Problems of dynamics of structures

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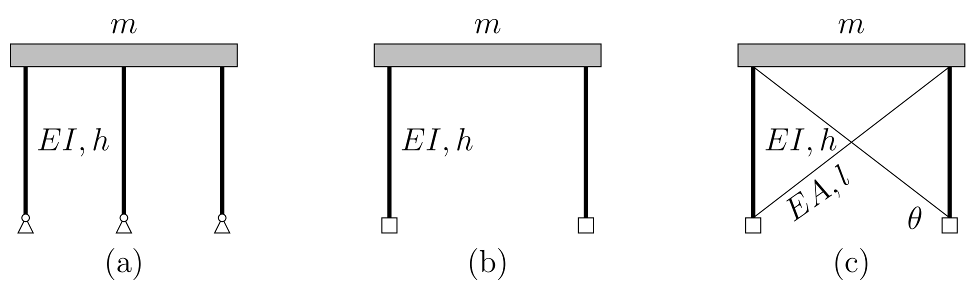
For steel take $E = 200\,\mathrm{GPa}$ and for concrete $E = 14\,\mathrm{GPa}$.

Free vibration of SDOF structures

Exercise 1 For the structures shown, determine the natural frequency of vibration using simple structural concepts. Consider the frames below. Compute the stiffness and natural frequencies.



Answer: (a) $\omega = \sqrt{\frac{9EI}{mh^3}}$, (b) $\omega = \sqrt{\frac{24EI}{mh^3}}$ and (c) $\omega = \sqrt{\frac{24EI}{mh^3} + \frac{EA}{ml}\cos^2\theta}$.

Exercise 2 The portal frame structure shown has a weight of $100\,\mathrm{kN}$. If the natural period of vibration is 0.9 seconds:

(a) determine the lateral stiffness of the structure;

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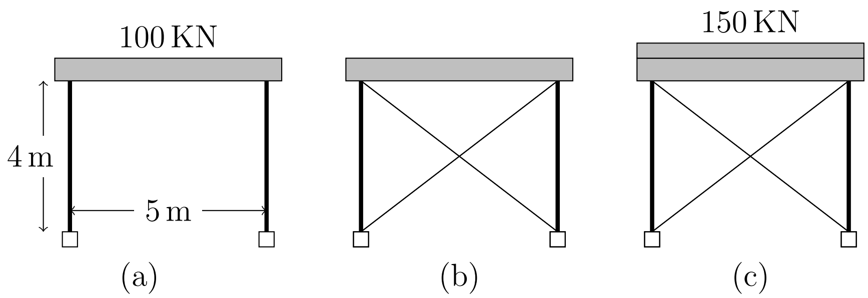
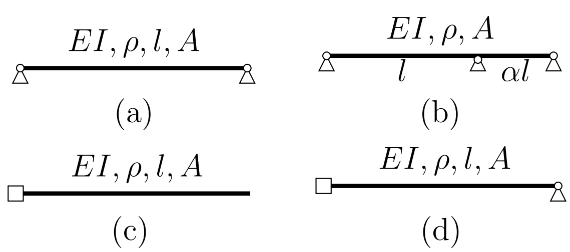
(b) determine the diameter of the steel cross-braces required to strengthen the structure by reducing the period to 0.3 seconds;

(c) determine the period if a further load of 50 kN is added to the strengthened structure.



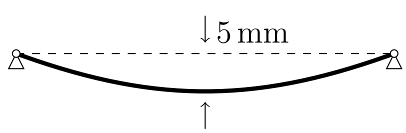
Answer: (a) $k = 487\,\mathrm{kN/m}$, (b) $D = 1.4\,\mathrm{cm}$, (c) $T = 0.37\,\mathrm{s}$.

Exercise 3 For the structures shown, determine the natural frequency of vibration using Rayleigh's method.



Answer: (a) $\omega \approx 10.95 \frac{1}{l^2}\sqrt{\frac{EI}{\rho A}}$ rad/s, (c) $\omega \approx 4.47 \frac{1}{l^2}\sqrt{\frac{EI}{\rho A}}$ rad/s.

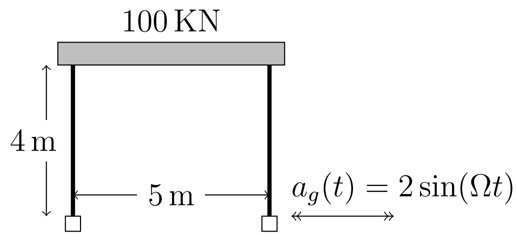
Exercise 4 In order to determine the dynamic properties of a simply supported bridge with a mass of 10^6 Kg, the midpoint is displaced 5 mm by a jack and then suddenly released. At the end of 20 complete cycles, the time is 3 seconds and the peak displacement measured is 1 mm. Determine the natural period and damping ratio of the bridge.



Answer: $T = 0.15\,\mathrm{s}$, $\xi = 1.28\,\%$.v
References: Chopra, *Dynamics of structures, SI Edition*, page 49; Blanco Díaz, *Análisis experimental de estructuras*, page 287

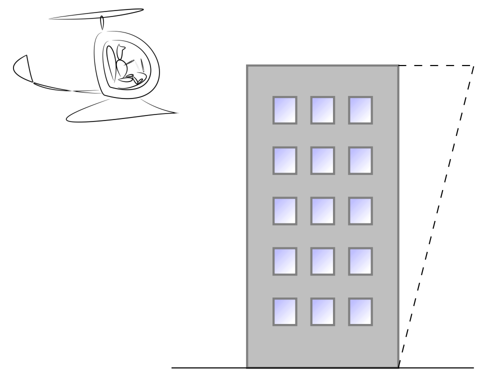
Forced vibration of SDOF structures

Exercise 5 The portal frame of example 2 (a) is subject to a sinusoidal ground vibration with horizontal acceleration amplitude of $2\,\mathrm{m/s^2}$. Assuming a damping ratio of $5\,\%$, determine the maximum displacement and maximum total acceleration of the frame when the period of floor vibration is: (a) 0.1 seconds; (b) 0.9 seconds and (c) 5 seconds.



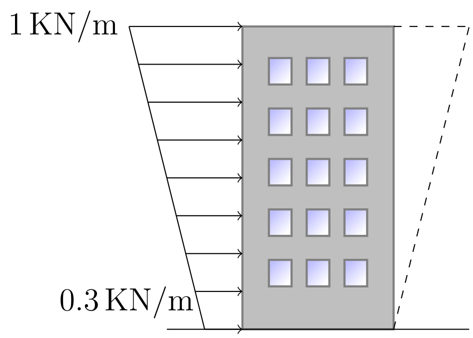
Answer: (a) $u_0 = 0.000\,64\,\mathrm{cm}$, $\ddot{u}_0 = 0.011\,\mathrm{m/s^2}$, (b) $u_0 = 41\,\mathrm{cm}$, $\ddot{u}_0 = 20\,\mathrm{m/s^2}$ and (c) $u_0 = 4.2\,\mathrm{cm}$, $\ddot{u}_0 = 0.066\,\mathrm{m/s^2}$.

