Multivariable Calculus

Vectors & Matrices

Vectors

Dot Product

Also called scalar product or inner product.

$$\underset{\text{vector}}{\begin{pmatrix}2\\3\\5\end{pmatrix}} \cdot \underset{\text{vector}}{\begin{pmatrix}7\\8\\9\end{pmatrix}} = (2)(7)+(3)(8)+(5)(9)$$

$$= 14+24+45$$

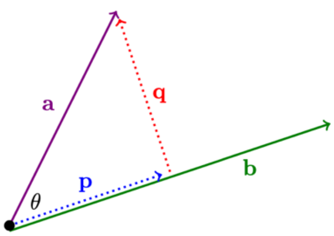
$$= \underset{\text{scalar}}{83}\ .$$

- Important case: $a \cdot a = |a|^2$
- (Geometric interpretation) If θ is the angle between nonzero vector a and b, then

$$\boxed{\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}|\ |\mathbf{b}|\cos\theta.}$$

- $a \cdot b = 0 \Leftrightarrow$ Vectors a and b are perpendicular

Scalar Component of a in the direction of b



$$\boxed{\operatorname{comp}_{\mathbf{b}}\mathbf{a} = \mathbf{a}\cdot\frac{\mathbf{b}}{|\mathbf{b}|} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|}.}$$

Matrices

Determinant

To each square matrix A is associated a number called its determinant.

$det(a) = a$
$= \pm$ length of segment in the real line R determined by a

$det\begin{pmatrix}a & b\\ c & d\end{pmatrix} = ad - bc$
$= \pm$ area of parallelogram in R^2 formed by $\langle a, c\rangle$ and $\langle b, d\rangle$

$$det\begin{pmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{pmatrix} = a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - c_1b_2a_3 - c_2b_3a_1 - c_3b_1a_2$$

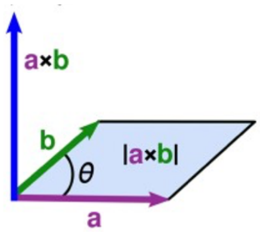
$= \pm$ volume of parallelepiped in R^3 formed by $\langle a_1, a_2, a_3\rangle$, $\langle b_1, b_2, b_3\rangle$ and $\langle c_1, c_2, c_3\rangle$

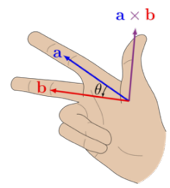
Geometrically, the factor by which a linear transformation changes in any area.

Cross Product

Defined only in R^3. **a x b** is:

- perpendicular to a and b
- with length equal the area of the parallelogram formed by a and b ($|a||b|sin\ \boldsymbol{\theta}$)
- The direction is given by the right-hand rule.





It is calculated as follows.

$$\mathbf{a}\times\mathbf{b} := \begin{vmatrix}\mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}$$

$$:= +\begin{vmatrix}a_2 & a_3\\ b_2 & b_3\end{vmatrix}\mathbf{e}_1 - \begin{vmatrix}a_1 & a_3\\ b_1 & b_3\end{vmatrix}\mathbf{e}_2 + \begin{vmatrix}a_1 & a_2\\ b_1 & b_2\end{vmatrix}\mathbf{e}_3$$

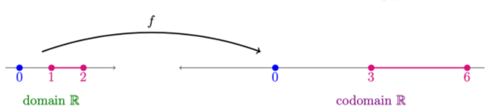
$$= \Big\langle a_2b_3 - a_3b_2,\quad a_3b_1 - a_1b_3,\quad a_1b_2 - a_2b_1\Big\rangle.$$

Matrices as Linear Transformations

1-Dimension

Given a number, say 3, we get a function

$$f\colon \mathbb{R} \longrightarrow \mathbb{R}$$
$$x \longmapsto 3x$$



Higher dimension

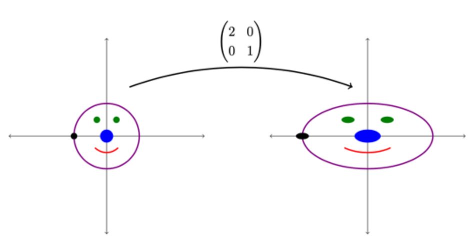
Given the 2×3 matrix $\begin{pmatrix}6 & 7 & 8\\ 2 & 3 & 5\end{pmatrix}$, we get a function

$$\mathbf{f}\colon \mathbb{R}^3 \longrightarrow \mathbb{R}^2$$
$$\mathbf{x} \longmapsto \begin{pmatrix}6 & 7 & 8\\ 2 & 3 & 5\end{pmatrix}\mathbf{x}$$

$$\mathbf{f}\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}6x+7y+8z\\ 2x+3y+5z\end{pmatrix}.$$

