CP - Ficha 8

Exercício 1

> A igualdade que se segue
>
> $$f \cdot \text{length} = (\![[\text{zero}, (2+) \cdot \pi_2]]\!)$$
>
> verifica-se para $f = (2*)$ ou $f = (2+)$? Use a lei de fusão-cata para justificar, por cálculo, a sua resposta.

Resolução 1

$$\text{length} = (\![[\text{zero}, \text{succ} \cdot \pi_2]]\!)$$

$$
\begin{array}{ll}
f \cdot \text{length} = (\![[\text{zero}, (2+) \cdot \pi_2]]\!) & \\
\equiv & \{\text{Def. length, Def. zero } (2\times)\} \\
f \cdot (\![[\underline{0}, \text{succ} \cdot \pi_2]]\!) = (\![[\underline{0}, (2+) \cdot \pi_2]]\!) & \\
\Leftarrow & \{\text{49: Fusão-cata, } \mathsf{F}_{\text{List}}\ f = id + id \times f\} \\
f \cdot [\underline{0}, \text{succ} \cdot \pi_2] = [\underline{0}, (2+) \cdot \pi_2] \cdot (id + id \times f) & \\
\equiv & \{\text{20: Fusão-+, 22: Absorção-+, 1: Natural-id}\} \\
[f \cdot \underline{0}, f \cdot \text{succ} \cdot \pi_2] = [\underline{0}, (2+) \cdot \pi_2 \cdot (id \times f)] & \\
\equiv & \{\text{13: Natural-}\pi_2\text{, 27: Eq-+}\} \\
\begin{cases} f \cdot \underline{0} = \underline{0} \\ f \cdot \text{succ} \cdot \pi_2 = (2+) \cdot f \cdot \pi_2 \end{cases} & \\
\equiv & \{\text{72: Ig. Ext., 73: Def-comp, 75: Def-const}\} \\
\begin{cases} f\ 0 = 0 \\ f\ (\text{succ}\ (\pi_2\ (\_, n))) = 2 + f\ (\pi_2\ (\_, n)) \end{cases} & \\
\equiv & \{\text{79: Def-proj, Def. succ}\} \\
\begin{cases} f\ 0 = 0 \\ f\ (n+1) = 2 + f\ n \end{cases} &
\end{array}
$$

Hipótese 1: $f = (2+)$

$$
\begin{cases} 2 + 0 = 2 \\ 2 + (n+1) = 2 + (2 + n) \end{cases}
$$

$\equiv$

False

Hipótese 2: $f = (2*)$

$$
\begin{cases} 2 * 0 = 2 \\ 2 * (n+1) = 2 + (2 * n) \end{cases}
$$

$\equiv$

True

Exercício 2

> As seguintes funções mutuamente recursivas testam a paridade de um número natural:
>
> $$\begin{cases} impar\ 0 = \text{FALSE} \\ impar\ (n+1) = par\ n \end{cases} \qquad \begin{cases} par\ 0 = \text{TRUE} \\ par\ (n+1) = impar\ n \end{cases}$$
>
> Assumindo o functor $\mathbf{F}\ f = id + f$, mostre que esse par de definições é equivalente ao sistema de equações
>
> $$\begin{cases} impar \cdot \text{in} = h \cdot \text{F}\ \langle impar, par \rangle \\ par \cdot \text{in} = k \cdot \text{F}\ \langle impar, par \rangle \end{cases}$$
>
> para um dado h e k (deduza-os). De seguida, recorra às leis da recursividade mútua e da troca para mostrar que
>
> $$imparpar = \langle impar, par \rangle = \text{for swap (FALSE, TRUE)}$$

