Multivariable Calculus

Vectors & Matrices

Vectors

Dot Product

Also called scalar product or inner product.

$$\underbrace{\begin{pmatrix}2\\3\\5\end{pmatrix}}_{\text{vector}} \cdot \underbrace{\begin{pmatrix}7\\8\\9\end{pmatrix}}_{\text{vector}} = (2)(7)+(3)(8)+(5)(9)$$

$$= 14+24+45$$

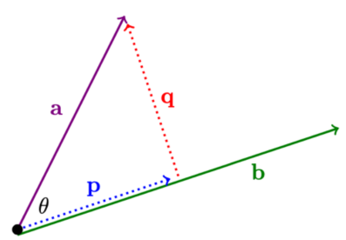
$$= \underbrace{83}_{\text{scalar}}.$$

- Important case: $a \cdot a = |a|^2$
- (Geometric interpretation) If θ is the angle between nonzero vector a and b, then

$$\boxed{\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}|\ |\mathbf{b}|\cos\theta.}$$

- $a \cdot b = 0 \Leftrightarrow$ Vectors a and b are perpendicular

Scalar Component of a in the direction of b



$$\boxed{\text{comp}_{\mathbf{b}}\,\mathbf{a} = \mathbf{a}\cdot\frac{\mathbf{b}}{|\mathbf{b}|} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|}.}$$

Matrices

Determinant

To each square matrix A is associated a number called its determinant.

$det(a) = a$

$= \pm$ length of segment in the real line R determined by a

$det\begin{pmatrix}a & b\\ c & d\end{pmatrix} = ad - bc$

$= \pm$ area of parallelogram in R^2 formed by $\langle a,c\rangle$ and $\langle b,d\rangle$

$$det\begin{pmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{pmatrix} = a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - c_1b_2a_3 - c_2b_3a_1 - c_3b_1a_2$$

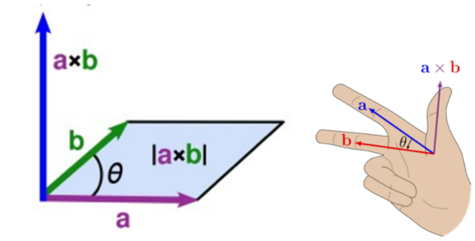
$= \pm$ volume of parallelepiped in R^3 formed by $\langle a_1,a_2,a_3\rangle$, $\langle b_1,b_2,b_3\rangle$ and $\langle c_1,c_2,c_3\rangle$

Geometrically, the factor by which a linear transformation changes in any area.

Cross Product

Defined only in $\mathbb{R}^3$. **a x b** is:

- perpendicular to a and b
- with length equal the area of the parallelogram formed by a and b ($|a||b|sin\,\boldsymbol{\theta}$)
- The direction is given by the right-hand rule.



It is calculated as follows.

$$\mathbf{a}\times\mathbf{b} := \begin{vmatrix}\mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}$$

$$:= +\begin{vmatrix}a_2 & a_3\\ b_2 & b_3\end{vmatrix}\mathbf{e}_1 - \begin{vmatrix}a_1 & a_3\\ b_1 & b_3\end{vmatrix}\mathbf{e}_2 + \begin{vmatrix}a_1 & a_2\\ b_1 & b_2\end{vmatrix}\mathbf{e}_3$$

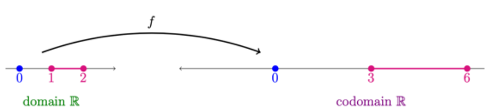
$$= \Big\langle a_2b_3 - a_3b_2,\quad a_3b_1 - a_1b_3,\quad a_1b_2 - a_2b_1\Big\rangle.$$

Matrices as Linear Transformations

1-Dimension

Given a number, say 3, we get a function

$$f\colon \mathbb{R} \longrightarrow \mathbb{R}$$
$$x \longmapsto 3x$$



Higher dimension

Given the 2×3 matrix $\begin{pmatrix}6 & 7 & 8\\ 2 & 3 & 5\end{pmatrix}$, we get a function

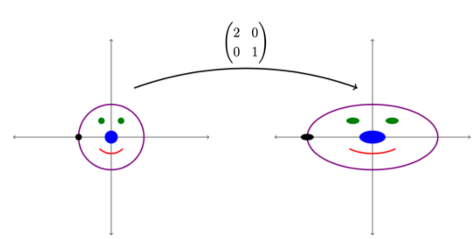
$$\mathbf{f}\colon \mathbb{R}^3 \longrightarrow \mathbb{R}^2$$

$$\mathbf{x} \longmapsto \begin{pmatrix}6 & 7 & 8\\ 2 & 3 & 5\end{pmatrix}\mathbf{x}$$

$$\mathbf{f}\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}6x+7y+8z\\ 2x+3y+5z\end{pmatrix}.$$

In general, an $m\times n$ matrix A gives rise to a function

$$\mathbf{f}\colon \mathbb{R}^n \longrightarrow \mathbb{R}^m$$
$$\mathbf{x} \longmapsto A\mathbf{x}$$



$$\boxed{\text{area scaling factor} = |\det A|.}$$

Systems of Equations

To solve $3x = 5$, multiply both sides by 3^{-1}.

