

CP - Ficha 8

Exercício 1

A igualdade que se segue

$$f \cdot \text{length} = ([\text{zero}, (2+) \cdot \pi_2])$$

verifica-se para $f = (2*)$ ou $f = (2+)$? Use a lei de fusão-cata para justificar, por cálculo, a sua resposta.

Resolução 1

$$\text{length} = ([\text{zero}, \text{succ} \cdot \pi_2])$$

$$\begin{aligned}
 f \cdot \text{length} &= ([\text{zero}, (2+) \cdot \pi_2]) \\
 &\equiv (\text{Def. length, Def. zero } (2\times)) \\
 f \cdot ([\underline{0}, \text{succ} \cdot \pi_2]) &= ([\underline{0}, (2+) \cdot \pi_2]) \\
 &\Leftarrow (49: \text{Fusão-cata}, F_{\text{List}} f = id + id \times f) \\
 f \cdot [\underline{0}, \text{succ} \cdot \pi_2] &= [\underline{0}, (2+) \cdot \pi_2] \cdot (id + id \times f) \\
 &\equiv (20: \text{Fusão-+}, 22: \text{Absorção-+}, 1: \text{Natural-id}) \\
 [f \cdot \underline{0}, f \cdot \text{succ} \cdot \pi_2] &= [\underline{0}, (2+) \cdot \pi_2 \cdot (id \times f)] \\
 &\equiv (13: \text{Natural-}\pi_2, 27: \text{Eq-+}) \\
 &\begin{cases} f \cdot \underline{0} = \underline{0} \\ f \cdot \text{succ} \cdot \pi_2 = (2+) \cdot f \cdot \pi_2 \end{cases} \\
 &\equiv (72: \text{Ig. Ext.}, 73: \text{Def-comp}, 75: \text{Def-const}) \\
 &\begin{cases} f \ 0 = 0 \\ f (\text{succ } (\pi_2 \ (-, n))) = 2 + f (\pi_2 \ (-, n)) \end{cases} \\
 &\equiv (79: \text{Def-proj}, \text{Def. succ}) \\
 &\begin{cases} f \ 0 = 0 \\ f (n + 1) = 2 + f \ n \end{cases}
 \end{aligned}$$

Hipótese 1: $f = (2+)$

$$\begin{cases} 2 + 0 = 2 \\ 2 + (n + 1) = 2 + (2 + n) \end{cases}$$

\equiv

False

Hipótese 2: $f = (2*)$

$$\begin{cases} 2 * 0 = 2 \\ 2 * (n + 1) = 2 + (2 * n) \end{cases}$$

\equiv

True

Exercício 2

As seguintes funções mutuamente recursivas testam a paridade de um número natural:

$$\begin{cases} \text{impar } 0 = \text{FALSE} \\ \text{impar } (n + 1) = \text{par } n \end{cases} \quad \begin{cases} \text{par } 0 = \text{TRUE} \\ \text{par } (n + 1) = \text{impar } n \end{cases}$$

Assumindo o functor $\mathbf{F} \, f = id + f$, mostre que esse par de definições é equivalente ao sistema de equações

$$\begin{cases} \text{impar} \cdot \text{in} = h \cdot \mathbf{F} \, \langle \text{impar}, \text{par} \rangle \\ \text{par} \cdot \text{in} = k \cdot \mathbf{F} \, \langle \text{impar}, \text{par} \rangle \end{cases}$$

para um dado h e k (deduza-os). De seguida, recorra às leis da recursividade mútua e da troca para mostrar que

$$\text{imparpar} = \langle \text{impar}, \text{par} \rangle = \text{for swap (FALSE, TRUE)}$$

Resolução 2

$$\begin{aligned} & \begin{cases} \text{impar } 0 = \text{FALSE} \\ \text{impar } (n + 1) = \text{par } n \end{cases} \\ & \equiv (73: \text{Def-comp}, 75: \text{Def-const}, \text{Def. succ}, 72: \text{Ig. Ext.}) \\ & \begin{cases} \text{impar} \cdot \underline{0} = \underline{\text{false}} \\ \text{impar} \cdot \text{succ} = \text{par} \end{cases} \\ & \equiv (27: \text{Eq-+}, 20 \text{ Fusão-+}) \\ & \text{impar} \cdot [\underline{0}, \text{succ}] = [\underline{\text{false}}, \text{par}] \\ & \equiv (\text{Def. in}_{\mathbb{N}_0}, 7: \text{Cancelamento-}\times, 3: \text{Fusão-const}) \\ & \text{impar} \cdot \text{in}_{\mathbb{N}_0} = [\underline{\text{false}}, \pi_2 \cdot \langle \text{impar}, \text{par} \rangle] \\ & \equiv (1: \text{Natural-id}, 22: \text{Absorção-+}) \\ & \text{impar} \cdot \text{in}_{\mathbb{N}_0} = [\underline{\text{false}}, \pi_2] \cdot (id + \langle \text{impar}, \text{par} \rangle) \\ & \equiv (\mathbf{F} \, f = id + f) \\ & \text{impar} \cdot \text{in}_{\mathbb{N}_0} = [\underline{\text{false}}, \pi_2] \cdot \mathbf{F} \, \langle \text{impar}, \text{par} \rangle \\ & \Leftarrow \\ & h = [\underline{\text{false}}, \pi_2] \end{aligned}$$

