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The Response of Test Masses to Gravitational Waves in the Coordinates of a Local Observer

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Abstract

The response of laser interferometers to gravitational waves has been calculated in a number of different ways, particularly in the transverse-traceless and the local Lorentz gauges. At first sight, it would appear that these calculations lead to different results when the separation between the test masses becomes comparable to the wavelength of the gravitational wave. In this paper this discrepancy is resolved. We describe the response of free test masses to plane gravitational waves in the coordinate frame of a local observer and show that it acquires contributions from three different effects: the displacement of the test masses, the apparent change in the photon velocity, and the variation in the clock speed of the local observer, all of which are induced by the gravitational wave. Only when taken together do these three effects represent a quantity which is translationally invariant. This translationally-invariant quantity is identical to the response function calculated in the transverse-traceless gauge. We thus resolve the well-known discrepancy between the two coordinates systems, and show that the results found in the coordinate frame of a local observer are valid for large separation between the masses.

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1 Introduction

Searches for gravitational waves are now conducted with laser interferometers in which the test masses for sensing gravitational waves are separated by distances of several kilometers [1, 2]. Variations in the proper distance between these test masses, which may be caused by gravitational waves, are measured with light. The response of the laser interferometers to gravitational waves is usually calculated in the transverse-traceless (TT) gauge [3]. The main assumption for all such calculations is that the test masses of the laser interferometers are inertial, i.e. accelerations of the test masses in the direction of the probe laser beam are negligible. A substantial engineering effort has been made to meet this requirement. Placed in ultra-high vacuum and isolated from the ground by multi-layer stacks and actively-controlled suspensions, these test masses become practically inertial at frequencies far above the suspension resonances.

A significant change in the attitude toward the laser interferometers took place with the introduction of optical springs in the last few years [4, 5, 6]. The optical spring is produced by the pressure of light on the test masses, which in the advanced interferometer configurations can lead to amplification of the gravitational wave signal. For this amplification the resonance frequency of the optical spring must be matched with the frequency of the expected gravitational waves. In this case, the main assumption of the TT-gauge – the requirement of test mass inertiality – can no longer be made. This problem can be overcome if one uses the coordinates of a local observer for which no requirement of test mass inertiality is needed. Although this coordinate system has been frequently used to describe the response of resonant bar detectors [7], its application to laser interferometers thus far has been only occasional.

The coordinates of a local observer, also known as the local Lorentz gauge, have a long history. Comparison of the TT coordinates and the coordinates of a local observer is given in [3] with the curious observation that they yield different answers for geodesic deviation when separation between the geodesics becomes comparable to the wavelength of the gravitational wave (Exercise 37.6 in [3]). As a result, the coordinates of a local observer were considered not suitable for large separation between the geodesics. Nonetheless, the studies of the effects of gravitational waves in the coordinates of a local observer continued and over the years led to a number of interesting results. Several insightful papers have been written about the role of the coordinate system in the detection of gravitational waves [8, 9, 10, 11]. Some of the calculations in these papers rely on the Fermi normal expansion as a means to build the coordinate frame of a local observer. Explicit transformations from the TT-coordinates to the coordinates of a local observer have been constructed and analyzed [12, 13, 14]. The response of the interferometric gravitational wave detector calculated in the TT-coordinates was transformed into the coordinates of the local observer [14, 15].

Despite of all these efforts, the coordinates of a local observer have remained an obscure gauge even to this day. One of the reasons for the lack of understanding is the avoidance of the local Lorentz gauge which is largely influenced by the disagreement between this gauge and the TT-gauge. Another drawback associated with the coordinates of a local observer is the lack of consistent mathematical formalism. To derive any nontrivial result in these coordinates, one usually starts with TT-gauge and then obtains the answer by complicated coordinate transformations. In this paper we show that it is possible to calculate the effects of the gravitational wave directly in the coordinates of a local observer. There is no need to start with the TT-gauge and no need to use the transformation rules to go from one coordinate frame to the other. In particular, we describe the response of test masses to gravitational waves when their separation is comparable to or greater than the wavelength of the gravitational wave. In this approach, several effects must be combined to obtain a consistent test mass response. In the end, the discrepancy between the two gauges is resolved.

The presentation in this paper is such that only a few concepts from differential geometry are used. Often, abstract mathematical derivations are replaced with those based on simple physical arguments, and many formulas are deliberately presented in the Newtonian form after they have been derived in general relativity. The motivation for this approach is two-fold. On one hand, it allows us to focus on physics of the problem and set aside mathematical details which can be overwhelming. On the other hand, such an approach allows us to assume the standpoint of a "Newtonian physicist" [3], conducting experiments in a laboratory environment and describing the outcomes of these experiments in the familiar Newtonian terms, even though they represent the effects in general relativity.

2 The Coordinates of Transverse Traceless Gauge

We begin with a brief overview of the TT-gauge. This digression will allow us to introduce the test mass response function which will be needed later for comparison. Subsequent calculations, however, do not rely on the TT-gauge in any way.

In the TT-gauge the metric which describes a plane polarized gravitational wave propagating in flat space-time is given by

$$g_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1+h & & \\ & & 1-h & \\ & & & 1 \end{pmatrix}, \tag{1}$$

where $h = h(t + z/c)$ represents the amplitude of the "+" polarization [3]. For all anticipated astrophysical sources, the amplitude of gravitational waves upon their arrival to Earth is expected to be extremely small, typically $|h| \sim 10^{-21}$ or less, which justifies the use of the perturbation method in the following calculations.

