Forcing with copies of the Rado and Henson graphs

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Abstract

If $\mathbb{B}$ is a relational structure, define $\mathbb{P}(\mathbb{B})$ the partial order of all substructures of $\mathbb{B}$ that are isomorphic to it. Improving a result of Kurilić and the second author, we prove that if $\mathcal{R}$ is the random graph, then $\mathbb{P}(\mathcal{R})$ is forcing equivalent to $\mathbb{S} * \dot{\mathbb{R}}$, where $\mathbb{S}$ is Sacks forcing and $\dot{\mathbb{R}}$ is an ω-distributive forcing that is not forcing equivalent to a σ-closed one. We also prove that $\mathbb{P}(\mathcal{H}_3)$ is forcing equivalent to a σ-closed forcing, where $\mathcal{H}_3$ is the generic triangle-free graph.

Introduction

Let $\mathbb{B}$ be a countable relational structure. By $\mathbb{P}(\mathbb{B})$ we denote the set of all copies of $\mathbb{B}$ in itself, i.e. the set of all substructures $\mathbb{A}$ of $\mathbb{B}$ such that $\mathbb{A}$ and $\mathbb{B}$ are isomorphic.[1] If $\mathbb{A}, \mathbb{C} \in \mathbb{P}(\mathbb{B})$ define $\mathbb{A} \leq \mathbb{C}$ if $\mathbb{A}$ is a substructure of $\mathbb{C}$. We are interested in the forcing properties of the partial order $(\mathbb{P}(\mathbb{B}), \leq)$. The study of $\mathbb{P}(\mathbb{B})$ is interesting since it give us information of "how the copies of $\mathbb{B}$ are placed inside $\mathbb{B}$". The more we understand $\mathbb{P}(\mathbb{B})$, the more we will understand $\mathbb{B}$ itself. Of course, it might be the case that no proper substructure of $\mathbb{B}$ is itself, so $\mathbb{P}(\mathbb{B})$ consists of a single element. The forcing $\mathbb{P}(\mathbb{B})$ is most interesting when $\mathbb{B}$ has many copies of itself, which is often the case for the *Fraïssé limits.* Although Fraïssé theory is not needed to understand the content of the paper, it is the motivation for several of the topics that are studied. The reader may consult [13], [19] or [47] to learn more about Fraïssé theory. The structure and forcing properties of $\mathbb{P}(\mathbb{B})$ has been previously studied in several papers, like in [39], [41], [40], [31], [32], [37], [33], [36], [35] and [30]. The reader may also consult the survey [34] to get a wide picture of this area of study.

Another motivation for the study of $\mathbb{P}(\mathbb{B})$ comes from the theory of ideals on countable sets. Assume $\mathbb{B}$ is a relational structure whose universe is ω. Define $\mathcal{I}_{\mathbb{B}}$ as the set of all $X \subseteq \omega$ that do not contain an isomorphic copy of $\mathbb{B}$. We say

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[1] The notation $\binom{\mathbb{B}}{\mathbb{B}}$ is also frequently used in the literature.

that $\mathbb{B}$ is *indivisible* if $\mathcal{I}_{\mathbb{B}}$ is an ideal, or equivalently, if whenever ω is splitted into two parts, one of the parts contains a copy of $\mathbb{B}$. In case $\mathbb{B}$ is indivisible, we get that $\mathcal{I}_{\mathbb{B}}$ is a tall coanalytic ideal (see [31]). Furthermore, $\mathbb{P}(\mathbb{B})$ is forcing equivalent to $\wp(\omega) \diagup \mathcal{I}_{\mathbb{B}}$ (If X is a set, by $\wp(X)$ we denote the power set of X). Boolean algebras and forcings of the type $\wp(\omega) \diagup \mathcal{I}$ (for $\mathcal{I}$ a definable ideal) have been extensively studied in the past, the reader may consult [11], [21] or [20] to learn more about this topic.

The starting points of this paper are the following theorems of Kurilić and the second author:

Theorem 1 (Kurilić, Todorcevic [39]) $\mathbb{P}(\mathbb{Q})$ *is forcing equivalent to a two step iteration of the form* $\mathbb{S} * \dot{\mathbb{R}}$ *where* $\mathbb{S}$ *denotes the Sacks forcing and* $\dot{\mathbb{R}}$ *is a* $\mathbb{S}$*-name for a* σ*-closed forcing*[2].

Theorem 2 (Kurilić, Todorcevic [41], [40]) *Let* $\mathcal{R}$ *be the random graph*[3]. $\mathbb{P}(\mathcal{R})$ *is forcing equivalent to a two step iteration of the form* $\mathbb{P} * \dot{\mathbb{R}}$ *such that* $\mathbb{P}$ *is a proper forcing that adds a real, has the* 2*-localization property (in particular, it has the Sacks property), does not add splitting reals and* $\dot{\mathbb{R}}$ *is a* $\mathbb{P}$*-name for a* ω*-distributive forcing.*

Probably the reader noted that the conclusions of both theorems are very similar; yet, there are some differences. Properness, the 2-localization property and not adding splitting reals are some of the main properties of Sacks forcing (furthermore, combining the results of [41] and [57], it is possible to prove that $\mathbb{P}(\mathcal{R})$ preserves P-points, which is another key property of Sacks forcing). It is then natural to ask the following:

Problem 3 *Is the first iterand of* $\mathbb{P}(\mathcal{R})$ *forcing equivalent to Sacks forcing?*

Another difference between the two theorems, is that in the case of the rationals, the quotient is σ-closed, while for the random graph it is only ω-distributive (recall that a σ-closed forcing is one in which every decreasing sequence of countable length has a lower bound, while a ω-distributive forcing is a forcing that does not add new sequences of ordinals. In this way, σ-closed forcings are ω-distributive, but there are ω-distributive forcings that are not σ-closed). We may wonder the following:

Problem 4 *Is the second iterand of* $\mathbb{P}(\mathcal{R})$ *forcing equivalent to a* σ*-closed forcing?*

Moreover, given the similarities between $\mathbb{P}(\mathbb{Q})$ and $\mathbb{P}(\mathcal{R})$, we can ask the following:

[2] In here, we are taking $\mathbb{Q}$ with its usual (linear) order.

[3] Unfamiliar concepts used in this introduction will be defined in the next section.

Problem 5 *Are $\mathbb{P}(\mathbb{Q})$ and $\mathbb{P}(\mathcal{R})$ forcing equivalent?*

In this note, we will provide answers to the previous questions. Mainly, we will prove that $\mathbb{P}(\mathcal{R})$ is forcing equivalent to a forcing of the form $\mathbb{S} * \dot{\mathbb{R}}$ where $\dot{\mathbb{R}}$ is a $\mathbb{S}$-name for a ω-distributive forcing that is not equivalent to a σ-closed one. In particular, it follows that $\mathbb{P}(\mathbb{Q})$ and $\mathbb{P}(\mathcal{R})$ are not forcing equivalent.

After that, we will shift our attention to the 3-*Henson graph* (also known as the *generic triangle-free graph*), here denoted by $\mathcal{H}_3$. We will prove that $\mathbb{P}(\mathcal{H}_3)$ although is not σ-closed, it is forcing equivalent to one, so the partial orders of copies of the random graph and of copies of the 3-Henson graph behave completely different.

Notation and Preliminaries

Let B be a set and $\sim$ a binary relation on B. We say that $G = (B, \sim)$ is a *graph* if $\sim$ is irreflexive and symmetric. Given $A \subseteq B$, we will often identify A with the subgraph it induces, $(A, \sim\restriction A)$. Given $x, y \in B$, we often say that x and y are *neighbors* or *x and y are connected* if $x \sim y$. Let $F \subseteq B$, we say that F is a *clique (or complete)* if every two points in F are connected. On the other hand, F is *discrete (anticlique or independent)* if no two points are connected in F.

If $(B, \sim_B)$, $(A, \sim_A)$ are graphs and $f : B \longrightarrow A$, we say that f is a *graph-monomorphism* (or just monomorphism) if f is injective and for every $x, y \in B$, we have that $x \sim_B y$ if and only if $f(x) \sim_A f(y)$. A *graph-isomorphism* (or just isomorphism) is a bijective monomorphism. If $G = (B, \sim)$ is a graph and $a \in \omega$, define $\mathcal{N}_G(a) = \{v \in B \mid v \sim a\}$ and $\overline{\mathcal{N}}_G(a) = \{v \in B \mid v \nsim a \wedge v \neq a\}$. In case the graph G is clear by context, we we will simply write $\mathcal{N}(a)$ and $\overline{\mathcal{N}}(a)$.

One of the most interesting graphs on a countable set is the *random graph* (also known as the *Rado graph* or the *Erdös–Rényi graph*), which is the Fraïssé limit of the class of all finite graphs. There is a very simple and nice characterization of the random graph. We say that a graph $G = (B, \sim)$ has the *Rado property* if for every disjoint $X, Y \in [B]^{<\omega}$, there is $b \in B$ such that b is connected with every element of X and not connected with every element of Y. The following is well known:

Proposition 6

1. *The random graph has the Rado property.*
2. *Two countable graphs with the Rado property are isomorphic.*

In this way, the random graph is the unique (up to isomorphism) countable graph with the Rado property. All the features of the random graph can be deduced from this property alone. Nevertheless, there are some very concrete models of the random graph (see [5]). Although we will not need them here, it is often useful to keep them in mind. For an introduction to the random graph, the reader may look at [5] or [19]. To learn more about it and see some of its applications, the reader may consult [9], [50], [43], [4], [26], [44], [15] or [1] among many others.

The following results are well known and will be often used implicitly (the reader may consult [5] or [19] for a proof, although none of them is hard to prove).

Proposition 7 *Let $\mathcal{R} = (\omega, \sim)$ be a copy of the random graph.*

1. *Every countable graph is isomorphic to a (induced) subgraph of $\mathcal{R}$.*
2. *$\mathcal{R}$ is indivisible, but even more is true: if ω is splitted into two parts, then one of the parts is a random graph.*

Let G and H be two graphs. We say that G *omits* H if there is no graph-monomorphism from H to G. Let $n > 0$, by K_n we denote the clique of n vertices. A very interesting family of graphs was constructed by Henson (see [18]), which we will review now. Let $p \geq 3$, the *p-Henson graph* (here denoted by $\mathcal{H}_p$) is the Fraïssé limit of all the finite graphs that omit K_p. The p-Henson graph has a simple combinatorial characterization, similar to the one of the Random graph. In [18], Henson showed that $\mathcal{H}_p$ is the unique (up to graph-isomorphism) countable graph with the following properties:

1. $\mathcal{H}_p$ omits K_p.
2. If X, Y are finite disjoint subsets of $\mathcal{H}_p$ such that X omits K_{p-1}, there is a vertex in $\mathcal{H}_p$ that is connected with every element of X and not connected with every element of Y.

In [18], there is an explicit construction of $\mathcal{H}_p$ from the random graph. The Henson graphs have been extensively studied recently. To learn more about the Henson graphs, the reader may consult [18], [28], [8], [6], [46] or [17].

If $s \in 2^{<\omega}$, define the *cone of* s as $\langle s \rangle = \{f \in 2^{\omega} \mid s \subseteq f\}$. This is an open set in the usual topology of 2^{ω}. If $T \subseteq \omega^{<\omega}$ is a tree, we denote by $[T]$ *the set of branches of* T, i.e. $[T] = \{f \in \omega^{\omega} \mid \forall n \in \omega\, (f \restriction n \in T)\}$. We say that a tree $p \subseteq 2^{<\omega}$ is a *Sacks tree* if for every $s \in p$ there is $t \in p$ extending s such that $t^\frown 0$, $t^\frown 1 \subseteq p$. The set of all Sacks trees is denoted by $\mathbb{S}$ and we order it by extension. We say that $s \in p$ is the *stem of* p if every $t \in p$ is comparable with s and s is maximal with this property. We denote the stem of p as $st\,(p)$. If $p \in \mathbb{S}$ and $s \in p$, we define $p_s = \{t \in p \mid t \subseteq s \vee s \subseteq t\}$, note that p_s is a Sacks tree. If $G \subseteq \mathbb{S}$ is a generic filter, the *Sacks real* is defined as $s_{gen} = \bigcap_{p \in G} [p]$. It is easy to see that $\dot{s}_{gen}$ is forced to be a new element of 2^ω. Sacks forcing is one of the most important and studied forcing notions for adding reals. Let p be a Sacks tree, by $split(p)$ we denote the set of all splitting nodes of p. Given $n \in \omega$, by $split_n(p)$ we denote the set of all splitting nodes of p that have exactly n splitting nodes before it. In this way, $split_0(p)$ consists only of the stem of p. To learn more about Sacks forcing, the reader may read [3], [2], [22], [16], [45], [7], [12] or [58].

The following definition is well-known:

Definition 8 *Let $\mathbb{P}$ be a partial order. $\mathbb{P}$ is* separative *if for every $p, q \in \mathbb{P}$, if $p \nleq q$, then there is $r \leq p$ that is incompatible with q.*

We need the following definitions[4]:

Definition 9 *Let $\mathbb{P}$ and $\mathbb{Q}$ be two forcing notions.*

1. *$\mathbb{P}$ and $\mathbb{Q}$ are* Solovay equivalent *if $\mathbb{B}(\mathbb{P})$ and $\mathbb{B}(\mathbb{Q})$ are isomorphic.*
2. *$\mathbb{P}$ and $\mathbb{Q}$ are* forcing equivalent *if they give the same forcing extensions.*

Clearly if $\mathbb{P}$ and $\mathbb{Q}$ are Solovay equivalent, then they are forcing equivalent. The converse is not true (although for trivial reasons). The following is an unpublished result of Solovay:

Proposition 10 (Solovay, see lemma 25.5 in [24]) *If $\mathbb{P}$ and $\mathbb{Q}$ are forcing equivalent, then there is $a \in \mathbb{B}(\mathbb{P})$ and $b \in \mathbb{B}(\mathbb{Q})$ such that $\mathbb{B}(\mathbb{P}) \restriction a$ and $\mathbb{B}(\mathbb{Q}) \restriction b$ are Solovay equivalent, i.e., are isomorphic as Boolean algebras.*

Let $\mathbb{P}$ be a partial order. The *distributivity game, or the Banach-Mazur game $\mathcal{DG}(\mathbb{P})$* is played as follows:

Empty	p_0		p_1		...
non-Empty		q_0		q_1	

Where $p_n, q_n \in \mathbb{P}$ and $p_{n+1} \leq q_n \leq p_n$ for every $n \in \omega$. The non-Empty player will *win the match* if there is $r \in \mathbb{P}$ such that $r \leq q_n$ for every $n \in \omega$.

Note that if $\mathbb{B}$ is a Boolean algebra and $\mathbb{P} \subseteq \mathbb{B}$ is dense, then the games $\mathcal{DG}(\mathbb{B})$ and $\mathcal{DG}(\mathbb{P})$ are equivalent. So, if $\mathbb{P}$ is forcing equivalent to a (separative) σ-closed, then the non-Empty player has a winning strategy in $\mathcal{DG}(\mathbb{P})$. For posets

[4] In [14] what we call Solovay equivalent and what we call forcing equivalent is called semantically forcing equivalent.

of size at most continuum this implication is reversible (see, [54], [53]). It is also worth pointing out the following reformulation of the classical result of Banach and Mazur.

Proposition 11 (see [48], [23]) *Let $\mathbb{P}$ be a separative partial order. Then the* Empty *player has a winning strategy in $\mathcal{DG}(\mathbb{P})$ if and only if $\mathbb{P}$ is not ω-distributive.*

Forcing with copies of the random graph

Preliminaries

For this section, we choose and fix $\mathcal{R} = (\omega, \sim)$ a random graph. We are interested in studying the forcing properties of $\mathbb{P}(\mathcal{R})$. As was mentioned in the introduction, this forcing has already been study by Kurilić and the second author in the papers [41], [40]. In there, it was proved that $\mathbb{P}(\mathcal{R})$ is a proper forcing and it decomposes as a two step iteration, where the first iterand adds reals, has the Sacks property (even the 2-localization property) and does not add splitting reals, while the second iterand does not add reals. In here, we aim to take a closer look at both iterands. For the convenience of the reader, we do not assume previous knowledge of [41], [40]. While we will need to repeat some of the arguments of [41] (although we will write them in a slightly different form), we believe some repetition is worth doing, specially since some of the ideas presented here will come back when working with copies of more complicated Fraïssé limits.

Definition 12 *Let $B \subseteq \omega$ and $X, Y \in [\omega]^{<\omega}$ with $X \subseteq Y$. Define $B_X^Y = B \cap \left(\bigcap\limits_{a \in X} \mathcal{N}(a) \cap \bigcap\limits_{b \in Y \setminus X} \overline{\mathcal{N}}(b) \right)$. In other words, B_X^Y is the set of all points in B that are connected with every element of X and not connected with every element of $Y \setminus X$. If $b \in B_X^Y$, we will say that b* realizes the type *(X, Y).*

It is clear that $B \subseteq \omega$ (or more formally, the subgraph induced by B) has the Rado property if and only if for every $X, Y \in [B]^{<\omega}$ with $X \subseteq Y$, the set B_X^Y is not empty. It follows that $B \in \mathbb{P}(\mathcal{R})$ if and only if $B_X^Y \neq \emptyset$ for every $X, Y \in [B]^{<\omega}$ with $X \subseteq Y$. Note that B_X^X is the set of points that are neighbors of every element of X, while $B_\emptyset^X$ is the set of points that are not connected to every element of X.

It is easy to see that if $B \in \mathbb{P}(\mathcal{R})$ and $X, Y \in [B]^{<\omega}$ with $X \subseteq Y$, then $B_X^Y \in \mathbb{P}(\mathcal{R})$.

