CP - Ficha 5

Exercício 1

> No cálculo de programas, as definições condicionais do tipo
>
> $$h\ x = \textbf{if}\ p\ x\ \textbf{then}\ f\ x\ \textbf{else}\ g\ x \tag{F1}$$
>
> são escritas usando o combinador ternário
>
> $$p \rightarrow f, g$$
>
> conhecido pelo nome de *condicional de McCarthy*, cuja definição
>
> $$p \rightarrow f, g = [f, g] \cdot p? \tag{F2}$$
>
> vem no formulário. Baseie-se em leis desse formulário para demonstrar a chamada 2ª-lei de fusão do condicional:
>
> $$(p \rightarrow f, g) \cdot h = (p \cdot h) \rightarrow (f \cdot h), (g \cdot h)$$

Resolução 1

$$
\begin{aligned}
& (p \rightarrow f, g) \cdot h \\
= & \qquad \{\text{30: Cond. McCharty (F2), 2: Assoc-comp}\} \\
& [f, g] \cdot (p? \cdot h) \\
= & \qquad \{\text{29: Natural-guarda}\} \\
& [f, g] \cdot (h + h) \cdot (p \cdot h)? \\
= & \qquad \{\text{22: Absorção-+}\} \\
& [f \cdot h, g \cdot h] \cdot (p \cdot h)? \\
= & \qquad \{\text{30: Cond. McCharty (F2)}\} \\
& (p \cdot h) \rightarrow f \cdot h, g \cdot h
\end{aligned}
$$

Exercício 2

> Numa máquina paralela pode fazer sentido, em (F1), não esperar por $p\ x$ para avaliar $f\ x$ ou $g\ x$, mas sim correr tudo em paralelo,
>
> $$parallel\ p\ f\ g = \langle\langle f, g\rangle, p\rangle$$
>
> e depois fazer a escolha do resultado:
>
> $$choose = \pi_2 \to \pi_1 \cdot \pi_1, \pi_2 \cdot \pi_1$$
>
> Mostre que, de facto:
>
> $$choose \cdot parallel\ p\ f\ g = p \to f, g$$

Resolução 2

$$
\begin{aligned}
& choose \cdot parallel\ p\ f\ g \\
= & \qquad \{\text{Def. } parallel\text{, Def. } choose\} \\
& (\pi_2 \to \pi_1 \cdot \pi_1, \pi_2 \cdot \pi_1) \cdot \langle\langle f, g\rangle, p\rangle \\
= & \qquad \{\text{32: 2ª lei de fusão condicional}\} \\
& \pi_2 \cdot \langle\langle f, g\rangle, p\rangle \to \pi_1 \cdot \pi_1 \cdot \langle\langle f, g\rangle, p\rangle, \pi_2 \cdot \pi_1 \cdot \langle\langle f, g\rangle, p\rangle \\
= & \qquad \{\text{7: Cancelamento-}\times\} \\
& p \to f, g
\end{aligned}
$$

Exercício 3

> Sabendo que as igualdades
>
> $$p \to k, k = k \tag{F3}$$
> $$(p? + p?) \cdot p? = (i_1 + i_2) \cdot p? \tag{F4}$$
>
> se verificam, demonstre as seguintes propriedades do mesmo combinador:
>
> $$\langle (p \to f, h), (p \to g, i) \rangle = p \to \langle f, g \rangle, \langle h, i \rangle \tag{F5}$$
> $$\langle f, (p \to g, h) \rangle = p \to \langle f, g \rangle, \langle f, h \rangle \tag{F6}$$
> $$p \to (p \to a, b), (p \to c, d) = p \to a, d \tag{F7}$$

