Linear (In)dependence.

Given a set of vectors $v_1, v_2, \cdots, v_k$, we look at their Linear Combinations: $c_1 v_1 + c_2 v_2 + \cdots + c_k v_k$. The trivial combination $c_i = 0$ produces the zero vector, since $0v_1 + 0v_2 + \cdots + 0v_k = 0$. The point is whether any other weights or scalars also produce it.

If all non-trivial combinations of vectors are *non-zero*, $c_1 v_1 + c_2 v_2 + \cdots + c_k v_k \neq 0$, unless $c_1 = c_2 = \cdots = c_k = 0$, then the vectors $v_1, v_2, \cdots, v_k$ are **Linearly Independent**. Otherwise they are **linearly dependent**, and one of them is a linear combination of the others. e.g.

> If one of the vectors, let's say, v_2, happen to be the zero vector, then we are certain that this combination is dependent. If we choose weights $c_2 = 4$ and $c_i = 0$, this is certainly a nontrivial combination that yields the zero vector.

Ex 2: Let A:

$$A = \begin{pmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 5 \\ -1 & -3 & 3 & 0 \end{pmatrix} \implies \begin{pmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 6 & 2 \end{pmatrix}$$

Here we clearly see that row 3 is a combination of the other rows, so A has *linearly dependent rows*, we can also see dependent columns since column 2 is three times column 1.

The rows of the $n \times n$ Identity Matrix I:

$$I = \begin{pmatrix} 1 & 0 & 0 & \cdot \\ 0 & 1 & 0 & \cdot \\ 0 & 0 & \cdot & 0 \\ \cdot & \cdot & 0 & 1 \end{pmatrix}$$

Are linearly independent. We give this vectors a special notation $e_1, e_2, \cdots, e_n$, they are the unit vectors in the coordinate directions, $e_1 = (1, 0, 0, \cdots, 0)$, $e_n = (0, 0, \cdots, 1)$.

Procedure for Proving Independence

Assume that the linear combination gives zero, and prove that all weight c_i must equal zero, for example:

$$c_1e_1 + c_2e_2 + \cdots + c_ne_n = (c_1, c_2, \cdots, c_n)$$

If the combination is the zero vector then obviously all $c_i = 0. \text{ e. g.}$

> Suppose U is an $n \times n$ Upper Triangular Matrix, with non-zero pivots in the diagonal. Then the rows of U are linearly independent.

Proof: We start by assuming that some linear combination of the rows is zero, $c_1v_1 + c_2v_2 + \cdots + c_kv_k = 0$. Then we head for the first non-zero entry in the diagonal u_{11}, since we know $c_1v_1 = 0$ and $v_1 \neq 0$, this implies that $c_1 = 0$, and $c_2 = 0$ since $c_1v_1 + c_2v_2 = 0$ and $v_2 \neq 0$, this applies to all u_{ij} pivots, since the only weights that make $c_1v_1 + c_2v_2 + \cdots + c_kv_k = 0$ are those of the trivial solution. Then U is linearly independent.

The r nonzero rows of an echelon matrix U are linearly independent, and so are the r columns that contain nonzero pivots. $\rightarrow$ An important reminder is that the definition of linear independence is "coordinate free". Given k points in n-dimensional space, the vectors from the origin to those points either can or cannot be combined to give zero, regardless of where we put the coordinate axes. A rotation will change the coordinates however it won't affect the question of dependent or independent whatsoever. $\rightarrow$ Given an arbitrary set of vectors, their verification of dependency or independency of course requires some calculation, $c_1v_1 + c_2v_2 + \cdots + c_kv_k$, the natural step is to form a matrix A, whose columns are the given vectors. Then if we write c for the vector of weights: $(c_1, c_2, \cdots, c_k)$:

$$Ac = \begin{pmatrix} . & . & . & . \\ . & . & . & . \\ v_1 & v_2 & \cdots & v_k \\ . & . & . & . \\ . & . & . & . \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{pmatrix} = c_1v_1 + c_2v_2 + \cdots + c_kv_k$$

The vectors are dependent **if and only if there is a nontrivial sollution for** $Ac = 0$. This is settled by Gaussian Elimination. If the rank of $A = k$, then there are no free variables and no Nullspace, (except for $c = 0$), then the vectors are linearly independent. If the rank is less than k then there's at least one free variable that can be chosen nonzero and the columns are linearly dependent. $\rightarrow$ A really important thing is that if we let the vectors have m components, so that A is a $m \times k$ matrix, and suppose now that $k > m$, it will be impossible for A to have rank k, since the number of pivots cannot surpass the number of rows. The rank must be less than k and a homogeneous system $Ax = 0$ with more unknowns than equations always has nontrivial solutions $x \neq 0$.

> A Set of vectors K in $\mathbb{R}^m$ with $K > m$ is always linearly dependent.