Conditional Expectation and Independence Exercises

Exercise 3.1. Show the four following properties : for any random variables X and Y and for any real numbers a and b,

$$\begin{array}{ll} (E1) & \mathrm{E}^{\mathbb{P}}\left[aX+bY\right]=a\mathrm{E}^{\mathbb{P}}\left[X\right]+b\mathrm{E}^{\mathbb{P}}\left[Y\right]; \\ (E2) & \text{If } \forall\omega\in\Omega,\ X\left(\omega\right)\leq Y\left(\omega\right) \text{ then } \mathrm{E}^{\mathbb{P}}\left[X\right]\leq\mathrm{E}^{\mathbb{P}}\left[Y\right]; \\ (E3) & \text{In general, } \mathrm{E}^{\mathbb{P}}\left[XY\right]\neq\mathrm{E}^{\mathbb{P}}\left[X\right]\mathrm{E}^{\mathbb{P}}\left[Y\right]; \\ (E4) & \text{If } X \text{ and } Y \text{ are independent then } \mathrm{E}^{\mathbb{P}}\left[XY\right]=\mathrm{E}^{\mathbb{P}}\left[X\right]\mathrm{E}^{\mathbb{P}}\left[Y\right]. \end{array}$$

Exercise 3.2. Show that for any random variable X and any real numbers a and b,

$$\begin{array}{ll} (V1) & \mathrm{Var}^{\mathbb{P}}\left[X\right]\geq 0; \\ (V2) & \mathrm{Var}^{\mathbb{P}}\left[X\right]=\mathrm{E}^{\mathbb{P}}\left[X^2\right]-\left(\mathrm{E}^{\mathbb{P}}\left[X\right]\right)^2; \\ (V3) & \forall a\in\mathbb{R},\ \mathrm{Var}^{\mathbb{P}}\left[aX+b\right]=a^2\mathrm{Var}^{\mathbb{P}}\left[X\right]. \\ (V4) & \mathrm{Var}^{\mathbb{P}}\left[X+Y\right]=\mathrm{Var}^{\mathbb{P}}\left[X\right]+\mathrm{Var}^{\mathbb{P}}\left[Y\right]+2\mathrm{Cov}^{\mathbb{P}}\left[X,Y\right] \end{array}$$

Exercise 3.3. Show that for any random variables X and Y and for any real numbers a and b,

$$\begin{array}{ll} (C1) & \mathrm{Cov}^{\mathbb{P}}\left[X,Y\right]=\mathrm{E}^{\mathbb{P}}\left[XY\right]-\mathrm{E}^{\mathbb{P}}\left[X\right]\mathrm{E}^{\mathbb{P}}\left[Y\right]; \\ (C2) & \text{If } X \text{ and } Y \text{ are independent then } \mathrm{Cov}^{\mathbb{P}}\left[X,Y\right]=0; \\ (C3) & \forall a,b\in\mathbb{R},\ \mathrm{Cov}^{\mathbb{P}}\left[aX_1+bX_2;Y\right]=a\mathrm{Cov}^{\mathbb{P}}\left[X_1;Y\right]+b\mathrm{Cov}^{\mathbb{P}}\left[X_2;Y\right]. \end{array}$$

Exercise 3.4. Let us revisit a problem treated in chapter 2. Today is Monday and you have one dollar in your piggy bank. Starting tomorrow, every morning until Friday (inclusively), you toss a coin. If the result is tails you withdraw one dollar (if possible) from the piggy bank. If the result is heads you deposit one dollar in it. What is the conditional expectation of the amount of money in the bank Friday noon, given the information by Wednesday noon?

Exercise 3.5. Recall exercise 2.4 on the gambler's ruin. Calculate $E\left[X_4 | \mathcal{F}_0\right]$, $E\left[X_4 | \mathcal{F}_2\right]$, $E\left[Y_4 | \mathcal{G}_2\right]$ and $E\left[Y_2 | \mathcal{G}_4\right]$ for an arbitrary p and for the specific case $p = \frac{1}{2}$.

Exercise 3.6. Justify all your answers.
Let the stochastic processes $X = \{X_t : t \in \{0, 1, 2, 3, 4\}\}$ represent the price of a share throughout time. We consider a time interval of 6 months.

	$X_0(\omega)$	$X_1(\omega)$	$X_2(\omega)$	$X_3(\omega)$	$X_4(\omega)$	$\mathbb{P}(\omega)$
ω_1	11	13	15	16	16	0,06
ω_2	11	13	15	16	14	0,065
ω_3	11	13	15	15	16	0,06
ω_4	11	13	15	15	14	0,065
ω_5	11	13	15	14	16	0,06
ω_6	11	13	15	14	14	0,065
ω_7	11	13	12	13	14	0,0625
ω_8	11	13	12	13	18	0,08
ω_9	11	13	12	13	12	0,04
ω_{10}	11	13	12	13	11	0,0675
ω_{11}	11	11	12	13	14	0,0625
ω_{12}	11	11	12	13	12	0,0625
ω_{13}	11	11	12	12	12	0,0625
ω_{14}	11	11	11	14	20	0,0625
ω_{15}	11	11	11	14	12	0,0625
ω_{16}	11	11	11	12	12	0,0625

- **a)** What filtration is generated by the process X ?
- **b)** Give the conditional distribution of X_4, knowing X_2.
- **c)** Calculate the conditional expectation of X_4, knowing X_2.
- **d)** Give the conditional distribution of X_4, knowing $\mathcal{F}_2$.
- **e)** Calculate the conditional expectation of X_4, knowing $\mathcal{F}_2$

Problem 3.7. Let us remain in the context of problem 3.6. Two investors, call them A and B, buy a share today (we could say a hundred thousands shares, but it would only add a bunch of zeros to our calculations) which they can sell at the end of this year ($t = 2$) or at the end of the next year ($t = 4$). If an investor chooses to sell his share at time $t = 2$, he will then place the money in a bank account yielding 15% in interest per year (the interest is compounded annually at times $t = 2$ and $t = 4$). If instead he chooses not to sell the share at time $t = 2$, he will have to sell it at time $t = 4$.