Exercise 6 A building has a height of $100\,\mathrm{m}$, a square base measuring $20 \times 20\,\mathrm{m^2}$, an average specific weight of $1500\,\mathrm{N/m^3}$ and a natural period of vibration of 5 seconds. The top floor is hit by an helicopter with a mass of $10\,000\,\mathrm{Kg}$ and traveling at $30\,\mathrm{m/s}$. Determine the maximum deflection at the top assuming conservation of linear momentum and a vibration shape function that increases linearly with the height.



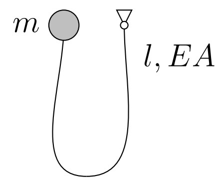
Answer: $u_0 = 12\,\mathrm{cm}$.

Exercise 7 The building of example 6 is hit by a sudden wind gust which results in the sudden application of horizontal forces distributed along the height of the building as shown in the picture. Assuming a vibration shape function that increases linearly with the height and neglecting damping, determine the maximum displacement at the top of the building.



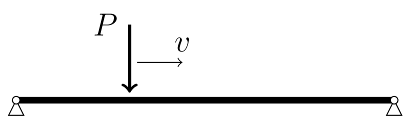
Answer: $u_0 = 2.5\,\mathrm{cm}$.

Exercise 8 A mass m is released from a given height attached to a massless cable of length l, area A and Young's modulus E. If the cable is fixed at the point from which the mass is released, describe the motion/vibration of the mass. Determine the maximum stress in the cable and the lowest point reached by the mass.



Answer: $u_{max} = \left(\frac{mg}{EA} + \sqrt{\frac{2mg}{EA}}\right) l$, $\sigma_{max} = \frac{mg}{A} + \sqrt{\frac{2mgE}{A}}$.

Exercise 9 A point load $P = 1\,\mathrm{KN}$ moves along with constant speed $v = 10\,\mathrm{m/s}$ on a simply supported beam of length $l = 10\pi\,\mathrm{m}$ as shown in the figure. The beam is made of concrete, has a rectangular section of 1 m width and 0.5 m height and an average density of $2800\,\mathrm{Kg/m^3}$. Determine the deflection of the beam as a function of time, the dynamic magnification factor and the maximum bending moment at the centre section.

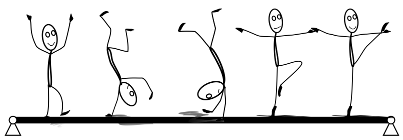


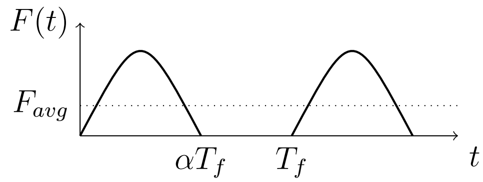
Answer: $H = 1.11$, $u = 0.5 \sin(t)\,\mathrm{cm}$, $M_{max} = 8.7\,\mathrm{KN\,m}$.
References: Chopra, *Dynamics of structures, SI Edition*, page 305

Exercise 10 A concrete ribbed slab floor spans 9 m and has an average mass of $500\,\mathrm{kg/m^2}$. The floor is simply supported on either side and has a natural frequency of vibration of 6.3 Hz. The floor is to be used for aerobics and other similar rhythmic activities at frequencies ranging from 1.5 Hz to 2.5 Hz and with contact ratios α between 0.5 and 1. During these activities the average imposed load will remain below $0.75\,\mathrm{kN/m^2}$ (before dynamic magnification) and the damping ratio can be taken to be 3 %.

(a) Determine the maximum possible resonant displacement and the resulting peak acceleration and bending moment per unit width.

(b) If the floor has been designed for a service load of $5\,\mathrm{kN/m^2}$, determine its suitability for the proposed use.





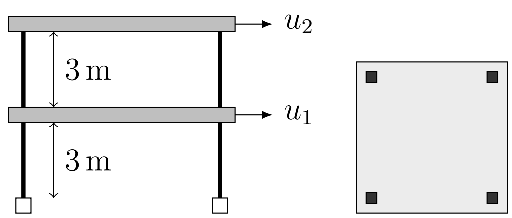
Answer: (a) $u_{max} = 1.2\,\mathrm{mm}$, $M_{max} = 20\,\mathrm{KN\,m/m}$. (b) The floor is suitable for aerobics.
References: Chopra, *Dynamics of structures, SI Edition*

Vibration of MDOF structures

Exercise 11 The two storey building shown is supported by four square concrete columns of dimensions $0.35 \times 0.35\,\mathrm{m}^2$. The total masses of the bottom and top floors are 150 and 100 Tn respectively.

(a) Determine the natural modes and frequencies of vibration in the horizontal direction shown.

(b) Determine the frequency of vibration that would be obtained making the assumption that the fundamental mode of vibration increases linearly with height.



Answer: (a) $\omega_1 = 10\,\mathrm{rad/s}$, $\omega_2 = 24.5\,\mathrm{rad/s}$, $\mathbf{v}_1 = [2\ 3]^T$, $\mathbf{v}_2 = [1\ {-1}]^T$ (b) $\omega_1 = 10.4\,\mathrm{rad/s}$, $\mathbf{v}_1 = [1\ 2]^T$.

Exercise 12 The two storey building from the previous exercise is hit by a helicopter with a mass of 10 Tn traveling at 20 m/s.

(a) Determine the resulting vibration and the maximum displacement at the top of the building using both modes of vibration.

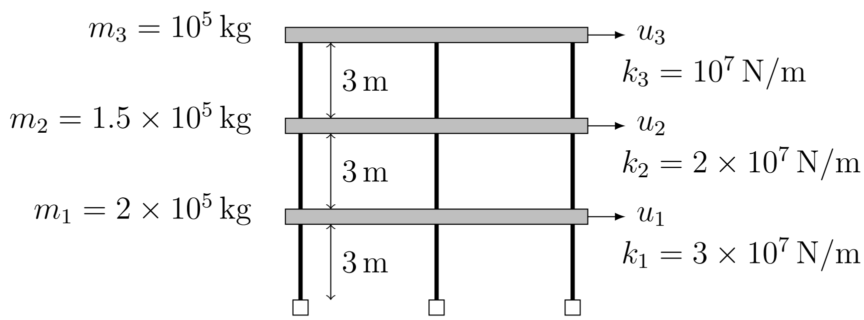
(b) Determine the resulting vibration and the maximum displacement at the top on the assumption that the linearly increasing mode absorbs the total momentum.

Answer: (a) $u_{max} = 15\,\text{cm}$, (b) $u_{max} = 14.5\,\text{cm}$.

Exercise 13 A three storey building has the mass and stiffness distribution shown.

(a) Approximate the first period of vibration using a linearly increasing mode.

(b) Using a linearly increasing mode together with a second Ritz vector increasing quadratically with height, approximate the first two modes and frequencies of vibration.



Answer: (a) $\omega_1 = 5.9\,\text{rad/s}$, (b) $\omega_1 = 5.9\,\text{rad/s}$, $\omega_2 = 12.8\,\text{rad/s}$.