Definition 13 *Let $B \subseteq \omega$. We will say that $L = \langle L_n \mid n \in \omega \rangle$ is a* labeling[5] of *B if for every $n \in \omega$, the following conditions hold:*

1. $L_n \in [B]^{<\omega}$.
2. $L_n \subseteq L_{n+1}$ and $B = \bigcup L_n$.
3. $L_0 = \emptyset$.
4. For every $K \subseteq L_n$, there is $q_K^{n+1} \in L_{n+1}$ such that $q_K^{n+1} \in B_K^{L_n}$.
5. $L_{n+1} = L_n \cup \left\{ q_K^{n+1} \mid K \subseteq L_n \right\}$.

The following lemma is easy, so we leave the proof for the reader:

Lemma 14 *Let $B \subseteq \omega$. B has a labeling if and only if B is a random graph*[6].

The following proposition is related to lemmas 4.3 and 4.4 of [41]. This is a key result for future arguments.

Proposition 15 *Let $D \in \mathbb{P}(\mathcal{R})$ and $s \in [D]^{<\omega}$. There are $B \in \mathbb{P}(\mathcal{R})$ and $\{f_z \mid z \subseteq s\}$ such that the following conditions hold:*

1. *$B \leq D_s^s$.*
2. *$f_z : B \longrightarrow D_z^s$ is a graph-monomorphism.*
3. *f_s is the identity on B.*
4. *If $a, b \in B$ and $z, t \subseteq s$, then $a \sim b$ if and only if $f_z(a) \sim f_t(b)$ (note that this implies point 2).*
5. *For every $A \leq B$, there is $\{A_z \mid z \subseteq s\}$ a partition of A, such that the set $\left(\bigcup_{z \subseteq s} f_z[A_z] \right) \cup s$ is in $\mathbb{P}(\mathcal{R})$.*

Proof. We will recursively construct $\{L_n \mid n \in \omega\}$ and $\{f_z^n \mid n \in \omega \wedge z \subseteq s\}$ such that the following holds for every $n \in \omega$ and $z \subseteq s$:

1. $L_n \subseteq D_s^s$ is finite and $L_n \subseteq L_{n+1}$.
2. $L_0 = \emptyset$.
3. $f_z^n : L_n \longrightarrow D_z^s$ is a graph-monomorphism.

[5] Our notion of labeling is formally different than the one of [41], yet the difference is unsubstancial.

[6] As mentioned before, we identify a set with the subgraph it induces. So "B is a random graph" means the same as "$(B, \sim\restriction B)$ is a random graph (i.e. $B \in \mathbb{P}(\mathcal{R})$)"

4. f_s^n is the identity on L_n.

5. $f_z^n \subseteq f_z^{n+1}$.

6. For every $K \subseteq L_n$, there is $q_K^{n+1} \in L_{n+1}$ such that if $z, t \subseteq s$ and $b \in L_n$, then:

$$b \in K \text{ if and only if } f_z^n(b) \sim f_t^{n+1}\left(q_K^{n+1}\right).$$

7. $L_{n+1} = L_n \cup \left\{q_K^{n+1} \mid K \subseteq L_n\right\}$.

8. $\left\{q_K^{n+1} \mid K \subseteq L_n\right\}$ is discrete.

9. If $K_1 \neq K_2$ then $f_z\left(q_{K_1}^{n+1}\right) \nsim f_t\left(q_{K_2}^{n+1}\right)$ for every $z, t \subseteq s$ (note that this implies the previous item).

We start with $L_0 = \emptyset$ and $f_z^0 = \emptyset$ for every $z \subseteq s$. Now, assume that L_n and $\{f_z^n \mid n \in \omega \wedge z \subseteq s\}$ have already been constructed, we will see what to do at step $n+1$. Let $l = 2^{|L_n|}$ and $\wp(L_n) = \{K_j \mid j < l\}$. We will recursively define $a_i \in D_s^s$ and $\left\{f_t^{n+1}(a_i) \mid t \subseteq s\right\}$ such that the following holds:

1. If $j < i$ and $z, t \subseteq s$, then $f_z^{n+1}(a_i) \nsim f_t^{n+1}(a_j)$.

2. If $b \in L_n$ and $z, t \subseteq s$, then $b \in K_i$ if and only if $f_z^n(b) \sim f_t^{n+1}(a_i)$.

Assume we have define all the items for all $j < i$, we will see how to do step i. Let $Y_i = \bigcup\limits_{t \subseteq s} f_t^n[K_i]$ (note that $K_i \subseteq Y_i$) and $X_i = \bigcup\limits_{t \subseteq s} f_t^{n+1}[L_n \cup \{a_j \mid j < i\}]$ (so $X_0 = \bigcup\limits_{t \subseteq s} f_t^n[L_n]$). It is clear that $Y_i \subseteq X_i$.

For every $t \subseteq s$, choose $a_t \in D_{Y_i \cup t}^{X_i \cup s}$ (a_t exists because D is a random graph). Note that $a_t \in D_t^s$. We now define $a_i = a_s$ and $f_t^{n+1}(a_i) = a_t$. It is easy to see that a_i has the desired properties. Finally, define $q_{K_i}^{n+1} = a_i$. This finishes the construction at step $n+1$.

Define $B = \bigcup\limits_{n \in \omega} L_n$ and $f_t = \bigcup\limits_{n \in \omega} f_t^n$. We will see that this objects have the desired properties. First, it is clear that $\{L_n \mid n \in \omega\}$ is a labelling of B, so $B \in \mathbb{P}(\mathcal{R})$ and $B \leq D_s^s$. It is clear that $f_s = Id_B$ and each f_t is injective. We will now prove that point 4 of the proposition holds.

Let $a, b \in B$ and $z, t \subseteq s$, we need to prove that $a \sim b$ if and only if $f_z(a) \sim f_t(b)$. Let $n, m \in \omega$ be the firsts such that $a \in L_{n+1}$ and $b \in L_{m+1}$. In case that $n = m$, by construction we have that both $\{a, b\}$ and $\{f_z(a), f_t(b)\}$ are discrete, so we are done. Now, assume that $m < n$. Let $K \subseteq L_n$ such that

$a = q_K^{n+1}$. By construction, we have that $b \in K$ if and only if $f_z(b) \sim f_t(a)$. In this way, we conclude that $a \sim b$ if and only if $f_z(a) \sim f_t(b)$.

Now, we only need to prove the last point of the proposition. Let $A \leq B$, choose $\{J_n \mid n \in \omega\}$ a labeling of A. Now, define $I_0 = J_0$ and $I_{n+1} = J_{n+1} \setminus J_n$. Choose $\{E_t \mid t \subseteq s\}$ a partition of ω such that each E_t is infinite. For every $t \subseteq s$, define $A_t = \bigcup_{n \in E_t} I_n$, it is easy to see that each A_t is a random graph. Let $C = \left(\bigcup_{z \subseteq s} f_z[A_z] \right) \cup s$, we must prove that $C \in \mathbb{P}(\mathcal{R})$.

Let $X, Y \in [C]^{<\omega}$ with $Y \subseteq X$, we need to prove that $C_Y^X \neq \emptyset$. Note that without lost of generality, we may assume that $s \subseteq X$. Define $w = s \cap Y$. For every $z \subseteq s$, define the following:

1. $Y_z = Y \cap D_z^s$ and $X_z = X \cap D_z^s$.
2. $\overline{Y}_z = f_z^{-1}(Y_z)$ and $\overline{X}_z = f_z^{-1}(X_z)$.

Note that $Y = \bigcup_{z \subseteq s} Y_z \cup w$ and $X = \bigcup_{z \subseteq s} X_z \cup s$. Since $\{A_t \mid t \subseteq s\}$ is a partition of A, we get that $\overline{Y}_t, \overline{X}_t \subseteq A_t$ for every $t \subseteq s$. In particular, for every $z, t \subseteq s$ with $z \neq t$, we get the following:

1. $\overline{X}_t \cap \overline{X}_z = \emptyset$.
2. $\overline{Y}_t \subseteq \overline{X}_t$.

Let $H = \bigcup_{t \subseteq s} \overline{Y}_t$ and $K = \bigcup_{t \subseteq s} \overline{X}_t$, it is clear that $H \subseteq K \subseteq A$. Now, find $n \in \omega$ such that $n+1 \in E_w$ and $K \subseteq J_n$ (this is possible since E_w is infinite). Let $a \in I_{n+1}$ such that $a \in A_H^{J_n}$, so $a \in A_w$. Let $b = f_w(a)$, note that $b \in C$. We claim that $b \in C_Y^X$.

Let $v \in Y$, we need to show that $v \sim b$. In case that $v \in s$, then $v \in w$ and since $b \in D_w^s$, we get that $v \sim b$. Assume that $v \notin s$, so there is $t \subseteq s$ such that $v \in Y_t$. In this way, $f_t^{-1}(v) \in \overline{Y}_t$, so $f_t^{-1}(v) \in H$. Since $a \in A_H^{I_n}$, we get that $a \sim f_t^{-1}(v)$, so $f_w(a) \sim f_t\left(f_t^{-1}(v)\right)$, hence $b \sim v$. We are done in this case.

Now, let $v \in X \setminus Y$. In case that $v \in s$, then $v \in s \setminus w$ and since $b \in D_w^s$, we get that $v \nsim b$. Assume that $v \notin s$, so there is $t \subseteq s$ such that $v \in Y_t$. In this way, $f_t^{-1}(v) \in \overline{X}_t \setminus \overline{Y}_t$, so $f_t^{-1}(v) \in K \setminus H$, in particular $f_t^{-1}(v) \in J_n \setminus H$. Since $a \in A_H^{J_n}$, we get that $a \nsim f_t^{-1}(v)$, so $f_w(a) \nsim f_t\left(f_t^{-1}(v)\right)$, hence $b \nsim v$. This finishes the proof. ■

Let $A, B \in \mathbb{P}(\mathcal{R})$ and L a finite subset of A. Define $B \leq_L A$ if $B \leq A$ and $L \subseteq B$. It is clear that this relation is transitive. The following lemma is related to theorem 4.1 and lemma 4.5 of [41].

Lemma 16 *Let $E \in \mathbb{P}(\mathcal{R})$, $D \subseteq \mathbb{P}(\mathcal{R})$ an open dense set and $s \in [E]^{<\omega}$. There is $C \leq_s E$ such that if $t \subseteq s$, then $C_t^s \in D$.*

Proof. By proposition 15, we can find $B \in \mathbb{P}(\mathcal{R})$ and $\{f_z \mid z \subseteq s\}$ such that the following conditions hold: ■

1. $B \leq E_s^s$.
2. $f_z : B \longrightarrow E_z^s$ is a graph-monomorphism.
3. f_s is the identity on B.
4. If $a, b \in B$ and $z, t \subseteq s$, then $a \sim b$ if and only if $f_z(a) \sim f_t(b)$.
5. For every $A \leq B$, there is $\{A_z \mid z \subseteq s\}$ a partition of A, such that the set $\left(\bigcup_{z \subseteq s} f_z[A_z]\right) \cup s$ is in $\mathbb{P}(\mathcal{R})$.

Proof. Let $l = 2^{|s|}$ and $\wp(s) = \{t_i \mid i < l\}$. We will build a sequence $\langle B_i \mid i \leq l \rangle$ with the following properties:

1. $B_0 = B$.
2. $B_{i+1} \leq B_i$.
3. $f_{t_i}[B_{i+1}] \in D$.

Building such sequence is easy: Given B_i, we know that $f_{t_i}[B_i] \in \mathbb{P}(\mathcal{R})$ (since $f_{t_i}[B_i]$ is isomorphic to B_i), so we can find $S \in D$ extending B_i, let $B_{i+1} = f_{t_i}^{-1}(S)$. Finally, define $A = B_l$. Since A extends B, we know that we can find $\{A_t \mid t \subseteq s\}$ a partition of A such that $C = \left(\bigcup_{z \subseteq s} f_z[A_z]\right) \cup s \in \mathbb{P}(\mathcal{R})$. We claim that C has the desired properties. It is clear that $C \leq_s E$. Now, let $i \leq l$, we have that $C_{t_i}^s = f_{t_i}[A_{t_i}] \leq f_{t_i}[B_{i+1}]$. Since $f_{t_i}[B_{i+1}] \in D$ and D is open dense, we conclude that $C_{t_i}^s \in D$. ■

Recall the following notion:

Definition 17 *Let $\mathbb{P}$ be a partial order. We say $\left(\mathbb{P}, \langle \leq_n \rangle_{n \in \omega}\right)$ is* axiom A *if the following holds:*

1. *If $n \in \omega$ then $\leq_n$ is a partial order on $\mathbb{P}$.*

2. *If $p \leq_0 q$ then $p \leq q$.*

3. *If $p \leq_{n+1} q$ then $p \leq_n q$.*

4. *(Fusion property) if $\langle p_n \rangle_{n \in \omega}$ is a sequence such that $p_{n+1} \leq_{n+1} p_n$ then there is $p_\omega \in \mathbb{P}$ such that $p_\omega \leq_n p_n$ for every $n \in \omega$.*

5. *(Freezing property) if $p \in \mathbb{P}$, $n \in \omega$ and $\mathcal{A}$ is a maximal antichain below p, then there is $q \leq_n p$ such that $\{r \in \mathcal{A} \mid r$ is compatible with $p\}$ is countable.*

Obviously if $\left(\mathbb{P}, \langle \leq_n \rangle_{n \in \omega}\right)$ satisfy axiom A then $\mathbb{P}$ is a proper forcing (see [51] or [52] for more on proper forcing). The axiom A structure is often very useful. We can give $\mathbb{P}(\mathcal{R})$ an axiom A structure as follows: Let $B \in \mathbb{P}(\mathcal{R})$, using the well order of ω, we can define a canonical labeling $L(B) = \{L_n(B) \mid n \in \omega\}$. Define $A \leq_n B$ if $A \leq_{L_n(B)} B$. Note that if $A \leq_n B$, then $L_i(B) = L_i(A)$ for every $i \leq n$. In this way, the following is a particular case of lemma 16:

Corollary 18 *Let $A \in \mathbb{P}(\mathcal{R})$, $D \subseteq \mathbb{P}(\mathcal{R})$ an open dense set and $n \in \omega$. There is $B \leq_n A$ such that if $K \subseteq L_n(B)$, then $B_K^{L_n(B)} \in D$.*

It is clear that this corollary implies a (strong form) of the freezing property of the Axiom A structure. The fusion property is taken care by the next lemma, that we leave as an exercise to the reader.

Lemma 19 *Let $\langle (A_n, L_n) \mid n \in \omega \rangle$ be a sequence such that for every $n \in \omega$, the following holds:*

1. *$A_n \in \mathbb{P}(\mathcal{R})$ and $L_n \in [A_n]^{<\omega}$.*

2. *$L_n \subseteq L_{n+1}$.*

3. *$A_{n+1} \leq_{L_n} A_n$.*

4. *For every $K \subseteq L_n$, there is $a \in L_{n+1}$ such that $a \in (A_{n+1})_K^{L_n}$.*

Define $B = \bigcup\limits_{n \in \omega} L_n$. Then $B \in \mathbb{P}(\mathcal{R})$ and $B \leq_{L_n} A_n$.

In this way, we get the following:

Corollary 20 ([41]) *$\mathbb{P}(\mathcal{R})$ has an Axiom A structure. In particular, it is a proper forcing.*

Let $B \in \mathbb{P}(\mathcal{R})$ and $s \subseteq B$. It is easy to see that the set $\{B_t^s \mid t \subseteq s\}$ is a maximal antichain below B. In particular, for every $a \in \omega$, the set $\left\{\mathcal{N}(a), \overline{\mathcal{N}}(a)\right\}$ is a maximal antichain.

Definition 21 *If $G \subseteq \mathbb{P}(\mathcal{R})$ is a generic filter, the* generic real *is defined as* $r_{gen} = \{a \in \omega \mid \mathcal{N}(a) \in G\}$.

Let $B \in \mathbb{P}(\mathcal{R})$ and $n \in \omega$. We will say that B *decides* n if either $B \Vdash$"$n \in \dot{r}_{gen}$" or $B \Vdash$"$n \notin \dot{r}_{gen}$". The following lemma is easy to check:

Lemma 22 *Let $B \in \mathbb{P}(\mathcal{R})$, $n \in \omega$ and $t, s \in [B]^{<\omega}$ with $t \subseteq s$.*

1. *If $n \in B$, then B does not decide n.*
2. *$B_t^s \Vdash$ "$\dot{r}_{gen} \cap s = t$".*
3. *If $n \in B$ and $n \notin s$, then B_t^s does not decide n.*
4. *If $A \leq B$ and $A \Vdash$ "$\dot{r}_{gen} \cap s = t$", then $A \leq B_t^s$.*

It is important to remark the following:

1. It is possible that $n \notin B$ and B does not determine n (in fact, it is very common).
2. It is possible that $n \notin B$, B does not determine n and B_t^s determine n.

Since no condition can decide if its elements are in the generic real or not, we get the following:

Corollary 23 ([41]) *$\dot{r}_{gen}$ is forced to be a new subset of ω.*

Since r_{gen} is added by the forcing $\mathbb{P}(\mathcal{R})$, we know that there is a forcing $\mathbb{P}_{ran}$ such that $\mathbb{P}(\mathcal{R}) = \mathbb{P}_{ran} * \dot{\mathbb{R}}_{ran}$ where $\mathbb{P}_{ran}$ adds the generic real r_{gen} (see [25] pages 246-247).