In general, an $m\times n$ matrix A gives rise to a function

$$\mathbf{f}\colon \mathbb{R}^n \longrightarrow \mathbb{R}^m$$
$$\mathbf{x} \longmapsto A\mathbf{x}$$



$$\boxed{\text{area scaling factor} = |\det A|.}$$

Systems of Equations

To solve $3x = 5$, multiply both sides by 3^{-1}.

Similarly, one way to solve

$$2x_1 + 3x_2 = 4$$
$$4x_1 + 5x_2 = 6,$$

is to rewrite as

$$\begin{pmatrix}2 & 3\\ 4 & 5\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} = \begin{pmatrix}4\\ 6\end{pmatrix},$$

which has the shape $A\mathbf{x} = \mathbf{b}$, and left multiply both sides by A^{-1} to get $\mathbf{x} = A^{-1}\mathbf{b}$.

Inverse

The inverse of a square matrix A is another matrix A^{-1}, such that

$$AA^{-1} = I \qquad \text{and} \qquad A^{-1}A = I.$$

It exists if and only if $det(A)\ \neq 0$; in that case, A is called invertible, or nonsingular.

Equation of Planes

The set of all vectors perpendicular to $\langle 1, 2, 3\ \rangle$ is a plane with equation

$$\langle 1,2,3\rangle \cdot \langle x,y,z\rangle = 0,$$

Which is

$$x + 2y + 3z = 0.$$

The vector $\boldsymbol{n} := \langle 1,2,3\rangle$ is called a **normal vector** to the plane.

Linear Algebra

Parametric Lines & Curves

There are two ways to describes lines in $\boldsymbol{R}^3$:

- Intersection of two planes
- Parametric equations

Think of the trajectory of an airplane moving at constant velocity. Let r_0 be the position vector of the airplane at time $t\ =\ 0$. Let v be the velocity.

$$r(t) := r_0 + tv$$

Partial Differentiation

Definition

The partial derivative of a function $f(x, y)$ with respect to x is a function $\boldsymbol{\frac{\partial f}{\partial x}}$ (also written as $\boldsymbol{f_x}$) whose value at (x_0, y_0) is:

- The rate of change of $f(x, y)$ when x is varying near $x = x_0$ and y is held constant at the value y_0, or:
- More precisely,

$$\frac{\partial f}{\partial x}(x_0, y_0) := lim_{h\to 0}\frac{f(x_0+h, y_0) - f(x_0, y_0)}{h}$$

Linear Approximation

Question: How do you approximate $f(x)$ for $x \coloneqq x_0 + \Delta x$?

1-Variable

$$\boxed{f(x_0+\Delta x) \approx \underbrace{f(x_0)}_{\text{starting value}} + \underbrace{f'(x_0)\,\Delta x}_{\text{adjustment}}}$$

2-Variables

$$f(x_0+\Delta x, y_0+\Delta y) \approx \underbrace{f(x_0,y_0)}_{\text{starting value}} +$$

$$\underbrace{\left(\frac{\partial f}{\partial x}\right)_0 \Delta x}_{\text{adjustment from }\Delta x} + \underbrace{\left(\frac{\partial f}{\partial y}\right)_0 \Delta y}_{\text{adjustment from }\Delta y} .$$

Max/Min Problem

Solving unconstrained max/min problems

Second derivative test

More on Derivatives of Multivariable Function

Differentials

The total differential of $f(x, y)$ is

$$df \coloneqq f_x dx + f_y dy$$

Chain Rule

In single variable calculus:

$$\frac{df}{dt} = \frac{df}{du} \cdot \frac{du}{dt}$$

In multi-variable calculus:

$$\frac{df}{dt} = f_x \frac{dx}{dt} + f_y \frac{dy}{dt}$$