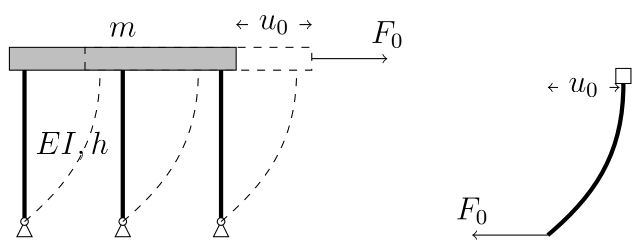
Solutions

Answer of exercise 1 The natural frequency of a structure is obtained from the solution of the differential equation governing the displacement of a *spring mass* system without excitation.

$$m\ddot{u} + ku = 0$$

where m is the mass of the idealized system and k is the stiffness. The natural frequency depends on both constants, $\omega^2 = k/m$.

Every single structure can be decomposed in its elements and each element, analyzed by any of the standard methods. Here, to obtain the stiffness of each element, we impose a unit displacement u_0 generated by the corresponding force F_0. The stiffness of the structure is the sum of the stiffness of its components.

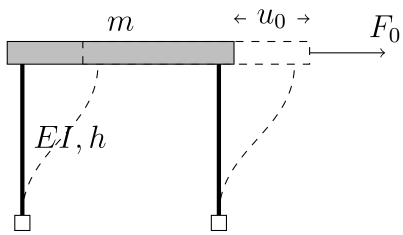


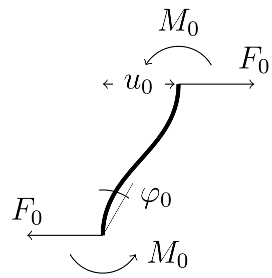
The displacement of the columns can be analyzed as a cantilever using static analysis concepts:

$$u_0 = \frac{F_0 h^3}{3EI} \quad \rightarrow \quad k_{column} = \frac{3EI}{h^3}$$

Finally, the stiffness and the frequency of the structure are

$$k = 3k_{column} = \frac{9EI}{h^3} \quad , \quad \omega = \sqrt{\frac{9EI}{mh^3}}$$





Analogously, the second structure can be analyzed combining the stiffness of the columns. In that case, rotation $\varphi_0 = u_0/h$ generated by the moment reaction M_0 has been imposed to the equivalent beams. The moment reaction must satisfy global equilibrium:

$$\sum M = 2M_0 - F_0 h = 0$$

And from static analysis, the rotation generated by the moment is

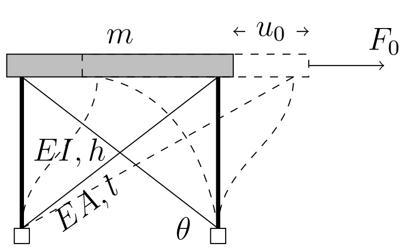
$$\varphi_0 = \frac{M_0 h}{6EI}$$

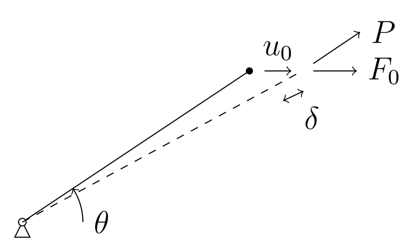
Substituting the moment and the rotation into the above expression gives

$$\frac{u_0}{h} = \frac{F_0 h}{12EI} \quad \rightarrow \quad k_{column} = \frac{12EI}{h^3}$$

The lateral stiffness and frequency of the structure are

$$k = 2k_{column} = \frac{24EI}{h^3} \quad , \quad \omega = \sqrt{\frac{24EI}{mh^3}}$$





The last structure adds two braces and its stiffness shall be added, but only one of them is contributing, since the bracing under compression buckles. The stiffness of a brace is

$$\delta = \frac{Pl}{EA} \quad \rightarrow \quad k_{brace} = \frac{EA}{l}\cos^2\theta$$

and the lateral stiffness and frequency of the structure are

$$k = 2k_{column} + k_{brace} = \frac{24EI}{h^3} + \frac{EA}{l}\cos^2\theta \quad , \quad \omega = \sqrt{\frac{24EI}{mh^3} + \frac{EA}{ml}\cos^2\theta}$$

Answer of exercise 2 Given that the natural period of vibration of the frame is $T = 0.9$ seconds, the frequency is computed as

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{0.9} = 6.98\,\mathrm{rad/s}$$

Then, the lateral stiffness is computed from the frequency and the mass of the structure,

$$\omega^2 = \frac{k}{m} \quad \rightarrow \quad k = \omega^2 m = \omega^2 \frac{P}{g} = 6.98^2 \frac{100}{10} = 487\,\mathrm{kN/m}$$

The goal of the second step is to determine the diameter of the steel cross-braces required to strengthen the structure by reducing the period to 0.3 seconds. First of all, the new stiffness is obtained following the same procedure,

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{0.3} = 20.94\,\mathrm{rad/s}$$
$$k = \omega^2 \frac{P}{g} = 20.94^2 \frac{100}{10} = 4384\,\mathrm{KN/m}$$

The bracing system should provide the additional stiffness,

$$k = k_{frame} + k_{bracing} \quad \rightarrow \quad k_{bracing} = 4384 - 487 = 3897\,\mathrm{KN/m}$$

We will consider the stiffness of one brace because the one under compression can buckle. Following the solution of example 1 (c), the area can be obtained from

$$k_{brace} = \frac{EA}{l}\cos^2\theta \quad \rightarrow \quad A = \frac{k_{brace} l}{E\cos^2\theta}$$
$$l = \sqrt{4^2 + 5^2} = 6.4\,\mathrm{m}$$
$$\cos\theta = \frac{5}{6.4} = 0.78$$
$$A = \frac{3897 \cdot 6.4}{2 \cdot 10^8 \cdot 0.78} = 1.6 \cdot 10^{-4} m^2 = 1.6\,\mathrm{cm}^2$$

and the diameter is obtained from the area,

$$D = 2\sqrt{\frac{A}{\pi}} = 2\sqrt{\frac{1.6}{\pi}} = 1.4\,\mathrm{cm}$$

The last part of the example consists on computing the new period of vibration if a further load of 50 kN is added to the strengthened structure. Using the new lateral stiffness, the frequency is

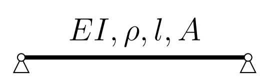
$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{4384}{15}} = 17.1\,\mathrm{rad/s}$$

and the new period is

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{17.1} = 0.37\,\mathrm{s}$$

Answer of exercise 3 The Rayleigh's method is based on assuming the vibration to be given in terms of a pre-determined shape function ψ as $u(x,t) = u_0(t)\psi(x)$. This assumption allows to find an equivalent mass and stiffness associated to that mode of vibration and thus, the frequency.

For the simple beam, a possible shape function would be a parabola satisfying the kynematic boundary conditions, $\psi(0) = \psi(l) = 0$.



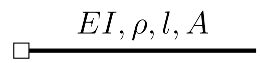
$$\psi(x) = \frac{4}{l^2}x(l-x)$$
$$\psi''(x) = \frac{8}{l^2}$$

where the term $4/l^2$ is a scaling parameter and does not modify the solution. The equivalent mass and stiffness and frequency are

$$m = \int_0^l \rho A \psi^2 dx = \int_0^l \rho A \left(\frac{4}{l^2}x(l-x)\right)^2 dx = \frac{8\rho A l}{15}$$
$$k = \int_0^l EI\psi''^2 dx = \int_0^l EI\left(\frac{8}{l^2}\right)^2 dx = \frac{64EI}{l^3}$$
$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{64EI}{l^3}\frac{15}{8\rho A l}} = \frac{1}{l^2}\sqrt{120\frac{EI}{\rho A}} \approx 10.95\frac{1}{l^2}\sqrt{\frac{EI}{\rho A}}\,\mathrm{rad/s}$$

The obtained result can be compared against the exact frequency, obtained by the same procedure applied to a fourth order polynomial satisfying both kinematic and dynamic boundary conditions. The analytical result is $\omega = \frac{1}{l^2}\sqrt{\frac{3024EI}{31\rho A}} \approx 9.86\frac{1}{l^2}\sqrt{\frac{EI}{\rho A}}$ rad/s.

