideal of* A as the set of all $X \subseteq 2^{<\omega}$ such that there are $b_1, \ldots, b_n \in A$ with the property that $X \subseteq \widehat{b_1} \cup \ldots \cup \widehat{b_n}$. Clearly, if $M \in [\widehat{b}]^\omega$ with $b \notin A$ then $M \in \mathscr{I}_A^+$, and also every infinite antichain, is positive.

Lemma 5 *$\mathscr{I}_A$ is P^+ for every $A \subseteq 2^\omega$.*

Proof This result follows since $\mathscr{I}_A$ is the ideal generated by an infinite almost disjoint family. □

We will now investigate when $\mathscr{I}_A$ is $P^+(tree)$ and Canjar.

Proposition 13 *If A is the union of less than $\mathfrak{d}$ compact sets, then $\mathscr{I}_A$ is Canjar.*

Proof Assume that $A = \bigcup_{\alpha<\kappa} C_\alpha$ where C_α is compact and $\kappa < \mathfrak{d}$ moreover, we may assume that for every $b_1, \ldots, b_n \in A$ there is a C_α such that $b_1, \ldots, b_n \in C_\alpha$. We will show that $\mathscr{I}_A^{<\omega}$ is a P^+-ideal. Before starting the proof we must do an important observation: assume that $Y \in \left(\mathscr{I}_A^{<\omega}\right)^+$ and for every $a \in Y$ define $U_a = \{b \in 2^\omega \mid a \cap \widehat{b} = \emptyset\}$ and since a is finite then U_a is open and $\langle U_a \mid a \in Y\rangle$ is an open cover of A. Therefore, every C_α is contained in only a finite number of U_a.

Let $\langle X_n \mid n \in \omega\rangle$ be a decreasing family of positive sets of $\mathscr{I}_A^{<\omega}$. For every $\alpha < \kappa$ we define $f_\alpha : \omega \longrightarrow [2^{<\omega}]^{<\omega}$ such that for every if $n \in \omega$ then $f_\alpha(n) \subseteq X_n$ and $C_\alpha \subseteq \bigcup_{a \in f_\alpha(n)} U_a$. Since $\kappa < \mathfrak{d}$, there is $f : \omega \longrightarrow [2^{<\omega}]^{<\omega}$ such that $f(n) \subseteq X_n$ and for all $\alpha < \kappa$ it happens that $f_\alpha(n) \subseteq f(n)$ for infinitely many $n \in \omega$. It is easy to see that $\bigcup_{n\in\omega} f(n)$ is a positive pseudointersection of $\langle X_n \mid n \in \omega\rangle$. $\square$

Given a topological space X, we say that an open cover $\mathscr{U}$ is an *ω-cover* if for every $x_0, \ldots, x_n \in X$ there is $U \in \mathscr{U}$ such that $x_0, \ldots, x_n \in \mathscr{U}$. We say that X is $S_{fin}(\Omega, \Omega)$ if for every sequence $\langle \mathscr{U}_n \mid n \in \omega\rangle$ of ω-covers, there are $F_n \in [\mathscr{U}_n]^{<\omega}$ such that $\bigcup_{n\in\omega} F_n$ is an ω-cover (see [18] for more information concerning this type of spaces). The following was noted by Ariet Ramos.

Proposition 14 *$\mathscr{I}_A$ is Canjar if and only if A is $S_{fin}(\Omega, \Omega)$.*

Proof First assume that A is $S_{fin}(\Omega, \Omega)$ and let $\langle X_n \mid n \in \omega\rangle \subseteq \left(\mathscr{I}_A^{<\omega}\right)^+$ be a decreasing sequence. Given any a we define $U_a = \{b \mid a \cap \widehat{b} = \emptyset\}$. Since each X_n is positive, $\mathscr{V}_n = \{U_a \mid a \in X_n\}$ is an ω-cover of A. In this way, $\langle \mathscr{V}_n \mid n \in \omega\rangle$ is a sequence of ω-covers, so there are $F_n \in [X_n]^{<\omega}$ such that $\{U_a \mid a \in \bigcup_{n\in\omega} F_n\}$ is an ω-cover. It is easy to see that $P = \bigcup_{n\in\omega} F_n$ is a positive pseudointersection of $\langle X_n \mid n \in \omega\rangle$.

Now, assume that $\mathscr{I}_A$ is Canjar and let $\langle \mathscr{U}_n \mid n \in \omega\rangle$ be a sequence of ω-covers. Given an open set U, define $Y_U = \{a \mid \forall b(\widehat{b} \cap a = \emptyset \longrightarrow b \in U)\}$. Define $X_n = \bigcup_{U \in \mathscr{U}_n} Y_U$. Since $\mathscr{U}_n$ is an ω-cover, each X_n is positive. Since $\mathscr{I}_A$ is Canjar, there are $F_n \in [X_n]^{<\omega}$ such that $P = \bigcup_{n\in\omega} F_n$ is a positive pseudointersection. For every $a \in F_n$ choose $U_a \in \mathscr{U}_n$ with the property that $a \in Y_{U_a}$. It is not difficult to check that $\{U_a \mid a \in F_n \wedge n \in \omega\}$ is an ω-cover. $\square$

Given an ideal $\mathscr{I}$ we define $\mathscr{LF}(\mathscr{I})$ the *Laflamme Game on $\mathscr{I}$* as follows,

I	X_0		X_1		X_2		X_3	$\cdots$
II		s_0		s_1		s_2		$\cdots$

where each $X_n \in \mathscr{I}^+$ and s_n is a finite subset of X_n. The player II wins the game if $\bigcup s_n \in \mathscr{I}^+$. Laflamme proved in [12] that $\mathscr{I}$ is a $P^+(tree)$ ideal if and only if player I does not have a winning strategy in $\mathscr{LF}(\mathscr{I})$. In case of branching ideals, the Laflamme game can be simplified. Given $A \subseteq 2^\omega$ define the game $\mathscr{LF}'(\mathscr{I})$ as follows,

I	b_0		b_1		b_2		b_3	$\cdots$
II		s_0		s_1		s_2		$\cdots$

where each $b_n \notin A$, s_n is an initial segment of b_n, $s_n \subsetneq s_{n+1}$ and $b_{n+1} \in \langle s_n \rangle$. The player II wins the game if $\bigcup s_n \notin A$. The analogue of the result of Laflamme is the following.

Proposition 15 *$\mathscr{I}_A$ is a $P^+(tree)$ ideal if and only if player I does not have a winning strategy in $\mathscr{LF}'(\mathscr{I})$.*

Proof It is easy to see that if I has a winning strategy in $\mathscr{LF}'(\mathscr{I})$ then she has one in $\mathscr{LF}(\mathscr{I})$ so $\mathscr{I}$ is not $P^+(tree)$. For the other direction, assume that I does not have a winning strategy and let T be a $\mathscr{I}_A^+$ tree. We will show that there is $b \in [T]$ such that $\bigcup b \restriction n \in \mathscr{I}_A^+$.

Case 1. For all $s \in T$ and $n \in \omega$ there is t an extension of s such that $\bigcup\limits_{i<|t|} t \restriction i$ can not be covered by n branches.

In this case, we simply choose $s_0, s_1, \ldots$ such that s_{n+1} extends s_n and it can not be covered by n branches. It is clear that $b = \bigcup s_n$ is as desired.

Case 2. Without loss of generality, there is $n \in \omega$ such that for every $t \in T$, the set $\bigcup\limits_{i<|t|} t \restriction i$ can be covered by n branches.

By an easy compactness argument, for every $s \in T$ there are $b_0^s, \ldots, b_{n-1}^s \in 2^\omega$ such that $X_s \subseteq \widehat{b}_0^s \cup \ldots \cup \widehat{b}_{n-1}^s$, $b_0^s \notin A$ and $X_s \cap \widehat{b}_0^s$ is infinite. Let $T' \subseteq T$ such that for every $t \in T'$ there is m_t with the property that $t = X_t \cap 2^{\leq m_t}$.

We say that *s prefers t* if s extends t, $m_s > m_t$ and $b_0^s \in \left\langle b_0^t \restriction m_t \right\rangle$. We also say that t is *totally preferred* if for all $s \leq t$ there is $s' \leq s$ such that s' prefers t. We first claim that there is $t \in T$ that is totally preferred. Assume this is not the case, then we do the following:

1. Let $t_\emptyset = \emptyset$.
2. Let $t_1 \leq t_0$ such that no extension of t_1 prefers t_0.
3. Let $t_2 \leq t_1$ such that no extension of t_2 prefers t_1.
4. $\vdots$

We keep this procedure until we find t_{n+1}, but then t_{n+1} must prefer some t_i (with $i \leq n$) which is a contradiction. Now assume t is totally preferred, we will describe π an strategy for player I.

1. First, player I plays b_0^t,
2. if player II plays s_0, then I finds $n_0 \geq |s_0|, \Delta(X_t)$ and let $t_0 = X_t \cap 2^{\leq n_0}$. Player I finds $t_0' \leq t_0$ such that t_0' prefers t and I plays $b_0^{t_0'}$.
3. if player II plays s_1, then I finds $n_1 \geq |s_1|, \Delta\left(X_{t_0'}\right)$ and let $t_1 = X_{t_0'} \cap 2^{\leq n_1}$. Player I finds $t_1' \leq t_1$ such that t_1' prefers t and I plays $b_0^{t_1'}$.
4. $\vdots$

since π is not a winning strategy, there are $s_0, s_1, s_2, \ldots$ such that if player II play s_n at round n then he will win in case I follows π. Let $d = (\pi(s_0, \ldots, s_i) \restriction n_i)$. Then $\bigcup d \notin A$ (since II won the game) and d is a branch through T. $\square$

We will now give a topological characterization of the sets such that its branching ideal is $P^+(tree)$. Recall that a topological space is a *Baire space* if no non-empty open sets are meager, and a space is called *completely Baire* if all of its closed subsets are Baire. Hurewicz proved that a space is completely Baire if and only if it does not contain a closed copy of $\mathbb{Q}$ (see [21] pages 78 and 79).

Proposition 16 *$\mathscr{I}_A$ is $P^+(tree)$ if and only if $2^\omega - A$ is completely Baire.*

Proof Assume that $\mathscr{I}_A$ is $P^+(tree)$ and suppose that $2^\omega - A$ is not completely Baire, so there is a perfect set C such that $A \cap C = \{d_n \mid n \in \omega\}$ is countable dense in C. Consider the following strategy π for I in $\mathscr{L}\mathscr{F}'(2^\omega - A)$.

1. I plays d_0,
2. if II plays s_0, then I plays d_{n_1} where $n_1 = \min\{i > 0 \mid d_i \in \langle s_0 \rangle\}$,
3. if II plays s_1, then I plays d_{n_2} where $n_1 = \min\{i > n_1 \mid d_i \in \langle s_1 \rangle\}$,
4. $\vdots$

Since this is not a winning strategy, there are $s_0, s_1, s_2, \dots$ such that if I follows π and II plays s_i at the round i, then II will win. Let $a = \bigcup_{n\in\omega} s_n$. Then $a \in A \cap C$ since C is compact and II won the game, however, a is different than all the d_n, which is a contradiction.

Now assume that $A \cap C$ is uncountable whenever C is perfect and $A \cap C$ is dense in C. Aiming for a contradiction, assume that I has π a winning strategy in $\mathscr{L}\mathscr{F}'(2^\omega - A)$. Let $D \subseteq 2^\omega$ be the set of all $b \in 2^\omega$ such that there are $s_0, s_1, \dots, s_n$ with the property that $\pi(s_0, s_1, \dots, s_n) = b$. Since π is a winning strategy, $D \subseteq A$ has no isolated points and $C = \overline{D}$ is perfect. Since D is countable, there is $b \in A \cap C - D$. Note that b corresponds to a legal play in $\mathscr{L}\mathscr{F}'(2^\omega - A)$ in which II won (since $b \in A$) which is a contradiction. $\square$

For our next result, we need to recall a result from Kechris, Louveau and Woodin ([9], see also [10] Theorem 21.22).

Proposition 17 ([9]) *If $A \subseteq 2^\omega$ is analytic and $A \cap B = \emptyset$ then one of the following holds,*

1. *there is F an F_σ set such that separates A from B or,*
2. *there is a perfect set $C \subseteq A \cup B$ such that $C \cap B$ is countable dense in C.*

With this we can easily prove the following.

Corollary 5 *If A is Borel and is not F_σ then $\mathscr{I}_A$ is not $P^+(tree)$.*

Proof If A is Borel but not F_σ then, by the Kechris-Louveau-Woodin theorem, there is a perfect set C such that $C \cap (2^\omega - A)$ is countable dense in C, which shows that $\mathscr{I}_A$ is not $P^+(tree)$. $\square$

An alternative proof of the previous corollary would be to note that if A is Borel but not F_σ then $\mathscr{I}_A$ will also be Borel but not F_σ, so it can not be $P^+(tree)$. The next result will give us an example of a non Canjar ideal that is $P^+(tree)$,

Proposition 18 *If B is Bernstein then $\mathscr{I}_B$ is $P^+(tree)$ but not Canjar.*

Proof Since the complement of a Bernstein set is Bernstein, it follows easily by the topological characterization of $P^+(tree)$ that $\mathscr{I}_B$ is $P^+(tree)$. We will now show it is not Canjar. Build an increasing sequence $\langle \mathscr{C}_n \mid n \in \omega \rangle$ of compact sets in the following way,

1. we choose $b_0^0 \notin B$ and let $\mathscr{C}_0 = \{\widehat{b_0^0}\}$,

2. we choose $\langle b_n^{01} \rangle_{n\in\omega} \subseteq 2^\omega - B$ a convergent sequence to b_0^0 and define $\mathscr{C}_1 = \mathscr{C}_0 \cup \{\widehat{b_n^{01}} \mid n \in \omega\}$,
3. for every b_n^{01} we choose $\langle b_n^{012} \rangle_{n\in\omega} \subseteq 2^\omega - B$ a convergent sequence to b_n^{01} and define $\mathscr{C}_2 = \mathscr{C}_1 \cup \{\widehat{b_n^{012}} \mid n \in \omega\}$,
4. $\vdots$

It is clear that each $\mathscr{C}_n \subseteq \mathscr{I}_B^+$ and $\langle \mathscr{C}_n : n \in \omega \rangle$ forms an increasing sequence of compact sets. Let $\mathscr{P} = \{P_n \mid n \in \omega\}$ be a finite partition of $2^{<\omega}$ and define D as the set of all $x \in 2^\omega$ such that there is $\langle d_n \mid n \in \omega \rangle$ with the coherence property with respect to $\mathscr{P}$ and $\widehat{x} \cap P_n = \widehat{d_n}$. It is easy to see that D is an uncountable closed set, so $B \cap D \neq \emptyset$ and hence $\mathscr{I}_B$ is not Canjar. $\square$

Recall that a *Luzin set* is an uncountable set that has countable intersection with every meager set. Luzin sets exist under CH or after adding at least ω_1 Cohen reals. However, it is easy to see that the existence of a Luzin set implies that non$(\mathscr{M})$ is ω_1, so their existence is not provable from ZFC. By a suitable modification of the previous argument, one can show the following.

Corollary 6 *If L is a (dense) Luzin set, then $\mathscr{I}_{\omega-L}$ is not Canjar.*

6 Open Questions

There are some questions we were unable to answer, probably the most interesting one is the following.

Problem 1 *Is there a Canjar MAD family? Is there one of cardinality continuum?*

We proved that if $\mathfrak{d} = \mathfrak{r} = \mathfrak{c}$ then there is a Canjar MAD family of size continuum, but we do not even know the answer to the following question.

Problem 2 *Does $\mathfrak{d} = \mathfrak{c}$ implies there is a Canjar MAD family?*

The characterization of Canjar ideals suggest the next questions.

Problem 3 *Are there coherent strong P^+-ideals that are not strong P^+?*

We know there are P^+-ideals that are not $P^+(tree)$, but we do not know the answer of the following question.

Problem 4 *Is there a Canjar ideal $\mathscr{I}$ such that $\mathscr{I}^{<\omega}$ is not $P^+(tree)$?*[4]

Notes

1. This connection has recently been further studied in [6].
2. We are using $\wp(Z)$ to denote the power set of Z.
3. We say that $\varphi : \wp(\omega) \longrightarrow [0,\infty]$ is a *lower semicontinuous submeasure* if $\varphi(\emptyset) = 0$, $\varphi(A) \leq \varphi(B)$ whenever $A \subseteq B$, $\varphi(A \cup B) \leq \varphi(A) + \varphi(B)$, and $\varphi(A) = lim_{n\longrightarrow\infty} \varphi(A \cap n)$
4. These questions except the first one have recently been answered by Chodounský, Repovš, and Zdomskyy, see [6]

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CANJAR FILTERS II: PROOFS OF $\mathfrak{b} < \mathfrak{s}$ AND $\mathfrak{b} < \mathfrak{a}$ REVISITED

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Abstract. It is a result of Shelah that both $\mathfrak{b} < \mathfrak{s}$ and $\mathfrak{b} < \mathfrak{a}$ are consistent. Using ideas of Brendle and Raghavan, we give alternative proofs of these results.

1. Introduction

For any filter $\mathcal{F}$ on the natural numbers, we can define two forcing notions that diagonalize it (i.e. adds a pseudointersection to it) the *Laver forcing* relative to $\mathcal{F}$, denoted by $\mathbb{L}(\mathcal{F})$, which consists of all trees of height ω that have a stem and above it the set of successors of every node is a member of $\mathcal{F}$, and there is also the *Mathias forcing* relative to $\mathcal{F}$, which is defined as $\mathbb{M}(\mathcal{F}) = \left\{(s, A) \mid s \in [\omega]^{<\omega} \wedge A \in \mathcal{F}\right\}$, the order is given by $(s, A) \leq (z, B)$ whenever z is an initial segment of s, $s - z \subseteq B$ and $A \subseteq B$. These two partial orders have many properties in common; however, in general these partial orders are not equivalent as forcing notions. For every filter $\mathcal{F}$, the Laver forcing associated with it adds a dominating real, but this may not be case for its Mathias forcing. A trivial example is when $\mathcal{F}$ is the filters of all cofinite subsets of ω, in this case $\mathbb{M}(\mathcal{F})$ is forcing equivalent to Cohen forcing, so it does not add a dominating real. A more interesting example was provided by Canjar in [8] (see also [9]) where under $\mathfrak{d} = \mathfrak{c}$, he constructed an ultrafilter which Mathias forcing does not add a dominating real. For this reason, we call such type of filters *Canjar filters*. We say that an ideal $\mathcal{I}$ is a *Canjar ideal* if its dual filter $\mathcal{I}^* = \{\omega - X \mid X \in \mathcal{I}\}$ is a Canjar filter. Canjar filters have been previously investigated in [10], [4] and [9] this paper can be seen as a continuation of that line of research (in fact this article was the last chapter of [9], but the referee of the paper suggested to publish this last chapter independently). No previous knowledge of the previous articles is needed here.

It is a result of Shelah that the unboundedness number $\mathfrak{b}$ can be smaller than the splitting number $\mathfrak{s}$. He achieved this result by using a countable support iteration of a creature forcing (see [1], [7] or [13]). Using a modification of the previous forcing, he also constructed a model where the unboundedness number is smaller than the almost disjointness number $\mathfrak{a}$. Brendle and Raghavan in [7] showed that the partial

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orders of Shelah can be decomposed as an iteration of two simpler forcings. In this note, we will show how to use this decomposition to give alternative proofs of Shelah's results. The consistency of $\mathfrak{b} < \mathfrak{s}$ and $\mathfrak{b} < \mathfrak{a}$ may also be achieved using finite support iteration, as was proved by Brendle [5], and Brendle and Fischer [6].

If $\mathcal{I}$ is an ideal we will denote by $\mathcal{I}^+$ the set of subsets of ω that are not in $\mathcal{I}$ and are called the *positive sets with respect to* $\mathcal{I}$ or $\mathcal{I}$-positive sets. Whenever a, b are two sets, $a - b$ will denote the set theoretic difference of a and b (and never the arithmetic difference, even if $a, b \in \omega$). If $\mathcal{A}$ is an almost disjoint family, we denote by $\mathcal{I}(\mathcal{A})$ the ideal generated by $\mathcal{A}$. If W is a countable set, we denote by $fin(W)$ the set of all non empty finite subsets of W. If $\mathcal{I}$ is an ideal on W, we define the ideal $\mathcal{I}^{<\omega}$ as the set of all $A \subseteq fin(W)$ such that there is $Y \in \mathcal{I}$ with the property that $a \cap Y \neq \emptyset$ for all $a \in A$. We will write fin instead of $fin(W)$ when it is clear from the context. Recall that $\mathcal{I}$ is a P^+*-ideal* if every decreasing sequence of positive sets has a positive pseudointersection. If $f, g \in \omega^\omega$ and $n \in \omega$ then $f <_n g$ means that $f(m) < g(m)$ for every $m \geq n$. If A is a set, we denote by $\wp(A)$ the collection of all subsets of A. We may identify $\wp(\omega)$ with 2^ω, which is homeomorphic to the Cantor set if we give it the product topology. In this way, we can talk about the topological properties (like being compact, F_σ or Borel) of families of subsets of ω. The rest of our notation is mostly standard and follows [3], where the definitions of $\mathfrak{a}, \mathfrak{b}$ and $\mathfrak{s}$ can be consulted as well as their basic properties.

2. Preliminaries

Let $\mathcal{B}$ be an unbounded $\leq^*$ well-ordered family of increasing functions. We call a filter $\mathcal{F}$ a $\mathcal{B}$*-Canjar filter* if $\mathbb{M}(\mathcal{F})$ preserves the unboundedness of $\mathcal{B}$. We will give a combinatorial characterization of this property. Given a decreasing sequence $\overline{X} = \{X_n \mid n \in \omega\} \subseteq fin$ and $f \in \mathcal{B}$, we define the set $\overline{X}_f = \bigcup\limits_{n\in\omega} (X_n \cap \wp(f(n)))$. Observe that $\overline{X}_f$ is a pseudointersection of $\overline{X}$. We say $\overline{X}$ has a *pseudointersection according to* $\mathcal{B}$ if there is $f \in \mathcal{B}$ such that $\overline{X}_f$ is positive. We call $\mathcal{F}^{<\omega}$ a P^+*-filter according to* $\mathcal{B}$ if every decreasing sequence $\overline{X}$ of positive sets has a pseudointersection according to $\mathcal{B}$. The following is a variant of the characterization of Canjar filters by Hrušák and Minami (see [10]).

Proposition 1. *A filter $\mathcal{F}$ is a $\mathcal{B}$-Canjar filter if and only if $\mathcal{F}^{<\omega}$ is a P^+-filter according to $\mathcal{B}$.*

Proof. Assume that $\mathcal{F}$ is not $\mathcal{B}$-Canjar, in other words, there is a name $\dot{g}$ for an increasing function such that $1_{\mathbb{M}(\mathcal{F})} \Vdash$ "$\dot{g}$ is an upper bound for $\mathcal{B}$". For every function $f \in \mathcal{B}$ let $s_f \in [\omega]^{<\omega}, n_f \in \omega$ and $F_f \in \mathcal{F}$ such that $(s_f, F_f) \Vdash$ "$f <_{n_f} \dot{g}$". Since $\mathcal{B}$ is an unbounded increasing chain there are $s \in [\omega]^{<\omega}, n \in \omega$ and a cofinal family $\mathcal{B}' \subseteq \mathcal{B}$ such that $s_f = s$ and $n_f = n$ for every $f \in \mathcal{B}'$.

For every $m \in \omega$ let X_m be the set of all $t \in [\omega - \bigcup s]^{<\omega}$ such that there is $F \in \mathcal{F}$ with the property that $(s \cup t, F)$ decides $\langle \dot{g}(0), \ldots, \dot{g}(m) \rangle$ and $(s \cup t, F) \Vdash$ "$\dot{g}(m) < max(t)$". It is easy to see that every $\overline{X} = \{X_m \mid m \in \omega\}$ is a decreasing sequence of positive sets. We will see that it has no pseudointersection according to $\mathcal{B}$. Since $\mathcal{B}'$ is cofinal in $\mathcal{B}$, it is enough to show that $\overline{X}$ has no pseudointersection according to $\mathcal{B}'$.

Aiming for a contradiction, assume that there is $f \in \mathcal{B}'$ such that $\overline{X}_f$ is positive. Since $\overline{X}_f \cap [F_f]^{<\omega}$ is infinite, pick $t \in \overline{X}_f \cap [F_f]^{<\omega}$ such that $t \in X_k \cap \wp(f(k))$ with $k > n$. Since $t \in X_k$ there is $F \in \mathcal{F}$ such that $(s \cup t, F) \Vdash$ "$\dot{g}(k) \leq max(t)$" and note that $(s \cup t, F) \Vdash$ "$\dot{g}(k) \leq f(k)$". In this way, $(s \cup t, F_h \cap F)$ forces both $f(k) < \dot{g}(k)$ and $\dot{g}(k) \leq f(k)$, which is a contradiction.

Now assume that that $\mathcal{F}$ is $\mathcal{B}$-Canjar, we will see that $\mathcal{F}^{<\omega}$ is P^+ according to $\mathcal{B}$. Let $\overline{X} = \langle X_n \mid n \in \omega \rangle$ be a decreasing sequence of positives. Let M be the Mathias real, observe that $[M]^{<\omega}$ intersect infinitely every member of $(\mathcal{F}^{<\omega})^+$. In this way, in $V[M]$ we may define an increasing function $g : \omega \longrightarrow \omega$ such that $(M - n) \cap g(n)$ contains a member of X_n. Since $\mathcal{F}$ preserves $\mathcal{B}$, then there is $f \in \mathcal{B}$ such that $f \nleq^* g$, we will see that $\overline{X}_f$ is positive. Let $F \in \mathcal{F}$ we must prove that $\overline{X}_f \cap [F]^{<\omega}$ is not empty. Since $F \in \mathcal{F}$ then $M \subseteq^* F$ so there is $k \in \omega$ such that $g(k) < f(k)$ and $M - k \subseteq F$ and hence $\overline{X}_f \cap [F]^{<\omega} \neq \emptyset$. □

Given $A \subseteq fin$, we denote by $\mathcal{C}(A)$ the set of all $X \subseteq \omega$ such that $a \cap X \neq \emptyset$ for all $a \in A$. It is easy to see that $\mathcal{C}(A)$ is a compact set and if $A \in (\mathcal{I}^{<\omega})^+$ then $\mathcal{C}(A) \subseteq \mathcal{I}^+$ for any ideal $\mathcal{I}$. The following lemma is well known and very easy to prove.

Lemma 2. *If $\mathcal{C}$ is a compact set and $A \in [\omega]^\omega$ intersects every element of $\mathcal{C}$, then there is $s \in [A]^{<\omega}$ such that s intersects every element of $\mathcal{C}$.*

The following lemma appears in [9]. We prove it here for the convenience of the reader.

Lemma 3. *Let $\mathcal{F}$ be a filter , let $X \subseteq fin$ be such that $\mathcal{C}(X) \subseteq \mathcal{F}$ and let $\mathcal{D}$ compact with $\mathcal{D} \subseteq \mathcal{F}$. Then, for every $n \in \omega$ there is $S \in [X]^{<\omega}$ such that if $A_0, \ldots, A_n \in \mathcal{C}(S)$ and $F \in \mathcal{D}$ then $A_0 \cap \ldots \cap A_n \cap F \neq \emptyset$.*

Proof. Given $s \in X$ define $K(s)$ as the set of all $(A_0, \ldots, A_n) \in \mathcal{C}(s)^{n+1}$ with the property that there is $F \in \mathcal{D}$ such that $A_0 \cap \ldots \cap A_n \cap F = \emptyset$. This is a compact set by the previous lemma. Note that if $(A_0, \ldots, A_n) \in \bigcap\limits_{s \in X} K(s)$ then $A_0, \ldots, A_n \in \mathcal{C}(X) \subseteq \mathcal{F}$ and there would be $F \in \mathcal{D} \subseteq \mathcal{F}$ such that $A_0 \cap \ldots \cap A_n \cap F = \emptyset$ which is clearly a contradiction. Since the $K(s)$ are compact, then there must be $S \in [F]^{<\omega}$ such that $\bigcap\limits_{s \in S} K(s) = \emptyset$. It is easy to see that this is the S we are looking for. □

We say $\mathcal{F}$ is *strongly Canjar* if $\mathcal{F}$ is $\mathcal{B}$-Canjar for every well-ordered and unbounded $\mathcal{B}$. We will show that all F_σ ideals are strongly Canjar.

Lemma 4. *Let $\mathcal{F}$ be a filter, $\mathcal{D} \subseteq \mathcal{F}$ a compact set and $X \in (\mathcal{F}^{<\omega})^+$. Then there is $n \in \omega$ such that if $F \in \mathcal{D}$ then $(X \cap \wp(n)) \cap [F]^{<\omega} \neq \emptyset$.*

Proof. For every $m \in \omega$ define U_m as the set of all $A \subseteq \omega$ such that $(X \cap \wp(m)) \cap [A]^{<\omega} \neq \emptyset$, clearly this is an open set. Since $\mathcal{D} \subseteq \mathcal{F}$ and $X \in (\mathcal{F}^{<\omega})^+$ we conclude that $\mathcal{D} \subseteq \bigcup\limits_{m\in\omega} U_m$. Finally, $\mathcal{D}$ is a compact set and $\langle U_m \rangle_{m\in\omega}$ is an increasing chain of open sets, so there must be an m such that $\mathcal{D} \subseteq U_m$. □

Now we can prove the following

Proposition 5. *Every F_σ ideal is strongly Canjar.*

Proof. Let $\mathcal{I} = \bigcup \mathcal{C}_n$ be an F_σ ideal with $\langle \mathcal{C}_n \mid n \in \omega \rangle$ an increasing sequence of compact sets and $\mathcal{B}$ be a well-ordered family of increasing functions. Let $\overline{X} = \{X_n \mid n \in \omega\} \subseteq (\mathcal{I}^{<\omega})^+$ be a decreasing sequence. By the previous lemma, we can construct $f : \omega \longrightarrow \omega$ such that if $m \in \omega$ then every element of $\mathcal{C}_m$ contains an element of $X_m \cap \wp(f(m))$. Since $\mathcal{B}$ is unbounded, there is $g \in \mathcal{B}$ that is not dominated by $\mathcal{B}$. It is easy to see that $\overline{X}_g$ is positive. □

3. A model of $\mathfrak{b} < \mathfrak{s}$

Shelah was the first to construct a model where $\mathfrak{b}$ is less than $\mathfrak{s}$ (see [1] or [13]). He achieved this by constructing a weakly ω^ω-bounding proper forcing[1] that adds a real not split by any ground model reals. Later Brendle and Raghavan in [7] showed that Shelah forcing is equivalent to a two step iteration of simpler forcings, we will work with this descomposition.