Those two investors will be away for the next five years because they are doing a (possibly metaphorical, this is up to you) trip to Venus. Hence they both appoint prosecutors (talented and trustworthy) to manage the operations in their place. But they're out of luck: both prosecutors are unfamiliar with money issues (we said talented ?), therefore the two investors must give precise instructions today to their respective prosecutors on what to do in the next year, depending on how the share will have evolved during that period. It goes without saying that both investors aim at maximizing their profit. The prosecutor of investor A will check the price of the stock at time $t = 1$, but the prosecutor of investor B will not (he might not be the most suited candidate for this job, don't you think?). Said otherwise, observation times for prosecutor A are $t = 1$ and $t = 2$, while for prosecutor B the only observation time is $t = 2$.

a) What instructions does investor A give to his prosecutor ?

b) What instructions does investor B give to his prosecutor ?

To facilitate comparisons, use the holding period yield (HPY). The HPY of a share on a given time interval (say we buy the share at the beginning of period t_1 and sell it at the end of period t_2) is

$$R_{t_1,t_2} = \frac{X_{t_2}}{X_{t_1}}.$$

Problem 3.8. This problem uses notations defined in problems 3.6 and 3.7.

a) Let τ_A and τ_B represent the random times at which the shares are sold, respectively by prosecutors A and B. Are τ_A and τ_B stopping times ?

b) Say we replace the probability measure $\mathbb{P}$ by $\mathbb{Q}$ where $\mathbb{Q}(\omega) = \frac{1}{16}$ for every $\omega \in \Omega$. Are τ_A and τ_B stopping times in this case ?

Problem 3.9. Let X_1, X_2, X_3 be independent random variables such that $\mathbb{P}(X_i = 1) = \mathbb{P}(X_i = -1) = \frac{1}{2}$. Let us define $A_1 = \{X_2 = X_3\}$, $A_2 = \{X_1 = X_3\}$ and $A_3 = \{X_1 = X_2\}$. Show that the events A_1, A_2 and A_3 are pairwise independent, but not mutually independent.

Problem 3.10. Let X and Y be two independent random variables, both of standard normal distribution. Let ε be a random variable independent of Y and such that: $\mathbb{P}\{\varepsilon = 1\} = \mathbb{P}\{\varepsilon = -1\} = \frac{1}{2}$.

a) Show that $Z = \varepsilon Y$ is also normal but that $Y + Z$ is not.

b) Show that Y and Z are not independent, although $Cov(Z, Y) = 0$.

Problem 3.11. In the case X and Y are independent and identically distributed random variables, show that $\mathrm{E}[X - Y \,|X + Y] = 0$.

Solutions

1 Exercise 3.1

Proof of $(E1)\quad E[aX+bY]=aE[X]+bE[Y]$.

Since

$$\begin{aligned}
\sum_{y\in\mathcal{S}_Y} f_{X,Y}(x,y) &= \sum_{y\in\mathcal{S}_Y} \mathbb{P}[X=x \text{ and } Y=y] \\
&= \mathbb{P}\left[\bigcup_{y\in\mathcal{S}_Y}\{X=x \text{ and } Y=y\}\right] \\
&\quad \text{because those events are disjoint} \\
&= \mathbb{P}\left[\bigcup_{y\in\mathcal{S}_Y}\{\{X=x\}\cap\{Y=y\}\}\right] \\
&= \mathbb{P}\left[\{X=x\}\cap\left\{\bigcup_{y\in\mathcal{S}_Y}\{Y=y\}\right\}\right] \\
&= \mathbb{P}[\{X=x\}\cap\Omega]=\mathbb{P}[\{X=x\}]=f_X(x) \qquad (1)
\end{aligned}$$

hence

$$\begin{aligned}
E[aX+bY] &= \sum_{x\in\mathcal{S}_X}\sum_{y\in\mathcal{S}_Y}(ax+by)\,f_{X,Y}(x,y) \\
&= \sum_{x\in\mathcal{S}_X}\sum_{y\in\mathcal{S}_Y} ax f_{X,Y}(x,y)+\sum_{y\in\mathcal{S}_Y}\sum_{x\in\mathcal{S}_X} by f_{X,Y}(x,y) \\
&= a\sum_{x\in\mathcal{S}_X} x\sum_{y\in\mathcal{S}_Y} f_{X,Y}(x,y)+b\sum_{y\in\mathcal{S}_Y} y\sum_{x\in\mathcal{S}_X} f_{X,Y}(x,y) \\
&= a\sum_{x\in\mathcal{S}_X} x f_X(x)+b\sum_{y\in\mathcal{S}_Y} y f_Y(y) \\
&= aE[X]+bE[Y]. \ \blacksquare
\end{aligned}$$

Proof of $(E2)\quad X\le Y\Rightarrow E[X]\le E[Y]$.

To begin with, note that if W is a non negative random variable then $E[W] \geq 0$. Indeed,

$$E[W] = \sum_{x \in \mathcal{S}_W} \underbrace{x}_{\geq 0} \underbrace{f_W(x)}_{\geq 0} \geq 0.$$

Let $W = Y - X$. Since $X \leq Y$, then $W \geq 0$. Hence

$$E[W] \geq 0.$$

But using $(E1)$, we have that

$$E[W] = E[Y - X] = E[Y] - E[X].$$

Therefore,

$$0 \leq E[Y] - E[X] \Leftrightarrow E[X] \leq E[Y]. \blacksquare$$

2 Exercise 3.3

Proof of $(C1) \quad Cov[X, Y] = E[XY] - E[X]E[Y]$.

