Velocity Profiles

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In concrete applications such as machine tools or industrial robots, where a path is being followed by means of a control algorithm. It is not advisable so select a sequence of interpolating points for which the arc length is large since the controller may generate a saturated control signal causing well-known issues such as forcing the servo motors beyond their working limits or an overshoot in the path— in the particular case of a servo system for the operation of a numerically controlled milling machine, the tool may undergo an excessive wear or even fracture or the machined part could exceed the required tolerances.

A solution for situations like these when the sequence of setpoints defining the tool path are quite separated is to use a generation path scheme involving a proper velocity profile to not only provide bounds for accelerations and decelerations but also to have a precise control of these. This, in turn, helps to keep any overshot within the required tolerances, to optimize the power consumption and to regulate, indirectly, the servo motor velocity.

The most common velocity profiles employed in industrial applications are carefully described next, allowing the reader to perform the same analysis for almost any other profile.

1 Triangular Velocity Profile

As its name suggests, the triangular velocity profile resembles a triangle as shown in figure 1. This is a piecewise defined function given by two linear segments corresponding to the acceleration and deceleration of the tool or end-effector, that is, during the first half of the motion the velocity increases linearly with respect to the elapsed time and during the second half of the motion the velocity decreases linearly with respect to the elapsed time.

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Fig. 1 Triangular velocity profile.

Since any linear (or affine) function has a constant derivative, the acceleration for the first half of the motion is a positive constant a and the deceleration for the second half of the motion is a negative constant $-a$ of the same magnitude, that is, the acceleration $\alpha\colon [0,T] \to \mathbb{R}$ is given by

$$\alpha(t) = \begin{cases} +a & 0 \le t < \frac{T}{2} \\ -a & \frac{T}{2} \le t < T \end{cases} \tag{1}$$

where T is the duration of the motion. The angular velocity $\omega(t)$ is defined by integrating equation (1) with respect to time t. Assuming, a static initial condition, that is, that $\omega(0) = 0$, for $0 \le t \le T/2$, we have

$$\omega(t) = \int_0^t d\omega(s) + \omega(0) = \int_0^t \alpha(s)\,ds + 0 = a\int_0^t ds = at. \tag{2}$$

At time $t = T/2$ we obtain the maximum angular velocity

$$\omega_{\max} = \omega\left(\frac{T}{2}\right) = \frac{aT}{2}. \tag{3}$$

Thus, for $0 < t \le T/2$, we must have

$$\omega(t) = \int_{T/2}^t d\omega(s) + \omega\left(\frac{T}{2}\right) = \int_{T/2}^t \alpha(s)\,ds + \frac{aT}{2} = -at + aT. \tag{4}$$

Hence,

$$\omega(t) = \begin{cases} at, & 0 \le t \le \frac{T}{2}, \\ -at + aT, & T/2 < t \le T. \end{cases} \tag{5}$$

The angular position $\theta(t)$ is computed as follows. For $0 \le t \le T/2$,

$$\theta(t) = \int_0^t d\theta(s) + \theta(0) = a\int_0^t s\,ds + \theta(0) = \frac{at^2}{2} + \theta(0); \tag{6}$$

assuming that $\theta(0) = 0$ this yields $\theta(t) = at^2/2$. Moreover, since

$$\theta\left(\frac{T}{2}\right) = \frac{aT^2}{8}, \tag{7}$$

for $T/2 < t \le T$ we have

$$\begin{aligned}\theta(t) &= \int_{T/2}^{t} d\theta(s) + \theta\left(\frac{T}{2}\right) \\ &= -a\int_{T/2}^{t} s\,ds + aT\int_{T/2}^{t} ds + \frac{aT^2}{8} \\ &= -\frac{at^2}{2} + aTt - \frac{aT^2}{4}.\end{aligned}$$

Therefore,

$$\theta(t) = \begin{cases} \dfrac{at^2}{2}, & 0 \le t \le \dfrac{T}{2}, \\ -\dfrac{at^2}{2} + aTt - \dfrac{aT^2}{4}, & \dfrac{T}{2} < t \le T. \end{cases} \tag{8}$$

This allows us to compute the final angular position θ_{f} occuring at time $t = T$ as

$$\theta_{\mathrm{f}} = \theta(T) = -\frac{aT^2}{2} + aT^2 - \frac{aT^2}{4} = \frac{aT^2}{4}. \tag{9}$$

A relation between the final position θ_{f}, the maximum angular velocity $\omega_{\max}$ and the time required to perform the motion T can be obtained by substituting equation (3) into equation (9), namely,

$$\theta_{\mathrm{f}} = \frac{\omega_{\max} T}{2}. \tag{10}$$

From this equation the elapsed time T can be computed from the final position θ_{f} and the maximum velocity $\omega_{\max}$ as

$$T = \frac{2\theta_{\mathrm{f}}}{\omega_{\max}}. \tag{11}$$

Moreover, we can solve for the required acceleration a in equation (3), which yields

$$a = \frac{2\omega_{\max}}{T}. \tag{12}$$

Example 1. In order to machine a carbon steel shaft using a CNC lathe 100 mm of the material must be grinded. The grinding tool manufacturer indicates that the maximum recommended feedrate is 25 mm/s. The ball bearing that converts rotational motion into linear displacement has a ratio of 8 mm/rev. Determine the time required for the CNC late to perform the grinding and the nominal value for the angular acceleration of the servo motor in $\mathrm{rad/s^2}$.

The first step is to compute the final angular position

$$\theta_{\mathrm{f}} = \frac{100(\mathrm{mm})}{8(\mathrm{mm/rev})} = 12.5(\mathrm{rev}),$$

that is,

$$\theta_{\mathrm{f}} = 12.5(\mathrm{rev}) \times \frac{2\pi(\mathrm{rad})}{1(\mathrm{rev})} = 78.5398(\mathrm{rad}).$$

Similarly, the maximum velocity is given by

$$\omega_{\max} = \frac{25(\mathrm{mm/s})}{8(\mathrm{mm/rev})} = 3.125\left(\frac{\mathrm{rev}}{\mathrm{s}}\right),$$

or equivalently,

$$\omega_{\max} = 3.125\left(\frac{\mathrm{rev}}{\mathrm{s}}\right) \times \frac{2\pi(\mathrm{rad})}{1(\mathrm{rev})} = 19.6349\left(\frac{\mathrm{rad}}{\mathrm{s}}\right).$$

Substituting into equation (11) yields

$$T = \frac{2 \times 78.5398(\mathrm{rad})}{19.6349(\mathrm{rad/s})} = 8(\mathrm{s}).$$

Moreover, by substituting into equation 12 we have that the angular acceleration is

$$a = \frac{2 \times 19.6349(\mathrm{rad/s})}{8(\mathrm{s})} = 4.9087\left(\frac{\mathrm{rad}}{\mathrm{s}^2}\right).$$

Figure 2 shows the curves for these values. Acceleration is depicted in figure 2(c) with a nominal value of almost five radians per second. The triangular behavior of the velocity profile can be seen in figure 2(b). The graphic of the angular position curve is shown in figure 2(a). The reader can check that these curves attain the corresponding maximum values.

