CP - Ficha 5

Exercício 1

> No cálculo de programas, as definições condicionais do tipo
>
> $$h\ x = \textbf{if}\ p\ x\ \textbf{then}\ f\ x\ \textbf{else}\ g\ x \tag{F1}$$
>
> são escritas usando o combinador ternário
>
> $$p \rightarrow f, g$$
>
> conhecido pelo nome de *condicional de McCarthy*, cuja definição
>
> $$p \rightarrow f, g = [f, g] \cdot p? \tag{F2}$$
>
> vem no formulário. Baseie-se em leis desse formulário para demonstrar a chamada 2ª-lei de fusão do condicional:
>
> $$(p \rightarrow f, g) \cdot h = (p \cdot h) \rightarrow (f \cdot h), (g \cdot h)$$

Resolução 1

$$
\begin{aligned}
& (p \rightarrow f, g) \cdot h \\
= & \qquad \{ \text{30: Cond. McCharty (F2), 2: Assoc-comp} \} \\
& [f, g] \cdot (p? \cdot h) \\
= & \qquad \{ \text{29: Natural-guarda} \} \\
& [f, g] \cdot (h + h) \cdot (p \cdot h)? \\
= & \qquad \{ \text{22: Absorção-+} \} \\
& [f \cdot h, g \cdot h] \cdot (p \cdot h)? \\
= & \qquad \{ \text{30: Cond. McCharty (F2)} \} \\
& (p \cdot h) \rightarrow f \cdot h, g \cdot h
\end{aligned}
$$

Exercício 2

> Numa máquina paralela pode fazer sentido, em (F1), não esperar por $p\ x$ para avaliar $f\ x$ ou $g\ x$, mas sim correr tudo em paralelo,
>
> $$parallel\ p\ f\ g = \langle\langle f, g\rangle, p\rangle$$
>
> e depois fazer a escolha do resultado:
>
> $$choose = \pi_2 \rightarrow \pi_1 \cdot \pi_1, \pi_2 \cdot \pi_1$$
>
> Mostre que, de facto:
>
> $$choose \cdot parallel\ p\ f\ g = p \rightarrow f, g$$

Resolução 2

$$
\begin{array}{ll}
choose \cdot parallel\ p\ f\ g & \\
= & \{\text{Def. } parallel\text{, Def. } choose\} \\
(\pi_2 \rightarrow \pi_1 \cdot \pi_1, \pi_2 \cdot \pi_1) \cdot \langle\langle f, g\rangle, p\rangle & \\
= & \{\text{32: 2ª lei de fusão condicional}\} \\
\pi_2 \cdot \langle\langle f, g\rangle, p\rangle \rightarrow \pi_1 \cdot \pi_1 \cdot \langle\langle f, g\rangle, p\rangle, \pi_2 \cdot \pi_1 \cdot \langle\langle f, g\rangle, p\rangle & \\
= & \{\text{7: Cancelamento-}\times\} \\
p \rightarrow f, g &
\end{array}
$$

Exercício 3

> Sabendo que as igualdades
>
> $$p \to k, k = k \tag{F3}$$
> $$(p? + p?) \cdot p? = (i_1 + i_2) \cdot p? \tag{F4}$$
>
> se verificam, demonstre as seguintes propriedades do mesmo combinador:
>
> $$\langle (p \to f, h), (p \to g, i) \rangle = p \to \langle f, g \rangle, \langle h, i \rangle \tag{F5}$$
> $$\langle f, (p \to g, h) \rangle = p \to \langle f, g \rangle, \langle f, h \rangle \tag{F6}$$
> $$p \to (p \to a, b), (p \to c, d) = p \to a, d \tag{F7}$$