Resolução 2

$$\begin{cases} impar\ 0 = \text{FALSE} \\ impar\ (n+1) = par\ n \end{cases}$$

$$\equiv \qquad \text{(73: Def-comp, 75: Def-const, Def. succ, 72: Ig. Ext.)}$$

$$\begin{cases} impar \cdot \underline{0} = \underline{\text{false}} \\ impar \cdot \text{succ} = par \end{cases}$$

$$\equiv \qquad \text{(27: Eq-+, 20 Fusão-+)}$$

$$impar \cdot [\underline{0}, \text{succ}] = [\underline{\text{false}}, par]$$

$$\equiv \qquad (\text{Def. in}_{\mathbb{N}_0}\text{, 7: Cancelamento-}\times\text{, 3: Fusão-const})$$

$$impar \cdot \text{in}_{\mathbb{N}_0} = [\underline{\text{false}}, \pi_2 \cdot \langle impar, par \rangle]$$

$$\equiv \qquad \text{(1: Natural-id, 22: Absorção-+)}$$

$$impar \cdot \text{in}_{\mathbb{N}_0} = [\underline{\text{false}}, \pi_2] \cdot (id + \langle impar, par \rangle)$$

$$\equiv \qquad (\mathbf{F}\ f = id + f)$$

$$impar \cdot \text{in}_{\mathbb{N}_0} = [\underline{\text{false}}, \pi_2] \cdot \mathbf{F}\ \langle impar, par \rangle$$

$$\Leftarrow$$

$$h = [\underline{\text{false}}, \pi_2]$$

$$
\begin{cases} par\ 0 = \text{TRUE} \\ par\ (n+1) = impar\ n \end{cases}
$$

$$\equiv \qquad (\ldots)$$

$$par \cdot \text{in}_{\mathbb{N}_0} = [\underline{\text{true}}, \pi_1] \cdot \mathbf{F}\ \langle impar, par \rangle$$

$$\Leftarrow$$

$$k = [\underline{\text{true}}, \pi_1]$$

$$
\begin{cases} impar \cdot \text{in} = h \cdot \mathbf{F}\ \langle impar, par \rangle \\ par \cdot \text{in} = k \cdot \mathbf{F}\ \langle impar, par \rangle \end{cases}
$$

$$\equiv \qquad \{\, 53\text{: Fokkinga} \,\}$$

$$\langle impar, par \rangle = (\!|\, \langle h, k \rangle \,|\!)$$

$$\equiv \qquad \{\, \text{Def. h, Def. k} \,\}$$

$$\langle impar, par \rangle = (\!|\, \langle [\underline{\text{false}}, \pi_2], [\underline{\text{true}}, \pi_1] \rangle \,|\!)$$

$$\equiv \qquad \{\, 28\text{: Lei da troca} \,\}$$

$$\langle impar, par \rangle = (\!|\, [\langle \underline{\text{false}}, \underline{\text{true}} \rangle, \langle \pi_2, \pi_1 \rangle] \,|\!)$$

$$\equiv \qquad \{\, \text{Def. swap}, \langle \underline{a}, \underline{b} \rangle = \underline{(a, b)} \,\}$$

$$\langle impar, par \rangle = (\!|\, [\underline{(\text{false}, \text{true})}, \text{swap}] \,|\!)$$

$$\equiv \qquad \{\, \text{for } b\ i = (\!|\, [\underline{i}, b] \,|\!) \,\}$$

$$\langle impar, par \rangle = \text{for swap (FALSE, TRUE)}$$

Exercício 3

> A seguinte função em Haskell lista os primeiros n números naturais por ordem inversa:
>
> $$\begin{cases} insg\ 0 = [\,] \\ insg\ (n+1) = (n+1) : insg\ n \end{cases}$$
>
> Mostre que $insg$ pode ser definida por recursividade mútua tal como se segue:
>
> $$\begin{cases} insg\ 0 = [] \\ insg\ (n+1) = (fsuc\ n) : insg\ n \end{cases}$$
>
> $$\begin{cases} fsuc\ 0 = 1 \\ fsuc\ (n+1) = fsuc\ n + 1 \end{cases}$$
>
> A seguir, usando a lei de recursividade mútua, derive:
>
> $$\begin{aligned} insg &= \pi_2 \cdot insgfor \\ insgfor &= \text{for}\ \langle (1+) \cdot \pi_1, \text{cons} \rangle\ (1, [\,]) \end{aligned}$$