A special property of the TT-coordinates is that an inertial test mass, which is initially at rest in these coordinates, remains at rest throughout the entire passage of the gravitational wave [3, 16]. Here, the use of words "at rest" requires clarification: they only mean that the coordinates of the test mass do not change in the presence of

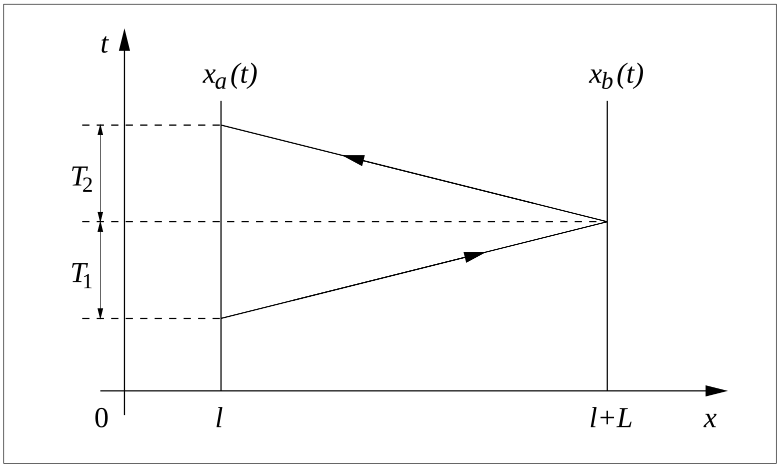

Figure 1: Bouncing photon in the coordinates of the TT-gauge.

the gravitational wave. The proper distance between any two test masses changes even though their coordinates remain the same. A convenient way to analyze variations in the proper distance is by means of "bouncing photons" [17]. For example, a photon can be launched from one test mass to be bounced back by the other, as shown in Fig. 1. For simplicity we assume that the test masses are located along the x-axis of the coordinate system. In this case, the interval takes the form:

$$ds^2 = -c^2\, dt^2 + [1 + h(t)]\, dx^2. \tag{2}$$

The condition for a null trajectory ($ds = 0$) gives us the coordinate velocity of the photon:

$$v^2 \equiv \left(\frac{dx}{dt}\right)^2 = \frac{c^2}{1 + h(t)}, \tag{3}$$

which is a convenient quantity for calculations of the photon propagation times between the test masses. As we know, the coordinates of the test masses, $x_a = l$ and $x_b = l + L$, do not change under the influence of gravitational wave. Therefore, the duration of the forward trip can be found as

$$T_1(t) = \int\limits_{l}^{l+L} \frac{dx}{v(t')}, \tag{4}$$

where $t' = t - (l + L - x)/c$. To first order in h, this integral can be approximated as

$$T_1(t) = T + \frac{1}{2c} \int\limits_{l}^{l+L} h(t')\, dx, \tag{5}$$

where $T = L/c$ is the light transit time in the absence of gravitational waves. Similarly, the duration of the return trip would be

$$T_2(t) = T + \frac{1}{2c} \int\limits_{l+L}^{l} h(t')\,(-dx), \tag{6}$$

though now the retardation time is given by $t' = t - (x - l)/c$.

The round trip time can then be found by adding $T_2(t)$ and $T_1[t - T_2(t)]$. The latter can be approximated by $T_1(t - T)$ because the difference between the exact and the approximate values is second order in h. Therefore, to first order in h, the duration of the round trip can be defined as

$$T_{\text{r.t.}}(t) = T_1(t - T) + T_2(t). \tag{7}$$

Deviations of this round-trip time from its unperturbed value, $2T$, are then given by

$$\delta T(t) = \frac{1}{2c} \int\limits_{l}^{l+L} \left[h\left(t - 2T + \frac{x - l}{c}\right) + h\left(t - \frac{x - l}{c}\right) \right] dx. \tag{8}$$

Even though l explicitly enters this equation, δT does not depend on l. This observation implies that the choice of the origin for this coordinate system does not affect δT; in other words, the result is translationally invariant.

The deviation in the round-trip time, Eq.(8), can also be written in the Laplace or Fourier domain. Laplace transformations are commonly used to analyze linear responses of interferometric gravitational-wave detectors [18], and sometimes are easier to interpret than their time-domain equivalents. Define the Laplace transform of an arbitrary function of time $h(t)$ by

$$\tilde{h}(s) = \int\limits_{0}^{\infty} e^{-st}\, h(t)\; dt. \tag{9}$$

Then the Laplace-domain version of Eq.(8) can be written as

$$\frac{\delta \tilde{T}(s)}{T} = C(s)\, \tilde{h}(s), \tag{10}$$

where $C(s)$ represents the response of test masses to gravitational waves:

$$C(s) = \frac{1 - e^{-2sT}}{2sT}. \tag{11}$$

A number of derivations of this result, some quite different from ours, can be found in literature, for example in [19, 20, 21, 22] and more recently in [23, 18].

There are several reasons why the above picture is not satisfactory from a physical point of view, even though it is mathematically sound. The main problem with the coordinates of the TT-gauge is that they can hardly be realized in the experiment. In fact, they cannot be implemented in the laboratory environment on Earth because the coordinate grid of the TT-frame must be changing in unison with the passing gravitational wave, the effect commonly known as "breathing of the frame." (They may, however, be realized in space with a network of freely-falling satellites.) Consequently, the application of the above calculations to ground-based gravitational-wave detectors becomes problematic. For physicists working with these detectors, it is sometimes not clear how the results derived in the TT-coordinates can be used in experiments when these coordinates are not available in practice.

Another problem, which is closely related to the previous one, comes from the assumption of test mass inertiality. Namely, the above derivation of the photon round-trip time was based on the premise that the coordinates of the test masses do not change under the influence of the gravitational wave, an assumption which is true only when the test masses are inertial. The test masses in laser gravitational-wave detectors constantly undergo accelerations in response to various forces acting on them and thus are never truly inertial. One can argue that these accelerations typically occur at frequencies of the suspension resonances which are well below the frequencies of the anticipated gravitational waves. However, in advanced interferometer configurations the accelerations of test masses will also be caused by the radiation-pressure variations which are intended to occur at the frequency of anticipated gravitational waves. Therefore, the assumption of test mass inertiality, which is most effective in the TT-gauge, becomes too restrictive for more realistic calculations. These problems do not occur if one uses the coordinates of a local observer.

3 The Coordinates of Local Observer

An observer in a laboratory environment on Earth typically uses the coordinate system in which the space-time is locally flat [17], and the distance between any two points is given simply by the difference in their coordinates in the usual sense of Newtonian physics [3]. In this reference frame, gravitational waves manifest themselves through the tidal forces which they exert on the masses [7]. To describe the tidal forces we consider a test mass which is free to move in the horizontal plane ($z = 0$). For simplicity, we assume that this plane coincides with the wavefront of the gravitational wave, and that the x and y directions of the coordinate system match the polarization of the gravitational wave. Then the tidal acceleration of the test mass caused by the gravitational wave [3] is given by

$$\ddot{x} = +\frac{1}{2}\,\ddot{h}\,x, \tag{12}$$

$$\ddot{y} = -\frac{1}{2}\,\ddot{h}\,y. \tag{13}$$

Equivalently [8, 12, 24], one can say that there is a gravitational potential:

$$\Phi(\mathbf{r}, t) = -\frac{1}{4}\,\ddot{h}(t)(x^2 - y^2), \tag{14}$$

which generates the tidal forces, and that the motion of the test mass is governed by the Newton equation:

$$\ddot{\mathbf{r}} = -\nabla\Phi. \tag{15}$$

The potential is not static, and therefore the energy of the test mass is not conserved. This is the Newtonian version of the theorem from general relativity which states that gravitational waves must supply energy to test masses to become detectable in experiments [25].

