

a) Comutatividad

Sea $|a\rangle = a^\alpha |q_\alpha\rangle$, $|m\rangle = m^\alpha |q_\alpha\rangle$ son cuaterniones con $a^\alpha, m^\alpha \in \mathbb{R}$

$$|a\rangle + |m\rangle = |m\rangle + |a\rangle$$

Con la cercadura bajo la cual se probó $|a\rangle + |m\rangle$ por lo tanto hace falta probar $|m\rangle + |a\rangle$

Para $|m\rangle + |a\rangle$

$$\begin{aligned} |m\rangle + |a\rangle &= (m^0 |q_0\rangle + m^1 |q_1\rangle + m^2 |q_2\rangle + m^3 |q_3\rangle) + (a^0 |q_0\rangle + a^1 |q_1\rangle + a^2 |q_2\rangle + a^3 |q_3\rangle) \\ &= (m^0 + a^0) |q_0\rangle + (m^1 + a^1) |q_1\rangle + (m^2 + a^2) |q_2\rangle + (m^3 + a^3) |q_3\rangle \\ &= C^\alpha |q_\alpha\rangle \end{aligned}$$

$$m^\alpha + a^\alpha = C^\alpha$$

a) Asociatividad

Sea $|a\rangle = a^\alpha |q_\alpha\rangle$, $|m\rangle = m^\alpha |q_\alpha\rangle$, $|Y\rangle = Y^\alpha |q_\alpha\rangle$ con $a^\alpha, m^\alpha, Y^\alpha \in \mathbb{R}$

$$\begin{aligned} (|a\rangle + |m\rangle) + |Y\rangle &= [(a_0 + m_0) |q_0\rangle + (a^1 + m^1) |q_1\rangle + (a^2 + m^2) |q_2\rangle + (a^3 + m^3) |q_3\rangle] + Y^0 |q_0\rangle + Y^1 |q_1\rangle + Y^2 |q_2\rangle + Y^3 |q_3\rangle \\ &= (a_0 + m_0 + Y_0) |q_0\rangle + (a^1 + m^1 + Y^1) |q_1\rangle + (a^2 + m^2 + Y^2) |q_2\rangle + (a^3 + m^3 + Y^3) |q_3\rangle \end{aligned}$$

Con la certidura entre la linea se ve que los $|q_i\rangle$ son; por lo tanto hace falta probar $|m\rangle + |q\rangle$

Para $|m\rangle + |q\rangle$

$$\begin{aligned}|m\rangle + |q\rangle &= (m^0|q_0\rangle + m^1|q_1\rangle + m^2|q_2\rangle + m^3|q_3\rangle) + (a^0|q_0\rangle + a^1|q_1\rangle + a^2|q_2\rangle + a^3|q_3\rangle) \\&= (m^0 + a^0)|q_0\rangle + (m^1 + a^1)|q_1\rangle + (m^2 + a^2)|q_2\rangle + (m^3 + a^3)|q_3\rangle \\&= |c^\alpha q_\alpha\rangle \quad m^0 + a^0 = c^\alpha\end{aligned}$$

a) Asociatividad

Sea $|q\rangle = a^\alpha|q_\alpha\rangle$, $|m\rangle = m^\alpha|q_\alpha\rangle$, $|Y\rangle = Y^\alpha|q_\alpha\rangle$ con $a^\alpha, m^\alpha, Y^\alpha \in \mathbb{R}$

$$\begin{aligned}(|q\rangle + |m\rangle) + |Y\rangle &= [(q_0 + m_0)|q_0\rangle + (a^1 + m^1)|q_1\rangle + (a^2 + m^2)|q_2\rangle + (a^3 + m^3)|q_3\rangle] + |Y^0|q_0\rangle + |Y^1|q_1\rangle + |Y^2|q_2\rangle + |Y^3|q_3\rangle \\&= (q_0 + m_0 + Y_0)|q_0\rangle + (a^1 + m^1 + Y^1)|q_1\rangle + (a^2 + m^2 + Y^2)|q_2\rangle + (a^3 + m^3 + Y^3)|q_3\rangle \\&= |c^\alpha q_\alpha\rangle\end{aligned}$$

$$\begin{aligned}(|Y\rangle + |q\rangle) + |m\rangle &= [(Y^0 + a^0)|q_0\rangle + (Y^1 + a^1)|q_1\rangle + (Y^2 + a^2)|q_2\rangle + (Y^3 + a^3)|q_3\rangle] + |m^0|q_0\rangle + |m^1|q_1\rangle + |m^2|q_2\rangle + |m^3|q_3\rangle \\&= (Y^0 + a^0 + m^0)|q_0\rangle + (Y^1 + a^1 + m^1)|q_1\rangle + (Y^2 + a^2 + m^2)|q_2\rangle + (Y^3 + a^3 + m^3)|q_3\rangle \\&= |c^\alpha q_\alpha\rangle\end{aligned}$$

• a) Existencia del neutro aditivo

Sea $|q\rangle = q^{\alpha}|q_{\alpha}\rangle$, $|m\rangle = m^{\alpha}|q_{\alpha}\rangle$ donde $q, m \in \mathbb{R}$ y $m^0 = m^1 + m^2 + m^3 = 0$

$$\begin{aligned} |q\rangle + |m\rangle &= (q^0 + m^0)|q_0\rangle + (q^1 + m^1)|q_1\rangle + (q^2 + m^2)|q_2\rangle + (q^3 + m^3)|q_3\rangle \\ &= (q^0 + 0)|q_0\rangle + (q^1 + 0)|q_1\rangle + (q^2 + 0)|q_2\rangle + (q^3 + 0)|q_3\rangle \\ &= q^0|q_0\rangle + q^1|q_1\rangle + q^2|q_2\rangle + q^3|q_3\rangle \\ &= q^{\alpha}|q_{\alpha}\rangle \end{aligned}$$

• a) Existencia de inverso aditivo

Sea $|q\rangle = q^{\alpha}|q_{\alpha}\rangle$, $| - q \rangle = -q^{\alpha}|q_{\alpha}\rangle$

$$\begin{aligned} |q\rangle + (-q) &= (q^0 - q^0)|q_0\rangle + (q^1 - q^1)|q_1\rangle + (q^2 - q^2)|q_2\rangle + (q^3 - q^3)|q_3\rangle \\ &= 0|q_0\rangle + 0|q_1\rangle + 0|q_2\rangle + 0|q_3\rangle \end{aligned}$$

• a) Cerradura respecto a la multiplicación por escalares

Sea $|q\rangle = q^{\alpha}|q_{\alpha}\rangle$ y $\alpha, \beta \in \mathbb{R}$

$$\beta|q\rangle = \beta(q^{\alpha}|q_{\alpha}\rangle) = \beta(q^0|q_0\rangle + q^1|q_1\rangle + q^2|q_2\rangle + q^3|q_3\rangle)$$

$$= a^0|q_0\rangle + a^1|q_1\rangle + a^2|q_2\rangle + a^3|q_3\rangle$$

$$= a^\alpha|q_\alpha\rangle$$

• a) existencia de inverso aditivo

$$\text{Sea } |q\rangle = a^\alpha|q_\alpha\rangle, \quad |-q\rangle = -a^\alpha|q_\alpha\rangle$$

$$|q\rangle + (-q\rangle) = (a^0 - a^0)|q_0\rangle + (a^1 - a^1)|q_1\rangle + (a^2 - a^2)|q_2\rangle + (a^3 - a^3)|q_3\rangle$$

$$= 0|q_0\rangle + 0|q_1\rangle + 0|q_2\rangle + 0|q_3\rangle$$

• a) cerradura respecto a la multiplicación por escalares

$$\text{Sea } |q\rangle = a^\alpha|q_\alpha\rangle \quad y \quad \alpha^\beta, \beta \in \mathbb{R}$$