Definition 6. *Define $\mathbb{F}_\sigma$ as the set of all F_σ filters and consider it as a forcing notion ordered by inclusion.*

It is easy to see that $\mathbb{F}_\sigma$ is σ-closed and if $G \subseteq \mathbb{F}_\sigma$ is a generic filter, then $\bigcup G$ is an ultrafilter. We denote the canonical name of this ultrafilter by $\dot{\mathcal{U}}_{gen}$. In [12] Laflamme showed that this is a Canjar ultrafilter, we will reprove this below. The following lemma is easy to verify.

Lemma 7. *If $\mathcal{U}$ is an ultrafilter and $X \subseteq fin$, then $X \in (\mathcal{U}^{<\omega})^+$ if and only if $\mathcal{C}(X) \subseteq \mathcal{U}$. It follows that if $\mathcal{F}$ is an F_σ filter then $\mathcal{F} \Vdash$"$X \in \left(\mathcal{U}_{gen}^{<\omega}\right)^+$ if and only if $\mathcal{C}(X) \subseteq \mathcal{F}$".*

With the aid of the previous lemmas, we can prove the following,

Proposition 8. *Let $\mathcal{B} \in V$ be an unbounded well-ordered family. Then $\mathbb{F}_\sigma$ forces that $\dot{\mathcal{U}}_{gen}$ is $\mathcal{B}$-Canjar.*

Proof. By the previous observation and since $\mathbb{F}_\sigma$ is σ-closed, it is enough to show that if $\mathcal{F} \Vdash$ "$\overline{X} = \langle X_n \rangle_{n\in\omega} \subseteq \dot{\mathcal{U}}_{gen}^{<\omega+}$" then there is $\mathcal{G} \leq \mathcal{F}$ and $f \in \mathcal{B}$ such that $\mathcal{C}\left(\overline{X}_f\right) \subseteq \mathcal{G}$.

Let $\mathcal{F} = \bigcup \mathcal{C}_n$ where each $\mathcal{C}_n$ is compact and they form an increasing chain. By lemma 3 there is $g : \omega \longrightarrow \omega$ such that if $n \in \omega$, $F \in \mathcal{C}_n$ and $A_0, \ldots, A_n \in \mathcal{C}(X_n \cap \wp(g(n)))$ then $A_0 \cap \ldots . \cap A_n \cap F \neq \emptyset$. Since $\mathcal{B}$ is unbounded, then there is $f \in \mathcal{B}$ such that $f \nleq^* g$. We claim that $\mathcal{F} \cup \mathcal{C}\left(\overline{X}_f\right)$ generates a filter. Let $F \in \mathcal{C}_n$

[1] Recall that a forcing notion $\mathbb{P}$ is *weakly ω^ω-bounding* if $\mathbb{P}$ does not add dominating reals.

and $A_0, \ldots, A_m \in \mathcal{C}\left(\overline{X}_f\right)$. We must show that $A_0 \cap \ldots . \cap A_m \cap F \neq \emptyset$. Since f is not bounded by g, we may find $r > n, m$ such that $f(r) > g(r)$. In this way, $A_0, \ldots, A_n \in \mathcal{C}(X_n \cap \wp(g(n)))$ and then $A_0 \cap \ldots . \cap A_m \cap F \neq \emptyset$. Finally, we can define $\mathcal{G}$ as the filter generated by $\mathcal{F} \cup \mathcal{C}\left(\overline{X}_f\right)$. $\square$

Unlike the ω^ω-bounding property, the weakly ω^ω-bounding property is not preserved under iteration. However, Shelah proved the following preservation result.

Proposition 9 (Shelah, see [1])**.** *If $\gamma \leq \omega_2$ is limit and $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha \leq \gamma \rangle$ is a countable support iteration of proper forcings and each $\mathbb{P}_\alpha$ is weakly ω^ω-bounding (over V) then $\mathbb{P}_\gamma$ is weakly ω^ω-bounding.*

Note that $\mathbb{P}$ is weakly ω^ω-bounding if and only if it preserves the unboundedness of every dominating family. By applying the result of Shelah we can easily conclude the following result.

Corollary 10. *If V satisfies* CH *(it is enough to assume that V has a well ordered dominating family) and $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha \leq \omega_2 \rangle$ is a countable support iteration of proper forcings such that $\mathbb{P}_\alpha$ forces that $\dot{\mathbb{Q}}_\alpha$ preserves the unboundedness of all well-ordered unbounded families, then $\mathbb{P}_{\omega_2}$ is weakly ω^ω-bounding.*

We are now in position to build a model where the unboundedness number is smaller than the splitting number.

Theorem 11 (Shelah)**.** *There is a model where $\mathfrak{b} < \mathfrak{s}$.*

Proof. Assume that V satisfies CH and let $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha \leq \omega_2 \rangle$ be the countable support iteration, where $\mathbb{P}_\alpha \Vdash$ "$\dot{\mathbb{Q}}_\alpha = \mathbb{F}_\sigma * \mathbb{M}(\dot{\mathcal{U}}_{gen})$". By the previous results, it follows that $\mathbb{P}_{\omega_2}$ is weakly ω^ω-bounding and then $\mathfrak{b} = \omega_1$ in the final model. On the other hand, since $\mathbb{F}_\sigma * \mathbb{M}(\dot{\mathcal{U}}_{gen})$ adds an ultrafilter and then diagonalice it, it follows that it destroys all splitting families of the ground model. Therefore $\mathfrak{s} = \omega_2$ in the extension. $\square$

Before constructing the model of $\mathfrak{b} < \mathfrak{a}$ we would like to make some remarks. Recall the definition of almost ω^ω-bounding forcings,

Definition 12. *We say that a forcing notion $\mathbb{P}$ is* almost ω^ω-bounding *if for every name $\dot{f}$ for a real and $p \in \mathbb{P}$, there is an increasing $g : \omega \longrightarrow \omega$ such that for all $A \in [\omega]^\omega$ there is $p_A \leq p$ with the property that $p_A \Vdash$ "$g \upharpoonright A \not\leq^* \dot{f} \upharpoonright A$."*

The following is well known.

Lemma 13. *If $\mathbb{P}$ is almost ω^ω-bounding then $\mathbb{P}$ preserves all unbounded families of the ground model.*

Proof. Let $\mathcal{B}$ be unbounded, let $\dot{f}$ a name for a real and let $p \in \mathbb{P}$. Find $g : \omega \longrightarrow \omega$ as above. Since $\mathcal{B}$ is unbounded, there is $h \in \mathcal{B}$ and $A \in [\omega]^\omega$ such that $g \upharpoonright A \leq h \upharpoonright A$. It then clearly follows that p_A forces that $\dot{f}$ does not dominate $\mathcal{B}$. $\square$

Given $A \in [\omega]^{\omega}$ denote by $e_A : \omega \longrightarrow A$ its enumerating function. It is a well known result of Talagrand (see [2]) that a filter $\mathcal{F}$ is non-meager if and only if $\{e_A \mid A \in \mathcal{F}\}$ is unbounded. It follows that no almost ω^{ω}-bounding forcing can diagonalize a non-meager filter. Since ultrafilters are non-meager, we conclude the following.

Corollary 14. *If $\mathcal{U}$ is an ultrafilter, then $\mathbb{M}(\mathcal{U})$ is not almost ω^{ω}-bounding.*

It follows by the theorems of Shelah, Brendle and Raghavan (see [1] and [7]) that $\mathbb{F}_{\sigma} * \mathbb{M}(\dot{\mathcal{U}}_{gen})$ is almost ω^{ω}-bounding, in spite the fact that $\mathbb{M}(\dot{\mathcal{U}}_{gen})$ is not.

4. A model of $\mathfrak{b} < \mathfrak{a}$

The first model where $\mathfrak{b} < \mathfrak{a}$ was constructed by Shelah using countable support iteration of proper forcings. Later, Brendle in [5] constructed a model of this result using finite support iteration. Although we will also use countable support iteration, the following proof was inspired by the work of Brendle.[2]

Given an AD family $\mathcal{A}$ define $\mathbb{F}_{\sigma}(\mathcal{A}) = \{\mathcal{F} \in \mathbb{F}_{\sigma} \mid \mathcal{I}(\mathcal{A}) \cap \mathcal{F} = \emptyset\}$ and order it by inclusion. As before, it is easy to see that $\mathbb{F}_{\sigma}(\mathcal{A})$ is a σ-closed filter and it adds an ultrafilter, which we will denote by $\dot{\mathcal{U}}_{\mathcal{A}}$. The *Brendle game* $\mathcal{BR}(\mathcal{A})$ is defined as follows,

I		Y_0		Y_1		Y_2		$\cdots$
II	$\mathcal{F}, X$		s_0		s_1		s_2	$\cdots$

Where

1. $\mathcal{F} \in \mathbb{F}_{\sigma}(\mathcal{A})$, $\mathcal{F} = \bigcup \mathcal{C}_n$, where the $\mathcal{C}_n$ are compact and increasing, $X \subseteq fin$ and $\mathcal{C}(X) \subseteq \langle \mathcal{I}(\mathcal{A})^{*} \cup \mathcal{F} \rangle$,
2. $Y_m \in \mathcal{I}(\mathcal{A})^{*}$, $s_m \in [Y_m]^{<\omega}$ intersects all the elements of $\mathcal{C}_m$ and $\max(s_m) < \min(s_{m+1})$.

The player I wins the game if $\bigcup\limits_{n\in\omega} s_n$ contains an element of X.

Note that this is an open game for I, i.e., if she wins, then she wins already in a finite number of steps. In the following, $V[C_{\omega_1}]$ denotes an extension of V by adding ω_1 Cohen reals.

Lemma 15. *If $\mathcal{A}$ is an AD family in V, then in $V[C_{\omega_1}]$ the player I has a winning strategy for $\mathcal{BR}(\mathcal{A})$.*

Proof. Assume this is not the case. Since $\mathcal{BR}(\mathcal{A})$ is an open game it follows from the Gale-Stewart theorem (see [11]) that II has a winning strategy, call it π. Let $\mathcal{F}, X,\ \mathcal{F} = \bigcup\limits_{n\in\omega} \mathcal{C}_n \in \mathbb{F}_{\sigma}(\mathcal{A})$ and $X \subseteq fin$ be the first play of II according to

[2]A similar but different approach has also been found recently by Andrew Brooke-Taylor and Joerg Brendle

π (so $\mathcal{C}(X) \subseteq \langle \mathcal{I}(\mathcal{A})^* \cup \mathcal{F} \rangle$). By standard Cohen forcing arguments, we may as well assume that $\mathcal{F}$, X and π are ground model sets. Call $\mathbb{P}$ the set of all $p = \langle s_0, \dots s_n \rangle$ such that there are $Y_0, \dots Y_n \in \mathcal{I}(\mathcal{A})^*$ with the property that $(\mathcal{F}, X, Y_0, s_0, \dots, Y_n, s_n)$ is a partial play and the s_n are chosen using π. We order $\mathbb{P}$ by extension, note that $\mathbb{P}$ is countable, therefore it is isomorphic to Cohen forcing and if $p = \langle s_0, \dots s_n \rangle \in \mathbb{P}$ then $\bigcup\limits_{i<n} s_i$ does not contain an element of X.

Given $Y \in \mathcal{I}(\mathcal{A})^*$ and $m \in \omega$ the set D_{Ym} of all conditions p such that p contains a response to Y and $|p| > m$ is open dense. Let $G \in V[C_{\omega_1}]$ be a $(\mathbb{P}, V)$ generic filter. By the above observation, we conclude that $D = \bigcup G$ is a legal play of the game, and it is a winning run for II, so D does not contain any element of X. By genericity $D \in \langle \mathcal{I}(\mathcal{A})^* \cup \mathcal{F} \rangle^+$ however, $\omega - D \in \mathcal{C}(X) \subseteq \langle \mathcal{I}(\mathcal{A})^* \cup \mathcal{F} \rangle$ which is obviously a contradiction. □

We will need the following important definition.

Definition 16. *We say a MAD family $\mathcal{A}$ is a* Laflamme family *if $\mathcal{I}(\mathcal{A})$ can not be extended to an F_σ ideal.*

Given $X \subseteq fin$ and $A \in [\omega]^\omega$ let $Catch(X, A) = \{s \in X \mid s \subseteq A\}$. With the previous lemma we can prove the following dichotomy.

Lemma 17. *Let $\mathcal{A} \in V$ be an AD family, then in $V[C_{\omega_1}]$ one of the following holds,*

1. *$\mathcal{A}$ is not a Laflamme family or,*
2. *For every $\mathcal{F} \in \mathbb{F}(\mathcal{A})$ and $\{X_n \mid n \in \omega\} \subseteq fin$ with the property that $\mathcal{C}(X_n) \subseteq \langle \mathcal{I}(\mathcal{A})^* \cup \mathcal{F} \rangle$ for all $n \in \omega$, there is $A \in \mathcal{A} \cap \mathcal{F}^+$ such that if $B \in \wp(A) \cap \mathcal{F}^+$ then $Catch(X_n, B) \in (\mathcal{F}^{<\omega})^+$ for every $n \in \omega$.*

Proof. Assume that $\mathcal{A}$ is a Laflamme family and let $\mathcal{F}$ and X_n as above. By the previous lemma, let π be a winning strategy for player I. Consider the games where II began by playing $\mathcal{F}, X_n$ and call $\mathcal{W}$ the countable set of elements of $\mathcal{I}(\mathcal{A})^*$ that were played by I following π in any of these games. Note that if $W \in \mathcal{W}$ then W almost contains every element of $\mathcal{A}$ except for finitely many. Let $\mathcal{A}' \subseteq \mathcal{A}$ be the countable set of all those elements of $\mathcal{A}$ that are not almost contained in every element of $\mathcal{W}$. Since $\mathcal{I}(\mathcal{A})^*$ can not be extended to an F_σ filter it is not contained in $\langle \mathcal{F} \cup \{\omega - B \mid B \in \mathcal{A}'\} \rangle$ so there is $A \in \mathcal{A}$ such that $\omega - A \notin \langle \mathcal{F} \cup \{\omega - B \mid B \in \mathcal{A}'\} \rangle$. This implies that $A \in \mathcal{F}^+$ and A is almost contain in every member of $\mathcal{W}$. Let $B \in \wp(A) \cap \mathcal{F}^+$ we will now show that $Catch(X_n, B)$ is positive for each $n \in \omega$. Let $F \in \mathcal{F}$ and consider the following play,

I		W_0		W_1		W_2		$\cdots$
II	$\mathcal{F}, X_n$		s_0		s_1		s_2	$\cdots$

where the W_n are played by I according to π and $s_i \in [B \cap F]^{<\omega}$ and intersects every element of $\mathcal{C}_i$. This is possible since $B \cap F$ is positive and is almost contained in every W_n. Since π is a winning strategy, this means that I wins the game,which entails that $\bigcup s_n \subseteq B \cap F$ contains an element of X_n. □

Given $A \in [\omega]^{\omega}$ and $l \in \omega$ define $Part_l(A)$ as the set of all sequences $\langle B_1, ..., B_l \rangle$ such that $A = \bigcup_{i \leq l} B_i$ and $B_i \cap B_j = \emptyset$ whenever $i \neq j$. Note that $Part_l(A_i)$ is a compact space with the natural topology. Also it is clear that if $A \in \mathcal{F}^+$ and $\langle B_1, ..., B_l \rangle \in Part_l(A)$ then there is $j \leq l$ such that $B_j \in \mathcal{F}^+$.

Lemma 18. *Let $\mathcal{F}$ be a filter, let $\mathcal{C} \subseteq \mathcal{F}$ be compact and let $X \in (\mathcal{F}^{<\omega})^+$. Assume that A is such that if $B \in \wp(A) \cap \mathcal{F}^+$ then $Catch(X, B) \in (\mathcal{F}^{<\omega})^+$ and let $l \in \omega$. Then there is $n \in \omega$ with the property that for all $\langle B_1, ..., B_l \rangle \in Part_l(A)$ there is $i \leq l$ such that if $F \in \mathcal{C}$ then $X \cap \wp(B_i \cap n)$ contains a subset of F.*

Proof. Let U_n be the set of all $\langle B_1, ..., B_l \rangle \in Part_l(A)$ such that there is $i \leq l$ with the property that if $F \in \mathcal{C}$ then $X \cap \wp(B_i \cap n)$ contains a subset of F. Note that $\{U_n \mid n \in \omega\}$ is an open set cover by lemma 4 and the result follows since $Part_l(A)$ is compact. □

It is easy to see that if $\mathcal{F} \in \mathbb{F}_\sigma(\mathcal{A})$ and $X \subseteq fin$, then $\mathcal{F} \Vdash$ "$X \in \dot{\mathcal{U}}_{\mathcal{A}}^{<\omega+}$" if and only if $\mathcal{C}(X) \subseteq \langle \mathcal{F} \cup \mathcal{I}(\mathcal{A})^* \rangle$. With this we may prove the following result.

Proposition 19. *Let $\mathcal{B} \in V$ be a well-ordered unbounded family and let $\mathcal{A}$ an AD family, then in $V[C_{\omega_1}]$ either $\mathcal{A}$ is not Laflamme or $\mathbb{F}_\sigma(\mathcal{A}) \Vdash$ "$\dot{\mathcal{U}}_{\mathcal{A}}$ is $\mathcal{B}$-Canjar".*

Proof. Assume that $\mathcal{A}$ is Laflamme after adding ω_1 Cohen reals. In $V[C_{\omega_1}]$ let $\mathcal{F} \in \mathbb{F}_\sigma(\mathcal{A})$ and let a sequence $\overline{X} = \langle X_n \mid n \in \omega \rangle$ be such that $\mathcal{F}$ forces that each X_n is in $\dot{\mathcal{U}}_{\mathcal{A}}^{<\omega+}$, so all the $\mathcal{C}(X_n)$ are contained $\langle \mathcal{F} \cup \mathcal{I}(\mathcal{A})^* \rangle$. We will find an extension of $\mathcal{F}$ that forces that the $\overline{X}$ has a positive pseudointersection. Applying the previous lemma ω times, we may find distinct $A_0, A_1, A_2, \ldots \in \mathcal{A}$ such that $Catch(X_m, B) \in (\mathcal{F}^{<\omega})^+$ for every $B \in \wp(A_n) \cap \mathcal{F}^+$ and $n, m \in \omega$.

Let $\mathcal{F} = \bigcup_{m \in \omega} \mathcal{C}_m$, where $\langle \mathcal{C}_m \rangle_{m \in \omega}$ is an increasing sequence of compact sets. Define an increasing function $g : \omega \longrightarrow \omega$ such that if $n \in \omega$ then for all $\langle B_1, ..., B_{2^n} \rangle \in Part_{2^n}(A_n)$ there is $j \leq 2^n$ such that if $F \in \mathcal{C}_n$ then $X_n \cap \wp(B_j \cap (g(n) - n))$ contains a subset of F. Since $\mathcal{B}$ is unbounded, we can find $f \in \mathcal{B}$ that is not dominated by g.

We will now show that $\mathcal{F} \cup \mathcal{C}(\overline{X}_f) \cup \mathcal{I}^*(\mathcal{A})$ generates a filter. Let $F \in \mathcal{F}$, $C_0, ..., C_n \in \mathcal{C}(\overline{X}_f)$ and $D_0, ..., D_n \in \mathcal{A}$, we must show $F \cap C_0 \cap ... \cap C_n \cap (\omega - D_0) \cap ... \cap (\omega - D_n) \neq \emptyset$. We first find $m \in \omega$ such that,

(1) $n \leq m$,
(2) $F \in \mathcal{C}_m$,
(3) $A_m \cap (D_0 \cup ... \cup D_n) \subseteq m$,
(4) $g(m) < f(m)$.

For every $s : m \longrightarrow 2$ define B_s as the set of all $a \in A_m$ such that $a \in C_i$ if and only if $s(i) = 1$. Clearly $\langle B_s \rangle_{s \in 2^m} \in Part_{2^m}(A_m)$ and then we conclude that there is s such that $X_m \cap \wp(B_s \cap (g(m) - m))$ contains an element of F and then so does $X_m \cap \wp(B_s \cap (f(m) - m))$. Since $C_0, ..., C_n \in \mathcal{C}(\overline{X}_f)$ we conclude that s must be the constant 1 function and this entails that $F \cap C_0 \cap ... \cap C_n \cap (\omega - D_0) \cap ... \cap (\omega - D_n) \neq \emptyset$.

Finally, if we define $\mathcal{G}$ as the filter generated by $\mathcal{F} \cup \mathcal{C}\left(\overline{X}_f\right)$ then $\mathcal{G} \in \mathbb{F}_\sigma(\mathcal{A})$ and it forces that $\overline{X}$ has a positive pseudointersection. □

We are now in position to prove the result of Shelah.

Theorem 20 (Shelah)**.** *There is a model where* $\mathfrak{b} < \mathfrak{a}$.

Proof. Assume that V satisfies CH, define the countable support iteration $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha \in \omega_2 \rangle$ such that (with a suitable bookkeeping device) we destroy every MAD $\mathcal{A}$ family either by adding Cohen reals, by forcing with the Mathias forcing of an F_σ filter or with $\mathbb{F}_\sigma(\mathcal{A}) * \mathbb{M}(\dot{\mathcal{U}}_A)$. It is clear that this construction works. □

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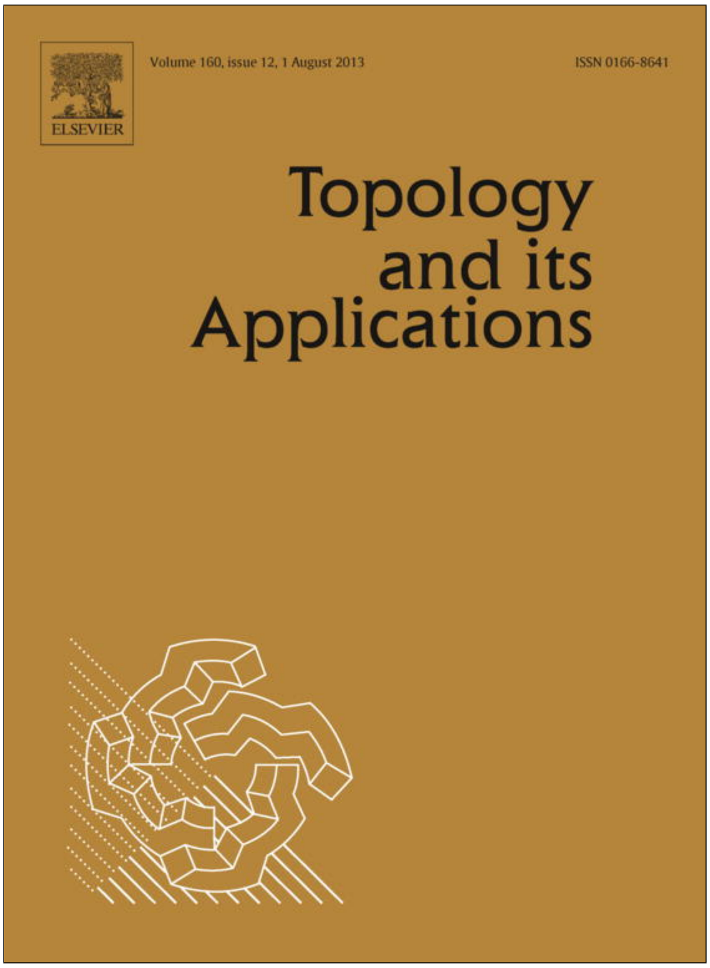
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n-Luzin gaps

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ABSTRACT

We introduce and study the notion of an n-Luzin gap, which is a natural generalization of a Luzin gap. We prove that under Martin's Axiom, every AD family $\mathcal{A}$ of size less than $\mathfrak{c}$ contains an n-Luzin gap or the corresponding Mrówka–Isbell space $\Psi(\mathcal{A})$ is normal.

0. Introduction

An infinite family $\mathcal{A} \subset \mathcal{P}(\omega)$ is *almost disjoint* (AD) if the intersection of any two distinct elements of $\mathcal{A}$ is finite. It is *maximal almost disjoint* (MAD) if it is not properly included in any larger AD family or, equivalently, if given an infinite $X \subseteq \omega$ there is an $A \in \mathcal{A}$ such that $|A \cap X| = \omega$. Given an almost disjoint family $\mathcal{A}$ and two subfamilies $\mathcal{B}, \mathcal{C}$ of $\mathcal{A}$ we say that a set $X \subseteq \omega$ *separates* $\mathcal{B}$ and $\mathcal{C}$ if $A \subseteq^* X$ for every $A \in \mathcal{B}$ and $A \cap X =^* \emptyset$ for every $A \in \mathcal{C}$.

One of the first constructions of almost disjoint families with special properties is the construction of Luzin [14] of an uncountable almost disjoint family $\mathcal{A}$ such that no two uncountable subfamilies of $\mathcal{A}$ can be separated. The ingenious property used in the proof deserves a name:

Definition 0.1. An almost disjoint family $\mathcal{A}$ is *Luzin* if it can be enumerated as $\{A_\alpha\colon \alpha < \omega_1\}$ so that $\forall \alpha < \omega_1\ \forall n \in \omega\ \{\beta < \alpha\colon A_\alpha \cap A_\beta \subseteq n\}$ is finite.

Abraham and Shelah [1] called (and so do we) an almost disjoint family $\mathcal{A}$ *inseparable* if no two uncountable subfamilies can be separated. It is easy to see that $\mathcal{A}$ is inseparable if and only if for every $\mathcal{B}, \mathcal{C} \in [\mathcal{A}]^{\omega_1}$ the set $\bigcup \mathcal{B} \cap \bigcup \mathcal{C}$ is infinite. The point of Luzin's proof was that, Luzin families are inseparable. Abraham and Shelah proved that (1) assuming CH, there is an inseparable AD family which contains no Luzin subfamily, while (2) under $\mathsf{MA} + \neg\mathsf{CH}$ every inseparable AD family is a countable union of Luzin subfamilies.

Roitman and Soukup in [17] introduced the notion of an anti-Luzin family: An AD family $\mathcal{A}$ is an *anti-Luzin* family if for every $\mathcal{B} \in [\mathcal{A}]^{\aleph_1}$ there are $\mathcal{C}, \mathcal{D} \in [\mathcal{B}]^{\aleph_1}$ which can be separated (or equivalently, $\mathcal{A}$ does not contain uncountable inseparable families) and proved that assuming $\mathsf{MA} + \neg\mathsf{CH}$, every AD family is either anti-Luzin or contains an uncountable Luzin subfamily, and assuming $\upharpoonright$,[3] there is an uncountable almost disjoint family which contains no uncountable anti-Luzin and no uncountable Luzin subfamilies.

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[3] Recall that $\upharpoonright$ is the following weakening of CH: There is a family $\mathcal{S} \subseteq [\omega_1]^\omega$ of size $\aleph_1$ such that every uncountable subset of ω_1 contains an element of $\mathcal{S}$.

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More recently, Dow [7] showed that PFA implies that every MAD family contains an uncountable Luzin subfamily. Dow and Shelah in [8] showed that Martin's Axiom does not suffice by showing that it is relatively consistent with $\mathsf{MA} + \neg\mathsf{CH}$ that there is a maximal almost disjoint family which is ω_1-*separated*, i.e. any disjoint pair of $\leqslant \omega_1$-sized subfamilies are separated.

To every almost disjoint family one can naturally associate the so-called *Mrówka–Isbell space*:

Definition 0.2. Given an AD family $\mathcal{A}$, define a space $\Psi(\mathcal{A})$ as follows: The underlying set is $\omega \cup \mathcal{A}$, all elements of ω are isolated and basic neighborhoods of $A \in \mathcal{A}$ are of the form $\{A\} \cup (A \setminus F)$ for some finite set F.

It follows immediately from the definition that $\Psi(\mathcal{A})$ is a separable, scattered, zero-dimensional, first countable, locally compact Moore space [16]. Normality of Ψ-spaces is characterized using separation as follows:

Proposition 0.3. *([20]) $\Psi(\mathcal{A})$ is normal if and only if $\mathcal{B}$ and $\mathcal{A} \setminus \mathcal{B}$ can be separated for every $\mathcal{B} \subseteq \mathcal{A}$.*

Slightly abusing notation we will call an AD family $\mathcal{A}$ *normal* if the space $\Psi(\mathcal{A})$ is normal. A natural choice would be to call $\mathcal{A}$ *completely separated*, but unfortunately a very similar term is already in use [19,9,5]. By the above proposition it follows that if $\Psi(\mathcal{A})$ is normal, then $2^{|\mathcal{A}|} = \mathfrak{c}$ so $\mathcal{A}$ must have size less than the continuum.

Luzin families are often referred to as Luzin gaps. However, that name has recently [21,10] been used to describe a weaker notion.

Definition 0.4. ([21]) A pair $\mathcal{A} = \{A_\alpha\colon \alpha < \omega_1\}$, $\mathcal{B} = \{B_\alpha\colon \alpha < \omega_1\}$ of subfamilies of $[\omega]^\omega$ is called a *Luzin gap* if there is an $m \in \omega$ such that

1. $A_\alpha \cap B_\alpha \subseteq m$ for all $\alpha < \omega_1$, and
2. $A_\alpha \cap B_\beta$ is finite yet $(A_\alpha \cap B_\beta) \cup (A_\beta \cap B_\alpha) \nsubseteq m$ for all $\alpha \neq \beta < \omega_1$.

Every Luzin family $\mathcal{A}$ contains many Luzin gaps: given a pair $\{A_\alpha\colon \alpha < \omega_1\}$, $\{B_\alpha\colon \alpha < \omega_1\}$ of disjoint subfamilies of $\mathcal{A}$, there is an uncountable $X \subseteq \omega_1$ such that $\{A_\alpha\colon \alpha \in X\}$, $\{B_\alpha\colon \alpha \in X\}$ forms a Luzin gap. The basic property of a Luzin gap is that the two families $\mathcal{A}$ and $\mathcal{B}$ cannot be separated, and the property of being a Luzin gap is indestructible by forcing preserving ω_1 (see [21,10] or Section 1). Hence, the space $\Psi(\mathcal{A})$ cannot be normal (in any forcing extension preserving ω_1) for any AD family $\mathcal{A}$ containing a Luzin gap.

