Theoretical Guide

Miguel Nogueira

1 Math

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MOD = 998'244'353
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PI = acos(-1)
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1.1 Logarithm

$$\log_b mn = \log_b m + \log_b n \qquad log_b \frac{m}{n} = \log_b m - \log_n n \qquad \log_b n^p = p \log_b n$$

$$\log_b \sqrt[q]{n} = \frac{1}{q} \log_b n \qquad \log_b n = \log_a n \log_b a \qquad b^{\log_b k} = k$$

$$\log_b a = \frac{\log_c a}{\log_c b} \qquad \log_b a = \frac{1}{\log_a b} \qquad \log_b a \ \log_a c = \log_b c$$

$$\log_b 1 = 0 \qquad \log_b b = 1$$

2 Identities

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$$

$$\sum_{i=1}^{n} \frac{1}{i} \approx \log n \qquad \sum_{i=0}^{\infty} \frac{1}{2^i} = 2$$

3 Number Theory

$$(a+b) \bmod m = (a \bmod m + b \bmod m) \bmod m$$

$$(a-b) \bmod m = (a \bmod m - b \bmod m) \bmod m$$

$$(a \times b) \bmod m = ((a \bmod m) \times (b \bmod m)) \bmod m$$

$$a^b \bmod m = (a \bmod m)^b \bmod m$$

$$a \equiv b \pmod{m} \iff (b-a) | m$$

$$\gcd(a_1, a_2, a_3, a_4) = \gcd(a_1, \gcd(a_2, \gcd(a_3, a_4)))$$

$$\operatorname{lcm}(a, b) \times \gcd(a, b) = a \times b$$

$$\operatorname{lcm}(a, b) = \frac{a \times b}{\gcd(a, b)} = \frac{a}{\gcd(a, b)} \times b$$

3.1 Some Primes

999999937 1000000007 1000000009 1000000021 1000000033

$10^{18} - 11$ $10^{18} + 3$ $2305843009213693951 = 2^{61} - 1$

$998244353 = 119 \times 2^{23} + 1$ $10^6 + 3$

1055244485953076591 1392181229391707271 1178970662972334411

257900257981 584598951247 989509930063 105539556781

998244353 754974721 167772161 188244827 205587737

555130769 809747989 572255561 396588799 327208423

773840099 207936359 952818871 935456867 670948771

3.2 Number of Divisors

The number of divisors of n is about $\sqrt[3]{n}$.

n	6	60	360	5040	55440	720720	4324320	21621600
$d(n)$	4	12	24	60	120	240	384	576

Given the prime factorization of some number n:

$$n = p_1^{a_1} \cdot p_2^{a_2} \cdot p_3^{a_3}$$

The number of divisors will be $(a_1+1)(a_2+1)(a_3+1)$.

3.3 Large Prime Gaps

For numbers until 10^9 the largest gap is 400.
For numbers until 10^{18} the largest gap is 1500.

3.4 Fermat's Theorems

Let P be a prime number and a an integer, then:

$$a^p \equiv a \pmod p$$

$$a^{p-1} \equiv 1 \pmod p$$

Lemma: Let p be a prime number and a and b integers, then:

$$(a+b)^p \equiv a^p + b^p \pmod p$$

Lemma: Let p be a prime number and a an integer. The inverse of a modulo p is a^{p-2}:

$$a^{-1} \equiv a^{p-2} \pmod p$$

3.5 Chicken McNugget Theorem

The Chicken McNugget Theorem states that for any two relatively prime positive integers m, n, the greatest integer that cannot be written in the form $am + bn$ for nonnegative integers a, b is $mn - m - n$.

A consequence of the theorem is that there are exactly $\frac{(m-1)(n-1)}{2}$ positive integers which cannot be expressed in the form $am + bn$. The proof is based on the fact that in each pair of the form $(k, mn - m - n - k)$, exactly one element is expressible.

4 Counting Problems

4.1 Counting

Combinatorics and counting principles:

- **Stars and Bars**:

  Number of ways to distribute n identical items into k distinct bins $= \binom{n+k-1}{k-1}$

- **Vandermonde's Identity**:

  $$\sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k} = \binom{m+n}{r}$$

- **Catalan Numbers**:

  $$C_n = \frac{1}{n+1}\binom{2n}{n}$$

  Applications: Valid parentheses, binary trees, Dyck paths.

- **Inclusion-Exclusion Principle**:

  $$|A \cup B| = |A| + |B| - |A \cap B|$$

- **Pigeonhole Principle**:

  If more than k items are put into k bins, then at least one bin must contain more than o

- **Permutations with Repetitions**:

  Number of ways to arrange n objects where there are k_1 objects of one kind, k_2 of anoth

- **r-Permutations**:

  $P(n, r) = \frac{n!}{(n-r)!}$ for the number of ways to arrange r items out of n distinct items.