$$\beta|q\rangle = \beta(a^\alpha|q_\alpha\rangle) = \beta(a^0|q_0\rangle + a^1|q_1\rangle + a^2|q_2\rangle + a^3|q_3\rangle)$$

$$= \beta a^0|q_0\rangle + \beta a^1|q_1\rangle + \beta a^2|q_2\rangle + \beta a^3|q_3\rangle \quad \boxed{\beta a^i = k^i}$$

$$= k^0|q_0\rangle + k^1|q_1\rangle + k^2|q_2\rangle + k^3|q_3\rangle$$

• a) asociatividad de la multiplicación escalar

$$\gamma(\beta|q\rangle) = (\gamma|q\rangle)\beta$$

$$= [\gamma(a^0|q_0\rangle + a^1|q_1\rangle + a^2|q_2\rangle + a^3|q_3\rangle)]\beta$$

$$= [\gamma a^0|q_0\rangle + \gamma a^1|q_1\rangle + \gamma a^2|q_2\rangle + \gamma a^3|q_3\rangle]\beta$$

$$= \beta \gamma a^0|q_0\rangle + \beta \gamma a^1|q_1\rangle + \beta \gamma a^2|q_2\rangle + \beta \gamma a^3|q_3\rangle$$

$$= \gamma(\beta a^0|q_0\rangle + \beta a^1|q_1\rangle + \beta a^2|q_2\rangle + \beta a^3|q_3\rangle)$$

$$= \gamma(\beta|q\rangle)$$

a) Elemento neutro del producto

$$\begin{aligned} |\odot|q\rangle &= |\odot(q^0 + q^1|q_1\rangle + q^2|q_2\rangle + q^3|q_3\rangle) \\ &= 1 \cdot q^0 + 1 \cdot q^1|q_1\rangle + 1 \cdot q^2|q_2\rangle + 1 \cdot q^3|q_3\rangle \\ &= q^0 + q^1|q_1\rangle + q^2|q_2\rangle + q^3|q_3\rangle \\ &= |q\rangle \end{aligned}$$

a) Distributividad respecto a la suma de vectores

$$\begin{aligned} B(|q\rangle + |m\rangle) &= B[(q_0 + q^1|q_1\rangle + q^2|q_2\rangle + q^3|q_3\rangle) + (m_0 + m^1|q_1\rangle + m^2|q_2\rangle + m^3|q_3\rangle)] \\ &= B[(q_0 + m_0) + (q^1 + m^1)|q_1\rangle + (q^2 + m^2)|q_2\rangle + (q^3 + m^3)|q_3\rangle] \\ &= B(q_0 + m_0) + B(q^1 + m^1)|q_1\rangle + B(q^2 + m^2)|q_2\rangle + B(q^3 + m^3)|q_3\rangle \\ &= \beta q_0 + \beta m_0 + \beta q^1|q_1\rangle + \beta m^1|q_1\rangle + \beta q^2|q_2\rangle + \beta m^2|q_2\rangle + \beta q^3|q_3\rangle + \beta m^3|q_3\rangle \\ &= \beta q_0 + \beta q^1|q_1\rangle + \beta q^2|q_2\rangle + \beta q^3|q_3\rangle + \beta(m^0 + m^1|q_1\rangle + m^2|q_2\rangle + m^3|q_3\rangle) \\ &= \beta(q_0 + q^1|q_1\rangle + q^2|q_2\rangle + q^3|q_3\rangle) + \beta(m^0 + m^1|q_1\rangle + m^2|q_2\rangle + m^3|q_3\rangle) \\ &= \beta|q\rangle + \beta|m\rangle \end{aligned}$$

a) Distributividad respecto a la suma de escalares

$$= q^0 + q^1 |q_1\rangle + q^2 |q_2\rangle + q^3 |q_3\rangle$$
$$= |q\rangle$$

Siendo la suma de vectores

$$\begin{aligned} \beta(|q\rangle + |m\rangle) &= \beta[(q_0 + \alpha|q_1\rangle + \alpha^2|q_2\rangle + \alpha^3|q_3\rangle) + (m_0 + \alpha|m_1\rangle + \alpha^2|m_2\rangle + \alpha^3|m_3\rangle)] \\ &= \beta[(q_0 + m_0) + (\alpha + \alpha^2 + \alpha^3)|q_1\rangle + (\alpha + \alpha^2 + \alpha^3)m_1\rangle] \\ &= \beta(q_0 + m_0) + \beta(\alpha + \alpha^2 + \alpha^3)|q_1\rangle + \beta(m_0 + \alpha^2 + \alpha^3)m_1\rangle \\ &= \beta q_0 + \beta m_0 + \beta\alpha|q_1\rangle + \beta\alpha^2|q_2\rangle + \beta\alpha^3|q_3\rangle + \beta m_0 + \beta\alpha^2|m_1\rangle + \beta\alpha^3|m_2\rangle \\ &= \beta q_0 + \beta m_0 + \beta\alpha|q_1\rangle + \beta\alpha^2|q_2\rangle + \beta\alpha^3|q_3\rangle + \beta(m_0 + \alpha^2|m_1\rangle + \alpha^3|m_2\rangle) \\ &= \beta|q\rangle + \beta|m\rangle \end{aligned}$$

a) Distributividad respecto a la suma de escalares

$$\begin{aligned} (\beta + \gamma)|q\rangle &= (\beta + \gamma)(q^0 + q^1|q_1\rangle + q^2|q_2\rangle + q^3|q_3\rangle) \\ &= (\beta + \gamma)q^0 + (\beta + \gamma)q^1|q_1\rangle + (\beta + \gamma)q^2|q_2\rangle + (\beta + \gamma)q^3|q_3\rangle \\ &= \beta q^0 + \gamma q^0 + \beta q^1|q_1\rangle + \gamma q^1|q_1\rangle + \beta q^2|q_2\rangle + \gamma q^2|q_2\rangle + \beta q^3|q_3\rangle + \gamma q^3|q_3\rangle \\ &= \beta q^0 + \beta q^1|q_1\rangle + \beta q^2|q_2\rangle + \beta q^3|q_3\rangle + \gamma q^0 + \gamma q^1|q_1\rangle + \gamma q^2|q_2\rangle + \gamma q^3|q_3\rangle \\ &= \beta(q^0 + q^1|q_1\rangle + q^2|q_2\rangle + q^3|q_3\rangle) + \gamma(q^0 + q^1|q_1\rangle + q^2|q_2\rangle + q^3|q_3\rangle) \\ &= \beta|q\rangle + \gamma|q\rangle \end{aligned}$$

$$b) |b\rangle \equiv (b^0, b) \quad y \quad |r\rangle \equiv (r^0, r)$$

$$|b\rangle \odot |r\rangle = (b^0 + b \cdot q)(r^0 + r \cdot q)$$

$$= b^0 r^0 + b^0 r \cdot q + r^0 b \cdot q + (b \cdot q)(r \cdot q)$$

$$= b^0 r^0 + b^0 \sum_j r^j q_j + r^0 \sum_i b^i q_i + \sum_j \sum_i r^j b^i q_j q_i$$

donde se puede expresar de la forma

$$= b^0 r^0 + b^0 r^j q_j + r^0 b^i q_i + r^j b^i (-\delta_{ij} 1 + E_{ijk} q_k)$$

$$= b^0 r^0 + b^0 r^j q_j + r^0 b^i q_i + r^j b^i (-\delta_{ij} 1) + r^j b^i E_{ijk} q_k$$

donde se puede identificar las partes vectorial y escalar

$$= b^0 r^0 - r^j b^i \delta_{ij} 1 + b^0 r^j q_j + r^0 b^i q_i + r^j b^i E_{ijk} q_k$$