The cantilever can be analyzed using a second order polynomial. In that case, the kinematic boundary conditions are $\psi(0) = \psi'(0) = 0$.



$$\psi(x) = \frac{1}{l^2}x^2$$
$$\psi''(x) = \frac{2}{l^2}$$

Again, the term $1/l^2$ is a scaling term providing consistency and does not modify the result of the calculations.

$$m = \int_0^l \rho A \psi^2 dx = \int_0^l \rho A \left(\frac{1}{l^2}x^2\right)^2 dx = \frac{\rho A l}{5}$$
$$k = \int_0^l EI \psi''^2 dx = \int_0^l EI \left(\frac{2}{l^2}\right)^2 dx = \frac{4EI}{l^3}$$
$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{4EI}{l^3}\frac{5}{\rho A l}} = \frac{1}{l^2}\sqrt{20\frac{EI}{\rho A}} \approx 4.47\frac{1}{l^2}\sqrt{\frac{EI}{\rho A}} \text{ rad/s}$$

On the other hand, the exact frequency of the first mode of vibration of a cantilever is $\omega \approx 3.52\frac{1}{l^2}\sqrt{\frac{EI}{\rho A}}$ rad/s.

The last two structures can be analyzed using a cubic polynomial, since the shape function must fulfill three kinematic boundary conditions.

For the application of Rayleigh's method with periodic shape functions, see the solution to example 9.

Answer of exercise 4 The solution to the differential equation governing free vibration of a *mass spring damper* system is governed by the natural and damped frequencies and the damping ratio:

$$u(t) = e^{-\xi\omega t}\sin(\omega_D t) \quad ; \quad \omega_D = \omega\sqrt{1-\xi^2}$$

For simplicity, we will assume a small damping and $\sqrt{1-\xi^2} \to 1$, thus, the natural period can be obtained directly from the measured values

$$T \approx T_D = \frac{3}{20} = 0.15\,\mathrm{s}$$

And the damping ratio ξ is directly related to the decay coefficient determined by the relation among two consecutive oscillations from a damped period

$$\frac{u(t)}{u(t+T_D)} = e^{\xi\omega T_D} = e^{\frac{2\pi\xi}{\sqrt{1-\xi^2}}} \quad \to \quad \frac{u_0}{u_1}\frac{u_1}{u_2}\dots\frac{u_{n-1}}{u_n} = \frac{u_0}{u_n} = e^{\frac{2n\pi\xi}{\sqrt{1-\xi^2}}}$$

Since ξ is small, the damping ratio can be found as

$$2n\pi\xi = \log\left(\frac{u_0}{u_n}\right) \quad \to \quad \xi = \frac{1}{2n\pi}\log\left(\frac{u_0}{u_n}\right) = \frac{1}{2\cdot 20\pi}\log\left(\frac{5}{1}\right) = 0.0128$$
$$\xi = 1.28\,\%$$

Answer of exercise 5 The maximum displacements and accelerations of a vibrating structure under sinusoidal excitation are obtained from the stationary or homogeneous solution to the differential equation,

$$m\ddot{u} + ku = F_0\sin(\Omega t + \phi)$$
$$u(t) = \frac{F_0}{k}H\sin(\Omega t + \phi - \Delta\phi)$$

where H is known as the magnification factor and is equal to

$$H = \frac{1}{\sqrt{(1-\gamma^2)^2 + 4\xi^2\gamma^2}} \quad ; \quad \gamma = \frac{\Omega}{\omega} = \frac{T_{struct}}{T_{force}}$$

From the solution to example 3 (a) we know that the period is $T = 0.9$ seconds and the frequency is $\omega = 2\pi/T \approx 7\,\mathrm{rad/s}$ and the lateral stiffness is $k = 487\,\mathrm{KN/m}$. The damping ratio is $\xi = 5\,\%$.

The amplitude F_0 of the external force is computed from the amplitude of the ground acceleration and the moving mass of the structure

$$F_0 = ma_0 = \frac{Pa_0}{g} = \frac{100\cdot 2}{10} = 20\,\mathrm{KN}$$

Finally, the maximum displacements will depend on the period Ω of the external force,

$$u_0 = \frac{F_0}{k} H = \frac{20}{487} H = 0.041 H \,\mathrm{m} = 4.1 H \,\mathrm{cm}$$
$$\ddot{u}_0 = \frac{F_0}{m} \gamma^2 H = \Omega^2 u_0$$

When the period of the ground acceleration is $T_g = 0.15\,\mathrm{s}$, the maximum displacements and accelerations are

$$\gamma = \frac{0.9}{0.15} = 9 \quad ; \quad H = \frac{1}{\sqrt{(1-9^2)^2 + 4 \cdot 0.05^2 \cdot 9^2}} = \frac{1}{\sqrt{80^4 + \cdots}} = 0.00015$$
$$u_0 = 4.1 \cdot 0.00015 = 0.000\,64\,\mathrm{cm}$$
$$\ddot{u}_0 = \left(\frac{2\pi}{0.15}\right)^2 \cdot 0.00064 = 1.1\,\mathrm{cm/s^2} = 0.011\,\mathrm{m/s^2}$$

This situation is *mass dominated*, also known as *vibration isolation*.

When the period of the ground acceleration is $T_g = 0.9\,\mathrm{s}$, the maximum values are

$$\gamma = \frac{0.9}{0.9} = 1 \quad ; \quad H = \frac{1}{\sqrt{(1-1^2)^2 + 4 \cdot 0.05^2 \cdot 1^2}} = \frac{1}{\sqrt{0 + 4 \cdot 0.05^2}} = 10$$
$$u_0 = 4.1 \cdot 10 = 41\,\mathrm{cm}$$
$$\ddot{u}_0 = \left(\frac{2\pi}{0.9}\right)^2 \cdot 41 = 2000\,\mathrm{cm/s^2} = 20\,\mathrm{m/s^2}$$

The *resonance* situation is *damping dominated*.

And for a period $T_g = 5\,\mathrm{s}$, the maximum values are

$$\gamma = \frac{0.9}{5} = 0.18 \quad ; \quad H = \frac{1}{\sqrt{(1-0.18^2)^2 + 4 \cdot 0.05^2 \cdot 0.18^2}} = 1.033$$
$$u_0 = 4.1 \cdot 1.033 = 4.2\,\mathrm{cm}$$
$$\ddot{u}_0 = \left(\frac{2\pi}{5}\right)^2 \cdot 4.2 = 6.6\,\mathrm{cm/s^2} = 0.066\,\mathrm{m/s^2}$$

Which is a practically *static* situation or *stiffness dominated*.