$$\begin{aligned}
& \begin{cases} \text{par } 0 = \text{TRUE} \\ \text{par } (n + 1) = \text{impar } n \end{cases} \\
& \equiv \quad \quad \quad (\dots) \\
& \text{par} \cdot \text{in}_{\mathbb{N}_0} = [\underline{\text{true}}, \pi_1] \cdot \mathbf{F} \langle \text{impar}, \text{par} \rangle \\
& \Leftarrow \\
& k = [\underline{\text{true}}, \pi_1]
\end{aligned}$$

$$\begin{aligned}
& \begin{cases} \text{impar} \cdot \text{in} = h \cdot \mathbf{F} \langle \text{impar}, \text{par} \rangle \\ \text{par} \cdot \text{in} = k \cdot \mathbf{F} \langle \text{impar}, \text{par} \rangle \end{cases} \\
& \equiv \quad \quad \quad (53: \text{Fokkinga}) \\
& \langle \text{impar}, \text{par} \rangle = (\llbracket \langle h, k \rangle \rrbracket) \\
& \equiv \quad \quad \quad (\text{Def. h, Def. k}) \\
& \langle \text{impar}, \text{par} \rangle = (\llbracket \langle [\underline{\text{false}}, \pi_2], [\underline{\text{true}}, \pi_1] \rangle \rrbracket) \\
& \equiv \quad \quad \quad (28: \text{Lei da troca}) \\
& \langle \text{impar}, \text{par} \rangle = (\llbracket \langle [\underline{\text{false}}, \underline{\text{true}}], \langle \pi_2, \pi_1 \rangle \rrbracket \rrbracket) \\
& \equiv \quad \quad \quad (\text{Def. swap, } \langle \underline{a}, \underline{b} \rangle = \underline{\langle a, b \rangle}) \\
& \langle \text{impar}, \text{par} \rangle = (\llbracket [\underline{(\text{false}, \text{true})}, \text{swap}] \rrbracket) \\
& \equiv \quad \quad \quad (\text{for } b \text{ } i = (\llbracket [\underline{i}, b] \rrbracket)) \\
& \langle \text{impar}, \text{par} \rangle = \text{for swap (FALSE, TRUE)}
\end{aligned}$$

Exercício 3

A seguinte função em Haskell lista os primeiros n números naturais por ordem inversa:

$$\begin{cases} \text{insg } 0 = [] \\ \text{insg } (n + 1) = (n + 1) : \text{insg } n \end{cases}$$

Mostre que insg pode ser definida por recursividade mútua tal como se segue:

$$\begin{cases} \text{insg } 0 = [] \\ \text{insg } (n + 1) = (\text{fsuc } n) : \text{insg } n \\ \text{fsuc } 0 = 1 \\ \text{fsuc } (n + 1) = \text{fsuc } n + 1 \end{cases}$$

A seguir, usando a lei de recursividade mútua, derive:

$$\begin{aligned} \text{insg} &= \pi_2 \cdot \text{insgfor} \\ \text{insgfor} &= \text{for } \langle (1+) \cdot \pi_1, \text{cons} \rangle (1, []) \end{aligned}$$

Resolução 3

$$\begin{aligned} &\begin{cases} \text{insg } 0 = [] \\ \text{insg } (n + 1) = (n + 1) : \text{insg } n \end{cases} \\ &\equiv (73, 75, \text{Def. succ}, \text{Def. cons}, \text{Def. nil}, 77, 72) \\ &\begin{cases} \text{insg} \cdot \underline{0} = \text{nil} \\ \text{insg} \cdot \text{succ} = \text{cons} \cdot \langle \text{succ}, \text{insg} \rangle \end{cases} \\ &\equiv (27: \text{Eq-+}, 20: \text{Fusão-+}) \\ &\text{insg} \cdot [\underline{0}, \text{succ}] = [\text{nil}, \text{cons} \cdot \langle \text{succ}, \text{insg} \rangle] \\ &\equiv (\text{Def. in}_{\mathbb{N}_0}, 1: \text{Natural-id}, 22: \text{Absorção-+}) \\ &\text{insg} \cdot \text{in}_{\mathbb{N}_0} = [\text{nil}, \text{cons}] \cdot (\text{id} + \langle \text{succ}, \text{insg} \rangle) \\ &\equiv (\mathbf{F} \ f = \text{id} + f) \\ &\text{insg} \cdot \text{in}_{\mathbb{N}_0} = [\text{nil}, \text{cons}] \cdot \mathbf{F} \ \langle \text{succ}, \text{insg} \rangle \\ &\Leftarrow \\ &k = [\text{nil}, \text{cons}] \end{aligned}$$