$$= b^0 r^0 - b \cdot r + r^0 b + b^0 r + b \times r$$

$$C) |d\rangle = |b\rangle \odot |r\rangle = b^\alpha |q_\alpha\rangle \odot r^\beta |q_\beta\rangle$$

$$= b^\alpha r^\beta (|q_\alpha\rangle |q_\beta\rangle)$$

$$= b^0 r^0 |q_0 q_0\rangle + b^0 r^i |q_0 q_i\rangle + r^0 b^i |q_i q_0\rangle + b^i r^j |q_i q_j\rangle$$

donde se puede expresar de la forma

$$= b^0 r^0 + b^0 r^j q_j + r^0 b^i q_i + r^{ij} b^i (-\delta_{ij} 1 + \epsilon_{ijk} q_k)$$

$$= b^0 r^0 + b^0 r^j q_j + r^0 b^i q_i + r^{ij} b^i (-8_{ij}) + r^{ij} b^i \epsilon_{ijk} q_k$$

donde se puede identificar las partes vectorial y escalar

$$= b^0 r^0 - r^{ij} b^i \delta_{ij} 1 + b^0 r^j q_j + r^0 b^i q_i + r^{ij} b^i \epsilon_{ijk} q_k$$

$$= b^0 r^0 - b \cdot r + r^0 b + b^0 r + b \times r$$

$$\text{C) } |d\rangle = |b\rangle \odot |r\rangle = b^0 |q_\alpha\rangle \odot r^0 |q_\beta\rangle$$

$$= b^\alpha r^\beta (|q_\alpha\rangle |q_\beta\rangle)$$

$$= b^0 r^0 |q_\alpha q_\alpha\rangle + b^0 r^j |q_\alpha q_j\rangle + r^0 b^i |q_\alpha q_\alpha\rangle + b^i r^j |q_\alpha q_j\rangle$$

$$= b^0 r^0 |q_\alpha\rangle + b^0 r^j |q_j\rangle + b^i r^j |q_i\rangle + b^i r^j (-\delta_{ij} |q_\alpha\rangle + \epsilon_{ijk} |q_k\rangle)$$

donde al agrupar se obtiene

$$q = b^0 r^0 - b^j r^j ; \quad d^i = b^0 r^i + r^0 b^i + \epsilon_{ijk} b^j r^k$$

se toma la parte simétrica y antisimétrica de d^i y la reescribimos

$$S^{(0)} = b^0 r^j + r^0 b^j ; \quad A^{[jk]i} = \epsilon_{ijk} b^j r^k$$

de forma que ahora se agrupa la parte escalar y vectorial

$$|d\rangle = q |q_0\rangle + S^{(0)} \delta_\alpha^0 |q_j\rangle + A^{[jk]i} b^j r_k |q_i\rangle$$

d) $a = b^0 i - b^1 j - b^2 k \rightarrow$ escalar

$$S^{(ij)} = b^0 j + b^1 k \rightarrow \text{parte simétrica del producto de las componentes vectoriales}$$

$$A^{(ijk)} = b^0 i + b^1 k \rightarrow \text{parte antisimétrica del producto vectorial}$$

• Se obtiene un predecesor por el comportamiento
antisimétrico del producto de dos cuaterniones

$$i \quad j \quad k \quad -i \quad -j \quad -k$$

e) Para que sean base se necesita independencia lineal, por lo tanto

$\alpha v_1 + \beta v_2 + \gamma v_3 + \lambda v_4 = 0$

$$\begin{pmatrix} 0 & x \\ x & 0 \end{pmatrix} + \begin{pmatrix} 0 & -\beta l \\ \alpha & 0 \end{pmatrix} + \begin{pmatrix} \gamma & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = 0$$

$$\begin{pmatrix} (\gamma+\lambda)x-\beta l \\ \alpha+\beta l \end{pmatrix} = 0 \Rightarrow \lambda+\gamma=0 \Rightarrow \lambda=-\gamma \quad \text{Por lo tanto } \alpha=\beta=\gamma=\lambda=0$$

$$\begin{pmatrix} \alpha+\beta l & \lambda-\gamma \\ \alpha-\beta l & \alpha+\beta l \end{pmatrix} = 0 \Rightarrow \alpha=\beta l$$

$$\lambda-\gamma=0 \Rightarrow \lambda=\gamma$$

• Para la matriz 2×2 se toma la relación

$$|b\rangle = \begin{pmatrix} z & w \\ -w^* & z^* \end{pmatrix} = \alpha \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \beta \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \gamma \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} z & w \\ -w^* & z^* \end{pmatrix} = \begin{pmatrix} \gamma+\lambda & x-\beta l \\ \alpha+\beta l & \lambda-\gamma \end{pmatrix}$$

$$z\alpha + z = 1 \cdot \alpha^* = 1 \cdot \alpha - \lambda \alpha$$

$$-w^* = \alpha + \beta \alpha - \lambda \alpha$$

• Se obtiene un pseudoscalar por el comportamiento antisimétrico del producto de dos cuaterniones

$$-i - i - ii - 4i$$

c) Para que sea base se necesita independencia lineal, por lo tanto

$$\alpha O_1 + \beta O_2 + \gamma O_3 + \lambda O_4 = 0$$

$$\begin{pmatrix} 0 & \alpha \\ \alpha & 0 \end{pmatrix} + \begin{pmatrix} 0 & -\beta i \\ \beta i & 0 \end{pmatrix} + \begin{pmatrix} \gamma & 0 \\ 0 & \gamma \end{pmatrix} + \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = 0$$

$$\begin{pmatrix} \alpha & \beta i \\ \beta i & \lambda - \gamma \end{pmatrix} = 0 \Rightarrow \begin{pmatrix} \alpha & \beta i \\ \beta i & \lambda - \gamma \end{pmatrix} = 0 \Rightarrow \alpha = \beta i \\ \alpha + \beta i = 0 \Rightarrow \alpha = -\beta i \\ \alpha - \beta i = 0 \Rightarrow \alpha = \beta i \\ \lambda - \gamma = 0 \Rightarrow \lambda = \gamma$$

• Para la matriz 2×2 se toma la relación

$$|b\rangle = \begin{pmatrix} z & w \\ -w^* & z^* \end{pmatrix} = \alpha \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \beta \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \gamma \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$$

$$\begin{pmatrix} z & w \\ -w^* & z^* \end{pmatrix} = \begin{pmatrix} \gamma + \lambda & \alpha - \beta i \\ \alpha + \beta i & \lambda - \gamma \end{pmatrix}$$

$$z = \gamma + \lambda \quad | \cdot z^* = \lambda - (z - \lambda)$$

$$w = \alpha - \beta i \quad | \quad \lambda = \frac{z^* + z}{2}$$

$$-w^* = \alpha + (\alpha - w)$$

$$\alpha = \frac{w - w^*}{2}$$

$$-w^* = \alpha + \beta i \quad | \quad z = \gamma + (z^* + \gamma)$$

$$z^* = \lambda - \gamma \quad | \quad \gamma = \frac{z - z^*}{2}$$

$$w = (-w^* - \beta i) - \beta i$$

$$\beta = i(w + w^*)$$

f) Para mostrar que es una posible representación para la base de cuaterniones analizaremos la independencia lineal:

$$\alpha|q_1\rangle + \beta|q_2\rangle + \gamma|q_3\rangle + \lambda|q_4\rangle = 0$$

$$\begin{bmatrix} \alpha & 1 & 0 & 0 \\ -\lambda & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} = 0$$