The following weakening of the notion of a Luzin gap is central for our considerations.

Definition 0.5. Let $n \in \omega$ and $\mathcal{B}_i = \{B^i_\alpha \mid \alpha \in \omega_1\}$ be disjoint subfamilies of an AD family $\mathcal{A}$ for $i < n$. We call $\langle \mathcal{B}_i \mid i < n\rangle$ an n-Luzin *gap* if there is $m \in \omega$ such that

1. $B^i_\alpha \cap B^j_\alpha \subseteq m$ for all $i \neq j$, $\alpha < \omega_1$ and
2. $\bigcup_{i \neq j}(B^i_\alpha \cap B^j_\beta) \nsubseteq m$ for all $\alpha \neq \beta < \omega_1$.

We say that $\mathcal{A}$ contains an n-Luzin gap if there is an n-Luzin gap $\langle \mathcal{B}_i \mid i < n\rangle$ where each $\mathcal{B}_i$ is a subfamily of $\mathcal{A}$. We will see that any family containing an n-Luzin gap is not normal, and our main theorem states that the converse is also true assuming Martin's Axiom:

Theorem 0.6. *Assume* MA. *Let $\mathcal{A}$ be an AD family. Then $\mathcal{A}$ is normal if and only if $|\mathcal{A}| < \mathfrak{c}$ and $\mathcal{A}$ does not contain n-Luzin gaps for any $n \in \omega$.*

Assuming PFA the theorem can be strenghtened (see Theorem 3.8). We also show that the result does not follow from MA(σ-centered), as

Theorem 0.7. *It is consistent with* MA(σ*-centered*) *that there is an inseparable AD family of size ω_1 which does not contain n-Luzin gaps for any $n \in \omega$.*

The situation is reminiscent of ω_1-trees and Hausdorff gaps, an inseparable family that does not contain n-Luzin gaps for any $n \in \omega$ being the equivalent of a Suslin tree or a ccc destructible gap. A Suslin tree can be destroyed by two different means: (1) one can force with the tree an add an uncountable branch and (2) one can specialize the tree by a ccc forcing making it a union of countably many antichains. Similar situation occurs with ccc destructible Hausdorff gaps ([13] see [18]) a destructible Hausdorff gaps can be either (1) filled or (2) frozen, both by ccc forcing. Here, an inseparable family with no n-Luzin gaps can be either (1) forced normal or (2) frozen by forcing it to contain a Luzin gap, both by a ccc forcing.

An early (probably the first) example of a Ψ-space appears in [3]: A topology of the real line is refined by declaring all rational points isolated. To each irrational point a convergent sequence is chosen and the cofinite subsets of the given convergent sequence are declared basic open neighborhoods of the irrational number.

We call an almost disjoint family $\mathcal{A}$ $\mathbb{R}$-*embeddable* (see [11]) if there is an injection $e:\omega \to \mathbb{Q}$ such that for every $A \in \mathcal{A}$ there is an $r_A \in \mathbb{R}$ such that $e[A]$ converges to r_A and, moreover, $r_A \neq r_B$ whenever $A \neq B$. Evidently, this is equivalent that there is an injective and continuous $f:\Psi(\mathcal{A}) \to \mathbb{R}$ such that $f(n) \in \mathbb{Q}$ for every $n \in \omega$. Using Tietze's theorem, it is easy to show that every normal family is $\mathbb{R}$-embeddable.

The notion of $\mathbb{R}$-embeddability together with a strengthening of the notion of an anti-Luzin family are some of the main tools used here.

Definition 0.8. An almost disjoint family $\mathcal{A}$ is *partially separated* if given a pair $\mathcal{B} = \{B_\alpha\colon\ \alpha < \omega_1\}$, $\mathcal{C} = \{C_\alpha\colon\ \alpha < \omega_1\}$ of disjoint subfamilies of $\mathcal{A}$ there is an uncountable $X \subseteq \omega_1$ such that the families $\{B_\alpha\colon\ \alpha \in X\}$, $\{C_\alpha\colon\ \alpha \in X\}$ are separated.

We call an AD family $\mathcal{A}$ *potentially* $\mathcal{P}$ (for a property $\mathcal{P}$) if there is a ccc forcing $\mathbb{P}$ such that $\Vdash_{\mathbb{P}}$ "$\mathcal{A}$ has $\mathcal{P}$". Similarly, we say that $\mathcal{A}$ is *indestructibly* $\mathcal{P}$, if $\mathcal{A}$ has property $\mathcal{P}$ in all ccc forcing extensions. We show that

Theorem 0.9. *The following are equivalent for an AD family* $\mathcal{A}$:

1. $\mathcal{A}$ *does not contain* n*-Luzin gaps for any* $n \in \omega$,
2. $\mathcal{A}$ *is potentially normal,*
3. $\mathcal{A}$ *is potentially* $\mathbb{R}$*-embeddable,*
4. $\mathcal{A}$ *is potentially partially separated.*

Dow and Shelah's [8] result mentioned above shows that it is consistent with MA that there is a MAD family which is potentially normal, while assuming PFA [7] all MAD families contain Luzin families, hence, also Luzin gaps. It is worth mentioning that Áviles and Todorčević also studied gaps of higher dimensions in [4], however their versions are strengthenings rather than weakenings of the classical notion.

1. Normality of AD families in ccc extensions

In the following, $\mathcal{A}$ will always be an AD family. Given $\mathcal{B}, \mathcal{C}$ disjoint subsets of $\mathcal{A}$, we will define a forcing that adds a set separating $\mathcal{B}$ from $\mathcal{C}$. Let $\mathbb{S}_{\mathcal{BC}}$ be the set of all $(s, \mathcal{F}, \mathcal{G})$ such that,

1. $s \in {}^{<\omega}2$, $\mathcal{F} \in [\mathcal{B}]^{<\omega}$, $\mathcal{G} \in [\mathcal{C}]^{<\omega}$.
2. If $B \in \mathcal{F}$ and $C \in \mathcal{G}$ then $B \cap C \subseteq |s|$.

We say $(s, \mathcal{F}, \mathcal{G}) \leqslant (s', \mathcal{F}', \mathcal{G}')$ if and only if,

1. $s' \subseteq s$, $\mathcal{F}' \subseteq \mathcal{F}$, $\mathcal{G}' \subseteq \mathcal{G}$.
2. If $i \in \mathrm{dom}(s) \setminus \mathrm{dom}(s')$ then,
   a) If $i \in \bigcup \mathcal{F}'$ then $s(i) = 1$.
   b) If $i \in \bigcup \mathcal{G}'$ then $s(i) = 0$.

It is easy to prove that for all $n \in \omega$, $B \in \mathcal{B}$ and $C \in \mathcal{C}$ the following sets $\{(s, \mathcal{F}, \mathcal{G}) \mid |s| \geqslant n\}$, $\{(s, \mathcal{F}, \mathcal{G}) \mid B \in \mathcal{F}\}$ and $\{(s, \mathcal{F}, \mathcal{G}) \mid C \in \mathcal{G}\}$ are dense, so $\mathbb{S}_{\mathcal{BC}}$ adds a set separating $\mathcal{B}$ from $\mathcal{C}$.

Lemma 1.1. *If* $\mathcal{A}$ *is partially separated, then* $\mathbb{S}_{\mathcal{BC}}$ *is ccc.*

Proof. Let $\{p_\alpha \mid \alpha \in \omega_1\}$ be a set of conditions, and write $p_\alpha = (s_\alpha, \mathcal{F}_\alpha, \mathcal{G}_\alpha)$. Without loss of generality, we may assume that there are $n, m \in \omega$ such that $|\mathcal{F}_\alpha| = n$ and $|\mathcal{G}_\alpha| = m$ for every $\alpha \in \omega_1$. Let us enumerate $\mathcal{F}_\alpha = \{\mathcal{F}_\alpha(i) \mid i < n\}$ and $\mathcal{G}_\alpha = \{\mathcal{G}_\alpha(i) \mid i < m\}$.

Let $\mathcal{B}_0 = \{\mathcal{F}_\alpha(0) \mid \alpha \in \omega_1\}$ and $\mathcal{C}_0 = \{\mathcal{G}_\alpha(0) \mid \alpha \in \omega_1\}$. Since $\mathcal{A}$ is partially separated, there are $Z_0 \in [\omega_1]^{\omega_1}$ and k_0 such that $\mathcal{F}_\alpha(0) \cap \mathcal{G}_\beta(0) \subseteq k_0$ for every $\alpha, \beta \in Z_0$. Now, let $\mathcal{B}_1 = \{\mathcal{F}_\alpha(0) \mid \alpha \in Z_0\}$, $\mathcal{C}_1 = \{\mathcal{G}_\alpha(1) \mid \alpha \in Z_0\}$ and find $Z_1 \in [Z_0]^{\omega_1}$, $k_1 \in \omega$ such that $\mathcal{F}_\alpha(0) \cap \mathcal{G}_\beta(1) \subseteq k_1$ for every $\alpha, \beta \in Z_1$. Repeating this process (mn times) we conclude there is $Z \in [\omega_1]^{\omega_1}$ and k such that $B_\alpha(i) \cap C_\beta(j) \subseteq k$ for every $\alpha, \beta \in Z$ and $i < n$, $j < m$.

For every $\alpha \in Z$, take s'_α such that $(s'_\alpha, \mathcal{F}_\alpha, \mathcal{G}_\alpha) \leqslant (s_\alpha, \mathcal{F}_\alpha, \mathcal{G}_\alpha)$ and $k < |s'_\alpha|$. Naturally, there are $s \in {}^{<\omega}2$ and $\alpha, \beta \in Z$ with the property that $s = s_\alpha = s_\beta$. We claim that $(s, \mathcal{F}_\alpha, \mathcal{G}_\alpha)$ and $(s, \mathcal{F}_\beta, \mathcal{G}_\beta)$ are compatible (and then, so are p_α and p_β). To prove this, we only need to realize that $(s, \mathcal{F}_\alpha \cup \mathcal{F}_\beta, \mathcal{G}_\alpha \cup \mathcal{G}_\beta)$ is a condition, but this is trivial since $k < |s'_\alpha|$. $\square$

We will prove that $\mathbb{R}$-embedabbility implies partial separability next.

Proposition 1.2. *If $\mathcal{A}$ is $\mathbb{R}$-embeddable, then it is partially separated.*

Proof. Let $h : \Psi(\mathcal{A}) \to \mathbb{R}$ witness that $\mathcal{A}$ is $\mathbb{R}$-embeddable and take $\mathcal{B} = \{B_\alpha \mid \alpha \in \omega_1\}$, $\mathcal{C} = \{C_\alpha \mid \alpha \in \omega_1\}$ disjoint subsets of $\mathcal{A}$. Fix D a countable base for $\mathbb{R}$ and for every $\alpha \in \omega_1$, find disjoint $U_\alpha, V_\alpha \in D$ such that $h(B_\alpha) \in U_\alpha$ and $h(C_\alpha) \in V_\alpha$. Choose also $m_\alpha \in \omega$ such that $h[B_\alpha \setminus m_\alpha] \subseteq U_\alpha$ and $h[C_\alpha \setminus m_\alpha] \subseteq V_\alpha$. Now, let $X \in [\omega_1]^{\omega_1}$ be such that there are $U, V \in D$ and m with the property that $U_\alpha = U$, $V_\alpha = V$ and $m_\alpha = m$ for all $\alpha \in X$. It is clear that if $\alpha, \beta \in X$ then $B_\alpha \cap C_\alpha \subseteq m$.

From the above we may conclude even more: note that being $\mathbb{R}$-embeddable is an indestructible property, so an $\mathbb{R}$-embeddable family is actually indestructibly partially separated.

Corollary 1.3. *The following are equivalent,*

1. *$\mathcal{A}$ is potentially $\mathbb{R}$-embeddable,*
2. *$\mathcal{A}$ is potentially indestructibly partially separated,*
3. *$\mathcal{A}$ is potentially normal.*

Proof. By the above comment it follows that 1 implies 2. Clearly 3 implies 1. In order to prove that 2 implies 3, let $\mathbb{P}$ be a ccc forcing such that $1_{\mathbb{P}}$ forces that an AD family $\mathcal{A}$ is indestructibly partially separated. Then, the forcing notions $\mathbb{S}_{\mathcal{BC}}$ will always be ccc (in any ccc extension) so we may iterate them and get a model where $\mathcal{A}$ is normal. □

As a consequence, assuming Martin's Axiom, small almost disjoint families which are potentially normal, are precisely those which are normal already.

Corollary 1.4. *Assume* MA. *Let $\mathcal{A}$ be an AD with $|\mathcal{A}| < \mathfrak{c}$, then $\mathcal{A}$ is potentially normal if and only if $\mathcal{A}$ is normal.*

Proof. Let $\mathcal{A}$ be potentially normal and of size less than $\mathfrak{c}$. We must prove that every $\mathcal{B}, \mathcal{C}$ disjoint subsets of $\mathcal{A}$ can be separated. Since we are assuming MA, it is enough to show that the forcing $\mathbb{S}_{\mathcal{BC}}$ is ccc (because we only need $|\mathcal{B}| + |\mathcal{C}| + \omega$ dense sets to do the job). Now, let $\mathbb{P}$ be a ccc forcing such that $\mathcal{A}$ is partially separated in $V[G]$ for every generic filter $G \subseteq \mathbb{P}$. Note that $\mathbb{S}_{\mathcal{BC}}$ is the same as $\mathbb{S}_{\mathcal{BC}}^{V[G]}$ and since $\mathcal{A}$ is partially separated, then it is ccc in $V[G]$. This implies that $\mathbb{S}_{\mathcal{BC}}$ is ccc in V (since any uncountable antichain in V would still be an uncountable antichain in $V[G]$). □

Assuming MA, we may get another equivalence of potential normality:

Corollary 1.5. *Assume* MA. *$\mathcal{A}$ is potentially normal if and only if $\mathcal{A}$ is indestructibly partially separated.*

Proof. Let $\mathcal{A}$ be potentially normal, let $\mathbb{P}$ be a ccc forcing and $G \subseteq \mathbb{P}$ a generic filter. We must prove that $\mathcal{A}$ is partially separated in $V[G]$. For this it is enough to see that every subfamily of $\mathcal{A}$ of size ω_1 is partially separated. To see this, in $V[G]$ choose $\mathcal{A}' \in [\mathcal{A}]^{\omega_1}$ and since $\mathbb{P}$ is ccc, then there is $\mathcal{A}'' \in V$ a subset of $\mathcal{A}$ of size ω_1 such that $\mathcal{A}' \subseteq \mathcal{A}''$. Since MA is true in V, $\mathcal{A}''$ is $\mathbb{R}$-embeddable, so it is partially separated in $V[G]$, hence so is $\mathcal{A}'$. □

The previous corollary cannot be proved in ZFC, as we will see in Section 3.

2. *n*-Luzin gaps

We start by proving some elementary facts about n-Luzin gaps.

Lemma 2.1. *If $\langle \mathcal{B}_i \colon i < n \rangle$ is an n-Luzin gap ($\mathcal{B}_i = \{B^i_\alpha \mid \alpha \in \omega_1\}$) then, for every $X \in [\omega_1]^{\omega_1}$ and every $k \in \omega$, there are $\alpha, \beta \in X$ such that*

$$\bigcup_{i \neq j} \left(B^i_\alpha \cap B^j_\beta\right) \nsubseteq k.$$

Proof. Let $m \in \omega$ testify that $\langle \mathcal{B}_i \mid i < n \rangle$ is n-Luzin. Without loss of generality $k > m$. First, find $Y \in [X]^{\omega_1}$ such that if $\alpha, \beta \in Y$ and $i < n$, then $B^i_\alpha \cap k = B^i_\beta \cap k$. Take $\alpha, \beta \in Y$ distinct. There are $i \neq j$ such that $B^i_\alpha \cap B^j_\beta \nsubseteq m$, but since $B^i_\alpha \cap k = B^i_\beta \cap k$ and $B^j_\beta \cap B^i_\beta \subseteq m \subseteq k$, B^i_α and B^j_β must intersect above k. □

With the aid of this lemma, we can prove the following:

Lemma 2.2. *If $\mathcal{A}$ is partially separated, then it does not contain n-Luzin gaps for any $n \in \omega$.*

Proof. Let $\mathcal{A}$ be partially separated and take $\{B^n_\alpha\colon n \in \omega\}$ such that $B^i_\alpha \cap B^j_\alpha \subseteq m$ when $i \neq j$. Since $\mathcal{A}$ is partially separated. There are $X \in [\omega_1]^{\omega_1}$ and $k \in \omega$ such that $B^i_\alpha \cap B^j_\alpha \subseteq k$ for all $\alpha, \beta \in X$. Then, by the previous lemma, $\mathcal{A}$ cannot contain n-Luzin gaps. $\square$

Since normal families are partially separated, we immediately conclude:

Corollary 2.3. *If $\mathcal{A}$ contains an n-Luzin gap, then it is not normal.*

Using this, we will be able to give a combinatorial reformulation of potential normality of AD families. First, we will introduce a forcing that makes $\mathcal{A}$ an $\mathbb{R}$-embeddable family. Instead of trying to embed $\Psi(\mathcal{A})$ into $\mathbb{R}$, we will try to embed it into the Cantor space ${}^{\omega}2$, identifying the rational numbers with the eventually 0 functions. It is easy to see that this suffices. Let $\mathcal{R}(\mathcal{A})$ be the set of all $(s, \mathcal{F})$ such that,

1. $s \in {}^{<\omega}\mathbb{Q}$ is injective and $\mathcal{F} \in [\mathcal{A}]^{<\omega}$.
2. If $A, B \in \mathcal{F}$ then $A \cap B \subseteq |s|$.

And $(s, \mathcal{F}) \leqslant (s', \mathcal{F}')$ if,

1. $s' \subseteq s$, $\mathcal{F}' \subseteq \mathcal{F}$.
2. If $i \in \mathrm{dom}(s) \setminus \mathrm{dom}(s')$ and there is $A \in \mathcal{F}'$ such that $i \in A$ and $j = \max\{A \cap \mathrm{dom}(s')\}$ then $\Delta(s(i), s'(j)) \geqslant |s'|$ (where $\Delta(x, y)$ is the first n such that $x(n) \neq y(n)$).

Note that the A is unique since $(s', \mathcal{F}')$ is a condition.

Lemma 2.4. *If $\mathcal{R}(\mathcal{A})$ is ccc, then $\mathcal{A}$ is potentially $\mathbb{R}$-embeddable.*

Proof. Given $n \in \omega$, it is easy to prove that the set $D_n = \{(s, \mathcal{F}) \mid n < |s|\}$ is dense (this is due to the fact that if $A, B \in \mathcal{F}$ then $A \cap B \subseteq \mathcal{F}$, so we may extend the condition $(s, \mathcal{F})$ without changing $\mathcal{F}$). Also, if $A \in \mathcal{A}$ then the set $E_A = \{(s, \mathcal{F}) \mid A \in \mathcal{F}\}$ is dense. Given $(s, \mathcal{F})$ we first find $m \in \omega$ such that $X \cap Y \subseteq m$ for every $X \neq Y \in \mathcal{F} \cup \{A\}$ and then we extend $(s, \mathcal{F})$ to a condition $(s', \mathcal{F})$ such that $m < |s'|$. In this way, $(s', \mathcal{F} \cup \{A\})$ is below $(s, \mathcal{F})$.

Fix G a generic filter for $\mathcal{R}(\mathcal{A})$, we will prove that $\mathcal{A}$ is $\mathbb{R}$-embeddable in $V[G]$. Let $e = \bigcup_{(s,\mathcal{F}) \in G} s$ since the D_n are dense, then e is a function from ω to $\mathbb{R}$. We will show that if $A \in \mathcal{A}$ then $e[A]$ is a convergent sequence. For this, just note that if $A \in \mathcal{F}$ and $A \cap \mathrm{dom}(s) \neq \emptyset$ then $(s, \mathcal{F}) \Vdash$"if $x, y \in A \setminus \mathrm{dom}(s)$, then $\dot{e}(x) \restriction |s| = \dot{e}(y) \restriction |s|$".

Let us call $r_A \in {}^{\omega}2$ the limit of $e[A]$. It remains to be shown that $r_A \neq r_B$ whenever $A \neq B$. Let D_{AB} be the set of those $(s, \mathcal{F})$ that force r_A to be different from r_B. It is enough to show that this set is dense. Take a condition $(s, \mathcal{F})$, without loss of generality, we may assume that $A, B \in \mathcal{F}$ and $A \cap \mathrm{dom}(s)$, $B \cap \mathrm{dom}(s)$ are not empty. Now, it is easy to extend this condition in such a way that r_A and r_B belong to different clopen sets. $\square$

We are finally ready to prove one of the main results of the paper.

Theorem 2.5. *$\mathcal{A}$ is potentially normal if and only if $\mathcal{A}$ does not contain n-Luzin gaps for any $n \in \omega$.*

Proof. If $\mathcal{A}$ contains an n-Luzin gap, then $\mathcal{A}$ still contains it in any forcing extension that preserves ω_1. Hence, we may conclude that $\mathcal{A}$ cannot be potentially normal. So, we only need to prove that if $\mathcal{A}$ does not contain n-Luzin gaps then it is potentially normal, or equivalently that it is potentially $\mathbb{R}$-embeddable. For this, we just need to see that $\mathcal{R}(\mathcal{A})$ is ccc.

Assume this is not the case, then there is a set $\{(s_\alpha, \mathcal{F}_\alpha) \mid \alpha \in \omega_1\}$ of pairwise incompatible conditions. We may assume that there is $s \in {}^{<\omega}\mathbb{R}$ such that $s_\alpha = s$ for all $\alpha \in \omega_1$ and $\{\mathcal{F}_\alpha \mid \alpha \in \omega_1\}$ forms a Δ-system with root R. Note that since $(s, \mathcal{F}_\alpha)$ and $(s, \mathcal{F}_\beta)$ are incompatible so are $(s, \mathcal{F}_\alpha \setminus R)$ and $(s, \mathcal{F}_\beta \setminus R)$. So, we may further assume that R is the empty set and all $\mathcal{F}_\alpha$ are of the same size, say n. We may also assume that if $i < n$ then $\mathcal{F}_\alpha(i) \cap m = \mathcal{F}_\beta(i) \cap m$ for all $\alpha, \beta \in \omega_1$.

Enumerate $\mathcal{F}_\alpha = \{\mathcal{F}_\alpha(i) \mid i < n\}$ and let $\mathcal{B}_i = \{\mathcal{F}_\alpha(i) \mid \alpha \in \omega_1\}$. Note that, since each $(s, \mathcal{F}_\alpha)$ is a condition, then $\mathcal{F}_\alpha(i) \cap \mathcal{F}_\alpha(j) \subseteq m$. Since $\mathcal{A}$ does not contain n-Luzin gaps, there are $\alpha \neq \beta$ such that if $i \neq j$ then $\mathcal{F}_\alpha(i) \cap \mathcal{F}_\beta(j) \subseteq m$. We claim that $(s, \mathcal{F}_\alpha)$ and $(s, \mathcal{F}_\beta)$ are compatible, which will be a contradiction. Note that $(s, \mathcal{F}_\alpha \cup \mathcal{F}_\beta)$ may fail to be a condition, since there could be $A, B \in \mathcal{F}_\alpha \cup \mathcal{F}_\beta$ such that $A \cap B \nsubseteq |s| = m$. However, in this case, A must be of the form $\mathcal{F}_\alpha(i)$ and B must be $\mathcal{F}_\beta(i)$ (because $(s, \mathcal{F}_\alpha)$ and $(s, \mathcal{F}_\beta)$ are conditions and $\mathcal{F}_\alpha(i) \cap \mathcal{F}_\beta(j) \subseteq m$ when $i \neq j$). However, since $\mathcal{F}_\alpha(i)$ and $\mathcal{F}_\beta(i)$ agree up to m, it is easy to extend $(s, \mathcal{F}_\alpha \cup \mathcal{F}_\beta)$ to a condition. $\square$

Evidently, we may conclude,

Corollary 2.6. *If $\mathcal{A}$ is partially separated then it is potentially $\mathbb{R}$-embeddable.*

The reader might wonder why we are only considering ccc extensions instead of extensions preserving ω_1. It turns out both concepts are equivalent, as the next result show,

Corollary 2.7. *$\mathcal{A}$ is potentially normal if and only if there is a forcing notion $\mathbb{P}$ that does not collapse ω_1 and forces $\mathcal{A}$ to be normal.*

Proof. Note that if $\mathcal{A}$ contains an n-Luzin gap, then it still contains an n-Luzin gap in any forcing extension that preserves ω_1, so the result follows by the previous theorem. $\square$

We may also prove the promised result,

Theorem 2.8. *Assume* MA. *Let $\mathcal{A}$ be an AD family. Then $\mathcal{A}$ is normal if and only if $|\mathcal{A}| < \mathfrak{c}$ and $\mathcal{A}$ does not contain n-Luzin gaps for any $n \in \omega$.*

Proof. The forward implication is clear, for the converse, just recall that under MA normality and potential normality are equivalent for families of size less than $\mathfrak{c}$. $\square$

We will show that, under the Proper Forcing Axiom, we may "remove the n" from the previous result. Assume $\mathcal{B} = \{B_\alpha \mid \alpha \in \omega_1\}$, $\mathcal{C} = \{C_\alpha \mid \alpha \in \omega_1\}$ are disjoint subfamilies of $\mathcal{A}$ and let $X = \{(B_\alpha, C_\alpha) \mid \alpha \in \omega_1\}$. For every $m \in \omega$ we define a coloring $c_m : [X]^2 \to 2$ by

$$c_m\big((B_\alpha, C_\alpha), (B_\beta, C_\beta)\big) = 1 \quad \text{iff} \quad (B_\alpha \cap C_\beta) \cup (B_\beta \cap C_\alpha) \subseteq m.$$

We may see X as a subset of the polish space ${}^{\omega}2 \times {}^{\omega}2$, so it carries a natural topology. In this way, note that $c^{-1}(\{0\}) \subseteq X^2$ is an open set. Let us recall the *Open Coloring Axiom* (see [22] and [15]),

OCA) If X is a separable metric space and $c : [X]^2 \to 2$ is such that $c^{-1}(\{0\})$ is open, then one of the following holds,
 $\star$) There is $M \in [X]^{\omega_1}$ that is monochromatic of color 0 (i.e. c restricted to $[M]^2$ is the constant 0).
 $\star\star$) X may be cover by ω-monochromatic sets of color 1.

For us, it will be enough to observe that OCA implies that (given X is uncountable) there is always an uncountable monochromatic set in one of the colors.

The following result is a consequence of Theorem 13.5 of [22]. We present its short proof for the sake of completeness. Later (Theorem 3.8) we will see that it follows also from the P-ideal dichotomy.

Proposition 2.9. *If* OCA *is true, then every almost disjoint family is partially separated or contains a Luzin gap.*

Proof. Assume $\mathcal{A}$ is not partially separated, so there are disjoint subfamilies $\mathcal{B} = \{B_\alpha \mid \alpha \in \omega_1\}$, $\mathcal{C} = \{C_\alpha \mid \alpha \in \omega_1\}$ of $\mathcal{A}$ such that for every $Y \in [\omega_1]^{\omega_1}$ and $n \in \omega$, there are $\alpha, \beta \in Y$ with the property that $B_\alpha \cap C_\beta \nsubseteq n$. We may assume there is $m \in \omega$ such that $B_\alpha \cap C_\alpha \subseteq m$ for all $\alpha \in \omega_1$.

Let X, and c_m be defined as above. The previous remark tells us that there are no uncountable 1-monochromatic sets, so OCA implies the existence of an uncountable 0-monochromatic set Y. Clearly $\{B_\alpha \mid \alpha \in Y\}$, $\{C_\alpha \mid \alpha \in Y\}$ is a Luzin gap. $\square$

Corollary 2.10. *Assume* PFA. *Let $\mathcal{A}$ be an AD family. Then $\mathcal{A}$ is normal if and only if $|\mathcal{A}| < \mathfrak{c}$ and $\mathcal{A}$ does not contain Luzin gaps.*

We do not know whether a version of PFA (OCA) is necessary for the conclusion of the corollary,

Question 2.11. *Does the previous corollary hold assuming* MA*?*

It cannot be proved in ZFC, since it is consistent with MA(σ-centered) that there are 3-Luzin gaps that do not contain Luzin gaps. In order to prove this, first recall that a family $\mathcal{D} \subseteq \mathcal{P}(\omega)$ is *independent* if for any distinct $A_0, \ldots, A_n$, $B_0, \ldots, B_m \in \mathcal{D}$ the set $A_0 \cap \cdots \cap A_n \cap (\omega \setminus B_0) \cap \cdots \cap (\omega \setminus B_m)$ is infinite. We say that $\mathcal{D}$ *separates points* if for every distinct $n, m \in \omega$, there is $D \in \mathcal{D}$ such that $\{n, m\} \cap D$ has size 1.

Given $\mathcal{D}$ an independent family that separates points, we define the topological space $(\omega, \tau_{\mathcal{D}})$ which has $\mathcal{D} \cup \{\omega - D \mid D \in \mathcal{D}\}$ as a subbase.