$$
\begin{aligned}
& Cov\left[X,Y\right] \\
= \ & \sum_{x\in\mathcal{S}_X}\ \sum_{y\in\mathcal{S}_Y}\left(x-E\left[X\right]\right)\left(y-E\left[Y\right]\right)\ f_{X,Y}\left(x,y\right) \\
= \ & \sum_{x\in\mathcal{S}_X}\ \sum_{y\in\mathcal{S}_Y}\left(xy-xE\left[Y\right]-yE\left[X\right]+E\left[X\right]E\left[Y\right]\right)\ f_{X,Y}\left(x,y\right) \\
= \ & \sum_{x\in\mathcal{S}_X}\ \sum_{y\in\mathcal{S}_Y} xy\ f_{X,Y}\left(x,y\right)-E\left[Y\right]\sum_{x\in\mathcal{S}_X} x\sum_{y\in\mathcal{S}_Y}\ f_{X,Y}\left(x,y\right) \\
& -E\left[X\right]\sum_{y\in\mathcal{S}_Y}\ y\sum_{x\in\mathcal{S}_X}\ f_{X,Y}\left(x,y\right)+E\left[X\right]E\left[Y\right]\sum_{x\in\mathcal{S}_X}\ \sum_{y\in\mathcal{S}_Y}\ f_{X,Y}\left(x,y\right) \\
= \ & \sum_{x\in\mathcal{S}_X}\ \sum_{y\in\mathcal{S}_Y} xy\ f_{X,Y}\left(x,y\right)-E\left[Y\right]\sum_{x\in\mathcal{S}_X} x\ f_X\left(x\right) \\
& -E\left[X\right]\sum_{y\in\mathcal{S}_Y}\ yf_Y\left(y\right)+E\left[X\right]E\left[Y\right]\underbrace{\sum_{x\in\mathcal{S}_X}\ f_X\left(x\right)}_{\substack{=1 \\ \text{by definition of a} \\ \text{density function}}} \\
= \ & E\left[XY\right]-E\left[Y\right]E\left[X\right]-E\left[X\right]E\left[Y\right]+E\left[X\right]E\left[Y\right] \\
= \ & E\left[XY\right]-E\left[X\right]E\left[Y\right].\ \blacksquare
\end{aligned}
$$

Proof of $(C2)$ If X and Y are independent then $Cov\left[X,Y\right]=0$.

Since independence of X and Y implies $E\left[XY\right]=E\left[X\right]E\left[Y\right]$, to obtain the result we only have to invoke $(C1)$:

$$
\begin{aligned}
Cov\left[X,Y\right] &= E\left[XY\right]-E\left[X\right]E\left[Y\right] \\
&= E\left[X\right]E\left[Y\right]-E\left[X\right]E\left[Y\right] \\
&= 0.\ \blacksquare
\end{aligned}
$$

Proof of $(C3) \quad Cov\left[a_1X_1 + a_2X_2, Y\right] = a_1Cov\left[X_1, Y\right] + a_2Cov\left[X_2, Y\right]$.
By $(C1)$, we have that

$$
\begin{aligned}
& Cov\left[a_1X_1 + a_2X_2, Y\right] \\
= \;& E\left[\left(a_1X_1 + a_2X_2\right)Y\right] - E\left[a_1X_1 + a_2X_2\right]E\left[Y\right] \\
= \;& E\left[a_1X_1Y + a_2X_2Y\right] - \left(a_1E\left[X_1\right] + a_2E\left[X_2\right]\right)E\left[Y\right] \text{ by } (E1) \\
= \;& a_1E\left[X_1Y\right] + a_2E\left[X_2Y\right] - a_1E\left[X_1\right]E\left[Y\right] + a_2E\left[X_2\right]E\left[Y\right] \text{ by } (E1) \\
= \;& a_1\left(E\left[X_1Y\right] - E\left[X_1\right]E\left[Y\right]\right) + a_2\left(E\left[X_2Y\right] - E\left[X_2\right]E\left[Y\right]\right) \\
= \;& a_1Cov\left[X_1, Y\right] + a_2Cov\left[X_2, Y\right] \;\blacksquare
\end{aligned}
$$

Proof of $(C4) \quad Var\left[X + Y\right] = Var\left[X\right] + Var\left[Y\right] + 2Cov\left[X, Y\right]$.

$$
\begin{aligned}
& Var\left[X + Y\right] \\
= \;& E\left[(X + Y)^2\right] - \left(E\left[X + Y\right]\right)^2 \text{ by } (V2) \\
= \;& E\left[X^2 + 2XY + Y^2\right] - \left(E\left[X\right] + E\left[Y\right]\right)^2 \text{ by } (E1) \\
= \;& E\left[X^2\right] + 2E\left[XY\right] + E\left[Y^2\right] - \left(E\left[X\right]\right)^2 - 2E\left[X\right]E\left[Y\right] - \left(E\left[Y\right]\right)^2 \\
= \;& E\left[X^2\right] - \left(E\left[X\right]\right)^2 + E\left[Y^2\right] - \left(E\left[Y\right]\right)^2 + 2\left(E\left[XY\right] - E\left[X\right]E\left[Y\right]\right) \\
= \;& Var\left[X\right] + Var\left[Y\right] + Cov\left[X, Y\right] \text{ using } (V2) \text{ and } (C1). \;\blacksquare
\end{aligned}
$$

3 Exercise 3.4

The blocks of $\mathcal{F}_2$ are

$$\begin{aligned}
B_1 &= \{TTTT, TTTH, TTHT, TTHH\}, \\
B_2 &= \{THTT, THTH, THHT, THHH\}, \\
B_3 &= \{HTTT, HTTH, HTHT, HTHH\}, \\
B_4 &= \{HHTT, HHTH, HHHT, HHHH\}.
\end{aligned}$$

Since

ω	$\mathbb{P}(\omega)$	$X_4(\omega)$	ω	$\mathbb{P}(\omega)$	$X_4(\omega)$
$TTTT$	p^4	0	$HTTT$	$p^3(1-p)$	0
$TTTH$	$p^3(1-p)$	1	$HTTH$	$p^2(1-p)^2$	1
$TTHT$	$p^3(1-p)$	0	$HTHT$	$p^2(1-p)^2$	1
$TTHH$	$p^2(1-p)^2$	2	$HTHH$	$p(1-p)^3$	3
$THTT$	$p^3(1-p)$	0	$HHTT$	$p^2(1-p)^2$	1
$THTH$	$p^2(1-p)^2$	1	$HHTH$	$p(1-p)^3$	3
$THHT$	$p^2(1-p)^2$	1	$HHHT$	$p(1-p)^3$	3
$THHH$	$p(1-p)^3$	3	$HHHH$	$(1-p)^4$	5

then

	$\mathbb{P}(B_i)$
B_1	$p^4+2p^3(1-p)+p^2(1-p)^2=p^2$
B_2	$p^3(1-p)+2p^2(1-p)^2+p(1-p)^3=p-p^2=p(1-p)$
B_3	$p^3(1-p)+2p^2(1-p)^2+p(1-p)^3=p-p^2=p(1-p)$
B_4	$p^2(1-p)^2+2p(1-p)^3+(1-p)^4=1-2p+p^2=(1-p)^2$