Resolução 3

$$(\text{F5})$$

$$\begin{aligned}
& \langle (p \to f, h), (p \to g, i) \rangle \\
= \quad & \{ \text{30: Cond. McCharty} \} \\
& \langle [f, h] \cdot p?, [g, i] \cdot p? \rangle \\
= \quad & \{ \text{9: Fusão-}\times \} \\
& \langle [f, h], [g, i] \rangle \cdot p? \\
= \quad & \{ \text{28: Lei da troca} \} \\
& [\langle f, g \rangle, \langle h, i \rangle] \cdot p? \\
= \quad & \{ \text{30: Cond. McCharty} \} \\
& p \to \langle f, g \rangle, \langle h, i \rangle
\end{aligned}$$

$$(\text{F6})$$

$$\begin{aligned}
& \langle f, (p \to g, h) \rangle \\
= \quad & \{ \text{F3} \} \\
& \langle (p \to f, f), (p \to g, h) \rangle \\
= \quad & \{ \text{F5} \} \\
& p \to \langle f, g \rangle, \langle f, h \rangle
\end{aligned}$$

$$(F7)$$

$$\begin{aligned}
& p \to (p \to a, b), (p \to c, d) \\
& = \qquad \{30\text{: Cond. McCharty } (3\times)\} \\
& [[a, b] \cdot p?, [c, d] \cdot p?] \cdot p? \\
& = \qquad \{22\text{: Absorção-}+\} \\
& [[a, b], [c, d]] \cdot (p? + p?) \cdot p? \\
& = \qquad \{F4\} \\
& [[a, b], [c, d]] \cdot (i_1 + i_2) \cdot p? \\
& = \qquad \{22\text{: Absorção-}+\text{, 18: Cancelamento-}+\} \\
& [a, d] \cdot p? \\
& = \qquad \{30\text{: Cond. McCharty}\} \\
& p \to a, d
\end{aligned}$$

Exercício 4

> Mostre que a propriedade de cancelamento da exponenciação
>
> $$\mathrm{ap} \cdot (\overline{f} \times id) = f \tag{F8}$$
>
> corresponde à definição
>
> $$\mathrm{curry}\ f\ a\ b = f\ (a, b)$$
>
> quando se escreve $\mathrm{curry}\ f$ em lugar de $\overline{f}$.

Resolução 4

$$\begin{aligned}
& \mathrm{ap} \cdot (\overline{f} \times id) = f \\
& \equiv \qquad \{72\text{: Ig. Ext., 73: Def-comp}\} \\
& \mathrm{ap}\ ((\overline{f} \times id)\ (a, b)) = f\ (a, b) \\
& \equiv \qquad \{78\text{: Def-}\times\} \\
& \mathrm{ap}\ (\overline{f}\ a, id) = f\ (a, b) \\
& \equiv \qquad \{84\text{: Def-ap, 1: Natural-id}\} \\
& \overline{f}\ a\ b = f\ (a, b) \\
& \equiv \qquad \{85\text{: Curry, Def. } curry\} \\
& curry\ f\ a\ b = f\ (a, b)
\end{aligned}$$

Exercício 5

> Mostre que a definição de uncurry se pode obter também de (F8) fazendo $f := \text{uncurry}\ g$, introduzindo variáveis e simplificando.

Resolução 5

$$
\begin{aligned}
& f := \text{uncurry}\ g = \widehat{g} \\
\equiv & \qquad \{\text{(F8)}\} \\
& \text{ap} \cdot (\overline{\widehat{g}} \times id) = \widehat{g} \\
\equiv & \qquad \{\text{72: Ig. Ext., 73: Def-comp, Iso.: } \overline{\widehat{g}} = g\} \\
& \text{ap}\ ((g \times id)\ (a, b)) = \widehat{g}\ (a, b) \\
\equiv & \qquad \{\text{78: Def-}\times\text{, 1: Natural-id}\} \\
& \text{ap}\ (g\ a, b) = \widehat{g}\ (a, b) \\
\equiv & \qquad \{\text{84: Def-ap, } \widehat{g} = \text{uncurry}\ g\} \\
& g\ a\ b = \text{uncurry}\ g\ (a, b)
\end{aligned}
$$

Exercício 6

> Prove a igualdade
>
> $$\overline{f \cdot (g \times h)} = \overline{\mathrm{ap} \cdot (\mathrm{id} \times h)} \cdot \overline{f} \cdot g \tag{F9}$$
>
> usando as leis das exponenciais e dos produtos.