$$\begin{bmatrix} \lambda - \alpha & -\gamma & \beta \\ -\kappa & \lambda & -\beta \\ \gamma & \beta & \lambda & \alpha \\ \beta & -\gamma & -\kappa & \lambda \end{bmatrix} = 0 \Rightarrow \begin{array}{l} \lambda = 0, \quad \kappa = 0, \quad \beta = 0, \quad \gamma = 0 \\ \alpha = 1 \end{array}$$

$$g) \langle a | b \rangle = |a\rangle^* \odot |b\rangle$$

$$\begin{aligned} 1. \langle a | a \rangle &= |a\rangle^* \odot |a\rangle \\ &= (a^0 - a^i |q_i\rangle)(a^0 + a^i |q_i\rangle) \\ &= a^0 a^0 |q_0\rangle + a^0 a^i |q_0\rangle |q_i\rangle - a^0 a^i |q_i\rangle |q_0\rangle - a^i a^i |q_i\rangle |q_i\rangle \\ &= a^0 a^0 1 - a^i a^i (-1) \\ &= a^0 a^0 + a^i a^i \end{aligned}$$

$$2. \langle a | b \rangle = \langle b | a \rangle^*$$

$$\begin{aligned} \langle a | b \rangle &= |a\rangle^* \odot |b\rangle = (a^0 |q_0\rangle - a^i |q_i\rangle)(b^0 |q_0\rangle + b^i |q_i\rangle) \\ &= a^0 b^0 |q_0\rangle^* |q_0\rangle + a^0 b^i |q_0\rangle^* |q_i\rangle - b^0 a^i |q_i\rangle^* |q_0\rangle - a^i b^i |q_i\rangle^* |q_i\rangle \\ &= a^0 b^0 1 + a^0 b^i |q_i\rangle - b^0 a^i |q_i\rangle - a^i b^i |q_i\rangle \end{aligned}$$

$$\begin{aligned} \langle b | a \rangle^* &= (|b\rangle^* \odot |a\rangle)^* = |b\rangle \odot |a\rangle^* \\ &= (b^0 |q_0\rangle + b^i |q_i\rangle)(a^0 |q_0\rangle - a^i |q_i\rangle) \\ &= b^0 a^0 |q_0\rangle^* |q_0\rangle + b^0 a^i |q_0\rangle^* |q_i\rangle - a^0 b^i |q_i\rangle^* |q_0\rangle - a^i b^0 |q_i\rangle^* |q_i\rangle \end{aligned}$$

$$\begin{bmatrix} \lambda & x & -y & p \\ -x & \lambda & -y & p \\ y & -y & \lambda & x \\ p & -y & -x & \lambda \end{bmatrix} = 0 \Rightarrow \text{dando } \lambda \neq 0, x=0, p=0, y=0$$

g) $\langle a|b\rangle = |a\rangle^* \circ |b\rangle$

$$\begin{aligned} 1. \langle a|a\rangle &= |a\rangle^* \circ |a\rangle \\ &= (a^0 - a^i|q_i\rangle)(a^0 + a^i|q_i\rangle) \\ &= a^0 a^0|q_0\rangle|q_0\rangle + a^0 a^i|q_0\rangle|q_i\rangle - a^i a^0|q_i\rangle|q_0\rangle - a^i a^i|q_i\rangle|q_i\rangle \\ &= a^0 a^0 - a^i a^i(-1) \\ &= a^0 a^0 + a^i a^i \end{aligned}$$

2. $\langle a|b\rangle = \langle b|a\rangle^*$

$$\begin{aligned} \langle a|b\rangle &= |a\rangle^* \circ |b\rangle = (a^0|q_0\rangle - a^i|q_i\rangle)(b^0|q_0\rangle + b^i|q_i\rangle) \\ &= a^0 b^0|q_0\rangle|q_0\rangle + a^0 b^i|q_0\rangle|q_i\rangle - b^0 a^i|q_i\rangle|q_0\rangle - a^i b^i|q_i\rangle|q_i\rangle \\ &= a^0 b^0 + a^0 b^i|q_i\rangle + b^0 a^i|q_i\rangle - a^i b^i|q_i\rangle|q_i\rangle \end{aligned}$$

$$\begin{aligned} \langle b|a\rangle^* &= (|b\rangle^* \circ |a\rangle)^* = |b\rangle \circ |a\rangle^* \\ &= (b^0|q_0\rangle + b^i|q_i\rangle)(a^0|q_0\rangle + a^i|q_i\rangle) \\ &= b^0 a^0|q_0\rangle|q_0\rangle + b^0 a^i|q_0\rangle|q_i\rangle + a^0 b^i|q_0\rangle|q_i\rangle - a^i b^i|q_i\rangle|q_i\rangle \end{aligned}$$

3. $\langle a|\alpha b + \beta c\rangle = \alpha \langle a|b\rangle + \beta \langle a|c\rangle$

$$\begin{aligned} = |a\rangle^* \circ (\alpha b + \beta c) &= (a^0 - a^i|q_i\rangle)(\alpha b + \beta c) \\ &= \alpha a^0(b^0 + b^i|q_i\rangle) + \beta a^0(c^0 + c^i|q_i\rangle) - \alpha a^i(b^0 + b^i|q_i\rangle) - \beta a^i(c^0 + c^i|q_i\rangle) \\ &= \alpha a^0 b^0 + \alpha a^0 b^i|q_i\rangle + \beta a^0 c^0 + \beta a^0 c^i|q_i\rangle - \alpha a^i b^0|q_i\rangle - \alpha a^i b^i|q_i\rangle - \beta a^i c^0|q_i\rangle - \beta a^i c^i|q_i\rangle \\ &= (\alpha a^0 b^0 + \alpha a^0 b^i|q_i\rangle - \alpha b^0 a^i|q_i\rangle - \alpha b^i a^i|q_i\rangle) + (\beta a^0 c^0 + \beta a^0 c^i|q_i\rangle - \beta c^0 a^i|q_i\rangle - \beta c^i a^i|q_i\rangle) \\ &= (\alpha a^0 b^0 + \alpha a^0 b^i|q_i\rangle - b^0 a^i|q_i\rangle - a^i b^i|q_i\rangle|q_i\rangle) + \beta (a^0 c^0 + a^0 c^i|q_i\rangle - c^0 a^i|q_i\rangle + a^i c^i|q_i\rangle|q_i\rangle) \\ &= \alpha \langle a|b\rangle + \beta \langle a|c\rangle \end{aligned}$$

$$\begin{aligned}
 4. & \langle \alpha a + \beta b | c \rangle = \alpha^* \langle a | c \rangle + \beta^* \langle b | c \rangle \\
 & = \langle a | a + b | c \rangle = |\alpha a + \beta b \rangle \circ |c\rangle \\
 & = (\alpha^*(a^0 - a^i | q_i \rangle) + (\beta^*(b^0 - b^i | q_i \rangle)) (c^0 + c^i | q_m \rangle) \\
 & = \alpha^* a^0 c^0 - \alpha^* a^i | q_i \rangle c^0 + \beta^* b^0 c^0 - \beta^* b^i | q_i \rangle c^0 + \alpha^* a^0 c^i | q_m \rangle + \beta^* b^0 c^i | q_m \rangle - \beta^* b^i c^i | q_j \rangle | q_m \rangle \\
 & = (\alpha^* a^0 c^0 - \alpha^* a^i | q_i \rangle c^0 + \alpha^* a^0 c^i | q_m \rangle - \alpha^* a^i | q_i \rangle c^i | q_m \rangle) + (\beta^* b^0 c^0 - \beta^* b^i | q_i \rangle c^0 + \beta^* b^0 c^i | q_m \rangle - \beta^* b^i c^i | q_j \rangle | q_m \rangle) \\
 & = \alpha^* (a^0 c^0 - a^i | q_i \rangle c^0 + a^0 c^i | q_m \rangle - a^i | q_i \rangle c^i | q_m \rangle) + \beta^* (b^0 c^0 - b^i | q_i \rangle c^0 + b^0 c^i | q_m \rangle - b^i c^i | q_j \rangle | q_m \rangle) \\
 & = \alpha^* \langle a | c \rangle + \beta^* \langle b | c \rangle
 \end{aligned}$$