Recall that if $\omega \in B_i$, then

$$\mathrm{E}^{\mathbb{P}}[X_4 \,|\mathcal{F}_2](\omega) = \frac{1}{\mathbb{P}(B_i)} \sum_{\omega^* \in B_i} X(\omega^*)\,\mathbb{P}(\omega^*)$$

Therefore,

$$
\begin{aligned}
& \text{If } \omega \in B_1 \text{ then} \\
& \mathrm{E}^{\mathbb{P}}\left[X_4 \left| \mathcal{F}_2\right.\right](\omega) \\
= \;& \frac{1}{\mathbb{P}(B_1)} \sum_{\omega^* \in B_1} X(\omega^*)\,\mathbb{P}(\omega^*) \\
= \;& \frac{1}{p^2}\left(0p^4 + 1p^3(1-p) + 0p^3(1-p) + 2p^2(1-p)^2\right) \\
= \;& (1-p)(2-p)
\end{aligned}
$$

$$
\begin{aligned}
& \text{If } \omega \in B_2 \text{ then} \\
& \mathrm{E}^{\mathbb{P}}\left[X_4 \left| \mathcal{F}_2\right.\right](\omega) \\
= \;& \frac{1}{\mathbb{P}(B_2)} \sum_{\omega^* \in B_2} X(\omega^*)\,\mathbb{P}(\omega^*) \\
= \;& \frac{1}{p(1-p)}\left(0p^3(1-p) + 1p^2(1-p)^2 + 1p^2(1-p)^2 + 3p(1-p)^3\right) \\
= \;& (3-p)(1-p)
\end{aligned}
$$

$$
\begin{aligned}
& \text{If } \omega \in B_3 \text{ then} \\
& \mathrm{E}^{\mathbb{P}}\left[X_4 \left| \mathcal{F}_2\right.\right](\omega) \\
= \;& \frac{1}{\mathbb{P}(B_3)} \sum_{\omega^* \in B_3} X(\omega^*)\,\mathbb{P}(\omega^*) \\
= \;& \frac{1}{p(1-p)}\left(0p^3(1-p) + 1p^2(1-p)^2 + 1p^2(1-p)^2 + 3p(1-p)^3\right) \\
= \;& (3-p)(1-p)
\end{aligned}
$$

$$
\begin{aligned}
& \text{If } \omega \in B_4 \text{ then} \\
& \mathrm{E}^{\mathbb{P}}\left[X_4 \left| \mathcal{F}_2\right.\right](\omega) \\
= \;& \frac{1}{\mathbb{P}(B_4)} \sum_{\omega^* \in B_4} X(\omega^*)\,\mathbb{P}(\omega^*) \\
= \;& \frac{1}{(1-p)^2}\left(1p^2(1-p)^2 + 3p(1-p)^3 + 3p(1-p)^3 + 5(1-p)^4\right) \\
= \;& 5-4p
\end{aligned}
$$

In the specific case of $p = \frac{1}{2}$,

$$\begin{aligned}
\text{If } \omega &\in B_1 \text{ then } \mathrm{E}^{\mathbb{P}}\left[X_4 \left| \mathcal{F}_2\right.\right](\omega) = \frac{3}{4} \\
\text{If } \omega &\in B_2 \text{ then } \mathrm{E}^{\mathbb{P}}\left[X_4 \left| \mathcal{F}_2\right.\right](\omega) = \frac{5}{4} \\
\text{If } \omega &\in B_3 \text{ then } \mathrm{E}^{\mathbb{P}}\left[X_4 \left| \mathcal{F}_2\right.\right](\omega) = \frac{5}{4} \\
\text{If } \omega &\in B_4 \text{ then } \mathrm{E}^{\mathbb{P}}\left[X_4 \left| \mathcal{F}_2\right.\right](\omega) = 3
\end{aligned}$$

4 Exercise 3.5

$$\begin{aligned}
E\left[X_4 \middle| \mathcal{F}_0\right] &= 2 \times (1-p)^4 + 4 \times 4p\,(1-p)^3 \\
&\quad + 6 \times 6p^2\,(1-p)^2 + 8 \times 4p^3\,(1-p) + 10 \times p^4 \\
&= 2 + 8p \\
\text{If } p &= \frac{1}{2}, \text{ then } E\left[X_4 \middle| \mathcal{F}_0\right] = 6.
\end{aligned}$$

$$
\begin{aligned}
\text{If } \omega &\in B_1 = \{TTTT, TTTH, TTHT, TTHH\}, \text{ then} \\
E\left[X_4 | \mathcal{F}_2\right](\omega) &= \frac{10 \times p^4 + 8 \times p^3 (1-p) + 8 \times p^3 (1-p) + 6 \times p^2 (1-p)^2}{p^4 + p^3 (1-p) + p^3 (1-p) + p^2 (1-p)^2} \\
&= 4p + 6
\end{aligned}
$$

$$
\begin{aligned}
\text{If } \omega &\in B_2 = \{THTT, THTH, THHT, THHH\}, \text{ then} \\
E\left[X_4 | \mathcal{F}_2\right](\omega) &= \frac{8 \times p^3 (1-p) + 6 \times p^2 (1-p)^2 + 6 \times p^2 (1-p)^2 + 4 \times p (1-p)^3}{p^3 (1-p) + p^2 (1-p)^2 + p^2 (1-p)^2 + p (1-p)^3} \\
&= 4p + 4
\end{aligned}
$$