Resolução 3

$$\text{(F5)}$$

$$\begin{aligned}
& \langle (p \to f, h), (p \to g, i) \rangle \\
& = && \text{(30: Cond. McCharty)} \\
& \langle [f, h] \cdot p?, [g, i] \cdot p? \rangle \\
& = && \text{(9: Fusão-}\times\text{)} \\
& \langle [f, h], [g, i] \rangle \cdot p? \\
& = && \text{(28: Lei da troca)} \\
& [\langle f, g \rangle, \langle h, i \rangle] \cdot p? \\
& = && \text{(30: Cond. McCharty)} \\
& p \to \langle f, g \rangle, \langle h, i \rangle
\end{aligned}$$

$$\text{(F6)}$$

$$\begin{aligned}
& \langle f, (p \to g, h) \rangle \\
& = && \text{(F3)} \\
& \langle (p \to f, f), (p \to g, h) \rangle \\
& = && \text{(F5)} \\
& p \to \langle f, g \rangle, \langle f, h \rangle
\end{aligned}$$

$$(F7)$$

$$
\begin{aligned}
& p \rightarrow (p \rightarrow a, b), (p \rightarrow c, d) \\
& = \qquad \{ \text{30: Cond. McCharty } (3\times) \} \\
& [[a, b] \cdot p?, [c, d] \cdot p?] \cdot p? \\
& = \qquad \{ \text{22: Absorção-+} \} \\
& [[a, b], [c, d]] \cdot (p? \cdot p?) \cdot p? \\
& = \qquad \{ \text{(F4)} \} \\
& [[a, b], [c, d]] \cdot (i_1 + i_2) \cdot p? \\
& = \qquad \{ \text{22: Absorção-+, 18: Cancelamento-+} \} \\
& [a, d] \cdot p? \\
& = \qquad \{ \text{30: Cond. McCharty} \} \\
& p \rightarrow a, d
\end{aligned}
$$

Exercício 4

> Mostre que a propriedade de cancelamento da exponenciação
>
> $$\mathrm{ap} \cdot (\overline{f} \times id) = f \tag{F8}$$
>
> corresponde à definição
>
> $$\mathrm{curry}\ f\ a\ b = f\ (a, b)$$
>
> quando se escreve $\mathrm{curry}\ f$ em lugar de $\overline{f}$.

Resolução 4

$$
\begin{aligned}
& \mathrm{ap} \cdot (\overline{f} \times id) = f \\
& \equiv \qquad \{ \text{72: Ig. Ext., 73: Def-comp} \} \\
& \mathrm{ap}\ ((\overline{f} \times id)\ (a, b)) = f\ (a, b) \\
& \equiv \qquad \{ \text{78: Def-}\times \} \\
& \mathrm{ap}\ (\overline{f}\ a, id) = f\ (a, b) \\
& \equiv \qquad \{ \text{84: Def-ap, 1: Natural-id} \} \\
& \overline{f}\ a\ b = f\ (a, b) \\
& \equiv \qquad \{ \text{85: Curry, Def. } curry \} \\
& curry\ f\ a\ b = f\ (a, b)
\end{aligned}
$$

Exercício 5

> Mostre que a definição de uncurry se pode obter também de (F8) fazendo $f :=$ $\text{uncurry}\ g$, introduzindo variáveis e simplificando.

Resolução 5

$$
\begin{aligned}
& f := \text{uncurry}\ g = \widehat{g} \\
\equiv \quad & \{\text{(F8)}\} \\
& \text{ap} \cdot (\overline{\widehat{g}} \times id) = \widehat{g} \\
\equiv \quad & \{\text{72: Ig. Ext., 73: Def-comp, Iso.: } \overline{\widehat{g}} = g\} \\
& \text{ap}\ ((g \times id)\ (a, b)) = \widehat{g}\ (a, b) \\
\equiv \quad & \{\text{78: Def-}\times\text{, 1: Natural-id}\} \\
& \text{ap}\ (g\ a, b) = \widehat{g}\ (a, b) \\
\equiv \quad & \{\text{84: Def-ap, } \widehat{g} = \text{uncurry}\ g\} \\
& g\ a\ b = \text{uncurry}\ g\ (a, b)
\end{aligned}
$$

Exercício 6

> Prove a igualdade
>
> $$\overline{f \cdot (g \times h)} = \overline{\mathrm{ap} \cdot (\mathrm{id} \times h)} \cdot \overline{f} \cdot g \tag{F9}$$
>
> usando as leis das exponenciais e dos produtos.