Answer of exercise 6 When the building is hit by the helicopter, the impulse or linear momentum is preserved. To compute the linear momentum of the building, we need to know the equivalent mass and we can use the Rayleigh's method for that purpose. As it is suggested, we will use a linear shape function

$$u = u_0\psi \quad ; \quad \psi = \frac{z}{h}$$
$$m = \int_0^h \rho A \psi^2 dz = \rho A \int_0^h \frac{z^2}{h^2} dz = \frac{1}{3}\rho A h = \frac{1}{3} 1500 \times 400 \times 100 =$$
$$= 2 \times 10^7\,\mathrm{N} = 2 \times 10^6\,\mathrm{Kg}$$

The impulse of the helicopter is

$$I = (mv)_{helicopter} = 10^4 \times 30 = 3 \times 10^5\,\mathrm{Kg\,m/s}$$

which is transferred to the building. Then, the maximum deflection at the top of the building is inferred according to the following expressions:

$$I = m\dot{u}_0 \quad \rightarrow \quad u_0 = \frac{I}{m\omega}$$
$$\omega = \frac{2\pi}{T} = \frac{2\pi}{5} = 1.25\,\mathrm{rad/s}$$
$$u_0 = \frac{3 \times 10^5}{2 \times 10^6 \times 1.25} = 0.12\,\mathrm{m} = 12\,\mathrm{cm}$$

Answer of exercise 7 Now, the building from example 6 is exposed to sudden wind gust. In that case, the dynamics of the structure is defined by a *mass spring damper* system under a constant force suddenly applied. The maximum amplitude is governed by the stationary or homogeneous solution to the differential equation

$$m\ddot{u} + c + ku = F_0$$
$$u(t) = \frac{F_0}{k}(1 - e^{\xi\omega t}\cos(\omega t))$$

First of all, we need to determine the equivalent lateral force using the Rayleigh's method from the wind pressure p_W, using the same shape function

ψ from example 6,

$$F_0 = \int_0^h p_W \psi dz = \int_0^h \left(0.3 + (1-0.3)\frac{z}{h}\right)\frac{z}{h} dz = \left(\frac{0.3}{2} + \frac{0.7}{3}\right) h = $$
$$= \left(\frac{0.3}{2} + \frac{0.7}{3}\right) 100 = 38.3\,\mathrm{KN}$$

The stiffness of the building can be estimated from the equivalent mass and the given period of the building

$$k = \omega^2 m = \left(\frac{2\pi}{T}\right)^2 m = \left(\frac{2\pi}{5}\right)^2 2\times 10^6 = 3.125\times 10^6\,\mathrm{N/m} = 3125\,\mathrm{KN/m}$$

Finally, the maximum displacement is obtained from the equation of motion

$$u_{max} = \frac{F_0}{k}(1+1) = \frac{38.3}{3125}2 = 0.025\,\mathrm{m} = 2.5\,\mathrm{cm}$$

Answer of exercise 8 The motion of the mass released from a given height and attached to a cable has two parts. Firstly, a free fall. Secondly, a vibration. The velocity at the end of the free fall is

$$\frac{1}{2} m \dot{u}_0^2 = mgl \quad \rightarrow \quad \dot{u}_0 = \sqrt{gl}$$

and may be taken as initial condition of the vibration. Lately, the elastic response of the cable is the superposition of the static elongation and the vibration. From the cable properties,

$$k = \frac{EA}{l}$$
$$m = \sqrt{\frac{k}{m}} = \sqrt{\frac{EA}{ml}}$$
$$u_{stat} = \frac{F}{M} = \frac{mgl}{EA}$$

Then, neglecting damping, we can consider the motion as the transient solution of the differential equation satisfying the appropriate initial conditions,

$$
\begin{aligned}
u(t) &= C\sin(\omega t + \phi) + u_{stat} \\
u(0) &= u_{stat} \rightarrow C\sin(\phi) = 0 \rightarrow \phi = 0 \\
\dot{u}(0) &= \sqrt{2gl} \rightarrow C\omega\cos(\phi) = \sqrt{2gl} \rightarrow C = \sqrt{\frac{2mg}{EA}}l
\end{aligned}
$$

And the maximum displacement u_{max} and stress σ are

$$
u_{max} = \left(\frac{mg}{EA} + \sqrt{\frac{2mg}{EA}}\right) l
$$

$$
\sigma = E\epsilon = E\frac{u_{max}}{l} = \frac{mg}{A} + \sqrt{\frac{2mgE}{A}}
$$

Note, the maximum displacement can also be obtained applying energy conservation.

Answer of exercise 9 In that case, the magnitude of the external force $P = 1\,\mathrm{KN}$ is not varying with time, but moving with constant speed $v = 10\,\mathrm{m/s}$. The Rayleigh's method allow us to compute the generalized force applied as a function of time. For that purpose, we choose a periodic shape function ψ satisfying the kinematic boundary conditions

$$
\begin{aligned}
\psi &= \sin\left(\frac{\pi x}{l}\right) \\
F &= P\psi(x) = P\psi(vt) = P\sin\left(\frac{\pi v t}{l}\right)
\end{aligned}
$$

thus, the period of the external excitation is $\Omega = \pi v/l = 1\,\mathrm{rad/s}$.

From the structure dimensions, we know that the mechanical properties of the section are:

$$
\begin{aligned}
A &= 0.5\,\mathrm{m}^2 \\
I &= \frac{1}{12}0.5^3 = \frac{0.125}{12} \approx 0.01\,\mathrm{m}^4 \\
E &= 14\,\mathrm{GPa}
\end{aligned}
$$

and the generalized mass and stiffness can be computed with the Rayleigh's method,

$$m = \int_0^l \rho A \psi^2 dx = \int_0^l \rho A \sin^2\left(\frac{\pi x}{l}\right) dx = \frac{\rho A l}{2} =$$
$$= \frac{2800 \cdot 0.5 \cdot 10\pi}{2} = 7000\pi Kg = 7\pi \text{ Tn}$$
$$k = \int_0^l EI \psi''^2 dx = \int_0^l EI \left(-\left(\frac{\pi}{l}\right)^2 \sin\left(\frac{\pi x}{l}\right)\right)^2 dx = \frac{EI\pi^4}{2l^3} =$$
$$= \frac{14 \cdot 10^6 \cdot 0.01 \cdot \pi^4}{2 \cdot (10\pi)^3} = 70\pi \text{ KN/m}$$

with frequency equal to

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{70\pi}{7\pi}} = \sqrt{10} = 3.16 \, \text{rad/s}$$

Finally, the structural response is obtained following the same steps from example 5, a structure under periodic loading. The deflection depends on the dynamic magnification factor H and assuming a small damping ratio,

$$\gamma = \frac{\Omega}{\omega} = \frac{\pi v}{l\omega}$$
$$H = \frac{1}{\sqrt{\left(1 - \frac{\pi^2 v^2}{l^2\omega^2}\right)^2 + 4\xi^2 \frac{\pi^2 v^2}{l^2 \omega^2}}} \approx \frac{1}{1 - \frac{\pi^2 v^2}{l^2\omega^2}} = \frac{l^2\omega^2}{l^2\omega^2 - \pi^2 v^2} =$$
$$= \frac{10^2 \cdot \pi^2 \cdot 3.16^2}{10^2 \cdot \pi^2 \cdot 3.16^2 - 10^2 \cdot \pi^2} = \frac{10}{10 - 1} = 1.11$$