$$
\begin{aligned}
\text{If } \omega &\in B_3 = \{HTTT, HTTH, HTHT, HTHH\}, \text{ then} \\
E\left[X_4 | \mathcal{F}_2\right](\omega) &= \frac{8 \times p^3 (1-p) + 6 \times p^2 (1-p)^2 + 6 \times p^2 (1-p)^2 + 4 \times p (1-p)^3}{p^3 (1-p) + p^2 (1-p)^2 + p^2 (1-p)^2 + p (1-p)^3} \\
&= 4p + 4
\end{aligned}
$$

$$
\begin{aligned}
\text{If } \omega &\in B_4 = \{HHTT, HHTH, HHHT, HHHH\}, \text{ then} \\
E\left[X_4 | \mathcal{F}_2\right](\omega) &= \frac{6 \times p^2 (1-p)^2 + 4 \times p (1-p)^3 + 4 \times p (1-p)^3 + 2 \times (1-p)^4}{p^2 (1-p)^2 + p (1-p)^3 + p (1-p)^3 + (1-p)^4} \\
&= 4p + 2
\end{aligned}
$$

$$
\begin{aligned}
\text{If } p &= \frac{1}{2}, \text{ then} \\
E\left[X_4 | \mathcal{F}_2\right](\omega) &= \begin{cases} 8 & \text{if } \omega \in B_1 = \{TTTT, TTTH, TTHT, TTHH\} \\ 6 & \text{if } \omega \in B_2 = \{THTT, THTH, THHT, THHH\} \\ 6 & \text{if } \omega \in B_3 = \{HTTT, HTTH, HTHT, HTHH\} \\ 4 & \text{if } \omega \in B_4 = \{HHTT, HHTH, HHHT, HHHH\} \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
\text{If } \omega &\in B_1=\left\{\begin{array}{c} TTTT,TTTH,TTHT,TTHH, \\ THTT,THTH,THHT,THHH \end{array}\right\},\ \text{then} \\
E\left[\left.Y_4\right|\mathcal{G}_2\right](\omega) &= 10
\end{aligned}
$$

$$
\begin{aligned}
\text{If } \omega &\in B_3=\{HTTT,HTTH,HTHT,HTHH\},\ \text{then} \\
E\left[\left.Y_4\right|\mathcal{G}_2\right](\omega) &= \frac{10\times p^3(1-p)+6\times p^2(1-p)^2+0\times p^2(1-p)^2+0\times p(1-p)^3}{p^3(1-p)+p^2(1-p)^2+p^2(1-p)^2+p(1-p)^3} \\
&= 2p(2p+3)
\end{aligned}
$$

$$
\begin{aligned}
\text{If } \omega &\in B_4=\{HHTT,HHTH,HHHT,HHHH\},\ \text{then} \\
E\left[\left.Y_4\right|\mathcal{G}_2\right](\omega) &= 0
\end{aligned}
$$

$$
\begin{aligned}
\text{If } p &= \frac{1}{2},\ \text{then} \\
E\left[\left.Y_4\right|\mathcal{G}_2\right](\omega) &= \begin{cases} 10 & \text{if } \omega\in B_1=\left\{\begin{array}{c} TTTT,TTTH,TTHT,TTHH, \\ THTT,THTH,THHT,THHH \end{array}\right\} \\ 4 & \text{if } \omega\in B_3=\{HTTT,HTTH,HTHT,HTHH\} \\ 0 & \text{if } \omega\in B_4=\{HHTT,HHTH,HHHT,HHHH\} \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
\text{If } \omega &\in C_1=\left\{\begin{array}{c} TTTT,TTTH,TTHT,TTHH, \\ THTT,THTH,THHT,THHH \end{array}\right\},\ \text{then} \\
E\left[\left.Y_2\right|\mathcal{G}_4\right](\omega) &= 10
\end{aligned}
$$

$$
\begin{aligned}
\text{If } \omega &\in C_2=\{HTTT\},\ \text{then} \\
E\left[\left.Y_2\right|\mathcal{G}_4\right](\omega) &= 4
\end{aligned}
$$

$$
\begin{aligned}
\text{If } \omega &\in C_3=\{HTTH\},\ \text{then} \\
E\left[\left.Y_2\right|\mathcal{G}_4\right](\omega) &= 4
\end{aligned}
$$

$$
\begin{aligned}
\text{If } \omega &\in C_4=\{HTHT,HTHH\},\ \text{then} \\
E\left[\left.Y_2\right|\mathcal{G}_4\right](\omega) &= 4
\end{aligned}
$$

$$
\begin{aligned}
\text{If } \omega &\in C_5=\{HHTT,HHTH,HHHT,HHHH\},\ \text{then} \\
E\left[\left.Y_2\right|\mathcal{G}_4\right](\omega) &= 0
\end{aligned}
$$

5 Exercise 3.6

a) *What filtration is generated by the process X ?*

$$\mathcal{F}_0 = \{\varnothing, \Omega\}$$

$$\mathcal{F}_1 = \sigma\left\{\{\omega_1, ..., \omega_{10}\}, \{\omega_{11}, ..., \omega_{16}\}\right\}$$

$$\mathcal{F}_2 = \sigma\left\{\{\omega_1, ..., \omega_6\}, \{\omega_7, ..., \omega_{10}\}, \{\omega_{11}, ..., \omega_{13}\}, \{\omega_{14}, \omega_{15}, \omega_{16}\}\right\}$$

$$\mathcal{F}_3 = \sigma\left\{\begin{array}{c} \{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}, \{\omega_5, \omega_6\}, \{\omega_7, ..., \omega_{10}\}, \\ \{\omega_{11}, \omega_{12}\}, \{\omega_{13}\}, \{\omega_{14}, \omega_{15}\} \{\omega_{16}\} \end{array}\right\}$$