Minimal real degree of constructibility

Let $B \in \mathbb{P}(\mathcal{R})$, define $Z(B) = \{t \in 2^{<\omega} \mid \exists A \leq B (A \Vdash \text{"}t \subseteq \dot{r}_{gen}\text{"})\}$. The following remarks are easy to check:

Lemma 24 *Let $B \in \mathbb{P}(\mathcal{R})$.*

1. *$Z(B)$ is a Sacks tree.*
2. *$B \Vdash$ "$\dot{r}_{gen} \in [Z(B)]$".*[7]
3. *If $A \leq B$, then $Z(A) \subseteq Z(B)$.*

[7] We will often identify $\dot{r}_{gen}$ with its characteristic function.

It is worth noting that it is possible that $Z(A) = Z(B)$, yet A and B are incompatible. For example, take $\{L_n \mid n \in \omega\}$ a labeling of $\mathcal{R}$. Define $L'_0 = L_0$ and $L'_{n+1} = L_{n+1} \setminus L_n$. Let $X, Y \in [\omega]^\omega$ be almost disjoint (i.e. $X \cap Y$ is finite), define $B = \bigcup_{n \in X} L'_n$ and $A = \bigcup_{n \in Y} L'_n$. It is easy to see that $Z(B) = Z(A) = 2^{<\omega}$, but A and B are incompatible as conditions in $\mathbb{P}(\mathcal{R})$.

Definition 25 *Let $B = \{b_n \mid n \in \omega\} \in \mathbb{P}(\mathcal{R})$ and $\dot{x}$ a $\mathbb{P}(\mathcal{R})$-name for an element of ω^ω.*

1. *We say that B is $\dot{x}$-*determined *if for every $n \in \omega$ and $F \subseteq \{b_i \mid i \leq n\}$, the condition $B_F^{\{b_i \mid i \leq n\}}$ knows $\dot{x} \upharpoonright (b_n + 1)$ (i.e. there is $t \in 2^{<\omega}$ such that $B_F^{\{b_i \mid i \leq n\}} \Vdash$ "$\dot{x} \upharpoonright (b_n + 1) = t$".*
2. *We say that B is* determined *if B is $\dot{r}_{gen}$-determined*

Now, we have the following:

Lemma 26 *Let $A \in \mathbb{P}(\mathcal{R})$ and $\dot{x}$ a $\mathbb{P}(\mathcal{R})$-name for an element of ω^ω. There is $B \leq A$ such that B is $\dot{x}$-determined.*

Proof. Let A be a condition in $\mathbb{P}(\mathcal{R})$, we will prove that A can be extended to a determined condition. In order to achieve this, we will recursively define $\langle b_n \rangle_{n \in \omega}$ and $\langle B_n \rangle_{n \in \omega}$ such that the following holds for every $n \in \omega$:

1. $B_0 \leq A$.
2. $b_0, ..., b_n \in B_n$.
3. $B_{n+1} \leq B_n$.
4. If $F \subseteq \{b_i \mid i \leq n\}$, then the condition $(B_n)_F^{\{b_i \mid i \leq n\}}$ knows $\dot{x} \upharpoonright (b_n + 1)$.
5. At step $n+1$, we choose $K_n \subseteq \{b_i \mid i \leq n\}$ and b_{n+1} such that b_{n+1} realizes $(K_n, \{b_i \mid i \leq n\})$.

We start at step 0. Let b_0 be the minimum of A, define D as the set of all $C \in \mathbb{P}(\mathcal{R})$ that knows $\dot{x} \upharpoonright (b_0 + 1)$. Clearly D is an open dense set. By lemma 16, there is $B_0 \leq_{\{b_0\}} A$ such that if $F \subseteq \{b_0\}$, then $(B_0)_F^{\{b_0\}} \in D$. The general case is similar, assume we are at step $n+1$. Let $K_n \subseteq \{b_i \mid i \leq n\}$, since B_n is a random graph, there is $b_{n+1} \in B_n$ realizing $(K_n, \{b_i \mid i \leq n\})$. Define D as the set of all $C \in \mathbb{P}(\mathcal{R})$ that knows $\dot{x} \upharpoonright (b_{n+1} + 1)$. Clearly D is an open dense set. By lemma 16, there is $B_{n+1} \leq B_n$ with $b_0, ..., b_{n+1} \in B_{n+1}$ such that if $F \subseteq \{b_0, ..., b_{n+1}\}$, then $(B_{n+1})_F^{\{b_0,...,b_{n+1}\}} \in D$.

Now, define $B = \{b_n \mid n \in \omega\}$. Moreover, by carefully choosing the sequence $\langle K_n \rangle_{n\in\omega}$, we can make sure that B is a random graph. It is easy to see that $B \leq A$ and it is $\dot{x}$-determined. ■

With the previous proof, we can also conclude the following result of Kurilić and the second author:

Corollary 27 (Kurilić, Todorcevic [41]) $\mathbb{P}(\mathcal{R})$ *is forcing equivalent to a two step iteration of the form* $\mathbb{P}_{ran} * \dot{\mathbb{Q}}$ *such that* $\mathbb{P}$ *adds a real and* $\dot{\mathbb{R}}$ *is a* $\mathbb{P}$*-name for a* ω*-distributive forcing.*

In particular, by lemma 26, we get that the determined conditions are dense. Let B be determined. It is natural to think that the height of the splitting points of $Z(B)$ belongs to B. But this is in general not the case, in fact, the conditions where this fails is dense (i.e. for every A, there is $B \leq A$ determined such that if $s \in Z(B)$ is a splitting point, then $|s| \notin Z(B)$).

Proposition 28 *Let* $A \in \mathbb{P}(\mathcal{R})$ *and* $s, w \in [A]^{<\omega}$ *with* $s \cap w = \emptyset$. *Let* $h : \wp(s) \longrightarrow \wp(w)$. *There is* $B \in \mathbb{P}(A)$ *such that the following holds:*

1. $B \leq A$.
2. $s \subseteq B$ *and* $B \cap w = \emptyset$.
3. *If* $t \subseteq s$, *then* $B^s_t = A^{s\cup w}_{t\cup h(t)}$, *hence* $B^s_t \Vdash$ "$\dot{r}_{gen} \cap (s \cup w) = t \cup h(t)$".

Proof. Let $B = s \cup (\bigcup\limits_{t\subseteq s} A^{s\cup w}_{t\cup h(t)})$ and note that $w \cap B = \emptyset$. We claim that $B \in \mathbb{P}(\mathcal{R})$. Let $X, Y \in [B]^{<\omega}$ with $Y \subseteq X$, we need to show that $B^X_Y \neq \emptyset$. We may assume that $s \subseteq X$, define $t = s \cap Y$. Let $Y_1 = Y \cup h(t)$ and $X_1 = X \cup w$, clearly $Y_1 \subseteq X_1$. Since A is a random graph, there is $v \in A^{X_1}_{Y_1}$. Since $A^{X_1}_{Y_1} \subseteq A^{s\cup w}_{t\cup h(t)}$, we get that $v \in B^X_Y$. It is clear that B has the desired properties. ■

In the proposition above, intuitively, under the condition B, if the "generic real chooses to be t in s", then it will "choose to be $h(t)$ in w".

Corollary 29 *Let* $A \in \mathbb{P}(\mathcal{R})$ *and* $n \in \omega$. *For every* $K \subseteq L_n(A)$, *let* $t_K \subseteq A^{L_n(A)}_K$ *be a finite set and* $m > n$ *such that* $t_K \subseteq L_m(A)$ *for every* $K \subseteq L_n(A)$. *There is* $B \in \mathbb{P}(\mathcal{R})$ *such that the following holds:*

1. $B \leq_n A$.
2. $B^{L_n(B)}_K \leq A^{L_m(A)}_{K\cup t_K}$.

Proof. Let $w = L_m(A) \setminus L_n(A)$ and $s = L_n(A)$. Define $h : \wp(s) \longrightarrow \wp(w)$ given by $h(K) = t_K$. We now just need to apply proposition 28. ■

Let $\dot{x}$ be a $\mathbb{P}(\mathcal{R})$-name for an element of ω^ω. Given $B \in \mathbb{P}(\mathcal{R})$, define $\dot{x}[B] = \bigcup \{t \in \omega^{<\omega} \mid B \Vdash \text{"}t \subseteq \dot{x}\text{"}\}$. It is easy to see that if $\dot{x}$ is forced to be a new real, then $\dot{x}[B] \in \omega^{<\omega}$. We will now prove that every new real in an extension by $\mathbb{P}(\mathcal{R})$, can be read continuously from r_{gen} in an injective way. The reader may consult [55] and [56] to learn more about the continuos reading of names on definable forcings.

Proposition 30 *Let $\dot{x}$ be a $\mathbb{P}(\mathcal{R})$-name for a new real of ω^ω. There is $B \in \mathbb{P}(\mathcal{R})$ and an injective continuous function $J : [Z(B)] \longrightarrow \omega^\omega$ such that $B \Vdash$ "$J(\dot{r}_{gen}) = \dot{x}$".*

Proof. Let $\dot{x}$ be a $\mathbb{P}(\mathcal{R})$-name for a new element of ω^ω. By the proof of lemma 26, we can find $A = \{a_n \mid n \in \omega\} \in \mathbb{P}(\mathcal{R})$ that is both determined and $\dot{x}$-determined. Define $L_n = \{a_i \mid i \leq n\}$.

We will recursively build $\langle b_n, m_n, B_n, h_n \rangle_{n\in\omega}$ such that for every $n \in \omega$, the following holds:

1. $b_n \in A$.
2. $\langle m_n \rangle_{n\in\omega}$ is an increasing sequence of natural numbers.
3. $B_0 \leq A$ and $b_0, ..., b_n \in B_n$.
4. $B_{n+1} \leq B_n$.
5. $h_n : \wp(P_n) \longrightarrow \wp(L_{m_n})$ where $P_n = \{b_0, ..., b_n\}$.
6. $B_n = P_n \cup \bigcup\limits_{t\in\wp(P_n)} A^{L_{m_n}}_{t\cup h_n(t)}$.
7. If $t_1, t_2 \in \wp(P_n)$ and $t_1 \neq t_2$, then $\dot{x}[(B_n)^{P_n}_{t_1}] \perp \dot{x}[(B_n)^{P_n}_{t_2}]$[8].
8. If $t \in \wp(P_n)$, then $(B_n)^{P_n}_t$ knows $\dot{x} \restriction (b_n + 1)$ and $\dot{r}_{gen} \restriction (b_n + 1)$.
9. At step $n + 1$, we choose $K_n \subseteq P_n$ and b_{n+1} such that b_{n+1} realizes (K_n, P_n).

We start at step 0, first, let $b_0 = a_0$. We know that both $A^{\{b_0\}}_{\{b_0\}}$ and $A^{\{b_0\}}_{\emptyset}$ force that $\dot{x}$ is a new real. In this way, we can find $m_0, z^0_0, z^1_0, z^0_1, z^1_1$ with the following properties:

1. $m_0 \in \omega$ and $z^0_0, z^1_0, z^0_1, z^1_1 \subseteq L_{m_0}$.
2. $b_0 \in z^0_1 \cap z^1_1$ while $b_0 \notin z^0_0 \cup z^1_0$.
3. $\dot{x}\left[A^{L_{m_0}}_{z^0_0}\right] \perp \dot{x}\left[A^{L_{m_0}}_{z^1_0}\right]$ and $\dot{x}\left[A^{L_{m_0}}_{z^0_1}\right] \perp \dot{x}\left[A^{L_{m_0}}_{z^1_1}\right]$.

[8] If $s, t \in \omega^{<\omega}$, by $s \perp t$ we denote that s and t are incompatible.

Now, note that one of $\dot{x}\left[A^{L_{m_0}}_{z^0_0}\right]$, $\dot{x}\left[A^{L_{m_0}}_{z^1_0}\right]$ must be incompatible with one of $\dot{x}\left[A^{L_{m_0}}_{z^0_1}\right]$, $\dot{x}\left[A^{L_{m_0}}_{z^1_1}\right]$. For concreteness and without lost of generality, we may assume that $\dot{x}\left[A^{L_{m_0}}_{z^0_0}\right]$ and $\dot{x}\left[A^{L_{m_0}}_{z^0_1}\right]$ are incompatible.

Let $w = L_{m_0} \setminus \{b_0\}$ and $s = \{b_0\}$. Define $h_n : \wp(s) \longrightarrow \wp(w)$ given by $h_0(\emptyset) = z^0_0 \setminus \{b_0\}$ and $h_0(\{b_0\}) = z^0_1 \setminus \{b_0\}$. Now, by lemma 28, we know that $B_0 = P_0 \cup \bigcup\limits_{t \in \wp(P_0)} A^{L_{m_0}}_{t \cup h_0(t)}$ is a random graph. Note that $\left(B^0_0\right)^{P_0}_{\{b_0\}} = A^{L_{m_0}}_{z^0_0}$ and $\left(B^0_0\right)^{P_0}_{\emptyset} = A^{L_{m_0}}_{z^0_1}$. It follows that $\dot{x}[\left(B^0_0\right)^{P_0}_{\{b_0\}}]$ and $\dot{x}[\left(B^0_0\right)^{P_0}_{\emptyset}]$ are incompatible. This concludes the base case.

The general case is similar (just notational more involved); assume we have constructed $\langle b_n, m_n, B_n, h_n \rangle$, we will see how to define the items for step $n+1$. Let $K_n \subseteq P_n$, since B_n is a random graph, there is $b_{n+1} \in B_n$ realizing (K_n, P_n). Let $\wp(P_n) = \{H_i \mid i \leq 2^n\}$. As in the base case we can find $m_{n+1}, z^0_0(i), z^1_0(i), z^0_1(i), z^1_1(i)$ such that for every $i \leq 2^n$, the following holds:

1. $m_{n+1} > m_n$ and $z^0_0(i), z^1_0(i), z^0_1(i), z^1_1(i) \subseteq L_{m_{n+1}}$.
2. $H_i \cap P_n = z^k_j(i)$ for all $j, k \in 2$.
3. $b_{n+1} \in z^0_1(i) \cap z^1_1(i)$ while $b_{n+1} \notin z^0_0(i) \cup z^1_0(i)$.
4. $\dot{x}\left[A^{L_{m_{n+1}}}_{z^0_0(i)}\right] \perp \dot{x}\left[A^{L_{m_{n+1}}}_{z^1_0(i)}\right]$ and $\dot{x}\left[A^{L_{m_{n+1}}}_{z^0_1(i)}\right] \perp \dot{x}\left[A^{L_{m_{n+1}}}_{z^1_1(i)}\right]$.

As in the base case, (by reenumerating if necessary), we may assume that $\dot{x}\left[A^{L_{m_{n+1}}}_{z^0_0(i)}\right]$ and $\dot{x}\left[A^{L_{m_{n+1}}}_{z^0_1(i)}\right]$ are incompatible. Let $w = L_{m_{n+1}} \setminus L_{m_n}$ and $s = P_{n+1}$. Define $h_{n+1} : \wp(s) \longrightarrow \wp(w)$ such that for every $i \leq 2^n$, we have that $h_{n+1}(H_i) = z^0_0(i) \setminus H_i$ and $h_{n+1}(H_i \cup \{b_{n+1}\}) = z^0_1(i) \setminus H_i$. Now, by lemma 28, we know that $B_{n+1} = P_{n+1} \cup \bigcup\limits_{t \in \wp(P_{n+1})} A^{L_{m_{n+1}}}_{t \cup h_{n+1}(t)}$ is a random graph. It is easy to see that $\langle b_{n+1}, m_{n+1}, B_{n+1}, h_{n+1} \rangle$ has the desired properties.

Let $B = \{b_n \mid n \in \omega\}$. Moreover, by carefully choosing the sequence $\langle K_n \rangle_{n \in \omega}$, we can make sure that B is a random graph. Note that for every $n \in \omega$ and $t \subseteq \{b_0, ..., b_n\}$, the condition $B^{\{b_0,...,b_n\}}_t$ knows both $\dot{x} \restriction (b_n+1)$ and $\dot{r}_{gen} \restriction (b_n+1)$. Furthermore, if $t, z \subseteq \{b_0, ..., b_n\}$ are different, the values of $\dot{x} \restriction (b_n+1)$ and $\dot{r}_{gen} \restriction (b_n+1)$ are forced to be different under the respective conditions. From this remarks, we can now define a continuous injective function that does as required. ∎

Recall the following notion:

Definition 31 *We say that a forcing* $\mathbb{P}$ *has* minimal real degree of constructibility *if for every generic filter* $G \subseteq \mathbb{P}$ *and every* $x \in \omega^{\omega} \cap V[G]$, *either* $x \in V$ *or* $V[x] = V[G]$.

It is well known that Sacks forcing has minimal real degree of constructibility (see [16]). By the injectivity in proposition 30, we get the following:

Corollary 32 $\mathbb{P}_{ran}$ *has minimal real degree of constructibility.*

This corollary will be help us in the future.

Main combinatorial result

We will now define the notion of flat graph, which in some sense, are the "simplest" conditions in $\mathbb{P}(\mathcal{R})$.

Definition 33 *Let* $B \subseteq \omega$. *We say that* B *is a* flat graph *if there are* $X, Y \in [\omega]^{<\omega}$ *and* $f : \wp(X) \longrightarrow \wp(Y)$ *such that the following holds:*

1. $X \neq \emptyset$.
2. $X \subseteq B$ *and* $Y \subseteq \omega \setminus B$.
3. $X \cup Y \in \omega$
4. $B = X \cup \bigcup_{s \subseteq X} \mathcal{R}^{X \cup Y}_{s \cup f(s)}$.

(Remember $\mathcal{R} = (\omega, \sim)$ is the random graph we started with). In the above situation, we say that B is (X, Y, f)*-flat.* We will say that a flat graph B is an X*-flat graph* if there is Y and f such that B is (X, Y, f)-flat (in the similar way, we will say that B is an (X, Y)*-flat graph* if there is f such that X is (X, Y, f)-flat).

