Technical University of Denmark

02685 Scientific Computing for Differential Equations
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Assignment 1

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1 Question 3

Write up the order conditions for an embedded Runge-Kutta method with 3 stages. The solution you advance must have order 3 and the embedded method used for error estimation must have order 2.

$$\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ c_2 & a_{21} & 0 & 0 \\ c_3 & a_{31} & a_{32} & 0 \\ \hline x & b_1 & b_2 & b_3 \\ \widehat{x} & \widehat{b}_1 & \widehat{b}_2 & \widehat{b}_3 \\ \hline e & d_1 & d_2 & d_3 \end{array}$$

Table 1: Butcher's tableau schema for Runge-Kutta method with 3 stages.

Order conditions derived from Butcher's table:

$$b^T e = 1 \qquad b_1 + b_2 + b_3 = 1 \tag{1a}$$

$$b^T Ce = \frac{1}{2} \qquad \underbrace{b_1 c_1}_{0} + b_2 c_2 + b_3 c_3 = \frac{1}{2} \tag{1b}$$

$$b^T C^2 e = \frac{1}{3} \qquad \underbrace{b_1 c_1^2}_{0} + b_2 c_2^2 + b_3 c_3^2 = \frac{1}{3} \tag{1c}$$

$$b^T ACe = \frac{1}{6} \qquad \underbrace{b_2 a_{21} c_1}_{0} + \underbrace{b_3 a_{31} c_1}_{0} + b_3 a_{32} c_2 = \frac{1}{6} \tag{1d}$$

values of c_2 and c_3 will be set to $\frac{1}{4}$ and 1 respecively. This leaves us with 6 unknown variables (3 as and 3 bs) and only 4 equations so we will add the so called consistency equations.

$$c_2 = a_{21} \tag{1e}$$

$$c_3 = a_{31} + a_{32} \tag{1f}$$

Using Matlab to solve the system we get the following results: $b_1 = -\frac{1}{6}$, $b_2 = \frac{8}{9}$, $b_3 = \frac{5}{18}$, $a_{21} = \frac{1}{4}$, $a_{31} = -\frac{7}{5}$, $a_{32} = \frac{12}{5}$.

Next we will solve the system where c_2 and c_3 are known thus giving 2 equations with 3 unknowns. In order to find a solution $\widehat{b}_2$ is set to be $\frac{1}{2}$.

$$\widehat{b}_1 + \widehat{b}_2 + \widehat{b}_3 = 1 \tag{2a}$$

$$\widehat{b}_2 c_2 + \widehat{b}_3 c_3 = \frac{1}{2} \tag{2b}$$

The above system yields $\widehat{b}_1 = \frac{1}{8}$ and $\widehat{b}_3 = \frac{3}{8}$. Going back to the Butcher's tableau we know that last row (d_1, d_2, d_3) is just the difference of the previous two rows by definition.

0	0	0	0
1/4	1/4	0	0
1	-7/5	12/5	0
x	-1/6	8/9	5/18
$\widehat{x}$	1/8	1/2	3/8

Table 2: Butcher's tableau with error estimators for our method.