Theoretical Guide

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1 Math

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MOD = 998'244'353

PI = acos(-1)
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1.1 Logarithm

$$\log_b mn = \log_b m + \log_b n \qquad log_b \frac{m}{n} = \log_b m - \log_n n \qquad \log_b n^p = p \log_b n$$

$$\log_b \sqrt[q]{n} = \frac{1}{q} \log_b n \qquad \log_b n = \log_a n \log_b a \qquad b^{\log_b k} = k$$

$$\log_b a = \frac{\log_c a}{\log_c b} \qquad \log_b a = \frac{1}{\log_a b} \qquad \log_b a \;\; \log_a c = \log_b c$$

$$\log_b 1 = 0 \qquad \log_b b = 1$$

2 Identities

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$$

$$\sum_{i=1}^{n} \frac{1}{i} \approx \log n \qquad \sum_{i=0}^{\infty} \frac{1}{2^i} = 2$$

3 Number Theory

$$(a+b) \bmod m = (a \bmod m + b \bmod m) \bmod m$$

$$(a-b) \bmod m = (a \bmod m - b \bmod m) \bmod m$$

$$(a \times b) \bmod m = ((a \bmod m) \times (b \bmod m)) \bmod m$$

$$a^b \bmod m = (a \bmod m)^b \bmod m$$

$$a \equiv b \pmod{m} \iff (b-a)|m$$

$$\gcd(a_1, a_2, a_3, a_4) = \gcd(a_1, \gcd(a_2, \gcd(a_3, a_4)))$$

$$\operatorname{lcm}(a,b) \times \gcd(a,b) = a \times b$$

$$\operatorname{lcm}(a,b) = \frac{a \times b}{\gcd(a,b)} = \frac{a}{\gcd(a,b)} \times b$$

3.1 Some Primes

999999937 1000000007 1000000009 1000000021 1000000033
$10^{18} - 11$ $10^{18} + 3$ $2305843009213693951 = 2^{61} - 1$
$998244353 = 119 \times 2^{23} + 1$ $10^6 + 3$
1055244485953076590 1392181229391707270 1178970662972334410
257900257981 584598951247 989509930063 105539556781
998244353 754974721 167772161 188244827 205587737
555130769 809747989 572255561 396588799 327208423
773840099 207936359 952818871 935456867 670948771

3.2 Number of Divisors

The number of divisors of n is about $\sqrt[3]{n}$.

n	6	60	360	5040	55440	720720	4324320	21621600
$d(n)$	4	12	24	60	120	240	384	576

Given the prime factorization of some number n:

$$n = p_1^{a_1} \cdot p_2^{a_2} \cdot p_3^{a_3}$$

The number of divisors will be $(a_1+1)(a_2+1)(a_3+1)$.

3.3 Large Prime Gaps

For numbers until 10^9 the largest gap is 400.
For numbers until 10^{18} the largest gap is 1500.

3.4 Fermat's Theorems

Let P be a prime number and a an integer, then:

$$a^p \equiv a \pmod p$$

$$a^{p-1} \equiv 1 \pmod p$$

Lemma: Let p be a prime number and a and b integers, then:

$$(a+b)^p \equiv a^p + b^p \pmod p$$

Lemma: Let p be a prime number and a an integer. The inverse of a modulo p is a^{p-2}:

$$a^{-1} \equiv a^{p-2} \pmod p$$

3.5 Chicken McNugget Theorem

The Chicken McNugget Theorem states that for any two relatively prime positive integers m, n, the greatest integer that cannot be written in the form $am + bn$ for nonnegative integers a, b is $mn - m - n$.

A consequence of the theorem is that there are exactly $\frac{(m-1)(n-1)}{2}$ positive integers which cannot be expressed in the form $am + bn$. The proof is based on the fact that in each pair of the form $(k, mn - m - n - k)$, exactly one element is expressible.

4 Graph

4.1 Graph

A Graph without an odd cycle is called an bipartite graph.

5 geometry

Geometry Identities and Transformations (Competitive Programming)

Basic Identities and Vector Operations

2D Geometry

- **Point Representation**: $P = (x, y)$
- **Vector from** P_1 **to** P_2: $\vec{P_1P_2} = (x_2 - x_1, y_2 - y_1)$
- **Distance between two points** $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$: $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
- **Midpoint of a segment** connecting $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$: $M = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$
- **Area of a triangle** with vertices $(x_1, y_1), (x_2, y_2), (x_3, y_3)$: $A = \frac{1}{2}|x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)|$
- **Signed Area of Triangle**: $\frac{1}{2}((x_2-x_1)(y_3-y_1)-(x_3-x_1)(y_2-y_1))$ (positive for CCW, negative for CW)
- **Cross Product (2D)** for vectors $\vec{A} = (x_1, y_1)$ and $\vec{B} = (x_2, y_2)$: $\vec{A} \times \vec{B} = x_1y_2 - x_2y_1$ (scalar value, positive if $\vec{B}$ is CCW from $\vec{A}$)
- **Dot Product (2D)** for vectors $\vec{A} = (x_1, y_1)$ and $\vec{B} = (x_2, y_2)$: $\vec{A} \cdot \vec{B} = x_1x_2 + y_1y_2 = |\vec{A}||\vec{B}|\cos\theta$