$$\mathcal{F}_4 = \sigma\left\{\begin{array}{c} \{\omega_1\}, \{\omega_2\}, \{\omega_3\}, \{\omega_4\}, \{\omega_5\}, \{\omega_6\}, \{\omega_7\}, \{\omega_8\} \\ \{\omega_9\}, \{\omega_{10}\}, \{\omega_{11}\}, \{\omega_{12}\}, \{\omega_{13}\}, \{\omega_{14}\}, \{\omega_{15}\}, \{\omega_{16}\} \end{array}\right\}$$

b) *Give the conditional distribution of X_4, knowing X_2.*

$$\mathbb{P}\left[X_4 = x \,|X_2 = 15\right] = \begin{cases} \frac{3\times 0.06}{3\times 0.06 + 3\times 0.065} = 0.48 & \text{if } x = 16 \\ \frac{3\times 0.065}{3\times 0.06 + 3\times 0.065} = 0.52 & \text{if } x = 14 \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbb{P}\left[X_4 = x \,|X_2 = 12\right] = \begin{cases} \frac{0.08}{4\times 0.0625 + 0.08 + 0.04 + 0.0675} = 0.18286 & \text{if } x = 18 \\ \frac{2\times 0.0625}{4\times 0.0625 + 0.08 + 0.04 + 0.0675} = 0.28571 & \text{if } x = 14 \\ \frac{0.04 + 2\times 0.0625}{4\times 0.0625 + 0.08 + 0.04 + 0.0675} = 0.37714 & \text{if } x = 12 \\ \frac{0.0675}{4\times 0.0625 + 0.08 + 0.04 + 0.0675} = 0,15429 & \text{if } x = 11 \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbb{P}\left[X_4 = x \,|X_2 = 11\right] = \begin{cases} \frac{0,0625}{3\times 0.0625} = \frac{1}{3} & \text{if } x = 20 \\ \frac{2\times 0,0625}{3\times 0.0625} = \frac{2}{3} & \text{if } x = 12 \\ 0 & \text{otherwise} \end{cases}$$

c) *Calculate the conditional expectation of X_4, knowing X_2.*

$$\begin{aligned}
\text{If } \omega &\in \{\omega_1, ..., \omega_6\} \\
\mathrm{E}^{\mathbb{P}}\left[X_4 \left| \sigma\left(X_2\right)\right.\right](\omega) &= 16\times\frac{3\times 0.06}{3\times 0.06 + 3\times 0.065}+14\times\frac{3\times 0.065}{3\times 0.06+3\times 0.065} \\
&= 14.96 \\
\text{If } \omega &\in \{\omega_7, ..., \omega_{13}\} \\
\mathrm{E}^{\mathbb{P}}\left[X_4 \left| \sigma\left(X_2\right)\right.\right](\omega) &= 18\times\frac{0.08}{4\times 0.0625+0.08+0.04+0.0675} \\
&\quad +14\times\frac{2\times 0.0625}{4\times 0.0625+0.08+0.04+0.0675} \\
&\quad +12\times\frac{0.04+2\times 0.0625}{4\times 0.0625+0.08+0.04+0.0675} \\
&\quad +11\times\frac{0.0675}{4\times 0.0625+0.08+0.04+0.0675} \\
&= 13.514 \\
\text{If } \omega &\in \{\omega_{14}, \omega_{15}, \omega_{16}\} \\
\mathrm{E}^{\mathbb{P}}\left[X_4 \left| \sigma\left(X_2\right)\right.\right](\omega) &= 20\times\frac{1}{3}+12\times\frac{2}{3} \\
&= 14.667
\end{aligned}$$

d) *Give the conditional distribution of X_4, knowing $\mathcal{F}_2$.*

$$\mathbb{P}\left[X_4 = x \,|\{\omega_1, ..., \omega_6\}\right] = \begin{cases} \frac{3\times 0.06}{3\times 0.06+3\times 0.065} = 0.48 & \text{if } x = 16 \\ \frac{3\times 0.065}{3\times 0.06+3\times 0.065} = 0.52 & \text{if } x = 14 \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbb{P}\left[X_4 = x \,|\{\omega_7, ..., \omega_{10}\}\right] = \begin{cases} \frac{0.08}{0.0625+0.08+0.04+0.0675} = 0.32 & \text{if } x = 18 \\ \frac{0.0625}{0.0625+0.08+0.04+0.0675} = 0.25 & \text{if } x = 14 \\ \frac{0.04}{0.0625+0.08+0.04+0.0675} = 0.16 & \text{if } x = 12 \\ \frac{0.0675}{0.0625+0.08+0.04+0.0675} = 0.27 & \text{if } x = 11 \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbb{P}\left[X_4 = x \,|\{\omega_{11}, ..., \omega_{13}\}\right] = \begin{cases} \frac{0.0625}{3\times 0.0625} = \frac{1}{3} & \text{if } x = 14 \\ \frac{2\times 0.0625}{3\times 0.0625} = \frac{2}{3} & \text{if } x = 12 \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbb{P}\left[X_4 = x \,|\{\omega_{14}, \omega_{15}, \omega_{16}\}\right] = \begin{cases} \frac{0{,}0625}{3\times 0.0625} = \frac{1}{3} & \text{if } x = 20 \\ \frac{2\times 0{,}0625}{3\times 0.0625} = \frac{2}{3} & \text{if } x = 12 \\ 0 & \text{otherwise} \end{cases}$$

