3–8 Relative Velocity

We now consider how observations made in different frames of reference are related to each other. For example, consider two trains approaching one another, each with a speed of 80 km/h with respect to the Earth. Observers on the Earth beside the train tracks will measure 80 km/h for the speed of each of the trains. Observers on either one of the trains (a different frame of reference) will measure a speed of 160 km/h for the train approaching them.

Similarly, when one car traveling 90 km/h passes a second car traveling in the same direction at 75 km/h, the first car has a speed relative to the second car of $90\text{ km/h} - 75\text{ km/h} = 15\text{ km/h}$.

When the velocities are along the same line, simple addition or subtraction is sufficient to obtain the **relative velocity**. But if they are not along the same line, we must make use of vector addition. We emphasize, as mentioned in Section 2–1, that when specifying a velocity, it is important to specify what the reference frame is.

When determining relative velocity, it is easy to make a mistake by adding or subtracting the wrong velocities. It is important, therefore, to draw a diagram and use a careful labeling process. Each velocity is labeled by *two subscripts: the first refers to the object, the second to the reference frame in which it has this velocity*. For example, suppose a boat heads directly across a river, as shown in Fig. 3–28. We let $\vec{\mathbf{v}}_{\mathrm{BW}}$ be the velocity of the **B**oat with respect to the **W**ater. (This is also what the boat's velocity would be relative to the shore if the water were still.) Similarly, $\vec{\mathbf{v}}_{\mathrm{BS}}$ is the velocity of the **B**oat with respect to the **S**hore, and $\vec{\mathbf{v}}_{\mathrm{WS}}$ is the velocity of the **W**ater with respect to the **S**hore (this is the river current). Note that $\vec{\mathbf{v}}_{\mathrm{BW}}$ is what the boat's motor produces (against the water), whereas $\vec{\mathbf{v}}_{\mathrm{BS}}$ is equal to $\vec{\mathbf{v}}_{\mathrm{BW}}$ plus the effect of the current, $\vec{\mathbf{v}}_{\mathrm{WS}}$. Therefore, the velocity of the boat relative to the shore is (see vector diagram, Fig. 3–28)

$$\vec{\mathbf{v}}_{\mathrm{BS}} = \vec{\mathbf{v}}_{\mathrm{BW}} + \vec{\mathbf{v}}_{\mathrm{WS}}. \tag{3–7}$$

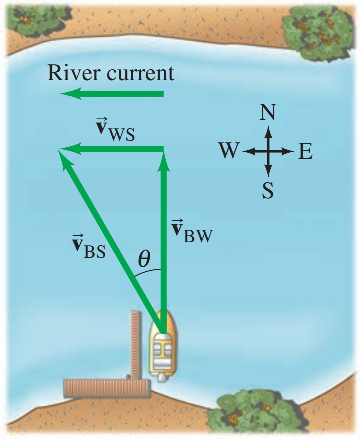


FIGURE 3–28 A boat heads north directly across a river which flows west. Velocity vectors are shown as green arrows:

$\vec{\mathbf{v}}_{\mathrm{BS}}$ = velocity of **B**oat with respect to the **S**hore,
$\vec{\mathbf{v}}_{\mathrm{BW}}$ = velocity of **B**oat with respect to the **W**ater,
$\vec{\mathbf{v}}_{\mathrm{WS}}$ = velocity of **W**ater with respect to the **S**hore (river current).

As it crosses the river, the boat is dragged downstream by the current.