Resolução 6

$$
\begin{aligned}
& \overline{f \cdot (g \times h)} = \overline{\mathrm{ap} \cdot (\mathrm{id} \times h)} \cdot \overline{f} \cdot g \\
\equiv & \qquad \{\text{35: Universal-exp}\} \\
& f \cdot (g \times h) = \mathrm{ap} \cdot ((\overline{\mathrm{ap} \cdot (id \times h)} \cdot \overline{f} \cdot g) \times id) \\
\equiv & \qquad \{\text{38: Fusão-exp, 1: Natural-id}\} \\
& f \cdot (g \times h) = \mathrm{ap} \cdot ((\overline{\mathrm{ap} \cdot (id \times h)} \cdot \overline{f \cdot (g \times id)}) \times (id \cdot id)) \\
\equiv & \qquad \{\text{2: Assoc-comp, 14: Functor-}\times\} \\
& f \cdot (g \times h) = \mathrm{ap} \cdot ((\overline{\mathrm{ap} \cdot (id \times h)} \times id) \cdot (\overline{f \cdot (g \times id)} \times id)) \\
\equiv & \qquad \{\text{36: Cancelamento-exp}\} \\
& f \cdot (g \times h) = (\mathrm{ap} \cdot (id \times h)) \cdot (\overline{f \cdot (g \times id)} \times id) \\
\equiv & \qquad \{\text{2: Assoc-comp, 14: Functor-}\times\} \\
& f \cdot (g \times h) = \mathrm{ap} \cdot ((id \cdot \overline{f \cdot (g \times id)}) \times (h \cdot id)) \\
\equiv & \qquad \{\text{1: Natural-id (2}\times\text{), 38: Fusão-exp}\} \\
& f \cdot (g \times h) = \mathrm{ap} \cdot ((\overline{f} \cdot g) \times (id \cdot h)) \\
\equiv & \qquad \{\text{14: Functor-}\times\} \\
& f \cdot (g \times h) = \mathrm{ap} \cdot ((\overline{f} \times id) \cdot (g \times h)) \\
\equiv & \qquad \{\text{2: Assoc-comp, 36: Cancelamento-exp}\} \\
& f \cdot (g \times h) = f \cdot (g \times h) \qquad \text{c.q.d.}
\end{aligned}
$$

Resolução 6 (Alternativa)

$$\overline{f \cdot (g \times h)}$$

$$= \qquad \text{(F9)}$$

$$\overline{\mathrm{ap} \cdot (\mathrm{id} \times h)} \cdot \overline{f} \cdot g$$

$$= \qquad \text{(38: Fusão-exp)}$$

$$\overline{\mathrm{ap} \cdot (\mathrm{id} \times h) \cdot (\overline{f} \times id)} \cdot g$$

$$= \qquad \text{(2: Assoc-comp, 14: Functor-}\times\text{)}$$

$$\overline{\mathrm{ap} \cdot ((\mathrm{id} \cdot \overline{f}) \times (h \cdot id))} \cdot g$$

$$= \qquad \text{(1: Natural-id)}$$

$$\overline{\mathrm{ap} \cdot (\overline{f} \times h)} \cdot g$$

$$= \qquad \text{(38: Fusão-exp)}$$

$$\overline{\mathrm{ap} \cdot (\overline{f} \times h) \cdot (g \times id)}$$

$$= \qquad \text{(14: Functor-}\times\text{, 1: Natural-id)}$$

$$\overline{\mathrm{ap} \cdot ((\overline{f} \cdot g) \times h)}$$

$$= \qquad \text{(38: Fusão-exp)}$$

$$\overline{\mathrm{ap} \cdot (\overline{f \cdot (g \times id)} \times h)}$$

$$= \qquad \text{(1: Natural-id (2}\times\text{))}$$

$$\overline{\mathrm{ap} \cdot ((\overline{f \cdot (g \times id)} \cdot id) \times (id \cdot h))}$$

$$= \qquad \text{(14: Functor-}\times\text{)}$$

$$\overline{\mathrm{ap} \cdot ((\overline{f \cdot (g \times id)} \times id) \cdot (id \times h))}$$