2 Trapezoidal Velocity Profile

Although the triangular velocity profile describes a smooth path, it is relatively inefficient in the sense that the angular velocity $\omega(t)$ begins to decrease right after reaching its maximum value $\omega_{\max}$ until it becomes zero again, which implies that the energy used to reach the maximum velocity is immediately wasted since the motor begins to slow down right away.

A much more efficient alternative is using a trapezoidal velocity profile, with allows us to exploit the energy employed to reach the maximum velocity for a period of time, as shown in figure 3.

In a similiar fashion as we did before, we can define the acceleration for the trapezoidal velocity profile as

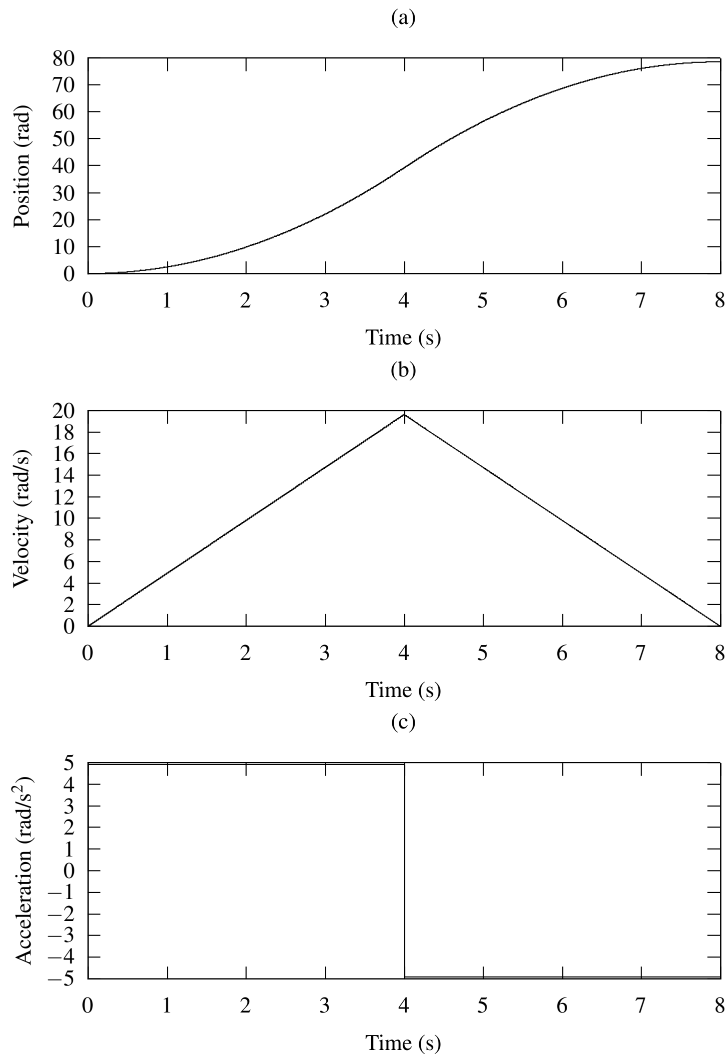

Fig. 2 Triangular velocity profile from example 1: (a) position, (b) velocity, and (c) angular acceleration.

Fig. 3 Trapezoidal velocity profile

$$\alpha(t) = \begin{cases} +a, & 0 \le t \le \frac{T}{3}, \\ 0, & \frac{T}{3} < t \le \frac{2T}{3}, \\ -a, & \frac{2T}{3} < t \le T, \end{cases} \tag{13}$$

where $a > 0$ is a constant acceleration and T is the duration of the motion. We can compute $\omega(t)$ as before by integrating $\alpha(t)$.

Assuming as above that $\omega(0) = 0$, for $0 \le t \le T/3$ we have

$$\omega(t) = \int_0^t d\omega(s) + \omega(0) = a\int_0^t ds + 0 = at. \tag{14}$$

The maximum speed is attained at time $t = T/3$, that is

$$\omega_{\max} = \omega\left(\frac{T}{3}\right) = \frac{aT}{3}. \tag{15}$$

For the interval $T/3 < t \le 2T/3$ velocity remains constant since

$$\omega(t) = \int_{T/3}^t d\omega(s) + \omega\left(\frac{T}{3}\right) = \int_{T/3}^t 0\, ds + \frac{aT}{3} = \frac{aT}{3}. \tag{16}$$

Finally, for $2T/3 < t \le T$ we have

$$\begin{aligned} \omega(t) &= \int_{2T/3}^t d\omega(s) + \omega\left(\frac{2T}{3}\right) \\ &= -a\int_{\frac{2T}{3}}^t ds + \omega\left(\frac{T}{3}\right) \\ &= -at + \frac{2aT}{3} + \frac{aT}{3} \\ &= -at + aT. \end{aligned}$$

Thus,

$$\omega(t) = \begin{cases} +at, & 0 \le t \le \frac{T}{3}, \\ \frac{aT}{3}, & \frac{T}{3} < t \le \frac{2T}{3}, \\ -at + aT, & \frac{2T}{3} < t \le T. \end{cases} \tag{17}$$

In order to compute the angular position, we again proceed interval by interval. For $0 \le t \le T/3$, assuming $\theta(0) = 0$,

$$\theta(t) = \int_0^t d\theta(s) + \theta(0) = a\int_0^t s\, ds = \frac{at^2}{2}. \tag{18}$$

For $T/3 < t \leq 2T/3$,

$$\begin{aligned}
\theta(t) &= \int_{T/3}^{t} d\theta(s) + \theta\left(\frac{T}{3}\right) \\
&= \frac{aT}{3}\int_{T/3}^{t} ds + \frac{aT^2}{18} \\
&= \frac{aT}{3}t - \frac{aT^2}{9} + \frac{aT^2}{18} \\
&= \frac{aT}{3}t - \frac{aT^2}{18}.
\end{aligned}$$

Finally, for $2t/3 < t \leq T$,

$$\begin{aligned}
\theta(t) &= \int_{2T/3}^{t} d\theta(s) + \theta\left(\frac{2T}{3}\right) \\
&= -a\int_{2T/3}^{t} s\,ds + aT\int_{2T/3}^{t} ds + \frac{aT^2}{6} \\
&= -\frac{a}{2}t^2 + \frac{4aT^2}{18} + aTt - \frac{2aT^2}{9} + \frac{aT^2}{6} \\
&= -\frac{a}{2}t^2 + aTt - \frac{5aT^2}{18}.
\end{aligned}$$

Then, the final position θ_{f}, that is, the position when $t = T$, is given by

$$\theta_{\mathrm{f}} = \theta(T) = \frac{2aT^2}{9}, \tag{19}$$

whence, by equation (15),

$$\theta_{\mathrm{f}} = \frac{2T}{3}\cdot\frac{aT}{3} = \frac{2T}{3}\omega_{\max}, \tag{20}$$

and

$$T = \frac{3\theta_{\mathrm{f}}}{2\omega_{\max}}. \tag{21}$$

Besides, equation (15) also implies

$$a = \frac{3\omega_{\max}}{T}. \tag{22}$$

Example 2. Determine the value for the angular acceleration and the time it will take for the CNC lathe from example 1 to carry out the motion using a trapezoidal velocity profile.