In the post-Newtonian approach to general relativity (see also Appendix A), the gravitational potential is related to the time component of the metric:

$$g_{00} = -1 - \frac{2}{c^2}\,\Phi. \tag{16}$$

In what follows we will frequently use a perturbation expansion, keeping only terms which are first order in h, and therefore rely on the assumption that $|\Phi|/c^2 \ll 1$. To satisfy this condition, we require that the spatial coordinates x and y do not extend indefinitely. This limitation, however, will not restrict us in any way. Indeed, for gravitational waves with the largest expected amplitudes ($|h| \sim 10^{-21}$) and the highest detectable frequencies ($\sim$ 10 kHz), the restriction on the spatial coordinates implies that $|x|, |y| \ll 10^{14}$ m, which is always satisfied in the laboratory environment on Earth.

The solution to Eqs.(12)–(13) is usually found using the perturbation method [3]. To first order in h, the displacements of the test mass caused by the gravitational waves are given by

$$\delta x(t) = +\frac{1}{2}\,x_0\;h(t), \tag{17}$$

$$\delta y(t) = -\frac{1}{2}\,y_0\;h(t), \tag{18}$$

where x_0 and y_0 are the initial (unperturbed) coordinates of the test mass. This notion is regarded as the major difference between the coordinates of a local observer and the coordinates of the TT-gauge, in which the test masses were not moving under the influence of the gravitational wave.

4 Requirement of Translational Invariance

An interesting feature of the local Lorentz gauge is the coordinate dependence of the tidal forces – they can be changed by a mere shift of the origin of the coordinate system:

$$x \to x + X, \qquad \text{and} \qquad y \to y + Y. \tag{19}$$

The same applies to the test mass displacements, Eqs.(17) and (18). This is the earliest indication that the coordinates of a local observer are not as simple as they may seem. However, at this point, the coordinate dependence seems to be quite harmless, and we can entertain the notion that it can be removed simply by considering the relative motion of test masses.

As before, we probe the geometry of space-time with a bouncing photon. Consider two test masses with coordinates x_a and x_b and assume that the photon is launched from one test mass and is bounced by the other. Let the unperturbed values for the test mass coordinates be

$$x_a = l, \qquad \text{and} \qquad x_b = l + L, \tag{20}$$

and the unperturbed propagation time between the masses be

$$T = \frac{L}{c}. \tag{21}$$

From Eq.(17) we find that the displacements of the test masses under the influence of the gravitational wave are

$$\delta x_a(t) = \frac{1}{2}\, l\, h(t), \tag{22}$$

$$\delta x_b(t) = \frac{1}{2}\, (l + L)\, h(t). \tag{23}$$

Then the relative displacement, commonly defined as

$$\begin{aligned} \delta L(t) &= \delta x_b(t) - \delta x_a(t) \\ &= \frac{1}{2}\, L\, h(t), \end{aligned} \tag{24}$$

would obviously be independent of l and therefore independent of the choice of the origin for these coordinates, as we anticipated. Equation (24), often written as

$$\frac{\delta L(t)}{L} = \frac{1}{2}\, h(t), \tag{25}$$

is widely used to describe the strain induced by gravitational waves on bar detectors. However, its application to laser interferometers immediately runs into a problem. Namely, the change in the round-trip time calculated from Eq.(25) would be

$$\frac{\delta \tilde{T}(s)}{T} = \tilde{h}(s), \tag{26}$$

which is different from the one obtained in the TT-gauge, Eq.(10). This is the precise origin of the well-known discrepancy between the two coordinate systems. One of the earliest accounts of this discrepancy appears in Exercise 37.6 of [3], which also suggests that the correct answer for the photon propagation time must be obtained in the coordinates of the TT-gauge. It is sometimes assumed that the discrepancy occurred because of the application of Eqs.(12)–(13) beyond their limits of validity. The actual cause of the discrepancy lies in the neglect of the effects of gravitational redshift, as will be shown below.

Historically, the discrepancy was not viewed as a serious problem when the searches for gravitational waves were conducted with bar detectors. The relatively small size of a bar detector (a few meters) implies small separation for its constituent parts, in which case the difference between the two coordinate systems becomes negligible. Indeed, for gravitational waves with wavelength much greater than the separation between the test masses, $|sT| \ll 1$ and therefore $C(s) \approx 1$, which makes Eq.(10) equivalent to Eq.(26). The situation became rather different with the arrival of long-baseline laser interferometers. In these detectors the test masses for sensing gravitational waves are separated by distances of several kilometers, and the long-wavelength approximation, $|sT| \ll 1$, becomes hard to justify. Furthermore, recent studies [26] have shown that these interferometers are capable of detecting gravitational waves with wavelengths comparable to their arm-length, $|sT| \sim 1$, thus operating entirely outside the long-wavelength regime.

5 Requirement of Causality

For large separation between the test masses, the definition for relative displacement, Eq.(24), becomes unphysical. In this definition the two test masses are taken at the same time and therefore cannot be in causal connection. The definitions for the relative test-mass displacement which are appropriate for the bouncing photon can be written as

$$\begin{aligned} \delta L_1(t) &= \delta x_b(t) - \delta x_a(t - T_1), && (27) \\ \delta L_2(t) &= \delta x_b(t - T_2) - \delta x_a(t), && (28) \end{aligned}$$

where T_1 and T_2 are the photon propagation times for the forward and return trip correspondingly. According to these definitions, the displacement of one test mass is compared with the displacement of the other at a later time to allow for finite delay from the light propagation, as can be seen from Fig. 2. Note that the propagation times T_1 and T_2 in Eqs.(27)-(28) can be replaced with their nominal value T because the test mass displacements are already first order in h.