Resolução 3

$$\begin{aligned}
& \begin{cases} insg\ 0 = [\,] \\ insg\ (n+1) = (n+1) : insg\ n \end{cases} \\
\equiv & \qquad \{ 73, 75, \text{Def. succ}, \text{Def. cons}, \text{Def. nil}, 77, 72 \} \\
& \begin{cases} insg \cdot \underline{0} = \text{nil} \\ insg \cdot \text{succ} = \text{cons} \cdot \langle \text{succ}, insg \rangle \end{cases} \\
\equiv & \qquad \{ 27\text{: Eq-+}, 20\text{: Fusão-+} \} \\
& insg \cdot [\underline{0}, \text{succ}] = [\text{nil}, \text{cons} \cdot \langle \text{succ}, insg \rangle] \\
\equiv & \qquad \{ \text{Def. in}_{\mathbb{N}_0}, 1\text{: Natural-id}, 22\text{: Absorção-+} \} \\
& insg \cdot \text{in}_{\mathbb{N}_0} = [\text{nil}, \text{cons}] \cdot (id + \langle \text{succ}, insg \rangle]) \\
\equiv & \qquad \{ \mathbf{F}\ f = id + f \} \\
& insg \cdot \text{in}_{\mathbb{N}_0} = [\text{nil}, \text{cons}] \cdot \mathbf{F}\ \langle \text{succ}, insg \rangle \\
\Leftarrow & \\
& k = [\text{nil}, \text{cons}]
\end{aligned}$$

$$
\begin{cases} fsuc\ 0 = 1 \\ fsuc\ (n+1) = fsuc\ n + 1 \end{cases}
$$

$$
\equiv \qquad \{ \ldots \}
$$

$$
fsuc \cdot \mathrm{in}_{\mathbb{N}_0} = [\underline{1}, \mathrm{succ} \cdot \pi_1] \cdot \mathbf{F}\ \langle \mathrm{succ}, insg \rangle
$$

$$
\Leftarrow
$$

$$
h = [\underline{1}, \mathrm{succ} \cdot \pi_1]
$$

$$
\begin{cases} fsuc \cdot \mathrm{in} = h \cdot \mathbf{F}\ \langle \mathrm{succ}, insg \rangle \\ insg \cdot \mathrm{in} = k \cdot \mathbf{F}\ \langle \mathrm{succ}, insg \rangle \end{cases}
$$

$$
\equiv \qquad \{ \text{53: Fokkinga} \}
$$

$$
\langle insg, fsuc \rangle = (\!|\, \langle h, k \rangle \,|\!)
$$

$$
\equiv \qquad \{ \text{Def. k, Def. h} \}
$$

$$
\langle insg, fsuc \rangle = (\!|\, \langle [\underline{1}, \mathrm{succ} \cdot \pi_1], [\mathrm{nil}, \mathrm{cons}] \rangle \,|\!)
$$

$$
\equiv \qquad \{ \text{28: Lei da troca, Def. nil} \}
$$

$$
\langle insg, fsuc \rangle = (\!|\, [\langle \underline{1}, \underline{[\,]} \rangle, \langle \mathrm{succ} \cdot \pi_1, \mathrm{cons} \rangle] \,|\!)
$$

$$
\equiv \qquad \{ \langle \underline{a}, \underline{b} \rangle = \underline{(a,b)}, \text{Def. for} \}
$$

$$
\langle insg, fsuc \rangle = \mathrm{for}\ \langle \mathrm{succ} \cdot \pi_1, \mathrm{cons} \rangle\ (1, [\,])
$$

$$
\equiv \qquad \{ \text{Def. succ} \}
$$

$$
\langle insg, fsuc \rangle = \mathrm{for}\ \langle (1+) \cdot \pi_1, \mathrm{cons} \rangle\ (1, [\,])
$$