$$5. \langle q | 0 \rangle = \langle 0 | q \rangle = 0$$

$$\begin{aligned}
 \langle q | 0 \rangle &= |q\rangle \circ |0\rangle \\
 &= (a^0 - a^i | q_i \rangle) (0) \\
 &= 0a^0 - 0a^i | q_i \rangle \\
 &= 0 + 0 \\
 &= 0
 \end{aligned}$$

$$\begin{aligned}
 \langle 0 | q \rangle &= |0\rangle \circ |q\rangle \\
 &= (0)(a^0 + a^i | q_i \rangle) \\
 &= 0a^0 + 0a^i | q_i \rangle \\
 &= 0 + 0 \\
 &= 0
 \end{aligned}$$

$$h) \langle q | b \rangle = \frac{1}{2} [\langle \widehat{q} | b \rangle - | q \rangle \circ \langle \widehat{q} | b \rangle \circ | q \rangle]$$

$$\begin{aligned}
 &= (K^* a^0 c^0 |q_1\rangle + K^* a^0 b^0 |q_2\rangle - K^* a^0 c^0 |q_3\rangle + K^* a^0 b^0 |q_4\rangle) + (P^* c^0 b^0 + P^* c^0 b^1 |q_1\rangle + P^* b^0 c^0 |q_2\rangle - P^* b^0 c^1 |q_3\rangle + P^* b^0 c^0 |q_4\rangle) \\
 &= \alpha^* (K^* a^0 - (c^0 b^0 + c^0 b^1) |q_1\rangle + a^0 c^0 |q_2\rangle - a^0 c^1 |q_3\rangle + a^0 c^0 |q_4\rangle) + \beta^* (c^0 b^0 - (c^0 b^1 + b^0 c^0) |q_1\rangle + b^0 c^0 |q_2\rangle - b^0 c^1 |q_3\rangle + b^0 c^0 |q_4\rangle) \\
 &= \alpha^* (q_1 |q\rangle + q_2 |q\rangle) + \beta^* (b_1 |q\rangle)
 \end{aligned}$$

$$5. \langle q | 0 \rangle = \langle 0 | q \rangle = 0$$

$$\begin{aligned}
 \langle q | 0 \rangle &= |q\rangle^\dagger \circ |0\rangle \\
 &= (a^0 - a^1 b^0) |0\rangle \\
 &= 0a^0 - 0a^1 b^0 |0\rangle \\
 &= 0 + 0 \\
 &= 0
 \end{aligned}$$

$$\begin{aligned}
 \langle 0 | q \rangle &= |0\rangle^\dagger \circ |q\rangle \\
 &= |0\rangle (a^0 + a^1 b^0) \\
 &= 0a^0 + 0a^1 b^0 |q\rangle \\
 &= 0 + 0 \\
 &= 0
 \end{aligned}$$

$$\begin{aligned}
 h | \langle q | b \rangle &= \frac{1}{2} [\langle q | b \rangle - |q\rangle \otimes (\langle q | b \rangle |q\rangle)] \\
 &= \frac{1}{2} [|q\rangle^\dagger \circ |b\rangle - |q\rangle \otimes (|q\rangle^\dagger \circ |b\rangle |q\rangle)] \\
 &= \frac{1}{2} [(a^0 - a^1 |q_1\rangle)(b^0 + b^1 |q_2\rangle) - |q_1\rangle \otimes (a^0 - a^1 |q_1\rangle)(b^0 + b^1 |q_2\rangle) \otimes |q_1\rangle] \\
 &= \frac{1}{2} [a^0 b^0 + a^0 b^1 |q_2\rangle - b^0 a^1 |q_1\rangle - a^1 b^1 |q_1\rangle |q_2\rangle - |q_1\rangle (a^0 b^0 + a^0 b^1 |q_2\rangle - b^0 a^1 |q_1\rangle - a^1 b^1 |q_1\rangle |q_2\rangle) |q_1\rangle] \\
 &= \frac{1}{2} [a^0 b^0 + a^0 b^1 |q_2\rangle - b^0 a^1 |q_1\rangle - a^1 b^1 |q_1\rangle |q_2\rangle + a^0 b^0 + a^0 b^1 |q_2\rangle - b^0 a^1 |q_1\rangle - a^1 b^1 |q_1\rangle |q_2\rangle] \\
 &= \frac{1}{2} [2a^0 b^0 + 2a^0 b^1 |q_2\rangle - 2b^0 a^1 |q_1\rangle - 2a^1 b^1 |q_1\rangle |q_2\rangle] \\
 &= \frac{1}{2} [2(a^0 + a^0 b^1) |q_2\rangle - 2b^0 a^1 |q_1\rangle - 2a^1 b^1 |q_1\rangle |q_2\rangle] \\
 &\cancel{=} a^0 b^0 + a^0 b^1 |q_2\rangle - b^0 a^1 |q_1\rangle - a^1 b^1 |q_1\rangle |q_2\rangle
 \end{aligned}$$

$$h.1) \langle a | a \rangle = | |a\rangle | |^2$$

$$\begin{aligned}
 \cdot \langle a | a \rangle &= \frac{1}{2} [\langle \tilde{a} | \tilde{a} \rangle - |a, \rangle \circ \langle \tilde{a} | \tilde{a} \rangle \odot |a, \rangle] \\
 &= \frac{1}{2} [|a\rangle^* \odot |a\rangle - |a, \rangle \circ |a\rangle^* \odot |a\rangle \odot |a, \rangle] \\
 &= \frac{1}{2} [(a^0 - a^1 |a, \rangle)(a^0 + a^1 |a, \rangle) - |a, \rangle \circ (a^0 - a^1 |a, \rangle)(a^0 + a^1 |a, \rangle) \odot |a, \rangle] \\
 &= \frac{1}{2} [a^0 a^0 + a^0 a^1 |a, \rangle - a^0 a^1 |a, \rangle - a^1 a^0 |a, \rangle - |a, \rangle \circ (a^0 a^0 + a^0 a^1 |a, \rangle) - a^1 a^0 |a, \rangle - a^1 a^1 |a, \rangle \odot |a, \rangle] \\
 &= \frac{1}{2} [a^0 a^0 + a^1 a^1 - |a, \rangle \circ (a^0 a^0 + a^1 a^1) \odot |a, \rangle] \\
 &= \frac{1}{2} [a^0 a^0 + a^1 a^1 + a^0 a^0 + a^1 a^1] \\
 &= \frac{1}{2} [2a^0 a^0 + 2a^1 a^1] \\
 &= \frac{1}{2} 2 [(a^0)^2 + (a^1)^2] = (a^0)^2 + (a^1)^2
 \end{aligned}$$

$$h.2) \langle a | b \rangle = \langle b | a \rangle^*$$

Ya se probó $\langle a | b \rangle$ por lo tanto se mostrará $\langle b | a \rangle^*$