Lemma 2.12. *(ω, τ_D) is homeomorphic to the rationals with the usual topology.*

Proof. This space is countable, first countable, zero-dimensional without isolated points, and this characterizes $\mathbb{Q}$ (this is an old result of Sierpiński, see [12]). $\square$

To construct our 3-Luzin gap, we will first construct (in ZFC) a special type of a Luzin gap, which is interesting on its own,

Lemma 2.13. *There is a Luzin gap $\mathcal{B} = \{B_\alpha \mid \alpha \in \omega_1\}$, $\mathcal{C} = \{C_\alpha \mid \alpha \in \omega_1\}$ such that $\mathcal{B}$ and $\mathcal{C}$ are $\mathbb{R}$-embeddable.*

Proof. Let $\mathcal{D} = \{D_n \mid n \in \omega\}$ and $\mathcal{E} = \{E_n \mid n \in \omega\}$ be disjoint families such that both separate points and $\mathcal{D} \cup \mathcal{E}$ is an independent family. As was remarked above, $(\omega, \tau_\mathcal{D})$ and $(\omega, \tau_\mathcal{E})$ are both homeomorphic to the rationals, and every open set of one topology is dense in the other. Identifying ω with $\mathbb{Q}$, we may view $\mathbb{R}$ as the metric completion of $(\omega, \tau_\mathcal{D})$ and $(\omega, \tau_\mathcal{E})$. Pick $\{r_\alpha \mid \alpha \in \omega_1\}$ a set of distinct irrationals, we will recursively build $\mathcal{B}$ and $\mathcal{C}$ such that,

1. In $(\omega, \tau_\mathcal{D})$, B_α is a convergent sequence to r_α and it is dense in $(\omega, \tau_\mathcal{E})$.
2. In $(\omega, \tau_\mathcal{E})$, C_α is a convergent sequence to r_α and it is dense in $(\omega, \tau_\mathcal{D})$.
3. $B_\alpha \cap C_\alpha = \emptyset$ while $B_\alpha \cap C_\beta$, $B_\beta \cap C_\alpha$ are non-empty finite sets for every $\beta < \alpha$.

It is clear that if the recursion could be carried out, we would have constructed the desired family. Assume B_ξ, C_ξ had been constructed for every $\xi < \alpha$, let's find B_α and C_α. Let $\{U_n \mid n \in \omega\}$ be a local base for r_α with $U_0 \supseteq U_1 \supseteq U_2 \supseteq \cdots$ in $(\omega, \tau_\mathcal{D})$ and $\{V_n \mid n \in \omega\}$ a base in $(\omega, \tau_\mathcal{E})$. Enumerate $\alpha = \{\xi_n \mid n \in \omega\}$ and we recursively build $B_\alpha = \{x_n \mid n \in \omega\} \cup \{y_n \mid n \in \omega\}$ such that:

1. $x_n, y_n \in U_n$,
2. $x_n \in V_n \setminus \bigcup_{m<n} C_{\xi_m}$,
3. $y_n \in C_{\xi_n} \setminus \bigcup_{m<n} C_{\xi_m}$.

It is easy to do that, since each U_n is dense in $(\omega, \tau_\mathcal{E})$ and all the V_n and C_{ξ_n} are dense in $(\omega, \tau_\mathcal{D})$. C_α is built in the same way, just making sure for it to be disjoint with B_α. $\square$

Proposition 2.14. MA(σ*-centered*) *is consistent with the existence of a* 3*-Luzin gap without Luzin subgaps.*

Proof. We will see that there is such a family after adding a Cohen real. Let $\mathcal{B} = \{B_\alpha \mid \alpha \in \omega_1\}$, $\mathcal{C} = \{C_\alpha \mid \alpha \in \omega_1\}$ be a Luzin gap with $B_a \cap C_\alpha = \emptyset$ such that both $\mathcal{B}$ and $\mathcal{C}$ are $\mathbb{R}$-embeddable. Assume D is a Cohen real. In $V[D]$, define $\mathcal{B}_1 = \{B_\alpha \cap D \mid \alpha \in \omega_1\}$, $\mathcal{B}_2 = \{B_\alpha \setminus D \mid \alpha \in \omega_1\}$, we will prove that $\langle \mathcal{B}_1, \mathcal{B}_2, \mathcal{C} \rangle$ is the family we are looking for. It is easy to see that it is indeed a 3-Luzin gap, so it remains to show that it has no Luzin gaps.

In $V[D]$, let $m \in \omega$ and $\mathcal{X} = \{X_\alpha \mid \alpha \in \omega_1\}$, $\mathcal{Y} = \{Y_\alpha \mid \alpha \in \omega_1\}$ be disjoint subfamilies of $\mathcal{A}$ such that $X_\alpha \cap Y_\alpha \subseteq m$. We may assume $\mathcal{X}$ is a subset of $\mathcal{B}_1, \mathcal{B}_2$ or $\mathcal{C}$ (similarly for $\mathcal{Y}$). However, since $\mathcal{B}$ and $\mathcal{C}$ are $\mathbb{R}$-embeddable and every member of $\mathcal{B}_1$ is disjoint from every member of $\mathcal{B}_2$, then we only need to consider the case where $\mathcal{X}$ is a subset of $\mathcal{B}_1$ or $\mathcal{B}_2$ and $\mathcal{Y}$ is a subset of $\mathcal{C}$. For concreteness, we will assume $\mathcal{X} \subseteq \mathcal{B}_1$, while the other case is similar. Find a function $h : \omega_1 \to \omega_1 \times \omega_1$ such that $X_\alpha = B_{h(\alpha)_0} \cap D$ and $Y_\alpha = C_{h(\alpha)_1}$. We know there is an uncountable $W \in V$ and $s \in \mathbb{C}$ such that s knows $h \restriction W$, we may assume $m < |s| = l$. Let $\alpha, \beta \in W$ distinct such that $B_{h(\alpha)_0} \cap l = B_{h(\beta)_0} \cap l$ and $C_{h(\alpha)_1} \cap l = C_{h(\beta)_1} \cap l$. Let $r > l$ such that $B_{h(\alpha)_0} \cap C_{h(\beta)_1}, B_{h(\beta)_0} \cap C_{h(\alpha)_1} \subseteq r$ and choose s' any extension of s such that $r < |s'|$ and if $x \in \operatorname{dom}(s') \setminus \operatorname{dom}(s)$ then $s'(x) = 0$. In this way, s' forces $X_\alpha \cap Y_\beta$, $X_\beta \cap Y_\alpha \subseteq m$ so $(\mathcal{X}, \mathcal{Y})$ is not a Luzin gap.

To finish the proof, assume MA holds in V, then MA(σ-centered) is still true after adding a Cohen real (by a theorem of Roitman, see [6, Theorem 3.3.8]). $\square$

3. Schizophrenic AD families

Recall that $\mathcal{A}$ is *inseparable* if for every $\mathcal{B}, \mathcal{C} \in [\mathcal{A}]^{\omega_1}$ the set $\bigcup \mathcal{B} \cap \bigcup \mathcal{C}$ is infinite or equivalently, for every $m \in \omega$ there are $B \in \mathcal{B}$ and $C \in \mathcal{C}$ such that $B \cap C \nsubseteq m$. Clearly, every uncountable subfamily of an inseparable family is inseparable and $\mathcal{A}$ is inseparable if and only if all of its subfamilies of size ω_1 are inseparable.

Let us introduce a forcing aiming to add a Luzin family to a given AD family $\mathcal{A}$. Assume that $\mathcal{A} = \{A_\alpha \mid \alpha \in \omega_1\}$ and for every $p \in [\omega_1]^{<\omega}$ let m_p be the smallest integer such that $A_\alpha \cap A_\beta \subseteq m_p$ for all $\alpha, \beta \in p$ distinct. We define the poset $\mathcal{SR}(\mathcal{A}) = [\omega_1]^{<\omega}$ (see [17]) and we say $p \leqslant q$ if and only if,

1. $q \subseteq p$,
2. If $\alpha \in p \setminus q$ and there is $\beta \in q$ with $\alpha < \beta$, then $A_\beta \cap A_\alpha \nsubseteq m_q$.

Lemma 3.1. *([17]) If $\mathcal{SR}(\mathcal{A})$ is ccc, then $\mathcal{A}$ potentially contains a Luzin family.*

Proof. For every $\alpha \in \omega_1$ define $D_\alpha = \{p \mid p \nsubseteq \alpha\}$. It is easy to see that this set is dense, since if $p \subseteq \alpha$ then $p \cup \{\alpha\} \leqslant p$. Let G be a generic filter and in $V[G]$ define $\mathfrak{B} = \{A_\alpha \mid \alpha \in \bigcup G\}$, then $\mathfrak{B}$ is uncountable (since the forcing is ccc) and it is easy to see that it is indeed a Luzin family. □

Using this result, we may obtain the following characterization due to Roitman and Soukup [17].

Proposition 3.2. *([17]) $\mathcal{A}$ is inseparable if and only if every uncountable subfamily of $\mathcal{A}$ potentially contains a Luzin family.*

Proof. First, assume every uncountable subfamily of $\mathcal{A}$ potentially contains a Luzin family. Let $\mathcal{B}, \mathcal{C}$ be uncountable subfamilies of $\mathcal{A}$ and define $\mathcal{A}' = \mathcal{B} \cup C$. We know there is $\mathbb{P}$ a ccc forcing such that $1_{\mathbb{P}}$ forces that $\mathcal{A}'$ contains a Luzin family. Aiming for a contradiction, assume there is $m \in \omega$ such that $B \cap C \subseteq m$ for every $B \in \mathcal{B}$ and $C \in \mathcal{C}$. Let $G \subseteq \mathbb{P}$ be a generic filter and in $V[G]$ find $\mathcal{D} = \{X_\alpha \mid \alpha \in \omega_1\} \subseteq \mathcal{A}'$ be a Luzin family. Clearly, there is $\alpha \in \omega_1$ such that $X_\alpha \in \mathcal{B}$ and $\{X_\xi \mid \xi < \alpha\} \cap \mathcal{C}$ is infinite, but then the set $\{\xi < \alpha \mid X_\alpha \cap X_\xi \subseteq m\}$ is infinite, which contradicts that $\mathcal{D}$ is a Luzin family.

For the other implication, it is enough to prove that if $\mathcal{A}$ is inseparable of size ω_1, then $\mathcal{SR}(\mathcal{A})$ is ccc. We will proceed by contradiction, suppose $\{p_\alpha \mid \alpha \in \omega_1\}$ is an antichain, we may assume it forms a Δ-system with root r, every $p_\alpha \setminus r$ has size n and there is $m \in \omega$ such that $m_{p_\alpha} = m$ for all $\alpha \in \omega_1$. Furthermore, thinning our family, we may assume that for all α, every member of r is below every member of $p_\alpha \setminus r$ and if $\alpha < \beta$, then every member of $p_\alpha \setminus r$ is below every member of $p_\beta \setminus r$. Write $p_\alpha \setminus r = \{p_\alpha(i) \mid i < n\}$ and we may suppose there is $k > m$ such that $p_\alpha(i) \cap (k \setminus m) \neq \emptyset$ for all $\alpha \in \omega_1$. Thinning our family again, we may assume $p_\alpha(i) \cap k = p_\beta(i) \cap k$ for all $\alpha, \beta \in \omega_1$.

We will now see that there are $X_0, Y_0 \in [\omega_1]^{\omega_1}$ such that if $\alpha \in X_0$ and $\beta \in Y_0$ then $p_\alpha(0) \cap p_\beta(1) \nsubseteq m$. Suppose this is false, then for every $x > m$, at least one of the following sets $B_x = \{\alpha \mid x \in p_\alpha(0)\}$, $C_x = \{\alpha \mid x \in p_\alpha(1)\}$ is countable (and they are disjoint, since x is bigger than m). Let $\mathcal{B}$ be the set of all the $p_\alpha(0)$ such that $\alpha \notin \bigcup_{|B_x| \leqslant \omega} B_x$ and $\mathcal{C}$ be the set of all $p_\alpha(1)$ such that $\alpha \notin \bigcup_{|C_x| \leqslant \omega} C_x$. In this way, $\mathcal{B}$ and $\mathcal{C}$ are two uncountable subfamilies of $\mathcal{A}$. However, if $B \in \mathcal{B}$ and $C \in \mathcal{C}$ then $B \cap C \subseteq m$, which contradicts that $\mathcal{A}$ was inseparable.

Repeating this process several times, we find there are $X, Y \in [\omega_1]^{\omega_1}$ such that if $\alpha \in X$ and $\beta \in Y$ then $p_\alpha(i) \cap p_\beta(j) \nsubseteq m$ when $i \neq j$. However, we already knew that $p_\alpha(i) \cap p_\beta(i) \nsubseteq m$, since $p_\alpha(i) \cap k = p_\beta(i) \cap k$ and $p_\alpha(i) \cap (k \setminus m) \neq \emptyset$. This implies that $p_\alpha \cup p_\beta$ is a common extension of p_α and p_β, which is a contradiction. □

Definition 3.3. We say that an AD family $\mathcal{A}$ is *schizophrenic* if it is inseparable and contains no n-Luzin gaps for any $n \in \omega$.

Recall that by Theorem 2.5 an AD family which contains no n-Luzin gaps for any $n \in \omega$ is potentially normal, hence there is a ccc forcing that makes it normal. On the other hand, by the previous result if $\mathcal{A}$ is inseparable there is another ccc forcing one that freezes it by adding a Luzin gap, so it become indestructibly not normal!

Corollary 3.4. *If $\mathcal{A}$ is schizophrenic, then $\mathcal{R}(\mathcal{A})$ and $\mathcal{SR}(\mathcal{A})$ are two ccc forcings such that $\mathcal{R}(\mathcal{A}) \times \mathcal{SR}(\mathcal{A})$ is not ccc.*

In particular, MA implies that there are no schizophrenic families (another way to prove this, is to remember that MA implies that potentially normal entails indestructibly partially separated, and partially separated families does not contain Luzin gaps).

We will now show that the existence of schizophrenic families is consistent with ZFC.

We will show that the existence of schizophrenic AD families is independent of the Martin Axiom for σ-centered partial orders and from CH (note that this is exactly the same situation with Suslin trees). We will use the Cohen forcing $\mathbb{C} = {}^{<\omega}2$.

Lemma 3.5. *If $\dot{A}$ is a $\mathbb{C}$ name for an uncountable subset of ordinals, then there is $s \in \mathbb{C}$ and $X \in V$ uncountable such that $s \Vdash$ "$X \subseteq \dot{A}$". In other words, any new uncountable set of ordinals contains an old uncountable set of ordinals.*

Proof. For every $s \in \mathbb{C}$, let $A_s = \{a \mid s \Vdash$ "$a \in A$"$\}$. Clearly, if $G \subseteq \mathbb{C}$ is generic, then $A = \bigcup_{s \in G} A_s$ and since A is uncountable, then one of the A_s must be uncountable. □

Now we will prove,

Theorem 3.6. *The existence of a schizophrenic family is consistent with* MA*(σ-centered).*

Proof. Let $\mathcal{A} = \{A_\alpha \mid \alpha \in \omega_1\}$ be an inseparable family (take a Luzin family, for example) and let $D \subseteq \omega$ be a Cohen real over V. In $V[D]$ define $\mathcal{A} \restriction \mathcal{D}$ to be the set of all $A_\alpha \cap D$ with $\alpha \in \omega_1$. We will show that this is a schizophrenic family (note first that $\mathcal{A} \restriction \mathcal{D}$ is still an almost disjoint family).

Let us see that it is inseparable. In $V[D]$, let $\mathcal{B} = \{B_\alpha \mid \alpha \in \omega_1\}$, and $\mathcal{C} = \{C_\alpha \mid \alpha \in \omega_1\}$ be uncountable subfamilies of $\mathcal{A}$. In this way, we may define $h : \omega_1 \to \omega_1 \times \omega_1$ in such a way that $B_\alpha = A_{h(\alpha)_0} \cap D$ and $C_\alpha = A_{h(\alpha)_1} \cap D$ where $h(\alpha) = (h(\alpha)_0, h(\alpha)_1)$. By the previous lemma, there is $s \in \mathbb{C}$ and $X \in [\omega_1]^{\omega_1}$ (in V) such that s knows $h \restriction X$. We will find an extension of s that forces what we need.

Fix $m \in \omega$, we need to show that there are $\alpha, \beta \in \omega_1$ such that $B_\alpha \cap C_\beta = (A_{h(\alpha)_0} \cap A_{h(\beta)_1}) \cap D$ is not contained in m. Let $\mathcal{B}' = \{A_{h(\alpha)_0} \mid \alpha \in X\}$ and $\mathcal{C}' = \{A_{h(\alpha)_1} \mid \alpha \in X\}$ since $\mathcal{A}$ is inseparable, there are $\alpha, \beta \in X$ and $k > m, |s|$ such that $k \in A_{h(\alpha)_0} \cap A_{h(\beta)_1}$. If s' is any extension of s such that $s'(k) = 1$, then $s' \Vdash$ "$k \in B_\alpha \cap C_\beta$" and we are done.

Now, we will prove that $\mathcal{A} \restriction \mathcal{D}$ contains no n-Luzin gaps for any $n \in \omega$. Let $n \in \omega$ and assume for every $i < n$ we have $\langle B^i_\alpha \mid \alpha \in \omega_1\rangle$ subfamilies of $\mathcal{A} \restriction \mathcal{D}$ such that there is $m \in \omega$ with the property that $B^i_\alpha \cap B^j_\alpha \subseteq m$ whenever $i \neq j$. As before, define a function $h : \omega_1 \to \omega_1^n$ such that $B^i_\alpha = A_{h(\alpha)_i} \cap D$ (with the same notation as before). Find $s \in \mathbb{C}$ that forces all of this, and an uncountable $X \in V$ such that s knows $h \restriction X$. Let $l = |s|$ and we may assume $m < l$.

Find $\alpha, \beta \in X$ distinct such that $A_{h(\alpha)_i} \cap l = A_{h(\beta)_i} \cap l$ for all $i < n$. Note that if $i \neq j$ then $A_{h(\alpha)_i} \cap A_{h(\beta)_j} \cap l = A_{h(\alpha)_i} \cap A_{h(\alpha)_j} \cap l \subseteq m$. Let $r > l$ such that $A_{h(\alpha)_i} \cap A_{h(\beta)_j} \subseteq r$ when $i \neq j$. Choose s' any extension of s such that $r < |s'|$ and if $x \in \mathrm{dom}(s') \setminus \mathrm{dom}(s)$ then $s'(x) = 0$. In this way, s' forces $B^i_\alpha \cap B^j_\beta \subseteq m$ for all $i \neq j$, so it is not an n-Luzin gap.

Again, by Roitman's theorem [6, Theorem 3.3.8], if we start with a model of MA then MA(σ-centered) holds in the extension. □

Inspired by the construction of a Suslin tree under $\diamondsuit$, we will construct a schizophrenic family under this axiom. Given $\mathcal{A} = \{A_\alpha \mid \alpha \in \omega_1\}$ we defined the poset $\mathcal{R}(\mathcal{A})$ which elements are pairs of the form $(s, \mathcal{F})$ with $\mathcal{F} \in [\mathcal{A}]^{<\omega}$. Evidently, we could instead define $\mathcal{R}(\mathcal{A})$ as pairs $(z, \mathcal{G})$ with $\mathcal{G} \in [\omega_1]^{<\omega}$. We use this reformulation in the next theorem.

Theorem 3.7. *$\diamondsuit$ implies the existence of a schizophrenic family.*

Proof. Using $\diamondsuit$ we will construct an inseparable $\mathcal{A}$ such that $\mathcal{R}(\mathcal{A})$ is ccc. Fix two sequences $\mathcal{D}_1 = \langle (X_\alpha, Y_\alpha) \mid \alpha \in \omega_1\rangle$ and $\mathcal{D}_2 = \langle D_\alpha \mid \alpha \in \omega_1\rangle$ such that X_α and Y_α are disjoint subsets of α and D_α is a collection of finite subsets of $\mathbb{Q}^{<\omega} \times [\alpha]^{<\omega}$. The idea is that $\mathcal{D}_1$ guesses pairs of disjoints subsets of ω_1 and $\mathcal{D}_2$ guesses subsets of $\mathbb{Q}^{<\omega} \times [\omega_1]^{<\omega}$. More precisely, if X, Y are disjoint subsets of ω_1 then there are stationary many α such that $(X \cap \alpha, Y \cap \alpha) = (X_\alpha, Y_\alpha)$ and if $\mathcal{B}$ is collection of finite subsets of $\mathbb{Q}^{<\omega} \times [\omega_1]^{<\omega}$ then there are stationary many α such that $\mathcal{B} \cap (\mathbb{Q}^{<\omega} \times [\alpha]^{<\omega}) = D_\alpha$. We will use $\mathcal{D}_1$ to get the inseparability and $\mathcal{D}_2$ to show that $\mathcal{R}(\mathcal{A})$ is ccc.

Recursively construct $\mathcal{A} = \{A_\alpha \mid \alpha \in \omega_1\}$ such that (denoting $\mathcal{A}_{<\alpha} = \{A_\xi \mid \xi < \alpha\}$ and $\mathcal{A}_{\leqslant\alpha} = \{A_\xi \mid \xi \leqslant \alpha\}$) if $\beta < \alpha$,

1. If $\mathcal{D}_\beta$ is a maximal antichain in $\mathcal{R}(\mathcal{A}_{<\beta})$ then it is still a maximal antichain in $\mathcal{R}(\mathcal{A}_{\leqslant\alpha})$.
2. If X_β and Y_β are infinite then for any $n \in \omega$ there are $\xi \in X_\beta$ and $\eta \in Y_\beta$ such that $A_\alpha \cap A_\xi$ and $A_\alpha \cap A_\eta$ are not contained in n.

We will first see that if the above construction can be carried out, then $\mathcal{A}$ is a schizophrenic family. We will prove the inseparability first. Assume X, Y are disjoint uncountable subsets of ω_1 and $n \in \omega$, we will see there are $\xi \in X$ and $\eta \in Y$ such that $A_\xi \cap A_\eta \nsubseteq n$. For the assumption of $\mathcal{D}_1$, there is β such that $(X \cap \beta, Y \cap \beta) = (X_\beta, Y_\beta)$ and X_β, Y_β are infinite. Since X is uncountable, there is $\alpha \in X$ such that $\beta < \alpha$. Using 2, there is $\eta \in Y_\beta \subseteq Y$ such that $A_\alpha \cap A_\eta \nsubseteq n$.

To prove that $\mathcal{R}(\mathcal{A})$ is ccc, let $\mathcal{B} \subseteq \mathcal{R}(\mathcal{A})$ be a maximal antichain. The set of β such that $\mathcal{B} \cap (\mathbb{Q}^{<\omega} \times [\beta]^{<\omega})$ is a maximal antichain in $\mathcal{R}(\mathcal{A}_{<\beta})$ contains a closed and unbounded set. So, there is a β such that $\mathcal{B} \cap (\mathbb{Q}^{<\omega} \times [\beta]^{<\omega}) = D_\beta$ and D_β is a maximal antichain of $\mathcal{R}(\mathcal{A}_{<\beta})$. Then using 1, we conclude that D_β is a maximal antichain in $\mathcal{R}(\mathcal{A})$ so $\mathcal{B} = D_\beta$ hence $\mathcal{B}$ is countable.

It remains to be seen that the construction can be carried out. Assume we have constructed everything up to $\alpha < \omega_1$.

Let $L_1 = \{\beta_n \mid n \in \omega\}$ enumerate the set of all $\beta < \alpha$ such that X_β and Y_β are infinite, also define $L_2 = \{\gamma_n \mid n \in \omega\}$ as the set of all $\gamma < \alpha$ such that $\mathcal{D}_\gamma$ is an infinite maximal antichain in $\mathcal{R}(\mathcal{A}_{<\gamma})$ and let $\mathcal{R}(\mathcal{A}_{<\alpha}) = \{(s_n, \mathcal{F}_n) \mid n \in \omega\}$, it will also be convenient to list $\alpha = \{\delta_n \mid n \in \omega\}$. Furthermore, we may assume that for every $\beta \in L_1, \gamma \in L_2, (s, F) \in \mathcal{R}(\mathcal{A}_{<\alpha})$ and $\delta \in \alpha$ there is $n \in \omega$ such that $\beta_n = \beta, \gamma_n = \gamma, (s_n, F_n) = (s, F)$ and $\delta_n = \delta$. For any $n \in \omega$ we will recursively define $\mathcal{P}_n$, m_n and A^n_α such that:

1. $\mathcal{P}_n \in [\mathcal{A}_{<\alpha}]^{<\omega}$, $m_n \in \omega$ and $A^n_\alpha \subseteq m_n$.
2. If $k < n$ then $\mathcal{P}_k \subseteq \mathcal{P}_n$, $m_k < m_n$ and $A^k_\alpha \sqsubseteq A^n_\alpha$ (where $\sqsubseteq$ denotes end extension).
3. If $A_\xi \in \mathcal{P}_k$ and $k < n$ then $A_\xi \cap A^n_\alpha \subseteq A^k_\alpha$.
4. $A_{\delta_n} \in \mathcal{P}_n$.
5. For every $n \in \omega$ there are $\xi \in X_{\beta_n}$ and $\eta \in Y_{\beta_n}$ such that $A^n_\alpha \cap A_\xi$, $A^n_\alpha \cap A_\eta \nsubseteq n$.
6. For every $n \in \omega$ either there is $B \in \mathcal{F}_n$ such that $B \cap A^n_\alpha \nsubseteq |s_n|$ or there is $(z, \mathcal{G}) \in \mathcal{R}(\mathcal{A}_{<\gamma_n})$ and $r \in^{<\omega} \mathbb{Q}$ with the property that:
   a) $s_n, z \subseteq r$ and $|r| \leqslant m_n$.
   b) If $B \in \mathcal{F}_n$ and $C \in \mathcal{G}$ then $B \cap C \subseteq |r|$.
   c) If $\xi \in \mathcal{G}$ then $A_\xi \in \mathcal{P}_n$ and $\mathcal{F} \subseteq \mathcal{P}_n$.

The idea is that A^n_α are finite approximations of A_α so at the end we will define $A_\alpha = \bigcup A^n_\alpha$. $\mathcal{P}_n$ are the elements we "promise" not to intersect anymore. In point 6 we are making sure that either $(s_n, \mathcal{F}_n \cup \{\alpha\})$ is not a condition of $\mathcal{R}(\mathcal{A}_{\leqslant\alpha})$

or it is compatible with an element of $\mathcal{R}(\mathcal{A}_{<\gamma_n})$ since if $(z,\mathcal{G}), r$ satisfy the conditions from above then $(r, \mathcal{G}\cup\mathcal{F}\cup\{\alpha\})$ will be a common extension of $(z,\mathcal{G})$ and $(s_n, \mathcal{F}_n\cup\{\alpha\})$.

Assume we have defined everything up to n, let's define $\mathcal{P}_n$, A^n_α and m_n. Let $\mathcal{P}'_n = \bigcup_{i<n}\mathcal{P}_i\cup\{A_{\delta_n}\}$, $A^{n\prime}_\alpha = \bigcup_{i<n} A^i_\alpha$. First, we find $\xi\in X_{\beta_n}$ and $\eta\in Y_{\beta_n}$ such that $A_\xi, A_\eta\notin\mathcal{P}'_n$ (this may be done easily since X_{β_n}, Y_{β_n} are infinite and $\mathcal{P}'_n$ is finite). Define $A^{n\prime\prime}_\alpha$ an end extension of $A^{n\prime}_\alpha$ such that $A^{n\prime\prime}_\alpha\cap A_\xi$, $A^{n\prime\prime}_\alpha\cap A_\eta\nsubseteq n$ and with the property that $A^{n\prime\prime}_\alpha\cap A_\nu\subseteq A^{n\prime}_\alpha$ for all $A_\nu\in\mathcal{P}'_n$. We must now take care of point 6. Take $(s_n,\mathcal{F}_n)$ and if there is $B\in\mathcal{F}_n$ such that $B\cap A^{n\prime\prime}_\alpha\nsubseteq|s_n|$ we just let $A^n_\alpha = A^{n\prime\prime}_\alpha$, $\mathcal{P}_n=\mathcal{P}'_n\cup\mathcal{F}_n$ and $m_n>\max(A^n_\alpha)$ and we are done. Moreover, if there is $B\in\mathcal{F}_n$ such that $B\notin\mathcal{P}'_n$ we just take A^n_α and end extension of $A^{n\prime\prime}_\alpha$ such that $B\cap A^n_\alpha\nsubseteq|s_n|$ and we define $\mathcal{P}_n=\mathcal{P}'_n\cup\mathcal{F}_n$ and $m_n>\max(A^n_\alpha)$ and the requirements are fulfilled.

So assume $B\cap A^{n\prime\prime}_\alpha\subseteq|s_n|$ for all $B\in\mathcal{F}_n$ and $\mathcal{F}_n\subseteq\mathcal{P}_n$. To simplify the notation name $s=s_n$ and $\mathcal{F}=\mathcal{F}_n$. With out loosing generality, we may assume $A^{n\prime\prime}_\alpha\subseteq|s|$ (if not, just extend s). Naturally, there is $\gamma_n\leqslant\nu<\alpha$ such that $(s,\mathcal{F})\in\mathcal{R}(\mathcal{A}_{\leqslant\nu})$ so by our recursion hypothesis, there is $(z,\mathcal{G})\in\mathcal{R}(\mathcal{A}_{<\gamma_n})$ that is compatible with $(s,\mathcal{F})$. Let r be such that $(r,\mathcal{F}\cup\mathcal{G})$ is a common extension and define $\mathcal{P}_n=\mathcal{P}'_n\cup\mathcal{G}$, $m_n>\max\{A^n_\alpha,|r|\}$ and $A^n_\alpha = A^{n\prime\prime}_\alpha$ and we are finally done. □

In particular, there may be schizophrenic families in models of CH, however we will now show that the continuum hypothesis is not sufficient for the existence of a schizophrenic family. This will be done with the aid of the *P-ideal dichotomy* (see [2] and [15]). Recall than an ideal $\mathcal{I}$ is a P-ideal if for every $\{Y_n \mid\in\omega\}\subseteq\mathcal{I}$ there is a $Y\in\mathcal{I}$ that contains mod fin every Y_n.

PID) If $\mathcal{I}\subseteq[\omega_1]^{\leqslant\omega}$ is a P-ideal then one of the following holds,
 ⋆) There is $X\in[\omega_1]^{\omega_1}$ such that $[X]^\omega\subseteq\mathcal{I}$.
 ⋆⋆) There is a partition $\omega_1=\bigcup S_n$ such that $[S_n]^\omega\cap\mathcal{I}=\emptyset$ for every $n\in\omega$.