Lemma 34 *If* B *is a flat graph, then* B *is a random graph.*

Proof. Let (X, Y) witness that B is flat. Let u, v be finite subsets of B with $u \subseteq v$. We may assume that $X \subseteq v$. Let $u_0 = u \cap X$, since $\mathcal{R}$ is a random graph, there is $a \in \mathcal{R}^{v \cup Y}_{u \cup f(u_0)}$. Note that $a \in \mathcal{R}^{X \cup Y}_{u_0 \cup f(u_0)}$, so $a \in B$. Furthermore, $a \in B^v_u$, so we are done. ■

The following is easy:

Lemma 35 *Let* B *be an* (X, Y, f)*-flat graph. If* u, v *are finite subsets of* $\omega \setminus (X \cup Y)$ *with* $u \subseteq v$, *then* $A = X \cup (B \cap \mathcal{R}^v_u) \in \mathbb{P}(\mathcal{R})$. *In particular,* B *and* $\mathcal{R}^v_u$ *are compatible.*

Proof. Let c, d be finite subsets of A with $c \subseteq d$. We may assume that $X \subseteq d$. Let $c_0 = c \cap X$, since $\mathcal{R}$ is a random graph, there is $a \in \mathcal{R}^{d \cup Y \cup v}_{c \cup f(c_0) \cup u}$. Note that $a \in \mathcal{R}^{X \cup Y}_{c_0 \cup f(u_0)}$ and $a \in \mathcal{R}^{v}_{u}$, so $a \in A$. Furthermore, $a \in A^{d}_{c}$, so we are done. ■

Let B be an (X, Y, f)-flat graph. For every $s \subseteq X$, let $e_s : X \cup Y \longrightarrow 2$ be the characteristic function of $s \cup f(s)$. By the above result, it follows that $[Z(B)] = \{z \in 2^{\omega} \mid \exists s \subseteq X \, (z \restriction (X \cup Y) = e_s)\}$.

If $T \subseteq 2^{<\omega}$ is a tree, we denote $ht(T)$ its height. For every $l \in \omega$, define $T_l = \{s \in T \mid |s| = l\}$. We also define $T_{\leq l} = \bigcup\limits_{i \leq l} T_i$. We say that a tree is *skew* if each T_l has at most one splitting node. A tree is called *well pruned* if for every $s \in T$ and $m \in \omega$ with $|s| \leq m \leq ht(T)$, there is $t \in T_m$ extending s.

Definition 36 *Let $T \subseteq 2^{<\omega}$ be a tree.*

1. *In case T is infinite, we say that T is* thin *if it is skew and there is $A = \{l_n \mid n \in \omega\} \subseteq \omega$ such that for every $n \in \omega$, the following holds:*
   (a) $l_0 = 0$.
   (b) $l_n < l_{n+1}$.
   (c) *If $s \in T_{l_n}$, then s has exactly two successors in $T_{l_{n+1}}$.*
   (d) *T is a well pruned tree.*
2. *In case T is finite, we say that T is* thin *if it is skew and there is $A = \{l_n \mid n \leq k\} \subseteq \omega$ if for every $n < k$, the following holds:*
   (a) *$l_0 = 0$ and $l_k = ht(T)$.*
   (b) $l_n < l_{n+1}$.
   (c) *If $s \in T_{l_n}$, then s has exactly two successors in $T_{l_{n+1}}$.*
   (d) *T is a well pruned tree.*

If T is a tree, denote by $[T]$ the set of branches through T, note that if T is finite, then $[T] = T_{ht(T)}$.

Definition 37 *Let $T, S \subseteq 2^{<\omega}$ trees.*

1. *By $split(T)$, we denote the set of all splitting nodes of T.*
2. *$Lev(T) = \{n \mid T_n \cap split(T) \neq \emptyset\}$.*
3. *$S \sqsubseteq T$ if $T \cap 2^{\leq ht(S)} = S$.*
4. *Let S and T be finite tree. Define $S \lhd T$ if $S \sqsubseteq T$ and every $s \in [S]$ has exactly two extensions in $[T]$.*

Let B be an (X,Y,f)-flat graph and $n = max(X \cup Y) + 1$. Define $E_B = Z(B)_{\leq n}$. By the previous results, E_B is an initial segment of $Z(B)$ and after that, every node is a splitting node. We will now prove several simple lemmas that will help us to prove that $\mathbb{P}(\mathcal{R})$ adds a Sacks real.

Lemma 38 *Let B be an (X,Y,f)-flat graph and $n = max(X \cup Y)+1$. Assume $T \subseteq 2^{<\omega}$ is a finite tree such that $E_B \lhd T$, and $a \in B$ with $a > ht(T)$. There is $A \in \mathbb{P}(\mathcal{R})$ with the following properties:*

1. *$A \leq B$.*
2. *$X \cup \{a\} \subseteq A$ and $[n,a) \cap A = \emptyset$.*
3. *A is an $(X \cup \{a\}, (Y \cup [n,a)))$-flat graph.*
4. *$T \sqsubseteq E_A$ and $split(T) = split(E_A)$ (i.e. every node in $[T]$ has exactly one successor in $[E_A]$).*

Proof. Define $X_1 = X \cup \{a\}$ and $Y_1 = Y \cup [n,a)$. We construct $g : \wp(X_1) \longrightarrow \wp(Y_1)$ as follows: Let $s \subseteq X$ and $z : n \longrightarrow 2$ be the characteristic function of $s \cup f(s)$. We know that $z \in [E_B]$ and z has exactly two successors in $[T]$, say z_0 and z_1. Define $g(s \cup \{a\}) = z_0^{-1}(1) \cap Y_1$ and $g(s) = z_1^{-1}(1) \cap Y_1$. Note that $g(s) \cap n = g(s \cup \{a\}) \cap n = f(s)$. We now define $A = X_1 \cup \bigcup\limits_{t \subseteq X_1} \mathcal{R}^{X \cup Y}_{t \cup g(t)}$.

We know that A is a flat graph. We claim that $A \subseteq B$. Clearly $X_1 \subseteq B$. Let $t \subseteq X_1$ and $s = t \cap X$. In this way, $(t \cup g(t)) \cap X = s \cup f(t)$, so $\mathcal{R}^{X_1 \cup Y_1}_{t \cup g(t)} \subseteq \mathcal{R}^{X \cup Y}_{s \cup f(s)}$, which is a subset of B. The other properties follow by construction. $\blacksquare$

We will also need the following:

Lemma 39 *Let B be an (X,Y,f)-flat graph, $T \subseteq 2^{<\omega}$ a finite well pruned tree such that $E_B \sqsubseteq T$ and every $t \in [E_B]$ has exactly one successor in $[T]$. There are A and Y_0 such that the following holds:*

1. *$A \leq B$ and is an (X, Y_0)-flat graph.*
2. *$Y \subseteq Y_0$.*
3. *$E_A = T$.*

Proof. Let $m = ht(T)$ and $Y_0 = m \setminus X$. For every $z \subseteq X$, let $e_z : ht(E_B) \longrightarrow 2$ be the characteristic function of $z \cup f(z)$. By the hypothesis of T, we know that there is a unique $\overline{z} \in [T]$ that is a successor of e_z. Define $g : \wp(X) \longrightarrow \wp(Y_0)$ as $g(z) = (\overline{z})^{-1}(1) \cap Y_0$. Note that $f(z)$ is an initial segment of $g(z)$. Define $A = X \cup \bigcup\limits_{z \subseteq X} \mathcal{R}^m_{z \cup g(z)}$, it is clear that A has the desired properties. $\blacksquare$

The following lemma is also easy:

Lemma 40 *Let T be a thin tree with only one splitting node and let $a \in \omega$ be the height of that node. There is an $\{a\}$-flat graph B such that $E_B = T$.*

Proof. Let $z_0, z_1 \in [T]$ such that $z_0(a) = 0$ and $z_1(a) = 1$. Denote $Y = ht(T) \setminus \{a\}$ and $X = \{a\}$. Define $g : \wp(X) \longrightarrow \wp(Y)$ given by $g(\emptyset) = z_0^{-1}(1) \cap Y$ and $g(\{a\}) = z_1^{-1}(1) \cap Y$. It is clear that $B = \{a\} \cup \bigcup_{s \subseteq X} \mathcal{R}^{X \cup Y}_{s \cup g(s)}$ has the desired properties. ■

By $\mathcal{H}$ we will denote the collection of all finite thin trees. For every $n \in \omega$, define $\mathcal{H}_{n+1} = \{T \in \mathcal{H} \mid ht(T) = n+1 \wedge n \in Lev(T)\}$, $\mathcal{H}_{\leq n} = \bigcup_{i \leq n} \mathcal{H}_i$ and $\mathcal{H}_{<n} = \bigcup_{i<n} \mathcal{H}_i$. Let $T \in \mathcal{H}_n$, we will say that T is a *successor* if there is $S \in \mathcal{H}_{<n}$ such that $S \lhd T$. Note that in this case, such S is unique, we will denote it by T^-. It is easy to see that $T \in \mathcal{H}$ is not a successor if and only if T has exactly one splitting node (recall that all thin trees must have at least one splitting point). Given $T \in \mathcal{H}_n$, define $Pred(T) = \{S \in \mathcal{H}_{\leq n} \mid S \sqsubseteq T\}$. Clearly, T is a successor if and only if $|Pred(T)| > 1$ (note that T is always in $Pred(T)$). Furthermore, $Pred(T)$ is linearly ordered by end-extension and the $\sqsubseteq$-least element has only one splitting node, while the $\sqsubseteq$-last element is T. The *degree* of T is defined as $|Pred(T)|$ and will be denoted by $deg(T)$.

Definition 41 *Let $g : \omega \longrightarrow [\omega]^{<\omega}$. We say that g is a* bookkeeping function for random graphs *if the following holds:*

1. *g is surjective.*
2. *$g(n) \subseteq n$ for every $n \in \omega$.*
3. *If $s \in [\omega]^{<\omega}$, then $g^{-1}(s)$ is infinite.*

The role of the function g is to keep track of the types we need to realize in order to build a random graph. The following simple lemma is left to the reader:

Lemma 42 *Let $g : \omega \longrightarrow [\omega]^{<\omega}$ be a bookkeeping function for random graphs. If $B = \{b_n \mid n \in \omega\} \subseteq \omega$ is such that for every $n \in \omega$, it is the case that b_n realizes the type $(\{b_i \mid i \in g(n)\}, \{b_i \mid i < n\})$, then B is a random graph.*

We are now in position to prove the main combinatorial result of this section:

Proposition 43 *There is an injective continuous function $F : 2^\omega \longrightarrow 2^\omega$ such that for every uncountable Borel set $X \subseteq 2^\omega$, there is $B \in \mathbb{P}(\mathcal{R})$ such that $[Z(B)] \subseteq F[X]$.*

Proof. Fix $g : \omega \longrightarrow [\omega]^{<\omega}$ a bookkeeping function for random graphs. We will recursively define $\langle f_n, L_n, K_n \rangle_{n \in \omega}$ such that for every $n \in \omega$, the following holds:

1. $f_n : 2^{\leq n} \longrightarrow 2^{<\omega}$ is injective.
2. $f_n \subseteq f_{n+1}$.
3. If $s, t \in 2^{\leq n}$ and $s \subseteq t$, then $f_n(s) \subseteq f_n(t)$.
4. If $s, t \in 2^n$, then $|f_n(s)| = |f_n(t)|$.
5. The downward closure[9] of $f_n\left[2^{\leq n}\right]$ is a (finite) skew tree.
6. $L_n = \{a_T \mid T \in \mathcal{H}_n\}$ is a finite subset of ω.
7. $K_n = \{B_T \mid T \in \mathcal{H}_n\}$ is a set of flat graphs.
8. If $T \in \mathcal{H}_n$ and $S \in Pred(T)$, then $a_S \in B_T$ (in particular, $a_T \in B_T$).
9. If $T \in \mathcal{H}_n$, then B_T is a $\{a_S \mid S \in Pred(T)\}$-flat graph.
10. If $T \in \mathcal{H}_n$ and is a successor, then $B_T \leq B_{T^-}$.
11. If $a_T \in L_n$ and $s \in 2^n$, then $a_T < |f_n(s)|$.
12. If $T \in \mathcal{H}_n$, then the downward closure of $f_n[T]$ is E_{B_T}.
13. Let $T \in \mathcal{H}_n$ be a successor, $m = deg(T)$ and $Pred(T) = \left\{S^i \mid i \leq m\right\}$ such that $S^i \lhd S^{i+1}$ for every $i < m$. Then, $a_T \sim a_{S^i}$ if and only if $i \in g(m)$ for $i < m$.

We start by defining $f_0(\emptyset) = \emptyset$, since there are no thin trees contained in 2^0, there is nothing more we need to do. Assume the items $\langle f_n, L_n, K_n\rangle$ have already been defined, we will see how to define f_{n+1}, L_{n+1} and K_{n+1}. First, let $h_0 : 2^{\leq n+1} \longrightarrow 2^{<\omega}$ be any function with the following properties:

1. h_0 is injective.
2. $f_n \subseteq h_0$.
3. If $s, t \in 2^{\leq n+1}$ and $s \subseteq t$, then $h_0(s) \subseteq h_0(t)$.
4. The downward closure of $h_0\left[2^{\leq n+1}\right]$ is a skew well pruned tree.

Fix $\left\{T^i \mid i \leq k\right\} \subseteq \mathcal{H}_{n+1}$ an enumeration of all successor trees in $\mathcal{H}_{n+1}$ (in case $n+1 = 1$, we skip this step, since there are no successor trees in $\mathcal{H}_1$). We start with T^0. Just for now, let $T = T^0$ and $h = h_0$.

We look at $h[T]$, which is a skew tree. Let $m = deg(T)$ and $Pred(T) = \left\{S^0, ..., S^m\right\}$. We know that $a_{S^i} \in B_{T^-}$ for every $i < m$. Since B_{T^-} is a random graph, we may find $a_T \in B_{T^-}$ such that for every $i < m$, we have that $a_T \sim a_{S^i}$

[9] If $A \subseteq 2^{<\omega}$, the *downward closure* of A is the set $\{s \in 2^{<\omega} \mid \exists t \in A\,(s \sqsubseteq t)\}$.

if and only if $i \in g(m)$. Furthermore, we may assume that a_T is larger than each a_{S^i}, every element of L_n and the height of $h[T]$.

By the recursive hypothesis, we know that $f_n[T^-] = E_{B_{T^-}}$. In this way, we get that $E_{B_{T^-}} \lhd h[T]$. By lemma 38, there is $A_T \leq B_{T^-}$ a $\{a_S \mid S \in Pred(T)\}$-flat graph such that $h[T] \sqsubseteq E_{A_T}$ and every node in $h[T]$ has exactly one successor in $[E_{A_T}]$. We now define a function $\overline{h} : 2^{\leq n+1} \longrightarrow 2^{<\omega}$ as follows:

1. $f_n \subseteq \overline{h}$.
2. If $s \in 2^{n+1}$, then $h(s) \subseteq \overline{h}(s)$.
3. If $s \in 2^{n+1}$, then $\overline{h}(s)$ has length $ht(E_{A_T})$.
4. If $s \in T_{n+1}$, then $\overline{h}(s) \in [E_{A_T}]$ and extends $h(s)$ (recall that there is only one node with this properties).
5. If $s \in 2^{n+1}$ but $s \notin T$, then $\overline{h}(s)$ is any element of height $ht(E_{A_T})$ extending $h(s)$.

We finished with T^0, define $h_0 = \overline{h}$.

We now repeat the construction above but with $T = T^1$ and $h = h_0$. We now obtain A_{T^1} and h_1. Now, we repeat the construction with $T = T^2$ and $h = h_1$ to obtain A_{T^2} and h_2. We continue this procedure until we finish with all the successor trees. At the end, let $f_{n+1} = h^k$ (recall that k was the number of successor trees).

Now, let $W \in \mathcal{H}_{n+1}$ be a tree that is not a successor. Let a_W larger than every element of L_n and smaller that than the height of $f_{n+1}[W]$. By lemma 40, we can find an $\{a_W\}$-flat graph B_W such that $E_{B_W} = f_{n+1}[W]$. We do this for every tree in $\mathcal{H}_{n+1}$ that is not a successor.

Finally, for every successor tree $T \in \mathcal{H}_{n+1}$, with the aid of lemma 39, we find a $\{a_S \mid S \in Pred(T)\}$-flat graph $B_T \leq A_T$ such that $E_{B_T} = f_{n+1}[T]$. This is possible since $f_{n+1}[T]$ is an end-extension of E_{B_T} and every node in $[E_{B_T}]$ has only one extension in $[f_{n+1}[T]]$. This finishes the construction at step $n+1$.

We now define the function $F : 2^\omega \longrightarrow 2^\omega$ given by $F(x) = \bigcup\limits_{n \in \omega} f_n(x \upharpoonright n)$ for every $x \in 2^\omega$. It is clear that F is injective and continuous. Let $X \subseteq 2^\omega$ be an uncountable Borel set, we must prove that there is $B \in \mathbb{P}(\mathcal{R})$ such that $[Z(B)] \subseteq F[X]$.