Exercício 4

> Considere o par de funções mutuamente recursivas
>
> $$\begin{cases} f_1 \; [\,] = [\,] \\ f_1 \; (h : t) = h : (f_2 \; t) \end{cases} \qquad \begin{cases} f_2 \; [\,] = [\,] \\ f_2 \; (h : t) = f_1 \; t \end{cases}$$
>
> Mostre por recursividade mútua que $\langle f_1, f_2 \rangle$ é um catamorfismo de listas (onde $\mathrm{F} \; f = id + id \times f$) e desenhe o respectivo diagrama. Que faz cada uma destas funções f_1 e f_2?

Resolução 4

TODO

Exercício 5

> Sejam dados os functores elementares seguintes:
>
> $$\begin{cases} \mathbf{F}\ X = \mathbb{Z} \\ \mathbf{F}\ f = id \end{cases} \qquad \begin{cases} \mathbf{G}\ X = X \\ \mathbf{G}\ f = f \end{cases}$$
>
> Mostre que $\mathbf{H}$ e $\mathbf{K}$ definidos por
>
> $$\mathbf{H}\ X = \mathbf{F}\ X + \mathbf{G}\ X$$
>
> $$\mathbf{K}\ X = \mathbf{G}\ X \times \mathbf{F}\ X$$
>
> são functores.

Resolução 5

$$\begin{cases} \mathbf{H}\ X = \mathbf{F}\ X + \mathbf{G}\ X \\ \mathbf{H}\ f = \mathbf{F}\ f + \mathbf{G}\ f \end{cases}$$
$$\equiv \qquad \{ \text{Def. } \mathbf{F} \text{ e } \mathbf{G} \}$$
$$\begin{cases} \mathbf{H}\ X = \mathbb{Z} + X \\ \mathbf{H}\ f = id + f \end{cases}$$

$$\text{Para } \mathbf{H} \text{ ser functor, tem de se verificar: } \begin{cases} \mathbf{H}\ (g \cdot f) = \mathbf{H}\ g \cdot \mathbf{H}\ f \\ \mathbf{H}\ id = id \end{cases}$$

$$\begin{aligned} (\mathbf{H}\ g) \cdot (\mathbf{H}\ h) &= (id + g) \cdot (id + h) && (\text{Def. } \mathbf{H}\ f = id + f) \\ &= id + g \cdot h && (\text{25: Functor-+, 1: Natural-id}) \\ &= \mathbf{H}\ (g \cdot h) \end{aligned}$$

$$\begin{aligned} \mathbf{H}\ id &= id + id && (\text{Def. } \mathbf{H}\ f = id + f) \\ &= id && (\text{1: Natural-id}) \end{aligned}$$

$$
\begin{cases} \mathbf{K}\ X = \mathbf{G}\ X \times \mathbf{F}\ X \\ \mathbf{K}\ f = \mathbf{G}\ f \times \mathbf{F}\ f \end{cases}
$$

$$
\equiv \qquad \{\text{Def. } \mathbf{F} \text{ e } \mathbf{G}\}
$$

$$
\begin{cases} \mathbf{K}\ X = X \times \mathbb{Z} \\ \mathbf{K}\ f = f \times id \end{cases}
$$

Para $\mathbf{K}$ ser functor, tem de se verificar: $\begin{cases} \mathbf{K}\ (g \cdot f) = \mathbf{K}\ g \cdot \mathbf{K}\ f \\ \mathbf{K}\ id = id \end{cases}$

$$
\begin{aligned}
(\mathbf{K}\ g) \cdot (\mathbf{K}\ h) &= (g \times id) \cdot (h \times id) && \{\text{Def. } \mathbf{K}\ f = f \times id\} \\
&= g \cdot h \times id && \{\text{25: Functor-}\times\text{, 1: Natural-id}\} \\
&= \mathbf{K}\ (g \cdot h)
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{K}\ id &= id \times id && \{\text{Def. } \mathbf{K}\ f = f \times id\} \\
&= id && \{\text{1: Natural-id}\}
\end{aligned}
$$