$$= \qquad \text{(2: Assoc-comp, 36: Cancelamento-exp)}$$

$$\overline{f \cdot (g \times id) \cdot (id \times h)}$$

$$= \qquad \text{(2: Assoc-comp, 14: Functor-}\times\text{)}$$

$$\overline{f \cdot ((g \cdot id) \times (id \cdot h))}$$

$$= \qquad \text{(1: Natural-id)}$$

$$\overline{f \cdot (g \times h)}$$

Exercício 7

> É dada a definição
>
> $$\text{flip } f = \overline{\hat{f} \cdot \text{swap}} \tag{F10}$$
>
> de acordo com:
>
> $$\begin{array}{ccccccc} (C^B)^A & \cong & C^{A\times B} & \cong & C^{B\times A} & \cong & (C^A)^B \\ f & \mapsto & \hat{f} & \mapsto & \hat{f} \cdot \text{swap} & \mapsto & \overline{\hat{f} \cdot \text{swap}} = \text{flip } f \end{array}$$
>
> Mostre que flip é um isomorfismo por ser a sua própria inversa:
>
> $$\text{flip } (\text{flip } f) = f \tag{F11}$$
>
> Mostre ainda que:
>
> $$\text{flip } f \ x \ y = f \ y \ x$$

Resolução 7

$$\begin{array}{ll} \text{flip } (\text{flip } f) & \\ = & \big(\text{Def. flip } (2\times)\big) \\ \overline{\widehat{\overline{\hat{f} \cdot \text{swap}}} \cdot \text{swap}} & \\ = & \big(\text{Isomorfismo curry/uncurry}\big) \\ \overline{\hat{f} \cdot \text{swap} \cdot \text{swap}} & \\ = & \big(\text{swap} \cdot \text{swap} = id, \text{ 1: Natural-id}\big) \\ \overline{\hat{f}} & \\ = & \big(\text{Isomorfismo curry/uncurry}\big) \\ f \qquad \text{c.q.m.} & \end{array}$$

Exercício 8

> Mostre que
>
> $$junc \cdot unjunc = id \tag{F12}$$
>
> $$unjunc \cdot junc = id \tag{F13}$$
>
> se verificam, onde
>
> $$\begin{tikzcd} A^{B+C} \arrow[rr, bend left, "unjunc"] \arrow[rr, phantom, "\cong"] & & A^B \times A^C \arrow[ll, bend left, "junc"] \end{tikzcd}$$
>
> $$\begin{cases} junc\ (f, g) = [f, g] \\ unjunc\ k = (k \cdot i_1, k \cdot i_2) \end{cases} \tag{F14}$$

Resolução 8

$$\begin{aligned}
& junc \cdot unjunc = id \\
\equiv & \qquad \{ 72\text{: Ig. Ext., } 73\text{: Def-comp} \} \\
& junc\ (unjunc\ k) = id\ k \\
\equiv & \qquad \{ \text{Def. } unjunc, 74\text{: Def-id} \} \\
& junc\ (k \cdot i_1, k \cdot i_2) = k \\
\equiv & \qquad \{ \text{Def. } junc \} \\
& [k \cdot i_1, k \cdot i_2] = k \\
\equiv & \qquad \{ 20\text{: Fusão-+, } 19\text{: Reflexão-+, } 1\text{: Natural-id} \} \\
& k = k
\end{aligned}$$

$$\begin{aligned}
& unjunc \cdot junc = id \\
\equiv & \qquad \{ 72\text{: Ig. Ext., } 73\text{: Def-comp, } 74\text{: Def-id} \} \\
& unjunc\ (junc\ (f, g)) = (f, g) \\
\equiv & \qquad \{ \text{Def. } junc \} \\
& unjunc\ [f, g] = (f, g) \\
\equiv & \qquad \{ \text{Def. } unjunc \} \\
& (f \cdot i_1, g \cdot i_2) = (f, g) \\
\equiv & \qquad \{ 18\text{: Cancelamento-+} \} \\
& (f, g) = (f, g)
\end{aligned}$$

