1 2 point Boundary Value Problems

2-point Boundary Value Problems (BVP) represent a special case of differential equations where the solution is limited by a pair of constraints at both ends of the interval. Hence, the task consists of finding the solution that apart from solving the differential equation satisfies the boundary conditions.

In the linear case, a numerical approximation to the solution to these problems can be obtained solving a linear system. However, for nonlinear differential equations the solution is not straightforward and an iterative process is needed. In this exercise, we shall study different methods for solving nonlinear BVPs by applying them to the following differential equation:

$$\begin{aligned} \epsilon u(t)'' + u(t)(u(t)' - 1) = 0 \qquad & 0 \leq t \leq 1 \\ u(0) = \alpha \qquad & u'(1) = \beta \end{aligned} \tag{1}$$

1.1 Newton's method

If the independent variable t is discretized and the second derivative in equation ?? is replaced by the centered difference method, an approximation to the solution at a series of equidistant points can be found solving the following scheme:

$$\begin{aligned} \epsilon \left(\frac{u_{i-1} - 2u_i + u_{i+1}}{h^2} \right) + u_i \left(\frac{u_{i+1} - u_{i-1}}{2h} - 1 \right) \\ u(0) = \alpha \qquad u(1) = \beta \end{aligned} \tag{2}$$

Where $i = 1, 2, ..., m$ and h is the distance between points in the interval. This can be expressed as a nonlinear system of the form:

$$G(U) = 0 \tag{3}$$

The centered approximation comes from a Taylor series expansion truncated to the second term. We can show that the approximation is second-order accurate by computing the local error:

The roots of the nonlinear system in equation ?? can be obtained using Newton's method.

$$U^{[k+1]} = U^{[k]} + J(U^{[k]})^{-1} G(U^{[k]}) \tag{4}$$

Where $U^{[k]}$ is a vector containing a discrete approximation to the solution after k iterations and $J(U^{[k]})$ is the Jacobian matrix of the system, which in our case:

$$J(U) = \begin{cases} \frac{\epsilon}{h^2} - \frac{u_i}{2h} & j = i - 1 \\ -\frac{2\epsilon}{h^2} - \frac{u_{i+1} - u_{i-1}}{2h} - 1 & j = i \\ \frac{\epsilon}{h^2} + \frac{u_i}{2h} & j = i - 1 \end{cases} \tag{5}$$

The convergence and accuracy of the method depend on the proximity