$$\begin{aligned}
& \begin{cases} f_{suc} 0 = 1 \\ f_{suc} (n + 1) = f_{suc} n + 1 \end{cases} \\
& \equiv \quad \quad \quad (\\dots) \\
& f_{suc} \cdot \text{in}_{\mathbb{N}_0} = [\underline{1}, \text{succ} \cdot \pi_1] \cdot \mathbf{F} \langle \text{succ}, \text{insg} \rangle \\
& \Leftarrow \\
& h = [\underline{1}, \text{succ} \cdot \pi_1]
\end{aligned}$$

$$\begin{aligned}
& \begin{cases} f_{suc} \cdot \text{in} = h \cdot \mathbf{F} \langle \text{succ}, \text{insg} \rangle \\ \text{insg} \cdot \text{in} = k \cdot \mathbf{F} \langle \text{succ}, \text{insg} \rangle \end{cases} \\
& \equiv \quad \quad \quad (53: \text{Fokkinga}) \\
& \langle \text{insg}, f_{suc} \rangle = (\langle h, k \rangle) \\
& \equiv \quad \quad \quad (\text{Def. k, Def. h}) \\
& \langle \text{insg}, f_{suc} \rangle = (\langle [\underline{1}, \text{succ} \cdot \pi_1], [\text{nil}, \text{cons}] \rangle) \\
& \equiv \quad \quad \quad (28: \text{Lei da troca, Def. nil}) \\
& \langle \text{insg}, f_{suc} \rangle = (\langle [\underline{1}, \underline{\quad}], \langle \text{succ} \cdot \pi_1, \text{cons} \rangle \rangle) \\
& \equiv \quad \quad \quad (\langle \underline{a}, \underline{b} \rangle = \underline{\langle a, b \rangle}, \text{Def. for}) \\
& \langle \text{insg}, f_{suc} \rangle = \text{for} \langle \text{succ} \cdot \pi_1, \text{cons} \rangle (1, [\quad]) \\
& \equiv \quad \quad \quad (\text{Def. succ}) \\
& \langle \text{insg}, f_{suc} \rangle = \text{for} \langle (1+) \cdot \pi_1, \text{cons} \rangle (1, [\quad])
\end{aligned}$$

Exercício 4

Considere o par de funções mutuamente recursivas

$$\begin{cases} f_1 [] = [] \\ f_1 (h : t) = h : (f_2 t) \end{cases} \quad \begin{cases} f_2 [] = [] \\ f_2 (h : t) = f_1 t \end{cases}$$

Mostre por recursividade mútua que $\langle f_1, f_2 \rangle$ é um catamorfismo de listas (onde $F f = id + id \times f$) e desenhe o respectivo diagrama. Que faz cada uma destas funções f_1 e f_2 ?

Resolução 4

TODO

Exercício 5

Sejam dados os funtores elementares seguintes:

$$\begin{cases} \mathbf{F} X = \mathbb{Z} \\ \mathbf{F} f = id \end{cases} \quad \begin{cases} \mathbf{G} X = X \\ \mathbf{G} f = f \end{cases}$$

Mostre que \mathbf{H} e \mathbf{K} definidos por

$$\mathbf{H} X = \mathbf{F} X + \mathbf{G} X$$

$$\mathbf{K} X = \mathbf{G} X \times \mathbf{F} X$$

são funtores.

Resolução 5

$$\begin{aligned} & \begin{cases} \mathbf{H} X = \mathbf{F} X + \mathbf{G} X \\ \mathbf{H} f = \mathbf{F} f + \mathbf{G} f \end{cases} \\ & \equiv \quad \quad \quad (\text{Def. } \mathbf{F} \text{ e } \mathbf{G}) \\ & \begin{cases} \mathbf{H} X = \mathbb{Z} + X \\ \mathbf{H} f = id + f \end{cases} \end{aligned}$$

