1–7 Order of Magnitude: Rapid Estimating

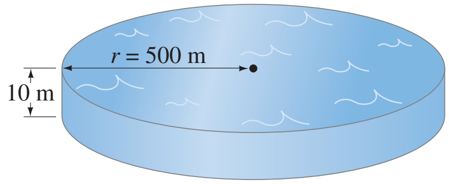
We are sometimes interested only in an approximate value for a quantity. This might be because an accurate calculation would take more time than it is worth or would require additional data that are not available. In other cases, we may want to make a rough estimate in order to check a calculation made on a calculator, to make sure that no blunders were made when the numbers were entered.

A rough estimate can be made by rounding off all numbers to one significant figure and its power of 10, and after the calculation is made, again keeping only one significant figure. Such an estimate is called an **order-of-magnitude estimate** and can be accurate within a factor of 10, and often better. In fact, the phrase "order of magnitude" is sometimes used to refer simply to the power of 10.

PROBLEM SOLVING
How to make a rough estimate

Let's do some Examples.

(a)



(b)

FIGURE 1–10 Example 1–6. (a) How much water is in this lake? (Photo is one of the Rae Lakes in the Sierra Nevada of California.) (b) Model of the lake as a cylinder. [We could go one step further and estimate the mass or weight of this lake. We will see later that water has a density of $1000\ \mathrm{kg/m^3}$, so this lake has a mass of about $(10^3\ \mathrm{kg/m^3})(10^7\ \mathrm{m^3}) \approx 10^{10}\ \mathrm{kg}$, which is about 10 billion kg or 10 million metric tons. (A metric ton is 1000 kg, about 2200 lb, slightly larger than a British ton, 2000 lb.)]

EXAMPLE 1–6 ESTIMATE **Volume of a lake.** Estimate how much water there is in a particular lake, Fig. 1–10a, which is roughly circular, about 1 km across, and you guess it has an average depth of about 10 m.

APPROACH No lake is a perfect circle, nor can lakes be expected to have a perfectly flat bottom. We are only estimating here. To estimate the volume, we can use a simple model of the lake as a cylinder: we multiply the average depth of the lake times its roughly circular surface area, as if the lake were a cylinder (Fig. 1–10b).

PHYSICS APPLIED
Estimating the volume (or mass) of a lake; see also Fig. 1–10

SOLUTION The volume V of a cylinder is the product of its height h times the area of its base: $V = h\pi r^2$, where r is the radius of the circular base.† The radius r is $\frac{1}{2}\ \mathrm{km} = 500\ \mathrm{m}$, so the volume is approximately

$$V = h\pi r^2 \approx (10\ \mathrm{m}) \times (3) \times (5 \times 10^2\ \mathrm{m})^2 \approx 8 \times 10^6\ \mathrm{m^3} \approx 10^7\ \mathrm{m^3},$$

where π was rounded off to 3. So the volume is on the order of $10^7\ \mathrm{m^3}$, ten million cubic meters. Because of all the estimates that went into this calculation, the order-of-magnitude estimate $(10^7\ \mathrm{m^3})$ is probably better to quote than the $8 \times 10^6\ \mathrm{m^3}$ figure.

NOTE To express our result in U.S. gallons, we see in the Table on the inside front cover that $1\ \text{liter} = 10^{-3}\ \mathrm{m^3} \approx \frac{1}{4}\ \text{gallon}$. Hence, the lake contains $(8 \times 10^6\ \mathrm{m^3})(1\ \text{gallon}/4 \times 10^{-3}\ \mathrm{m^3}) \approx 2 \times 10^9$ gallons of water.

†Formulas like this for volume, area, etc., are found inside the back cover of this book.

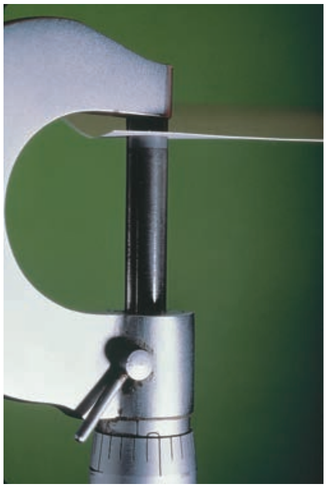


FIGURE 1–11 Example 1–7. Micrometer used for measuring small thicknesses.