3D Geometry

- **Point Representation**: $P = (x, y, z)$
- **Vector from** P_1 **to** P_2: $\vec{P_1P_2} = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$
- **Distance between two points** $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$: $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$
- **Dot Product (3D)** for vectors $\vec{A} = (x_1, y_1, z_1)$ and $\vec{B} = (x_2, y_2, z_2)$: $\vec{A} \cdot \vec{B} = x_1x_2 + y_1y_2 + z_1z_2 = |\vec{A}||\vec{B}|\cos\theta$
- **Cross Product (3D)** for vectors $\vec{A} = (x_1, y_1, z_1)$ and $\vec{B} = (x_2, y_2, z_2)$: $\vec{A} \times \vec{B} = (y_1z_2 - y_2z_1, x_2z_1 - x_1z_2, x_1y_2 - x_2y_1)$

- **Volume of Tetrahedron** with vertices P_0, P_1, P_2, P_3: $\frac{1}{6}|\det(\vec{P_0P_1}, \vec{P_0P_2}, \vec{P_0P_3})|$ (scalar triple product)

Lines and Segments (2D)

- **Line Equation**: $Ax + By + C = 0$ (normal vector (A, B))
- **Slope**: $m = (y_2 - y_1)/(x_2 - x_1)$
- **Perpendicular Slope**: $-1/m$
- **Point-Slope Form**: $y - y_1 = m(x - x_1)$
- **Intersection of two lines** ($A_1x + B_1y + C_1 = 0$, $A_2x + B_2y + C_2 = 0$): $x = \frac{B_1C_2 - B_2C_1}{A_1B_2 - A_2B_1}$, $y = \frac{A_2C_1 - A_1C_2}{A_1B_2 - A_2B_1}$ (check for $A_1B_2 - A_2B_1 = 0$ for parallel/coincident lines)
- **Distance from a point** $P_0(x_0, y_0)$ **to line** $Ax + By + C = 0$: $d = \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}$
- **Orientation of three points** P_1, P_2, P_3: Use cross product $\vec{P_1P_2} \times \vec{P_1P_3}$.
  - > 0: Counter-clockwise (left turn)
  - < 0: Clockwise (right turn)
  - $= 0$: Collinear
- **Segment Intersection**: Check orientation of (P_1, P_2, P_3), (P_1, P_2, P_4), (P_3, P_4, P_1), (P_3, P_4, P_2). Special handling for collinear segments.

Circles (2D)

- **Equation**: $(x - h)^2 + (y - k)^2 = r^2$, center (h, k), radius r.
- **Area**: $A = \pi r^2$
- **Circumference**: $C = 2\pi r$
- **Distance between two circle centers**: $d = \sqrt{(h_2 - h_1)^2 + (k_2 - k_1)^2}$
- **Intersection of two circles**:
  - $d > r_1 + r_2$: No intersection (disjoint)
  - $d = r_1 + r_2$: One intersection point (external tangent)
  - $|r_1 - r_2| < d < r_1 + r_2$: Two intersection points
  - $d = |r_1 - r_2|$: One intersection point (internal tangent)
  - $d < |r_1 - r_2|$: No intersection (one inside other)
  - $d = 0$ and $r_1 = r_2$: Coincident circles (infinite points)

Polygon Properties (2D)

- **Area of a simple polygon** with vertices $(x_0, y_0), \ldots, (x_{n-1}, y_{n-1})$ (Shoelace Formula): $A = \frac{1}{2}\left|\sum_{i=0}^{n-1}(x_iy_{i+1} - x_{i+1}y_i)\right|$, where $(x_n, y_n) = (x_0, y_0)$. (Sum of cross products: $\sum_{i=0}^{n-1} \vec{OP_i} \times \vec{OP_{i+1}}$ where O is origin)
- **Convex Polygon**: All internal angles are less than or equal to 180°.
- **Concave Polygon**: Has at least one internal angle greater than 180°.
- **Point in Polygon Test (Ray Casting)**: Draw a ray from the point in any direction (e.g., positive x-axis) and count intersections with polygon edges. If odd, point is inside; if even, outside. Handle horizontal edges and vertices carefully.
- **Point in Convex Polygon Test**: Check if the point is consistently on one side (e.g., left) of all directed edges of the polygon.

Convex Hull (2D)

- The smallest convex polygon enclosing a given set of points.
- Algorithms: Graham Scan ($O(N \log N)$), Monotone Chain ($O(N \log N)$).
- **Graham Scan steps**:
  1. Find lowest-most (and leftmost if ties) point P_0.
  2. Sort all other points by angle with P_0 (or cross product $\vec{P_0P_i} \times \vec{P_0P_j}$).
  3. Iterate through sorted points, maintaining a stack. If new point creates a right turn with top two stack points, pop from stack until left turn.

