CP - Ficha 5

Exercício 1

No cálculo de programas, as definições condicionais do tipo

$$h\ x = \textbf{if}\ p\ x\ \textbf{then}\ f\ x\ \textbf{else}\ g\ x \qquad\qquad \text{(F1)}$$

são escritas usando o combinador ternário

$$p \to f, g$$

conhecido pelo nome de *condicional de McCarthy*, cuja definição

$$p \to f, g = [f, g] \cdot p? \qquad\qquad \text{(F2)}$$

vem no formulário. Baseie-se em leis desse formulário para demonstrar a chamada 2ª-lei de fusão do condicional:

$$(p \to f, g) \cdot h = (p \cdot h) \to (f \cdot h), (g \cdot h)$$

Resolução 1

$$(p \to f, g) \cdot h$$
$$= \qquad\qquad (30: \text{Cond. McCharty (F2)}, 2: \text{Assoc-comp})$$
$$[f, g] \cdot (p? \cdot h)$$
$$= \qquad\qquad (29: \text{Natural-guarda})$$
$$[f, g] \cdot (h + h) \cdot (p \cdot h)?$$
$$= \qquad\qquad (22: \text{Absorção-+})$$
$$[f \cdot h, g \cdot h] \cdot (p \cdot h)?$$
$$= \qquad\qquad (30: \text{Cond. McCharty (F2)})$$
$$(p \cdot h) \to f \cdot h, g \cdot h$$

Exercício 2

Numa máquina paralela pode fazer sentido, em (F1), não esperar por $p\ x$ para avaliar $f\ x$ ou $g\ x$, mas sim correr tudo em paralelo,

$$parallel\ p\ f\ g = \langle\langle f,g\rangle, p\rangle$$

e depois fazer a escolha do resultado:

$$choose = \pi_2 \to \pi_1 \cdot \pi_1, \pi_2 \cdot \pi_1$$

Mostre que, de facto:

$$choose \cdot parallel\ p\ f\ g = p \to f, g$$

Resolução 2

$choose \cdot parallel\ p\ f\ g$

$=$ $(\text{Def. } parallel, \text{Def. } choose)$

$(\pi_2 \to \pi_1 \cdot \pi_1, \pi_2 \cdot \pi_1) \cdot \langle\langle f,g\rangle, p\rangle$

$=$ $(32\text{: 2ª lei de fusão condicional})$

$\pi_2 \cdot \langle\langle f,g\rangle, p\rangle \to \pi_1 \cdot \pi_1 \cdot \langle\langle f,g\rangle, p\rangle, \pi_2 \cdot \pi_1 \cdot \langle\langle f,g\rangle, p\rangle$

$=$ $(7\text{: Cancelamento-}\times)$

$p \to f, g$

Exercício 3

Sabendo que as igualdades

$$p \to k, k = k \tag{F3}$$

$$(p? + p?) \cdot p? = (i_1 + i_2) \cdot p? \tag{F4}$$

se verificam, demonstre as seguintes propriedades do mesmo combinador:

$$\langle (p \to f, h), (p \to g, i) \rangle = p \to \langle f, g \rangle, \langle h, i \rangle \tag{F5}$$

$$\langle f, (p \to g, h) \rangle = p \to \langle f, g \rangle, \langle f, h \rangle \tag{F6}$$

$$p \to (p \to a, b), (p \to c, d) = p \to a, d \tag{F7}$$

Resolução 3

$$(\text{F5})$$

$$\langle (p \to f, h), (p \to g, i) \rangle$$

$$= \qquad\qquad (30: \text{Cond. McCharty})$$

$$\langle [f, h] \cdot p?, [g, i] \cdot p? \rangle$$

$$= \qquad\qquad (9: \text{Fusão-}\times)$$

$$\langle [f, h], [g, i] \rangle \cdot p?$$

$$= \qquad\qquad (28: \text{Lei da troca})$$

$$[\langle f, g \rangle, \langle h, i \rangle] \cdot p?$$

$$= \qquad\qquad (30: \text{Cond. McCharty})$$

$$p \to \langle f, g \rangle, \langle h, i \rangle$$

$$(\text{F6})$$

$$\langle f, (p \to g, h) \rangle$$

$$= \qquad\qquad (\text{F3})$$

$$\langle (p \to f, f), (p \to g, h) \rangle$$

$$= \qquad\qquad (\text{F5})$$

$$p \to \langle f, g \rangle, \langle f, h \rangle$$

$$(\text{F7})$$

$$p \to (p \to a, b), (p \to c, d)$$
$$= \qquad\qquad (30: \text{Cond. McCharty } (3\times))$$
$$[[a, b] \cdot p?, [c, d] \cdot p?] \cdot p?$$
$$= \qquad\qquad (22: \text{Absorção-+})$$
$$[[a, b], [c, d]] \cdot (p? + p?) \cdot p?$$
$$= \qquad\qquad (\text{F4})$$
$$[[a, b], [c, d]] \cdot (i_1 + i_2) \cdot p?$$
$$= \qquad\qquad (22: \text{Absorção-+}, 18: \text{Cancelamento-+})$$
$$[a, d] \cdot p?$$
$$= \qquad\qquad (30: \text{Cond. McCharty})$$
$$p \to a, d$$