EXAMPLE 1–7 ESTIMATE **Thickness of a sheet of paper.** Estimate the thickness of a page of this book.

APPROACH At first you might think that a special measuring device, a micrometer (Fig. 1–11), is needed to measure the thickness of one page since an ordinary ruler can not be read so finely. But we can use a trick or, to put it in physics terms, make use of a *symmetry*: we can make the reasonable assumption that all the pages of this book are equal in thickness.

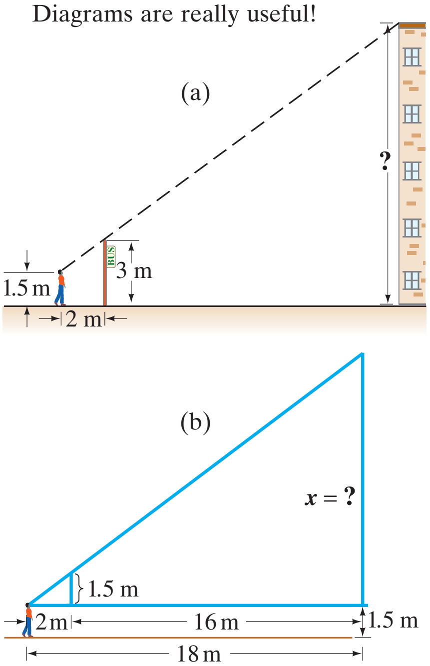
SOLUTION We can use a ruler to measure hundreds of pages at once. If you measure the thickness of the first 500 pages of this book (page 1 to page 500), you might get something like 1.5 cm. Note that 500 numbered pages, counted front and back, is 250 separate pieces of paper. So one sheet must have a thickness of about

$$\frac{1.5\text{ cm}}{250\text{ sheets}} \approx 6 \times 10^{-3}\text{ cm} = 6 \times 10^{-2}\text{ mm},$$

or less than a tenth of a millimeter (0.1 mm).

It cannot be emphasized enough how important it is to draw a diagram when solving a physics Problem, as the next Example shows.

FIGURE 1–12 Example 1–8. Diagrams are really useful!



EXAMPLE 1–8 ESTIMATE **Height by triangulation.** Estimate the height of the building shown in Fig. 1–12, by "triangulation," with the help of a bus-stop pole and a friend.

APPROACH By standing your friend next to the pole, you estimate the height of the pole to be 3 m. You next step away from the pole until the top of the pole is in line with the top of the building, Fig. 1–12a. You are 5 ft 6 in. tall, so your eyes are about 1.5 m above the ground. Your friend is taller, and when she stretches out her arms, one hand touches you, and the other touches the pole, so you estimate that distance as 2 m (Fig. 1–12a). You then pace off the distance from the pole to the base of the building with big, 1-m-long steps, and you get a total of 16 steps or 16 m.

SOLUTION Now you draw, to scale, the diagram shown in Fig. 1–12b using these measurements. You can measure, right on the diagram, the last side of the triangle to be about $x = 13$ m. Alternatively, you can use similar triangles to obtain the height x:

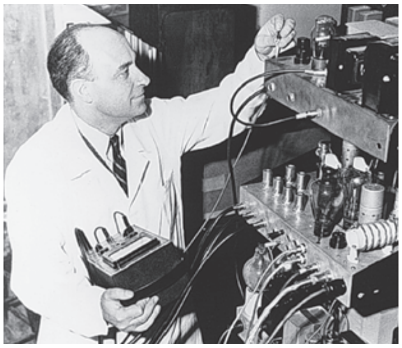
$$\frac{1.5\text{ m}}{2\text{ m}} = \frac{x}{18\text{ m}},$$

so

$$x \approx 13\tfrac{1}{2}\text{ m}.$$

Finally you add in your eye height of 1.5 m above the ground to get your final result: the building is about 15 m tall.