Sweep Line Algorithms

- Used for problems involving geometric objects (segments, rectangles) where a vertical (or horizontal) line sweeps across the plane.
- Often involves a ‘std::set‘ or segment tree to maintain active objects on the sweep line.

Floating Point Precision Issues

- Use a small epsilon ($\varepsilon \approx 10^{-9}$ to 10^{-12}) for comparisons instead of direct equality:
  - $a == b \Rightarrow |a - b| < \varepsilon$
  - $a < b \Rightarrow a < b - \varepsilon$
  - $a > b \Rightarrow a > b + \varepsilon$
- Prefer integer arithmetic where possible (e.g., use cross products for collinearity/orientation instead of slopes).

Geometric Transformations (2D Homogeneous Coordinates)

A point (x, y) is represented as a column vector $\begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$.

Translation by (t_x, t_y)

$$T = \begin{pmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{pmatrix}$$

Scaling by (s_x, s_y) relative to origin

$$S = \begin{pmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Rotation by angle θ relative to origin (counter-clockwise)

$$R = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Shear

- X-Shear (parallel to x-axis, factor k_x): $Sh_x = \begin{pmatrix} 1 & k_x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
- Y-Shear (parallel to y-axis, factor k_y): $Sh_y = \begin{pmatrix} 1 & 0 & 0 \\ k_y & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Reflection

- Across X-axis: $Ref_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
- Across Y-axis: $Ref_y = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
- Across origin: $Ref_O = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
- Across line $y = x$: $Ref_{y=x} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Geometric Transformations (3D Homogeneous Coordinates)

A point (x, y, z) is represented as a column vector $\begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$.

Translation by (t_x, t_y, t_z)

$$T = \begin{pmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Scaling by (s_x, s_y, s_z) relative to origin

$$S = \begin{pmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Rotation by angle θ (counter-clockwise)

- About X-axis: $R_x(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$
- About Y-axis: $R_y(\theta) = \begin{pmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$
- About Z-axis: $R_z(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

6 Counting Problems

Math, Counting, and Number Theory (Competitive Programming)

Combinatorics and Counting

- **Factorial**: $n! = n \times (n-1) \times \cdots \times 1$; $0! = 1$.
- **Permutations (arrangements without repetition)**: $P(n,k) = \frac{n!}{(n-k)!}$
- **Combinations (selections without repetition)**: $C(n,k) = \binom{n}{k} = \frac{n!}{k!(n-k)!}$
- **Binomial Theorem**: $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$
- **Pascal's Identity**: $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$
- **Stars and Bars**: Number of ways to distribute n identical items into k distinct bins (or non-negative integer solutions to $x_1 + \cdots + x_k = n$):

$$\binom{n+k-1}{k-1} \quad \text{or} \quad \binom{n+k-1}{n}$$

Use Case: Find the number of ways to put 10 identical candies into 3 distinct bags. ($\binom{10+3-1}{3-1} = \binom{12}{2}$)

- **Vandermonde's Identity**:

$$\sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k} = \binom{m+n}{r}$$

Use Case: Choosing r people from a group of m men and n women. The LHS sums ways to choose k men and $r-k$ women.

- **Catalan Numbers (C_n)**: Explicit formula:

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

Recursive formula:

$$C_0 = 1, \quad C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i} \quad \text{for } n \geq 0$$

Applications:

  - Number of Dyck paths of length $2n$ (paths from $(0,0)$ to (n,n) using only right/up steps that don't go above $y = x$).
  - Number of ways to correctly parenthesize n pairs of parentheses.
  - Number of full binary trees with $n+1$ leaves.

Use Case: How many ways to form a balanced sequence of 3 pairs of parentheses? ($C_3 = 5$)

- **Inclusion-Exclusion Principle**: For two sets:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

For n sets: $|\bigcup_{i=1}^{n} A_i| = \sum |A_i| - \sum_{i<j} |A_i \cap A_j| + \sum_{i<j<k} |A_i \cap A_j \cap A_k| - \cdots + (-1)^{n-1}|A_1 \cap \cdots \cap A_n|$ *Use Case*: Count numbers up to 100 divisible by 2 or 3. ($|A_2 \cup A_3| = |A_2| + |A_3| - |A_2 \cap A_3|$).

- **Pigeonhole Principle**: If N items are put into K bins, then at least one bin must contain $\lceil N/K \rceil$ items. *Use Case*: In any group of 13 people, at least two must share the same birth month.
- **Permutations with Repetitions**: Number of distinct permutations of n objects where there are k_1 identical objects of type 1, k_2 identical objects of type 2, ..., k_r identical objects of type r (and $k_1 + \cdots + k_r = n$):

$$\frac{n!}{k_1! k_2! \cdots k_r!}$$

Use Case: How many distinct permutations of the letters in "MISSISSIPPI"? ($\frac{11!}{1!4!4!2!}$)

- **r-Permutations**: Number of ways to arrange r items chosen from n distinct items ($P(n,r)$):
$$P(n,r) = \frac{n!}{(n-r)!}$$
Use Case: Number of ways to award gold, silver, and bronze medals to 10 runners. ($P(10,3) = \frac{10!}{7!}$)