The total change in the distance between the masses in one round-trip of light would be

$$\begin{aligned} \delta L_{\text{r.t.}}(t) &= \delta L_1(t - T) + \delta L_2(t) \\ &= 2\,\delta x_b(t - T) - \delta x_a(t) - \delta x_a(t - 2T). && (29) \end{aligned}$$

An explicit formula for this length change written in terms of the amplitude of the gravitational wave is

$$\delta L_{\text{r.t.}}(t) = (l + L)h(t - T) - \frac{1}{2}\,l\,h(t) - \frac{1}{2}\,l\,h(t - 2T). \tag{30}$$

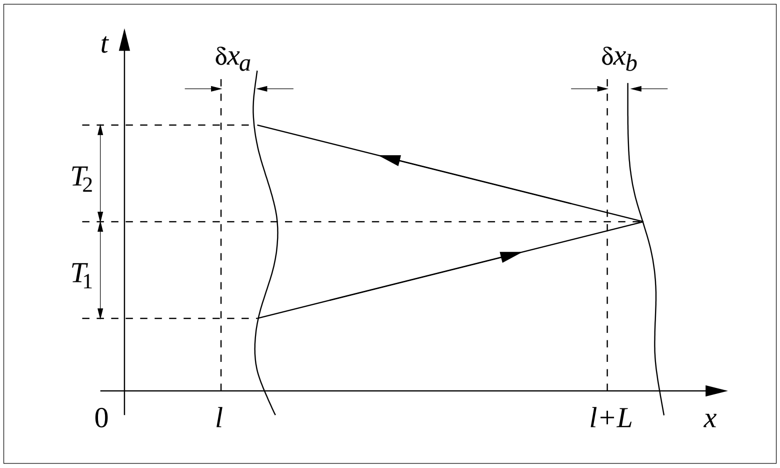

Figure 2: Bouncing photon in the coordinates of a local observer.

This quantity represents causal length variations in laser interferometers for gravitational wave detection. Note that $\delta L_{\rm r.t.}$ is not translationally invariant, despite the fact that it represents the relative displacement of the test masses. This is the price one has to pay for satisfying the causality condition.

Changes in the distance, Eq.(30), lead to changes in the round-trip time for photons propagating between the masses:

$$\frac{\delta_x T(t)}{T} = h(t-T) - \mu\left[h(t) - 2h(t-T) + h(t-2T)\right], \tag{31}$$

where we introduced a dimensionless parameter

$$\mu = \frac{l}{2L}. \tag{32}$$

The presence of this parameter in the subsequent formulas will indicate the loss of translational invariance. The Laplace-domain version of Eq.(31) can be written in a manner similar to Eq.(10), namely

$$\frac{\delta_x \tilde{T}(s)}{T} = D_x(s)\,\tilde{h}(s). \tag{33}$$

where $D_x(s)$ is the corresponding response function

$$D_x(s) = e^{-sT} - \mu\left(1 - e^{-sT}\right)^2. \tag{34}$$

Note that $D_x(s)$ depends on the choice of the origin for this coordinate system. At first it may seem that this loss of translational invariance is natural. After all, the potential explicitly depends on coordinates, which in classical mechanics usually means that the symmetry with respect to translations is lost. However, such a conclusion would contradict our physical intuition which maintains that all locations on the wavefront of the plane gravitational wave must be equivalent. This implies that physical quantities must be the same no matter where on this plane they are measured, even though the potential explicitly discriminates between different locations on the plane. We will see shortly that this is indeed the case and that translational invariance is restored, but only when another significant effect is added to the picture: the gravitational redshift of light propagating between the masses.

6 Distributed Gravitational Redshift

We have calculated variations in the photon round-trip time which come from the motion of the test masses induced by the gravitational wave. In this calculation, we implicitly assumed that the propagation of the photon between the test masses is uniform, as if it were moving in flat space-time. However, the presence of the tidal forces indicates that space-time is curved. As a result, the bouncing photon will experience a gravitational redshift. There will be two such effects in the following calculations. The first will be called the distributed gravitational redshift because it requires spacial separation, the second will be called the localized gravitational redshift because it occurs at a single point in space.

The distributed gravitational redshift can be calculated as follows. Consider the interval for photons propagating along the x-axis:

$$ds^2 = g_{00}\, c^2\, dt^2 + dx^2, \tag{35}$$

where g_{00} is the time component of the metric, Eq.(16). The condition for a null trajectory ($ds = 0$) gives us the coordinate velocity of the photons:

$$v^2 \equiv \left(\frac{dx}{dt}\right)^2 = c^2 + 2\Phi(t, x). \tag{36}$$

To first order in h, the velocity can be approximated by

$$v \approx \pm\, c \left[1 + \frac{1}{c^2}\Phi(t, x)\right], \tag{37}$$

where $+$ and $-$ correspond to the forward and return trip, respectively.

Knowing the coordinate velocity of the photons, we can define the propagation time for the photon traveling between the masses:

$$T_1(t) = \int\limits_{x_a(t-T_1)}^{x_b(t)} \frac{dx}{v}, \qquad \text{and} \qquad T_2(t) = \int\limits_{x_b(t-T_2)}^{x_a(t)} \frac{(-dx)}{v}, \tag{38}$$

in accordance with Fig. 2. We will not attempt to calculate these integrals directly. Such calculations would be complicated because the boundaries of these integrals are changing with time:

$$
\begin{aligned}
x_a(t) &= l + \delta x_a(t), &(39)\\
x_b(t) &= l + L + \delta x_b(t). &(40)
\end{aligned}
$$

Fortunately, we do not need to calculate the contributions of the boundary terms to these integrals. To first order in h, these contributions can be approximated by $\delta L_1(t)/c$ and $\delta L_2(t)/c$ (see Eqs.(27) and (28)). Therefore, the combined effect of the varying boundaries is given by $\delta_x T$, previously found in Eq.(31). Thus, we only calculate the times for photon propagation between the fixed boundaries: l and $l+L$. Such propagation times will be denoted here by $\Delta T_{1,2}$ to be distinguished from $T_{1,2}$.