$$\begin{aligned}
 \langle b | a \rangle^* &= \frac{1}{2} [\langle \tilde{b} | \tilde{a} \rangle^* - |b, \rangle \circ (\langle \tilde{b} | \tilde{a} \rangle)^* \odot |a, \rangle] \\
 &= \frac{1}{2} [(|b\rangle^* \odot |a\rangle)^* - |b, \rangle \circ (|b\rangle^* \odot |a\rangle)^* \odot |a, \rangle] \\
 &= \frac{1}{2} [(b^0 + b^1 |a, \rangle)(a^0 - a^1 |a, \rangle) - |b, \rangle \circ (b^0 + b^1 |a, \rangle)(a^0 - a^1 |a, \rangle) \odot |a, \rangle]
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{2} [a^0 a^0 + a^1 a^1 - 19_7 \odot (a^0 a^0 + a^1 a^1) \odot 19_7] \\
 &= \frac{1}{2} [a^0 a^0 + a^1 a^1 + a^0 a^0 + a^1 a^1] \\
 &= \frac{1}{2} [2a^0 a^0 + 2a^1 a^1] \\
 &= \frac{1}{2} 2 [(a^0)^2 + (a^1)^2] = (a^0)^2 + (a^1)^2
 \end{aligned}$$

$$h. 2) \langle a | b \rangle = \langle b | a \rangle^*$$

ya se probó $\langle a | b \rangle$ por lo tanto se mostrará $\langle b | a \rangle^*$

$$\begin{aligned}
 \langle b | a \rangle^* &= \frac{1}{2} [\langle \widehat{b} | \widehat{a} \rangle^* - 19_7 \odot (\langle \widehat{b} | \widehat{a} \rangle^* \odot 19_7)] \\
 &= \frac{1}{2} [(\langle b | \odot a \rangle^* - 19_7 \odot ((\langle b | \odot a \rangle^* \odot 19_7)] \\
 &= \frac{1}{2} [(b^0 + b^1 | 19_7 \rangle)(a^0 - a^1 | 19_7 \rangle) - 19_7 \odot (b^0 + b^1 | 19_7 \rangle)(a^0 - a^1 | 19_7 \rangle) \odot 19_7] \\
 &= \frac{1}{2} [b^0 a^0 - b^0 a^1 | 19_7 \rangle + a^0 b^1 | 19_7 \rangle - a^1 b^1 | 19_7 \rangle - 19_7 \odot (b^0 a^0 - b^0 a^1 | 19_7 \rangle + a^0 b^1 | 19_7 \rangle - a^1 b^1 | 19_7 \rangle) \odot 19_7] \\
 &= \frac{1}{2} [b^0 a^0 - b^0 a^1 | 19_7 \rangle + a^0 b^1 | 19_7 \rangle - a^1 b^1 | 19_7 \rangle | 19_7 \rangle + b^0 a^0 + b^0 a^1 | 19_7 \rangle + a^0 b^1 | 19_7 \rangle - a^1 b^1 | 19_7 \rangle | 19_7 \rangle] \\
 &= \frac{1}{2} [2(b^0 a^0) - 2(b^0 a^1 | 19_7 \rangle) + 2(a^0 b^1 | 19_7 \rangle) - 2(a^1 b^1 | 19_7 \rangle | 19_7 \rangle)] \\
 &= \frac{1}{2} 2 [b^0 a^0 - b^0 a^1 | 19_7 \rangle + a^0 b^1 | 19_7 \rangle - a^1 b^1 | 19_7 \rangle | 19_7 \rangle] \\
 &= b^0 a^0 - b^0 a^1 | 19_7 \rangle + a^0 b^1 | 19_7 \rangle - a^1 b^1 | 19_7 \rangle | 19_7 \rangle
 \end{aligned}$$

$$\cdot \langle ab \rangle = \frac{1}{2} [\langle a|b\rangle - \langle b|a\rangle] + \langle a|b\rangle \otimes \langle b|a\rangle$$

$$3. \langle a|\alpha b + \beta c \rangle =$$

$$= \frac{1}{2} [\langle a|ab + bc \rangle - \langle b|a + \langle a|bc \rangle \otimes \langle b|c \rangle]$$

$$= \frac{1}{2} [(\alpha^0 - \alpha^1|q_1\rangle) (\alpha b^0 + \beta b^1|q_1\rangle) + \langle q_1| \otimes (\alpha^0 - \alpha^1|q_1\rangle) (\alpha b^0 + \beta b^1|q_1\rangle)]$$

$$= \frac{1}{2} [(\alpha^0 - \alpha^1|q_1\rangle) (\alpha b^0 + \beta b^1|q_1\rangle) (\alpha^0 - \alpha^1|q_1\rangle) (\alpha b^0 + \beta b^1|q_1\rangle) + \dots]$$

$$= \frac{1}{2} [(\alpha^0 - \alpha^1|q_1\rangle) (\alpha b^0 + \beta b^1|q_1\rangle) (\alpha^0 - \alpha^1|q_1\rangle) (\beta c^0 + \beta c^1|q_1\rangle) - \langle q_1| \otimes (\alpha^0 - \alpha^1|q_1\rangle) (\alpha b^0 + \beta b^1|q_1\rangle) + \dots]$$

$$\dots (\alpha^0 - \alpha^1|q_1\rangle) (\beta c^0 + \beta c^1|q_1\rangle) \otimes |q_1\rangle]$$

$$= \frac{1}{2} [(\alpha^0 b^0 + \alpha^0 b^1|q_1\rangle - \alpha b^0 a^1|q_1\rangle - \alpha b^1 a^1|q_1\rangle) + (\beta a^0 c^0 + \beta a^0 c^1|q_1\rangle - \beta c^0 a^1|q_1\rangle - \beta c^1 a^1|q_1\rangle) + \dots]$$

$$- \langle q_1| \otimes [(\alpha^0 b^0 + \alpha^0 b^1|q_1\rangle - \alpha b^0 a^1|q_1\rangle - \alpha b^1 a^1|q_1\rangle + \beta a^0 c^0 + \beta a^0 c^1|q_1\rangle - \beta c^0 a^1|q_1\rangle - \beta c^1 a^1|q_1\rangle \otimes |q_1\rangle]$$

$$= \frac{1}{2} [(\alpha^0 b^0 + \alpha^0 b^1|q_1\rangle - \alpha b^0 a^1|q_1\rangle - \alpha b^1 a^1|q_1\rangle + \beta a^0 c^0 + \beta a^0 c^1|q_1\rangle - \beta c^0 a^1|q_1\rangle - \beta c^1 a^1|q_1\rangle + \dots]$$

$$\dots (\alpha^0 b^0 + \alpha^0 b^1|q_1\rangle - \alpha b^0 a^1|q_1\rangle - \alpha b^1 a^1|q_1\rangle + \beta a^0 c^0 + \beta a^0 c^1|q_1\rangle - \beta c^0 a^1|q_1\rangle - \beta c^1 a^1|q_1\rangle)]$$

$$= \frac{1}{2} [(\alpha^0 b^0 + \alpha^0 b^1|q_1\rangle - \alpha b^0 a^1|q_1\rangle - \alpha b^1 a^1|q_1\rangle + \alpha^0 b^0 + \alpha^0 b^1|q_1\rangle - \alpha b^0 a^1|q_1\rangle - \alpha b^1 a^1|q_1\rangle) + \dots]$$

$$\dots + (\beta a^0 c^0 + \beta a^0 c^1|q_1\rangle - \beta c^0 a^1|q_1\rangle - \beta c^1 a^1|q_1\rangle + \beta a^0 c^0 + \beta a^0 c^1|q_1\rangle - \beta c^0 a^1|q_1\rangle - \beta c^1 a^1|q_1\rangle)]$$

$$= \frac{1}{2} [(2\alpha^0 b^0 + 2\alpha^0 b^1|q_1\rangle - 2\alpha b^0 a^1|q_1\rangle - 2\alpha b^1 a^1|q_1\rangle) + (2\beta a^0 c^0 + 2\beta a^0 c^1|q_1\rangle - 2\beta c^0 a^1|q_1\rangle - 2\beta c^1 a^1|q_1\rangle)]$$