Para \mathbf{H} ser functor, tem de se verificar: $\begin{cases} \mathbf{H} (g \cdot f) = \mathbf{H} g \cdot \mathbf{H} f \\ \mathbf{H} id = id \end{cases}$

$$\begin{aligned} (\mathbf{H} g) \cdot (\mathbf{H} h) &= (id + g) \cdot (id + h) \quad (\text{Def. } \mathbf{H} f = id + f) \\ &= id + g \cdot h \quad (25: \text{Functor-}, 1: \text{Natural-id}) \\ &= \mathbf{H} (g \cdot h) \end{aligned}$$

$$\begin{aligned} \mathbf{H} id &= id + id & (\text{Def. } \mathbf{H} f = id + f) \\ &= id & (1: \text{Natural-id}) \end{aligned}$$

$$\begin{aligned}
& \begin{cases} \mathbf{K} X = \mathbf{G} X \times \mathbf{F} X \\ \mathbf{K} f = \mathbf{G} f \times \mathbf{F} f \end{cases} \\
& \equiv \quad \quad \quad (\text{Def. } \mathbf{F} \text{ e } \mathbf{G}) \\
& \begin{cases} \mathbf{K} X = X \times \mathbb{Z} \\ \mathbf{K} f = f \times id \end{cases}
\end{aligned}$$

Para \mathbf{K} ser functor, tem de se verificar: $\begin{cases} \mathbf{K} (g \cdot f) = \mathbf{K} g \cdot \mathbf{K} f \\ \mathbf{K} id = id \end{cases}$

$$\begin{aligned}
(\mathbf{K} g) \cdot (\mathbf{K} h) &= (g \times id) \cdot (h \times id) && (\text{Def. } \mathbf{K} f = f \times id) \\
&= g \cdot h \times id && (25: \text{Functor-}\times, 1: \text{Natural-id}) \\
&= \mathbf{K} (g \cdot h)
\end{aligned}$$

$$\begin{aligned}
\mathbf{K} id &= id \times id && (\text{Def. } \mathbf{K} f = f \times id) \\
&= id && (1: \text{Natural-id})
\end{aligned}$$

Exercício 6

Mostre que, sempre que \mathbf{F} e \mathbf{G} são funtores, então a sua composição $\mathbf{H} = \mathbf{F} \cdot \mathbf{G}$ é também um functor.

Resolução 6

Se \mathbf{F} e \mathbf{G} são funtores, então:

$$\begin{cases} \mathbf{F} (g \cdot f) = \mathbf{F} g \cdot \mathbf{F} f \\ \mathbf{F} id = id \end{cases} \quad \text{e} \quad \begin{cases} \mathbf{G} (g \cdot f) = \mathbf{G} g \cdot \mathbf{G} f \\ \mathbf{G} id = id \end{cases}$$

Para $\mathbf{H} = \mathbf{F} \cdot \mathbf{G}$ ser functor, tem de se verificar:

$$\begin{cases} \mathbf{H} (g \cdot f) = \mathbf{H} g \cdot \mathbf{H} f \\ \mathbf{H} id = id \end{cases}$$

$$\begin{aligned} \mathbf{H} (g \cdot h) &= \mathbf{F} \cdot \mathbf{G} (g \cdot h) && (\text{Def. } \mathbf{H} = \mathbf{F} \cdot \mathbf{G}) \\ &= \mathbf{F} (\mathbf{G} g \cdot \mathbf{G} h) && (\mathbf{G} (g \cdot f) = \mathbf{G} g \cdot \mathbf{G} f) \\ &= (\mathbf{F} \cdot \mathbf{G} g) \cdot (\mathbf{F} \cdot \mathbf{G} h) && (\mathbf{F} (g \cdot f) = \mathbf{F} g \cdot \mathbf{F} f) \\ &= (\mathbf{H} g) \cdot (\mathbf{H} h) && (\text{Def. } \mathbf{H} = \mathbf{F} \cdot \mathbf{G}) \end{aligned}$$

$$\begin{aligned} \mathbf{H} id &= (\mathbf{F} \cdot \mathbf{G}) id && (\text{Def. } \mathbf{H} = \mathbf{F} \cdot \mathbf{G}) \\ &= \mathbf{F} (\mathbf{G} id) && (\text{Def. comp}) \\ &= \mathbf{F} id && (\mathbf{G} id = id) \\ &= id && (\mathbf{F} id = id) \end{aligned}$$

Exercício 7

Questão prática

Problem definition: Page [UNZIP IN ONE PASS?](#) of Stack Overflow addresses the question as to whether

$$\text{unzip } xs = (\text{map } \pi_1 \ xs, \text{map } \pi_2 \ xs)$$

can do one traversal only. The answer is affirmative:

$$\text{unzip } [] = ([], [])$$

$$\text{unzip } ((a, b) : xs) = (a : as, b : bs) \textbf{ where } (as, bs) = \text{unzip } xs$$

What is missing from Stack Overflow is the explanation of how the two steps of `unzip` merge into one. Show that the banana-split law is what needs to be known for the one traversal version to be derived from the two traversal one.

Resolução 7

TODO