The required time is given by

$$T = \frac{3 \times 78.5398(\mathrm{rad})}{2 \times 19.6349(\mathrm{rad/s})} = 6(\mathrm{s}),$$

and the acceleration is computed as

$$a = \frac{3 \times 19.6349(\mathrm{rad/s})}{6(\mathrm{s})} = 9.81745 \left(\frac{\mathrm{rad}}{\mathrm{s}^2}\right).$$

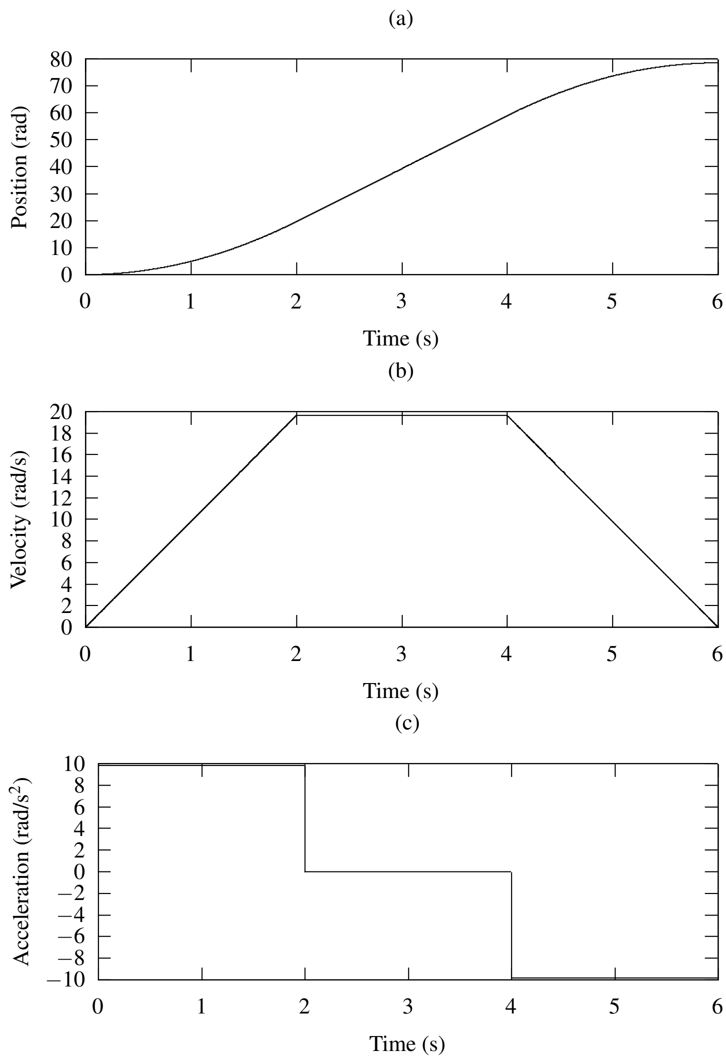

Fig. 4 Trapezoidal velocity profile for example 2: (a) position, (b) velocity, and (c) angular acceleration.

The above data suggests that the trapezoidal velocity profile performs the motion in less time in comparison with the triangular profile, though, certainly, acceleration is almost the double in this case since the servo motor must attain the same angular velocity in less time. Figure 4 depicts the results for the implementation of this example.

3 Parabolic Velocity Profile

An even lower power consumption can be attained by means of the parabolic velocity profile, shown in figure 5, which provides a smoother velocity curve than the one for the trapezoidal velocity profile since its second derivative has no discontinuities.

Fig. 5 Parabolic velocity profile.

The parabolic velocity profile has an angular acceleration α defined as a line with negative slope m as shown in figure 6, namely,

$$\alpha(t) = mt + \alpha(0) \tag{23}$$

where

$$m = \frac{-a-(+a)}{T-0} = \frac{-2a}{T}, \tag{24}$$

that is,

$$\alpha(t) = -\frac{2a}{T}t + a. \tag{25}$$

Fig. 6 Acceleration for a parabolic velocity profile.

Assuming, as before that $\omega(0) = 0$, the angular velocity can be obtained as

$$\omega(t) = \int_0^t d\omega(s) + \omega(0) = -\frac{2a}{T}\int_0^t s\,ds + a\int_0^t ds = -\frac{a}{T}t^2 + at. \tag{26}$$

The maximum velocity is attained at time $t = T/2$, since $\alpha(T/2) = 0$ and $m < 0$,

$$\omega_{\max} = \omega\left(\frac{T}{2}\right) = \frac{aT}{4}, \tag{27}$$

and the end position is reached at time $t = T$, as shown in figure 5.

Similarly, in order to compute the angular position function, we have

$$\theta(t) = \int_0^t d\theta(t) + \theta(0) = -\frac{a}{T}\int_0^t s^2\,ds + a\int_0^t s\,ds = -\frac{at^3}{3T} + \frac{at^2}{2}. \tag{28}$$

Then, the final position is

$$\theta_{\mathrm{f}} = \theta(T) = \frac{aT^2}{6}, \tag{29}$$

or equivalently, using (27),

$$\theta_{\mathrm{f}} = \frac{2aT^2}{12} = \frac{2T}{3}\frac{aT}{4} = \frac{2T}{3}\omega_{\mathrm{max}}. \tag{30}$$

Thus, the total duration of the motion is

$$T = \frac{3\theta_{\mathrm{f}}}{2\omega_{\mathrm{max}}}. \tag{31}$$

Besides, solving for the acceleration in (27) yields

$$a = \frac{4\omega_{\mathrm{max}}}{T}. \tag{32}$$

Example 3. Determine the angular acceleration and the time it takes for the lathe from example 1 using a parabolic velocity profile.

The time required to perform the motion is

$$T = \frac{3 \times 78.5398(\mathrm{rad})}{2 \times 19.6349(\mathrm{rad/s})} = 6(\mathrm{s}).$$

Besides, the acceleration is given by

$$a = \frac{4 \times 19.6349(\mathrm{rad/s})}{6(\mathrm{s})} = 13.08993\left(\frac{\mathrm{rad}}{\mathrm{s}^2}\right).$$

Figure 7 shows the motion for the setup discussed in example 2 using a parabolic velocity profile. Remark that, unlike other velocity profiles, the initial value of the acceleration curve is positive, but it decreases with time instead of being a piecewise constant.

4 Sinusoidal Velocity Profile

The acceleration curve for the velocity profiles discussed in previous sections is not smooth, that is, its derivatives have discontinuities which can introduce unwanted vibrations if they exceed certain system-dependent limits; for example, the parabolic velocity profile changes from an acceleration of $-a$ to an acceleration of a at each setpoint. This kind of vibrations are not an issue for the motion control system, but do reduce the lifetime of the system components.