$$= \frac{1}{2} \cdot 2\alpha (\alpha^0 b^0 + 2\alpha^0 b^1|q_1\rangle - b^0 a^1|q_1\rangle - b^1 a^1|q_1\rangle) + \frac{1}{2} \cdot 2\beta (\beta a^0 c^0 + 2\beta a^0 c^1|q_1\rangle - c^0 a^1|q_1\rangle - c^1 a^1|q_1\rangle)$$

$$= \alpha \langle a|b \rangle + \beta \langle a|c \rangle$$

$$4. \langle \alpha a + \beta b | c \rangle = \alpha^* \langle a | c \rangle + \beta^* \langle b | c \rangle$$

$$= \frac{1}{2} \left[\langle \widehat{\alpha a + \beta b} | c \rangle - |c\rangle \otimes \langle \alpha a + \beta b | c \rangle \right]$$

$$= \frac{1}{2} \left[(\alpha a + \beta b)^* |c\rangle - |c\rangle \otimes (\alpha a + \beta b) \right]$$

$$= \frac{1}{2} \left[(\alpha^* a^* + \beta^* b^*) (c^* + |q_2\rangle) - |q_2\rangle \otimes (\alpha^* a^* + \beta^* b^*) (c^* + |q_2\rangle) \otimes |q_1\rangle \right]$$

$$= \frac{1}{2} \left[(\alpha^* a^* - \alpha^* a^* |q_2\rangle) (c^* + |q_2\rangle) + (\beta^* b^* - \beta^* b^* |q_2\rangle) (c^* + |q_2\rangle) - |q_2\rangle \otimes (\alpha^* a^* - \alpha^* a^* |q_2\rangle) (c^* + |q_2\rangle) + (\beta^* b^* - \beta^* b^* |q_2\rangle) (c^* + |q_2\rangle) \right. \\ \left. |q_1\rangle \right]$$

$$= \frac{1}{2} \left[\alpha^* a^* c^* + \alpha^* a^* |q_2\rangle - \alpha^* a^* |q_2\rangle - \alpha^* a^* |q_2\rangle \otimes |q_2\rangle + \beta^* b^* c^* + \beta^* b^* |q_2\rangle - \beta^* b^* |q_2\rangle - \beta^* b^* |q_2\rangle \otimes |q_2\rangle - |q_2\rangle \otimes \dots \right. \\ \left. \dots (\alpha^* a^* c^* + \alpha^* a^* |q_2\rangle - \alpha^* a^* |q_2\rangle - \alpha^* a^* |q_2\rangle \otimes |q_2\rangle + \beta^* b^* c^* + \beta^* b^* |q_2\rangle - \beta^* b^* |q_2\rangle \otimes |q_2\rangle) \otimes |q_1\rangle \right]$$

$$= \frac{1}{2} \left[\alpha^* a^* c^* + \alpha^* a^* |q_2\rangle - \alpha^* c^* |q_2\rangle - \alpha^* a^* |q_2\rangle \otimes |q_2\rangle + \beta^* b^* c^* + \beta^* b^* |q_2\rangle - \beta^* b^* |q_2\rangle \otimes |q_2\rangle + \dots \right. \\ \left. \dots \alpha^* a^* c^* + \alpha^* a^* |q_2\rangle - \alpha^* c^* |q_2\rangle - \alpha^* a^* |q_2\rangle \otimes |q_2\rangle + \beta^* b^* c^* + \beta^* b^* |q_2\rangle - \beta^* b^* |q_2\rangle \otimes |q_2\rangle - \beta^* b^* |q_2\rangle \otimes |q_1\rangle \right]$$

$$= \frac{1}{2} [(2\alpha^* a^* c^* + 2\alpha^* a^* |q_2\rangle - 2\alpha^* c^* |q_2\rangle - 2\alpha^* a^* |q_2\rangle \otimes |q_2\rangle) + (2\beta^* b^* c^* + 2\beta^* b^* |q_2\rangle - 2\beta^* c^* |q_2\rangle - 2\beta^* b^* |q_2\rangle \otimes |q_2\rangle)]$$

$$= \frac{1}{2} (\alpha^* (a^* c^* + a^* |q_2\rangle - c^* |q_2\rangle - a^* |q_2\rangle \otimes |q_2\rangle) + \frac{1}{2} \cdot 2\beta^* (b^* c^* + b^* |q_2\rangle - c^* |q_2\rangle - b^* |q_2\rangle \otimes |q_2\rangle))$$

$$= \alpha^* \langle a | c \rangle + \beta^* \langle b | c \rangle$$

5. $\langle a | \alpha \rangle, \langle b | \alpha \rangle \in W$ se tiene:

$$= \frac{1}{2} \left[(\alpha^* a^0 + \alpha^* a^1 |q_1\rangle) (|0\rangle \langle 1| q_2\rangle) + (\beta^0 + \beta^1 |q_1\rangle) (|0\rangle \langle 1| q_2\rangle) - |q_1\rangle \langle 1| q_2\rangle (|0\rangle \langle 1| q_1\rangle) + (\beta^0 b^0 + \beta^1 b^1 |q_1\rangle) (|0\rangle \langle 1| q_2\rangle) \right]$$

$|0|q_1\rangle$

$$= \frac{1}{2} \left[\alpha^* a^0 + \alpha^* a^1 |q_1\rangle - \alpha^* a^2 |q_1\rangle - \alpha^* a^3 |q_1\rangle + \beta^0 + \beta^1 |q_1\rangle - \beta^2 b^0 + \beta^3 b^1 |q_1\rangle - \beta^4 b^2 + \beta^5 b^3 |q_1\rangle - |q_1\rangle \langle 0| \dots \right.$$

$$\left. \dots (\alpha^* a^0 + \alpha^* a^1 |q_1\rangle - \alpha^* a^2 |q_1\rangle - \alpha^* a^3 |q_1\rangle - \beta^0 + \beta^1 |q_1\rangle - \beta^2 b^0 + \beta^3 b^1 |q_1\rangle - \beta^4 b^2 + \beta^5 b^3 |q_1\rangle - |q_1\rangle \langle 0| \dots) \right]$$

$$= \frac{1}{2} \left[\alpha^* a^0 + \alpha^* a^1 |q_1\rangle - \alpha^* a^2 |q_1\rangle - \alpha^* a^3 |q_1\rangle + \beta^0 + \beta^1 |q_1\rangle - \beta^2 b^0 + \beta^3 b^1 |q_1\rangle - \beta^4 b^2 + \beta^5 b^3 |q_1\rangle + \dots \right.$$

$$\left. \dots \alpha^* a^0 + \alpha^* a^1 |q_1\rangle - \alpha^* a^2 |q_1\rangle - \alpha^* a^3 |q_1\rangle - \beta^0 + \beta^1 |q_1\rangle - \beta^2 b^0 + \beta^3 b^1 |q_1\rangle - \beta^4 b^2 + \beta^5 b^3 |q_1\rangle \right]$$

$$= \frac{1}{2} \left[[2\alpha^* a^0 + 2\alpha^* a^1 |q_1\rangle - 2\alpha^* a^2 |q_1\rangle - 2\alpha^* a^3 |q_1\rangle] + (2\beta^0 + 2\beta^1 |q_1\rangle - 2\beta^2 b^0 + 2\beta^3 b^1 |q_1\rangle - 2\beta^4 b^2 + 2\beta^5 b^3 |q_1\rangle) \right]$$

$$= \frac{1}{2} \cdot 2\alpha^* (a^0 + a^1 |q_1\rangle - a^2 |q_1\rangle - a^3 |q_1\rangle) + \frac{1}{2} \cdot 2\beta^* (b^0 + b^1 |q_1\rangle - b^2 |q_1\rangle - b^3 |q_1\rangle)$$