Since X is an uncountable Borel set, we may find an infinite thin tree $p \in \mathbb{S}$ such that $[p] \subseteq X$ (recall that every uncountable Borel set contains the branches of a Sacks tree, see [27]). Let $\{l_n \mid n \in \omega\}$ witness that p is thin, we may assume that $l_n - 1 \in Lev(p)$ for every $n > 0$. In this way, we get that $p_{\leq l_n} \in \mathcal{H}_n$ and $p_{\leq l_n} \lhd p_{\leq l_{n+1}}$ for every $n \in \omega$. Define $B = \left\{ a_{p_{\leq l_n}} \mid n \in \omega \right\}$ and it is easy to see that B is a random graph (because of the function g) and $B = \bigcap_{n \in \omega} B_{p_{\leq l_n}}$. Furthermore, we get that $[Z(B)] = F[[p]]$, so $[Z(B)] \subseteq F[X]$. ■

The next lemma is easy:

Lemma 44 *Let $A \in \mathbb{P}(\mathcal{R})$, there is a $B = \{b_n \mid n \in \omega\} \in \mathbb{P}(\mathcal{R})$ with the following properties:*

1. *$b_n < b_{n+1}$ for every $n \in \omega$.*
2. *$B \leq A$.*
3. *B is determined.*
4. *The function $f : \omega \longrightarrow B$ given by $f(n) = b_n$ is a graph-isomorphism between $\mathcal{R}$ and B.*

Proof. We will recursively construct $(L_n, B_n)_{n \in \omega}$ such that for every $n \in \omega$, the following properties hold:

1. $L_n = \{b_0, ..., b_n\}$ is a finite subset of A.
2. $L_n \subseteq L_{n+1}$.
3. $B_n \leq A$.
4. $B_{n+1} \leq B_n$.
5. The function $f_n : n+1 \longrightarrow B_n$ given by $f_n(i) = b_i$ is a graph-monomorphism.
6. If $s \subseteq L_n$, then $(B_n)_s^{L_n}$ knows $\dot{r}_{gen} \restriction (b_n + 1)$.

Let b_0 be any element of A, we start with $L_0 = \{b_0\}$. By lemma 16, we can find $B_0 \leq A$ such that $b_0 \in B_0$ and both $B_0 \cap \mathcal{N}(b_0)$ and $B_0 \cap \overline{\mathcal{N}}(b_0)$ decide (possibly in different ways) $\dot{r}_{gen}(b_0 + 1)$. This finishes the first step of the construction.

Assume we have constructed L_n and B_n, we will see how to construct L_{n+1} and B_{n+1}. Let $X = \{i < n+1 \mid i \sim n+1\}$, since B_n is a random graph, there is $b_{n+1} \in (B_n)_{f_n[X]}^{L_n}$, define $L_{n+1} = L_n \cup \{b_{n+1}\}$. Note that the function $f_{n+1} : n+2 \longrightarrow B_n$ given by $f_{n+1}(i) = b_i$ is a graph-monomorphism. We can now find $B_{n+1} \leq B_n$ such that $L_{n+1} \subseteq B_{n+1}$ and for every $s \subseteq L_{n+1}$, the condition $(B_{n+1})_s^{L_{n+1}}$ decides $\dot{r}_{gen} \restriction (b_{n+1} + 1)$ (again, by lemma 16). This finishes the construction at step $n + 1$.

Finally, define $B = \{b_n \mid n \in \omega\}$. It is clear that B has the desired properties.
■

The following notion will be very important:

Definition 45 *Let $B \in \mathbb{P}(\mathcal{R})$ and $X \subseteq 2^{\omega}$ an uncountable Borel set. We say that X is a* kernel *for B if for every uncountable Borel set $Y \subseteq X$, there is $A \leq B$ such that $[Z(A)] \subseteq Y$.*

Given $X, Y \subseteq 2^{\omega}$, define $Y \subseteq_{ctble} X$ if $Y \setminus X$ is countable. We now have the following:

Lemma 46 *Let $B \in \mathbb{P}(\mathcal{R})$ and X an uncountable Borel set. If X is a kernel for B, then $X \subseteq_{ctble} [Z(B)]$.*

Proof. Assume this is not the case, so $Y = X \setminus [Z(B)]$ is an uncountable Borel set. Since X is a kernel for B, there is $A \leq B$ such that $[Z(A)] \subseteq Y$. But this implies that $Z(A)$ is not contained in $Z(B)$, which is a contradiction. ■

At first glance, it would be natural to think that $[Z(B)]$ is a kernel for B; but unfortunately, this is not true. We will see an example of this. Let $B = \{b_n \mid n \in \omega\}$ be a decided random graph. It is easy to see that there is a Sacks tree p with the following properties:

1. $p \subseteq Z(B)$.
2. If s is the stem of p, then $|s| > b_0$ and $s(b_0) = 1$.
3. For every $n \in \omega$ and $t \in p$, if $b_0 \sim b_n$ and $|t| > b_n$, then $t(b_n) = 0$.

It follows that there is no $A \leq B$ such that $Z(A) \subseteq p$. Note however, that it might be possible that there is C incompatible with B such that $Z(C) \subseteq p$.

Some basic properties about the kernels are the following:

Lemma 47 *Let $A, B \in \mathbb{P}(\mathcal{R})$ and $X, Y \subseteq 2^{\omega}$ uncountable Borel sets.*

1. *If $B \leq A$ and X is a kernel for B, then X is a kernel for A.*
2. *If X is a kernel for B and $Y \subseteq X$, then Y is a kernel for B.*

We can now prove that every random graph has a kernel:

Proposition 48 *If $A \in \mathbb{P}(\mathcal{R})$, then A has a kernel.*

Proof. By lemma 44, there is a random graph $B = \{b_n \mid n \in \omega\} \subseteq A$ determined and the function $g : \omega \longrightarrow B$ given by $g(n) = b_n$ is an isomorphism. We will prove that B has a kernel, which will imply that A has a kernel.

For every $f \in 2^{\leq\omega}$, define the function $\overline{f}$ such that $dom(\overline{f}) = \{b_n \mid n \in dom(f)\}$ and $\overline{f}(b_n) = f(n)$ for every $n \in dom(f)$. Since B is determined, for each $f \in 2^{<\omega}$ ($f \in 2^{\omega}$) there is a unique $\widehat{f} \in Z(B)$ ($\widehat{f} \in [Z(B)]$) such that $\overline{f} \subseteq \widehat{f}$. Let $H : 2^{\omega} \longrightarrow [Z(B)]$ be the function given by $H(f) = \widehat{f}$. It is easy to see that H is injective and continuous. We also define $H_1 : 2^{<\omega} \longrightarrow Z(B)$ where $H_1(s) = \widehat{s}$.

Claim 49 *If $C \in \mathbb{P}(\mathcal{R})$, then $[Z(g[C])] \subseteq H[[Z(C)]]$.* [10]

We will prove the claim, but first note that $g[C]$ is a random graph, since g is an isomorphism. We will prove that $Z(g[C]) \subseteq H_1[Z(C)]$, this will be enough since H_1 is injective and preserves initial segments.

Let $s \in Z(g[C])$. By extending if necessary, we may assume that there is $n \in \omega$ such that $dom(s) = b_n + 1$. Define $t \in 2^{n+1}$ such that $t(m) = s(b_m)$ for every $m < n+1$, we have that $H_1(t) = s$. We need to prove that $t \in Z(C)$. Let $X = s^{-1}(1) \cap \{b_0, ..., b_n\}$, in order to prove that $t \in Z(C)$, we need to show that $C \cap \mathcal{R}^{n+1}_{g^{-1}(X)}$ contains a random graph.

We claim that $g^{-1}(g[C] \cap \mathcal{R}^{\{b_0,...,b_n\}}_X) \subseteq C \cap \mathcal{R}^{n+1}_{g^{-1}(X)}$. Let $a \in g[C] \cap \mathcal{R}^{\{b_0,...,b_n\}}_X$ and find $c \in C$ such that $a = g(c)$. In this way, we have that $g(c) \sim b_i$ if and only if $b_i \in X$, hence $g(c) \sim g(i)$ if and only if $i \in g^{-1}(X)$, since g is a graph-isomorphism, we conclude that $c \sim i$ if and only if $i \in g^{-1}(X)$, so $c \in C \cap \mathcal{R}^{n+1}_{g^{-1}(X)}$. Since g^{-1} is a graph-isomorphism, we get that $C \cap \mathcal{R}^{n+1}_{g^{-1}(X)}$ contains a random graph and we are done.

By the proposition 43, we know there is an injective continuous function $F : 2^{\omega} \longrightarrow 2^{\omega}$ such that for every uncountable Borel set $W \subseteq 2^{\omega}$, there is $C \in \mathbb{P}(\mathcal{R})$ such that $[Z(C)] \subseteq F[W]$. Define $G = HF : 2^{\omega} \longrightarrow [Z(B)]$, clearly G is an injective and continuous function. Let $X = im(G)$, which is an uncountable closed set. We claim that X is a kernel for B (so it is also a kernel for A).

Let $Y \subseteq X$ be an uncountable Borel set. In this way, $G^{-1}(Y)$ is an uncountable Borel set, so there is a random graph C such that $[Z(C)] \subseteq F\left[G^{-1}(Y)\right]$. We now get that $[Z(C)] \subseteq H^{-1}(Y)$, so $H[[Z(C)]] \subseteq Y$ and by the claim before, we conclude that $[Z(g[C])] \subseteq Y$. Since $g[C] \leq B$, we are done. ■

[10] In fact, a very similar argument than the one below shows that $[Z(g[C])] = H[[Z(C)]]$, but we only need one inclusion.

If B is a random graph, define $Ker\,(B)$ as the set of all $p \in \mathbb{S}$ such that $[p]$ is a kernel for B, which we now know it is always non-empty.

$\mathbb{P}(\mathcal{R})$ and Sacks forcing

After all our hard work, we can finally prove that the first iterand of $\mathbb{P}(\mathcal{R})$ is Sacks forcing. First we show the following:

Proposition 50 *$\mathbb{P}(\mathcal{R})$ adds a Sacks real.*

Proof. Let $\mathbb{B}$ be the Boolean completion of Sacks forcing. We now define a function $\pi : \mathbb{P}(\mathcal{R}) \longrightarrow \mathbb{B}$ given by $\pi\,(B) = \bigvee Ker\,(B)$. By proposition 48, we know that $\pi\,(B)$ is a non-zero element of $\mathbb{B}$. Note that if $B \leq A$, then $\pi\,(B) \leq \pi\,(A)$.

Claim 51 *Let $A \in \mathbb{P}(\mathcal{R})$ and $X \in \mathbb{B}$ with $X \leq \pi\,(A)$. There is $B \leq A$ such that $\pi\,(B) \leq X$.*

We will prove the claim. By extending X if necessary, we may assume that there is $p \in Ker\,(A)$ such that $X \leq p$. Since $\mathbb{S}$ is dense in $\mathbb{B}$, we may find $q \in \mathbb{S}$ such that $q \leq X$. In this way, q is also an extension of p, so $[q] \subseteq [p]$. Since p is a kernel for A and $[q]$ is an uncountable closed set, there is $B \leq A$ such that $[Z\,(B)] \subseteq [q]$, hence $Z\,(B) \leq q$. By lemma 46, we know that $Z\,(B)$ is an upper bound for $Ker\,(B)$, so $\pi\,(B) \leq Z\,(B)$, which implies that $\pi\,(B) \leq X$. This finishes the proof of the claim.

The rest of the proof is now standard. We claim that forcing with $\mathbb{P}(\mathcal{R})$ adds a generic filter to $\mathbb{B}$. Let $G \subseteq \mathbb{P}(\mathcal{R})$ be a generic filter. In $V\,[G]$, define $H \subseteq \mathbb{B}$ as the upward closure of $\pi\,[G]$. It is clear that H is a filter, we will prove that is $\mathbb{B}$-generic. Let $D \subseteq \mathbb{B}$ be an open dense set, Take any $B \in G$, by the previous claim, we know that $E = \{A \leq B \mid \pi\,(A) \in D\}$ is an open dense set below B. Since $B \in G$, there is $A \in G \cap E$, which implies that $\pi\,(A) \in H \cap D$. ■

With this, we finally get the following:

Theorem 52 *There is $\mathbb{S}$-name $\dot{\mathbb{Q}}$ for an ω-distributive forcing such that $\mathbb{P}\,(\mathcal{R})$ is forcing equivalent to $\mathbb{S} * \dot{\mathbb{Q}}$.*

Proof. Recall that $\mathbb{P}\,(\mathcal{R})$ is equivalent to an iteration $\mathbb{P}_{ran} * \dot{\mathbb{Q}}$ where $\dot{\mathbb{Q}}$ is ω-distributive. Let $G \subseteq \mathbb{P}(\mathcal{R})$ be a generic filter. Let s_{gen} be a Sacks real added by $\mathbb{P}(\mathcal{R})$. Since $\dot{\mathbb{Q}}$ does not add reals, we get that $s_{gen} \in V\,[r_{gen}]$. By corollary 32, we know that $\mathbb{P}_{ran}$ has minimal real degree of constructibility, so $V\,[r_{gen}] = V\,[s_{gen}]$, hence an extension with $\mathbb{P}_{ran}$ is the same as a Sacks extension. ■

Now, we aim to get a more explicit description of $\mathbb{P}(\mathcal{R})$ as an iteration. We took some inspiration from [39]. Define $\mathbb{K} = \bigcup_{B \in \mathbb{P}(\mathcal{R})} Ker\,(B)$ and order it by inclusion. In this way, $\mathbb{K}$ is a suborder of Sacks forcing. Furthermore, $\mathbb{K}$ is an open (but not dense) suborder of $\mathbb{S}$. Since Sacks forcing is a homogenous forcing, in terms of forcing, $\mathbb{K}$ and $\mathbb{S}$ are equivalent. The definition of kernel was done for elements of $\mathbb{P}(\mathcal{R})$, we naturally extend the definition for all subsets of ω that contain a random graph. If A does not contain a random graph, define $Ker\,(A) = \emptyset$.

Let $G \subseteq \mathbb{K}$ be a generic filter and s_{gen} be the generic real added by $\mathbb{K}$. In $V\,[s_{gen}]$, we define $\mathbb{R} = \{B \in \mathbb{P}(\mathcal{R}) \cap V | G \cap Ker\,(B) \neq \emptyset\}$. Note that a ground model $B \in \mathbb{P}(\mathcal{R})$ if and only if there is $p \in Ker\,(B)$ such that $w_{gen} \in [p]$. Given $A, B \in \mathbb{R}$, define $A \leq_{\mathbb{R}} B$ if $G \cap Ker\,(A \setminus B) = \emptyset$. Equivalently, if $p \in Ker\,(A \setminus B)$, then $s_{gen} \notin [p]$.

Lemma 53 *$V\,[w_{gen}] \models \mathbb{R}$ is a preorder.*

Proof. We only need to check transitivity. First we have the following:

Claim 54 *If $G \subseteq \mathbb{K}$ is a generic filter and $A, B \in \mathbb{P}(\mathcal{R})$, then $G \cap Ker\,(A \cup B) \neq \emptyset$ if and only if $G \cap Ker\,(A) \neq \emptyset$ or $G \cap Ker\,(B) \neq \emptyset$.*

Clearly, if $G \cap Ker\,(A) \neq \emptyset$ or $G \cap Ker\,(B) \neq \emptyset$, then $G \cap Ker\,(A \cup B) \neq \emptyset$. For the other implication, it will be enough to prove that if $p \in Ker\,(A \cup B)$, then $Ker\,(A) \cup Ker\,(B)$ is open dense below p. It is clearly open. Let $q \leq p$, in case $q \notin Ker\,(A)$, there will be $r \leq q$ such that $Z\,(C)$ does not extend r for every $C \leq A$. If $r \notin Ker\,(B)$, we do the same and contradict that $p \in Ker\,(A \cup B)$.

We are now in position to prove the lemma. Let $A, B, C \in \mathbb{R}$ such that $A \leq_{\mathbb{R}} B \leq_{\mathbb{R}} C$, we must show that $A \leq_{\mathbb{R}} C$, or in other words, we must prove that $G \cap Ker(A \setminus C) = \emptyset$. Since $A \leq_{\mathbb{R}} B$, we know that $G \cap Ker\,(A \setminus B) = \emptyset$, and since $B \leq_{\mathbb{R}} C$, we know that $G \cap Ker\,(B \setminus C) = \emptyset$. By the last claim, we get that$G$ has empty intersection with $Ker\,((A \setminus B) \cup (B \setminus C))$. Since $A \setminus C \subseteq (A \setminus B) \cup (B \setminus C)$, we get that $G \cap Ker(A \setminus C) = \emptyset$. ■

We now recall the following well-known definition:

Definition 55 *Let $\mathbb{P}$ and $\mathbb{Q}$ be two partial orders. We say that $i : \mathbb{P} \longrightarrow \mathbb{Q}$ is a* dense embedding *if the following conditions hold for every $p_1, p_2 \in \mathbb{P}$:*

1. *If $p_1 \leq p_2$, then $i\,(p_1) \leq i\,(p_2)$.*
2. *If p_1 and p_2 are incompatible, then $i\,(p_1)$ and $i\,(p_2)$ are incompatible (or equivalently, if $i\,(p_1)$ and $i\,(p_2)$ are compatible, then p_1 and p_2 are compatible).*

3. $i[\mathbb{P}]$ *is a dense subset of* $\mathbb{Q}$.

If there is a dense embedding $i : \mathbb{P} \longrightarrow \mathbb{Q}$, then $\mathbb{P}$ and $\mathbb{Q}$ yield the same generic extensions. To learn more about dense embeddings, the reader may consult [29]. Now we get the following representation of $\mathbb{P}(\mathcal{R})$:

Proposition 56 *$\mathbb{P}(\mathcal{R})$ is forcing equivalent to $\mathbb{K} * \dot{\mathbb{R}}$. Furthermore, we get that $r_{gen} = s_{gen}$ (where r_{gen} is the generic real added by $\mathbb{P}(\mathcal{R})$ and s_{gen} is the Sacks generic added by $\mathbb{K}$).*

Proof. Let $\mathbb{B}(\mathbb{K})$ be the Boolean completion of $\mathbb{K}$. Define $F : \mathbb{P}(\mathcal{R}) \longrightarrow \mathbb{B}(\mathbb{K}) * \dot{\mathbb{R}}$ where $F(B) = (\bigvee Ker(B), B)$. First, note that $\bigvee Ker(B)$ forces that the generic filter intersects $Ker(B)$, so $\bigvee Ker(B)$ forces that $B \in \mathbb{R}$. We will show that F is a dense embedding. It is clear that if $A \leq B$, then $F(A) \leq F(B)$.