Then, the deflection of the beam as a function of time is

$$u = \frac{P}{k} H \sin(\Omega t) = \frac{1}{70\pi} 1.11 \sin(t) = 0.005 \sin(t) \, f\text{m} = 0.5 \sin(t) \, \text{cm}$$

And the maximum bending moment at the centre section is

$$M_{max} = \frac{Pl}{4} H = \frac{1 \cdot 10\pi}{4} 1.11 = 8.7 \, \text{KN m}$$

Solution of the differential equation Given that the load is a piecewise function in time, the dynamic magnification factor H may not be accurate. The full solution to the differential equation is also a piecewise function of the form

$$u = \begin{cases} C_1 \sin(\omega t + \varphi_1) + \frac{P}{k} H \sin(\Omega t + \phi - \Delta\phi) & \text{if } t < \frac{l}{v} \\ C_2 \sin(\omega t + \varphi_2) & \text{if } t \geq \frac{l}{v} \end{cases}$$

The first part of the solution must satisfy homogeneous boundary conditions

$$\begin{array}{l} u_1(0) = 0 \\ \dot{u}_1(0) = 0 \end{array} \rightarrow \begin{array}{l} \varphi_1 = 0 \\ C_1 = -\frac{P}{k} H\gamma \end{array}$$

with $\gamma = \frac{\Omega}{\omega} = \frac{\pi v}{\omega l}$. For the second part of the motion, the expression must satisfy continuity at $t = l/v$,

$$u_2(l/v) = C_2 \sin\left(\frac{\omega l}{v} + \varphi_2\right) = u_1(l/v) = -\frac{P}{k} H\gamma \sin\frac{\omega l}{v}$$
$$\dot{u}_2(l/v) = C_2 \omega \cos\left(\frac{\omega l}{v} + \varphi_2\right) = \dot{u}_1(l/v) = -\frac{P}{k} H\gamma\omega \cos\frac{\omega l}{v} - \frac{P}{k} H\Omega$$

Note that $\Omega l/v = \pi$.

The values of the integration constants are obtained after applying some trigonometric identities:

$$\omega \cot\left(\frac{\omega l}{v} + \varphi_2\right) = \omega \cot\frac{\omega l}{v} + \omega \csc(\omega l/v)$$
$$\cot\left(\frac{\omega l}{v} + \varphi_2\right) = \cot\frac{\omega l}{2v}$$
$$\varphi_2 = -\frac{\omega l}{2v}$$

and

$$C_2 = -\frac{P}{k} H\gamma \frac{\sin(\omega l/v)}{\sin(\omega l/v + \varphi_2)} = -\frac{P}{k} H\gamma \frac{\sin(\omega l/v)}{\sin(\omega l/v - \omega l/(2v))}$$
$$C_2 = -\frac{P}{k} H\gamma \frac{2\sin(\omega l/(2v))\cos(\omega l/(2v))}{\sin(\omega l/(2v))}$$
$$C_2 = -2\frac{P}{k} H\gamma \cos\frac{\omega l}{2v}$$

The result is

$$u = \frac{P}{k}\frac{\omega^2 l^2}{\omega^2 l^2 - \pi^2 v^2}\begin{cases} \sin\frac{\pi v t}{l} - \frac{\pi v}{\omega l}\sin(\omega t) & \text{if } t < \frac{l}{v} \\ -\frac{2\pi v}{\omega l}\cos\frac{\omega l}{2v}\sin(\omega t - \frac{\omega l}{2v}) & \text{if } t \geq \frac{l}{v} \end{cases}$$

Answer of exercise 10 The structure properties per unit width using a sine shape function are

$$m = \int_0^l w\psi^2 dx = \int_0^l w\sin^2\left(\frac{\pi x}{l}\right)dx = \frac{wl}{2} = \frac{500 \times 9}{2} = 2250\,\text{kg/m}$$

$$k = \omega^2 m = \left(\frac{6.3}{2\pi}\right)^2 \times 2250 = 9000\,\text{KN/m}^2$$

and the generalized force is

$$\tilde{F} = \int_0^l F\psi dx = \frac{2Fl}{\pi}$$

The considered transient walking force is a periodic loading with a piecewise definition. Since the action is periodic, a different approach from exercise 9 can be followed, the action will be approximated using a Fourier series, which is valid for any periodic action.

Pre computed Fourier terms Each component of a Fourier series is also known as harmonic. For a half sine activity, the amplitudes of the first four harmonics are

Activity	α	F_1/F_{avg}	F_2/F_{avg}	F_3/F_{avg}	F_4/F_{avg}
Walking	2/3	1.29	0.16	0.13	0.04
Exercise	1/2	1.57	0.67	0.00	0.13
Jumping	1/3	1.80	1.29	0.67	0.16
High jumping	1/4	1.89	1.57	1.13	0.67

then, the dynamic force is the superposition of the static force and the harmonics multiplied by the corresponding dynamic magnification factor,

$$F_e = F_{avg} + H_1F_1 + H_2F_2 + H_3F_3 + \dots$$

For the frequency of vibration of 1.5 Hz

$$\gamma_1 = \frac{6.3}{1.5} = 4.2 \;\rightarrow\; H_1 = \frac{1}{\sqrt{(1-4.2^2)^2 + 4\times 0.03^2 \times 4.2^2}} = 0.06$$
$$\gamma_2 = \frac{6.3}{3} = 2.1 \;\rightarrow\; H_1 = \frac{1}{\sqrt{(1-2.1^2)^2 + 4\times 0.03^2 \times 2.1^2}} = 0.29$$
$$\gamma_3 = \frac{6.3}{4.5} = 1.4 \;\rightarrow\; H_1 = \frac{1}{\sqrt{(1-1.4^2)^2 + 4\times 0.03^2 \times 1.4^2}} = 1.03$$
$$\gamma_4 = \frac{6.3}{6} = 1.05 \;\rightarrow\; H_1 = \frac{1}{\sqrt{(1-1.05^2)^2 + 4\times 0.03^2 \times 1.05^2}} = 8.3$$

if walking is considered, the contact ratio α is 2/3 and the distributed force

$$F_e = F_{avg}(1 + 0.06\cdot 1.29 + 0.29\cdot 0.16 + 1.03\cdot 0.13 + 8.3\cdot 0.04) = 1.59F_{avg}$$

and if exercise is considered, the contact ratio α is 0.5 and the distributed force

$$F_e = F_{avg}(1 + 0.06\cdot 1.57 + 0.29\cdot 0.67 + 8.3\cdot 0.13) = 2.37F_{avg}$$