In the forward trip, the propagation time between the fixed boundaries is

$$
\begin{aligned}
\Delta T_1(t) &= \int_l^{l+L} \frac{dx}{v(t',x)} &(41)\\
&\approx T - \frac{1}{c^3}\int_l^{l+L} \Phi(t',x)\,dx, &(42)
\end{aligned}
$$

where t' is the retardation time which corresponds to the unperturbed photon trajectory: $t' = t - (l+L-x)/c$. Similarly, the propagation time between the fixed boundaries in the return trip is

$$
\Delta T_2(t) = T - \frac{1}{c^3}\int_{l+L}^{l} \Phi(t',x)\,(-dx), \tag{43}
$$

though now the retardation time is given by $t' = t - (x-l)/c$. The round-trip time for photons traveling between the fixed boundaries can be found by adding $\Delta T_2(t)$ and $\Delta T_1(t-T)$. Deviations of this round-trip time from its unperturbed value, $2T$, are given by

$$
\delta_v T(t) = -\frac{1}{c^3}\int_l^{l+L}\left[\Phi\left(t-2T+\frac{x-l}{c},x\right)+\Phi\left(t-\frac{x-l}{c},x\right)\right]dx. \tag{44}
$$

After replacing the potential with its explicit form, Eq.(14), we obtain the formula for $\delta_v T$ in terms of the amplitude of the gravitational wave:

$$
\delta_v T(t) = \frac{1}{4c^3}\int_l^{l+L}\left[\ddot{h}\left(t-2T+\frac{x-l}{c}\right)+\ddot{h}\left(t-\frac{x-l}{c}\right)\right]x^2\,dx. \tag{45}
$$

This quantity represents the effect of the distributed gravitational redshift [6].

Equation (45) bears close similarity with Eq.(8), as both formulas represent cumulative effects of the gravitational wave. However, unlike Eq.(8), which is translationally invariant, Eq.(45) is not, as can be seen from the presence of x^2-factor in the integrand. A better way to analyze the loss of translational invariance would be to rewrite the result in the Laplace domain:

$$\frac{\delta_v \tilde{T}(s)}{T} = D_v(s)\,\tilde{h}(s), \tag{46}$$

where $D_v(s)$ is the corresponding response function:

$$\begin{aligned} D_v(s) &= \frac{1}{2sT}\left(1-e^{-2sT}\right) - e^{-sT} + \\ &\quad \mu\left(1-e^{-sT}\right)^2 + \mu^2\left(1-e^{-2sT}\right) sT. \end{aligned} \tag{47}$$

The terms proportional to μ and μ^2 represent the dependence of the response function on the choice of the origin for this coordinate system.

We can now combine the variations in the photon propagation time which are caused by the motion of the test masses with those caused by the distributed gravitational redshift. The resulting round-trip time would be

$$T_{\text{r.t.}} = 2T + \delta_x T + \delta_v T. \tag{48}$$

To this point, the combined effect of the gravitational wave is given by the sum:

$$D_x(s) + D_v(s) = \left(\frac{1}{2sT} + \mu^2 sT\right)\left(1-e^{-2sT}\right). \tag{49}$$

By adding the two response functions we cancel the terms proportional to μ. However, the term proportional to μ^2 remains. As will be shown next, this term is related to the localized gravitational redshift.

7 Localized Gravitational Redshift

The last contribution to the photon round-trip time is also related to the gravitational redshift, although it is somewhat different from the distributed effect described above. In the presence of gravitational waves the clocks at different places run differently. The rate (dt^*) of the clock which is located at x is related to the rate (dt) of the clock at the origin by

$$dt^{*2} = -g_{00}(t,x)\; dt^2, \tag{50}$$

which is the proper time at this location. In the above calculation of the photon round-trip time, Eq.(48), we implicitly assumed that the time is measured with the clock at the origin: $x = 0$. The photon trajectories, however, begin and end at the location of the first test mass, a finite distance (l) away from the origin. As a result, the readings of time become dependent on this distance. To avoid this problem, we shall measure time with the clock located at $x = l$. For this clock, the round-trip time is different from $T_{\rm r.t.}$, Eq.(48). The presence of the time-dependent gravitational potential affects the reading of this clock, causing it to register the round-trip time as

$$\begin{aligned} T^*_{\rm r.t.}(t) &= \int\limits_{t-T_{\rm r.t.}}^{t} \sqrt{-g_{00}(t', l)}\, dt' \\ &\approx T_{\rm r.t.}(t) + \frac{1}{c^2} \int\limits_{t-T_{\rm r.t.}}^{t} \Phi(t', l)\, dt'. \end{aligned} \tag{51}$$

To first order in h, the variation of the round-trip time due to this effect can be estimated as

$$\delta_t T(t) \approx \frac{1}{c^2} \int\limits_{t-2T}^{t} \Phi(t', l)\, dt' \tag{52}$$

$$= -\frac{l^2}{4c^2} \left[\dot{h}(t) - \dot{h}(t-2T)\right]. \tag{53}$$

This contribution to the round-trip propagation time comes from the non-uniformity of time flow caused by the presence of the gravitational wave. It will be called here the localized gravitational redshift. In the Laplace domain it can be written as

$$\frac{\delta_t \tilde{T}(s)}{T} = D_t(s)\, \tilde{h}(s), \tag{54}$$

where $D_t(s)$ is the corresponding response function

$$D_t(s) = -\mu^2 \left(1 - e^{-2sT}\right) sT. \tag{55}$$

Addition of this response function to Eq.(49) will cancel the μ^2-terms, giving us a translationally invariant result.

We can now conclude that the change in the round-trip time caused by the gravitational wave consists of three contributions:

$$\delta T = \delta_x T + \delta_v T + \delta_t T, \tag{56}$$

which come from displacement of the test masses, changes in the coordinate velocity of the photons and variations in the clock rate. The combined result of these effects is given by the sum:

$$D_x(s) + D_v(s) + D_t(s) = \frac{1 - e^{-2sT}}{2sT}, \tag{57}$$

which is translationally invariant. Furthermore, the sum gives us a response function which is identical to $C(s)$, Eq. (11), which proves that the two coordinate systems indeed yield the same answer for the observable photon round-trip time.

The requirement of translational invariance played a special role in the above analysis. The coordinate transformations, Eq.(19), are a particular case of transformations known as changes of the origin, which in general relativity are accomplished with the help of parallel transports [17]. Following the Newtonian style of our presentation, we referred to these transformations as *translations* and assumed that they represent a symmetry. The origin of this symmetry is related to the planeness of the gravitational wave [25].