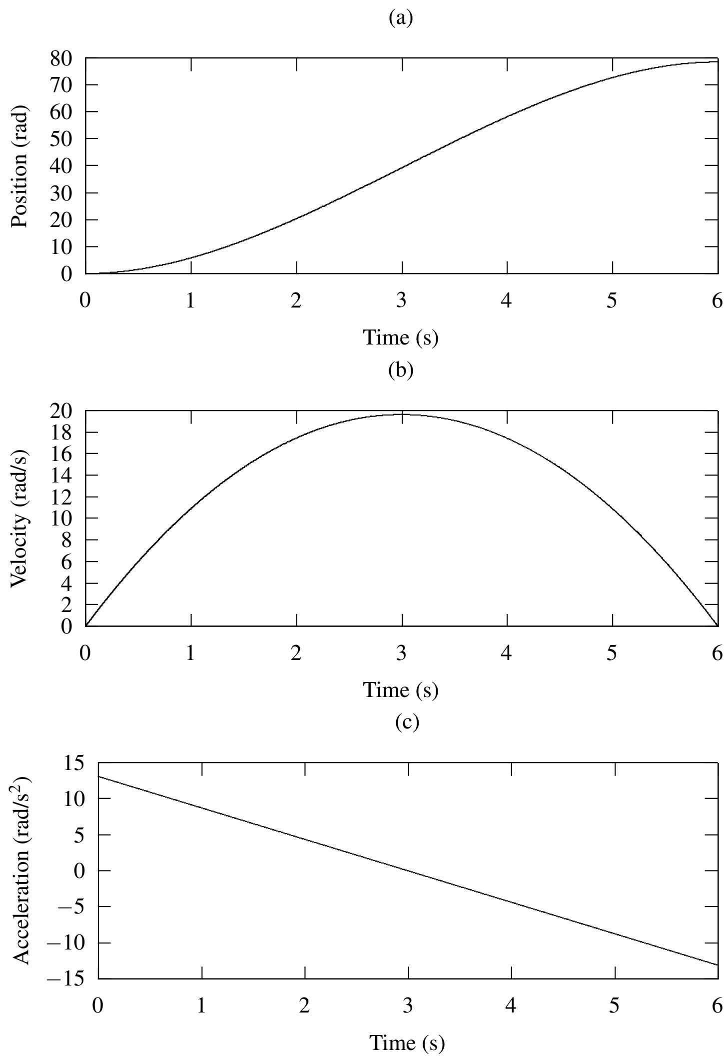

Fig. 7 Parabolic velocity profile for example 3: (a) position, (b) velocity, and (c) angular acceleration.

A great improvement over the previous velocity profiles is to combine the variable acceleration for the parabolic velocity profile with the continuous acceleration for the cases of the triangular and trapezoidal profiles. Trigonometric functions offer both of these properties, and their derivatives are also trigonometric functions, unlike polynomials which become simpler but less versatile functions.

The curve shown in figure 8 allow us to appreciate the properties of a sinusoidal acceleration curve, namely, continuity and smoothness; its derivatives share the same properties.

Fig. 8 Sinusoidal velocity profile.

The acceleration curve is of the form

$$\alpha(t) = a \sin(\omega_0 t) \tag{33}$$

subject to

$$\omega_0 T = 2\pi. \tag{34}$$

The velocity function is, assuming $\omega(0) = 0$,

$$\begin{aligned} \omega(t) &= \int_0^t d\omega(s) + \omega(0) \\ &= a \int_0^t \sin(\omega_0 s)\, ds \\ &= -\frac{a}{\omega_0} \cos(\omega_0 t) + \frac{a}{\omega_0}. \end{aligned}$$

The maximum velocity occurs at $t = T/2$ since $\alpha(T/2) = 0$ and $\alpha'(T/2) < 0$. Besides, since by equation (34)

$$\frac{\omega_0 T}{2} = \pi, \tag{35}$$

we have

$$\omega_{\text{max}} = \omega\left(\frac{T}{2}\right) = \frac{2a}{\omega_0}. \tag{36}$$

Assuming, as before, that $\theta(0) = 0$, we have

$$\begin{aligned} \theta(t) &= \int_0^t d\theta(s) \\ &= -\frac{a}{\omega_0} \int_0^t \cos(\omega_0 s)\, ds + \frac{a}{\omega_0} \int_0^t \\ &= -\frac{a}{\omega_0^2} \sin(\omega_0 t) + \frac{a}{\omega_0} t. \end{aligned}$$

The end position θ_{f} is reached at time $t = T$, so that

$$\theta_{\text{f}} = \theta(T) = -\frac{a}{\omega_0^2} \sin(\omega_0 T) + \frac{a}{\omega_0} T. \tag{37}$$

Substitution of equation (34) into (37) yields

$$\theta_{\text{f}} = \frac{aT}{\omega_0} = \frac{2a}{\omega_0} \frac{T}{2}, \tag{38}$$

that is,

$$\theta_{\mathrm{f}} = \omega_{\max} \frac{T}{2}. \tag{39}$$

Thus, the time it takes to perform the motion is

$$T = \frac{2\theta_{\mathrm{f}}}{\omega_{\max}}. \tag{40}$$

Given T, the frequency for the sinusuidal curve ω_0 is computed as

$$\omega_0 = \frac{2\pi}{T}. \tag{41}$$

Therefore, the acceleration a is given by

$$a = \frac{\omega_{\max}\,\omega_0}{2} \tag{42}$$

Example 4. Determine the time required, the acceleration and the frequency for a motion using a sinusoidal velocity profile for the case discussed in example 1.

The time required to perform the motion is given by

$$T = \frac{2 \times 78.5398(\mathrm{rad})}{19.6349\,(\mathrm{rad/s})} = 8(\mathrm{s}).$$

The sinusoidal frequency is

$$\omega_0 = \frac{2\pi(\mathrm{rad})}{8(\mathrm{s})} = 0.7853\left(\frac{\mathrm{rad}}{\mathrm{s}}\right).$$

The acceleration is computed as

$$a = \frac{19.6349\,(\mathrm{rad/s}) \times 0.7853\,(\mathrm{rad/s})}{2} = 15.41928697\left(\frac{\mathrm{rad}}{\mathrm{s}^2}\right)$$

(remark radians are real numbers, i.e., are adimensional, so $\mathrm{rad}^2 = \mathrm{rad}$).

Figure 9 depicts the resulting curves for example 4. The curve describing the angular position is a sigmoid, suggesting that the motion is smoother than in the other velocity profiles.

5 Evaluation of Power Consumption

The selection of the profile depends on the application requirements, that is, the criteria may be minimizing the power consumption or reducing chattering, or may even been depend on the specific platform used for the implementation of the system.