For the highest frequency of vibration of 2.5 Hz

$$\gamma_1 = \frac{6.3}{2.5} = 2.5 \;\rightarrow\; H_1 = \frac{1}{\sqrt{(1-2.5^2)^2 + 4\times 0.03^2 \times 2.5^2}} = 0.19$$
$$\gamma_1 = \frac{6.3}{5} = 1.26 \;\rightarrow\; H_1 = \frac{1}{\sqrt{(1-1.26^2)^2 + 4\times 0.03^2 \times 1.26^2}} = 1.7$$
$$\gamma_1 = \frac{6.3}{7.5} = 0.84 \;\rightarrow\; H_1 = \frac{1}{\sqrt{(1-0.84^2)^2 + 4\times 0.03^2 \times 0.84^2}} = 3.4$$
$$\gamma_1 = \frac{6.3}{10} = 0.63 \;\rightarrow\; H_1 = \frac{1}{\sqrt{(1-0.63^2)^2 + 4\times 0.03^2 \times 0.63^2}} = 1.6$$

if walking is considered, the contact ratio α is 2/3 and the distributed force

$$F_e = F_{avg}(1 + 0.19\cdot 1.29 + 1.7\cdot 0.16 + 3.4\cdot 0.13 + 1.6\cdot 0.04) = 2.02F_{avg}$$

and if exercise is considered, the contact ratio α is 0.5 and the distributed force

$$F_e = F_{avg}(1 + 0.19 \cdot 1.57 + 1.7 \cdot 0.67 + 1.6 \cdot 0.13) = 2.65 F_{avg}$$

It can be seen that the higher contact ratio generates a higer dynamic action and that a frequency closer to the structure frequency has a hyger dynamic magnification factor. Then, we will consider $\alpha = 0.5$ and 2.5 Hz as the worst scenario. Finally, the maximum ressonant displacement is

$$u_{max} = \frac{\tilde{F}_e}{k} = \frac{2F_e l}{\pi k} = \frac{2 \times 2.65 \times 0.75 \times 9}{9000\pi} = 0.0012\,\text{m} = 1.2\,\text{mm}$$

and the maximum bending moment per unit width for a uniformly distributed load is

$$M_{max} = \frac{F_e l^2}{8} = \frac{2.65 \times 0.75 \times 9^2}{8} = 20\,\text{KN m/m}$$

Calculation of the Fourier terms Most commonly, Fourier series are expressed as a linear combination of sin and cos,

$$F(t) \approx \frac{a_0}{2} + \sum_{n=0}^{\infty} \left(a_n \cos(n\Omega t) + b_n \sin(n\Omega t)\right)$$

with

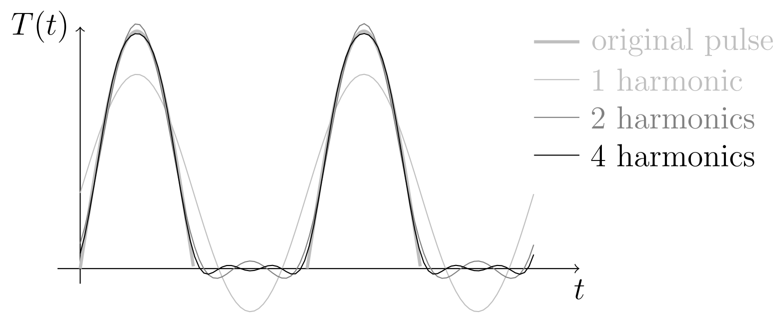
$$\begin{aligned} a_0 &= \frac{2}{T}\int_0^T F(t)dt \\ a_n &= \frac{2}{T}\int_0^T F(t)\cos(n\Omega t)dt \\ b_n &= \frac{2}{T}\int_0^T F(t)\sin(n\Omega t)dt \end{aligned}$$

An alternative form is to compute the combination of the trigonometric functions as

$$F(t) \approx F_0 + \sum_{n=0}^{\infty} F_n \sin(n\Omega t + \phi_n)$$

with

$$F_0 = \frac{a_0}{2}$$
$$F_n = \sqrt{a_n^2 + b_n^2}$$
$$\tan \phi_n = \frac{b_n}{a_n}$$



Answer of exercise 11 The inter-storey stiffness is obtained from the concrete columns properties,

$$E = 14\,\text{GPa} = 14 \times 10^6\,\text{KPa}$$
$$I = \frac{1}{12}bc^3 = \frac{1}{12}0.35^4 = \frac{15}{12} \times 10^{-3}\text{m}^4$$
$$k_{column} = \frac{12EI}{h^3} = \frac{12 \times 14 \times 15 \times 10^3}{12 \times 27} = 7.77 \times 10^3\,\text{KN/m}$$
$$k = 4k_{column} = 31.1\,\text{KN/m}$$

Using those values, the mass and stiffness matrices are

$$\mathbf{M} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} = \begin{bmatrix} 150 & 0 \\ 0 & 100 \end{bmatrix} \text{Tn}$$

$$\mathbf{K} = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \approx \begin{bmatrix} 60 & -30 \\ -30 & 30 \end{bmatrix} \times 10^3\text{KN/m}$$

(a) Natural modes of vibration The natural modes of vibration can be obtained from the solution to the generalized eigenvalue problem

$$
\begin{aligned}
|\mathbf{K} - \omega^2 \mathbf{M}| &= 0 \\
\begin{vmatrix} 60000 - \omega^2 150 & -30000 \\ -30000 & 30000 - \omega^2 100 \end{vmatrix} &= 0 \\
(6000 - 15\omega^2)(3000 - 1\omega^2) - 3000 \times 3000 &= 0 \\
15\omega^2 - 10500\omega^2 + 900000 &= 0
\end{aligned}
$$

$$
\omega^2 = \begin{cases} 100 \rightarrow \omega_1 = 10\,\mathrm{rad/s} \\ 600 \rightarrow \omega_2 = 24.5\,\mathrm{rad/s} \end{cases}
$$

Since the the determinant of the system is zero for the eigenvalues, the eigenvectors are found assigning an arbitrary value to a component,

$$
\begin{aligned}
(\mathbf{K} - \omega_1^2 \mathbf{M})\mathbf{v}_1 &= \mathbf{0} \\
\begin{bmatrix} 4500 & -3000 \\ -3000 & 2000 \end{bmatrix} \mathbf{v}_1 &= \mathbf{0} \\
\mathbf{v}_1 &= \begin{bmatrix} 2 \\ 3 \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
(\mathbf{K} - \omega_2^2 \mathbf{M})\mathbf{v}_2 &= \mathbf{0} \\
\begin{bmatrix} -3000 & -3000 \\ -3000 & -3000 \end{bmatrix} \mathbf{v}_2 &= \mathbf{0} \\
\mathbf{v}_2 &= \begin{bmatrix} 1 \\ -1 \end{bmatrix}
\end{aligned}
$$

(b) Ritz-Rayleigh method In order to show the possibilities of the Ritz-Rayleigh method, a linearly increasing mode of vibration $\mathbf{r}_1$ is choosen. If it were the case of a more complex structure, more trial modes of vibration $\mathbf{r}_i$

could be provided.

$$\mathbf{R} = [\mathbf{r}_1] = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\hat{\mathbf{M}} = \mathbf{R}^T\mathbf{M}\mathbf{R} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}^T \begin{bmatrix} 150 & 0 \\ 0 & 100 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 550\,\text{Tn}$$

$$\hat{\mathbf{K}} = \mathbf{R}^T\mathbf{K}\mathbf{R} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}^T \begin{bmatrix} 60000 & -30000 \\ -30000 & 30000 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 6 \times 10^4\,\text{KN/m}$$

And the eigenvalue problem is reduced to a scalar equation,

$$|\hat{\mathbf{K}} - \omega^2\hat{\mathbf{M}}| = 0 \quad \rightarrow \quad \omega_1^2 = \frac{6 \times 10^4}{550} \quad \rightarrow \quad \omega_1 = 10.4\,\text{rad/s}$$