FIGURE 1–13 Enrico Fermi. Fermi contributed significantly to both theoretical and experimental physics, a feat almost unique in modern times.



Another approach, this one made famous by Enrico Fermi (1901–1954, Fig. 1–13), was to show his students how to estimate the number of piano tuners in a city, say, Chicago or San Francisco. To get a rough order-of-magnitude estimate of the number of piano tuners today in San Francisco, a city of about 800,000 inhabitants, we can proceed by estimating the number of functioning pianos, how often each piano is tuned, and how many pianos each tuner can tune. To estimate the number of pianos in San Francisco, we note that certainly not everyone has a piano. A guess of 1 family in 3 having a piano would correspond to 1 piano per 12 persons, assuming an average family of 4 persons.

As an order of magnitude, let's say 1 piano per 10 people. This is certainly more reasonable than 1 per 100 people, or 1 per every person, so let's proceed with the estimate that 1 person in 10 has a piano, or about 80,000 pianos in San Francisco. Now a piano tuner needs an hour or two to tune a piano. So let's estimate that a tuner can tune 4 or 5 pianos a day. A piano ought to be tuned every 6 months or a year—let's say once each year. A piano tuner tuning 4 pianos a day, 5 days a week, 50 weeks a year can tune about 1000 pianos a year. So San Francisco, with its (very) roughly 80,000 pianos, needs about 80 piano tuners. This is, of course, only a rough estimate.† It tells us that there must be many more than 10 piano tuners, and surely not as many as 1000.

PROBLEM SOLVING
Estimating how many piano tuners there are in a city

A Harder Example—But Powerful

EXAMPLE 1–9 ESTIMATE **Estimating the radius of Earth.** Believe it or not, you can estimate the radius of the Earth without having to go into space (see the photograph on page 1). If you have ever been on the shore of a large lake, you may have noticed that you cannot see the beaches, piers, or rocks at water level across the lake on the opposite shore. The lake seems to bulge out between you and the opposite shore—a good clue that the Earth is round. Suppose you climb a stepladder and discover that when your eyes are 10 ft (3.0 m) above the water, you can just see the rocks at water level on the opposite shore. From a map, you estimate the distance to the opposite shore as $d \approx 6.1$ km. Use Fig. 1–14 with $h = 3.0$ m to estimate the radius R of the Earth.

APPROACH We use simple geometry, including the theorem of Pythagoras,

$$c^2 = a^2 + b^2,$$

where c is the length of the hypotenuse of any right triangle, and a and b are the lengths of the other two sides.

SOLUTION For the right triangle of Fig. 1–14, the two sides are the radius of the Earth R and the distance $d = 6.1\text{ km} = 6100\text{ m}$. The hypotenuse is approximately the length $R + h$, where $h = 3.0$ m. By the Pythagorean theorem,

$$\begin{aligned} R^2 + d^2 &\approx (R + h)^2 \\ &\approx R^2 + 2hR + h^2. \end{aligned}$$

We solve algebraically for R, after cancelling R^2 on both sides:

$$\begin{aligned} R \approx \frac{d^2 - h^2}{2h} &= \frac{(6100\text{ m})^2 - (3.0\text{ m})^2}{6.0\text{ m}} \\ &= 6.2 \times 10^6\text{ m} \\ &= 6200\text{ km}. \end{aligned}$$

NOTE Precise measurements give 6380 km. But look at your achievement! With a few simple rough measurements and simple geometry, you made a good estimate of the Earth's radius. You did not need to go out in space, nor did you need a very long measuring tape.

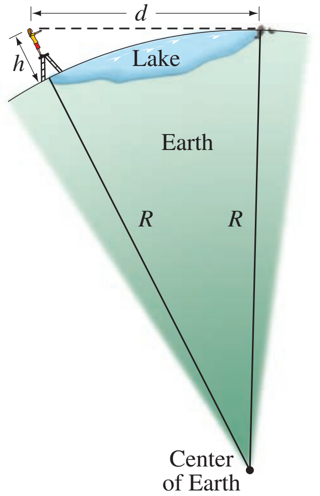


FIGURE 1–14 Example 1–9, but not to scale. You can just barely see rocks at water level on the opposite shore of a lake 6.1 km wide if you stand on a stepladder.