Resolução 6

$$
\begin{aligned}
& \overline{f \cdot (g \times h)} = \overline{\mathrm{ap} \cdot (\mathrm{id} \times h)} \cdot \overline{f} \cdot g \\
\equiv & \qquad \{ \text{35: Universal-exp} \} \\
& f \cdot (g \times h) = \mathrm{ap} \cdot ((\overline{\mathrm{ap} \cdot (id \times h)} \cdot \overline{f} \cdot g) \times id) \\
\equiv & \qquad \{ \text{38: Fusão-exp, 1: Natural-id} \} \\
& f \cdot (g \times h) = \mathrm{ap} \cdot ((\overline{\mathrm{ap} \cdot (id \times h)} \cdot \overline{f \cdot (g \times id)}) \times (id \cdot id)) \\
\equiv & \qquad \{ \text{2: Assoc-comp, 14: Functor-}\times \} \\
& f \cdot (g \times h) = \mathrm{ap} \cdot ((\overline{\mathrm{ap} \cdot (id \times h)} \times id) \cdot (\overline{f \cdot (g \times id)} \times id)) \\
\equiv & \qquad \{ \text{36: Cancelamento-exp} \} \\
& f \cdot (g \times h) = (\mathrm{ap} \cdot (id \times h)) \cdot (\overline{f \cdot (g \times id)} \times id) \\
\equiv & \qquad \{ \text{2: Assoc-comp, 14: Functor-}\times \} \\
& f \cdot (g \times h) = \mathrm{ap} \cdot ((id \cdot \overline{f \cdot (g \times id)}) \times (h \cdot id)) \\
\equiv & \qquad \{ \text{1: Natural-id } (2\times), \text{ 38: Fusão-exp} \} \\
& f \cdot (g \times h) = \mathrm{ap} \cdot ((\overline{f} \cdot g) \times (id \cdot h)) \\
\equiv & \qquad \{ \text{14: Functor-}\times \} \\
& f \cdot (g \times h) = \mathrm{ap} \cdot ((\overline{f} \times id) \cdot (g \times h)) \\
\equiv & \qquad \{ \text{2: Assoc-comp, 36: Cancelamento-exp} \} \\
& f \cdot (g \times h) = f \cdot (g \times h) \qquad \text{c.q.d.}
\end{aligned}
$$

Resolução 6 (Alternativa)

