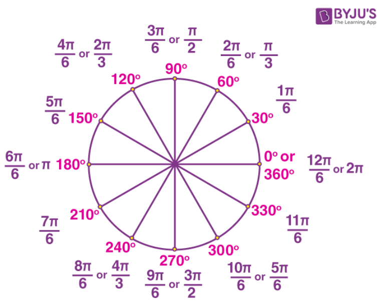
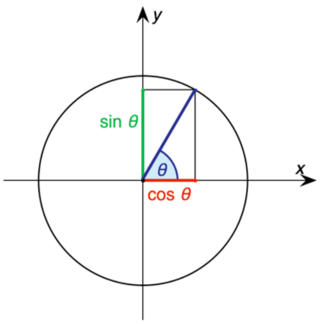
Calculus

Trigonometry





Pythagorean identity

- $\sin^2(x) + \cos^2(x) = 1$

Quotient identities

- $\tan(x) = \frac{\sin(x)}{\cos(x)}$, $\cot(x) = \frac{\cos(x)}{\sin(x)}$

Reciprocal identities

- $\csc(x) = \frac{1}{\sin(x)}$
- $\sec(x) = \frac{1}{\cos(x)}$
- $cot(x) = \frac{1}{\tan(x)}$

Even-Odd identities

- $\sin(-x) = -\sin(x)$
- $\cos(-x) = \cos(x)$

Limits

L'Hopital's Rule

If $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{0}{0}$ or $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\pm\infty}{\pm\infty}$ then,

$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$, a is a number or $\pm\infty$

Derivatives

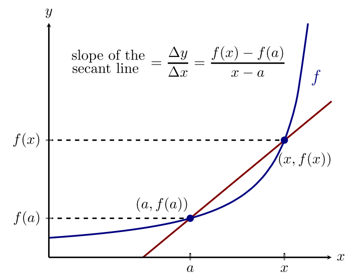
Definition

The derivative of $f(x)$ is defined as:

$$f'(x) = lim_{(h\to 0)} \frac{f(x+h) - f(x)}{h}$$

Interpretation

1. $f'(a)$ is the slope of the tangent line to $y = f(x)$ at $x = a$



"We seek to calculate the slope at point x by calculating the slope of a secant line that passes through f(x+e) and f(x) for e -> 0"

2. $f'(a)$ is the rate of change of $f(x)$ at $x = a$
3. If $f(t)$ is the position of an object at time t then $f'(a)$ is the velocity of the object at time $t = a$

Rules of Computation

Derivatives of powers

- $\frac{d}{dx} x^a = ax^{a-1}$ *power rule*

Exponential and logarithmic functions

- $\frac{d}{dx} x^e = x^e$
- $\frac{d}{dx} a^x = a^x ln(a)$, $a > 0$
- $\frac{d}{dx} ln(x) = \frac{1}{x}$, $x > 0$
- $\frac{d}{dx} log_{a(x)} = \frac{1}{xln(a)}$, $x, a > 0$

Trigonometric functions

- $\frac{d}{dx} sin(x) = cos(x)$
- $\frac{d}{dx} cos(x) = -sin(x)$
- $\frac{d}{dx} tan(x) = sec^2(x) = \frac{1}{cos^2(x)} = 1 + tan^2(x)$

Inverse trigonometric functions

- $\frac{d}{dx} arcsin(x) = \frac{1}{\sqrt{1-x^2}}$, $-1 < x < 1$
- $\frac{d}{dx} arccos(x) = -\frac{1}{\sqrt{1-x^2}}$, $-1 < x < 1$
- $\frac{d}{dx} arctan(x) = \frac{1}{1+x^2}$

Rules for combining functions

- $f'(x) = 0$
- $(\alpha f + \beta g)' = \alpha f' + \beta g'$
- $(fg)' = f'g + fg'$ *product rule*

  Also in Linear Algebra:

$$\frac{d}{dt}(\mathbf{r} \cdot \mathbf{s}) = \frac{d\mathbf{r}}{dt} \cdot \mathbf{s} + \mathbf{r} \cdot \frac{d\mathbf{s}}{dt};$$

$$\frac{d}{dt}(\mathbf{r} \times \mathbf{s}) = \frac{d\mathbf{r}}{dt} \times \mathbf{s} + \mathbf{r} \times \frac{d\mathbf{s}}{dt}.$$

- $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$ *quotient rule*

Chain Rule

$$\frac{df}{dx} = \frac{df}{du} \cdot \frac{du}{dx}$$

Example:

Given $f(x) = (2x+3)^2$, then define

$u = 2x + 3$ and $f(u) = u^2$.