We will now prove that if $F(A)$ and $F(B)$ are compatible, then A and B are compatible. Assume there is a condition (p, C) that extends both $F(A)$ and $F(B)$. Let $D = A \cap B \cap C$, we claim that D contains a random graph. For this, note that $C = (A \cap B \cap C) \cup (C \setminus A) \cup (C \setminus B)$. We know that p forces that $\dot{G} \cap Ker(C)$ is not empty (where $\dot{G}$ is the $\mathbb{K}$-name for the generic filter), while it forces that both $\dot{G} \cap Ker(C \setminus A)$ and $\dot{G} \cap Ker(C \setminus B)$ are empty. It follows that p forces that $\dot{G}$ has non-empty intersection with D, so D must contain a random graph. The result then follows.

We will now prove that the image of F is dense. Let (p, B) be an element of $\mathbb{B}(\mathbb{K}) * \mathbb{R}$. We may assume that $p \in Ker(B)$, so there is $A \leq B$ such that $Z(A) \subseteq p$. Let $q \in Ker(A)$, it follows that (q, A) is a condition. Furthermore, $F(A) \leq (Z(A), B) \leq (p, B)$ so we are done.

Finally, we prove that $r_{gen} = s_{gen}$. Assume this is not the case, so we can find incompatible $s, t \in 2^{<\omega}$ such that $r_{gen} \in \langle s \rangle$ and $s_{gen} \in \langle t \rangle$. But this is a contradiction, because no condition in the filter can be contained in $\langle t \rangle$. ■

The quotient is not σ-closed

By the results in the last section, we know that $\mathbb{P}(\mathcal{R})$ is forcing equivalent to $\mathbb{K} * \dot{\mathbb{R}}$, $\mathbb{K}$ is forcing equivalent to Sacks forcing and $\dot{\mathbb{R}}$ is forced to be ω-distributive. In this section, we will prove that it is not σ-closed. Our main motivation was to compare the forcing of to the random graph with the forcing of the rational numbers (see [39]).

The next result is related to the well-known theorem that Sacks forcing does not add splitting reals (see [42] or [11]).

Lemma 57 *Let $p \in \mathbb{S}$, $\dot{X}$ a $\mathbb{S}$-name such that $p \Vdash$ "$\dot{X} \subseteq \omega$" and $A \in \mathbb{P}(\mathcal{R})$. There is $B \leq A$ and $q \leq p$ such that either $q \Vdash$ "$B \subseteq \dot{X}$" or $q \Vdash$ "$B \cap \dot{X} = \emptyset$".*

Proof. Let $\dot{Y}$ be the $\mathbb{S}$-name for $\dot{X} \cap A$. For every q extending p, define $\dot{Y}(q) = \{a \in A \mid q \Vdash \text{“}a \in \dot{Y}\text{”}\}$. Obviously, q forces that $\dot{Y}(q)$ is a subset of $\dot{Y}$. We now proceed by cases:

Case 58 *There is $q \leq p$ such that $\dot{Y}(q)$ contains a random graph.*

Let $B \in \mathbb{P}(\mathcal{R})$ such that $B \subseteq \dot{Y}(q)$. It is clear that $q \Vdash$“$B \subseteq \dot{X}$” and we are done.

Case 59 *If $q \leq p$, then q does not contain a random graph.*

Fix $g : \omega \longrightarrow [\omega]^{<\omega}$ a bookkeeping function for random graphs. We will recursively define two sequences $\{p_n \mid n \in \omega\}$ and $\{b_n \mid n \in \omega\}$ such that the following holds for every $n \in \omega$:

1. $p_0 \leq p$.
2. $p_{n+1} \leq_{n+1} p_n$.
3. $b_n \in A$ (and if $m \neq n$, then $b_n \neq b_m$).
4. $p_n \Vdash$“$b_n \notin \dot{X}$”.
5. If $i < n$, then $b_n \sim b_i$ if and only if $i \in g(n)$.

We will start by defining p_0 and b_0. Let s be the stem of p. We know that $L = \dot{Y}(p_{s^\frown 0}) \cup \dot{Y}(p_{s^\frown 1})$ does not contain a random graph, so we can find $b_0 \in A \backslash L$. Since $b_0 \notin \dot{Y}(p_{s^\frown 0})$, there must be $q_0 \leq p_{s^\frown 0}$ such that $q_0 \Vdash$“$b_0 \notin \dot{Y}$”. By the same argument, there is $q_1 \leq p_{s^\frown 1}$ such that $q_1 \Vdash$“$b_0 \notin \dot{Y}$”. Let $p_0 = q_0 \cup q_1$, it is straightforward to check that q has the following properties:

1. $p_0 \leq p$.
2. The stem of p_0 is s.
3. $p_0 \Vdash$“$b_0 \notin \dot{Y}$”.

In this way, p_0 and b_0 have the desired properties. Now, assume we are at step $n+1$ and p_n, b_n have already been defined, we will see how to define p_{n+1} and b_{n+1}. The idea is similar to the base case. We know that $L = \bigcup \left\{ \dot{Y}\left((p_n)_{s^\frown i}\right) \mid s \in split_n(p_n) \wedge i < 2 \right\}$ does not contain a random graph (since random graphs are indivisible), so we can find $b_{n+1} \in A^{\{b_i \mid i \leq n\}}_{\{b_i \mid i \in g(n+1)\}} \setminus L$. By the same argument as before, for every $s \in split_n(p_n)$ and $i \in 2$, there is $q^{s,i}$ extending $(p_n)_{s^\frown i}$ such that $q^{s,i} \Vdash$“$b_{n+1} \notin \dot{Y}$”. Define the condition

$p_{n+1} = \bigcup \{ q^{s,i} \mid s \in split_n(p_n) \wedge i < 2 \}$. It is easy to see that p_{n+1} and b_{n+1} have the desired properties. This concludes the recursive construction.

Let $B = \{ b_n \mid n \in \omega \}$ and $q = \bigcap_{n \in \omega} p_n$. By lemma 42, B is a random graph extending A, q is a Sacks tree extending p and $q \Vdash$"$B \cap \dot{X} = \emptyset$". ■

Let $A, B \in \mathbb{P}(\mathcal{R})$, we say that the pair $\langle A, B \rangle$ *is mutually decided* if the following conditions hold:

1. Either $A \Vdash$"$B \subseteq \dot{r}_{gen}$" or $A \Vdash$"$B \cap \dot{r}_{gen} = \emptyset$" and,
2. Either $B \Vdash$"$A \subseteq \dot{r}_{gen}$" or $B \Vdash$"$A \cap \dot{r}_{gen} = \emptyset$".

By the previous lemma and the decomposition theorem, we have the following:

Corollary 60 *Let $A, B \in \mathbb{P}(\mathcal{R})$. There are $A' \leq A$ and $B' \leq B$ such that $\langle A', B' \rangle$ is mutually decided.*

The following is easy:

Lemma 61 *Let $A, B \in \mathbb{P}(\mathcal{R})$ such that $\langle A, B \rangle$ is mutually decided, $a \in A$ and $b \in B$. The following holds:*

1. *Either $A \cap \mathcal{N}(b)$ or $A \cap \overline{\mathcal{N}}(b)$ does not contain a random graph.*
2. *Either $B \cap \mathcal{N}(a)$ or $B \cap \overline{\mathcal{N}}(a)$ does not contain a random graph.*

Proof. The proof follows by definitions. In case $A \Vdash$"$B \subseteq \dot{r}_{gen}$", then $A \cap \overline{\mathcal{N}}(b)$ can not contain a random graph, because if there was a random graph $C \subseteq A \cap \overline{\mathcal{N}}(b)$, we would have that $C \Vdash$"$b \notin r_{gen}$", but this is a contradiction since $A \Vdash$"$B \subseteq \dot{r}_{gen}$". The other cases are similar. ■

We need the following:

Lemma 62 *Let r_{gen} be a $\mathbb{P}(\mathcal{R})$-generic real. The following holds in $V[r_{gen}]$: For every $A, B \in \mathbb{R}$, if $B \nleq_{\mathbb{R}} A$, then there is $C \leq_{\mathbb{R}} B$ such that C and A are incompatible in $\mathbb{R}$.*

Proof. Since $B \nleq_{\mathbb{R}} A$, we know that $G \cap Ker(B \setminus A) \neq \emptyset$ (where G is the $\mathbb{K}$-generic filter). In this way, there is $C \in \mathbb{R}$ such that $C \subseteq B \setminus A$. It follows that A and C are incompatible. ■

Formally, $\mathbb{R}$ is not a separative partial order since it is not antisymmetric. However, it becomes separative when we identify equivalent conditions (i.e. conditions A and B such that $A \leq_{\mathbb{R}} B$ and $B \leq_{\mathbb{R}} A$). We will not bother with this technical detail. We will now prove the following:

Proposition 63 *If s_{gen} is a $\mathbb{K}$-generic real over V, then $V[s_{gen}] \models$ "The non-Empty player does not have a winning strategy in $\mathcal{DG}(\mathbb{R})$".*

Proof. Let $p \in \mathbb{K}$ and $\dot{\sigma}$ be a $\mathbb{K}$-name for a strategy of the non-Empty player in the game $\mathcal{DG}(\mathbb{R})$, we will prove that p has an extension that forces that $\dot{\sigma}$ is not a winning strategy.

Lets say that the Empty player decides he will play $\omega \in \mathbb{P}(\mathcal{R})$ in his first turn (note that ω is the largest condition, essentially he is giving the non-Empty player a free turn). In this way, the first move of the non-Empty player is $\dot{\sigma}(\langle\omega\rangle)$. By extending p if necessary, we may assume that there is a random graph $B \in \mathbb{P}(\mathcal{R})$ such that $p \Vdash$ "$\dot{\sigma}(\langle\omega\rangle) = B$" and p is a kernel for B.

We will recursively construct $\langle p_n, \mathcal{B}_n, \mathcal{A}_n \rangle_{n\in\omega}$ such that for every $n \in \omega$, the following holds:

1. $p_0 \leq p$ and $p_{n+1} \leq_n p_n$.
2. $\mathcal{B}_n = \{B_s \mid s \in split_n(p_n)\}$ and $\mathcal{A}_n = \{A_{s^\frown i} \mid s \in split_n(p_n) \wedge i \in 2\}$ are collection of random graphs.
3. $B_{st(p_0)} = B$ (so $\mathcal{B}_0 = \{B\}$).
4. If $s \in split_n(p_n)$, then $A_{s^\frown 0}, A_{s^\frown 1} \subseteq B_s$.
5. $(p_n)_{s^\frown i}$ is a kernel for $A_{s^\frown i}$ for $i \in 2$.
6. The pair $\langle A_{s^\frown 0}, A_{s^\frown 1}\rangle$ is mutually decided.
7. Let $s \in split_n(p_n)$, denote $L_s = \{l_1, ..., l_{n-1}, l_n\} \subseteq \omega$ such that $z_j = s \restriction l_j$ is a splitting node. We have that the condition $(p_n)_s$ forces that $(\omega, B_{z_1}, A_{z_1{}^\frown s(l_1)}, B_{z_2}, A_{z_2{}^\frown s(l_2)}, ..., B_{z_{n-1}}, A_{z_{n-1}{}^\frown s(l_{n-1})})$ is a legal partial play in the game $\mathcal{DG}(\mathbb{R})$ and that B_s is equal to:
$$\dot{\sigma}(\omega, B_{z_1}, A_{z_1{}^\frown s(l_1)}, B_{z_2}, A_{z_2{}^\frown s(l_2)}, ..., B_{z_{n-1}}, A_{z_{n-1}{}^\frown s(l_{n-1})})$$

We will define $p_0, \mathcal{B}_0$ and $\mathcal{A}_0$. We know that p is a kernel for B, so we may find $q \leq p$ and a determined $C \leq B$ such that $Z(C) = q$. Let c_0 be the smallest element of C. We know that both $\mathcal{N}(c_0) \cap C$ and $\overline{\mathcal{N}}(c_0) \cap C$ are random graphs, so by corollary 60, we can find $A_0 \leq \mathcal{N}(c_0) \cap C$ and $A_1 \leq \overline{\mathcal{N}}(c_0) \cap C$ such that the pair $\langle A_0, A_1 \rangle$ is mutually decided. Let s be the stem of q. Since $q = Z(C)$ and $c_0 = \min(C)$, there are i_0 and i_1 such that $q_{s^\frown i_0} \Vdash$ "$c_0 \in \dot{r}_{gen}$" and $q_{s^\frown i_1} \Vdash$ "$c_0 \notin \dot{r}_{gen}$". We can now find p_0 with the following properties:

1. $p_0 \leq q$ (so $p_0 \leq p$).
2. $st(p_0) = s$.
3. $(p_0)_{s^\frown i_0}$ is a kernel for A_0 and $(p_0)_{s^\frown i_1}$ is a kernel for A_1.

Define $B_{st(p_0)} = B$, $A_{s^\frown i_0} = A_0$ and $A_{s^\frown i_1} = A_1$. It is easy to see that this items have the desired properties.

Assume we are now at step $n+1$. We have already defined $\langle p_n, \mathcal{B}_n, \mathcal{A}_n \rangle$, we will see how to define $\langle p_{n+1}, \mathcal{B}_{n+1}, \mathcal{A}_{n+1} \rangle$. Let $s \in split_n(p_n)$, define $L_s = \{l_1, ..., l_{n-1}, l_n\} \subseteq \omega$ such that $z_j = s \restriction l_j$ is a splitting node and denote $u_s = \left\langle \omega, B_{z_1}, A_{z_1 {}^\frown s(l_1)}, B_{z_2}, A_{z_2 {}^\frown s(l_2)}, ..., B_{z_{n-1}}, A_{z_{n-1} {}^\frown s(l_{n-1})} \right\rangle$. We also know that $A_{s^\frown i} \subseteq B_s$, (for every $i \in 2$) furthermore, $(p_n)_{s^\frown i}$ is a kernel for $A_{s^\frown i}$, hence $(p_n)_{s^\frown i} \Vdash$"$A_{s^\frown i} \in \dot{\mathbb{R}}$". In this way, $(p_n)_{s^\frown i}$ forces that $A_{s^\frown i}$ is a valid move for the Empty player and if the non-Empty player follows her strategy, she will play $\dot{\sigma}(u_s {}^\frown A_{s^\frown i})$.

Let $q^{s,i} \leq (p_n)_{s^\frown i}$ and $B^{s,i} \in \mathbb{P}(\mathcal{R})$ such that $q^{s,i} \Vdash$"$\dot{\sigma}(u_s {}^\frown A_{s^\frown i}) = B^{s,i}$". We may also assume that $q^{s,i}$ is a kernel for $B^{s,i}$. The rest of the construction is essentially the same as in the case of $n = 0$. The notation is a little bit messy, but the reader should note that we are just applying the same procedure as in the base case below the respective nodes. We know that $q^{s,i}$ is a kernel for $B^{s,i}$, so we may find $q \leq q^{s,i}$ and a determined $C \leq B^{s,i}$ such that $Z(C) = q$. Let c_0 be the smallest element of C. We know that both $\mathcal{N}(c_0) \cap C$ and $\overline{\mathcal{N}}(c_0) \cap C$ are random graphs, so by corollary 60, we can find $A_0^{s,i} \leq \mathcal{N}(c_0) \cap C$ and $A_1^{s,i} \leq \overline{\mathcal{N}}(c_0) \cap C$ such that the pair $\left\langle A_0^{s,i}, A_1^{s,i} \right\rangle$ is mutually decided. Let $t^{s,i}$ be the stem of q. Since $q = Z(C)$ and $c_0 = \min(C)$, there are j_0 and j_1 such that $q_{s^\frown j_0} \Vdash$"$c_0 \in \dot{r}_{gen}$" and $q_{s^\frown j_1} \Vdash$"$c_0 \notin \dot{r}_{gen}$". We can now find $r^{s,i}$ with the following properties:

1. $r^{s,i} \leq q$.
2. $st\left(r^{s,i}\right) = t^{s,i}$.
3. $\left(r^{s,i}\right)_{t^{s,i} {}^\frown j_0}$ is a kernel for $A_0^{s,i}$ and $\left(r^{s,i}\right)_{t^{s,i} {}^\frown j_1}$ is a kernel for $A_1^{s,i}$.

Let $p_{n+1} = \bigcup \left\{ r^{s,i} \mid s \in split_n(p_n) \wedge i \in 2 \right\}$, note that $p_{n+1} \leq_n p_n$ and $split_{n+1}(p_{n+1}) = \left\{ t^{s,i} \mid s \in split_n(p_n) \wedge i \in 2 \right\}$. Define $B_{t^{s,i}} = B^{s,i}$, $A_{t^{s,i} {}^\frown j_0} = A_0^{s,i}$ and $A_{t^{s,i} {}^\frown j_1} = A_1^{s,i}$. It is clear that this items have the desired properties.

Now, we define $q = \bigcap\limits_{n \in \omega} p_n$. By construction, we have the following properties for every $n \in \omega$:

1. $q \leq p$.
2. $q \leq_n p_n$, so $split_n(q) = split_n(p_n)$.
3. If $s \in split_n(q)$, then $q_s \Vdash$"$\dot{\sigma}(u_s) = B_s$" (where
$$u_s = \left\langle \omega, B_{z_1}, A_{z_1 {}^\frown s(l_1)}, B_{z_2}, A_{z_2 {}^\frown s(l_2)}, ..., B_{z_{n-1}}, A_{z_{n-1} {}^\frown s(l_{n-1})} \right\rangle).$$

4. $q_{s^\frown i}$ is a kernel for $A_{s^\frown i}$.