Exercício 9

> Considere a seguinte sintaxe concreta em Haskell para um tipo que descreve pontos no espaço tridimensional:
>
> ```haskell
> data Point a = Point {x :: a, y :: a, z :: a } deriving (Eq, Show)
> ```
>
> Pelo GHCi apura-se:
>
> ```haskell
> Point :: a -> a -> a -> Point a
> ```
>
> Raciocinando apenas em termos de tipos, conjecture a definição de in na seguinte conversão dessa sintaxe concreta para abstracta:
>
> $$\begin{tikzcd} Point\ A \arrow[r, bend left, "\mathsf{out} = \langle\langle x, y\rangle, z\rangle"] \arrow[r, phantom, "\cong"] & (A \times A) \times A \arrow[l, bend left, "\mathsf{in} = \cdots"] \end{tikzcd}$$

Resolução 9

$$
\begin{array}{l}
\mathrm{in} \cdot \langle x, \langle y, z\rangle\rangle = id \\
\equiv \qquad \{ \text{72: Ig. Ext., 74: Def-id} \} \\
\mathrm{in} \cdot \langle x, \langle y, z\rangle\rangle\ (\mathrm{Point}\ a\ b\ c) = \mathrm{Point}\ a\ b\ c \\
\equiv \qquad \{ p = \mathrm{Point}\ a\ b\ c,\ \text{73: Def-comp, 77: Def-split } (2\times) \} \\
\mathrm{in}\ (x\ p, (y\ p, z\ p)) = p \\
\equiv \qquad \{ \text{Def. } x, y \text{ e } z \} \\
\mathrm{in}\ (a, (b, c)) = \mathrm{Point}\ a\ b\ c \\
\equiv \qquad \{ \text{86: Uncurry } (2\times) \} \\
\mathrm{in}\ (a, (b, c)) = \widehat{\widehat{\mathrm{Point}}}\ ((a, b), c) \\
\equiv \qquad \{ \text{72: Ig. Ext.} \} \\
\mathrm{in} = \widehat{\widehat{\mathrm{Point}}}
\end{array}
$$

Exercício 10

> **Questão prática**
>
> ***Problem requirements***: *The solution given for a previous problem,*
>
> $$store\ c = \text{take}\ 10 \cdot nub \cdot (c{:}) \tag{F15}$$
>
> *calls the standard function*
>
> ```
> nub :: (Eq a) => [a] -> [a]
> ```
>
> *available from the Data.List library in Haskell.*
>
> *After inspecting the standard implementation of this function, define f so that*
>
> $$\text{nub} = [nil, cons] \cdot f$$
>
> *is an alternative to the standard definition, where $nil\ \_ = [\,]$ and $cons\ (h, t) = h : t$.*
>
> *Check that $store\ c$ (F15) works properly once the standard nub is replaced by yours.*
>
> ***Important***: *Structure your solution across the $f \cdot g$, $\langle f, g\rangle$, $f \times g$, $[f, g]$ and $f + g$ combinators that can be found in library Cp.hs. Use **diagrams** to plan your proposed solution, which should avoid re-inventing functions over lists already available in the Haskell standard libraries.*

Resolução 10

```
import Cp (i1, i2, (-|-), (><), nil, cons)

-- Point-wise aux
aux :: Eq a => [a] -> Either () (a, [a])
aux [] = i1 ()
aux (h:t) = i2 (h, filter (h /=) t)

nub :: Eq a => [a] -> [a]
nub = either nil cons . (id -|- (id >< nub)) . aux

-- Tests
test = null (nub ([] :: [Int])) && -- equivalent: nub ([] :: [Int]) == []
       nub [1,2,3,4,1,2,3,4,1,2,3,4] == [1,2,3,4]

store c = take 10 . nub . (c:)
```

```haskell
-- Versão alternativa da função aux point-free
import Cp (i1, i2, p1, p2, split, (-|-), (><), nil, cons)

aux :: Eq a => [a] -> Either () (a, [a])
aux = either (const (i1 ())) (i2 . split p1 (uncurry filter')) . unconsToEither
  where
    unconsToEither [] = Left ()
    unconsToEither (h:t) = Right (h, t)
    filter' h = filter (h /=)
```

$$
\begin{tikzcd}
A^* \arrow[r, "aux"] \arrow[d, "\mathrm{nub}"'] & 1 + A \times A^* \arrow[d, "id + id \times \mathrm{nub}"] \\
A^* & 1 + A \times A^* \arrow[l, "{[nil, cons]}"']
\end{tikzcd}
$$

□