$$= \alpha^* \langle q_1 | c \rangle + \beta^* \langle b | c \rangle$$

S. Sean $|q_1\rangle, |10\rangle \in \mathbb{H}$ se tiene

$$\langle q_1 | 10 \rangle = \frac{1}{2} [\langle \widetilde{q_1} | 10 \rangle - |q_1\rangle \langle \widetilde{q_1} | 10 \rangle + \langle \widetilde{q_1} | 10 \rangle \langle 0 | q_1 \rangle]$$

$$= \frac{1}{2} [a^0 \langle 0 | - |q_1\rangle \langle 0 | (a^* \langle 0 |) + \langle 0 | q_1 \rangle]$$

$$= \frac{1}{2} [0 - |q_1\rangle \langle 0 | 0 | q_1 \rangle]$$

$$= 0$$

$$\langle 0 | q_1 \rangle = \frac{1}{2} [\langle \widetilde{0} | q_1 \rangle - |q_1\rangle \langle \widetilde{0} | q_1 \rangle + \langle \widetilde{0} | q_1 \rangle \langle 0 | q_1 \rangle] = \frac{1}{2} [\widetilde{0} \langle 0 | - |q_1\rangle \langle 0 | (0^* \langle 0 |) + \langle 0 | q_1 \rangle]$$

$$= 0$$

$$\langle 0 | q_1 \rangle = \langle q_1 | 10 \rangle = 0$$

$$\begin{aligned}
 i) \|n(\vec{a})\| &= \|\vec{a}\| = \sqrt{|\vec{a}|^2} = \sqrt{|\vec{a}^0|^2 + |\vec{a}^i|^2} \\
 &= \sqrt{(q^0 - q^i \langle \vec{a}_i \rangle)^2 + (q^i + q^0 \langle \vec{a}_i \rangle)^2} \\
 &= \sqrt{q^0 q^0 + (q^0)^2 \langle \vec{a}_i \rangle^2 - 2q^0 q^i \langle \vec{a}_i \rangle + q^i q^i + (q^i)^2 \langle \vec{a}_i \rangle^2} \\
 &= \sqrt{q^0 q^0 + q^i q^i} \\
 &= \sqrt{|\vec{a}|^2 + (\vec{a}_i)^2}
 \end{aligned}$$

note que se cumple con:

- $\|\vec{a}\| \geq 0$
- $\sqrt{(\vec{a}^0)^2 + (\vec{a}^i)^2} = K$; $K \in \mathbb{R}$
- Homogeneidad $\|\lambda \vec{a}\| = \sqrt{(\lambda \vec{a}^0)^2 + (\lambda \vec{a}^i)^2} = |\lambda| \|\vec{a}\|$
- Desigualdad triangular al ser norma euclídea

$$j) \overline{|\vec{a}|} = \frac{|\vec{a}|}{\|\vec{a}\|^2}$$

$$\begin{aligned}
 |\vec{a}| \odot \overline{|\vec{a}|} &= (q^0 + q^i \langle \vec{a}_i \rangle) \left(\frac{q^0 - q^i \langle \vec{a}_i \rangle}{(q^0)^2 + (q^i)^2} \right) \\
 &= \frac{q^0 q^0}{(q^0)^2 + (q^i)^2} - \frac{q^0 q^i \langle \vec{a}_i \rangle}{(q^0)^2 + (q^i)^2} + \frac{q^0 q^i \langle \vec{a}_i \rangle}{(q^0)^2 + (q^i)^2} - \frac{q^i q^i \langle \vec{a}_i \rangle}{(q^0)^2 + (q^i)^2} \\
 &= \frac{(q^0)^2}{(q^0)^2 + (q^i)^2} + \frac{(q^i)^2}{(q^0)^2 + (q^i)^2} = \frac{(q^0)^2 + (q^i)^2}{(q^0)^2 + (q^i)^2} = 1
 \end{aligned}$$

DD MM AA

$$K) |a\rangle = a_0 + a_i \quad |b\rangle = b_0 + b_i$$

$$|a\rangle \otimes |b\rangle = [a_0 b_0 - a_i b_i, a^0 b + b^0 a + a \times b] = (c_0, c_i) = |c\rangle \in \mathbb{H}$$

$$= |a\rangle = \begin{bmatrix} a_0 + a_1 i & a_2 + a_3 i \\ -a_2 + a_3 i & a_0 - a_1 i \end{bmatrix} = M_1 \quad |b\rangle = \begin{bmatrix} b_0 + b_1 i & b_2 + b_3 i \\ -b_2 + b_3 i & b_0 - b_1 i \end{bmatrix} = M_2$$

$$|d\rangle = \begin{bmatrix} d_0 + d_1 i & d_2 + d_3 i \\ -d_2 + d_3 i & d_0 - d_1 i \end{bmatrix} = M_3$$

$$|a\rangle \otimes (|b\rangle \otimes |d\rangle) = (|a\rangle \otimes |b\rangle) \otimes |d\rangle$$

$$M_1(M_2 M_3) = (M_1 M_2) M_3 \quad \text{note que de ayer se sabe que la multiplicación de matrices es asociativa por lo que la de los cuaterniones también}$$

$$I) |V'\rangle = \overline{|a\rangle} \otimes |V\rangle \otimes |a\rangle$$

$$= \frac{|a\rangle^* \otimes |V\rangle \otimes |a\rangle}{\| |a\rangle \|^2}$$

$$= \frac{(a^0 - a^i |a_i\rangle)}{(a^0)^2 + (a^i)^2} (V^j |a_j\rangle) \otimes |a\rangle$$

$$\begin{bmatrix} a_1 - a_3 a_4 & a_2 + a_3 a_5 \\ -a_2 + a_3 a_6 & a_4 - a_3 a_7 \end{bmatrix}$$

$$a \otimes (1b \otimes 1d) = (1a \otimes 1b \otimes 1d)$$

$M_1(M_2 M_3) = (M_1 M_2) M_3$ note que de anterior se sabe que la multiplicación de matrices es asociativa por lo que la de los cuaterniones también

$$I) |V'| = |\bar{a} \otimes V \otimes a|$$

$$= \frac{|a|^2}{||a||^2} \otimes |V| \otimes |a|$$

$$= \frac{(a^0 - a^i |q_i\rangle)}{(a^0)^2 + (a^i)^2} (V^j |q_j\rangle) \otimes |a\rangle$$

$$= \frac{(a^0 V^j |q_i\rangle - a^i V^j |q_i\rangle |q_i\rangle)}{(a^0)^2 + (a^i)^2} (a^0 + a^k |q_k\rangle)$$

$$= \frac{a^0 a^0 V^j |q_i\rangle}{(a^0)^2 + (a^i)^2} - \frac{a^0 a^i V^j |q_i\rangle |q_i\rangle}{(a^0)^2 + (a^i)^2} + \frac{a^0 a^k V^j |q_j\rangle |q_k\rangle}{(a^0)^2 + (a^i)^2} - \frac{a^i V^j |q_i\rangle |q_i\rangle |q_k\rangle}{(a^0)^2 + (a^i)^2}$$

$$= \frac{(a^0)^2 V^j |q_j\rangle}{(a^0)^2 + (a^i)^2} + \frac{(a^i)^2 V^j |q_j\rangle}{(a^0)^2 + (a^i)^2}$$

$$= \frac{(a^0)^2 + (a^i)^2}{(a^0)^2 + (a^i)^2} (V^j |q_j\rangle) = V^j |q_j\rangle$$

$$\|V'\|^2 = ((V^1)^2 + (V^2)^2 + (V^3)^2) \equiv ((V^1)^2 + (V^2)^2 + (V^3)^2) = \|V\|^2$$