We claim that q forces that $\dot{\sigma}$ can be defeated by the Empty player. Moreover, we claim that if r_{gen} is a $\mathbb{K}$-generic real with $r_{gen} \in [q]$, the following holds in $V[r_{gen}]$:

* Let $J = \{j_n \mid n \in \omega\}$ such that $r_{gen} \upharpoonright j_n \in split_n(q)$. We claim that if Empty player plays $A_n = A_{r_{gen} \upharpoonright (j_{n+1})}$ in his $(n+1)$-turn (recall that he played ω in his 0-turn), then he will win.

We argue by contradiction. In this way, there is $r \leq q$ forcing that the non-Empty player won the match. By extending r if necessary, we may assume that there is $D \in \mathbb{P}(\mathcal{R})$ such that $r \Vdash$"$D \in \dot{\mathbb{R}}$" and $r \Vdash$"$D \leq_{\mathbb{R}} \dot{A}_n$" for every $n \in \omega$ (recall that $\dot{A}_n$ is a name for $\dot{A}_{r_{gen} \upharpoonright (j_{n+1})}$ where $r_{gen} \upharpoonright j_n \in split_n(q)$). Furthermore, we may assume that r is a kernel for D. In this way, we may find a determined $E \leq D$ and $\overline{r} \leq r$ such that $Z(E) = \overline{r}$.

Let d be the smallest member of E and $s = st(\overline{r})$. We can find $n \in \omega$ such that $s \in split_n(q)$. Note that we have the following:

1. $\overline{r}_{s^\frown 0} \Vdash$"$\dot{A}_n = A_{s^\frown 0}$".
2. $\overline{r}_{s^\frown 1} \Vdash$"$\dot{A}_n = A_{s^\frown 1}$".

Recall that $\langle A_{s^\frown 0}, A_{s^\frown 1}\rangle$ is mutually decided. For concreteness, lets assume that $A_{s^\frown 1} \Vdash$"$A_{s^\frown 0} \subseteq \dot{r}_{gen}$" and $A_{s^\frown 0} \Vdash$"$A_{s^\frown 1} \cap \dot{r}_{gen} = \emptyset$" (the other cases are similar). By corollary 61, we get the following:

1. If $a_0 \in A_{s^\frown 0}$, then $\overline{\mathcal{N}}(a_0) \cap A_{s^\frown 1}$ does not contain a random graph.
2. If $a_1 \in A_{s^\frown 1}$, then $\mathcal{N}(a_1) \cap A_{s^\frown 0}$ does not contain a random graph.

Since $\overline{r}_{s^\frown 1} \Vdash$"$E \leq_{\mathbb{R}} A_{s^\frown 1}$", we get that $\overline{r}_{s^\frown 1}$ forces that the generic real is not in any element of $ker(E \setminus A_{s^\frown 1})$. Note that this entails that $E \cap A_{s^\frown 1}$ must be infinite. In the same way, $\overline{r}_{s^\frown 0} \Vdash$"$E \leq_{\mathbb{R}} A_{s^\frown 0}$" and $E \cap A_{s^\frown 0}$ is infinite.

Choose distinct $e_0 \in E \cap A_{s^\frown 0}$ and $e_1 \in E \cap A_{s^\frown 1}$ both larger that d. Define $u = \{e_0, e_1\}$ and $v = \{e_1\}$. Obviously, E^u_v is a random graph. However, we claim that both $E^u_v \cap A_{s^\frown 0}$ and $E^u_v \cap A_{s^\frown 1}$ do not contain a random graph. To prove this, simply note that $E^u_v \cap A_{s^\frown 0} \subseteq \mathcal{N}(e_1) \cap A_{s^\frown 0}$ and $E^u_v \cap A_{s^\frown 1} \subseteq \overline{\mathcal{N}}(e_0) \cap A_{s^\frown 1}$, and we already know that neither of them contains a random graph.

Define $W = E_v^u$, we clearly have that $Z(W) \subseteq \overline{r}$. We can now find $\widehat{r} \subseteq Z(W)$ that is a kernel for W (hence, $\widehat{r} \Vdash$"$W \in \dot{\mathbb{R}}$"). Since $W \cap A_{s^\frown 0}$ and $W \cap A_{s^\frown 1}$ do not contain random graphs, we get that $\widehat{r} \Vdash$"$W \nleq_{\mathbb{R}} A_{s^\frown 0}$" and $\widehat{r} \Vdash$"$W \nleq_{\mathbb{R}} A_{s^\frown 1}$". We conclude that $\widehat{r} \Vdash$"$W \nleq_{\mathbb{R}} \dot{A}_n$". However, $W \subseteq D$ and we knew that $\widehat{r} \Vdash$"$D \leq \dot{A}_n$", which is a contradiction. This finishes the proof. ■

If $\mathbb{P}$ is a partial order, we denote by $\mathbb{B}(\mathbb{P})$ the Boolean completion of $\mathbb{P}$. We already know that the quotient is ω-distributive. Now, by the theorem of [53], we conclude the following:

Theorem 64 *If s_{gen} is a Sacks real over V, then $V[s_{gen}] \models$"$\mathbb{B}(\mathbb{R})$ is a ω-distributive boolean algebra that does not contain a σ-closed dense set".*

We have proved that the quotient $\mathbb{R}$ is not Solovay equivalent to a σ-closed forcing. We will now prove the following:

Corollary 65 *If s_{gen} is $\mathbb{K}$-generic real over V, then $V[s_{gen}] \models$"$\mathbb{R}$ is a ω-distributive boolean algebra that is not forcing equivalent to a σ-closed forcing".*

Proof. The argument of proposition 63 actually shows that $\mathbb{B}(\mathbb{R})$ is nowhere σ-closed (to show this, we do the same proof but instead that Empty player plays ω in his first move, he plays any condition of $\mathbb{R}$). The conclusion follows by theorem 10. ■

In particular, we get the following (see [39]):

Theorem 66 *$\mathbb{P}(\mathcal{R})$ and $\mathbb{P}(\mathbb{Q})$ are not forcing equivalent.*

Forcing with copies of the 3-Henson graph

For this section, we fix $\mathcal{H}_3 = (\omega, \sim)$ a copy of the 3-Henson graph (which from now on, we will simply call it Henson graph, for simplicity). In this section, we will prove that, unlike the random graph, $\mathbb{P}(\mathcal{H}_3)$ is σ-closed. We will start by recalling some of the most important properties of $\mathcal{H}_3$. First, we have the following:

Proposition 67 (Henson, [18]) *$\mathcal{H}_3$ is the unique (up to isomorphism) countable graph with the following properties:*

1. *$\mathcal{H}_3$ has no triangles (i.e. it omits K_3).*
2. *If $X, Y \in [\omega]^{<\omega}$ are disjoint and X is discrete, then there is $a \in \omega$ that realizes the type (X, Y) (i.e. a has a connection with every element of X and is not connected with every element of Y).*

While the indivisibility of the random graph is essentially trivial, the case for the Henson graph is much more difficult. This was settle by Komjath and Rödl, when they proved the following:

Theorem 68 (Komjath, Rödl [28]) *$\mathcal{H}_3$ is indivisible.*

The reader may also read the proof of the above result in [17] or in the book [49]. Very recently, Hubička found a very elegant and simple proof of using Ramsey theory. It is worth mentioning, that (with a very hard proof), El-Zahar and Sauer proved in [10] that if $p \geq 3$, then $\mathcal{H}_p$ is indivisible.

Recall that $\mathcal{I}_{\mathcal{H}_3}$ is defined as the collection of all sets $X \subseteq \omega$ that do not contain a copy of $\mathcal{H}_3$. Since $\mathcal{H}_3$ is indivisible, it follows that $\mathcal{I}_{\mathcal{H}_3}$ is an ideal. The forcing $\wp(\omega) \diagup \mathcal{I}_{\mathcal{H}_3}$ is the set of all $B \subseteq \omega$ such that $B \notin \mathcal{I}_{\mathcal{H}_3}$ (i.e. B contains a copy of $\mathcal{H}_3$). Given $A, B \in \wp(\omega) \diagup \mathcal{I}_{\mathcal{H}_3}$, define $B \leq A$ if $B \setminus A \in \mathcal{I}_{\mathcal{H}_3}$. From a theorem of [31] (see also[34]), it follows that $\mathbb{P}(\mathcal{H}_3)$ and $\wp(\omega) \diagup \mathcal{I}_{\mathcal{H}_3}$ are forcing equivalent, so we may work with any of them. In this section, it will be convenient to work with $\wp(\omega) \diagup \mathcal{I}_{\mathcal{H}_3}$, but for ease of writing, we will continue to denote it by $\mathbb{P}(\mathcal{H}_3)$ (as mentioned before, this causes no problems since this two partial orders are the same from the forcing point of view).

The following lemma is easy and it is left to the reader,

Lemma 69 *Let $a \in \omega$ and $B \in \mathbb{P}(\mathcal{H}_3)$.*

1. *Both $\mathcal{N}(a) \cap B$ and $\overline{\mathcal{N}}(a) \cap B$ are infinite.*
2. *$\mathcal{N}(a)$ is discrete.*
3. *If $a \in B$, then $B \setminus \mathcal{N}(a)$ is a copy of $\mathcal{H}_3$.*

We will need the following notion:

Definition 70 *Let $A, B \subseteq \omega$. We say that B is* Henson over *A if for every disjoint $X, Y \in [A]^{<\omega}$ with X discrete, there is $b \in B$ realizing the type (X, Y).*

Note that B is a Henson graph[11] if and only if B is Henson over itself.

Lemma 71 *Let $B \in \mathbb{P}(\mathcal{H}_3)$ and a finite $H \subseteq [\omega]^{<\omega}$ with $\emptyset \in H$. There is $\{C_s \mid s \in H\}$ such that for every $s \in H$, the following holds:*

[11] As in the case for the random graph, we identify a set with the subgraph it induces. So "B is a Henson graph" means the same as "$(B, \sim\restriction B)$ is a Henson graph".

1. *$C_s \subseteq B$ and $\{C_s \mid s \in H\}$ is pairwise disjoint.*
2. *$C_\emptyset$ is Henson and if $s \neq \emptyset$, then C_s is discrete.*
3. *C_s is Henson over $\bigcup \{C_t \mid t \in H \wedge s \cap t = \emptyset\}$. In particular, $C_\emptyset$ is Henson over $\bigcup\limits_{t \in H} C_t$.*
4. *If $t \in H$ and $s \cap t \neq \emptyset$, then $C_s \cup C_t$ is discrete.*

Proof. Given $s \in H$, define $W(s) = \{t \in H \mid s \cap t = \emptyset\}$. Note that $W(\emptyset) = H$. We will recursively define $\{a_s^n \mid s \in H\}$ and C_s^n such that for every $n \in \omega$ and $s \in H$, the following will hold:

1. $C_s^n = \left\{a_s^i \mid i \leq n\right\}$.
2. If $s \neq \emptyset$, then C_s^n is discrete.
3. If $s \cap t \neq \emptyset$, then $C_s^n \cup C_t^n$ is discrete for every $t \in H$.
4. There are disjoint $X_s^n, Y_s^n \subseteq \bigcup \{C_t^n \mid t \in W(s)\}$ with X_s^n discrete, such that a_s^{n+1} realizes the type (X_s^n, Y_s^n).

Once concluded the construction, we will define $C_s = \bigcup\limits_{n \in \omega} C_s^n$. Moreover, we arrange the choice of the sets X_s^n, Y_s^n such that at the end, C_s will be Henson over $\bigcup \{C_t \mid t \in W(s)\}$.

We start by choosing $a_s^\emptyset$ in such a way that $\left\{a_s^\emptyset \mid s \in H\right\}$ is a discrete set. Assume we successfully performed step n and we are now at step $n+1$. For every $s \in H$, we have two disjoint sets $X_s^n, Y_s^n \subseteq \bigcup \{C_t^n \mid t \in W(s)\}$ with X_s^n discrete. Since B is a Henson graph, we can find a discrete set $\left\{a_s^{n+1} \mid s \in H\right\}$ such that a_s^{n+1} realizes the type $(X_n^s, Y_n^s \cup \bigcup \{C_t^n \mid s \cap t \neq \emptyset\})$. Note that if $s \neq \emptyset$, then a_s^{n+1} has no neighbors in C_s^n. This finishes the construction and the proof. ■

The following notion will be key in the future:

Definition 72 *Let $F \in [\omega]^{<\omega}$ and $B \in \mathbb{P}(\mathcal{H}_3)$. We say that F* can be resurrected below B *if for every $A \leq B$, there is a Henson graph $C \leq A$ such that $F \subseteq C$.*

In some sense, it means that we can "always recover F" when forcing below B. This notion will be important for us in order to do some sort of fusion later on. It is worth noting that it is not true that every finite set can be resurrected below any condition. We will see an example and in order to do that, we will need the following definition:

Definition 73 *Let $B \in \mathbb{P}(\mathcal{H}_3)$, we say that B is a* far graph *if for every $a \notin B$, the set $\mathcal{N}(a) \cap B$ is finite.*

We took the above notion from the paper of Hasson, Kojman, Onshuus [17], where far graphs play an important role in proving the *symmetric indivisibility* of the Henson graph. They are important for us because of the following:

Proposition 74 *Let $B \in \mathbb{P}(\mathcal{H}_3)$ and $a \in \omega$. If B is far and $a \notin B$, then $\{a\}$ can not be resurrected below B.*

Proof. Let A be a Henson graph with $a \in A$, we will prove that $A \nleq B$, this will be enough to prove the proposition. Define $C = (\overline{\mathcal{N}}(a) \cap A) \setminus B$. We first claim that C is infinite. Since A is a Henson graph and $a \in A$, then $\mathcal{N}(a) \cap A$ is infinite. Since B is far, we can find $x \in \mathcal{N}(a) \cap A$ such that $x \notin B$. In the same way, $\mathcal{N}(x) \cap A$ is infinite and only finitely many of this elements are in B. Finally, note that $\mathcal{N}(a)$ and $\mathcal{N}(x)$ are disjoint, which implies that C is infinite.

We will now prove that C is a Henson graph. Let X, Y be two disjoint subsets of C with X discrete. We may further assume that $X \neq \emptyset$. Let $Y_1 = Y \cup \{a\}$, since A is a Henson graph, there are infinitely many vertices in A realizing the type (X, Y_1). Since $X \neq \emptyset$, it follows that only finitely many of them are in B, so we can find $d \in A$ realizing (X, Y_1) with $d \notin B$, it is clear that $d \in C$.

In this way, $C \subseteq A \setminus B$, so we conclude that $A \nleq B$. ■

Evidently, the whole $\mathcal{H}_3$ is a far graph, but there are other examples. The following is a particular case of lemma 5.26 of [17]:

Lemma 75 ([17]) *Let $a \in \omega$ and $A = \overline{\mathcal{N}}(a)$. There is a Henson graph $B \subseteq A$ such that B is a far graph.*

In particular, we get that there many non-trivial far graphs (although by our results, they are not dense). We can now prove the following:

Proposition 76 *$\mathbb{P}(\mathcal{H}_3)$ is not σ-closed.*

Proof. We will recursively build a sequence of Henson graphs $\{B_n \mid n \in \omega\}$ such that for every $n \in \omega$, the following holds:

1. $B_{n+1} \subseteq B_n$.
2. B_n is a far graph.
3. $n \notin B_n$.

In order to find B_0, we do the following: Let $A = \overline{\mathcal{N}}(0)$, by lemma 75, there is a far graph $B_0 \subseteq A$. it is clear that $0 \notin B_0$. Assume we are at step n and have defined B_n, we will see how to define B_{n+1}. If $n+1 \in B_n$, let $a = n+1$;

otherwise, let a be any element of B_n and let $A = \overline{\mathcal{N}}(a) \cap B_n$. We now apply lemma 75 relativized to B_n, so we find $B_{n+1} \subseteq A$ that is *far in* B_n i.e. if $b \in B_n$ and $b \notin B_{n+1}$, then $(\mathcal{N}(b) \cap B_n) \cap B_{n+1}$ is finite. We claim that B_{n+1} is far. Let $b \notin B_{n+1}$, if $b \in B_n$ we are done since B_{n+1} is far in B_n. In case $b \notin B_n$, the result follows since B_n is far and $B_{n+1} \subseteq B_n$.

Clearly $\{B_n \mid n \in \omega\}$ is a decreasing sequence of conditions, we claim that it has no lower bound. Let A be a Henson graph, we will prove that it is not a lower bound of the sequence. Let m be the smallest element of A, we know that $m \in A \setminus B_m$. By proposition 74, it follows that $A \nleq B_m$. ■

Nevertheless, we will prove that $\mathbb{P}(\mathcal{H}_3)$ is forcing equivalent to a σ-closed forcing. The following result will be key for this:

Proposition 77 *Let $B \in \mathbb{P}(\mathcal{H}_3)$ and $F \in [B]^{<\omega}$. There is $A \leq B$ such that F can be resurrected below A.*

Proof. Let $L = \{s \subseteq F \mid s \text{ is discrete}\}$ and for every $s \in L$, let $B(s)$ the set of all $v \in B$ that realizes the type $(s, F \setminus s)$. Note that if $s \neq \emptyset$, then $B(s)$ is a discrete set.