Exercício 4

Mostre que a propriedade de cancelamento da exponenciação

$$\mathrm{ap} \cdot (\overline{f} \times id) = f \qquad\qquad (\text{F8})$$

corresponde à definição

$$\mathrm{curry}\ f\ a\ b = f\ (a, b)$$

quando se escreve $\mathrm{curry}\ f$ em lugar de $\overline{f}$.

Resolução 4

$$\mathrm{ap} \cdot (\overline{f} \times id) = f$$
$$\equiv \qquad\qquad (72: \text{Ig. Ext.}, 73: \text{Def-comp})$$
$$\mathrm{ap}\ ((\overline{f} \times id)\ (a, b)) = f\ (a, b)$$
$$\equiv \qquad\qquad (78: \text{Def-}\times)$$
$$\mathrm{ap}\ (\overline{f}\ a, id) = f\ (a, b)$$
$$\equiv \qquad\qquad (84: \text{Def-ap}, 1: \text{Natural-id})$$
$$\overline{f}\ a\ b = f\ (a, b)$$
$$\equiv \qquad\qquad (85: \text{Curry}, \text{Def. } curry)$$
$$curry\ f\ a\ b = f\ (a, b)$$

Exercício 5

Mostre que a definição de $\mathrm{uncurry}$ se pode obter também de (F8) fazendo $f :=$ uncurry g, introduzindo variáveis e simplificando.

Resolução 5

$$f := \mathrm{uncurry}\ g = \widehat{g}$$
$$\equiv \qquad\qquad\qquad\qquad\qquad\qquad (\mathrm{F8})$$
$$\mathrm{ap} \cdot (\overline{\widehat{g}} \times id) = \widehat{g}$$
$$\equiv \qquad\qquad\qquad (72\text{: Ig. Ext., } 73\text{: Def-comp, Iso.: } \overline{\overline{\widehat{g}}} = g)$$
$$\mathrm{ap}\ ((g \times id)\ (a, b)) = \widehat{g}\ (a, b)$$
$$\equiv \qquad\qquad\qquad\qquad (78\text{: Def-}\times\text{, } 1\text{: Natural-id})$$
$$\mathrm{ap}\ (g\ a, b) = \widehat{g}\ (a, b)$$
$$\equiv \qquad\qquad\qquad\qquad (84\text{: Def-ap, } \widehat{g} = \mathrm{uncurry}\ g)$$
$$g\ a\ b = \mathrm{uncurry}\ g\ (a, b)$$

Exercício 6

> Prove a igualdade
>
> $$\overline{f \cdot (g \times h)} = \overline{\mathrm{ap} \cdot (\mathrm{id} \times h) \cdot \overline{f} \cdot g} \tag{F9}$$
>
> usando as leis das exponenciais e dos produtos.

Resolução 6

$$\overline{f \cdot (g \times h)} = \overline{\mathrm{ap} \cdot (\mathrm{id} \times h) \cdot \overline{f} \cdot g}$$

$\equiv \qquad\qquad\qquad\qquad\qquad\qquad$ (35: Universal-exp)

$$f \cdot (g \times h) = \mathrm{ap} \cdot ((\overline{\mathrm{ap} \cdot (id \times h) \cdot \overline{f} \cdot g}) \times id)$$

$\equiv \qquad\qquad\qquad\qquad\qquad$ (38: Fusão-exp, 1: Natural-id)

$$f \cdot (g \times h) = \mathrm{ap} \cdot ((\overline{\mathrm{ap} \cdot (id \times h) \cdot \overline{f \cdot (g \times id)}}) \times (id \cdot id))$$

$\equiv \qquad\qquad\qquad\qquad\qquad$ (2: Assoc-comp, 14: Functor-$\times$)

$$f \cdot (g \times h) = \mathrm{ap} \cdot ((\overline{\mathrm{ap} \cdot (id \times h)} \times id) \cdot (\overline{f \cdot (g \times id)} \times id))$$

$\equiv \qquad\qquad\qquad\qquad\qquad$ (36: Cancelamento-exp)

$$f \cdot (g \times h) = (\mathrm{ap} \cdot (id \times h)) \cdot (\overline{f \cdot (g \times id)} \times id)$$