PID is known to be consistent with CH (see [2, Section 3]). Given $\mathcal{B}=\{B_\alpha\mid\alpha\in\omega_1\}$ and $\mathcal{C}=\{C_\alpha\mid\alpha\in\omega_1\}$ we call $X\subseteq\omega_1$ a *partial separator* if there is $m\in\omega$ such that if $\alpha,\beta\in X$ then $B_\alpha\cap C_\beta\subseteq n$. Therefore, $\mathcal{A}$ is partially separated if any two disjoint uncountable subsets have an uncountable partial separator. Applying the same ideas as in [2, Theorem 2.2] we prove:

Theorem 3.8. *If* PID *holds and $\mathcal{A}$ is an AD of size ω_1, then either $\mathcal{A}$ is partially separated or it contains a Luzin gap.*

Proof. Let $\mathcal{B}=\{B_\alpha\mid\alpha\in\omega_1\}$ and $\mathcal{C}=\{C_\alpha\mid\alpha\in\omega_1\}$ be two disjoint subsets of $\mathcal{A}$. We will prove that either there is a Luzin gap contained in $\mathcal{B},\mathcal{C}$ or they have an uncountable partial separator. Given $X\subseteq\omega_1$ and $n\in\omega$ define

$$X_n(\alpha)=\bigl\{\xi\in X\cap\alpha \bigm| (B_\xi\cap C_\alpha)\cup(B_\alpha\cap C_\xi)\subseteq n\bigr\}.$$

Now, let $\mathcal{I}$ be the set of all $X\in[\omega_1]^{\leqslant\omega}$ such that if $\alpha\leqslant\sup(X)$ and $n\in\omega$, then $X_n(\alpha)$ is finite. It is easy to see that $\mathcal{I}$ is an ideal and for the moment assume it is a P-ideal. Using PID either there is an uncountable $X\subseteq\omega_1$ such that $[X]^\omega\subseteq\mathcal{I}$ or there is an uncountable S such that $[S]^\omega\cap\mathcal{I}=\emptyset$. We will see that if the first option holds, then $\mathcal{B},\mathcal{C}$ contains a Luzin gap and if the second then we get an uncountable partial separator. Without loss of generality, we may assume that there is $n\in\omega$ such that $B_\alpha\cap C_\alpha\subseteq n$ for all $\alpha\in\omega_1$.

Assume first that there is $X\in[\omega_1]^{\omega_1}$ such that $[X]^\omega\subseteq\mathcal{I}$ and define $h: X\to[X]^{<\omega}$ by $h(\alpha)=X_n(\alpha)\subseteq\alpha$. By a standard use of the pressing down lemma, there is $S\subseteq X$ stationary (stationary in X, not necessary in ω_1) such that h is constant on S. It is immediate that $\{B_\alpha\mid\alpha\in S\}$, $\{C_\alpha\mid\alpha\in S\}$ form a Luzin gap.

Now assume there is an uncountable S such that $[S]^\omega\cap\mathcal{I}=\emptyset$, we want to show that S contains an uncountable partial separator for $\mathcal{B}$ and $\mathcal{C}$. Assume this is not the case, let $M\subseteq S$ be a maximal partial separator, so it is countable. Pick $\gamma\in S$ such that $M\subseteq\gamma$. Since $\gamma\notin M$ then there is $\alpha_m\in M$ such that $(B_{\alpha_m}\cap C_\gamma)\cup(B_\gamma\cap C_{\alpha_m})$ is not contained in m. Let $X=\{\alpha_m\mid m\in\omega\}\cup\{\gamma\}\subseteq S$, since $X\notin\mathcal{I}$ then there is $m\in\omega$ such that $X_m(\gamma)$ is infinite, but this is clearly a contradiction. So we conclude there must be an uncountable partial separator.

To finish the proof, we only need to show that $\mathcal{I}$ is indeed a P-ideal. Let $Y^0\subseteq Y^1\subseteq Y^2\subseteq\cdots\in\mathcal{I}$ and $\alpha=\sup(\bigcup_{n\in\omega}Y^n)$. Let $\alpha+1=\{\alpha_n\mid n\in\omega\}$ and define the set:

$$\begin{array}{ll} & Y^0\setminus Y^0_0(\alpha_0)\\ \bigcup & \bigl(Y^1\setminus Y^1_1(\alpha_0)\cup Y^1_1(\alpha_1)\bigr)\\ \bigcup & \bigl(Y^2\setminus Y^2_2(\alpha_0)\cup Y^2_2(\alpha_1)\cup Y^2_2(\alpha_2)\bigr)\\ \vdots & \vdots \end{array}$$

It is easy to see that this set belongs to $\mathcal{I}$ and it is a pseudounion of the Y^n. □

So we may conclude the following,

Corollary 3.9. *There is a model of CH without schizophrenic families.*

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How to drive our families mad

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Abstract

Given a family $\mathcal{F}$ of pairwise almost disjoint (ad) sets on a countable set S, we study maximal almost disjoint (mad) families $\tilde{\mathcal{F}}$ extending $\mathcal{F}$.

We define $\mathfrak{a}^+(\mathcal{F})$ to be the minimal possible cardinality of $\tilde{\mathcal{F}} \setminus \mathcal{F}$ for such $\tilde{\mathcal{F}}$ and $\mathfrak{a}^+(\kappa) = \max\{\mathfrak{a}^+(\mathcal{F}) : |\mathcal{F}| \leq \kappa\}$. We show that all infinite cardinals less than or equal to the continuum $\mathfrak{c}$ can be represented as $\mathfrak{a}^+(\mathcal{F})$ for some ad $\mathcal{F}$ (Theorem 4.6) and that the inequalities $\aleph_1 = \mathfrak{a} < \mathfrak{a}^+(\aleph_1) = \mathfrak{c}$ (Corollary 4.3) and $\mathfrak{a} = \mathfrak{a}^+(\aleph_1) < \mathfrak{c}$ (Theorem 4.4) are both consistent.

We also give several constructions of mad families with some additional properties.

1 Introduction

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Keywords: cardinal invariants – almost disjoint number – Cohen model – destructible maximal almost disjoint family

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Given a family $\mathcal{F}$ of pairwise almost disjoint countable sets, we can ask what the maximal almost disjoint (mad) families extending $\mathcal{F}$ look like. In this note and [5], we address some instances of this question and other related problems.

Let us begin with the definition of some notions and notation about almost disjointness we shall use here. Two countable sets A, B are said to be *almost disjoint* (*ad* for short) if $A \cap B$ is finite. A family $\mathcal{F}$ of countable sets is said to be *pairwise almost disjoint* (*ad* for short) if any two distinct A, $B \in \mathcal{F}$ are ad.

If $\mathcal{X} \subseteq [S]^{\aleph_0}$ and $S = \bigcup \mathcal{X}$, $\mathcal{F} \subseteq \mathcal{X}$ is said to be *mad in* $\mathcal{X}$ if $\mathcal{F}$ is ad and there is no ad $\mathcal{F}'$ such that $\mathcal{F} \subsetneqq \mathcal{F}' \subseteq \mathcal{X}$. Thus an ad family $\mathcal{F}$ is mad in $\mathcal{X}$ if and only if there is no $X \in \mathcal{X}$ which is ad from every $Y \in \mathcal{F}$. If $\mathcal{F}$ is mad in $[S]^{\aleph_0}$ for $S = \bigcup \mathcal{F}$, we say simply that $\mathcal{F}$ is a mad family (on S). S as above is called the *underlying set* of $\mathcal{F}$.

Let

(1.1) $\mathfrak{a}(\mathcal{X}) = \min\{|\,\mathcal{F}\,| \,:\, |\,\mathcal{F}\,| \geq \aleph_0 \text{ and } \mathcal{F} \text{ is mad in } \mathcal{X}\}.$

Clearly, the cardinal invariant $\mathfrak{a}$ known as the almost disjoint number ([2]) can be characterized as:

Example 1.1 $\mathfrak{a} = \mathfrak{a}([S]^{\aleph_0})$ *for any countable* S.

In this paper we concentrate on the case where the underlying set $S = \bigcup \mathcal{X}$ (or $S = \bigcup \mathcal{F}$) is countable. In [5] and the forthcoming continuation of this paper, we will deal with the cases where S may be also uncountable.

As the countable $S = \bigcup \mathcal{X}$, we often use ω or $T = {}^{\omega >}2$ where T is considered as a tree growing downwards. That is, for b, $b' \in T$, we write $b' \leq_T b$ if $b \subseteq b'$. Each $f \in {}^{\omega}2$ induces the (maximal) branch

(1.2) $B(f) = \{f \restriction n \,:\, n \in \omega\} \subseteq T$

in T.

In Section 2, we consider several cardinal invariants of the form $\mathfrak{a}(\mathcal{X})$ for some $\mathcal{X} \subseteq [T]^{\aleph_0}$.

For $\mathcal{X} \subseteq [S]^{\aleph_0}$ with $S = \bigcup \mathcal{X}$, let

(1.3) $\mathcal{X}^{\perp} = \{Y \in [S]^{\aleph_0} \,:\, \forall X \in \mathcal{X} \; |\,X \cap Y\,| < \aleph_0\}.$

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If $Y \in \mathcal{X}^\perp$ we shall say that Y is *almost disjoint* (ad) *to* $\mathcal{X}$.

For an ad family $\mathcal{F}$, let

(1.4) $\mathfrak{a}^+(\mathcal{F}) = \mathfrak{a}(\mathcal{F}^\perp)$.

For a cardinal κ, let

(1.5) $\mathfrak{a}^+(\kappa) = \sup\{\mathfrak{a}^+(\mathcal{F}) : \mathcal{F}$ is an ad family on ω of cardinality $\leq \kappa\}$.

Clearly, $\mathfrak{a}^+(\omega) = \mathfrak{a}$ and $\mathfrak{a}^+(\kappa) \leq \mathfrak{a}^+(\lambda) \leq \mathfrak{c}$ for any $\kappa \leq \lambda \leq \mathfrak{c}$. In Section 3 we give several constructions of ad families $\mathcal{F}$ for which $\mathcal{F}^\perp$ has some particular property. Using these constructions, we show in Section 4 that $\mathfrak{a}^+(\mathfrak{c}) = \mathfrak{c}$ (actually we have $\mathfrak{a}^+(\bar{\mathfrak{o}}) = \mathfrak{c}$, see Theorem 4.1) and the consistency of the inequalities $\mathfrak{a} = \aleph_1 < \mathfrak{a}^+(\aleph_1) = \mathfrak{c}$ (see Corollary 4.3). We also show the consistency of $\mathfrak{a}^+(\aleph_1) < \mathfrak{c}$ (Theorem 4.4).

For notions in the theory of forcing, the reader may consult [7] or [8]. We mostly follow the notation and conventions set in [7] and/or [8]. In particular, elements of posets $\mathbb{P}$ are considered in such a way that stronger conditions are smaller. We assume that $\mathbb{P}$-names are constructed just as in [8] for a poset $\mathbb{P}$ but we use alphabets with a tilde below them like $\underset{\sim}{a}$, $\underset{\sim}{b}$ etc. to denote the $\mathbb{P}$-names corresponding to the sets a, b etc. in the generic extension. V denotes the ground model (in which we live). The canonical $\mathbb{P}$-names of elements a, b etc. of V are denoted by the same symbols with hat like $\hat{a}$, $\hat{b}$ etc. For a poset $\mathbb{P}$ (in V) we use $V^{\mathbb{P}}$ to denote a "generic" generic extension $V[G]$ of V by some $(V, \mathbb{P})$-generic filter G. Thus $V^{\mathbb{P}} \models \cdots$ is synonymous to $\Vdash_{\mathbb{P}}$ " $\cdots$ " or $V \models \Vdash_{\mathbb{P}}$ " $\cdots$ " and a phrase like: "Let $W = V^{\mathbb{P}}$" is to be interpreted as saying: "Let W be a generic extension of V by some/any $(V, \mathbb{P})$-generic filter".

For the notation connected to the set theory of reals see [1] and [2]. By $\mathfrak{c}$ we denote the size of the continuum $2^{\aleph_0}$. $\mathcal{M}$ and $\mathcal{N}$ are the ideals of meager sets and null sets (e.g. over the Cantor space ${}^\omega 2$ or the Baire space ${}^\omega\omega$) respectively. For $I = \mathcal{M}, \mathcal{N}$ etc., $\mathsf{cov}(I)$ and $\mathsf{non}(I)$ are *covering number* and *uniformity* of I.

For an infinite cardinal κ let $\mathcal{C}_\kappa = \mathrm{Fn}(\kappa, 2)$ or, more generally $\mathcal{C}_X = \mathrm{Fn}(X, 2)$ for any set X. $\mathcal{C}_\kappa$ is the Cohen forcing for adding κ many Cohen reals. $\mathcal{R}_\kappa$ denotes the random forcing for adding κ many random reals. $\mathcal{R}_\kappa$ is the poset consisting of Borel sets of positive measure in ${}^\kappa 2$, which corresponds to the homogeneous measure algebra of Maharam type κ.

For a poset $\mathbb{P} = \langle \mathbb{P}, \leq_{\mathbb{P}} \rangle$, $X \subseteq \mathbb{P}$ and $p \in \mathbb{P}$, let

$$X \downarrow p = \{q \in X : q \leq_{\mathbb{P}} p\}.$$

2 Mad families and almost disjoint numbers

One of the advantages of using $T = {}^{\omega>}2$ as the countable underlying set is that we can define some natural subfamilies of $[T]^{\aleph_0}$ such as $\mathcal{O}_T$, $\mathcal{A}_T$, $\mathcal{B}_T$ below.

For $X \subseteq T$, let

(2.1) $[X] = \{f \in {}^{\omega}2 \,:\, B(f) \subseteq X\}$, and

(2.2) $\lceil X \rceil = \{f \in {}^{\omega}2 \,:\, |\, B(f) \cap X \,| = \aleph_0\}$.

Clearly, we have $[X] \subseteq \lceil X \rceil$. For $X \subseteq T$, let $X^{\uparrow}$ be the upward closure of X, that is:

(2.3) $X^{\uparrow} = \{t \restriction n \,:\, t \in X,\, n \leq \ell(t)\}$.

Then we have $\lceil X \rceil \subseteq [X^{\uparrow}]$ for any $X \subseteq T$.

Definition 2.1 (Off-binary sets, [9]) *Let*

$$\mathcal{O}_T = \{X \in [T]^{\aleph_0} \,:\, \lceil X \rceil = \emptyset\}.$$

T. Leathrum [9] called elements of $\mathcal{O}_T$ off-binary sets. Note that $\lceil X \rceil = \emptyset$ if and only if there is no branch in T with infinite intersection with X.

Definition 2.2 (Antichains) *Let*

$$\mathcal{A}_T = \{X \in [T]^{\aleph_0} \,:\, X \textit{ is an antichain in } T\}.$$

Clearly, we have $\mathcal{A}_T \subseteq \mathcal{O}_T$.

Using the notation above, the cardinal invariants $\mathfrak{o}$ and $\bar{\mathfrak{o}}$ introduced by Leathrum [9] can be characterized as:

(2.4) $\mathfrak{o} = \mathfrak{a}(\mathcal{O}_T)$,

(2.5) $\bar{\mathfrak{o}} = \mathfrak{a}(\mathcal{A}_T)$

(see [9]). Leathrum also showed $\mathfrak{a} \leq \mathfrak{o} \leq \bar{\mathfrak{o}}$. J. Brendle [3] proved $\mathsf{non}(\mathcal{M}) \leq \mathfrak{o}$.

Definition 2.3 (Sets without infinite antichains) *Let*

$$\mathcal{B}_T = \{X \in [T]^{\aleph_0} \,:\, X \textit{ does not contain any infinite antichain}\}.$$

Note that $\mathcal{B}_T = {\mathcal{A}_T}^{\perp}$. Elements of $\mathcal{B}_T$ are those infinite subsets of T which can be covered by finitely may branches:

Lemma 2.1 (K. Kunen) *Let $X \in [T]^{\aleph_0}$. Then $X \in \mathcal{B}_T$ if and only if X is covered by finitely may branches in T.*

Proof. If X is covered by finitely many branches in T then X clearly does not contain any infinite antichain since otherwise one of the finitely many branches would contain an infinite antichain.

Suppose now that X cannot be covered by finitely many branches. By induction on n, we choose $t_n \in 2^n$ such that $t_0 = \emptyset$, $t_{n+1} = t_n {}^\frown i$ for some $i \in 2$ and

(2.6) $X_{n+1} = X \downarrow t_{n+1}$ can not be covered by finitely many branches.

This is possible since $X_0 = X$ and $X_n \subseteq (X_n \downarrow (t_n {}^\frown 0)) \cup (X_n \downarrow (t_n {}^\frown 1)) \cup \{t_n\}$.

By (2.6), the branch $B = \{t_n : n < \omega\}$ does not cover X_n for each $n \in \omega$. So we can pick $s_n \in X_n \setminus B$. Let $S = \{s_n : n \in \omega\}$. S is an infinite subset of X since $\ell(s_n) \geq n$ for all $n \in \omega$. If C is a branch in T different from B then $t_n \notin C$ for some $n \in \omega$ and so $s_m \notin C$ for all $m \geq n$. Hence $S \cap C$ is finite. Moreover $S \cap B = \emptyset$. So we have $\lceil S \rceil = \emptyset$. Thus S should contain an infinite antichain by König's Lemma. □

Theorem 2.2 (K. Kunen) $\mathfrak{a}(\mathcal{B}_T) = \mathfrak{c}$.

Proof. Suppose that $\mathcal{F} \subseteq \mathcal{B}_T$ is an ad family of cardinality $< \mathfrak{c}$. We show that $\mathcal{F}$ is not mad. For each $X \in \mathcal{F}$ there is $b_X \in [{}^\omega 2]^{<\aleph_0}$ such that $X \subseteq \bigcup_{f \in b_X} B(f)$ by Lemma 2.1. Since $\mathcal{S} = \bigcup\{b_X : X \in \mathcal{F}\}$ has cardinality $\leq |\mathcal{F}| \cdot \aleph_0 < \mathfrak{c}$, there is $f^* \in {}^\omega 2 \setminus \mathcal{S}$. We have $B(f^*) \in \mathcal{B}_T$ and $B(f^*)$ is ad to $\mathcal{F}$. □ Let us say $X \subseteq T$ is *nowhere dense* if $\lceil X \rceil$ is nowhere dense in the Cantor space ${}^\omega 2$. It can be easily shown that X is nowhere dense if and only if

(2.7) $\forall t \in T\ \exists t' \leq_T t\ \forall t'' \leq_T t'\ (t'' \notin X)$.

Note that, if $X \subseteq T$ is not nowhere dense, then X is dense below some $t \in T$ (in terms of forcing). Also note that from (2.7) it follows that the property of being nowhere dense is absolute.

Definition 2.4 (Nowhere dense sets) *Let*

$\mathcal{ND}_T = \{X \in [T]^{\aleph_0} : X$ *is nowhere dense*$\}$.

Note that, for $X \in [T]^{\aleph_0}$ with $X = \{t_n : n \in \omega\}$, we have

$\lceil X \rceil = \bigcap_{n \in \omega} \bigcup_{m > n} [T \downarrow t_m]$.

In particular $\lceil X \rceil$ is a G_δ subset of ${}^\omega 2$. Hence by Baire Category Theorem we have

$\mathcal{ND}_T = \{X \in [T]^{\aleph_0} : \lceil X \rceil$ is a meager subset of ${}^\omega 2\}$.

Lemma 2.3 *If $X \in [T]^{\aleph_0}$ then there is $X' \in [X]^{\aleph_0}$ such that $X' \in \mathcal{ND}_T$.*

Proof. If $\lceil X \rceil = \emptyset$ then $X \in \mathcal{ND}_T$. Thus we can put $X' = X$. Otherwise let $f \in \lceil X \rceil$ and let $X' = X \cap B(f)$. □

Theorem 2.4 *$\mathsf{cov}(\mathcal{M})$, $\mathfrak{a} \leq \mathfrak{a}(\mathcal{ND}_T)$.*

Proof. For the inequality $\mathsf{cov}(\mathcal{M}) \leq \mathfrak{a}(\mathcal{ND}_T)$, suppose that $\mathcal{F} \subseteq \mathcal{ND}_T$ is an ad family of cardinality $< \mathsf{cov}(\mathcal{M})$. Then $\bigcup\{\lceil X\rceil : X \in \mathcal{F}\} \neq {}^{\omega}2$. Let $f \in {}^{\omega}2 \setminus \bigcup\{\lceil X\rceil : X \in \mathcal{F}\}$. Then $B(f) \in \mathcal{ND}_T$ and $B(f)$ is ad from all $X \in \mathcal{F}$.

To show $\mathfrak{a} \leq \mathfrak{a}(\mathcal{ND}_T)$ suppose that $\mathcal{F} \subseteq \mathcal{ND}_T$ is an ad family of cardinality $< \mathfrak{a}$. Then $\mathcal{F}$ is not a mad family in $[T]^{\aleph_0}$. Hence there is some $X \in [T]^{\aleph_0}$ ad to $\mathcal{F}$. By Lemma 2.3, there is $X' \subseteq X$ such that $X' \in \mathcal{ND}_T$. Since X' is also ad to $\mathcal{F}$, it follows that $\mathcal{F}$ is not mad in $\mathcal{ND}_T$. $\square$

Let σ be the measure on Borel sets of the Cantor space ${}^{\omega}2$ defined as the product measure of the probability measure on 2. For $X \subseteq T$, let $\mu(X) = \sigma(\lceil X\rceil)$.

Definition 2.5 (Null sets) *Let*

$$\mathcal{N}_T = \{X \in [T]^{\aleph_0} : \mu(X) = 0\}.$$

Theorem 2.5 *$\mathsf{cov}(\mathcal{N})$, $\mathfrak{a} \leq \mathfrak{a}(\mathcal{N}_T)$.*

Proof. Similarly to the proof of Theorem 2.4. $\square$

Definition 2.6 (Nowhere dense null sets) *Let*

$$\mathcal{NDN}_T = \mathcal{ND}_T \cap \mathcal{N}_T.$$

Lemma 2.6 *$\mathfrak{a}(\mathcal{ND}_T) \leq \mathfrak{a}(\mathcal{NDN}_T)$ and $\mathfrak{a}(\mathcal{N}_T) \leq \mathfrak{a}(\mathcal{NDN}_T)$.*

Proof. For the first inequality, suppose that $\mathcal{F}$ is a mad family in $\mathcal{NDN}_T$. Then $\mathcal{F}$ is an ad family in $\mathcal{ND}_T$. It is also mad in $\mathcal{ND}_T$. Suppose not. Then there is an $X \in \mathcal{ND}_T$ ad to $\mathcal{F}$. Let $X' \in [X]^{\aleph_0}$ be as in the measure analog of Lemma 2.3. Then $X' \in \mathcal{NDN}_T$. Hence $\mathcal{F}$ is not mad in $\mathcal{NDN}_T$. This is a contradiction. The second inequality can be also proved similarly. $\square$

The diagram Fig. 1 summarizes the inequalities obtained in this section integrated into the cardinal diagram given in Brendle [4]. "$\kappa \to \lambda$" in the diagram means that "$\kappa \leq \lambda$ is provable in ZFC". There are still some open questions concerning the (in)completeness of this diagram. In particular:

Problem 2.7 (a) *Are the inequalities between $\mathfrak{a}(\mathcal{N}_T)$, $\mathfrak{a}(\mathcal{ND}_T)$, $\mathfrak{a}(\mathcal{NDN}_T)$ consistently strict and complete?*

(b) *Are $\mathfrak{a}(\mathcal{ND}_T)$ etc. independent from $\mathfrak{o}$, $\bar{\mathfrak{o}}$, $\mathfrak{a}_{\mathfrak{s}}$?*

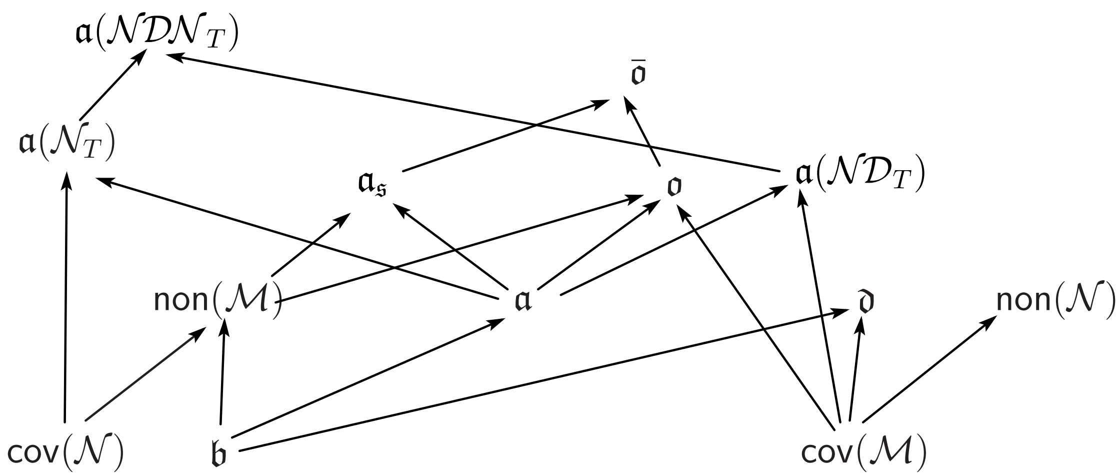

Figure 1:

3 Ad families $\mathcal{F}$ for which $\mathcal{F}^\perp$ is contained in a certain subfamily of $[T]^{\aleph_0}$

In this section we give several constructions of ad families with the property that the sets ad to them in a given generic extension are necessarily in a certain subfamily of $[T]^{\aleph_0}$. The constructions in this section are used in the proof of some results in the next sections.

Theorem 3.1 *There is an ad family $\mathcal{F} \subseteq \mathcal{A}_T$ of cardinality $\mathsf{non}(\mathcal{M})$ such that, for any poset $\mathbb{P}$ preserving the non-meagerness of ground-model non-meager sets, we have*

(3.1) $\Vdash_{\mathbb{P}}$ "$\mathcal{F}^\perp \subseteq \mathcal{ND}_T$".

The following assertion was originally proved under CH:

Corollary 3.2 *There is an ad family $\mathcal{F} \subseteq \mathcal{A}_T$ of cardinality $\mathsf{non}(\mathcal{M})$ such that, for any cardinal κ, we have*

(3.2) $V^{\mathcal{C}_\kappa} \models \mathcal{F}^\perp \subseteq \mathcal{ND}_T$.

Proof. The corollary follows from Theorem 3.1 since the Cohen forcing $\mathcal{C}_\kappa$ preserves the non-meagerness of ground-model non-meager sets (see e.g. 11.3 in [2]) $\square$

For the proof of Theorem 3.1, we use the following lemma.

Let

(3.3) $\mathcal{P} = \{f \,:\, f : X \to \omega$ for some $X \in [\omega]^{\aleph_0}\}$.

Lemma 3.3 *There is a mapping $F : {}^\omega\omega \to \mathcal{P}$ such that*

(3.4) *If* $f,\ g \in {}^{\omega}\omega$, $f \neq g$, *then* $|\, F(f) \cap F(g)\,| < \aleph_0$.

(3.5) *If* $h \in {}^{\omega}\omega$ *and* $X \subseteq {}^{\omega}\omega$ *is non-meager, then there is* $f \in X$ *such that* $|\, h \cap F(f)\,| = \aleph_0$.

Furthermore, F as above can be chosen such that it is definable and absolute in the sense that (3.4) *and* (3.5) *hold for the extension of F with the same definition in any generic extension of the ground model.*

Proof. Let $\langle s_n : n \in \omega\rangle$ be a one to one recursive enumeration of ${}^{\omega>}\omega$.

For $f \in {}^{\omega}\omega$, let $\mathrm{dom}(F(f)) = \{n \in \omega : s_n \subseteq f\}$. Let $F(f) : \mathrm{dom}(F(f)) \to \omega$ be defined by

(3.6) $F(f)(n) = f(|\, s_n\,|)$

for $n \in \mathrm{dom}(F(f))$.

Claim 3.3.1 *This F is as desired.*

$\vdash$ It is clear that F satisfies (3.4) — note that it is crucial here that the enumeration $\langle s_n : n \in \omega\rangle$ is chosen to be one to one.

To show that F also satisfies (3.5), suppose $h \in {}^{\omega}\omega$. It is enough to show that

(3.7) $N(h) = \{g \in {}^{\omega}\omega : |\, h \cap F(g)\,| < \aleph_0\}$ is a meager subset of ${}^{\omega}\omega$.

For $k \in \omega$, let $N_k(h) = \{g \in {}^{\omega}\omega : |\, h \cap F(g)\,| < k\}$.

Since $N(h) = \bigcup_{k\in\omega} N_k(h)$, it is enough to show that $N_k(h)$ is a nowhere dense subset of ${}^{\omega}\omega$ for each $k \in \omega$.

For this, we prove, by induction on k,

(3.8) For any $s \in {}^{\omega>}\omega$, there are $s' \in {}^{\omega>}\omega$ and $m' \in \omega$ such that such that $s' \subseteq s$ and $|\,(h \restriction m') \cap F(g)\,| \geq k$ for all $g \in [s']$.

Suppose that (3.8) holds for $k = \ell$ and let $s \in {}^{\omega>}\omega$. By the induction hypothesis we may assume without loss of generality that there is an $m \in \omega$ such that $|\,(h \restriction m) \cap F(f)\,| \geq \ell$ for all $g \in [s]$.

Let $n \in \omega$ be such that $n \geq m, |\, s\,|$ and $s_n \supseteq s$. Let

(3.9) $s' = s_n \cup \{\langle |\, s_n\,|, h(n)\rangle\}$.

For any $g \in [s']$, we have $n \in \mathrm{dom}(F(g))$ by $s_n \subseteq s' \subseteq g$, and $F(g)(n) = g(|\, s_n\,|) = h(n)$. Letting $m' = n+1$, we have $|\,(h \restriction m') \cap F(g)\,| \geq \ell+1$. Thus, (3.8) holds for $k = \ell + 1$ with these s' and m'. $\dashv$ (Claim 3.3.1)

The definability and the absoluteness of F is clear from the construction given above. $\square$

Proof of Theorem 3.1: Let

(3.10) $Q = \{q \in T \,:\, q(n) \text{ is eventually } 0\}$.

That is, for $q \in T$, $q \in Q$ if and only if $|\,\{n \in \omega \,:\, q(n) = 1\}\,| < \aleph_0$.

For $q \in Q$, let

(3.11) $\ell_q = \min\{\ell \in \omega \,:\, \forall m\ (\ell \leq m \rightarrow q(m) = 0)\}$.

Let $\langle q_n \,:\, n \in \omega\rangle$ be a one to one enumeration of Q.