$$
\begin{aligned}
& \overline{f \cdot (g \times h)} \\
= & \qquad \{\text{(F9)}\} \\
& \overline{\mathrm{ap} \cdot (\mathrm{id} \times h)} \cdot \overline{f} \cdot g \\
= & \qquad \{\text{(38: Fusão-exp)}\} \\
& \overline{\mathrm{ap} \cdot (\mathrm{id} \times h) \cdot (\overline{f} \times id)} \cdot g \\
= & \qquad \{\text{(2: Assoc-comp, 14: Functor-}\times\text{)}\} \\
& \overline{\mathrm{ap} \cdot ((\mathrm{id} \cdot \overline{f}) \times (h \cdot id))} \cdot g \\
= & \qquad \{\text{(1: Natural-id)}\} \\
& \overline{\mathrm{ap} \cdot (\overline{f} \times h)} \cdot g \\
= & \qquad \{\text{(38: Fusão-exp)}\} \\
& \overline{\mathrm{ap} \cdot (\overline{f} \times h) \cdot (g \times id)} \\
= & \qquad \{\text{(14: Functor-}\times\text{, 1: Natural-id)}\} \\
& \overline{\mathrm{ap} \cdot ((\overline{f} \cdot g) \times h)} \\
= & \qquad \{\text{(38: Fusão-exp)}\} \\
& \overline{\mathrm{ap} \cdot (\overline{f \cdot (g \times id)} \times h)} \\
= & \qquad \{\text{(1: Natural-id (2}\times\text{))}\} \\
& \overline{\mathrm{ap} \cdot ((\overline{f \cdot (g \times id)} \cdot id) \times (id \cdot h))} \\
= & \qquad \{\text{(14: Functor-}\times\text{)}\} \\
& \overline{\mathrm{ap} \cdot ((\overline{f \cdot (g \times id)} \times id) \cdot (id \times h))} \\
= & \qquad \{\text{(2: Assoc-comp, 36: Cancelamento-exp)}\} \\
& \overline{f \cdot (g \times id) \cdot (id \times h)} \\
= & \qquad \{\text{(2: Assoc-comp, 14: Functor-}\times\text{)}\} \\
& \overline{f \cdot ((g \cdot id) \times (id \cdot h))} \\
= & \qquad \{\text{(1: Natural-id)}\} \\
& \overline{f \cdot (g \times h)}
\end{aligned}
$$

Exercício 7

> É dada a definição
>
> $$\text{flip}\ f = \overline{\hat{f} \cdot \text{swap}} \tag{F10}$$
>
> de acordo com:
>
> $$\begin{array}{ccccccc} (C^B)^A & \cong & C^{A\times B} & \cong & C^{B\times A} & \cong & (C^A)^B \\ f & \mapsto & \hat{f} & \mapsto & \hat{f} \cdot \text{swap} & \mapsto & \overline{\hat{f} \cdot \text{swap}} = \text{flip}\ f \end{array}$$
>
> Mostre que flip é um isomorfismo por ser a sua própria inversa:
>
> $$\text{flip}\ (\text{flip}\ f) = f \tag{F11}$$
>
> Mostre ainda que:
>
> $$\text{flip}\ f\ x\ y = f\ y\ x$$

Resolução 7

$$\begin{array}{ll}
\text{flip}\ (\text{flip}\ f) & \\
= & \{\text{Def. flip } (2\times)\} \\
\overline{\widehat{\overline{\hat{f} \cdot \text{swap}}} \cdot \text{swap}} & \\
= & \{\text{Isomorfismo curry/uncurry}\} \\
\overline{\hat{f} \cdot \text{swap} \cdot \text{swap}} & \\
= & \{\text{swap} \cdot \text{swap} = id,\ 1\text{: Natural-id}\} \\
\overline{\hat{f}} & \\
= & \{\text{Isomorfismo curry/uncurry}\} \\
f \qquad \text{c.q.m.} &
\end{array}$$

Exercício 8

> Mostre que
>
> $$junc \cdot unjunc = id \tag{F12}$$
>
> $$unjunc \cdot junc = id \tag{F13}$$
>
> se verificam, onde
>
> $$\begin{tikzcd} A^{B+C} \arrow[rr, bend left, "unjunc"] \arrow[rr, phantom, "\cong"] & & A^B \times A^C \arrow[ll, bend left, "junc"] \end{tikzcd}$$
>
> $$\begin{cases} junc\ (f, g) = [f, g] \\ unjunc\ k = (k \cdot i_1, k \cdot i_2) \end{cases} \tag{F14}$$