$\equiv \qquad\qquad\qquad\qquad\qquad$ (2: Assoc-comp, 14: Functor-$\times$)

$$f \cdot (g \times h) = \mathrm{ap} \cdot ((id \cdot \overline{f \cdot (g \times id)}) \times (h \cdot id))$$

$\equiv \qquad\qquad\qquad\qquad$ (1: Natural-id ($2\times$), 38: Fusão-exp)

$$f \cdot (g \times h) = \mathrm{ap} \cdot ((\overline{f} \cdot g) \times (id \cdot h))$$

$\equiv \qquad\qquad\qquad\qquad\qquad\qquad$ (14: Functor-$\times$)

$$f \cdot (g \times h) = \mathrm{ap} \cdot ((\overline{f} \times id) \cdot (g \times h))$$

$\equiv \qquad\qquad\qquad$ (2: Assoc-comp, 36: Cancelamento-exp)

$$f \cdot (g \times h) = f \cdot (g \times h) \qquad \text{c.q.d.}$$

Resolução 6 (Alternativa)

$$\overline{f \cdot (g \times h)}$$

$$= \qquad (\text{F9})$$

$$\overline{\text{ap} \cdot (\text{id} \times h) \cdot \overline{f} \cdot g}$$

$$= \qquad (38: \text{Fusão-exp})$$

$$\overline{\text{ap} \cdot (\text{id} \times h) \cdot (\overline{f} \times id) \cdot g}$$

$$= \qquad (2: \text{Assoc-comp}, 14: \text{Functor-} \times)$$

$$\overline{\text{ap} \cdot ((\text{id} \cdot \overline{f}) \times (h \cdot id)) \cdot g}$$

$$= \qquad (1: \text{Natural-id})$$

$$\overline{\text{ap} \cdot (\overline{f} \times h) \cdot g}$$

$$= \qquad (38: \text{Fusão-exp})$$

$$\overline{\text{ap} \cdot (\overline{f} \times h) \cdot (g \times id)}$$

$$= \qquad (14: \text{Functor-} \times, 1: \text{Natural-id})$$

$$\overline{\text{ap} \cdot ((\overline{f} \cdot g) \times h)}$$

$$= \qquad (38: \text{Fusão-exp})$$

$$\overline{\text{ap} \cdot (\overline{f \cdot (g \times id)} \times h)}$$

$$= \qquad (1: \text{Natural-id } (2\times))$$

$$\overline{\text{ap} \cdot ((\overline{f \cdot (g \times id)} \cdot id) \times (id \cdot h))}$$

$$= \qquad (14: \text{Functor-} \times)$$

$$\overline{\text{ap} \cdot ((\overline{f \cdot (g \times id)} \times id) \cdot (id \times h))}$$

$$= \qquad (2: \text{Assoc-comp}, 36: \text{Cancelamento-exp})$$

$$\overline{f \cdot (g \times id) \cdot (id \times h)}$$

$$= \qquad (2: \text{Assoc-comp}, 14: \text{Functor-} \times)$$

$$\overline{f \cdot ((g \cdot id) \times (id \cdot h))}$$

$$= \qquad (1: \text{Natural-id})$$

$$\overline{f \cdot (g \times h)}$$

Exercício 7

É dada a definição

$$\text{flip } f = \overline{\hat{f} \cdot \text{swap}} \tag{F10}$$

de acordo com:

$$
\begin{array}{ccccccc}
(C^B)^A & \cong & C^{A \times B} & \cong & C^{B \times A} & \cong & (C^A)^B \\
f & \mapsto & \hat{f} & \mapsto & \hat{f} \cdot \text{swap} & \mapsto & \overline{\hat{f} \cdot \text{swap}} = \text{flip } f
\end{array}
$$

Mostre que flip é um isomorfismo por ser a sua própria inversa:

$$\text{flip } (\text{flip } f) = f \tag{F11}$$

Mostre ainda que:

$$\text{flip } f \; x \; y = f \; y \; x$$

Resolução 7

$$
\begin{array}{ll}
\text{flip } (\text{flip } f) & \\
= & (\text{Def. flip } (2\times)) \\
\overline{\widehat{\overline{\hat{f} \cdot \text{swap}}} \cdot \text{swap}} & \\
= & (\text{Isomorfismo curry/uncurry}) \\
\overline{\hat{f} \cdot \text{swap} \cdot \text{swap}} & \\
= & (\text{swap} \cdot \text{swap} = id,\ 1\text{: Natural-id}) \\
\overline{\hat{f}} & \\
= & (\text{Isomorfismo curry/uncurry}) \\
f \quad \text{c.q.m.} &
\end{array}
$$