For n, $k \in \omega$ let

(3.12) $T_{n,k} = \{s \in T \,:\, q_n \restriction (\ell_q + k) \cup \{\langle \ell_q + k, 1\rangle\} \subseteq s\}$

and let $\langle s_{n,k,i} \,:\, i \in \omega\rangle$ be a one to one enumeration of $T_{n,k}$. Let F be as in Lemma 3.3. For $n \in \omega$ and $f \in {}^{\omega}\omega$, let

(3.13) $F_n(f) = \{s_{n,k,i} \,:\, k \in \mathrm{dom}(F(f)),\, i = F(f)(k)\}$.

Let $N \subseteq {}^{\omega}\omega$ be a non-meager set with $|\,N\,| = \mathsf{non}(\mathcal{M})$. Let $\mathcal{F}_n = F_n{}''N$ and $\mathcal{F} = \bigcup_{n\in\omega} \mathcal{F}_n$.

We show that this $\mathcal{F}$ is as desired:

Claim 3.3.2 (1) $\mathcal{F} \subseteq \mathcal{A}_T$.

(2) $\mathcal{F}$ *is ad.*

(3) (3.1) *holds for all poset* $\mathbb{P}$ *preserving non-meagerness of ground-model non-meager sets.*

$\vdash$ (1): Suppose that $A \in \mathcal{F}$ and $A = F_n(f)$ for some $n \in \omega$ and $f \in N$. If s_0, s_1 are two different elements of A, then there are k_0, $k_1 \in \mathrm{dom}(F(f))$, $k_0 \neq k_1$ and i_0, $i_1 \in \omega$ such that $s_0 = s_{n,k_0,i_0}$ and $s_1 = s_{n,k_1,i_1}$. Since $s_0 \in T_{n,k_0}$ and $s_1 \in T_{n,k_1}$, it follows that s_0 and s_1 are incompatible.

(2): Suppose that A_0, $A_1 \in \mathcal{F}$ with $A_0 \neq A_1$. Let $A_0 = F_{n_0}(f_0)$ and $A_1 = F_{n_1}(f_1)$. If $n_0 \neq n_1$ then we have $|\,A_0 \cap A_1\,| \leq 1$. Then $f_0 \neq f_1$. Thus, by (3.4), $|\,A_0 \cap A_1\,| = |\,F(f_0) \cap F(f_1)\,| < \aleph_0$.

(3): Let G be a $(V, \mathbb{P})$-generic set and we work in $V[G]$. Note, that by our assumption, N is still non-meager in $V[G]$.

Suppose that $B \in [T]^{\aleph_0} \setminus \mathcal{ND}_T$. We have to show that $|\,A \cap B\,| = \aleph_0$ for some $A \in \mathcal{F}$.

Since $B \notin \mathcal{ND}_T$ there is $n \in \omega$ such that $B \downarrow (q_n \restriction \ell_{q_n})$ is dense below $q_n \restriction \ell_{q_n}$. It follows that, for each $k \in \omega$, there is $h(k) \in \omega$ such that $s_{n,k,h(k)} \in B$. By (3.5) (which still holds in the generic extension $V[G]$), there is $f \in M$ such that $|\,h \cap F(f)\,| = \aleph_0$.

By the definition of h and $F_n(f)$, it follows that $|\,B \cap F_n(f)\,| = \aleph_0$. $\dashv$ (Claim 3.3.2)

$\square$

We can also obtain a variation of Theorem 3.1 if our ground model is a generic extension of some inner model by adding uncountably may Cohen reals. Note that $\mathsf{non}(\mathcal{M}) = \aleph_1$ holds in such a ground model.

Theorem 3.4 *Suppose that $W = V^{\mathcal{C}}_{\omega_1}$. Then, in W, there is an ad family $\mathcal{F} \subseteq \mathcal{A}_T$ of cardinality $\aleph_1$ such that*

(3.14) *for any c.c.c. poset $\mathbb{P}$ with $\mathbb{P} \in V$, we have $W^{\mathbb{P}} \models \mathcal{F}^{\perp} \subseteq \mathcal{ND}_T$.*

Proof. Let $A \in [T]^{\aleph_0} \cap V$ be an antichain and let $\langle t^*_n : n \in \omega\rangle$ be a one to one enumeration of A.

Let G be a $(V, \mathcal{C}_{\omega_1})$-generic filter and $W = V[G]$. For $p \in \mathcal{C}_{\omega_1}$, $\alpha < \omega_1$ and $k \in \omega$, let

$$f^p_\alpha = \{\langle n, i\rangle \in \omega \times \omega : \langle \omega\alpha + 3n, i\rangle \in p\};$$

$$n^p_{\alpha,k} = \begin{cases} n, & \text{if } [\omega\alpha, \omega\alpha + 3n + 1] \subseteq \mathrm{dom}(p), \\ & \quad p(\omega\alpha + 3n + 1) = 1 \text{ and} \\ & \quad |\{m < n : p(\omega\alpha + 3m + 1) = 1\}| = k, \\ \text{undefined}, & \text{if there is no such } n \text{ as above}; \end{cases}$$

$$t^p_\alpha = \begin{cases} \{\langle n, i\rangle \in \omega \times \omega : n < n^p_{\alpha,0},\ \langle \omega\alpha + 3n + 2, i\rangle \in p\}, \\ \qquad\qquad \text{if } n^p_{\alpha,0} \text{ is defined}, \\ \text{undefined}, \qquad \text{otherwise} \end{cases}$$

and

$$t^p_{\alpha,k} = \begin{cases} \{\langle n, i\rangle \in \omega \times \omega : n < n^p_{\alpha,k+1},\ \langle \omega\alpha + 3n + 2, i\rangle \in p\}, \\ \qquad\qquad \text{if } n^p_{\alpha,k+1} \text{ is defined}, \\ \text{undefined}, \qquad \text{otherwise}. \end{cases}$$

Let

$f^G_\alpha = \bigcup_{p \in G} f^p_\alpha$,
$t^G_\alpha = t^p_\alpha$ for some $p \in G$ such that t^p_α is defined, and
$t^G_{\alpha,k} = t^p_{\alpha,k}$ for some $p \in G$ such that $t^p_{\alpha,k}$ is defined.

For $\alpha \in \omega_1$, let

(3.15) $A_\alpha = \{t^G_\alpha {}^\frown t^*_k {}^\frown t^G_{\alpha,k} : k \in \omega\}$.

Clearly each A_α is an antichain in T.

A_α, $\alpha < \omega_1$ are pairwise almost disjoint: Suppose that $\alpha < \beta < \omega_1$. Then there is $k_0 < \omega$ such that $t^G_{\alpha,k} \neq f^G_{\beta,k}$ for all $k \in \omega \setminus k_0$. It follows that $A_\alpha \cap A_\beta \subseteq \{t^G_\alpha {}^\frown t^*_k {}^\frown t^G_{\alpha,k} : k < k_0\}$.

We show that $\mathcal{F} = \{A_\alpha : \alpha < \omega_1\}$ satisfies (3.14).

Suppose that $\mathbb{P}$ is a c.c.c. poset (in W) and $\mathbb{P} \in V$. Let H be a $(W, \mathbb{P})$-generic filter. It is enough to show that, in $W[H]$, if $X \in [T]^{\aleph_0}$ is not nowhere dense then X is not almost ad to $\mathcal{F}$.

By the c.c.c. of $\mathcal{C}_{\omega_1} * \hat{\mathbb{P}} \sim \mathcal{C}_{\omega_1} \times \mathbb{P}$, there is an $\alpha^* \in \omega_1$ such that $X \in V[G \restriction \mathcal{C}_{\omega\alpha^*}][H]$. Let $t \in T$ be such that X is dense below t. Then

$$D = \{p \in \mathcal{C}_{\omega_1 \setminus \omega\alpha^*} : t^p_\alpha \supseteq t \text{ for some } \alpha \in \omega_1 \setminus \omega\alpha^*\}$$

is dense in $\mathcal{C}_{\omega_1 \setminus \omega\alpha^*}$.

For $p \in D$ and $\alpha \in \omega_1 \setminus \omega\alpha^*$ such that $t^p_\alpha \supseteq t$, letting $\underset{\sim}{A}_\alpha$ a $\mathcal{C}_{\omega_1 \setminus \alpha^*}$-name of A_α, we have $p \Vdash_{\mathcal{C}_{\omega_1 \setminus \omega\alpha^*}}$ "$|\,\underset{\sim}{A}_\alpha \cap X \downarrow t\,| = \aleph_0$" by (3.15) and since X is dense below t.

By genericity, it follows that, in $W[G]$, there is $\alpha < \omega_1$ such that $|\,A_\alpha \cap X\,| = \aleph_0$.

$\square$ $\square$ A measure version of Theorem 3.4 also holds:

Theorem 3.5 *Let $W = V^{\mathcal{C}}_{\omega_1}$. Then, in W, there is an ad family $\mathcal{F}$ in $\mathcal{N}_T$ of cardinality $\aleph_1$ such that for any c.c.c. poset $\mathbb{P}$ with $\mathbb{P} \in V$, we have $W^{\mathbb{P}} \models \mathcal{F}^\perp \subseteq \mathcal{O}_T$.*

For the proof of Theorem 3.5 we note first the following:

Lemma 3.6 *Suppose that $X \subseteq T$ is such that $X = \{t_k : k \in \omega\}$ for some enumeration t_k, $k \in \omega$ of X with $\ell(t_k) \geq k$ for all $k \in \omega$. Then $X \in \mathcal{N}_T$.*

Proof. For all $n \in \omega$, we have $\lceil X \rceil \subseteq \bigcup_{k \in \omega \setminus n} \lceil T \downarrow t_k \rceil$. Hence

$$\mu(X) = \sigma(\lceil X \rceil) \leq \textstyle\sum_{k \in \omega \setminus n} \sigma(\lceil T \downarrow t_k \rceil) \leq \sum_{k \in \omega \setminus n} 2^k = 2^{-n}.$$

It follows that $\mu(X) = 0$. $\square$ $\square$ *Proof.* [of *Theorem 3.5*] Let G be a $(V, \mathcal{C}_{\omega_1})$-generic filter and $W = V[G]$. In W, let

$$f^G_\alpha = \{\langle n, i \rangle : \langle \omega\alpha + n, i \rangle \in p \text{ for some } p \in G\}$$

for $\alpha < \omega_1$ and let $g^G_\alpha \in {}^\omega\omega$ be the increasing enumeration of $\left(f^G_\alpha\right)^{-1}[\{1\}]$.

Further in W, we construct inductively $A_\alpha \in \mathcal{N}_T$, $\alpha < \omega_1$ as follows.

For $n \in \omega$, let $A_n \in \mathcal{N}_T$ be such that $\langle A_n : n \in \omega \rangle$ is a partition of T in V. We can be easily find such A_n's by Lemma 3.6.

For $\omega \leq \alpha < \omega_1$, suppose that pairwise almost disjoint A_β, $\beta < \alpha$ have been constructed. Let $\langle B_\ell : \ell \in \omega \rangle$ be an enumeration of $\{A_\beta : \beta < \alpha\}$ and, for each $n \in \omega$, let $\langle b_{n,m} : m \in \omega \rangle$ be an enumeration of

(3.16) $C_n = T \setminus ({}^{n>}2 \cup \{B_\ell : \ell < n\})$.

Let

(3.17) $A_\alpha = \{b_{n,g^G_\alpha(n)} : n \in \omega\}$.

$A_\alpha \in \mathcal{N}_T$ by (3.16) and Lemma 3.6. A_α is ad to $\{A_\beta : \beta < \alpha\}$ by (3.16) and (3.17).

We show that $\mathcal{F} = \{A_\alpha : \alpha < \omega_1\}$ is as desired. Suppose that $\mathbb{P}$ is c.c.c. (in W) and $\mathbb{P} \in V$. Let H be a $(W, \mathbb{P})$-generic filter. It is enough to show that, in $W[H]$, if $X \in [T]^{\aleph_0} \setminus \mathcal{O}_T$ then X is not ad to $\mathcal{F}$. So suppose that (in $W[H]$) $X \in [T]^{\aleph_0} \setminus \mathcal{O}_T$ and $f \in \lceil X \rceil$. Let $B = X \cap B(f)$. By the c.c.c. of $\mathcal{C}_{\omega_1} * \hat{\mathbb{P}} \sim \mathcal{C}_{\omega_1} \times \mathbb{P}$, there is an $\alpha^* \in \omega_1 \setminus \omega$ such that $B \in V[(G \restriction \mathcal{C}_{\omega\alpha^*})][H]$. If $B \cap A_\alpha$ is infinite for some $\alpha < \alpha^*$ then we are done. So assume that B is ad to all A_α, $\alpha < \alpha^*$. Then $B \cap C_n$ is infinite for all $n \in \omega$. Since $f^G_{\alpha^*}$ is a Cohen real generic over $V[(G \restriction \mathcal{C}_{\omega\alpha^*})][H]$, it follows that $B \cap A_{\alpha^*}$ is infinite. ❏ $\square$

4 Almost disjoint numbers over ad families

In this section we turn to questions on the possible values of $\mathfrak{a}^+(\cdot)$.

Theorem 4.1 (K. Kunen) $\mathfrak{a}^+(\bar{\mathfrak{o}}) = \mathfrak{c}$.

Proof. Let $\mathcal{F}$ be any mad family in $\mathcal{A}_T$ of cardinality $\bar{\mathfrak{o}}$. By maximality of $\mathcal{F}$ we have $\mathcal{F}^\perp = \mathcal{B}_T$. If $\mathcal{G} \subseteq [T]^{\aleph_0}$ is disjoint from $\mathcal{F}$ and $\mathcal{F} \cup \mathcal{G}$ is mad then $\mathcal{G}$ is mad in $\mathcal{B}_T$ and hence $|\,\mathcal{G}\,| = \mathfrak{c}$ by Theorem 2.2. ❏ $\square$

Theorem 4.2 *$V^{\mathcal{C}_\kappa} \models \mathfrak{a}^+(\aleph_1) \geq \kappa$ for all regular κ.*

Proof. If $\kappa = \omega_1$ this is trivial. So suppose that $\kappa > \omega_1$. Let $W = V^{\mathcal{C}_{\omega_1}}$. Then $V^{\mathcal{C}_\kappa} = W^{\mathcal{C}_{\kappa \setminus \omega_1}}$. Let $\mathcal{F}$ be as in the proof of Theorem 3.4. Suppose that $\tilde{\mathcal{F}} \supseteq \mathcal{F}$ is mad on T in $V^{\mathcal{C}_\kappa}$. Then $\tilde{\mathcal{F}} \subseteq (\mathcal{ND}_T)^{V^{\mathcal{C}_\kappa}}$. Since $V^{\mathcal{C}}_\kappa \models \mathsf{cov}(\mathcal{M}) \geq \kappa$, it follows that $|\,\tilde{\mathcal{F}}\,| \geq \kappa$ by Theorem 2.4. ❏ $\square$

Corollary 4.3 *The inequality $\mathfrak{a} = \aleph_1 < \mathfrak{a}^+(\aleph_1) = \mathfrak{c}$ is consistent.*

Proof. Start from a model V of CH. Since there is a $\mathcal{C}_\kappa$-indestructible mad family in V it follows that $V^{\mathcal{C}_{\omega_2}} \models \mathfrak{a} = \aleph_1$ (see e.g. [8], Theorem 2.3). On the other hand we have $V^{\mathcal{C}_{\omega_2}} \models \mathfrak{a}^+(\aleph_1) = \aleph_2 = \mathfrak{c}$ by Theorem 4.2. ❏ $\square$

Theorem 4.4 *The inequality $\mathfrak{a}^+(\aleph_1) < \mathfrak{c}$ is consistent.*

For the proof of the theorem we use the following forcing notions: for a family $\mathcal{I} \subseteq \{A \in [\omega]^{\aleph_0} : |\,\omega \setminus A\,| = \aleph_0\}$ closed under union, let $\mathbb{Q}_\mathcal{I} = \langle \mathbb{Q}_\mathcal{I}, \leq_{\mathbb{Q}_\mathcal{I}} \rangle$ be the poset defined by

$$\mathbb{Q}_\mathcal{I} = \mathcal{C}_\omega \times \mathcal{I}\,;$$

For all $\langle s, A \rangle$, $\langle s', A' \rangle \in \mathbb{Q}_\mathcal{I}$

$$(4.1)\quad \langle s', A'\rangle \leq_{\mathbb{Q}_{\mathcal{I}}} \langle s, A\rangle \quad \Leftrightarrow \quad s \subseteq s',\ A \subseteq A' \text{ and } \forall n \in \mathrm{dom}(s') \setminus \mathrm{dom}(s)\ (n \in A \ \rightarrow\ s'(n) = 0).$$

Clearly $\mathbb{Q}_{\mathcal{I}}$ is σ-centered.

For a $(V, \mathbb{Q}_{\mathcal{I}})$-generic G, let

$$f_G = \bigcup\{s \,:\, \langle s, A\rangle \in G \text{ for some } A \in \mathcal{I}\} \text{ and}$$
$$A_G = f_G^{-1}{}''\{1\}.$$

Let $\tilde{\mathcal{I}}$ be the ideal in $[\omega]^{\aleph_0}$ generated from $\mathcal{I}$ (i.e. the downward closure of $\mathcal{I}$ with respect to $\subseteq$). By the genericity of G and the definition of $\leq_{\mathbb{Q}_{\mathcal{I}}}$ it is easy to see that A_G is infinite and

(4.2) for every $B \in ([\omega]^{\aleph_0})^V$, A_G is almost disjoint from $B \ \Leftrightarrow\ B \in \tilde{\mathcal{I}}$.

Proof. [of *Theorem 4.4*] Working in a ground model V of $2^{\aleph_0} = 2^{\aleph_1} = \aleph_3$, let

$$\langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta \,:\, \alpha \leq \omega_2,\ \beta < \omega_2\rangle$$

be the finite support iteration of c.c.c. posets defined as follows: for $\beta < \omega_2$, let $\underset{\sim}{\mathbb{Q}}_\beta$ be the $\mathbb{P}_\beta$-name of the finite support (side-by-side) product of

(4.3) $\mathbb{Q}_{\tilde{\mathcal{F}}}$, $\tilde{\mathcal{F}} \in \Phi$

where

$$\Phi = \{\tilde{\mathcal{F}} \,:\, \tilde{\mathcal{F}} \text{ is an ideal in } [\omega]^{\aleph_0} \text{ generated from an ad family in } [\omega]^{\aleph_0} \text{ of cardinality } \aleph_1\}$$

in $V^{\mathbb{P}_\beta}$. We have

$$V^{\mathbb{P}_\beta} \models \underset{\sim}{\mathbb{Q}}_\beta \text{ satisfies the c.c.c.}$$

since $V^{\mathbb{P}_\beta} \models \mathbb{Q}_{\tilde{\mathcal{F}}}$ is σ-centered for all $\tilde{\mathcal{F}} \in \Phi$. By induction on $\alpha \leq \omega_2$, we can show that $\mathbb{P}_\alpha$ satisfies the c.c.c. and $|\,\mathbb{P}_\alpha\,| \leq 2^{\aleph_1} = \aleph_3$ for all $\alpha \leq \omega_2$. It follows that

(4.4) $V^{\mathbb{P}_{\omega_2}} \models 2^{\aleph_0} = 2^{\aleph_1} = \aleph_3$.

Thus the following claim finishes the proof:

Claim 4.4.1 $V^{\mathbb{P}_{\omega_2}} \models \mathfrak{a} = \mathfrak{a}^+(\aleph_1) = \aleph_2$.

$\vdash$ Working in $V^{\mathbb{P}_{\omega_2}}$, suppose that $\mathcal{F}$ is an ad family in $[\omega]^{\aleph_0}$ of cardinality $\aleph_1$. By the c.c.c. of $\mathbb{P}_{\omega_2}$, there is some $\alpha^* < \omega_2$ such that $\mathcal{F} \in V^{\mathbb{P}_{\alpha^*}}$. By (4.3) and (4.2), there are A_α, $\alpha \in \omega_2 \setminus \alpha^*$ such that

(4.5) for every $B \in ([\omega]^{\aleph_0})^{V^{\mathbb{P}_\alpha}}$, A_α is ad from $B \ \Leftrightarrow\ B \in$ the ideal generated from $\mathcal{F} \cup \{A_\beta \,:\, \beta \in \alpha \setminus \alpha^*\}$.

Since $([\omega]^{\aleph_0})^{V^{\mathbb{P}_{\omega_2}}} = \bigcup_{\alpha<\omega_2}([\omega]^{\aleph_0})^{V^{\mathbb{P}_\alpha}}$, it follows that $\mathcal{F} \cup \{A_\alpha : \alpha \in \omega_2 \setminus \alpha^*\}$ is a mad family in $V^{\mathbb{P}_{\omega_2}}$. This shows that $V^{\mathbb{P}_{\omega_2}} \models \mathfrak{a}^+(\aleph_1) \leq \aleph_2$.

We also have $V^{\mathbb{P}_{\omega_2}} \models \mathfrak{a} \geq \aleph_2$: for any ad family $\mathcal{G} \subseteq ([\omega]^{\aleph_0})^{V^{\mathbb{P}_{\omega_2}}}$ of cardinality $\leq \aleph_1$, there is some $\alpha^* < \omega_2$ such that $\mathcal{G} \in V^{\mathbb{P}_{\alpha^*}}$. But $\underset{\sim}{\mathbb{Q}}_{\alpha^*}$ adds an infinite subset of ω almost disjoint to every element of $\mathcal{G}$. Hence $\mathcal{G}$ is not mad. $\dashv$ (Claim 4.4.1)

□ Clearly, the method of the proof of Theorem 4.4 cannot produce a model of $\mathfrak{a}^+(\aleph_1) = \aleph_1 < \mathfrak{c}$.

Problem 4.5 *Is $\mathfrak{a}^+(\aleph_1) = \aleph_1 < \mathfrak{c}$ consistent?*

All infinite cardinals less than or equal to the continuum $\mathfrak{c}$ can be represented as $\mathfrak{a}^+(\mathcal{F})$ for some $\mathcal{F}$.

Theorem 4.6 *For any infinite $\kappa \leq \mathfrak{c}$, there is an ad family $\mathcal{F} \subseteq [T]^{\aleph_0}$ of cardinality $\mathfrak{c}$ such that $\mathfrak{a}^+(\mathcal{F}) = \kappa$.*

Proof. Let $\mathcal{F}'$ be a mad family in $\mathcal{A}_T$. Then by Lemma 2.1, we have

(4.6) $\mathcal{F}'^\perp = \mathcal{B}_T$.

Let X and X' be disjoint with ${}^\omega 2 = X \cup X'$, $|\,X\,| = \mathfrak{c}$ and $|\,X'\,| = \kappa$. Let

$$\mathcal{F} = \mathcal{F}' \cup \{B(f) : f \in X\}.$$

Clearly $\mathcal{F}$ is an ad family. By (4.6) we have $\mathcal{F}^\perp \subseteq \mathcal{B}_T$.

We claim $\mathfrak{a}^+(\mathcal{F}) = \kappa$: Since $\mathcal{F} \cup \{B(f) : f \in X'\}$ is a mad family by Lemma 2.1, we have $\mathfrak{a}^+(\mathcal{F}) \leq \kappa$. Again by Lemma 2.1, if $\mathcal{G} \subseteq \mathcal{F}^\perp$ is an ad family of cardinality $< \kappa$, then there is $f \in X'$ such that $B(f)$ is ad from every $B \in \mathcal{G}$. Thus $\mathfrak{a}^+(\mathcal{F}) \geq \kappa$. □

5 Destructibility of mad families

For a poset $\mathbb{P}$, a mad family $\mathcal{F}$ in $[T]^{\aleph_0}$ is said to be $\mathbb{P}$-*destructible* if

$$V^{\mathbb{P}} \models \mathcal{F} \text{ is not mad in } [T]^{\aleph_0}.$$

Otherwise it is $\mathbb{P}$-*indestructible*.

The results in Section 3 can be also formulated in terms of destructibility of mad families.

Theorem 5.1 (1) *There is an ad family $\mathcal{F} \subseteq \mathcal{A}_T$ of size $\mathsf{non}(\mathcal{M})$ which cannot be extended to a $\mathcal{C}_\omega$-indestructible mad family in any generic extension of the ground model $V^{\mathbb{P}}$ as long as non-meager sets in V remain non-meager in $V^{\mathbb{P}}$.*

(2) *Let $W = V^{\mathcal{C}_{\omega_1}}$. Then, in W, there is an ad family $\mathcal{F} \subseteq \mathcal{ND}_T$ of cardinality $\aleph_1$ such that, in any generic extension of W by a c.c.c. poset $\mathbb{P}$ with $\mathbb{P} \in V$, $\mathcal{F}$ cannot be extended to a $\mathcal{C}_\omega$-indestructible mad family.*

(3) *Let $W = V^{\mathcal{C}_{\omega_1}}$. Then, in W, there is an ad family $\mathcal{F} \subseteq \mathcal{N}_T$ of cardinality $\aleph_1$ such that, in any generic extension of W by a c.c.c. poset $\mathbb{P}$ with $\mathbb{P} \in V$, $\mathcal{F}$ cannot be extended to a $\mathcal{R}_\omega$-indestructible mad family.*

Proof. (1): The family $\mathcal{F}$ as in Theorem 3.1 will do. Since we have $\mathcal{F}' \subseteq \mathcal{ND}_T$ for any mad $\mathcal{F}'$ extending $\mathcal{F}$ in $V^{\mathbb{P}}$, a further Cohen real over $V^{\mathbb{P}}$ introduces a branch almost avoiding all elements of $\mathcal{F}'$. Thus $\mathcal{F}'$ is no longer mad in $V^{\mathbb{P}*\mathcal{C}_\omega}$.

(2): By Theorem 3.4 and by an argument similar to the proof of (1).

(3): In W, let $\mathcal{F}$ be as in the proof of Theorem 3.5. Then any mad $\mathcal{F}' \supseteq \mathcal{F}$ on T in any $W^{\mathbb{P}}$ for $\mathbb{P}$ as above is included in $\mathcal{N}_T$ by $\mathcal{O}_T \subseteq \mathcal{N}_T$. Hence, in $W^{\mathbb{P}*\mathcal{R}_\omega}$, the random real f over $W^{\mathbb{P}}$ introduces the branch $B(f)$ almost avoiding all elements of $\mathcal{F}'$. Thus $\mathcal{F}'$ is no longer mad in $W^{\mathbb{P}*\mathcal{R}_\omega}$. $\Box$

6 κ-almost decided and λ-minimal mad families

In this final section we collect several other constructions of mad families with some additional properties.

Given an ad family $\mathcal{F}$ on T let $\mathcal{I}(\mathcal{F})$ be the ideal on T generated by $\mathcal{F} \cup [T]^{<\omega}$, i.e. for $S \subset T$ we have $S \in \mathcal{I}(\mathcal{F})$ if $S \subset^* \cup\mathcal{F}'$ for some finite subfamily $\mathcal{F}'$ of $\mathcal{F}$.

Let $\mathcal{F}$ be a mad family on T and $\mathcal{B} \subseteq \mathcal{F}$. Clearly $\mathcal{B}^\perp \supseteq \mathcal{I}(\mathcal{F} \setminus \mathcal{B}) \setminus [T]^{<\aleph_0}$. We say that $\mathcal{B}$ *almost decides* $\mathcal{F}$ if $\mathcal{B}^\perp = \mathcal{I}(\mathcal{F} \setminus \mathcal{B}) \setminus [T]^{<\aleph_0}$. A mad family $\mathcal{F}$ is said to be *κ-almost decided* if every $\mathcal{B} \in [\mathcal{F}]^\kappa$ almost decides $\mathcal{F}$.

Theorem 6.1 *Assume that* MA(σ-*centered*) *holds. Then there is a $\mathfrak{c}$-almost decided mad family $\mathcal{F}$ on T.*

Proof. Let $\langle B_\beta : \beta < \mathfrak{c} \rangle$ be an enumeration of $[T]^{\aleph_0}$. We define A_α, $\alpha < \mathfrak{c}$ inductively such that

(6.1) $\{A_n : n \in \omega\}$ is a partition of T into infinite subsets;

For all $\alpha \in \mathfrak{c} \setminus \omega$

(6.2) A_α is ad from A_β for all $\beta < \alpha$;

(6.3) For $\beta < \alpha$, if $B_\beta \notin \mathcal{I}(\{A_\delta : \delta < \alpha\})$ then $|\, A_\alpha \cap B_\beta \,| = \aleph_0$;

Claim 6.1.1 *The construction of A_α, $\alpha < \mathfrak{c}$ as above is possible.*

$\vdash$ Suppose that $\alpha \in \mathfrak{c} \setminus \omega$ and A_β, $\beta < \alpha$ have been constructed according to (6.1), (6.2) and (6.3). Let

$$S_\alpha = \{\beta < \alpha \,:\, B_\beta \notin \mathcal{I}(\{A_\delta \,:\, \delta < \alpha\})\}.$$

Let $\mathbb{P}_\alpha = \{\langle \varphi, s\rangle \,:\, \varphi \in \mathrm{Fn}(T,2),\ s \in [\alpha]^{<\aleph_0}\}$ be the poset with the ordering defined by

$$\begin{aligned}&\langle \varphi', s'\rangle \leq_{\mathbb{P}_\alpha} \langle \varphi, s\rangle \ \Leftrightarrow\\ &\qquad \varphi \subseteq \varphi',\ s \subseteq s' \text{ and}\\ &\qquad \forall t \in \mathrm{dom}(\varphi') \setminus \mathrm{dom}(\varphi)\ (\varphi'(t) = 1 \ \rightarrow\ t \notin A_\delta \text{ for all } \delta \in s)\end{aligned}$$

for $\langle \varphi, s\rangle$, $\langle \varphi', s'\rangle \in \mathbb{P}_\alpha$.

$\mathbb{P}_\alpha$ is σ-centered since $\langle \varphi, s\rangle$, $\langle \varphi', s'\rangle \in \mathbb{P}_\alpha$ are compatible if $\varphi = \varphi'$.