By writing the subscripts using this convention, we see that the inner subscripts (the two W's) on the right-hand side of Eq. 3–7 are the same; also, the outer subscripts on the right of Eq. 3–7 (the B and the S) are the same as the two subscripts for the sum vector on the left, $\vec{\mathbf{v}}_{\mathrm{BS}}$. By following this convention (first subscript for the object, second for the reference frame), you can write down the correct equation relating velocities in different reference frames.†

Equation 3–7 is valid in general and can be extended to three or more velocities. For example, if a fisherman on the boat walks with a velocity $\vec{\mathbf{v}}_{\mathrm{FB}}$ relative to the boat, his velocity relative to the shore is $\vec{\mathbf{v}}_{\mathrm{FS}} = \vec{\mathbf{v}}_{\mathrm{FB}} + \vec{\mathbf{v}}_{\mathrm{BW}} + \vec{\mathbf{v}}_{\mathrm{WS}}$. The equations involving relative velocity will be correct when adjacent inner subscripts are identical and when the outermost ones correspond exactly to the two on the velocity on the left of the equation. But this works only with plus signs (on the right), not minus signs.

It is often useful to remember that for any two objects or reference frames, A and B, the velocity of A relative to B has the same magnitude, but opposite direction, as the velocity of B relative to A:

$$\vec{\mathbf{v}}_{\mathrm{BA}} = -\vec{\mathbf{v}}_{\mathrm{AB}}. \tag{3–8}$$

For example, if a train is traveling 100 km/h relative to the Earth in a certain direction, objects on the Earth (such as trees) appear to an observer on the train to be traveling 100 km/h in the opposite direction.

†We thus can see, for example, that the equation $\vec{\mathbf{v}}_{\mathrm{BW}} = \vec{\mathbf{v}}_{\mathrm{BS}} + \vec{\mathbf{v}}_{\mathrm{WS}}$ is wrong: the inner subscripts are not the same, and the outer ones on the right do not correspond to the subscripts on the left.

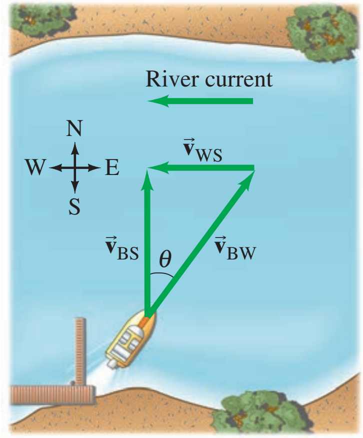


FIGURE 3–29 Example 3–10.

EXAMPLE 3–10 **Heading upstream.** A boat's speed in still water is $v_{BW} = 1.85\ \mathrm{m/s}$. If the boat is to travel north directly across a river whose westward current has speed $v_{WS} = 1.20\ \mathrm{m/s}$, at what upstream angle must the boat head? (See Fig. 3–29.)

APPROACH If the boat heads straight across the river, the current will drag the boat downstream (westward). To overcome the river's current, the boat must have an upstream (eastward) component of velocity as well as a cross-stream (northward) component. Figure 3–29 has been drawn with $\vec{\mathbf{v}}_{BS}$, the velocity of the **B**oat relative to the **S**hore, pointing directly across the river because this is where the boat is supposed to go. (Note that $\vec{\mathbf{v}}_{BS} = \vec{\mathbf{v}}_{BW} + \vec{\mathbf{v}}_{WS}$.)

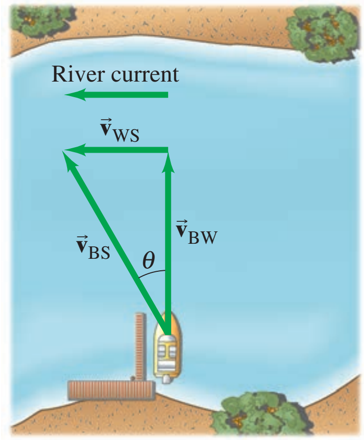
SOLUTION Vector $\vec{\mathbf{v}}_{BW}$ points upstream at angle θ as shown. From the diagram,

$$\sin\theta = \frac{v_{WS}}{v_{BW}} = \frac{1.20\ \mathrm{m/s}}{1.85\ \mathrm{m/s}} = 0.6486.$$

Thus $\theta = 40.4°$, so the boat must head upstream at a 40.4° angle.