Table 1 lists in a comparative way the results of the four different kinds of profiles being used to carry out a motion of one radian and at a maximum velocity of one

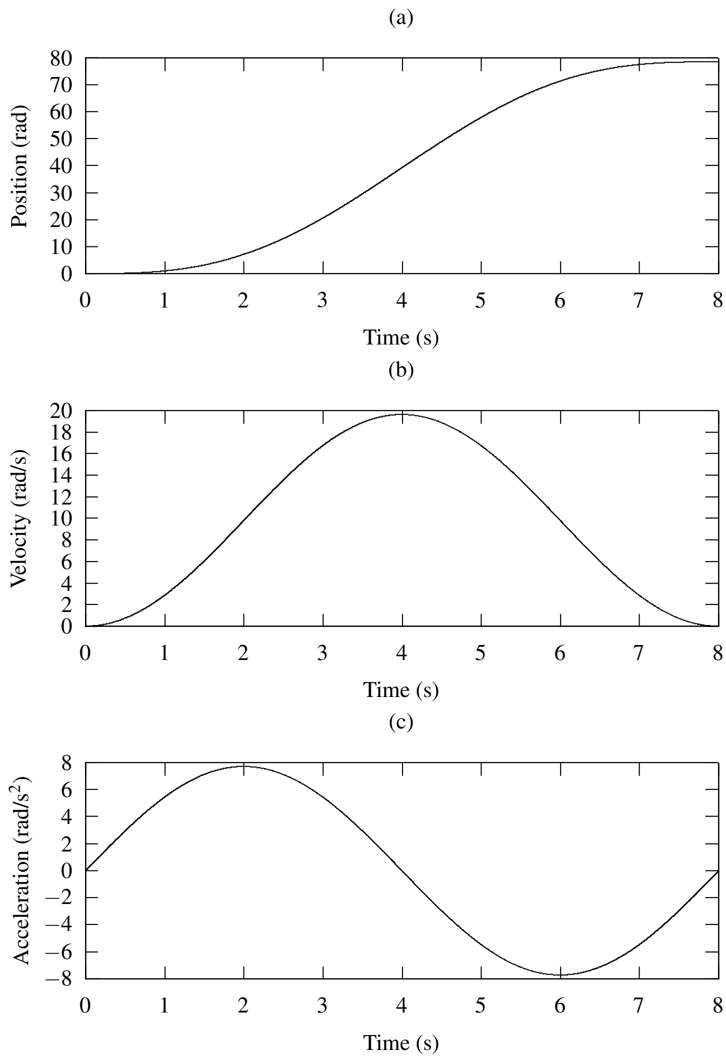

Fig. 9 Sinusoidal velocity profile for example 4: (a) position, (b) velocity, and (c) angular acceleration.

radian per second. These results can give us an idea of how fast are the motions performed using the above profiles; moreover, they also give us a simple way to grasp which is the maximum value for the acceleration for each case. This allow us to approximate the required torque the servo motor needs in order to handle a given load. The third column in table 1 above is an upper bound for the derivative of the acceleration, also known as *jerk*. This value is important to consider when a smooth motion is needed.

Table 1 Experimental results for a motion of $\theta = 1\,(\mathrm{rad})$ at a velocity of $\omega_{\max} = 1$ (rad/s) using a sampling period of $T_s = 0.001\,(\mathrm{s})$.

Profile	$a\,(\mathrm{rad/s^2})$	$T\,(\mathrm{s})$	$\max\{\frac{\Delta a}{T_s}\}\,(\mathrm{rad/s^3})$
Triangular	2.0	2.0	2000.0
Trapezoidal	2.0	1.5	2000.0
Parabolic	2.6667	1.5	2666.7
Sinusoidal	1.5708	2.0	12.1

In order to determine the consumed energy for each profile it is necessary to model a few relationships inherent to servo mechanisms. First, by Newton's Second Law, it is possible to compute the torque generated by the motor as

$$\tau_g(t) = J\frac{d\omega(t)}{dt} + \tau_f, \tag{43}$$

where J and τ_f are the inertia and the friction torque, respectively.

On the other hand, the generated torque $\tau_g(t)$ is a function of the current passing through the monitor, that is

$$\tau_g(t) = K_t i(t), \tag{44}$$

or equivalently,

$$i(t) = \frac{\tau_g(t)}{K_t}. \tag{45}$$

When this current flow through the motor windings, consumes a power that can be described as

$$P(t) = i^2(t)R, \tag{46}$$

where R is the electric resistance of the motor. Therefore it is possible to calculate the power consumption required to perform the motion with a specific velocity profile by means of

$$E = R\int_0^T i^2(t)dt = \frac{R}{K_t^2}\int_0^T \tau^2(t)dt. \tag{47}$$

Using the identity of equation (43) yields

$$E = \frac{R}{K_t^2}\int_0^T \left[J\frac{d\omega(t)}{dt} + \tau_f\right]^2 dt. \tag{48}$$

By expanding the terms of the binomial, the above equation takes the form

$$E = \frac{R}{K_t^2}\left\{J^2\int_0^T \left[\frac{d\omega(t)}{dt}\right]^2 dt + 2J\tau_f\int_0^T \frac{d\omega(t)}{dt}dt + \tau_f^2\int_0^T dt\right\}. \tag{49}$$

Separating term by term we rewrite

$$E = E_1 + E_2 + E_3 \tag{50}$$

where

$$\begin{aligned} E_1 &= \frac{RJ^2}{K_t^2}\int_0^T \left[\frac{d\omega(t)}{dt}\right]^2 dt \\ E_2 &= \frac{2RJ\tau_f}{K_t^2}\int_0^T d\omega(t) = 0 \\ E_3 &= \frac{R\tau_f^2}{K_t^2}\int_0^T dt = \frac{R\tau_f^2 T}{K_t^2} \end{aligned} \tag{51}$$

First, we notice that the second term E_2 is null since both the initial ($t = 0$) and final ($t = T$) velocities are both zero. Besides, the third term E_3 does not depend on the particular velocity function being used—it is just the energy required to oppose the static friction. Thus, the only term that can be optimized is E_1. In order to determine the energy consumption for each profile, it is necessary to characterize the velocity function by means of the final position and the time the system will take to carry out the motion.