Exercício 6

> Mostre que, sempre que $\mathbf{F}$ e $\mathbf{G}$ são functores, então a sua composição $\mathbf{H} = \mathbf{F} \cdot \mathbf{G}$ é também um functor.

Resolução 6

$$\text{Se } \mathbf{F} \text{ e } \mathbf{G} \text{ são functores, então:}$$

$$\begin{cases} \mathbf{F}\ (g \cdot f) = \mathbf{F}\ g \cdot \mathbf{F}\ f \\ \mathbf{F}\ id = id \end{cases} \quad \text{e} \quad \begin{cases} \mathbf{G}\ (g \cdot f) = \mathbf{G}\ g \cdot \mathbf{G}\ f \\ \mathbf{G}\ id = id \end{cases}$$

$$\text{Para } \mathbf{H} = \mathbf{F} \cdot \mathbf{G} \text{ ser functor, tem de se verificar:}$$

$$\begin{cases} \mathbf{H}\ (g \cdot f) = \mathbf{H}\ g \cdot \mathbf{H}\ f \\ \mathbf{H}\ id = id \end{cases}$$

$$\begin{aligned} \mathbf{H}\ (g \cdot h) &= \mathbf{F} \cdot \mathbf{G}\ (g \cdot h) && (\text{Def. } \mathbf{H} = \mathbf{F} \cdot \mathbf{G}) \\ &= \mathbf{F}\ (\mathbf{G}\ g \cdot \mathbf{G}\ h) && (\mathbf{G}\ (g \cdot f) = \mathbf{G}\ g \cdot \mathbf{G}\ f) \\ &= (\mathbf{F} \cdot \mathbf{G}\ g) \cdot (\mathbf{F} \cdot \mathbf{G}\ h) && (\mathbf{F}\ (g \cdot f) = \mathbf{F}\ g \cdot \mathbf{F}\ f) \\ &= (\mathbf{H}\ g) \cdot (\mathbf{H}\ h) && (\text{Def. } \mathbf{H} = \mathbf{F} \cdot \mathbf{G}) \end{aligned}$$

$$\begin{aligned} \mathbf{H}\ id &= (\mathbf{F} \cdot \mathbf{G})\ id && (\text{Def. } \mathbf{H} = \mathbf{F} \cdot \mathbf{G}) \\ &= \mathbf{F}\ (\mathbf{G}\ id) && (\text{Def. comp}) \\ &= \mathbf{F}\ id && (\mathbf{G}\ id = id) \\ &= id && (\mathbf{F}\ id = id) \end{aligned}$$

Exercício 7

> **Questão prática**
>
> ***Problem definition:*** *Page [UNZIP IN ONE PASS?] of* Stack Overflow *addresses the question as to whether*
>
> $$\mathrm{unzip}\ xs = (\mathrm{map}\ \pi_1\ xs, \mathrm{map}\ \pi_2\ xs)$$
>
> *can do one traversal only. The answer is affirmative:*
>
> $$\begin{array}{l} \mathrm{unzip}\ [\,] = ([\,],[\,]) \\ \mathrm{unzip}\ ((a,b) : xs) = (a : as, b : bs)\ \mathbf{where}\ (as, bs) = \mathrm{unzip}\ xs \end{array}$$
>
> *What is missing from* Stack Overflow *is the explanation of how the two steps of* unzip *merge into one. Show that the* banana-split *law is what needs to be known for the one traversal version to be derived from the two traversal one.*

Resolução 7

TODO

□