Answer of exercise 12 The impulse of the helicopter from exercise 6 is fully transmitted to the building with dynamic properties caluclated in exercise 11. The impulse of the helicopter is applied at the top level of the building,

$$I = (mv)_{helicopter} = 10 \times 20 = 200\,\text{Tn m/s}$$

$$\mathbf{I} = \begin{bmatrix} 0 \\ 200 \end{bmatrix} \text{Tn m/s}$$

Both modes of vibration When several modes are present, the resulting vibration is calculated using modal decomposition. Finally, the full vibration os obtained by superposition of each mode of vibration. For the first mode of vibration,

$$I_1 = \mathbf{v}_1^T\mathbf{I} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}^T \begin{bmatrix} 0 \\ 200 \end{bmatrix} = 600\,\text{Tn m/s}$$

$$m_1 = \mathbf{v}_1^T\mathbf{M}\mathbf{v}_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}^T \begin{bmatrix} 150 & 0 \\ 0 & 100 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = 1500\,\text{Tn}$$

$$\omega_1 = 10\,\text{rad/s}$$

$$x_1 = \frac{I_1}{m_1\omega_1} = \frac{600}{1500 \times 10}\text{m}$$

And for the second mode of vibration

$$
\begin{aligned}
I_2 &= \mathbf{v}_1^T \mathbf{I} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}^T \begin{bmatrix} 0 \\ 200 \end{bmatrix} = -200\,\text{Tn}\,\text{m/s} \\
m_2 &= \mathbf{v}_1^T \mathbf{M} \mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}^T \begin{bmatrix} 150 & 0 \\ 0 & 100 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 250\,\text{Tn} \\
\omega_2 &= 24.5\,\text{rad/s} \\
x_2 &= \frac{I_2}{m_2 \omega_2} = \frac{-200}{250 \times 24.5}\text{m}
\end{aligned}
$$

Then, the resulting vibration is

$$
\mathbf{u}(t) = \sum x_i(t)\mathbf{v}_i = \begin{bmatrix} 0.08 \\ 0.12 \end{bmatrix} \sin(10t) + \begin{bmatrix} -0.032 \\ 0.032 \end{bmatrix} \sin(24.5t)
$$

being $u_{max} = 0.12 + 0.032 \approx 15\,\text{cm}$.

Linearly increasing mode When the Ritz-Rayleigh method is applied, the equivalent action shall be computed. The values from exercise 11 are considered.

$$
\begin{aligned}
\hat{\mathbf{I}} &= \mathbf{R}^T \mathbf{I} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}^T \begin{bmatrix} 0 \\ 200 \end{bmatrix} = 400\,\text{Tn}\,\text{m/s} \\
\hat{\mathbf{M}} &= 550\,\text{Tn} \\
\omega_1 &= 10.4\,\text{rad/s} \\
x_1 &= \frac{\hat{I}_1}{\hat{m}_1 \omega_1} = \frac{400}{550 \times 10.4}\text{m}
\end{aligned}
$$

And the resulting vibration is

$$
\mathbf{u}(t) = x_1(t)\mathbf{r}_1 = \begin{bmatrix} 0.072 \\ 0.145 \end{bmatrix} \sin(10.4t)
$$

being $u_{max} = 14.5\,\text{cm}$.

Answer of exercise 13 The first step before using the Ritz-Rayleigh method to approximate the periods of vibration is to construct the mass and stiffness matrices:

$$\mathbf{M} = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1.5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times 10^5\,\mathrm{kg}$$

$$\mathbf{K} = \begin{bmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_2 + k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix} = \begin{bmatrix} 5 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix} \times 10^7\,\mathrm{N/m}$$

1 mode We start applying the Ritz-Rayleigh method with a mode linearly increasing with height. The matrix of deformation modes will contain only one vector:

$$\mathbf{R} = [\mathbf{r}_1] = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}^T$$

And the reduced system is obtained by the following arithmetics:

$$\hat{\mathbf{M}} = \mathbf{R}^T\mathbf{M}\mathbf{R} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1.5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 17 \times 10^5\,\mathrm{kg}$$

$$\hat{\mathbf{K}} = \mathbf{R}^T\mathbf{K}\mathbf{R} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 5 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 6 \times 10^7\,\mathrm{N/m}$$

Thus, the eigenvalue problem is reduced to a scalar equation:

$$|\hat{\mathbf{K}} - \omega^2\hat{\mathbf{M}}| = 0 \quad \rightarrow \quad \omega_1^2 = \frac{6 \times 10^7}{17 \times 10^5} = 35.3 \quad \rightarrow \quad \omega_1 = 5.9\,\mathrm{rad/s}$$

2 modes The Ritz-Rayleigh method can be enriched with more modes of vibration, in this case, with a mode varying quadratically with height,

$$\mathbf{R} = \begin{bmatrix} r_1 & r_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}^T$$

The reduced system is then obtained by the algebraic multiplications

$$\hat{\mathbf{M}} = \mathbf{R}^T\mathbf{M}\mathbf{R} = \begin{bmatrix}1 & 2 & 3\\ 1 & 4 & 9\end{bmatrix}\begin{bmatrix}2 & 0 & 0\\ 0 & 1.5 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 1\\ 2 & 4\\ 3 & 9\end{bmatrix} = \begin{bmatrix}2 & 3 & 3\\ 2 & 6 & 9\end{bmatrix}\begin{bmatrix}1 & 1\\ 2 & 4\\ 3 & 9\end{bmatrix} =$$

$$= \begin{bmatrix}17 & 41\\ 41 & 107\end{bmatrix} \times 10^5\,\mathrm{kg}$$

$$\hat{\mathbf{K}} = \mathbf{R}^T\mathbf{K}\mathbf{R} = \begin{bmatrix}1 & 2 & 3\\ 1 & 4 & 9\end{bmatrix}\begin{bmatrix}5 & -2 & 0\\ -2 & 3 & -1\\ 0 & -1 & 1\end{bmatrix}\begin{bmatrix}1 & 1\\ 2 & 4\\ 3 & 9\end{bmatrix} = \begin{bmatrix}1 & 1 & 1\end{bmatrix}\begin{bmatrix}1 & 1\\ 2 & 4\\ 3 & 9\end{bmatrix} =$$

$$= \begin{bmatrix}6 & 14\\ 14 & 46\end{bmatrix} \times 10^7\,\mathrm{N/m}$$

and the eigenvalue problem involves the calculation of the determinant of a 2×2 matrix,

$$|\hat{\mathbf{K}} - \omega^2\hat{\mathbf{M}}| = \begin{vmatrix}600 - \omega^2 17 & 1400 - \omega^2 41\\ 1400 - \omega^2 41 & 4600 - \omega^2 107\end{vmatrix} \times 10^5 = 0$$

The result is the second order polynomial $138\omega^4 - 27600\omega^2 + 800000 = 0$ with the following roots:

$$\begin{aligned}\omega_1^2 &= 35.1 \quad\rightarrow\quad \omega_1 = 5.9\,\mathrm{rad/s}\\ \omega_2^2 &= 164.8 \quad\rightarrow\quad \omega_2 = 12.8\,\mathrm{rad/s}\end{aligned}$$

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