5.1 Triangular velocity profile

The final position is defined by equation (9) as a function of acceleration and time; from this equation we have

$$a = \frac{4\theta_{\mathrm{f}}}{T^2}, \tag{52}$$

and by substituting into equation (2) we obtain the angular velocity as a function of the final position and the time, that is,

$$\omega(t) = \begin{cases} \dfrac{4\theta_{\mathrm{f}}}{T^2}t, & 0 \le t \le \dfrac{T}{2}, \\[2ex] \dfrac{4\theta_{\mathrm{f}}}{T^2}(T-t), & \dfrac{T}{2} < t \le T. \end{cases} \tag{53}$$

Thus, by differentiation,

$$\frac{d\omega(t)}{dt} = \begin{cases} \dfrac{4\theta_{\mathrm{f}}}{T^2}, & 0 \le t \le \dfrac{T}{2}, \\[2ex] -\dfrac{4\theta_{\mathrm{f}}}{T^2}, & \dfrac{T}{2} < t \le T. \end{cases} \tag{54}$$

Squaring both sides yields

$$\left[\frac{d\omega(t)}{dt}\right]^2 = \begin{cases} \dfrac{16\theta_\mathrm{f}^2}{T^4}, & 0 \le t \le \dfrac{T}{2}, \\ \\ \dfrac{16\theta_\mathrm{f}^2}{T^4}, & \dfrac{T}{2} < t \le T. \end{cases} \tag{55}$$

Therefore, the energy dissipated for the motion can be written as

$$E_1 = 16\frac{RJ^2\theta_\mathrm{f}^2}{K_t^2T^4}\left(\int_0^{T/2} dt + \int_{T/2}^{T} dt\right), \tag{56}$$

that is,

$$E_1 = 16\frac{RJ^2\theta_\mathrm{f}^2}{K_t^2T^3}. \tag{57}$$

Remark that E_1 is just the energy required to carry out the motion at the maximum velocity. The total energy used comprises, besides E_1, the term E_3 from equation (51), that is

$$E = 16\frac{RJ^2\theta_\mathrm{f}^2}{K_t^2T^3} + \frac{R\tau_\mathrm{f}^2T}{K_t^2}[\mathrm{J}], \tag{58}$$

even tough the second term, as mentioned before, corresponds to the energy dissipated by friction. This is the reason why, for the rest of this chapter, we will only regard the energy used to perform the motion, namely, E_1.

5.2 Trapezoidal velocity profile

The maximum acceleration for this velocity profile is given by

$$a = \frac{9\theta_\mathrm{f}}{2T^2}. \tag{59}$$

Sustituting into equation (17) yields

$$\omega(t) = \begin{cases} \dfrac{9\theta_\mathrm{f}}{2T^2}t, & 0 \le t \le \dfrac{T}{3}, \\ \\ \dfrac{3\theta_\mathrm{f}}{2T}, & \dfrac{T}{3} < t \le \dfrac{2T}{3}, \\ \\ \dfrac{9\theta_\mathrm{f}}{2T^2}(T-t), & \dfrac{2T}{3} < t \le T. \end{cases} \tag{60}$$

Differentiation with respect to time yields

$$\frac{d\omega(t)}{dt} = \begin{cases} \frac{9\theta_{\mathrm{f}}}{2T^2}, & 0 \le t \le \frac{T}{3}, \\ 0, & \frac{T}{3} < t \le \frac{2T}{3}, \\ -\frac{9\theta_{\mathrm{f}}}{2T^2}, & \frac{2T}{3} < t \le T. \end{cases} \tag{61}$$

Squaring both sides of the equation above results

$$\left[\frac{d\omega(t)}{dt}\right]^2 = \begin{cases} \frac{81\theta_{\mathrm{f}}^2}{4T^4}, & 0 \le t \le \frac{T}{3}, \\ 0, & \frac{T}{3} < t \le \frac{2T}{3}, \\ \frac{81\theta_{\mathrm{f}}^2}{4T^4}, & \frac{2T}{3} < t \le T. \end{cases} \tag{62}$$

Thus, the energy consumption for the motion is given by

$$E_1 = \frac{81RJ^2\theta_{\mathrm{f}}^2}{4K_t^2T^4}\left(\int_0^{\frac{T}{3}} dt + \int_{\frac{2T}{3}}^{T} dt\right), \tag{63}$$

that is,

$$E_1 = 13.5\frac{RJ^2\theta_{\mathrm{f}}^2}{K_t^2T^3} \tag{64}$$

5.3 Parabolic velocity profile

Now, consider the parabolic velocity profile. The acceleration can be described using equation (29) as

$$a = \frac{6\theta_{\mathrm{f}}}{T^2}. \tag{65}$$

Sustitution into (26) yields

$$\omega(t) = \frac{6\theta_{\mathrm{f}}}{T^3}(Tt - t^2); \tag{66}$$

by differentiating with respect to time, we have

$$\frac{d\omega(t)}{dt} = \frac{6\theta_{\mathrm{f}}}{T^3}(T - 2t). \tag{67}$$

whence, by squaring both sides of the equality,

$$\left[\frac{d\omega(t)}{dt}\right]^2 = \frac{36\theta_{\mathrm{f}}^2}{T^6}(T^2 - 4Tt + 4t^2). \tag{68}$$

Thus, the energy is described as

$$E_1 = \frac{36RJ^2\theta_{\mathrm{f}}^2}{K_t^2T^6}\left(T^2\int_0^T dt - 4T\int_0^T t\,dt + 4\int_0^T t^2 dt\right), \tag{69}$$

or, by integrating term by term,

$$E_1 = \frac{36RJ^2\theta_{\mathrm{f}}^2}{K_t^2T^6}\left(T^3 - 2T^3 + \frac{4T^3}{3}\right) = 12\frac{RJ^2\theta_{\mathrm{f}}^2}{K_t^2T^3}. \tag{70}$$

5.4 Sinusoidal velocity profile

For the sinusoidal velocity profile, equation (38) becomes

$$\frac{a}{\omega_0} = \frac{\theta_{\mathrm{f}}}{T}, \tag{71}$$

and substitution into equation (**??**) yields

$$\omega(t) = \frac{\theta_{\mathrm{f}}}{T}\left[1 - \cos(\omega_0 t)\right]; \tag{72}$$

therefore, by differentiation,

$$\frac{d\omega(t)}{dt} = \frac{\theta_{\mathrm{f}}\omega_0}{T}\sin(\omega_0 t), \tag{73}$$

whence, by squaring both sides of the equality,

$$\left[\frac{d\omega(t)}{dt}\right]^2 = \frac{\theta_{\mathrm{f}}^2\omega_0^2}{T^2}\sin^2(\omega_0 t), \tag{74}$$

that is,

$$\left[\frac{d\omega(t)}{dt}\right]^2 = \frac{\theta_{\mathrm{f}}^2\omega_0^2}{T^2}\left[\frac{1 - \cos(2\omega_0 t)}{2}\right]. \tag{75}$$

Thus, we have

$$E_1 = \frac{RJ^2\theta_{\mathrm{f}}^2\omega_0^2}{2K_t^2T^2}\left[\int_0^T dt - \int_0^T \cos(2\omega_0 t)dt\right]. \tag{76}$$

Taking into account equation (34), by integrating the equation above the second term becomes zero, so that

$$E_1 = \frac{RJ^2 \theta_{\mathrm{f}}^2 \omega_0^2}{2K_t^2 T} \tag{77}$$

In order to compare the energy consumption with that of other profiles, E_1 can be rewritten as

$$E_1 = \frac{RJ^2 \theta_{\mathrm{f}}^2}{K_t^2 T^3} \frac{\omega_0^2 T^2}{2}, \tag{78}$$

or, using equation (34),

$$E_1 = 2\pi^2 \frac{RJ^2 \theta_{\mathrm{f}}^2}{K_t^2 T^3}. \tag{79}$$

From these equations we can conclude that the parabolic velocity profile is the profile that requires less energy while, at the same time, allows the motor to perform the motion in less time. Using the same techniques discussed in this chapter the reader can do the same analysis for any velocity profile not discussed here, e.g., for polynomial profiles of higher order or those based on convolution.

