2–5 Motion at Constant Acceleration

We now examine motion in a straight line when the magnitude of the acceleration is constant. In this case, the instantaneous and average accelerations are equal. We use the definitions of average velocity and acceleration to derive a set of valuable equations that relate x, v, a, and t when a is constant, allowing us to determine any one of these variables if we know the others. We can then solve many interesting Problems.

Notation in physics varies from book to book; and different instructors use different notation. We are now going to change our notation, to simplify it a bit for our discussion here of motion at **constant acceleration**. First we choose the initial time in any discussion to be zero, and we call it t_0. That is, $t_1 = t_0 = 0$. (This is effectively starting a stopwatch at t_0.) We can then let $t_2 = t$ be the elapsed time. The initial position (x_1) and the initial velocity (v_1) of an object will now be represented by x_0 and v_0, since they represent x and v at $t = 0$. At time t the position and velocity will be called x and v (rather than x_2 and v_2). The average velocity during the time interval $t - t_0$ will be (Eq. 2–2)

$$\bar{v} = \frac{\Delta x}{\Delta t} = \frac{x - x_0}{t - t_0} = \frac{x - x_0}{t}$$

since we chose $t_0 = 0$. The acceleration, assumed constant in time, is $a = \Delta v/\Delta t$ (Eq. 2–4), so

$$a = \frac{v - v_0}{t}.$$

FIGURE 2–13 An accelerating motorcycle.

A common problem is to determine the velocity of an object after any elapsed time t, when we are given the object's constant acceleration. We can solve such problems[†] by solving for v in the last equation: first we multiply both sides by t,

$$at = v - v_0 \qquad \text{or} \qquad v - v_0 = at.$$

Then, adding v_0 to both sides, we obtain

$$v = v_0 + at. \qquad \text{[constant acceleration]} \qquad \textbf{(2–6)}$$

If an object, such as a motorcycle (Fig. 2–13), starts from rest $(v_0 = 0)$ and accelerates at $4.0\,\mathrm{m/s^2}$, after an elapsed time $t = 6.0\,\mathrm{s}$ its velocity will be $v = 0 + at = (4.0\,\mathrm{m/s^2})(6.0\,\mathrm{s}) = 24\,\mathrm{m/s}$.

Next, let us see how to calculate the position x of an object after a time t when it undergoes constant acceleration. The definition of average velocity (Eq. 2–2) is $\bar{v} = (x - x_0)/t$, which we can rewrite by multiplying both sides by t:

$$x = x_0 + \bar{v}t. \qquad \textbf{(2–7)}$$

Because the velocity increases at a uniform rate, the average velocity, $\bar{v}$, will be midway between the initial and final velocities:

CAUTION

Average velocity, but only if a = constant

$$\bar{v} = \frac{v_0 + v}{2}. \qquad \text{[constant acceleration]} \qquad \textbf{(2–8)}$$

(Careful: Equation 2–8 is not necessarily valid if the acceleration is not constant.) We combine the last two Equations with Eq. 2–6 and find, starting with Eq. 2–7,

$$\begin{aligned} x &= x_0 + \bar{v}t \\ &= x_0 + \left(\frac{v_0 + v}{2}\right)t \\ &= x_0 + \left(\frac{v_0 + v_0 + at}{2}\right)t \end{aligned}$$

or

$$x = x_0 + v_0 t + \tfrac{1}{2}at^2. \qquad \text{[constant acceleration]} \qquad \textbf{(2–9)}$$

Equations 2–6, 2–8, and 2–9 are three of the four most useful equations for motion at constant acceleration. We now derive the fourth equation, which is useful

[†]Appendix A–4 summarizes simple algebraic manipulations.

in situations where the time t is not known. We substitute Eq. 2–8 into Eq. 2–7:

$$x = x_0 + \bar{v}t = x_0 + \left(\frac{v + v_0}{2}\right)t.$$

Next we solve Eq. 2–6 for t, obtaining (see Appendix A–4 for a quick review)

$$t = \frac{v - v_0}{a},$$

and substituting this into the previous equation we have

$$x = x_0 + \left(\frac{v + v_0}{2}\right)\left(\frac{v - v_0}{a}\right) = x_0 + \frac{v^2 - v_0^2}{2a}.$$

We solve this for v^2 and obtain

$$v^2 = v_0^2 + 2a(x - x_0), \qquad \text{[constant acceleration]} \quad \textbf{(2–10)}$$

which is the other useful equation we sought.

We now have four equations relating position, velocity, acceleration, and time, when the acceleration a is constant. We collect these *kinematic equations for constant acceleration* here in one place for future reference (the tan background screen emphasizes their usefulness):

$$v = v_0 + at \qquad [a = \text{constant}] \quad \textbf{(2–11a)}$$

$$x = x_0 + v_0 t + \tfrac{1}{2}at^2 \qquad [a = \text{constant}] \quad \textbf{(2–11b)}$$

$$v^2 = v_0^2 + 2a(x - x_0) \qquad [a = \text{constant}] \quad \textbf{(2–11c)}$$

$$\bar{v} = \frac{v + v_0}{2}. \qquad [a = \text{constant}] \quad \textbf{(2–11d)}$$

Kinematic equations for constant acceleration (we'll use them a lot)

These useful equations are not valid unless a is a constant. In many cases we can set $x_0 = 0$, and this simplifies the above equations a bit. Note that x represents position (not distance), also that $x - x_0$ is the displacement, and that t is the elapsed time. Equations 2–11 are useful also when a is approximately constant to obtain reasonable estimates.

EXAMPLE 2–7 **Runway design.** You are designing an airport for small planes. One kind of airplane that might use this airfield must reach a speed before takeoff of at least 27.8 m/s (100 km/h), and can accelerate at $2.00\ \text{m/s}^2$. (a) If the runway is 150 m long, can this airplane reach the required speed for takeoff? (b) If not, what minimum length must the runway have?

PHYSICS APPLIED
Airport design

APPROACH Assuming the plane's acceleration is constant, we use the kinematic equations for constant acceleration. In (a), we want to find v, and what we are given is shown in the Table in the margin.

Known	Wanted
$x_0 = 0$	v
$v_0 = 0$	
$x = 150\ \text{m}$	
$a = 2.00\ \text{m/s}^2$	

SOLUTION (a) Of the above four equations, Eq. 2–11c will give us v when we know v_0, a, x, and x_0:

$$\begin{aligned} v^2 &= v_0^2 + 2a(x - x_0) \\ &= 0 + 2(2.00\ \text{m/s}^2)(150\ \text{m}) = 600\ \text{m}^2/\text{s}^2 \\ v &= \sqrt{600\ \text{m}^2/\text{s}^2} = 24.5\ \text{m/s}. \end{aligned}$$

PROBLEM SOLVING
Equations 2–11 are valid only when the acceleration is constant, which we assume in this Example

This runway length is *not* sufficient, because the minimum speed is not reached. (b) Now we want to find the minimum runway length, $x - x_0$, for a plane to reach $v = 27.8\ \text{m/s}$, given $a = 2.00\ \text{m/s}^2$. We again use Eq. 2–11c, but rewritten as

$$(x - x_0) = \frac{v^2 - v_0^2}{2a} = \frac{(27.8\ \text{m/s})^2 - 0}{2(2.00\ \text{m/s}^2)} = 193\ \text{m}.$$

A 200-m runway is more appropriate for this plane.

NOTE We did this Example as if the plane were a particle, so we round off our answer to 200 m.