EXERCISE F Return to the second Chapter-Opening Question, page 1, and answer it again now. Try to explain why you may have answered differently the first time.

†A check of the San Francisco Yellow Pages (done after this calculation) reveals about 60 listings. Each of these listings may employ more than one tuner, but on the other hand, each may also do repairs as well as tuning. In any case, our estimate is reasonable.

*1–8 Dimensions and Dimensional Analysis

When we speak of the **dimensions** of a quantity, we are referring to the type of base units or base quantities that make it up. The dimensions of area, for example, are always length squared, abbreviated $[L^2]$, using square brackets; the units can be square meters, square feet, cm^2, and so on. Velocity, on the other hand, can be measured in units of km/h, m/s, or mi/h, but the dimensions are always a length $[L]$ divided by a time $[T]$: that is, $[L/T]$.

The formula for a quantity may be different in different cases, but the dimensions remain the same. For example, the area of a triangle of base b and height h is $A = \frac{1}{2}bh$, whereas the area of a circle of radius r is $A = \pi r^2$. The formulas are different in the two cases, but the dimensions of area are always $[L^2]$.

Dimensions can be used as a help in working out relationships, a procedure referred to as **dimensional analysis**. One useful technique is the use of dimensions to check if a relationship is *incorrect*. Note that we add or subtract quantities only if they have the same dimensions (we don't add centimeters and hours); and the quantities on each side of an equals sign must have the same dimensions. (In numerical calculations, the units must also be the same on both sides of an equation.)

For example, suppose you derived the equation $v = v_0 + \frac{1}{2}at^2$, where v is the speed of an object after a time t, v_0 is the object's initial speed, and the object undergoes an acceleration a. Let's do a dimensional check to see if this equation could be correct or is surely incorrect. Note that numerical factors, like the $\frac{1}{2}$ here, do not affect dimensional checks. We write a dimensional equation as follows, remembering that the dimensions of speed are $[L/T]$ and (as we shall see in Chapter 2) the dimensions of acceleration are $[L/T^2]$:

$$\begin{aligned}\left[\frac{L}{T}\right] &\stackrel{?}{=} \left[\frac{L}{T}\right] + \left[\frac{L}{T^2}\right][T^2] \\ &\stackrel{?}{=} \left[\frac{L}{T}\right] + [L].\end{aligned}$$

The dimensions are incorrect: on the right side, we have the sum of quantities whose dimensions are not the same. Thus we conclude that an error was made in the derivation of the original equation.

A dimensional check can only tell you when a relationship is wrong. It can't tell you if it is completely right. For example, a dimensionless numerical factor (such as $\frac{1}{2}$ or 2π) could be missing.

Dimensional analysis can also be used as a quick check on an equation you are not sure about. For example, consider a simple pendulum of length ℓ. Suppose that you can't remember whether the equation for the period T (the time to make one back-and-forth swing) is $T = 2\pi\sqrt{\ell/g}$ or $T = 2\pi\sqrt{g/\ell}$, where g is the acceleration due to gravity and, like all accelerations, has dimensions $[L/T^2]$. (Do not worry about these formulas—the correct one will be derived in Chapter 11; what we are concerned about here is a person's recalling whether it contains ℓ/g or g/ℓ.) A dimensional check shows that the former (ℓ/g) is correct:

$$[T] = \sqrt{\frac{[L]}{[L/T^2]}} = \sqrt{[T^2]} = [T],$$

whereas the latter (g/ℓ) is not:

$$[T] \neq \sqrt{\frac{[L/T^2]}{[L]}} = \sqrt{\frac{1}{[T^2]}} = \frac{1}{[T]}.$$

The constant 2π has no dimensions and so can't be checked using dimensions.

*Some Sections of this book, such as this one, may be considered *optional* at the discretion of the instructor, and they are marked with an asterisk (*). See the Preface for more details.