Similarly, one way to solve

$$2x_1 + 3x_2 = 4$$
$$4x_1 + 5x_2 = 6,$$

is to rewrite as

$$\begin{pmatrix}2 & 3\\ 4 & 5\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} = \begin{pmatrix}4\\6\end{pmatrix},$$

which has the shape $A\mathbf{x} = \mathbf{b}$, and left multiply both sides by A^{-1} to get $\mathbf{x} = A^{-1}\mathbf{b}$.

Inverse

The inverse of a square matrix A is another matrix A^{-1}, such that

$$AA^{-1} = I \qquad \text{and} \qquad A^{-1}A = I.$$

It exists if and only if $det(A)\ \neq 0$; in that case, A is called invertible, or nonsingular.

Equation of Planes

The set of all vectors perpendicular to $\langle 1,2,3\rangle$ is a plane with equation

$$\langle 1,2,3\rangle \cdot \langle x,y,z\rangle = 0,$$

Which is

$$x + 2y + 3z = 0.$$

The vector $\boldsymbol{n} := \langle 1,2,3\rangle$ is called a **normal vector** to the plane.

Linear Algebra

Parametric Lines & Curves

There are two ways to describes lines in $\boldsymbol{R}^3$:

- Intersection of two planes
- Parametric equations

Think of the trajectory of an airplane moving at constant velocity. Let r_0 be the position vector of the airplane at time $t\ =\ 0$. Let v be the velocity.

$$r(t) := r_0 + tv$$

Partial Differentiation

Definition

The partial derivative of a function $f(x,y)$ with respect to x is a function $\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}$ (also written as $\boldsymbol{f_x}$) whose value at (x_0,y_0) is:

- The rate of change of $f(x,y)$ when x is varying near $x = x_0$ and y is held constant at the value y_0, or:
- More precisely,

$$\frac{\partial f}{\partial x}(x_0,y_0) := lim_{h\to 0}\frac{f(x_0+h,y_0) - f(x_0,y_0)}{h}$$

Linear Approximation

Question: How do you approximate $f(x)$ for $x := x_0 + \Delta x$?

1-Variable

$$\boxed{f(x_0+\Delta x) \approx \underbrace{f(x_0)}_{\text{starting value}} + \underbrace{f'(x_0)\,\Delta x}_{\text{adjustment}}}$$

2-Variables

$$f(x_0+\Delta x, y_0+\Delta y) \approx \underbrace{f(x_0,y_0)}_{\text{starting value}} + \underbrace{\left(\frac{\partial f}{\partial x}\right)_0 \Delta x}_{\text{adjustment from } \Delta x} + \underbrace{\left(\frac{\partial f}{\partial y}\right)_0 \Delta y}_{\text{adjustment from } \Delta y}.$$

Max/Min Problem

Solving unconstrained max/min problems

Second derivative test

More on Derivatives of Multivariable Function

Differentials

The total differential of $f(x,y)$ is

$$df := f_x dx + f_y dy$$

Chain Rule

In single variable calculus:

$$\frac{df}{dt} = \frac{df}{du} \cdot \frac{du}{dt}$$

In multi-variable calculus:

$$\frac{df}{dt} = f_x \frac{dx}{dt} + f_y \frac{dy}{dt}$$

Example (polar coordinates)

Find $\frac{df}{dx}$ given

$$f(x,y) = \cos\left(5 \tan^{-1}\frac{y}{x}\right) + ln\left(\sqrt{x^2+y^2}\right)$$

Which is equivalent to

$$f(r,\theta) = \cos(5\,\theta) + \ln(r)$$

Because

$$r = \sqrt{x^2+y^2}$$
$$\theta = tan^{-1}\frac{y}{x}$$

Using the chain rule:

$$\frac{df}{dx} = \frac{df}{dr}\frac{dr}{dx} + \frac{df}{d\theta}\frac{d\theta}{dx}$$

$$\frac{df}{dy} = \frac{df}{dr}\frac{dr}{dy} + \frac{df}{d\theta}\frac{d\theta}{dy}$$

Gradient ∇f

The direction of ∇f is the direction in which f is increasing the fastest (perpendicular to level curve/surface)

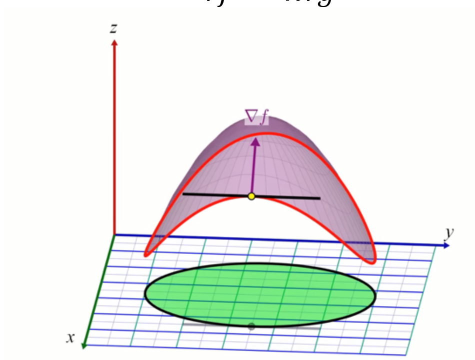
$$\nabla f = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k$$

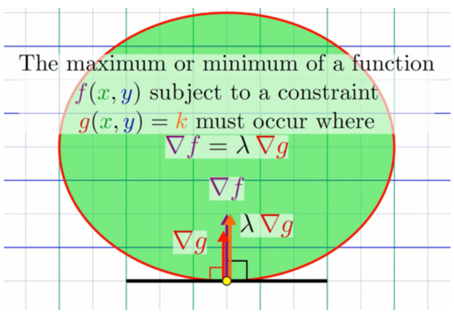
Lagrange Multipliers

A method for finding max/min of $f(x,y)$ when x and y are required to satisfy a constraint $g(x,y) = c$.