Exercício 8

Mostre que

$$junc \cdot unjunc = id \tag{F12}$$

$$unjunc \cdot junc = id \tag{F13}$$

se verificam, onde

$$A^{B+C} \xrightarrow[\text{junc}]{\text{unjunc}} \cong A^B \times A^C$$

$$\begin{cases} junc\,(f,g) = [f,g] \\ unjunc\,k = (k \cdot i_1, k \cdot i_2) \end{cases} \tag{F14}$$

Resolução 8

$$junc \cdot unjunc = id$$
$$\equiv \qquad \qquad (\text{72: Ig. Ext., 73: Def-comp})$$
$$junc\,(unjunc\,k) = id\,k$$
$$\equiv \qquad \qquad (\text{Def. } unjunc, \text{74: Def-id})$$
$$junc\,(k \cdot i_1, k \cdot i_2) = k$$
$$\equiv \qquad \qquad (\text{Def. } junc)$$
$$[k \cdot i_1, k \cdot i_2] = k$$
$$\equiv \qquad \qquad (\text{20: Fusão-+, 19: Reflexão-+, 1: Natural-id})$$
$$k = k$$

$$unjunc \cdot junc = id$$
$$\equiv \qquad \qquad (\text{72: Ig. Ext., 73: Def-comp, 74: Def-id})$$
$$unjunc\,(junc\,(f,g)) = (f,g)$$
$$\equiv \qquad \qquad (\text{Def. } junc)$$
$$unjunc\,[f,g] = (f,g)$$
$$\equiv \qquad \qquad (\text{Def. } unjunc)$$
$$(f \cdot i_1, g \cdot i_2) = (f,g)$$
$$\equiv \qquad \qquad (\text{18: Cancelamento-+})$$
$$(f,g) = (f,g)$$

Exercício 9

Considere a seguinte sintaxe concreta em Haskell para um tipo que descreve pontos no espaço tridimensional:

```haskell
data Point a = Point {x :: a, y :: a, z :: a } deriving (Eq, Show)
```

Pelo GHCi apura-se:

```haskell
Point :: a -> a -> a -> Point a
```

Raciocinando apenas em termos de tipos, conjecture a definição de in na seguinte conversão dessa sintaxe concreta para abstracta:

$$Point\ A \overset{\text{out}=\langle\langle x,y\rangle,z\rangle}{\underset{\text{in}=\cdots}{\cong}} (A \times A) \times A$$
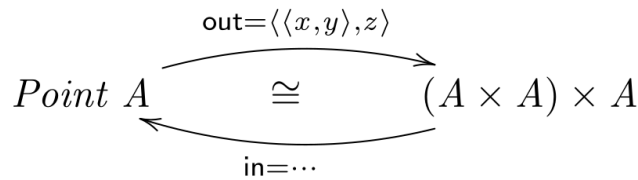
Resolução 9

$$\text{in} \cdot \langle x, \langle y, z \rangle \rangle = id$$
$$\equiv \qquad\qquad\qquad (72\text{: Ig. Ext., } 74\text{: Def-id})$$
$$\text{in} \cdot \langle x, \langle y, z \rangle \rangle\ (\text{Point}\ a\ b\ c) = \text{Point}\ a\ b\ c$$
$$\equiv \qquad\qquad (p = \text{Point}\ a\ b\ c, 73\text{: Def-comp}, 77\text{: Def-split}\ (2\times))$$
$$\text{in}\ (x\ p, (y\ p, z\ p)) = p$$
$$\equiv \qquad\qquad\qquad (\text{Def. } x, y \text{ e } z)$$
$$\text{in}\ (a, (b, c)) = \text{Point}\ a\ b\ c$$
$$\equiv \qquad\qquad\qquad (86\text{: Uncurry}\ (2\times))$$
$$\text{in}\ (a, (b, c)) = \widehat{\widehat{\text{Point}}}\ ((a, b), c)$$
$$\equiv \qquad\qquad\qquad (72\text{: Ig. Ext.})$$
$$\text{in} = \widehat{\widehat{\text{Point}}}$$

Exercício 10

Questão prática

Problem requirements: The solution given for a previous problem,

$$store\ c = \text{take } 10 \cdot nub \cdot (c :) \tag{F15}$$

calls the standard function

```
nub :: (Eq a) => [a] -> [a]
```

available from the Data.List library in Haskell.

After inspecting the standard implementation of this function, define f so that

$$\text{nub} = [nil, cons] \cdot f$$

is an alternative to the standard definition, where $nil\ \_ = [\ ]$ and $cons\ (h, t) = h : t$.

Check that $store\ c$ (F15) works properly once the standard nub is replaced by yours.

Important: Structure your solution across the $f \cdot g$, $\langle f, g \rangle$, $f \times g$, $[f, g]$ and $f + g$ combinators that can be found in library Cp.hs. Use **diagrams** to plan your proposed solution, which should avoid re-inventing functions over lists already available in the Haskell standard libraries.

Resolução 10

TODO

☐