e) *Calculate the conditional expectation of X_4, knowing $\mathcal{F}_2$.*

$$\begin{aligned}
\text{if } \omega &\in \{\omega_1, ..., \omega_6\}, \\
\mathrm{E}^{\mathbb{P}}\left[X_4 \,|\mathcal{F}_2\right](\omega) &= 16\times\frac{3\times 0.06}{3\times 0.06 + 3\times 0.065} + 14\times\frac{3\times 0.065}{3\times 0.06 + 3\times 0.065} \\
&= 14.96 \\
\text{if } \omega &\in \{\omega_7, ..., \omega_{10}\}, \\
\mathrm{E}^{\mathbb{P}}\left[X_4 \,|\mathcal{F}_2\right](\omega) &= 18\times\frac{0.08}{0.0625+0.08+0.04+0.0675} + 14\times\frac{0.0625}{0.0625+0.08+0.04+0.0675} \\
&\quad +12\times\frac{0.04}{0.0625+0.08+0.04+0.0675} + 11\times\frac{0.0675}{0.0625+0.08+0.04+0.0675} \\
&= 14.15 \\
\text{if } \omega &\in \{\omega_{11}, \omega_{12}, \omega_{13}\}, \\
\mathrm{E}^{\mathbb{P}}\left[X_4 \,|\mathcal{F}_2\right](\omega) &= 14\times\frac{1}{3} + 12\times\frac{2}{3} \\
&= 12.667 \\
\text{if } \omega &\in \{\omega_{14}, \omega_{15}, \omega_{16}\}, \\
\mathrm{E}^{\mathbb{P}}\left[X_4 \,|\mathcal{F}_2\right](\omega) &= 20\times\frac{1}{3} + 12\times\frac{2}{3} = 14.667
\end{aligned}$$

6 Problem 3.7

The conditional expectations of the yields are :

$$
\begin{aligned}
\mathrm{E}^{\mathbb{P}}\left[R_{2,4}\left|\{\omega_1,...,\omega_6\}\right.\right] &= \frac{14.96}{15} = .997\,33 \\
\mathrm{E}^{\mathbb{P}}\left[R_{2,4}\left|\{\omega_7,...,\omega_{10}\}\right.\right] &= \frac{14.15}{12} = 1.1792 \\
\mathrm{E}^{\mathbb{P}}\left[R_{2,4}\left|\{\omega_{11},\omega_{12},\omega_{13}\}\right.\right] &= \frac{12.67}{12} = 1.0558 \\
\mathrm{E}^{\mathbb{P}}\left[R_{2,4}\left|\{\omega_{14},\omega_{15},\omega_{16}\}\right.\right] &= \frac{14.667}{11} = 1.3334 \\
\mathrm{E}^{\mathbb{P}}\left[R_{2,4}\left|\{\omega_7,...,\omega_{13}\}\right.\right] &= \frac{13.514}{12} = 1.1262
\end{aligned}
$$

Possible values for the random variable $R_{2,4}$ are :

$$
\begin{aligned}
\frac{16}{15} &= 1.0667 \text{ and } \frac{14}{15} = 0.93333 \\
\frac{18}{12} &= 1.5 \text{ and } \frac{14}{12} = 1.1667 \text{ and } \frac{12}{12} = 1 \text{ and } \frac{11}{12} = 0.91667 \\
\frac{20}{11} &= 1.8182 \text{ and } \frac{12}{11} = 1.0909.
\end{aligned}
$$

Recall that the bank account has a yield of 1.15 for times $t = 2$ to $t = 4$.

$$\begin{aligned}
\mathbb{P}\left[R_{2,4} < 1.15 \,|\{\omega_1, ..., \omega_6\}\right] &= \mathbb{P}\left[X_4 = 16 \text{ or } X_4 = 14 \,|\{\omega_1, ..., \omega_6\}\right] \\
&= 1
\end{aligned}$$

$$\begin{aligned}
\mathbb{P}\left[R_{2,4} < 1.15 \,|\{\omega_7, ..., \omega_{10}\}\right] &= \mathbb{P}\left[X_4 = 12 \text{ or } X_4 = 11 \,|\{\omega_7, ..., \omega_{10}\}\right] \\
&= 0.43
\end{aligned}$$

$$\begin{aligned}
\mathbb{P}\left[R_{2,4} < 1.15 \,|\{\omega_{11}, \omega_{12}, \omega_{13}\}\right] &= \mathbb{P}\left[X_4 = 12 \,|\{\omega_{11}, \omega_{12}, \omega_{13}\}\right] \\
&= 0.66667
\end{aligned}$$

$$\begin{aligned}
\mathbb{P}\left[R_{2,4} < 1.15 \,|\{\omega_{14}, \omega_{15}, \omega_{16}\}\right] &= \mathbb{P}\left[X_4 = 12 \,|\{\omega_{14}, \omega_{15}, \omega_{16}\}\right] \\
&= 0.66667
\end{aligned}$$

$$\begin{aligned}
\mathbb{P}\left[R_{2,4} < 1.15 \,|\{\omega_7, ..., \omega_{13}\}\right] &= \mathbb{P}\left[X_4 = 12 \text{ or } X_4 = 11 \,|\{\omega_7, ..., \omega_{13}\}\right] \\
&= 0.37714 + 0.15429 = 0.53143
\end{aligned}$$

a) *What instructions does investor A give to his prosecutor ? Justify your answer.*

If we observe the process from time $t = 0$, at time $t = 2$ we will be able to determine which of the following four events happened : $\{\omega_1, ..., \omega_6\}, \{\omega_7, ..., \omega_{10}\}, \{\omega_{11}, ..., \omega_{13}\}, \{\omega_{14}, \omega_{15}, \omega_{16}\}$.

If event $\{\omega_1, ..., \omega_6\}$ happened (the price of the share followed trajectories 11, 13, 15), then the probability to obtain a holding period yield (for the period from $t = 2$ to $t = 4$) superior to that of the bank account is zero. Hence in that case we would sell the share and invest the 15 dollars in the bank account. Then, the value of our portfolio at time $t = 4$ would be $15 \times 1.15 = 17.25$.

If event $\{\omega_7, ..., \omega_{10}\}$ happened (the price of the share followed trajectories 11, 13, 12), then the probability to obtain a HPY (for the period from $t = 2$ to $t = 4$) inferior to that of the bank account is around 43%. However, the expected HPY is 1.18 which is superior to that of the bank account. Hence we would keep the share.

If event $\{\omega_{11}, ..., \omega_{13}\}$ happened (the price of the share followed trajectories 11, 12, 12), then the probability to obtain a HPY (for the period from $t = 2$ to $t = 4$) inferior to that of the bank account is around 67%. In addition, the expected HPY is 1.0558 which is inferior to the 1.15 of the bank account. Hence we would sell the share and invest the 12 dollars in the bank account. Then, the value of our portfolio at time $t = 4$ would be $12 \times 1.15 = 13.8$.