EXAMPLE 3–11 **Heading across the river.** The same boat $(v_{BW} = 1.85\ \mathrm{m/s})$ now heads directly across the river whose current is still 1.20 m/s. (*a*) What is the velocity (magnitude and direction) of the boat relative to the shore? (*b*) If the river is 110 m wide, how long will it take to cross and how far downstream will the boat be then?

FIGURE 3–30 Example 3–11. A boat heading directly across a river whose current moves at 1.20 m/s.



APPROACH The boat now heads directly across the river and is pulled downstream by the current, as shown in Fig. 3–30. The boat's velocity with respect to the shore, $\vec{\mathbf{v}}_{BS}$, is the sum of its velocity with respect to the water, $\vec{\mathbf{v}}_{BW}$, plus the velocity of the water with respect to the shore, $\vec{\mathbf{v}}_{WS}$: just as before,

$$\vec{\mathbf{v}}_{BS} = \vec{\mathbf{v}}_{BW} + \vec{\mathbf{v}}_{WS}.$$

SOLUTION (*a*) Since $\vec{\mathbf{v}}_{BW}$ is perpendicular to $\vec{\mathbf{v}}_{WS}$, we can get v_{BS} using the theorem of Pythagoras:

$$v_{BS} = \sqrt{v_{BW}^2 + v_{WS}^2} = \sqrt{(1.85\ \mathrm{m/s})^2 + (1.20\ \mathrm{m/s})^2} = 2.21\ \mathrm{m/s}.$$

We can obtain the angle (note how θ is defined in Fig. 3–30) from:

$$\tan\theta = v_{WS}/v_{BW} = (1.20\ \mathrm{m/s})/(1.85\ \mathrm{m/s}) = 0.6486.$$

A calculator with a key INV TAN or ARC TAN or TAN^{-1} gives $\theta = \tan^{-1}(0.6486) = 33.0°$. Note that this angle is not equal to the angle calculated in Example 3–10.

(*b*) The travel time for the boat is determined by the time it takes to cross the river. Given the river's width $D = 110\ \mathrm{m}$, we can use the velocity component in the direction of D, $v_{BW} = D/t$. Solving for t, we get $t = 110\ \mathrm{m}/1.85\ \mathrm{m/s} = 59.5\ \mathrm{s}$. The boat will have been carried downstream, in this time, a distance

$$d = v_{WS}t = (1.20\ \mathrm{m/s})(59.5\ \mathrm{s}) = 71.4\ \mathrm{m} \approx 71\ \mathrm{m}.$$

NOTE There is no acceleration in this Example, so the motion involves only constant velocities (of the boat or of the river).

Summary

A quantity such as velocity, that has both a magnitude and a direction, is called a **vector**. A quantity such as mass, that has only a magnitude, is called a **scalar**. On diagrams, vectors are represented by arrows.

Addition of vectors can be done graphically by placing the tail of each successive arrow at the tip of the previous one. The sum, or **resultant vector**, is the arrow drawn from the tail of the first vector to the tip of the last vector. Two vectors can also be added using the parallelogram method.

Vectors can be added more accurately by adding their **components** along chosen axes with the aid of trigonometric functions. A vector of magnitude V making an angle θ with the $+x$ axis has components

$$V_x = V\cos\theta, \qquad V_y = V\sin\theta. \qquad \textbf{(3–3)}$$

Given the components, we can find a vector's magnitude and direction from

$$V = \sqrt{V_x^2 + V_y^2}, \qquad \tan\theta = \frac{V_y}{V_x}. \qquad \textbf{(3–4)}$$

Projectile motion is the motion of an object in the air near the Earth's surface under the effect of gravity alone. It can be analyzed as two separate motions if air resistance can be ignored. The horizontal component of motion is at constant velocity, whereas the vertical component is at constant acceleration, $\vec{\mathbf{g}}$, just as for an object falling vertically under the action of gravity.

The velocity of an object relative to one frame of reference can be found by vector addition if its velocity relative to a second frame of reference, and the **relative velocity** of the two reference frames, are known.