1. Compute ∇f and ∇g.
2. Solve the system

$$g = c$$
$$\nabla f = \lambda \nabla g$$





Double Integrals & Line Integrals

Integrals

Double Integrals

Let R be a region in $\mathbb{R}^2$ cut into tiny regions $R_1, \dots, R_n$. Choose (x_1, y_1) in $R_1, \dots, (x_n, y_n)$ in R_n, Then

$$\iint_R f(x,y)\,dA \approx f(x_1,y_1)\,\text{Area}(R_1) + \cdots$$
$$+ f(x_n,y_n)\,\text{Area}(R_n).$$

Double Integrals as iterated integrals

Suppose that R is a rectangle $[a, b] \times [c, d]$, then

$$\iint_R f(x,y)\,dA = \int_{x=a}^{x=b} \left(\int_{y=c}^{y=d} f(x,y)\,dy \right) dx$$
$$=: \int_a^b \int_c^d f(x,y)\,dy\,dx.$$

Applications of Double Integrals

The average value of $f(x,y)$ on a region R

$$\frac{\iint_R f\,dA}{\text{Area}(R)}$$

The mass of a 2-dimensional object is

$$m := \iint_R \underbrace{\delta(x,y)\,dA}_{dm}.$$

The centroid of a 2-dimensional object is the point $(\bar{x}, \bar{y})$ where

$$\bar{x} := \frac{\iint_R x\,dm}{m} = \frac{\iint_R x\,\delta(x,y)\,dA}{\iint_R \delta(x,y)\,dA}$$
$$\bar{y} := \frac{\iint_R y\,dm}{m} = \frac{\iint_R y\,\delta(x,y)\,dA}{\iint_R \delta(x,y)\,dA}.$$

The moment of inertia of an object with respect to an axis measures how difficult it is to rotate it

$$I = \iint_R (\text{distance to axis})^2\,dm.$$

Change of variables

When integrating over a region in the plane, we may want to transform our coordinate system to make the integral easier to compute.

When we change of variables, we are applying a linear transformation to the space and the areas are scaled by a factor that corresponds to the absolute value of the **Jacobian matrix** (J):

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left|\det\begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix}\right|.$$

Example (gaussian integral)

$$I = \int_{-\infty}^{\infty} e^{-x^2} dx$$

(It's the PDF of the standard normal distribution)

1. Square it to make a double integral

$$I^2 = \left(\int_{-\infty}^{\infty} e^{-x^2} dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2} dy\right)$$

2. Switch to polar coordinates by making the substitution

$$x = r\,cos(\theta)$$
$$y = r\,sin(\theta)$$

Our integral becomes

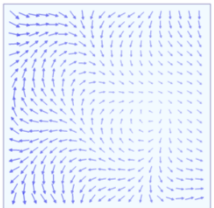
$$I^2 = \int_0^{\infty} \int_0^{\frac{\pi}{2}} e^{-r^2} r\,dr\,d\theta = \frac{\pi}{4}$$

And

$$I = \frac{\sqrt{\pi}}{2}$$

Vector Fields

A vector field is a function whose value at each point of a region is a vector.



$$\mathbf{F} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

$$(x, y) \longmapsto \begin{pmatrix} P(x,y) \\ Q(x,y) \end{pmatrix}$$

$$= P(x,y)\,\mathbf{e}_1 + Q(x,y)\,\mathbf{e}_2,$$

Line Integrals

Integral of curve C in vector field F:

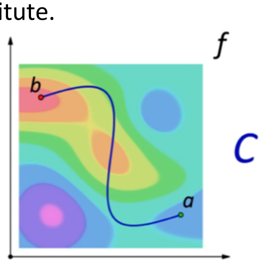
$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}\big(x(t), y(t)\big) \cdot \vec{r}\,'(t)\, dt$$

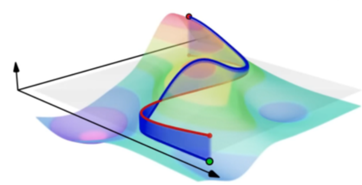
And also:

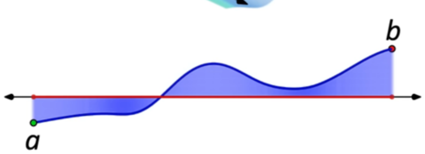
$$= \int_C \vec{F} \cdot \hat{T}\, ds$$

Where $\hat{T}$ is the unit tangent vector and s the arc length.

Evaluate x, y in terms of a single variable and substitute.







$$\int_C f\, ds = \int_a^b f(\mathbf{r}(t))|\mathbf{r}'(t)|\, dt$$

<u>Example</u>

Triple Integrals & Surface Integrals