6 Experimentation

In this section the results of the implementation of each of the profiles introduced above are presented. The servo motor used to carry out the tests has these characteristics: 12V input, 2.1A nominal current, a transmission ratio of 171.79:1, a nominal speed of 7386.97rev, and an incremental encoder of 48 counts per revolution.

Fig. 10 Experimental setup for testing the velocity profiles.

The system used to implement the velocity profiles is shown in figure 10. The profiles are generated by means of a Cortex ARM-9 processor embedded in a Zynq chip which also includes an FPGA used to implement a position control loop—based on a 16 bit module used to compute and saturate the position error, a digital PID filter with 9.16-format fixed point coefficients, a serial communications module for the DAC and a 32 bit quadrature decoder for feedback purposes.

The first system tests are meant for plant identification in the frequency domain. The results return a system cut frequency of $\omega_c = 100[\mathrm{rad/s}]$ and a phase delay of $\phi = -192.678°$. Using this information, and in order to compute a phase margin of $\theta_m = 45°$, the controller gains must be $K_p = 0.7077$, $T_i = 0.0768$, and $T_d = 0.0192$.

The figure 11 shows the response of the system described above under a triangular velocity profile. Position is the only meassured variable; hence, in order to obtain the velocity and acceleration, a derivative filter has been used. Such filter has the form:

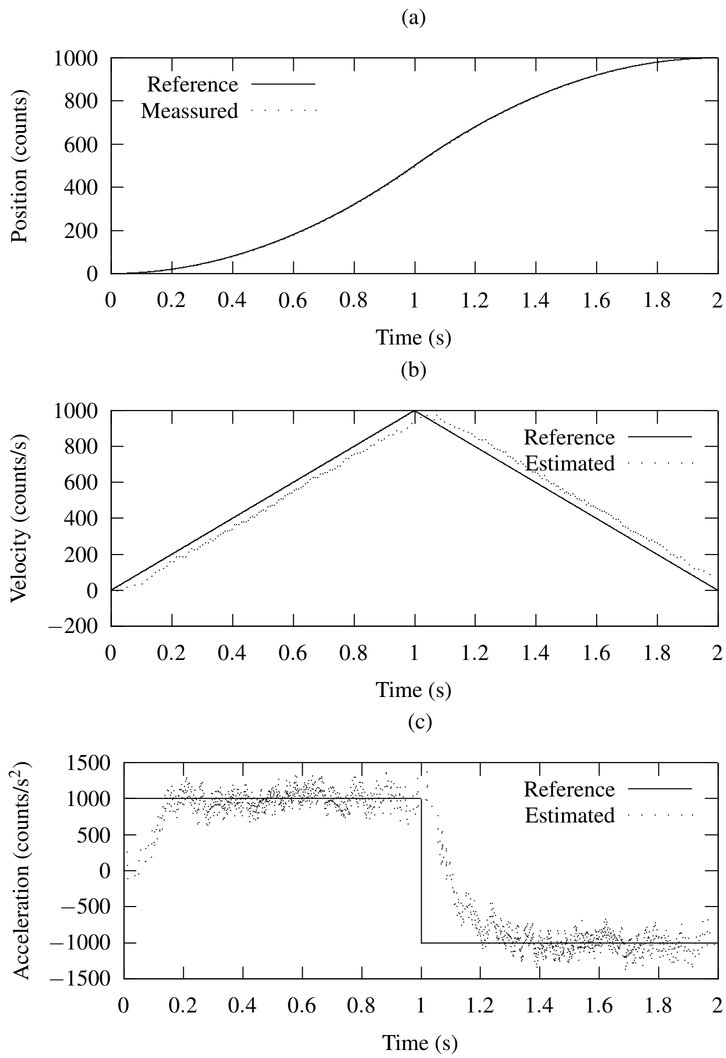

Fig. 11 Experimental results of the triangular velocity profile: a) position, b) velocity, and c) acceleration.

$$\Omega(s) = \frac{s}{\tau s + 1}\Theta(s). \tag{80}$$

By using the *Tustin* aproximation, avobe equation can be rewriten in a discrete form as:

$$\omega(k) = \frac{2}{T_s + 2\tau}[\theta(k) - \theta(k-1)] - \frac{T_s - 2\tau}{T_s + 2\tau}\omega(k-1). \tag{81}$$

where T_s and τ are the sampling and filtering times. For sake of this experiment they are $T_s = 0.001$ y $\tau = 0.05$.

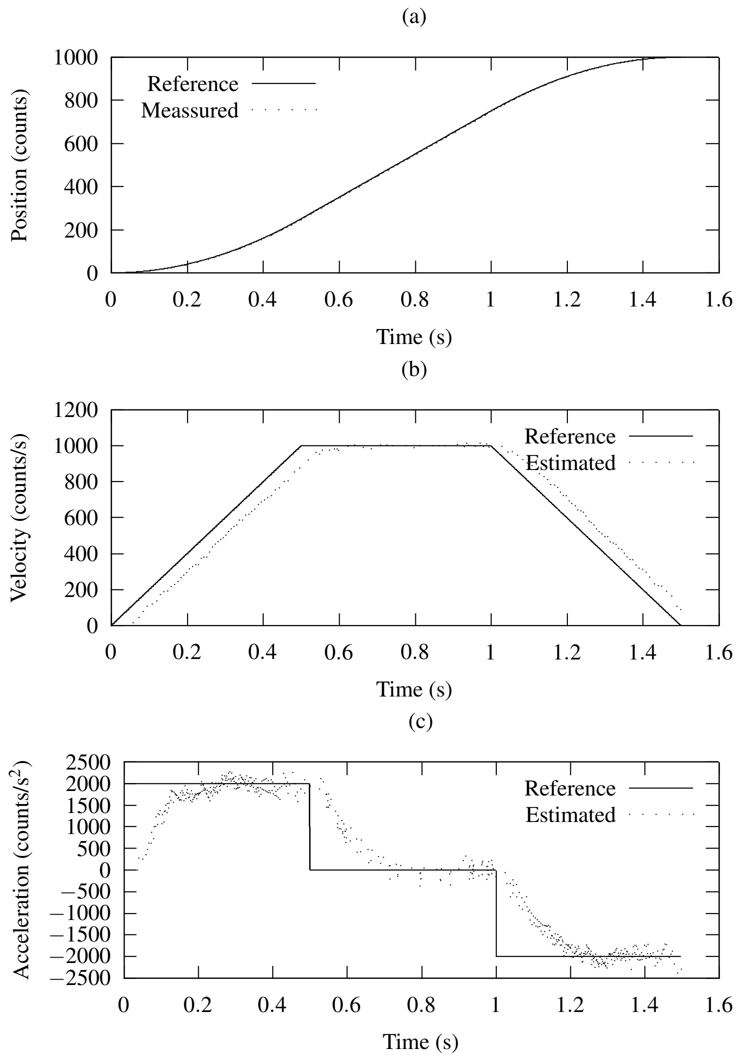

Fig. 12 Experimental results of the trapezoidal velocity profile: a) position, b) velocity, and c) acceleration.