Resolução 8

$$\begin{array}{ll}
junc \cdot unjunc = id \\
\equiv & \{ \text{72: Ig. Ext., 73: Def-comp} \} \\
junc\ (unjunc\ k) = id\ k \\
\equiv & \{ \text{Def. } unjunc\text{, 74: Def-id} \} \\
junc\ (k \cdot i_1, k \cdot i_2) = k \\
\equiv & \{ \text{Def. } junc \} \\
[k \cdot i_1, k \cdot i_2] = k \\
\equiv & \{ \text{20: Fusão-+, 19: Reflexão-+, 1: Natural-id} \} \\
k = k
\end{array}$$

$$\begin{array}{ll}
unjunc \cdot junc = id \\
\equiv & \{ \text{72: Ig. Ext., 73: Def-comp, 74: Def-id} \} \\
unjunc\ (junc\ (f, g)) = (f, g) \\
\equiv & \{ \text{Def. } junc \} \\
unjunc\ [f, g] = (f, g) \\
\equiv & \{ \text{Def. } unjunc \} \\
(f \cdot i_1, g \cdot i_2) = (f, g) \\
\equiv & \{ \text{18: Cancelamento-+} \} \\
(f, g) = (f, g)
\end{array}$$

Exercício 9

> Considere a seguinte sintaxe concreta em Haskell para um tipo que descreve pontos no espaço tridimensional:
>
> ```haskell
> data Point a = Point {x :: a, y :: a, z :: a } deriving (Eq, Show)
> ```
>
> Pelo GHCi apura-se:
>
> ```haskell
> Point :: a -> a -> a -> Point a
> ```
>
> Raciocinando apenas em termos de tipos, conjecture a definição de in na seguinte conversão dessa sintaxe concreta para abstracta:
>
> $$\begin{tikzcd} Point\ A \arrow[r, bend left, "\mathsf{out} = \langle\langle x, y\rangle, z\rangle"] \arrow[r, phantom, "\cong"] & (A \times A) \times A \arrow[l, bend left, "\mathsf{in} = \cdots"] \end{tikzcd}$$

Resolução 9

$$
\begin{aligned}
& \mathrm{in} \cdot \langle x, \langle y, z\rangle\rangle = id \\
\equiv \quad & \{ \text{72: Ig. Ext., 74: Def-id} \} \\
& \mathrm{in} \cdot \langle x, \langle y, z\rangle\rangle\ (\mathrm{Point}\ a\ b\ c) = \mathrm{Point}\ a\ b\ c \\
\equiv \quad & \{ p = \mathrm{Point}\ a\ b\ c\text{, 73: Def-comp, 77: Def-split } (2\times) \} \\
& \mathrm{in}\ (x\ p, (y\ p, z\ p)) = p \\
\equiv \quad & \{ \text{Def. } x, y \text{ e } z \} \\
& \mathrm{in}\ (a, (b, c)) = \mathrm{Point}\ a\ b\ c \\
\equiv \quad & \{ \text{86: Uncurry } (2\times) \} \\
& \mathrm{in}\ (a, (b, c)) = \widehat{\widehat{\mathrm{Point}}}\ ((a, b), c) \\
\equiv \quad & \{ \text{72: Ig. Ext.} \} \\
& \mathrm{in} = \widehat{\widehat{\mathrm{Point}}}
\end{aligned}
$$

Exercício 10

Questão prática

> ***Problem requirements****: The solution given for a previous problem,*
>
> $$store\ c = \mathrm{take}\ 10 \cdot nub \cdot (c\ :) \tag{F15}$$
>
> *calls the standard function*
>
> ```
> nub :: (Eq a) => [a] -> [a]
> ```
>
> *available from the Data.List library in Haskell.*
>
> *After inspecting the standard implementation of this function, define f so that*
>
> $$\mathrm{nub} = [nil, cons] \cdot f$$
>
> *is an alternative to the standard definition, where $nil\ \_ = [\,]$ and $cons\ (h, t) = h : t$.*
>
> *Check that $store\ c$ (F15) works properly once the standard nub is replaced by yours.*
>
> ***Important****: Structure your solution across the $f \cdot g$, $\langle f, g \rangle$, $f \times g$, $[f, g]$ and $f + g$ combinators that can be found in library Cp.hs. Use* ***diagrams*** *to plan your proposed solution, which should avoid re-inventing functions over lists already available in the Haskell standard libraries.*

Resolução 10

TODO

□