$\frac{du}{dx} = 2$; $\frac{df}{du} = 2u$

$$\frac{df}{dx} = \frac{df}{du} \cdot \frac{du}{dx} = 2u \cdot 2$$

Shape of Curve

c is a **critical point** of $f(x)$ provided either:

- $f'(c) = 0$ or
- $f'(c)$ doesn't exist

For all x in an interval I, the interval

- increases if $f'(x) > 0$
- decreases if $f'(x) < 0$
- is constant if $f'(x) = 0$

For all x in an interval I, the interval is

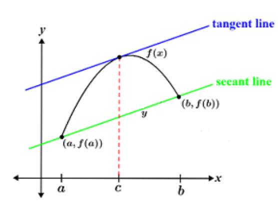
- concave up if $f''(x) > 0$
- concave down if $f''(x) < 0$

c is an **inflection point** of $f(x)$ if the concavity changes at $x = c$

Extrema

Mean Value Theorem

If $f(x)$ is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b) then there is a number $a < c < b$ such that $f'(c) = \frac{f(b) - f(a)}{b - a}$



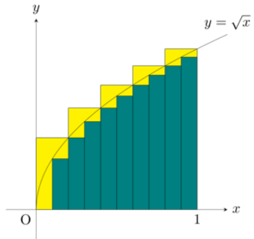
Integrals

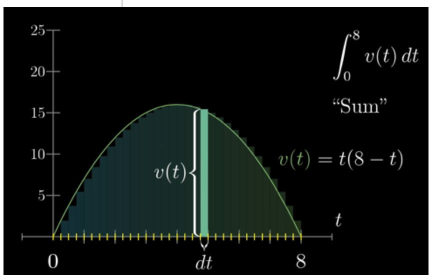
Definition

Continuous analog of a sum which is used to calculate areas, volumes, and their generalizations

Conceptually:

1. **Definite integrals**: the signed area of the region in the plane that is bounded by the graph of a given function between two points in the real line





"To get the area under the curve, divide it into infinitely many infinitesimal pieces, then sum the pieces to achieve an accurate approximation"

2. **Antiderivative**: a function whose derivative is the given function; in this case, they are also called *indefinite integrals*

Rules of Computation

Basic Integrals

- $\int x\ dx = \frac{1}{2}x^2 + C$
- $\int e^x\ dx = e^x + C$
- $\int \sin(x)\ dx = -\cos(x) + C$
- $\int \cos(x)\ dx = \sin(x) + C$

Power rule

- $\int x^n\ dx = \frac{1}{n+1}x^{n+1} + C, \quad (n \neq -1)$

Exponential and logarithmic functions

- $\int \frac{1}{x}\ dx = \ln(|x|) + C$
- $\int e^{ax}\ dx = \frac{1}{a}e^{ax} + C$

Trigonometric functions

- $\int\ tan\ (x)\ dx\ =\ -\ ln\ |\ cos\ (x)|\ +\ C$
- $\int \sec^2(x)\ dx = \tan(x) + C$
- $\int \csc(x)\cot(x)\ dx = -\csc(x) + C$

Trigonometric substitutions

- $\int \frac{1}{\sqrt{1-x^2}}\ dx = \arcsin(x) + C$
- $\int \frac{1}{1+x^2}\ dx = \arctan(x) + C$

Partial fractions

- $\int \frac{1}{(x-a)(x-b)}\ dx = \frac{\ln|x-a| - \ln|x-b|}{a-b} + C$

Substitution

$$\int f(g(x)) \cdot g'(x)\, dx = \int f(u)\ du$$

Where $u = g(x)$ and $du = g'(x)\, dx$

Example:

$$\int 2x\ cos(x^2)dx$$

Let $u = x^2$ and therefore $du = 2x\ dx$

Then

$$\int cos(u)\ du = sin(u) + C = sin(x^2) + C$$

Finally

$$\int 2x\ cos(x^2)dx = sin(x^2) + C$$

Integration by parts

$$\int u\, dv = uv - \int v\, du$$

"Un Día Ví Una Vaca y un soldado Vestido De Uniforme"

Applications of Integrals

Sequences

Series

Absolute Convergence

Conditional Convergence

Important Series

Geometric Series

$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$ Converges if $|r| < 1$

p-Series

$\sum_{n=1}^{\infty} \frac{1}{n^p}$ Converges if $p > 1$