Similarly, the figure 12 illustrates the response of the system under a trapezoidal velocity profile. These graphics includes both: the reference path and its response for position, velocity and acceleration.

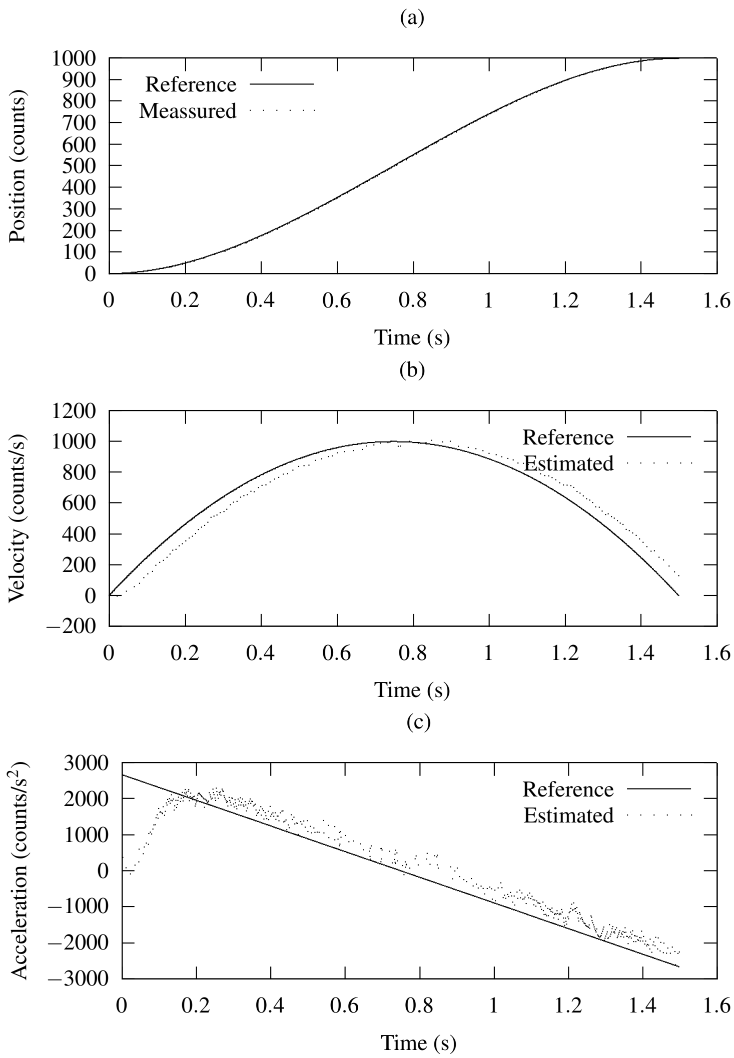

Fig. 13 Experimental results of the parabolic velocity profile: a) position, b) velocity, and c) acceleration.

The figure 13 depicts the response of the parabolic velocity profile. It is important to note that the velocity and acceleration are estimated values; therefore, these curves presents high frequency noise.

Finally, figure 14 shows the curves obtained for the sinusoidal velocity profile; notice that, since a derivative filter is used for estimating both velocity and acceleration, these curves show a displacement in the time domain.

Table 2 list the RMS value of the position tracking error in the path. Notice that all profiles where evaluated under the same conditions.

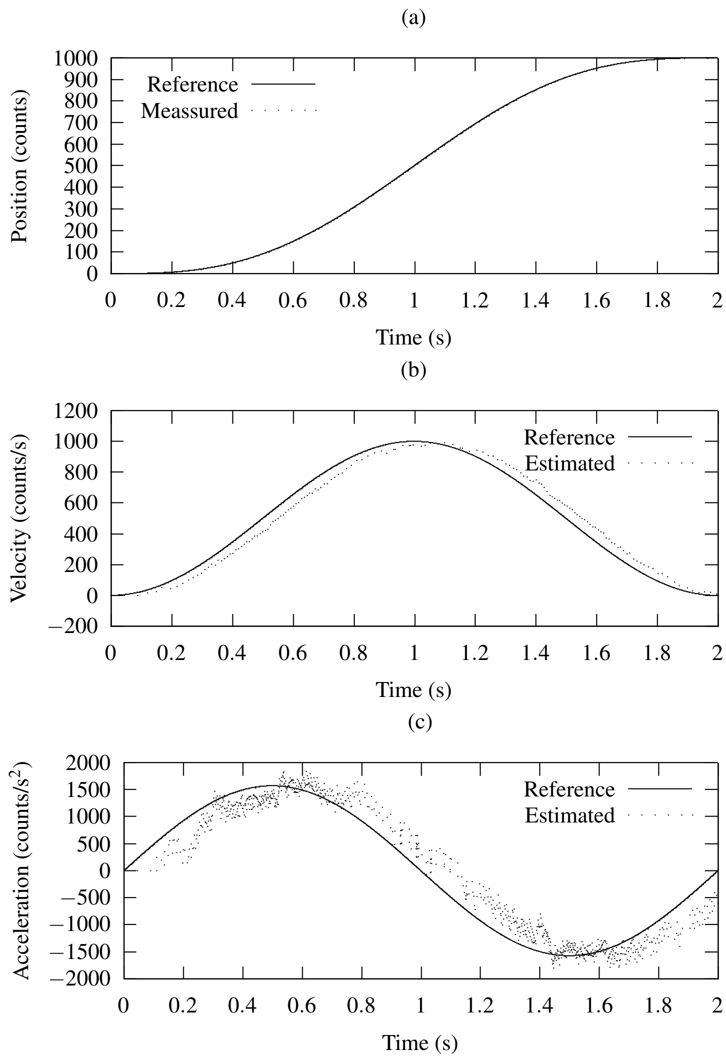

Fig. 14 Experimental results of the sinusoidal velocity profile: a) position, b) velocity, and c) acceleration.

Table 2 Position tracking error.

	Triangular	Trapezoidal	Parabolic	Sinusoidal
RMS error	4.643366	7.290751	7.238560	5.020558

7 Conclusions

Results from table 2 suggest that, by using a PID controller like the one proposed in the experimental platform, those profiles with a better path tracking are the ones that take longer to carry out the motion.

If the main goal is to minimize the energy consumption, the best option is the parabolic velocity profile; however, if its implementation is not feasible, an almost equally efficient alternative that is easier to implement is the trapezoidal profile. On the other hand, if the goal is to avoid any acceleration derivative (jerk) discontinuities and their corresponding effects on the generated torque, the sinusoidal profile is the best option since its acceleration (an all of its derivatives) and the generated torque are smooth.