For $\beta < \alpha$, let

$$C_\beta = \{\langle \varphi, s\rangle \in \mathbb{P}_\alpha \,:\, \beta \in s\}$$

and, for $\beta \in S_\alpha$ and $n \in \omega$, let

$$D_{\beta,n} = \{\langle \varphi, s\rangle \in \mathbb{P}_\alpha \,:\, \exists t \in \mathrm{dom}(\varphi)\ (\ell(t) \geq n \ \wedge\ \varphi(t) = 1 \ \wedge\ t \in B_\beta)\}.$$

It is easy to see that C_β, $\beta < \alpha$ and $D_{\beta,n}$, $\beta \in S_\alpha$, $n \in \omega$ are dense in $\mathbb{P}_\alpha$. Let

$$\mathcal{D} = \{C_\beta \,:\, \beta < \alpha\} \cup \{D_{\beta,n} \,:\, \beta \in S_\alpha,\ n \in \omega\}.$$

Since $|\,\mathcal{D}\,| < \mathfrak{c}$, we can apply MA(σ-centered) to obtain a $(\mathcal{D}, \mathbb{P}_\alpha)$-generic filter G. Let

$$A_\alpha = \{t \in T \,:\, \varphi(t) = 1 \text{ for some } \langle \varphi, s\rangle \in G\}.$$

Then this A_α is as desired. $\dashv$ (Claim 6.1.1)

Let $\mathcal{F} = \{A_\alpha \,:\, \alpha < \mathfrak{c}\}$. $\mathcal{F}$ is infinite by (6.2) and mad by (6.3).

We show that $\mathcal{F}$ is $\mathfrak{c}$-almost decided. First, note that we have $\mathfrak{a} = \mathfrak{c}$ by the assumptions of the theorem. By (6.3), we have:

(6.4) For any $B \in [T]^{\aleph_0}$, if $B \notin \mathcal{I}(\{A_\alpha \,:\, \alpha < \mathfrak{c}\})$ then
$|\,\{\alpha < \mathfrak{c} \,:\, |\,A_\alpha \cap B\,| < \aleph_0\}\,| < \mathfrak{c}$.

Suppose that $\mathcal{H} \in [\mathcal{F}]^{\mathfrak{c}}$ and $B \in \mathcal{H}^\perp$. Then $|\,\{\alpha < \mathfrak{c} \,:\, |\,A_\alpha \cap B\,| < \aleph_0\}\,| = \mathfrak{c}$ and so $B \in \mathcal{I}(\mathcal{F})$ by (6.4). Thus there is a finite $\mathcal{F}' \subset \mathcal{F}$ such that $B \subset^* \bigcup\mathcal{F}'$ and $F \cap B$ is infinite for each $F \in \mathcal{F}'$. But $B \in \mathcal{H}^\perp$ so $\mathcal{F}' \cap \mathcal{H} = \emptyset$. Thus $\mathcal{F}'$ witnesses that $B \in \mathcal{I}(\mathcal{F} \setminus \mathcal{H})$ which was to be proved. ❏ □

For a mad family $\mathcal{F}$ on T, $\mathcal{C} \subseteq \mathcal{F}$ is said to be *minimal in* $\mathcal{F}$ if $\mathfrak{a}^+(\mathcal{F} \setminus \mathcal{C}) = |\,\mathcal{C}\,|$. A mad family $\mathcal{F}$ is said to be λ-*minimal* if every $\mathcal{C} \in [\mathcal{F}]^\lambda$ is minimal in $\mathcal{F}$.

Lemma 6.2 *Suppose that $\mathcal{F}$ is a mad family on T.*
(1) *If $\mathcal{F}$ is $|\,\mathcal{F}\,|$-minimal then $|\,\mathcal{F}\,| = \mathfrak{a}$.*
(2) *If $\mathcal{B} \subseteq \mathcal{F}$ almost decides $\mathcal{F}$ and $\mathcal{F} \setminus \mathcal{B}$ is infinite then $\mathcal{F} \setminus \mathcal{B}$ is minimal in $\mathcal{F}$.*
(3) *If $\mathcal{F}$ is κ-almost decided for $\kappa = |\,\mathcal{F}\,|$ then $\mathcal{F}$ is λ-minimal for all $\omega \leq \lambda < \kappa$.*
(4) *If $|\,\mathcal{F}\,| = \mathfrak{a}$ and $\mathcal{F}$ is $\mathfrak{a}$-almost decided then $\mathcal{F}$ is $\mathfrak{a}$-minimal.*

Proof. (1): If $\mathcal{F}$ is $|\mathcal{F}|$-minimal then $\mathcal{F}$ itself is minimal in $\mathcal{F}$. Thus $\mathfrak{a} = \mathfrak{a}^+(\emptyset) = \mathfrak{a}^+(\mathcal{F} \setminus \mathcal{F}) = |\mathcal{F}|$.

(2): First, note that, for any infinite ad $\mathcal{F}$, we have $\mathfrak{a}(\mathcal{I}(\mathcal{F})) = |\mathcal{F}|$.

Suppose that $\mathcal{F}$ is a mad family on T and $\mathcal{B} \subseteq \mathcal{F}$ almost decides $\mathcal{F}$, i.e. $\mathcal{B}^\perp = \mathcal{I}(\mathcal{F} \setminus \mathcal{B})$. Hence

$$\mathfrak{a}^+(\mathcal{F} \setminus (\mathcal{F} \setminus \mathcal{B})) = \mathfrak{a}^+(\mathcal{B}) = \mathfrak{a}(\mathcal{B}^\perp) = \mathfrak{a}(\mathcal{I}(\mathcal{F} \setminus \mathcal{B})) = |\mathcal{F} \setminus \mathcal{B}|.$$

(3): Suppose that $\kappa = |\mathcal{F}|$ and $\mathcal{F}$ is κ-almost decided. If $\mathcal{C} \in [\mathcal{F}]^\lambda$ for some $\omega \leq \lambda < \kappa$ then $|\mathcal{F} \setminus \mathcal{C}| = \kappa$ and hence $\mathcal{F} \setminus \mathcal{C}$ almost decides $\mathcal{F}$. By (2) it follows that $\mathcal{C} = \mathcal{F} \setminus (\mathcal{F} \setminus \mathcal{C})$ is minimal in $\mathcal{F}$.

(4): Suppose that $|\mathcal{F}| = \mathfrak{a}$ and $\mathcal{F}$ is $\mathfrak{a}$-almost decided. Suppose that $\mathcal{C} \in [\mathcal{F}]^{\mathfrak{a}}$. If $|\mathcal{F} \setminus \mathcal{C}| < \mathfrak{a}$, then clearly $\mathfrak{a}^+(\mathcal{F} \setminus \mathcal{C}) = \mathfrak{a} = |\mathcal{C}|$. Hence $\mathcal{C}$ is minimal in $\mathcal{F}$. If $|\mathcal{F} \setminus \mathcal{C}| = \mathfrak{a}$ then $\mathcal{F} \setminus \mathcal{C}$ almost decides $\mathcal{F}$. Thus, by (2), $\mathcal{C} = \mathcal{F} \setminus (\mathcal{F} \setminus \mathcal{C})$ is again minimal in $\mathcal{F}$. ❏ $\square$

Corollary 6.3 *Assume that* MA(σ-*centered*) *holds. Then there is a mad family $\mathcal{F}$ on T which is λ-minimal for all $\omega \leq \lambda \leq \mathfrak{c}$.*

Proof. By Theorem 6.1 and Lemma 6.2, (3), (4). ❏$\square$ Theorem 6.1 can be further improved to the following theorem:

Theorem 6.4 *Assume that* MA(σ-*centered*) *holds. Let $\kappa = \mathfrak{c}$. Then there is a $\mathcal{C}_\omega$-indestructible mad family $\mathcal{F}$ (of size κ) such that*

(6.5) $V_\omega^{\mathcal{C}} \models \mathcal{F}$ *is κ-almost decided on T.*

Proof. Let $\langle\langle t_\beta, \underset{\sim}{B}_\beta\rangle : \beta < \kappa\rangle$ be an enumeration of

$$T \times \{\underset{\sim}{B} : \underset{\sim}{B} \text{ is a nice } \mathcal{C}_\omega\text{-name of an element of } [T]^{\aleph_0} \text{ in } V^{\mathcal{C}_\omega}\}.$$

Let A_α, $\alpha < \kappa$ be then defined inductively just as in the proof of Theorem 6.1 with

(6.3)′ For $\beta < \alpha$, if $t \Vdash_{\mathcal{C}_\omega}$ "$\underset{\sim}{B}_\alpha \notin \mathcal{I}(\{A_\delta : \delta < \alpha\})$" then $t \Vdash_{\mathcal{C}_\omega}$ "$|A_\alpha \cap \underset{\sim}{B}_\beta| = \aleph_0$"

in place of (6.3). ❏ $\square$

Corollary 6.5 *For any cardinal $\kappa \geq \mathfrak{c}$ in the ground model V there is a cardinal preserving generic extension W of V such that, in W, $\kappa < \mathfrak{c}$ and there is a κ-almost decided mad family $\mathcal{F}$ of size κ (furthermore $\mathcal{F}$ is λ-minimal for all $\omega \leq \lambda \leq \kappa$).*

Proof. First extend V to a model V' of $\kappa = \mathfrak{c}$ and MA(σ-centered). In V', let $\mathcal{F}$ be as in Theorem 6.4. Then $\mathcal{F}$ is as desired in $V_\mu^{\mathcal{C}}$ for any $\mu > \kappa$. The claim in the parentheses follows from Lemma 6.2, (3) and (6.3)′. ❏ $\square$

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SACKS FORCING AND THE SHRINK WRAPPING PROPERTY

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Abstract. We consider a property stronger than the Sacks property, called the *shrink wrapping* property, which holds between the ground model and each Sacks forcing extension. Unlike the Sacks property, the shrink wrapping property does not hold between the ground model and a Silver forcing extension. We also show an application of the shrink wrapping property.

1. The Shrink Wrapping Property

Within this section, we will define the *shrink wrapping property*, which is a strengthening of the Sacks property, which holds between each Sacks forcing extension and the ground model, but not between any Silver forcing extension and the ground model.

Definition 1.1. Given a tree $T \subseteq {}^{<\omega}\omega$ (or $T \subseteq {}^{<\omega}2$) and a node $t \in T$, by $T|t$ we mean the tree T restricted to t:

$$T|t := \{s \in T : s \sqsubseteq t \text{ or } s \sqsupseteq t\}.$$

Definition 1.2. Given a function $f : \omega \to (\omega - \{0\})$, we say that a tree $T \subseteq {}^{<\omega}\omega$ *obeys* f iff for each $l \in \omega$, the set

$$\{n \in \omega : t^\frown n \in T \text{ for some } t \text{ on level } l \text{ of } T\}$$

has size $\leq f(l)$.

Definition 1.3. Let M be a transitive model of ZF. The *Sacks property* holds between V and M iff given any function $f : \omega \to (\omega - \{0\})$ in M satisfying $\lim_{l\to\omega} f(l) = \omega$ and given any $x \in {}^{\omega}\omega$ (in V), there is some tree $T \in M$ which obeys f such that $x \in [T]$.

The Sacks property as we have just defined it is equivalent to the version where we only consider a single such function f in M, instead of all such functions. Suppose that V is a Sacks forcing extension of a model M. Then the *Sacks property* holds between V and M. Now, if V is a Sacks forcing extension of M and $\langle x_n \in {}^{\omega}\omega : n < \omega\rangle$ is a sequence of reals (in V), then we cannot expect there to be a single function f in M and a sequence of trees $\langle T_n \subseteq {}^{<\omega}\omega : n \in \omega\rangle$ in M such that for each n, T_n obeys f and $x_n \in [T_n]$. Because of this, fix the following sequence of functions:

Definition 1.4. Fix $\langle f_i : i < \omega \rangle$, where each $f_i : \omega \to (\omega - \{0\})$ satisfies $\lim_{l \to \omega} f_i(l) = \omega$, and the sequence itself is such that $(i, l) \mapsto f_i(l)$ is an injection.

We can take this sequence to be computable, so that it is contained in every model of ZF. We have that if the Sacks property holds between V and M and $\langle x_n \in {}^\omega\omega : n \in \omega \rangle$ is any sequence of reals, then there is a sequence of trees $\langle T_n : n < \omega \rangle \in M$ such that $(\forall n < \omega)\ T_n$ obeys f_n and $x_n \in [T_n]$.

The following is a stronger property that we might want to hold between V and M: for every sequence $\mathcal{X} = \langle x_n \in {}^\omega\omega : n < \omega \rangle$ there exists a sequence of trees $\langle T_n \subseteq {}^{<\omega}\omega : n < \omega \rangle \in M$ such that

1) $(\forall n \in \omega)\ T_n$ obeys f_n and $x_n \in [T_n]$;
2) $(\forall n_1, n_2 \in \omega)$ one of the following holds:
   a) $x_{n_1} = x_{n_2}$;
   b) $[T_{n_1}] \cap [T_{n_2}] = \emptyset$.

Unfortunately, if the sequence $\mathcal{X}$ satisfies

$$\langle (n_1, n_2) : x_{n_1} = x_{n_2} \rangle \notin M,$$

then there can be no such sequence of trees in M. Thus, we need a weaker notion: a *shrink wrapper*.

Definition 1.5. Fix a canonical bijection $\eta : \omega \to [\omega]^2$ so that for each $\tilde{n} \in \omega$, we may talk about the $\tilde{n}$-th pair $\eta(\tilde{n}) \in [\omega]^2$.

The idea of a shrink wrapper is that for each $\{n_1, n_2\} = \eta(\tilde{n}) \in [\omega]^2$, the functions $F_{\tilde{n},n_1}$ and $F_{\tilde{n},n_2}$, together with the finite sets $I(n_1)$ and $I(n_2)$, will separate x_{n_1} and x_{n_2} as much as possible. For $n \in \eta(\tilde{n})$, the function $F_{\tilde{n},n} : {}^{\tilde{n}}2 \to \mathcal{P}({}^{<\omega}\omega)$ is shrink-wrapping $2^{\tilde{n}}$ possibilities for the value of x_n. We need to make sure that what contains one possibility for x_{n_1} is sufficiently disjoint from what contains another possibility for x_{n_2}, even if it is not possible that simultaneously both x_{n_1} and x_{n_2} are in the respective containers.

Fix $\tilde{n} \in \omega$ and consider the $\tilde{n}$-th pair $\{n_1, n_2\}$. If $x_{n_1} = x_{n_2}$, they certainly cannot be separated and this is a special case. Also, there are finitely many "isolated" points which might prevent the separation of x_{n_1} from x_{n_2}. In fact, we can get a finite set $I(k)$ of isolated points associated to each x_k as opposed to each pair $\{x_{n_1}, x_{n_2}\}$.

When we construct a shrink wrapper for a sequence of reals in a Sacks forcing extension, we can easily get the trees that occur in the shrink wrapper to obey functions in the ground model. To facilitate this, we do the following:

Definition 1.6. Fix an injection $\Phi : {}^{<\omega}2 \times \omega \to \omega$.

In the definition of a shrink wrapper, we will have each $F_{\tilde{n},n}(s)$ be a tree which obeys $f_{\Phi(s,n)}$. Thus, the definition of a shrink wrapper depends on the injection Φ and the injection $(i,l) \mapsto f_i(l)$. However, the reader can check that the choice of these two injections is not important, as long as they are both in the ground model.

Definition 1.7. A *shrink wrapper* $\mathcal{W}$ for $\mathcal{X} = \langle x_n \in {}^{\omega}\omega : n \in \omega \rangle$ is a pair $\langle \mathcal{F}, I \rangle$ such that $I : \omega \to [{}^{\omega}\omega]^{<\omega}$ and $\mathcal{F}$ is a collection of functions $F_{\tilde{n},n}$ for $\tilde{n} \in \omega$ and $n \in \eta(\tilde{n})$ which satisfy the following conditions.

1) Given $\tilde{n}$ and $n \in \eta(\tilde{n})$, $F_{\tilde{n},n} : {}^{\tilde{n}}2 \to \mathcal{P}({}^{<\omega}\omega)$ and for each $s \in {}^{\tilde{n}}2$, $F_{\tilde{n},n}(s) \subseteq {}^{<\omega}\omega$ is a leafless tree that obeys $f_{\Phi(s,n)}$.
2) Given $\tilde{n}$ and $n \in \eta(\tilde{n})$, $(\exists s \in {}^{\tilde{n}}2)\, x_n \in [F_{\tilde{n},n}(s)]$.
3) Given $\{n_1, n_2\} = \eta(\tilde{n})$, $(\forall s_1, s_2 \in {}^{\tilde{n}}2)$ one of the following relationships holds between the sets $C_1 := [F_{\tilde{n},n_1}(s_1)]$ and $C_2 := [F_{\tilde{n},n_2}(s_2)]$:
   3a) $C_1 = C_2$ and if either $x_{n_1} \in C_1$ or $x_{n_2} \in C_2$, then $x_{n_1} = x_{n_2}$;
   3b) $(\exists x \in I(n_1) \cap I(n_2))\, C_1 = C_2 = \{x\}$;
   3c) $C_1 \cap C_2 = \emptyset$, and therefore $(\exists l \in \omega)(\forall (y_1, y_2) \in C_1 \times C_2)\, y_1$ and y_2 differ before level l.

The therefore part of 3c) is because if for each l there was a node on level l of the tree $T := F_{\tilde{n},n_1}(s_1) \cap F_{\tilde{n},n_2}(s_2)$, then because T has finite branching, by Konig's lemma it would have an infinite branch. When we construct a shrink wrapper, we can usually ensure that it satisfies the following additional property:

4) Given $\tilde{n}$ and $n \in \eta(\tilde{n})$, $(\forall s_1, s_2 \in {}^{\tilde{n}}2)$ one of the following relationships holds between the sets $C_1 := [F_{\tilde{n},n}(s_1)]$ and $C_2 := [F_{\tilde{n},n}(s_2)]$:
   4a) $(\exists x \in I(n))\, C_1 = C_2 = \{x\}$;
   4b) $C_1 \cap C_2 = \emptyset$, and therefore $(\exists l \in \omega)(\forall (y_1, y_2) \in C_1 \times C_2)\, y_1$ and y_2 differ before level l.

Note this is a requirement on the single function $F_{\tilde{n},n}$ where $n \in \eta(\tilde{n})$, and not a requirement on the pair of functions $(F_{\tilde{n},n_1}, F_{\tilde{n},n_2})$ where $\{n_1, n_2\} = \eta(\tilde{n})$.

Definition 1.8. Given a model M of ZFC, we say that the *shrink wrapping property* holds between M and V iff every sequence $\mathcal{X} = \langle x_n \in {}^{\omega}\omega : n \in \omega \rangle$ has a shrink wrapper $\mathcal{W}$ in M. A forcing $\mathbb{P}$ has the shrink wrapping property iff the shrink wrapping property holds between the ground model and each forcing extension.

In Theorem 3.7 we will show that Sacks forcing has the shrink wrapping property. If a forcing has the shrink wrapping property, then it automatically has the Sacks property. That is, consider any real x in the forcing extension. Now consider any sequence $\mathcal{X} = \langle x_n : n \in \omega$ such that $x_0 = x$. Let $\tilde{n}$ be such that $0 \in \eta(\tilde{n})$. Let $\mathcal{W}$ be a shrink wrapper for $\mathcal{X}$ in the ground model. We have that x_0 is a path through the tree

$$\bigcup\{F_{\tilde{n},0}(s) : s \in {}^{\tilde{n}}2\},$$

and this tree obeys the function

$$l \mapsto \sum_{s \in {}^{\tilde{n}}2} f_{\Phi(s,0)}(l),$$

which is in M (and does not depend on $\mathcal{X}$).

2. Application to Pointwise Eventual Domination

Before we show that there is always a shrink wrapper in the ground model after doing Sacks forcing, let us discuss an application of shrink wrappers themselves. Given two functions $f, g : {}^{\omega}\omega \to {}^{\omega}\omega$, let us write $f \leq^* g$ and say that g *pointwise eventually dominates* f iff

$$(\forall x \in {}^{\omega}\omega)(\forall^\infty n)\, f(x)(n) \leq g(x)(n).$$

One may ask what is the cofinality of the set of Borel functions from ${}^{\omega}\omega$ to ${}^{\omega}\omega$ ordered by $\leq^*$. The answer is 2^ω, which follows from the result in [1] that given any $A \subseteq \omega$, there is a Baire class one (and therefore Borel) function $f_A : {}^{\omega}\omega \to {}^{\omega}\omega$ such that given any Borel $g : {}^{\omega}\omega \to {}^{\omega}\omega$ satisfying $f_A \leq^* g$, we have that A is Δ^1_1 in any code for g. One may ask what functions f_A have such a property.

Being precise, say that a function $f : {}^{\omega}\omega \to {}^{\omega}\omega$ *sufficiently encodes* $A \subseteq \omega$ iff whenever $g : {}^{\omega}\omega \to {}^{\omega}\omega$ is Borel and satisfies $f \leq^* g$, then $A \in L[c]$ where c is any code for g. What must a function do to sufficiently encode A? Given a sequence $\mathcal{X} = \langle x_n \in {}^{\omega}\omega : n < \omega\rangle$, let us write $f_{\mathcal{X}} : {}^{\omega}\omega \to {}^{\omega}\omega$ for the function

$$f_{\mathcal{X}}(x)(n) := \begin{cases} \min\{l : x(l) \neq x_n(l)\} & \text{if } x \neq x_n, \\ 0 & \text{otherwise.} \end{cases}$$

Given $A \subseteq \omega$, is there always some $\mathcal{X}$ such that $f_{\mathcal{X}}$ sufficiently encodes A? It might seem like the answer is yes, because if a Borel function $g : {}^{\omega}\omega \to \omega$ everywhere dominates one of the sections $x \mapsto f_{\mathcal{X}}(x)(n)$, then x_n is Δ^1_1 in any code for g [1].

However, using a shrink wrapper, we can show that consistently there is not always a function of the form $f_{\mathcal{X}}$ that sufficiently encodes A. Specifically, suppose V is a Sacks forcing extension of an inner model M, $A \notin M$, and $\mathcal{X}$ is a sequence of reals. In the next section, we will show that there is a shrink wrapper $\mathcal{W} \in M$ for $\mathcal{X}$. In this section we will show how to build from $\mathcal{W}$ a Borel function $g : {}^\omega\omega \to {}^\omega\omega$, with a code $c \in M$, satisfying $f_{\mathcal{X}} \leq^* g$. Since $c \in M$, also $L[c] \subseteq M$, which implies $A \notin L[c]$. Hence, $f_{\mathcal{X}}$ does not sufficiently encode A.

To facilitate the discussion, let us make the following definitions:

Definition 2.1. Give $x \in {}^\omega\omega$, $[[x]] \subseteq {}^{<\omega}\omega$ is the set of all finite initial segments of x.

Definition 2.2. Given a tree $T \subseteq {}^{<\omega}\omega$, $\mathrm{Exit}(T) : {}^\omega\omega \to \omega$ is the function

$$\mathrm{Exit}(T)(x) := \begin{cases} \min\{l : x \restriction l \notin T\} & \text{if } x \notin [T], \\ 0 & \text{if } x \in [T]. \end{cases}$$

For the remainder of this section we will show that if M is a transitive model of ZF and a sequence $\mathcal{X}$ of reals has a shrink wrapper in M, then there is a Borel function g with a code in M such that $f_{\mathcal{X}} \leq^* g$. We will illustrate the main ideas by considering a situation where M contains something stronger than a shrink wrapper for $\mathcal{X}$.

Proposition 2.3. *Let M be a transitive model of ZF. Let*

$$\mathcal{X} = \langle x_n \in {}^\omega\omega : n \in \omega \rangle.$$

Suppose

$$\mathcal{T} = \langle T_n : n \in \omega \rangle \in M$$

is a sequence of subtrees of ${}^{<\omega}\omega$ satisfying the following:

1) $(\forall n \in \omega)\, x_n \in [T_n]$.
2) $(\forall n_1, n_2 \in \omega)$ *one of the following holds:*
   a) $x_{n_1} = x_{n_2}$;
   b) $[T_{n_1}] \cap [T_{n_2}] = \emptyset$.

Then there is a Borel function $g : {}^\omega\omega \to {}^\omega\omega$ that has a Borel code in M satisfying

$$(\forall x \in {}^\omega\omega)\, f_{\mathcal{X}}(x) \leq^* g(x).$$

Proof. Let $g : {}^\omega\omega \to {}^\omega\omega$ be defined by

$$g(x)(n) := \max\{\mathrm{Exit}(T_n)(x), n\}.$$

Certainly g is Borel, with a code in M (because $\mathcal{T} \in M$). The "$\mathrm{Exit}(T_n)(x)$" part of the definition is doing most of the work. Specifically, for any $n \in \omega$ and $x \notin [T_n]$,

$$f_{\mathcal{X}}(x)(n) = \mathrm{Exit}([[x_n]])(x) \leq \mathrm{Exit}(T_n)(x).$$

This is because since x_n is a path through the tree T_n, $x \notin [T_n]$ implies the level where x exits T_n is not before the level where x differs from x_n. Thus, we have

$$(\forall n \in \omega)\, x \notin [T_n] \Rightarrow f_{\mathcal{X}}(x)(n) \leq g(x)(n).$$

Suppose, towards a contradiction, that there is some $x \in {}^{\omega}\omega$ satisfying $f_{\mathcal{X}}(x) \not\leq^* g(x)$. Fix such an x. Let A be the infinite set

$$A := \{n \in \omega : f_{\mathcal{X}}(x)(n) > g(x)(n)\}.$$

It must be that $x \in [T_n]$ for each $n \in A$. By hypothesis, this implies $x_{n_1} = x_{n_2}$ for all $n_1, n_2 \in A$. Thus, $f_{\mathcal{X}}(x)(n)$ is the same constant for all $n \in A$. This is a contradiction, because $g(x)(n) \geq n$ for all n. $\square$

Here is the stronger result where we only assume that M has a shrink wrapper for $\mathcal{X}$:

Theorem 2.4. *Let M be a transitive model of ZF. Let*

$$\mathcal{X} = \langle x_n \in {}^{\omega}\omega : n \in \omega \rangle.$$

Suppose $\mathcal{W} = \langle \mathcal{F}, I \rangle \in M$ is a shrink wrapper for $\mathcal{X}$. Then there is a Borel function $g : {}^{\omega}\omega \to {}^{\omega}\omega$ that has a Borel code in M satisfying

$$(\forall x \in {}^{\omega}\omega)\, f_{\mathcal{X}}(x) \leq^* g(x).$$

Proof. For each $n \in \omega$, let $T_n \subseteq {}^{<\omega}\omega$ be the tree

$$T_n := \bigcap \{ \bigcup \mathrm{Im}(F_{\tilde{n},n}) : \tilde{n} \in \omega \wedge n \in \eta(\tilde{n}) \}.$$

That is, for each $t \in {}^{<\omega}\omega$, $t \in T_n$ iff

$$(\forall \tilde{n} \in \omega)[n \in \eta(\tilde{n}) \Rightarrow t \in \bigcup_{s \in {}^{\tilde{n}}2} F_{\tilde{n},n}(s)].$$

By part 2) of the definition of a shrink wrapper,

$$(\forall n \in \omega)\, x_n \in [T_n].$$

Let $e(n_2)$ be the least level l such that if $n_1 < n_2$, $\tilde{n}$ satisfies $\eta(\tilde{n}) = \{n_1, n_2\}$, and $s_1, s_2 \in {}^{\tilde{n}}2$ satisfy $[F_{\tilde{n},n_1}(s_1)] \cap [F_{\tilde{n},n_2}(s_2)] = \emptyset$, then all elements of $[F_{\tilde{n},n_1}(s_1)]$ differ from all elements of $[F_{\tilde{n},n_2}(s_2)]$ before level l.

Let $g : {}^{\omega}\omega \to {}^{\omega}\omega$ be defined by

$$g(x)(n) := \max\{\mathrm{Exit}(T_n)(x), e(n), n\}.$$

Certainly g is Borel, with a code in M (because $\mathcal{W} \in M$). Just like in the previous proposition, since $x_n \in [T_n]$, for all $x \in {}^\omega\omega$ and $n \in \omega$ we have

$$x \notin [T_n] \Rightarrow f_{\mathcal{X}}(x)(n) \leq g(x)(n).$$

Suppose, towards a contradiction, that there is some $x \in {}^\omega\omega$ satisfying $f_{\mathcal{X}}(x) \not\leq^* g(x)$. Fix such an x. Let A be the infinite set

$$A := \{n \in \omega : f_{\mathcal{X}}(x)(n) > g(x)(n)\}.$$

It must be that $x \in [T_n]$ for each $n \in A$. Since A is infinite, we may fix $n_1, n_2 \in A$ satisfying the following:

i) $n_1 < n_2$;
ii) $f_{\mathcal{X}}(x)(n_1) \leq n_2$.

Let $\tilde{n}$ satisfy $\eta(\tilde{n}) = \{n_1, n_2\}$. Since $x \in [T_{n_1}]$, fix some $s_1 \in {}^{\tilde{n}}2$ satisfying

$$x \in [F_{\tilde{n},n_1}(s_1)] =: C_1.$$

Also, since $x_{n_2} \in [T_{n_2}]$, fix some $s_2 \in {}^{\tilde{n}}2$ satisfying

$$x_{n_2} \in [F_{\tilde{n},n_2}(s_2)] =: C_2.$$

Because $f_{\mathcal{X}}(x)(n_2) > g(x)(n_2)$, we have $f_{\mathcal{X}}(x)(n_2) > e(n_2)$, so

$$\mathrm{Exit}([[x_{n_2}]])(x) > e(n_2).$$

This, combining with the definition of $e(n_2)$ and the fact that $x \in C_1$ and $x_{n_2} \in C_2$ tells us that $C_1 \cap C_2 \neq \emptyset$ (because otherwise $x \in C_1$ and $x_{n_2} \in C_2$ would differ before level $e(n_2)$, which by definition of $e(n_2)$ would mean that $\mathrm{Exit}([[x_{n_2}]])(x) \leq e(n_2)$). Thus, by part 3) of the definition of a separation device, one of the following holds:

a) $x_{n_1} = x_{n_2}$;
b) $C_1 = C_2 = \{x\}$.

Now, b) cannot be the case because $C_2 = \{x\}$ implies $x_{n_2} = x$, which implies $f_{\mathcal{X}}(x)(n_2) = 0$, which contradicts the fact that $f_{\mathcal{X}}(x)(n_2) > g(x)(n_2)$. On the other hand, a) cannot be the case because $x_{n_1} = x_{n_2}$ implies $f_{\mathcal{X}}(x)(n_1) = f_{\mathcal{X}}(x)(n_2)$, which by ii) implies

$$f_{\mathcal{X}}(x)(n_2) = f_{\mathcal{X}}(x)(n_1) \leq n_2 \leq g(x)(n_2) < f_{\mathcal{X}}(x)(n_2),$$

which is impossible. $\square$

3. Sacks Forcing

In this section, we will show that the shrink wrapping property holds between the ground model and any Sacks forcing extension.