8 The Round-Trip Phase of Light

In the above picture, we considered the bouncing photon as a particle, assuming that there is a beginning and an end for the photon round trips. In practice, measurements of photon propagation times are usually done with optical interferometry in which photons are represented by continuous electromagnetic waves. We shall therefore briefly describe how the above calculations can be modified to become applicable to continuous waves. To be specific, we assume that the light is represented by a plane monochromatic wave with frequency ω and wavenumber k. In the absence of gravitational waves, such a wave is given by $\exp[i(\omega t \mp kx)]$. Then the photon trajectory introduced above would describe advancement of a surface of constant phase, whereas the photon velocity becomes the phase velocity of the wave. In this approach, the quantity of interest would be the round-trip phase, or more precisely, its variation caused by the gravitational wave.

The first contribution to the round-trip phase comes from the motion of the test masses:

$$\psi_x = -k\, \delta L_{\mathrm{r.t.}} = -\omega\ \delta_x T, \tag{58}$$

where $\delta L_{\mathrm{r.t.}}$ represents variations in the distance between the test masses, Eq.(30), and $\delta_x T$ represents the corresponding time variations, Eq.(31). The second contribution comes from the variations in the phase velocity of the wave:

$$\psi_k = \frac{k}{c^2} \int\limits_{\mathcal{C}} \Phi\, dx. \tag{59}$$

Here we give a brief derivation of this result based on simple physical arguments. (Another derivation, based on the solution of the eikonal equation, is given in Appendix B.)

In the presence of gravitational waves, the frequency and wavenumber are no longer constant; they become functions of position and time: $\Omega(x,t)$ and $K(x,t)$. Then the dispersion relation for the electromagnetic wave would read

$$\Omega^2 = v^2\, K^2, \tag{60}$$

where v is the phase velocity of the light, defined in Eq.(36). To first order in h, Eq.(60) can be written as

$$\Omega - c\,K = \frac{k}{c}\,\Phi. \tag{61}$$

For a plane electromagnetic wave moving in the positive x-direction, an infinitesimal phase change is given by $(\Omega\,dt - K\,dx)$. Then the accumulated phase change can be found by integrating this quantity along the trajectory of a given wavefront. In doing so, we would find that the accumulated phase change vanishes because $dx/dt = \Omega/K$. This result is quite natural, as traveling with the wavefront means following the surface of constant phase for which no phase change ensues. However, we must remember that we are not interested in the absolute phase change along the photon trajectory. Rather, we are interested in the variation of this phase change with respect to the unperturbed wave. Such a phase variation would be given by

$$\psi_k = \int_{\mathcal{C}} (\Omega\,dt - K\,dx), \tag{62}$$

where $\mathcal{C}$ denotes the unperturbed trajectory: $dx/dt = \pm c$. Taken along the unperturbed photon trajectory, such an integral would yield a non-zero answer, which is equivalent to Eq.(59). Note that the integral over the photon trajectory, Eq.(59), has already been calculated, see Eq.(44). Therefore, the phase change can then be written as

$$\psi_k = -\omega\;\delta_v T, \tag{63}$$

where $\delta_v T$ is the corresponding variation in the round-trip time.

We now can add this phase change to the phase change produced by the motion of the test masses, Eq.(58). There is no need to worry about the difference between k and K for this part. The displacements of the test masses are first order in h, and therefore any correction to k would result in second order terms. Thus, the combined effect is given by

$$\psi = \psi_x + \psi_k = -\omega\,(\delta_x T + \delta_v T). \tag{64}$$

As we already know from Eqs.(48) and (49), this quantity is not translationally invariant, which means that it cannot be observed in the experiment. The change in the phase shift ψ represents the difference of the round-trip phases for two different electromagnetic waves: with and without the gravitational wave. Such a phase change cannot be measured in the experiment as the two waves cannot exist in the same space-time. To form an observable quantity, we shall compare the phase change of the probe electromagnetic wave with that of a reference wave. The natural reference is the source itself, and therefore we need to find the phase change of the source.

In flat space-time, the phase of the source would simply be ωt, and the phase shift of the source $2\omega T$. In the presence of gravitational waves, the phase of the source becomes ωt^*, where t^* is the proper time at the location of the source. Then the phase shift of the source can be found as

$$\omega\,[t^*(t) - t^*(t - T_{\mathrm{r.t.}})] = \omega\,T^*_{\mathrm{r.t.}}(t). \tag{65}$$

Therefore, the change in this phase shift which is produced by the gravitational wave is

$$\psi_{\rm so} = \omega \ \delta_t T, \tag{66}$$

where $\delta_t T$ is given by Eq.(52).

We can now compare the phase change of the moving wave, Eq.(64), with that of a static source, Eq.(66). The difference between the phase change of the wavefront for the electromagnetic wave returning to the source and the phase change of the source at that moment is

$$\delta\psi = \psi - \psi_{\rm so}. \tag{67}$$

In the explicit form this phase difference is given by

$$\delta\psi = -\omega(\delta_x T + \delta_v T + \delta_t T). \tag{68}$$

As we already know from Eqs.(56) and (57), this phase is translationally invariant and therefore represents an observable quantity. It is not surprising that this phase deviation is related to the time deviation by a simple formula:

$$\delta\psi = -\omega \ \delta T. \tag{69}$$

This result could have been guessed from simple dimensional analysis and the requirement of translational invariance. The above derivation serves to explain the physical meaning of the relative phase shift and its constituent parts. In short, the motion of the test masses and the distributed gravitational redshift appear now as the phase shift of the traveling wave, whereas the localized gravitational redshift appears as the phase shift of the static source.

9 Concluding Remarks

We have shown that the response of test masses to gravitational waves in the local Lorentz gauge acquires contributions from three different effects: the motion of the test masses and the distributed and localized gravitational redshifts. Only when taken together do these effects yield an observable quantity. The approach developed in this paper has allowed us to calculate physical quantities directly in the coordinates of the local observer. In these coordinates, the assumption of the test mass inertiality is not required, and various forces acting on the masses can be added at will. We have provided a consistent framework for doing calculations in the coordinate system which is more natural for ground-based laser gravitational-wave detectors than the TT-gauge.

