TECHNICAL UNIVERSITY OF DENMARK

02685 SCIENTIFIC COMPUTING FOR DIFFERENTIAL EQUATIONS
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Assignment 1

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1 Question 3

1.1 Order conditions, coefficients for the error estimator and the Butcher tableau

Using the excerpt from the book provided in the lecture 10 folder we will write up the order conditions for an embedded Runge-Kutta method with 3 stages. The solution will have order 3 and the embedded method used for error estimation will have order 2.

Firstly the Butcher tableau for our ERK will have the following schema (following the lower triangular shape for the a coefficients and $c_1 = 0$):

$$
\begin{array}{c|ccc}
0 & 0 & 0 & 0 \\
c_2 & a_{21} & 0 & 0 \\
c_3 & a_{31} & a_{32} & 0 \\
\hline
x & b_1 & b_2 & b_3 \\
\widehat{x} & \widehat{b}_1 & \widehat{b}_2 & \widehat{b}_3 \\
\hline
e & d_1 & d_2 & d_3
\end{array}
$$

Table 1: Butcher's tableau for ERK with 3 stages and embedded method.

Order conditions (one for first order, one for second order and two for third order) derived from our Butcher tableau:

$$b^T e = 1 \qquad b_1 + b_2 + b_3 = 1 \tag{1a}$$

$$b^T C e = \frac{1}{2} \qquad \underbrace{b_1 c_1}_{0} + b_2 c_2 + b_3 c_3 = \frac{1}{2} \tag{1b}$$

$$b^T C^2 e = \frac{1}{3} \qquad \underbrace{b_1 c_1^2}_{0} + b_2 c_2^2 + b_3 c_3^2 = \frac{1}{3} \tag{1c}$$

$$b^T A C e = \frac{1}{6} \qquad \underbrace{b_2 a_{21} c_1}_{0} + \underbrace{b_3 a_{31} c_1}_{0} + b_3 a_{32} c_2 = \frac{1}{6} \tag{1d}$$

values of c_2 and c_3 will be set to $\frac{1}{4}$ and 1 respecively. This leaves us with 6 unknown variables (3 as and 3 bs) and only 4 equations so we will add the so called consistency conditions in order for the system to be solvable.

$$c_2 = a_{21} \tag{1e}$$

$$c_3 = a_{31} + a_{32} \tag{1f}$$

Using Matlab to solve the system we get the following results:

$b_1 = -\frac{1}{6}$, $b_2 = \frac{8}{9}$, $b_3 = \frac{5}{18}$, $a_{21} = \frac{1}{4}$, $a_{31} = -\frac{7}{5}$, $a_{32} = \frac{12}{5}$.

Next we will solve the system defined for second order embedded method with one first order and one second order condition where c_2 and c_3 are known thus giving 2 equations with 3 unknowns. In order to find a solution $\widehat{b}_2$ is set to be $\frac{1}{2}$[1].

$$\widehat{b}_1 + \widehat{b}_2 + \widehat{b}_3 = 1 \tag{2a}$$

$$\widehat{b}_2 c_2 + \widehat{b}_3 c_3 = \frac{1}{2} \tag{2b}$$

The above system yields $\widehat{b}_1 = \frac{1}{8}$ and $\widehat{b}_3 = \frac{3}{8}$. Going back to the Butcher's tableau we know that last row $e = (d_1, d_2, d_3)$ is just the difference of the previous two rows by definition.

$c_1 = 0$	0	0	0
$c_2 = \frac{1}{4}$	$1/4$	0	0
$c_3 = 1$	$-7/5$	$12/5$	0
x	$-1/6$	$8/9$	$5/18$
$\widehat{x}$	$1/8$	$1/2$	$3/8$
e	$-7/24$	$7/18$	$-7/72$

Table 2: Butcher's tableau with error estimators for our method.

1.2 Testing on the test equation

1.3 Verifying the order

1.4 R(λh) and stability plot

The solution to the test equation obtained by a Runge-Kutta method is defined as $x(t_n + h) = R(\lambda h)x(t_n)$ and $R(z) = 1 + zb^T(I - zA)^{-1}e$. From the Butcher's tableau with error estimators for our method. b vector and A matrix are plugged in to $R(z)$ resulting in $R(z) = 1 + z + \frac{1}{2}z^2 + \frac{3}{18}z^3$ where $z = \lambda h$ for the third order method.

1.5 Testinging on the Van der Pol problem and comparison with `ode15s`

[1] According to the book excerpt given in lecture 10 folder.