Claim 78 *There are $\{f_s \mid s \in L\}$ and $\{Z_s \mid s \in L\}$ such that for every $s \in L$, the following holds:*

1. $Z_s \subseteq B(s)$.
2. $f_s : Z_\emptyset \longrightarrow Z_s$ *is a bijection.*
3. $f_\emptyset$ *is the identity.*
4. $Z_\emptyset$ *is a Henson graph.*
5. *Let $t \in L$ such that $s \cap t = \emptyset$. For every $x, y \in Z_\emptyset$, the following holds:*
$$x \sim y \text{ if and only if } f_s(x) \sim f_t(y)$$
6. *Let $t \in L$. If $s \cap t \neq \emptyset$ and $x \in Z_s$, $y \in Z_t$, then $x \nsim y$.*
7. *If $x \in Z_\emptyset$, then $x \nsim f_s(x)$.*

Before proving the claim, we would like to remark that points 6 and 7 are redundant (but we wrote them since it is useful to keep them in mind). If $s \cap t \neq \emptyset$, then $B(s) \cup B(t)$ is contained in a discrete set, so point 6 follows. For point 7, if $x \sim f_s(x)$, then $f_\emptyset(x) \sim f_s(x)$, which would imply (by point 5) that $x \sim x$, which is impossible. It is also worth pointing out that f_s (with $s \neq \emptyset$) is not a graph-embedding (it can not be, since $Z_\emptyset$ is Henson and Z_s is discrete).

Now we are able to prove the claim. We will now recursively define the set $\{(Z^n_s, f^n_s) \mid n \in \omega \wedge s \in L\}$ such that for every $n \in \omega$ and $s, t \in L$, the following conditions hold:

1. $Z_s^n \subseteq B(s)$.

2. $f_s^n : Z_\emptyset^n \longrightarrow Z_s^n$ is bijective.

3. $f_\emptyset^n$ is the identity mapping.

4. $Z_s^n \subseteq Z_s^{n+1}$ and $f_s^n \subseteq f_s^{n+1}$.

5. If $x \in Z_\emptyset^n$, then $f_t^n(x) \nsim f_s^n(x)$.

6. If $x, y \in Z_\emptyset^n$ and $s \cap t = \emptyset$, then the following holds:
$$x \sim y \text{ if and only if } f_s^n(x) \sim f_t^n(y)$$

7. There is a partition $\langle X_n, Y_n \rangle$ of $Z_\emptyset^n$ with X_n discrete such that there is $a \in Z_\emptyset^{n+1}$ realizing the type (X_n, Y_n).

At the first step, we choose $z_s \in B(s)$ (for all $s \in L$) such that $\{z_s \mid s \in L\}$ is discrete. Define $Z_s^0 = \{z_s\}$ and $f_s^0(z_\emptyset) = z_s$. Now, assume we just performed step n, we will see how to do step $n+1$. Let $\langle X_n, Y_n \rangle$ be a partition of $Z_\emptyset^n$ with X_n discrete. For ease of writing, let $X = X_n$ and $Y = Y_n$.

Define $\overline{X} = \bigcup\limits_{s \in L} f_s^n[X]$ and $\overline{Y} = \bigcup\limits_{s \in L} f_s^n[Y]$. Since $f_\emptyset^n$ is the identity, it follows that $X \subseteq \overline{X}$ and $Y \subseteq \overline{Y}$. We now have the following:

Claim 79 *$\overline{X}$ is a discrete set.*

There are several cases to consider, all being trivial except one:

1. $f_\emptyset^n[X] = X$ is discrete by hypothesis.

2. If $s \neq \emptyset$, then $f_s^n[X]$ is contained in $B(s)$, which is a discrete set.

3. If $s, t \in L$ and $s \cap t \neq \emptyset$, then there are no connections between Z_s and Z_t.

It remains to prove that there are no connections between $f_s^n[X]$ and $f_t^n[X]$ where $s \cap t = \emptyset$ (of course, $s = \emptyset$ or $t = \emptyset$ is allowed). We argue by contradiction, assume that there are $a \in f_s^n[X]$ and $c \in f_t^n[X]$ with $a \sim c$. We now find $x, y \in X$ such that $f_s(x) = a$ and $f_t(y) = c$. Since $a \sim c$, it follows that $x \neq y$. Now, since $a \sim c$, we get that $f_s^n(x) \sim f_t^n(y)$ and by the recursive hypothesis, we get that $x \sim y$. However, this is a contradiction since X is a discrete set. We conclude that $\overline{X}$ is a discrete set.

Given $w \in L$, define $\overline{X}(w) = \bigcup\{f_s^n[X] \mid s \in L \wedge s \cap w = \emptyset\}$ and $\overline{Y}(w) = \bigcup\{f_s^n[Y] \mid s \in L \wedge s \cap w = \emptyset\}$. Clearly, we have that $\overline{X}(\emptyset) = \overline{X}$ and $\overline{Y}(\emptyset) = \overline{Y}$. We now have the following:

Claim 80 *If $w \in L$, then $\overline{X}(w) \cup w$ is discrete.*

We already know that both w and $\overline{X}(w)$ are discrete. Furthermore, if $s \in L$ and $s \cap w = \emptyset$, it follows by the definition that there are no connections between $B(s)$ and w, since $f_s^n[X] \subseteq B(s)$, the claim follows.

Now, since B is a Henson graph, we may find a discrete set $\{a_s \mid s \in L\}$ such that a_s realizes the type $(\overline{X}(s) \cup s, \overline{Y}(s))$. In particular, we get that $a_s \in B(s)$. Define $Z_s^{n+1} = Z_s^n \cup \{a_s\}$ and let $f_s^{n+1} : Z_\emptyset^{n+1} \longrightarrow Z_s^{n+1}$ extend f_s^n such that $f_s^{n+1}(a_\emptyset) = a_s$.

We now only need to prove that if $x \in Z_\emptyset^n$ and $s, t \in L$ are such that $s \cap t = \emptyset$, then $a_\emptyset \sim x$ if and only if $f_s^{n+1}(a_\emptyset) = a_s \sim f_t^n(x)$. On one hand, $a_\emptyset \sim x$ if and only if $x \in \overline{X} \cap Z_\emptyset = X$. While on the other hand, $a_s \sim f_t^n(x)$ if and only if $f_t^n(x) \in \overline{X}(s) \cap Z_t = f_t[X]$, or in other words, if $x \in X$. It follows that $a_\emptyset \sim x$ if and only if $a_s \sim f_t^n(x)$. This finishes the recursive construction.

For every $s \in L$, define $Z_s = \bigcup\limits_{n \in \omega} Z_s^n$. Furthermore, by carefully choosing the partitions (X_n, Y_n) at each step, we make sure that $Z_\emptyset$ is a Henson graph. This finishes the proof of claim 78.

Let $A = Z_\emptyset$, we will now prove that F can be resurrected below A. Let $C \leq A$, we need to prove that C can be extended to a condition that contains F. We may assume that $C \subseteq A$. Now, by lemma 71, we know that there is a family $\{C_s \mid s \in L\}$ such that for every $s \in L$, the following conditions hold:

1. $C_s \subseteq A$ and $\{C_s \mid s \in L\}$ is pairwise disjoint.
2. $C_\emptyset$ is Henson and if $s \neq \emptyset$, then C_s is discrete.
3. C_s is Henson over $\bigcup \{C_t \mid t \in L \wedge s \cap t = \emptyset\}$.
4. If $t \in L$ and $s \cap t \neq \emptyset$, then $C_s \cup C_t$ is discrete.

For every $s \in L$, let $D_s = f_s[C_s]$ (so $D_\emptyset = C_\emptyset$) and define $D = F \cup \bigcup\limits_{s \in L} D_s$. We have the following:

Claim 81 *D is a Henson graph.*

Let X, Y be two disjoint finite subsets of D, with X discrete. We may assume that $F \subseteq X \cup Y$. For every $\in L$, let $X_s = X \cap D_s$ and $Y_s = Y \cap D_s$, define $w = F \cap X$. We will need the following claim:

Claim 82 *Let $s \in L$. If $X_s \neq \emptyset$, then $w \cap s = \emptyset$.*

We argue by contradiction, so assume that $X_s \neq \emptyset$ and $w \cap s \neq \emptyset$. Let $a \in w \cap s$ and $b \in X_s$. Since $b \in X_s$, it follows that $b \in Z_s$, so $b \sim a$. However, $a \in X$, so both a and b are in X, but this is a contradiction since X was assumed to be discrete. In particular, if $w \neq \emptyset$, then $X_w = \emptyset$.

Define $\overline{X}_s = f_s^{-1}(X_s)$, note that $\overline{X}_s \subseteq C_s$ and $\overline{X}_\emptyset = X_\emptyset$. Now, we have the following:

Claim 83 *The set* $\bigcup\{\overline{X}_s \mid s \in L \wedge X_s \neq \emptyset\}$ *is discrete.*

We start by noting the following:

1. $\overline{X}_\emptyset \subseteq X$, which is discrete.
2. If $s \neq \emptyset$, then $\overline{X}_s$ is contained in C_s, which is discrete.
3. If $s, t \in L$ and $s \cap t \neq \emptyset$, then $C_s \cup C_t$ is discrete, so $\overline{X}_s \cup \overline{X}_t$ is discrete.

It only remains to prove that if $s, t \in L$ and $s \cap t = \emptyset$, then $\overline{X}_s \cup \overline{X}_t$ is discrete. Assume that this is not the case, so there are $x \in \overline{X}_s$ and $y \in \overline{X}_t$ such that $x \sim y$. Let $a = f_s(x)$ and $b = f_t(y)$, note that $a, b \in X$. We know that $x \sim y$ and since $s \cap t = \emptyset$, this implies that $f_s(x) \sim f_t(y)$, so $a \sim b$, which is a contradiction since X is a discrete set. This concludes the proof that $\{\overline{X}_s \mid s \in L \wedge X_s \neq \emptyset\}$ is discrete.

Note that C_w is Henson over $\bigcup\{C_s \mid X_s \neq \emptyset\}$ (in case $w = \emptyset$ this is trivial and if $w \neq \emptyset$, then (by claim 82) $X_w = \emptyset$). In this way, we can find $z \in C_w$ realizing the type $(\bigcup \overline{X}_s, \bigcup \overline{Y}_s)$. We will prove that $b = f_w(z)$ realizes the type (X, Y).

Since $b \in C_w \subseteq B(w)$ and $w = F \cap X$, it follows that b is connected with every element of $F \cap X$ and not connected with every element of $F \cap Y$, so at least in F, we are fine.

Now, let $a \in X \setminus F$, so there is $s \in L$ such that $a \in D_s \cap X = f_s[C_s] \cap X$. Let $x \in \overline{X}_s$ such that $f_s(x) = a$. Note that since $X_s \neq \emptyset$, it follows that $w \cap s = \emptyset$. In this way, we have that $z \sim x$, so $f_w(z) \sim f_s(x)$, which implies that $b \sim a$.

In a similar way, let $a \in Y \setminus F$. There is $s \in L$ such that $a \in D_s \cap Y = f_s[C_s] \cap Y$. Let $y \in \overline{Y}_s$ such that $f_s(y) = a$. If $w \cap s = \emptyset$, then $f_w(z) \nsim f_s(y)$, so $b \nsim a$. Assume that $w \cap s = \emptyset$. In this way, we have that $z \nsim y$, so $f_w(z) \nsim f_s(y)$, which implies that $b \nsim a$. This finishes the proof that D is a Henson graph.

It only remains to prove that $D \leq C$, i.e. that $D \setminus C$ does not contain a copy of the Henson graph. Note that $D \setminus C = F \cup \bigcup\{D_s \mid s \in L \wedge s \neq \emptyset\}$. In this way, $D \setminus C$ is the union of a finite set and finitely many discrete sets. Since $\mathcal{H}_3$ is indivisible, $D \setminus C$ can not contain a copy of the Henson graph. ■

With the previous result, we can easily prove the following:

Proposition 84 *The non-Empty player has a winning strategy in $\mathcal{DG}(\mathbb{P}(\mathcal{H}_3))$.*

Proof. We describe a winning strategy for the non-Empty player as follows:

0. Let A_0 be the first move of the Empty player. Let b_0 be the smallest element of A_0. By proposition 77, there is $\overline{A_0} \leq A_0$ such that $\{b_0\}$ can be resurrected below A_0. Let $B_0 \leq \overline{A_0}$ such that $b_0 \in B_0$. The non-Empty player will play B_0.

1. Let A_1 be the response of the Empty player. Since $\{b_0\}$ can be resurrected below $\overline{A_0}$ and $A_1 \leq B_0$, there is $C_1 \leq A_1$ with $b_0 \in C_1$. Choose a point $b_1 \in C_1$ and by proposition 77, we can find $\overline{A_1} \leq C_1$ such that $\{b_0, b_1\}$ can be resurrected below $\overline{A_1}$. Let $B_1 \leq \overline{A_1}$ such that $b_0, b_1 \in B_1$. The non-Empty player will play B_1.

2. Let A_2 be the response of the Empty player. Since $\{b_0, b_1\}$ can be resurrected below $\overline{A_1}$ and $A_2 \leq B_1$, there is $C_2 \leq A_2$ with $b_0, b_1 \in C_1$. Choose a point $b_2 \in C_1$ and by proposition 77, we can find $\overline{A_2} \leq C_2$ such that $\{b_0, b_1, b_2\}$ can be resurrected below $\overline{A_2}$. Let $B_2 \leq \overline{A_2}$ such that $b_0, b_1, b_2 \in B_2$. The non-Empty player will play B_2.

$\vdots \qquad\qquad\qquad\qquad \vdots$

By playing this way, the set $D = \{b_n \mid n \in \omega\}$ is a pseudointersection of the conditions played by the non-Empty player. By carefully choosing each b_n, the non-Empty player can make D to be a Henson graph, giving her the victory in the match. ■

Finally, by theorem of [53], we conclude the following:

Theorem 85 *$\mathbb{P}(\mathcal{H}_3)$ is forcing equivalent to a σ-closed forcing (in fact, the Boolean completion of $\mathbb{P}(\mathcal{H}_3)$ is σ-closed).*

Forcing with copies of other Fraïssé limits

We believe that the study of $\mathbb{P}(\mathbb{B})$ where $\mathbb{B}$ is a natural Fraïssé limit is very interesting. There are still many open questions. For the convenience of reader, we list here some of the known results, we also take the opportunity to announce some results that will appear in a future paper. The reader may consult [34] and [38] for more information and other results.

1. If we take an empty signature (or only containing a symbol for equality), then ω as a set is the Fraïssé limit of the finite sets. Clearly, $\mathbb{P}(\omega) = \wp(\omega) \diagup \mathsf{fin}$ and it is σ-closed.

2. (Kurilić, Todorcevic [39]) $\mathbb{P}(\mathbb{Q})$ is forcing equivalent to $\mathbb{S} * \dot{\mathbb{R}}$, where $\dot{\mathbb{R}}$ is forced to be a σ-closed forcing ($\mathbb{Q}$ is taken as a linearly ordered set).

3. $\mathbb{P}(\mathcal{R})$ is forcing equivalent to an iteration $\mathbb{S} * \dot{\mathbb{R}}$, where $\dot{\mathbb{R}}$ is forced to be an ω-distributive not σ-closed forcing.

4. $\mathbb{P}(\mathcal{H}_3)$ is a σ-closed forcing.

5. (Guzmán, Todorcevic) If $\mathbb{U}_{\mathbb{Q}}$ is the rational Urysohn space, then $\mathbb{P}(\mathbb{U}_{\mathbb{Q}})$ collapses the continuum.

6. (Guzmán, Todorcevic) If $\mathbb{U}_3$ is the Fraïssé limit of the finite metric spaces with distances in $\{0, 1, 2, 3\}$, then $\mathbb{P}(\mathbb{U}_3)$ is forcing equivalent to an iteration $\mathbb{S} * \dot{\mathbb{R}}$, where $\dot{\mathbb{R}}$ is forced to be an ω-distributive not σ-closed forcing.

7. (Kurilić, Todorcevic [38]) If $\mathbb{T}^{\infty}$ is the random tournament, then $\mathbb{P}(\mathbb{T}^{\infty})$ is forcing equivalent to $\mathbb{P}(\mathcal{R})$.

8. (Kurilić, Todorcevic [38]) Let $\mathbb{S}(2)$ be the dense local order. $\mathbb{P}(\mathbb{S}(2))$ is forcing equivalent to $\mathbb{S} * \dot{\mathbb{R}}$, where $\dot{\mathbb{R}}$ is forced to be a σ-closed forcing.

9. (Kurilić, Todorcevic [38]) $\mathbb{P}(\mathbb{S}(3))$ is forcing equivalent to $\mathbb{S} * \dot{\mathbb{R}}$, where $\dot{\mathbb{R}}$ is forced to be a σ-closed forcing (see [38] for the definition of $\mathbb{S}(3)$).

Open questions and problems

Recall that $\mathbb{P}(\mathbb{Q})$ is forcing equivalent to an iteration $\mathbb{S} * \dot{\mathbb{R}}$, where $\dot{\mathbb{R}}$ is forced to be a σ-closed forcing. In [39], Kurilić and the second author proved that under $\mathfrak{b} = \omega_1$ or PFA, the partial order $\mathbb{P}(\mathbb{Q})$ is forcing equivalent to $\mathbb{S} * \wp(\omega) \diagup \mathsf{fin}$. This raises the following question:

Problem 86 *Is it consistent that $\mathbb{P}(\mathbb{Q})$ and $\mathbb{S} * \wp(\omega) \diagup \mathsf{fin}$ are not forcing equivalent?*

We proved that $\mathbb{P}(\mathcal{H}_3)$ is forcing equivalent to a σ-closed forcing. We can ask the following:

Problem 87 *Let $p > 3$, is $\mathbb{P}(\mathcal{H}_p)$ forcing equivalent to a σ-closed forcing?*

We conjecture that the problem has a positive answer. In that case, we can ask the following:

Problem 88 *Let $p, q \geq 3$ with $p \neq q$. Is it consistent that $\mathbb{P}(\mathcal{H}_p)$ and $\mathbb{P}(\mathcal{H}_q)$ are not forcing equivalent?*

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