To simplify the calculations, we introduced the three effects of the gravitational wave in a step-by-step fashion. At each step, mathematical derivations took advantage of the previous step. In retrospect, it is clear that a more direct way of doing the calculations would be to start with an abstract definition:

$$T^*_{\rm r.t.} = \int dt^*, \tag{70}$$

and then to proceed with integration over the photon trajectory

$$T^*_{\text{r.t.}} = \int \sqrt{-g_{00}(l,t')}\, dt' = \int\limits_{\mathcal{C}^*} \sqrt{-g_{00}(l,t')}\, \frac{dx}{v(x,t')}. \tag{71}$$

In this approach the contour $\mathcal{C}^*$ would represent the actual photon trajectory: $dx/dt = \pm v(x,t)$ which extends to the actual test mass positions: $l + \delta x_a$ and $l + L + \delta x_b$. By evaluating various terms in the contour integral to first order in h, one would reproduce the above three contributions to the round-trip time. Although this approach may seem different from the one described in this paper, the mathematical equations and their physical interpretations would be essentially the same.

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A Coordinate and Metric Transformations

For completeness, we present here the transformations from the TT-gauge to the gauge of a local observer. Denote the coordinates of a local observer by x^μ and the metric in these coordinates by $g_{\mu\nu}$. Also, denote the coordinates of the TT-gauge by $\bar{x}^\mu$ and the corresponding metric by $\bar{g}_{\alpha\beta}$. The components of the metric in the TT-gauge, Eq.(1), can be written as

$$\bar{g}_{\mu\nu} = \eta_{\mu\nu} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h & 0 & 0 \\ 0 & 0 & -h & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{72}$$

where $\eta_{\mu\nu} = \mathrm{diag}\{-1, 1, 1, 1\}$ is the Minkowski metric, and $h = h(\bar{t} + \bar{z}/c)$. From general relativity we know that the coordinate transformations, $\bar{x}^\mu \to x^\mu$, induce the transformations of the metric:

$$g_{\mu\nu} = \frac{\partial \bar{x}^\alpha}{\partial x^\mu} \frac{\partial \bar{x}^\beta}{\partial x^\nu} \; \bar{g}_{\alpha\beta}. \tag{73}$$

By definition, $g_{\mu\nu}$ must become the Minkowski metric at the origin, and all its derivatives must vanish at this point. There are a number of metrics which satisfy these conditions. Here we consider one such choice [8, 12, 9]. It can be obtained with the coordinate transformations, which to first order in h, are given by

$$\bar{t} = t - \frac{1}{4c^2} \dot{h} \, (x^2 - y^2), \tag{74}$$

$$\bar{x} = x - \frac{1}{2} h \, x, \tag{75}$$

$$\bar{y} = y + \frac{1}{2} h \, y, \tag{76}$$

$$\bar{z} = z + \frac{1}{4c} \dot{h} \, (x^2 - y^2). \tag{77}$$

The corresponding metric tensor can be obtained by performing the induced transformation, Eq.(73). To first order in h, the result is

$$g_{\mu\nu} = \eta_{\mu\nu} - \frac{2}{c^2} \begin{pmatrix} \Phi & 0 & 0 & \Phi \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \Phi & 0 & 0 & \Phi \end{pmatrix}, \tag{78}$$

where Φ is a function of the new coordinates:

$$\Phi = -\frac{1}{4} \ddot{h}(t + z/c) \, (x^2 - y^2). \tag{79}$$

In the post-Newtonian approach, this function becomes the potential for the tidal forces, which for $z = 0$ appeared in Eq.(14).

It is interesting to note that although the transformation rules Eqs.(74)-(77) are approximate, the metric Eq.(78) is an exact solution of Einstein equations [27]. This metric is generally known as the plane-front solution for strong gravitational waves [28, 29]. Further discussion of the relationship between the metric of the local observer and the exact solution can be found in [24].

The propagation of an electromagnetic wave in curved space-time is described by the eikonal Ψ [30], which satisfies the equation:

$$g^{\mu\nu} \frac{\partial \Psi}{\partial x^\mu} \frac{\partial \Psi}{\partial x^\nu} = 0, \tag{80}$$

where $g^{\mu\nu}$ stands for the contravariant metric tensor. Its components are given by

$$g^{\mu\nu} = \eta^{\mu\nu} - \frac{2}{c^2} \begin{pmatrix} -\Phi & 0 & 0 & \Phi \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \Phi & 0 & 0 & -\Phi \end{pmatrix}. \tag{81}$$

For light propagating along the x-axis, the eikonal equation becomes

$$\left(1 - \frac{2}{c^2}\,\Phi\right) \left(\frac{\partial \Psi}{\partial ct}\right)^2 = \left(\frac{\partial \Psi}{\partial x}\right)^2. \tag{82}$$

The solution of this equation is described next.

B Solution to the Eikonal Equation

The eikonal equation can reduced to a linear differential equation by taking the square root of both sides of Eq.(82) and by keeping only the terms which are first order in h:

$$\left(\frac{\partial}{\partial ct} \pm \frac{\partial}{\partial x}\right) \Psi = \frac{1}{c^2}\,\Phi\,\left(\frac{\partial \Psi}{\partial ct}\right), \tag{83}$$

where $\pm$ correspond to the wave propagation in the positive and negative x-directions. The large unperturbed value of the eikonal satisfies Eq. (83) in the absence of the gravitational waves ($\Phi = 0$), and is given by $\omega t \mp kx$ up to an additive constant. Therefore, to first order in h, the solution of the eikonal equation can be found as

$$\Psi_1 = \omega t - kx + kl + \delta\Psi_1, \tag{84}$$

$$\Psi_2 = \omega t + kx - k(l + 2L) + \delta\Psi_2, \tag{85}$$

where $\delta\Psi_{1,2}$ are the first order perturbations. For convenience we introduce the light-cone coordinates:

$$\xi = (ct + x)/2, \tag{86}$$

$$\eta = (ct - x)/2, \tag{87}$$

in which the photon world-lines become collinear with the ξ and η axes, as shown in Fig. 3. In these coordinates, the first order perturbations satisfy the equations:

$$\frac{\partial}{\partial \xi}\,\delta\Psi_1 = \frac{k}{c^2}\,\Phi, \tag{88}$$

$$\frac{\partial}{\partial \eta}\,\delta\Psi_2 = \frac{k}{c^2}\,\Phi. \tag{89}$$

These equations allow direct integration:

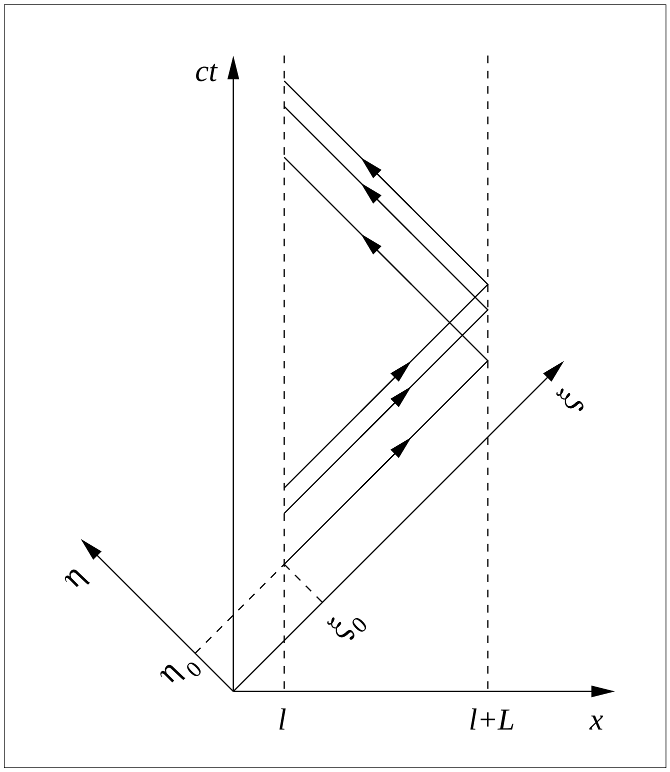

Figure 3: Propagation of the electromagnetic waves in the light cone coordinates. The source is located at $x = l$ and the turning point at $x = l + L$.

$$\delta\Psi_1(\xi,\eta) = \frac{k}{c^2}\int\limits_{\xi_0}^{\xi} \Phi(\xi',\eta)\, d\xi' + f_1(\eta), \tag{90}$$

$$\delta\Psi_2(\xi,\eta) = \frac{k}{c^2}\int\limits_{\eta_0}^{\eta} \Phi(\xi,\eta')\, d\eta' + f_2(\xi), \tag{91}$$

where $f_1(\eta)$ and $f_2(\xi)$ are arbitrary at this point. Transforming back to the coordinates x and t, we obtain the solution:

$$\delta\Psi_1(x,t) = \frac{k}{c^2}\int\limits_{l}^{x} \Phi\left(x', t - \frac{x-x'}{c}\right) dx' + f_1(x,t), \tag{92}$$

$$\delta\Psi_2(x,t) = \frac{k}{c^2}\int\limits_{x}^{l+L} \Phi\left(x', t + \frac{x-x'}{c}\right) dx' + f_2(x,t). \tag{93}$$

The function f_1 is defined by the value of the eikonal at the source for the electromagnetic wave: $\Psi_1(l,t) = \omega\, t^*$. To first order in h, this value can be found as

$$\Psi_1(l,t) = \omega \int_0^t \sqrt{-g_{00}(l,t')}\, dt' \tag{94}$$

$$\approx \omega \int_0^t \left[1 + \frac{1}{c^2}\Phi(l,t')\right] dt', \tag{95}$$

which must be the same as $\omega t + f_1(l,t)$, according to the definition Eq.(84). We thus find the function f_1 at the location of the source. Knowing that f_1 is a function of $ct - x$, we can extend the values of f_1 from the source location to the entire xt-plane:

$$f_1(x,t) = \frac{k}{c} \int_0^{t-\frac{x-l}{c}} \Phi(l,t')\, dt'. \tag{96}$$

The function f_2 is defined by the value of the eikonal Ψ_1 at the turning point. The continuity of the eikonal implies that

$$\delta\Psi_1(l+L,t) = \delta\Psi_2(l+L,t). \tag{97}$$

From this condition we can find f_2 at the turning point. Knowing that f_2 is a function of $ct + x$, we can extend the values of f_2 from the turning point to the entire xt-plane:

$$\begin{aligned} f_2(x,t) = & \frac{k}{c} \int_0^{t-2T+\frac{x-l}{c}} \Phi(l,t')\, dt' + \\ & \frac{k}{c^2} \int_l^{l+L} \Phi\left(x', t - 2T + \frac{x + x' - 2l}{c}\right) dx'. \end{aligned} \tag{98}$$

The phase shift acquired by the electromagnetic wave in one round trip is given by the difference between the value of the eikonal at the beginning and the end of the propagation. To first order in h, this phase shift is given by

$$\psi_k(t) = \delta\Psi_2(l,t) - \delta\Psi_1(l, t-2T). \tag{99}$$

Simple algebra shows that this definition leads to

$$\begin{aligned} \psi_k(t) = & \frac{k}{c^2} \int_l^{l+L} \Phi\left(x, t - \frac{x-l}{c}\right) dx + \\ & \frac{k}{c^2} \int_l^{l+L} \Phi\left(x, t - 2T + \frac{x-l}{c}\right) dx, \end{aligned} \tag{100}$$

which is an extended form of Eq.(59).

Finally, we give explicit formulas for Ω and K in terms of the gravitational potential. These two quantities can be derived from the eikonal:

$$\Omega = \frac{\partial \Psi}{\partial t}, \qquad \text{and} \qquad K = \mp \frac{\partial \Psi}{\partial x}, \tag{101}$$

where $-$ corresponds to the forward trip and $+$ to the return trip. For example, in the forward propagation

$$\begin{aligned} \Omega(x,t) &= \omega + \frac{k}{c}\,\Gamma(x,t) + \frac{k}{c}\,\Phi\left(l, t - \frac{x-l}{c}\right), && (102)\\ K(x,t) &= k + \frac{k}{c^2}\,\Gamma(x,t) + \frac{k}{c^2}\,\Phi\left(l, t - \frac{x-l}{c}\right) - \frac{k}{c^2}\,\Phi(x,t), && (103) \end{aligned}$$

where Γ represents the non-stationary effect of the gravitational redshift:

$$\Gamma(x,t) = \frac{1}{c}\int_l^x \frac{\partial}{\partial t}\left[\Phi\left(x', t - \frac{x-x'}{c}\right)\right]\,dx'. \tag{104}$$

Note that K can also be written as

$$K(x,t) = \frac{1}{c}\,\Omega(x,t) - \frac{k}{c^2}\,\Phi(x,t), \tag{105}$$

which leads directly to the dispersion relation, Eq.(61).