If event $\{\omega_{14}, \omega_{15}, \omega_{16}\}$ happened (the price of the share followed trajectories 11, 11, 11), then the probability to obtain a HPY (for the period from $t = 2$ to $t = 4$) inferior to that of the bank account is around 67%. However, the expected HPY is 1.3334 which is superior to the 1.15 of the bank account. This is because when the HPY exceeds the yield of the bank account, it does it by a huge margin (1.82 versus 1.15). People with a taste for risky investment would choose to keep the share, especially because even when the HPY is inferior to the yield of the bank account, it is not drastically inferior (1.09 versus 1.15).

Trajectory	Decision at time $t=2$	Value of the portfolio at time $t=4$ if the share is sold	Value of the portfolio at time $t=4$ if the share is kept
$11, 13, 15$	sell the share	**17.25**	16 with prob. 18%
		17.25	14 with prob. 19.5%
$11, 13, 12$	keep the share	13.8	**18** with prob. **8%**
		13.8	**14** with prob. **6.25%**
		13.8	**12** with prob. **4%**
		13.8	**11** with prob. **6.75%**
$11, 11, 12$	sell the share	**13.8**	14 with prob. 6.25%
		13.8	12 with prob. 12.5%
$11, 11, 11$	keep the share	12.65	**20** with prob. **6.25%**
		12.65	**12** with prob. **12.5%**

b) *What instructions does investor B give to his prosecutor ? Justify your answer.*
If we observe the process only at time $t = 2$, we will be able to determine which of the following three events happened : $\{\omega_1, ..., \omega_6\}, \{\omega_7, ..., \omega_{13}\}, \{\omega_{14}, \omega_{15}, \omega_{16}\}$.

If event $\{\omega_1, ..., \omega_6\}$ happened (the price of the share at time $t = 2$ is 15), then the probability to obtain a HPY (for the period from $t = 2$ to $t = 4$) superior to that of the bank account is zero. Hence in that case we would sell the share and invest the 15 dollars in the bank account. Then, the value of our portfolio at time $t = 4$ would be $15 \times 1.15 = 17.25$.

If event $\{\omega_7, ..., \omega_{13}\}$ happened (the price of the share at time $t = 2$ is 12), then the probability to obtain a HPY (for the period from $t = 2$ to $t = 4$) inferior to that of the bank account is around 53% $(0.37714 + 0.15429)$. In addition, the expected HPY is 1.1262, which is lower then the yield of the bank account. Hence in that case we would sell the share and invest the 12 dollars in the bank account. Then, the value of our portfolio at time $t = 4$ would be $12 \times 1.15 = 13.8$.

If event $\{\omega_{14}, \omega_{15}, \omega_{16}\}$ happened (the price of the share at time $t = 2$ is 11), then the probability to obtain a HPY (for the period from $t = 2$ to $t = 4$) inferior to that of the bank account is around 67%. However, the expected HPY is 1.3334 which is superior to the 1.15 of the bank account. This is because when the HPY exceeds the yield of the bank account, it does it by a huge margin (1.82 versus 1.15). People with a taste for risky investment would choose to keep the share, especially because even when the HPY is inferior to the yield of the bank account, it is not drastically inferior (1.09 versus 1.15).

Value of X_2	Decision at time $t = 2$	Value of the portfolio at time $t = 4$ if the share is sold	Value of the portfolio at time $t = 4$ if the share is kept
15	sell the share	**17.25**	16 with prob. 18%
		17.25	14 with prob. 19.5%
12	sell the share	**13.8**	18 with prob. 8%
		13.8	14 with prob. 12.5%
		13.8	12 with prob. 16.5%
		13.8	11 with prob. 6.75%
11	keep the share	12.65	**20** with prob. **6.25**%
		12.65	**12** with prob. **12.5**%

7 Problem 3.8

a) *Let τ_A and τ_B represent the random times at which the shares are sold, respectively by prosecutors A and B. Are τ_A and τ_B stopping times ? Justify your answer.*

τ_A is a stopping time because

$$
\begin{aligned}
\{\omega \in \Omega : \tau_A(\omega) = 0\} &= \varnothing \in \mathcal{F}_0 \\
\{\omega \in \Omega : \tau_A(\omega) = 1\} &= \varnothing \in \mathcal{F}_1 \\
\{\omega \in \Omega : \tau_A(\omega) = 2\} &= \{\omega_1, ..., \omega_6\} \cup \{\omega_{11}, ..., \omega_{13}\} \in \mathcal{F}_2 \\
\{\omega \in \Omega : \tau_A(\omega) = 3\} &= \varnothing \in \mathcal{F}_3 \\
\{\omega \in \Omega : \tau_A(\omega) = 4\} &= \{\omega_7, ..., \omega_{10}\} \cup \{\omega_{14}, ..., \omega_{15}\} \in \mathcal{F}_4
\end{aligned}
$$

τ_B is a stopping time because

$$
\begin{aligned}
\{\omega \in \Omega : \tau_B(\omega) = 0\} &= \varnothing \in \mathcal{F}_0 \\
\{\omega \in \Omega : \tau_B(\omega) = 1\} &= \varnothing \in \mathcal{F}_1 \\
\{\omega \in \Omega : \tau_B(\omega) = 2\} &= \{\omega_1, ..., \omega_{13}\} \in \mathcal{F}_2 \\
\{\omega \in \Omega : \tau_B(\omega) = 3\} &= \varnothing \in \mathcal{F}_3 \\
\{\omega \in \Omega : \tau_B(\omega) = 4\} &= \{\omega_{14}, ..., \omega_{15}\} \in \mathcal{F}_4
\end{aligned}
$$

b) *Say we replace the probability measure* $\mathbb{P}$ *by* $\mathbb{Q}$ *where* $\mathbb{Q}(\omega) = \frac{1}{16}$ *for every* $\omega \in \Omega$. *Are* τ_A *and* τ_B *stopping times in this case ? Justify your answer.*

Yes since being (or not being) a stopping time does not depend in any way on the probability measure used on the filtrated measurable space.