Definition 3.1. A tree $p \subseteq {}^{<\omega}2$ is *perfect* iff it is nonempty and for each $t \in p$, there are incompatible $t_1, t_2 \in p$ extending t. Sacks forcing $\mathbb{S}$ is the poset of all perfect trees $p \subseteq {}^{<\omega}2$, where $p_1 \leq p_2$ iff $p_1 \subseteq p_2$.

Given $p_1, p_2 \in \mathbb{S}$, $p_1 \perp p_2$ means that p_1 and p_2 are incompatible.

Definition 3.2. Let $p \subseteq {}^{<\omega}2$ be a perfect tree. A node $t \in p$ is called a *branching node* iff $t^\frown 0, t^\frown 1 \in p$. Stem$(p)$ is the unique branching node t of p such that all elements of p are comparable to t. A node $t \in p$ is said to be an *n-th branching node* iff it is a branching node and there are exactly n branching nodes that are proper initial segments of it. In particular, Stem(p) is the unique 0-th branching node of p. Given Sacks conditions p, q, we write $q \leq_n p$ iff $q \leq p$ and all of the k-th branching nodes, for $k \leq n$, of p are in q and are branching nodes.

Lemma 3.3 (Fusion Lemma)**.** *Let $\langle p_n : n \in \omega \rangle$ be a sequence of Sacks conditions such that*

$$p_0 \geq_0 p_1 \geq_1 p_2 \geq_2 \ldots .$$

Then $p_\omega := \bigcap_{n\in\omega} p_n$ is a Sacks condition below each p_n.

Proof. This is standard and can be found in introductory presentations of Sacks forcing. See, for example, [2]. □

The sequence $\langle p_n : n \in \omega \rangle$ in the lemma above is known as a *fusion sequence*. The following will help in the construction of fusion sequences.

Lemma 3.4 (Fusion Helper Lemma)**.** *Let $R : {}^{<\omega}2 \to \mathbb{S}$ be a function with the following properties:*

1) $(\forall s_1, s_2 \in {}^{<\omega}2)$ $s_2 \sqsupseteq s_1$ *implies* $R(s_2) \leq R(s_1)$*;*
2) $(\forall s \in {}^{<\omega}2)$ *Stem*$(R(s^\frown 0)) \perp$ *Stem*$(R(s^\frown 1))$*.*

For each $n \in \omega$, let p_n be the Sacks condition

$$p_n := \bigcup \{R(s) : s \in {}^{n}2\}.$$

Then

$$R(\emptyset) = p_0 \geq p_1 \geq_0 p_2 \geq_1 p_3 \geq_2 \ldots$$

is a fusion sequence.

Proof. Consider any $n \geq 1$. Certainly $p_n \supseteq p_{n+1}$, because for each $s \in {}^{n}2$, $R(s) \supseteq R(s^\frown 0) \cup R(s^\frown 1)$. To show that $p_n \geq_{n-1} p_{n+1}$, consider a k-th branching node t of p_n for some $k \leq n-1$. One can check that there is some $s \in {}^{k}2$ such that t is the largest common initial segment of Stem$(R(s^\frown 0))$ and Stem$(R(s^\frown 1))$. Since

$$\mathrm{Stem}(R(s^\frown 0)) \cup \mathrm{Stem}(R(s^\frown 1)) \subseteq R(s^\frown 0) \cup R(s^\frown 1) \subseteq p_{n+1},$$

we have that t is a branching node of p_{n+1}. Thus, we have shown that for each $k \leq n-1$, each k-th branching node of p_n is a branching node of p_{n+1}. Hence, $p_n \geq_{n-1} p_{n+1}$. $\square$

We present a forcing lemma that is a basic building block for separating x_{n_1} from x_{n_2}. Combining this with a fusion argument gives us the result.

Lemma 3.5. *Let $\mathbb{P}$ be any forcing. Let $p_0, p_1 \in \mathbb{P}$ be conditions. Let $\dot{\tau}_0, \dot{\tau}_1$ be names for elements of ${}^\omega\omega$. Suppose that there is no $x \in {}^\omega\omega$ satisfying the following two statements:*

1) $p_0 \Vdash \dot{\tau}_0 = \check{x}$;
2) $p_1 \Vdash \dot{\tau}_1 = \check{x}$.

Then there exist $p_0' \leq p_0$; $p_1' \leq p_1$; and $t_0, t_1 \in {}^{<\omega}\omega$ satisfying the following:

3) $t_0 \perp t_1$,
4) $p_0' \Vdash \dot{\tau}_0 \sqsupseteq \check{t}_0$,
5) $p_1' \Vdash \dot{\tau}_1 \sqsupseteq \check{t}_1$.

Proof. There are two cases to consider. The first is that there exists some $x \in {}^\omega\omega$ such that 1) is true. When this happens, 2) is false. Hence, there exist $t_1 \in {}^{<\omega}\omega$ and $p_1' \leq p_1$ such that 5) is true and $x \not\sqsupseteq t_1$. Letting $p_0' := p_0$ and t_0 be some initial segment of x incompatible with t_1, we see that 3) and 4) are true.

The second case is that there is no $x \in {}^\omega\omega$ satisfying 1). When this happens, there exist conditions $p_0^a, p_0^b \leq p_0$ and incompatible nodes $s_a, s_b \in {}^{<\omega}\omega$ satisfying both $p_0^a \Vdash \dot{\tau}_0 \sqsupseteq \check{s}_a$ and $p_0^b \Vdash \dot{\tau}_0 \sqsupseteq \check{s}_b$. Now, it cannot be that both $p_1 \Vdash \dot{\tau}_1 \sqsupseteq \check{s}_a$ and $p_1 \Vdash \dot{\tau}_1 \sqsupseteq \check{s}_b$. Assume, without loss of generality, that $p_1 \nVdash \dot{\tau}_1 \sqsupseteq \check{s}_a$. This implies that there exist $p_1' \leq p_1$ and $t_1 \in {}^{<\omega}\omega$ such that $s_a \perp t_1$ and $p_1' \Vdash \dot{\tau}_1 \sqsupseteq \check{t}_1$. Letting $p_0' := p_0^a$ and $t_0 := s_a$, we are done. $\square$

At this point, the reader may want to think about how to use this lemma to prove that if V is a Sacks forcing extension of a transitive model M of ZF and $\mathcal{X} = \langle x_n \in {}^\omega\omega : n \in \omega \rangle$ satisfies

$$(\forall n \in \omega)\, x_n \notin M$$

and

$$\{(n_1, n_2) : x_{n_1} = x_{n_2}\} \in M,$$

then there is a sequence $\mathcal{T}$ of subtrees of ${}^{<\omega}\omega$ satisfying the hypotheses of Proposition 2.3.

The next lemma explains the appearance of I in the definition of a shrink wrapper. We are intending the name $\dot{\tau}$ to be such that $\dot{\tau}(n)$ refers to the x_n in the sequence $\mathcal{X} = \langle x_n : n \in \omega \rangle$.

Lemma 3.6. *Consider Sacks forcing $\mathbb{S}$. Let $p \in \mathbb{S}$ be a condition and $\dot\tau$ a name satisfying $p \Vdash \dot\tau : \omega \to {}^\omega\omega$. Then there exists a condition $p' \leq p$ and there exists a function $I : \omega \to [{}^\omega\omega]^{<\omega}$ satisfying*

$$p' \Vdash (\forall n \in \omega)\, \dot\tau(n) \in \check{V} \to \dot\tau(n) \in \check{I}(n).$$

Proof. We may easily construct a function $R : \omega \to \mathbb{S}$ that satisfies the conditions of Lemma 3.4 such that $R(\emptyset) \leq p$ and for each $s \in {}^n 2$, either $R(s) \Vdash \dot\tau(n) \notin \check{V}$ or $(\exists x \in {}^\omega\omega)\, R(s) \Vdash \dot\tau(n) = \check{x}$. Define I as follows:

$$I(n) := \{x \in {}^\omega\omega : (\exists s \in {}^n 2)\, R(s) \Vdash \dot\tau(n) = \check{x}\}.$$

Let $p' := \bigcap_n \bigcup\{R(s) : s \in {}^n 2\}$. The condition p' and the function I are as desired. □

We are now ready for the main forcing argument of this section.

Theorem 3.7. *Consider Sacks forcing $\mathbb{S}$. Let $p \in \mathbb{S}$ be a condition and $\dot\tau$ be a name satisfying $p \Vdash \dot\tau : \omega \to {}^\omega\omega$. Then there exists a condition $q \leq p$ and there exists $\mathcal{W} = \langle \mathcal{F}, I \rangle$ satisfying*

$$q \Vdash \check{\mathcal{W}} \textit{ is a shrink wrapper for } \langle \dot\tau(n) : n \in \omega \rangle.$$

Proof. First, let $p' \leq p$ and $I : \omega \to [{}^\omega\omega]^{<\omega}$ be given by the lemma above. That is, for each $n \in \omega$,

$$p' \Vdash \dot\tau(\check{n}) \in \check{V} \to \dot\tau(\check{n}) \in \check{I}(\check{n}).$$

We will define a function $R : {}^{<\omega}2 \to \mathbb{S}$ with $R(\emptyset) \leq p'$ satisfying conditions 1) and 2) of Lemma 3.4. At the same time, we will construct a family of functions

$$\mathcal{F} = \langle F_{\tilde n, n} : \tilde n \in \omega, n \in \eta(\tilde n) \rangle.$$

Let $\mathcal{W} = \langle \mathcal{F}, I \rangle$. Our q will be

$$q := \bigcap_{\tilde n} \bigcup_{s \in {}^{\tilde n}2} R(s).$$

The function $F_{\tilde n, n}$ will return leafless subtrees of ${}^{<\omega}\omega$. Moreover, each tree $F_{\tilde n, n}(s)$ will obey the function $f_{\Phi(s,n)}$. We will have it so for all $n \in \omega$ and all $\tilde n$ satisfying $n \in \eta(\tilde n)$,

$$(\forall s \in {}^{\tilde n}2)\, R(s) \Vdash \dot\tau(\check{n}) \in [\check{F}_{\tilde n, n}(\check{s})].$$

Thus, q will force that $\mathcal{W}$ satisfies conditions 1) and 2) of the definition of a shrink wrapper. To show that q forces condition 3) of that definition, it suffices to show that for all $\{n_1, n_2\} = \eta(\tilde n)$ and all $s_1, s_2 \in {}^{\tilde n}2$, one of the following holds, where $T_1 := F_{\tilde n, n_1}(s_1)$ and $T_2 := F_{\tilde n, n_2}(s_2)$:

3a′) $T_1 = T_2$ and $(\forall s \in {}^{\tilde n}2)$,

$$R(s) \Vdash (\dot\tau(\check{n}_1) \in [\check{T}_1] \vee \dot\tau(\check{n}_2) \in [\check{T}_2]) \to \dot\tau(\check{n}_1) = \dot\tau(\check{n}_2);$$

3b′) $(\exists x \in I(n_1) \cap I(n_2))\, [T_1] = [T_2] = \{x\}$;
3c′) $[T_1] \cap [T_2] = \emptyset$, and moreover $\mathrm{Stem}(T_1) \perp \mathrm{Stem}(T_2)$.

We will define the functions $F_{\tilde{n},n}$ and the conditions $R(s)$ for $s \in {}^{\tilde{n}}2$ by induction on $\tilde{n}$. Beginning at $\tilde{n} = 0$, let $\{n_1, n_2\} = \eta(0)$. We will define F_{0,n_1}, F_{0,n_2}, and $R(\emptyset) \le p'$. If $p' \Vdash \dot{\tau}(\check{n}_1) = \dot{\tau}(\check{n}_2)$, then let $R(\emptyset) := p'$ and define $F_{0,n_1}(\emptyset) = F_{0,n_2}(\emptyset) = T$, where $T \subseteq {}^{<\omega}\omega$ is a tree that obeys both $f_{\Phi(\emptyset,n_1)}$ and $f_{\Phi(\emptyset,n_2)}$, such that $p' \Vdash \dot{\tau}(\check{n}_1) \in [\check{T}]$. Such a T is guaranteed to exist because $\mathbb{S}$ has the Sacks property. This causes 3a′) to be satisfied. If $p' \not\Vdash \dot{\tau}(\check{n}_1) = \dot{\tau}(\check{n}_2)$, then let $t_1, t_2 \in {}^{<\omega}\omega$ be incomparable nodes and let $R(\emptyset) \le p'$ satisfy $R(\emptyset) \Vdash \dot{\tau}(\check{n}_1) \sqsupseteq \check{t}_1$ and $R(\emptyset) \Vdash \dot{\tau}(\check{n}_2) \sqsupseteq \check{t}_2$. Then we may define $F_{0,n_1}(\emptyset) = T_1$ and $F_{0,n_2}(\emptyset) = T_2$ where T_1 and T_2 are leafless trees that obey $f_{\Phi(\emptyset,n_1)}$ and $f_{\Phi(\emptyset,n_2)}$ respectively such that $\mathrm{Stem}(T_1) \sqsupseteq t_1$, $\mathrm{Stem}(T_2) \sqsupseteq t_2$, $R(\emptyset) \Vdash \dot{\tau}(\check{n}_1) \in [\check{T}_1]$, and $R(\emptyset) \Vdash \dot{\tau}(\check{n}_2) \in [\check{T}_2]$. This causes 3c′) to be satisfied.

We will now handle the successor step of the induction. Let $\{n_1, n_2\} = \eta(\tilde{n})$ for some $\tilde{n} > 0$. We will define $R(s)$ for each $s \in {}^{\tilde{n}}2$, and we will define both $F_{\tilde{n},n_1}$ and $F_{\tilde{n},n_2}$ assuming $R(s')$ has been defined for each $s' \in {}^{<\tilde{n}}2$. To keep the construction readable, we will start with initial values for the $R(s)$'s and the $F_{\tilde{n},n}$'s, and we will modify them as the construction progresses until we arrive at their final values. That is, we will say "replace $R(s)$ with a stronger condition..." and "shrink the tree $F_{\tilde{n},n}(s)$...". When we make these replacements, it is understood that still $R(s) \Vdash \dot{\tau}(\check{n}) \in [\check{F}_{\tilde{n},n}(\check{s})]$. The construction consists of 5 steps.

Step 1: First, for each $s \in {}^{(\tilde{n}-1)}2$, let $R(s^\frown 0)$ and $R(s^\frown 1)$ be arbitrary extensions of $R(s)$ such that $\mathrm{Stem}(R(s^\frown 0)) \perp \mathrm{Stem}(R(s^\frown 1))$. Also, for each $n \in \{n_1, n_2\}$ and $s \in {}^{\tilde{n}}2$, let $F_{\tilde{n},n}(s)$ be a leafless subtree of ${}^{<\omega}\omega$ that obeys $f_{\Phi(s,n)}$ and satisfies $R(s) \Vdash \dot{\tau}(\check{n}) \in [\check{F}_{\tilde{n},n}(\check{s})]$.

Step 2: For each $s \in {}^{\tilde{n}}2$ and $n \in \{n_1, n_2\}$, strengthen $R(s)$ so that either $R(s) \Vdash \dot{\tau}(\check{n}) \notin \check{V}$ or $(\exists x \in I(n))\, R(s) \Vdash \dot{\tau}(\check{n}) = \check{x}$. If the latter case holds, shrink $F_{\tilde{n},n}(s)$ so that it has only one path.

Step 3: For this step, fix $n \in \{n_1, n_2\}$. For each pair of distinct $s_1, s_2 \in {}^{\tilde{n}}2$, strengthen each $R(s_1)$ and $R(s_2)$ and shrink each $F_{\tilde{n},n}(s_1)$ and $F_{\tilde{n},n}(s_2)$ so that one of the following holds:

i) $(\exists x \in I(n))\, [F_{\tilde{n},n}(s_1)] = [F_{\tilde{n},n}(s_2)] = \{x\}$;
ii) $\mathrm{Stem}(F_{\tilde{n},n}(s_1)) \perp \mathrm{Stem}(F_{\tilde{n},n}(s_2))$.

That is, if i) cannot be satisfied, then we may use Lemma 3.5 to satisfy ii).

Step 4: For each pair of distinct $s_1, s_2 \in {}^{\tilde{n}}2$ such that either $R(s_1) \Vdash \dot{\tau}(\check{n}_1) \notin \check{V}$ or $R(s_2) \Vdash \dot{\tau}(\check{n}_2) \notin \check{V}$, use Lemma 3.5 to strengthen $R(s_1)$

and $R(s_2)$ and shrink $F_{\tilde{n},n_1}(s_1)$ and $F_{\tilde{n},n_1}(s_1)$ so that

$$\mathrm{Stem}(F_{\tilde{n},n_1}(s_1)) \perp \mathrm{Stem}(F_{\tilde{n},n_2}(s_2)).$$

Step 5: For each $s \in {}^{\tilde{n}}2$, do the following: If $R(s) \Vdash \dot{\tau}(\check{n}_1) = \dot{\tau}(\check{n}_2)$, then replace both $F_{\tilde{n},n_1}(s)$ and $F_{\tilde{n},n_2}(s)$ with $F_{\tilde{n},n_1}(s) \cap F_{\tilde{n},n_2}(s)$. Otherwise, strengthen $R(s)$ and shrink $F_{\tilde{n},n_1}(s)$ and $F_{\tilde{n},n_2}(s)$ so that

$$\mathrm{Stem}(F_{\tilde{n},n_1}(s)) \perp \mathrm{Stem}(F_{\tilde{n},n_2}(s)).$$

This completes the construction of $\{R(s) : s \in {}^{\tilde{n}}2\}$, $F_{\tilde{n},n_1}$, and $F_{\tilde{n},n_2}$. We will now prove that it works. Fix $\tilde{n} \in \omega$ and $s_1, s_2 \in {}^{\tilde{n}}2$. Let $T_1 := F_{\tilde{n},n_1}(s_1)$ and $T_2 := F_{\tilde{n},n_2}(s_2)$. We must show that one of 3a′), 3b′), or 3c′) holds. The cleanest way to do this is to break into cases depending on whether or not $s_1 = s_2$.

Case $s_1 \neq s_2$: If either $R(s_1) \Vdash \dot{\tau}(\check{n}_1) \notin \check{V}$ or $R(s_2) \Vdash \dot{\tau}(\check{n}_2) \notin \check{V}$, then by Step 4, we see that 3c′) holds. Otherwise, by Step 2, $(\exists x \in I(n_1))\,[T_1] = \{x\}$ and $(\exists x \in I(n_1))\,[T_2] = \{x\}$. Hence, either 3b′) or 3c′) holds.

Case $s_1 = s_2$: If $R(s_1) \not\Vdash \dot{\tau}(\check{n}_1) = \dot{\tau}(\check{n}_2)$, then by Step 5, we see that 3c′) holds. Otherwise, we are in the case that

$$R(s_1) \Vdash \dot{\tau}(\check{n}_1) = \dot{\tau}(\check{n}_2).$$

By Step 5, $T_1 = T_2$. Now, if $R(s_1) \Vdash \dot{\tau}(\check{n}_1) \in \check{V}$, then of course also $R(s_1) \Vdash \dot{\tau}(\check{n}_2) \in \check{V}$, and by Step 2) we see that 3b′) holds. Otherwise, $R(s_1) \Vdash \dot{\tau}(\check{n}_1) \notin \check{V}$. Hence, $[T_1]$ is not a singleton. We will show that 3a′) holds. Consider any $s \in {}^{\tilde{n}}2$. We must show

$$R(s) \Vdash (\dot{\tau}(\check{n}_1) \in [\check{T}_1] \vee \dot{\tau}(\check{n}_2) \in [\check{T}_1]) \to \dot{\tau}(\check{n}_1) = \dot{\tau}(\check{n}_2).$$

If $s = s_1$, we are done. Now suppose $s \neq s_1$. It suffices to show

$$R(s) \Vdash \neg(\dot{\tau}(\check{n}_1) \in [\check{T}_1] \vee \dot{\tau}(\check{n}_2) \in [\check{T}_1]).$$

That is, it suffices to show $R(s) \Vdash \dot{\tau}(\check{n}_1) \notin [\check{T}_1]$ and $R(s) \Vdash \dot{\tau}(\check{n}_2) \notin [\check{T}_1]$. Since $s \neq s_1$ and $[T_1]$ is not a singleton, by Step 3, $\mathrm{Stem}(F_{\tilde{n},n}(s)) \perp \mathrm{Stem}(T_1)$. Recall that

$$R(s) \Vdash \dot{\tau}(\check{n}_1) \in [\check{F}_{\tilde{n},n}(\check{s})].$$

Hence, since $[\check{F}_{\tilde{n},n}(\check{s})] \cap [T_1] = \emptyset$, $R(s) \Vdash \dot{\tau}(\check{n}_1) \notin [\check{T}_1]$. By a similar argument, $R(s) \Vdash \dot{\tau}(\check{n}_2) \notin [\check{T}_1]$. This completes the proof. $\square$

4. Silver Forcing

In this section, we will show that the shrink wrapping property does not hold between the ground model and any Silver forcing extension.

Definition 4.1. A tree $T \subseteq {}^{<\omega}2$ is a *Silver* tree iff it is leafless and the following are satisfied. There is an infinite set of levels $L \subseteq \omega$ such that for each $t \in T$, if $\mathrm{Dom}(t) \in L$, then both $t^\frown 0$ and $t^\frown 1$ are in T, and if $\mathrm{Dom}(t) \notin L$, then exactly one of $t^\frown 0$ or $t^\frown 1$ is in T. Also, if $x_1, x_2 \in [T]$ are two paths through T and $l \notin L$, then $x_1(l) = x_2(l)$. The poset of all Silver trees ordered by inclusion is called Silver forcing $\mathbb{V}$.

Fact 4.2. *Suppose G is $\mathbb{V}$-generic over V. Let $g = \bigcap G$. Then*

$$\{T \in \mathbb{V} : g \in [T]\} = G.$$

For this reason, we will sometimes say that g is $\mathbb{V}$-generic over V.

Definition 4.3. Let $p \subseteq {}^{<\omega}2$ be a tree. Let $t, s \in p$ be such that $\mathrm{Dom}(t) = \mathrm{Dom}(s)$. When we say "replace p below t with p below s", we mean replace p with

$$\{u \in p : u \not\supseteq t\} \cup \{t^\frown w \in p : s^\frown w \in p\}.$$

That is, the subtree of p below s is replacing the subtree of p below t.

In the following we will talk about elementary submodels of V, but we might as well be talking about elementary submodels of $V_\Theta \subseteq V$ for some large enough ordinal Θ.

Lemma 4.4. *Let M be a countable elementary submodel of V and let $p \in \mathbb{V}$ be in M. Then there is some $p' \leq p$ (not in M) such that each branch through p' is $\mathbb{V}$-generic over M.*

Proof. Let $\langle U_n : n \in \omega \rangle$ be an enumeration of the dense subsets of $\mathbb{V}$ that are in M. We will define a decreasing sequence of conditions $p = p_{-1} \geq p_0 \geq p_1 \geq \ldots$ in M. Now fix $n \geq 0$ and suppose we have defined this sequence for $p_{-1} \geq \ldots \geq p_{n-1}$. We will define p_n. Let $\langle t_n^i : i < 2^n \rangle$ be the nodes on the n-th splitting level of p_{n-1}. First shrink $p_{n-1}|t_n^0$ to be within U_n, calling the resulting condition p_{n-1}^0. This shrinking is possible because p_{n-1} is in M. Then for each $i \neq 0$, replace p_{n-1}^0 below t_n^i with p_{n-1}^0 below t_n^0. Call the resulting condition $\tilde{p}_{n-1}^0$. Then shrink $\tilde{p}_{n-1}^0|t_n^1$ to be within U_n, calling the resulting condition p_{n-1}^1. Then for each $i \neq 1$, replace p_{n-1}^1 below t_n^i with p_{n-1}^1 below t_n^1. Call the resulting condition $\tilde{p}_{n-1}^1$. Continue this for all $i < 2^n$. After all this shrinking, let $p_n := \tilde{p}_{n-1}^{2^n-1}$. Now unfix n. Note that $p_n \in \mathbb{V}$.

We have now constructed the sequence $p = p_{-1} \geq p_0 \geq p_1 \geq \ldots$ with the property that for each $n \in \omega$, each branch through p_n is a path through some element of U_n. Let $p' = \bigcap_{n \in \omega} p_n$. Then each branch through p' is a branch through an element of each U_n. Hence, each branch through p' is $\mathbb{V}$-generic over M. $\square$

Theorem 4.5. *Consider Silver forcing $\mathbb{V}$. There is some $\dot{\mathcal{X}}$ such that there is no p and $\mathcal{W}$ such that $p \Vdash \check{\mathcal{W}}$ is a shrink wrapper for $\dot{\mathcal{X}}$.*

Proof. Given a function $r : \omega \to 2$ and $n \in \omega$, let $\text{Flatten}(r, n) : \omega \to 2$ be the function

$$\text{Flatten}(r, n)(i) := \begin{cases} 0 & \text{if } i \leq n, \\ r(i) & \text{otherwise.} \end{cases}$$

Let $\dot{r}$ be the canonical name for the generic real. We have $1 \Vdash \dot{r} : \omega \to 2$. Let $\vec{0} \in {}^\omega 2$ be the constant zero function. Let $\langle \dot{x}_n \in {}^\omega 2 : n \in \omega \rangle$ be a sequence of names such that for each $n \in \omega$,

$$1 \Vdash \dot{x}_{2n} = \begin{cases} \text{Flatten}(\dot{r}, n) & \text{if } \dot{r}(n) = 0, \\ \vec{0} & \text{if } \dot{r}(n) = 1, \end{cases}$$

and

$$1 \Vdash \dot{x}_{2n+1} = \begin{cases} \vec{0} & \text{if } \dot{r}(n) = 0, \\ \text{Flatten}(\dot{r}, n) & \text{if } \dot{r}(n) = 1. \end{cases}$$

That is, one of $\dot{x}_{2n}$ and $\dot{x}_{2n+1}$ will be a final segment of the generic real with initial zeros, and the other will the constant zero function. Define $\dot{\mathcal{X}}$ such that

$$1 \Vdash \dot{\mathcal{X}} = \langle \dot{x}_n : n \in \omega \rangle.$$

Suppose there is some condition p and some $\mathcal{W} = \langle \mathcal{F}, I \rangle$ such that

$$p \Vdash \check{\mathcal{W}} \text{ is a shrink wrapper for } \dot{\mathcal{X}}.$$

We will find a contradiction.

Let M be a countable elementary substructure of V such that $p, \mathcal{W}, \dot{\mathcal{X}} \in M$. By Lemma 4.4, let $p' \leq p$ be such that all branches through p' are $\mathbb{V}$-generic over M. Let $n := |\text{Stem}(p')|$. Let $\tilde{n} \in \omega$ be such that $\{2n, 2n+1\} = \eta(\tilde{n})$. That is, $\{2n, 2n+1\}$ is the $\tilde{n}$-th pair.

Let r_0 be the leftmost branch through $p' | (\text{Stem}(p')^\frown 0)$ and let r_1 be the leftmost branch through $p' | (\text{Stem}(p')^\frown 1)$. Hence, $r_0(l) = r_1(l)$ for all $l \neq n$. Let $u : \omega \to 2$ be such that

$$u = \text{Flatten}(r_0, n) = \text{Flatten}(r_1, n).$$

Note that $u \notin M$.

Now,

$$M \models p \Vdash \check{W} \text{ is a shrink wrapper for } \dot{\mathcal{X}}.$$

Given a name $\dot{\tau}$ and a generic filter G, let $\dot{\tau}_G$ refer to the valuation of $\dot{\tau}$ with respect to G. Since r_0 and r_1 are both paths through p', they are generic over M. Thus, we have

$$M[r_0] \models \mathcal{W} \text{ is a shrink wrapper for } \dot{\mathcal{X}}_{r_0}.$$

By part 2) of Definition 1.7, we have

$$M[r_0] \models (\exists s_1 \in {}^{\tilde{n}}2)(\dot{x}_{2n})_{r_0} \in [F_{\tilde{n},2n}(s_1)].$$

Fix this s_1. Let $T_1 := F_{\tilde{n},2n}(s_1)$. We have

$$(\dot{x}_{2n})_{r_0}^{M[r_0]} \in [T_1].$$

Similarly, we have

$$M[r_1] \models (\exists s_2 \in {}^{\tilde{n}}2)(\dot{x}_{2n+1})_{r_1} \in [F_{\tilde{n},2n+1}(s_2)].$$

Fix this s_2. Let $T_2 = F_{\tilde{n},2n+1}(s_2)$. We have

$$(\dot{x}_{2n+1})_{r_1}^{M[r_1]} \in [T_2].$$

Here is the crucial part: by the definition of r_0, r_1, and $\dot{\mathcal{X}}$, since $r_0(n) = 0$ and $r_1(n) = 1$, we have

$$(\dot{x}_{2n})_{r_0}^{M[r_0]} = \mathrm{Flatten}(r_0, n) = u = \mathrm{Flatten}(r_1, n) = (\dot{x}_{2n+1})_{r_1}^{M[r_1]}.$$

Thus, we have

$$[T_1] \cap [T_2] \neq \emptyset.$$

Since $\mathcal{W} \in M[r_0]$, by absoluteness we have

$$M[r_0] \models [T_1] \cap [T_2] \neq \emptyset.$$

Working in $M[r_0]$, by part 3) of Definition 1.7 applied to $C_1 := [T_1]$ and $C_2 := [T_2]$, it must be that either 3a) or 3b) holds. Note that

$$(\dot{x}_{2n})_{r_0}^{M[r_0]} = u \notin M,$$

which implies that 3b) cannot hold. Since $(\dot{x}_{2n})_{r_0}^{M[r_0]} \in [T_1]$, using 3a) we have that

$$M[r_0] \models (\dot{x}_{2n})_{r_0} = (\dot{x}_{2n+1})_{r_0}.$$

Thus,

$$(\dot{x}_{2n})_{r_0}^{M[r_0]} = (\dot{x}_{2n+1})_{r_0}^{M[r_0]}.$$

We already know that the left hand side of this equation is u. On the other hand, by definition, the right hand side must be $\vec{0}$. This is a contradiction. $\square$

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