Modelling the transition from homogeneous to columnar states in locust hopper bands

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Abstract

Many biological systems form structured swarms, for instance in locusts, whose swarms are known as hopper bands. There is growing interest in applying mathematical models to understand the dynamics of these biological and social systems. We model the locusts of a hopper band as point particles interacting through repulsive and attractive social forces on a one dimensional periodic domain. We assume that repulsion is local and attraction is global, such that only an individual's nearest neighbors interact repulsively and everyone interacts attractively. This is a biologically inspired assumption because repulsion in swarms is mainly a collision avoidance mechanism. From a discrete model of pairwise additive social forces, we derive an analogous continuum model to examine the linear stability of the constant mass density of the swarm using variational calculus. We expect from locust behavior that there is a transition from constant density to clumped density that depends on the total mass of the swarm. We show that with local repulsion and global attraction there is a stability boundary of constant density that depends on the average density of the swarm.

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Chapter 1

Introduction

A swarm is a group of biological individuals that have emergent collective behaviors usually using simple interaction rules. There are many evolutionary benefits to sociality, for example in locust hopper bands. Locusts organize themselves into planar fronts when food is abundant and into foraging columns when food is sparse to make consumption of resources most efficient. We are interested in the transition from homogeneous states to columnar states in these locusts; in other words, states in which the locusts are evenly spaced to states in which some are clumped together. This transition corresponds to the emergence of foraging columns in locust hopper bands.

In the case of collective motion, the interactions between individuals in the swarm can be modelled as "social forces" of attraction and repulsion. Because swarming individuals tend to have a strongly aligned direction of motion, we can reduce the problem to one dimension in which we consider the particles interacting perpendicular to the average direction of motion of the swarm. We treat social forces that are pairwise and additive such that the velocity of a particle is proportional to the sum of its interaction force with every other particle. Because locusts are large and experience strong drag forces relative to their weight, the first order model is justifiable.

Often, edge effects in swarms on an infinite line are ignored and we will do the same. To appropriately account for this, we assume the individuals in our model lie on a periodic boundary such that, as the swarm size becomes large, we can approximate well the behavior of swarms on an infinite line without edge effects. Like in physics, we use an energy formulation of this problem to evaluate the stability of the evenly spaced configuration of particles. Because of the periodic boundary conditions, we are able to find

an analytic expression for the eigenvalues determining stability. Previous work detailed by Barth (2019), works through a stability analysis of this model on periodic domain. Quentin shows where the model transitions from even spacing to clumped spacing between particles.

The discrete formulation of our swarming model has a continuum analogue that can be found by considering the limit of infinite number of individuals. As the number of individuals approaches infinity, an evenly spaced configuration of particles becomes a constant density solution. We use a framework developed by Bernoff and Topaz (2011) for transitioning from discrete to continuum formulations of the model and conduct a stability analysis on the resulting energy functional using calculus of variations. However, in biology, repulsion functions as a collision avoidance mechanism. For this reason, we later adapt our model such that only an individual's nearest neighbors interact repulsively while all particles interact attractively. With a continuum model of nearest neighbor repulsion, find test the stability as a function of total mass. Burger et al. (2014) investigate the properties of a continuum swarming model with local repulsion and so our analysis is inspired by and compared to their own.

We analyze the stability of various local repulsion energy functions such as a Morse potential and power-law with general exponent. Previous work on continuum swarming models shows that, with global repulsion, the stability of the constant density solution is independent of total mass. There are empirical reasons to believe that this is nonbiological. As the total mass of the swarm increases, the support of the swarm should increase to keep the density constant and when a density threshold is crossed the system should become unstable. In locusts, higher densities are associated with the formation of columnar structures that we are modelling as clumping.

Chapter 2

Discrete Formulation

2.1 Equations of motion

Consider N point particles interacting via a "social force" of attraction and repulsion, $q(r)$, in one dimension. The position of each particle, x_i, changes according to a first order equation of motion proportional to the sum of all the interaction forces,

$$\frac{dx_i}{dt} = \sum_{\substack{j=1 \\ j \neq i}}^{N} m q(x_i - x_j), \quad m = \frac{M}{N}, \tag{2.1}$$

where M is the total mass of all the particles in the system and m is a "social mass". The social mass ensures that the strength of the superposition of forces remains bounded as $N \to \infty$. If the force, $q(r)$, is an integralable function then we can rewrite the equations of motion as the negative gradient of a pairwise potential energy of each interaction. The potential energy is given by,

$$Q(r) = -\int^{r} q(r)\, dr. \tag{2.2}$$

Under this assumption, the equations of motion can be rewritten as,

$$\frac{dx_i}{dt} = -m \sum_{\substack{j=1 \\ j \neq i}}^{N} \frac{\partial Q}{\partial x_i}(x_i - x_j). \tag{2.3}$$

2.2 Energy in periodic domain

We consider a class of potential called the Morse potential which is characterized by short ranged repulsion and long ranged attraction while being asymptotic to zero. These characteristics make it such that two particles interacting flow to the minimum of the potential at intermediate values of r. One such function is,

$$Q(r_{ij}) = e^{-|r_{ij}|} - FAe^{-|r_{ij}|/A}, \tag{2.4}$$

where $r_{ij} = x_i - x_j$, F is a parameter of the strength of attraction, and A is a parameter for its range. The function has two terms: the first is the repulsive contribution, $Q_{rep}(r_{ij}) = e^{-|r_{ij}|}$ and the second is the attractive contribution, $Q_{att}(r_{ij}) = -FAe^{-|r_{ij}|/A}$.

In a previous thesis by Barth (2019), this interaction potential was periodicitized to the interval between $[0, L]$. His result is that the potential between any two individuals i and j is

$$\begin{aligned}\tilde{Q}(r_{ij}) &= \sum_{n=-\infty}^{\infty} Q(x_i - (x_j + nL)) \\ &= \frac{\cosh\left(L/2 - \mid r_{ij} \mid\right)}{\sinh(L/2)} - FA\frac{\cosh\left(\frac{L/2-|r_{ij}|}{A}\right)}{\sinh\left(\frac{L}{2A}\right)} \qquad r_{ij} \in [0, L],\end{aligned} \tag{2.5}$$

such that i is effected by an infinite amount of mirrors of particle j outside the boundary. This allows us to simulate large swarms while ignoring edge effects. With a substitution of $r = r - L\lfloor r/L \rfloor$, the function $\tilde{Q}(r)$ is fully periodic for use in numerical simulation. Note that $\tilde{Q}(r)$ is even such that $\tilde{Q}(r) = \tilde{Q}(-r)$ and periodic such that $\tilde{Q}(r + L) = \tilde{Q}(r)$.

The total energy of the system is the superposition of all pairwise interactions excluding self-interaction,

$$E(\vec{x}) = \frac{1}{2}\sum_{i=1}^{N}\sum_{\substack{j=1 \\ j\neq i}}^{N} m^2\tilde{Q}(x_i - x_j) \tag{2.6}$$

$$= \frac{m^2}{2}\sum_{i=1}^{N}\sum_{\substack{j=1 \\ j\neq i}}^{N} \frac{\cosh\left(L/2 - \mid r_{ij} \mid\right)}{\sinh(L/2)} - FA\frac{\cosh\left(\frac{L/2-|r_{ij}|}{A}\right)}{\sinh\left(\frac{L}{2A}\right)}. \tag{2.7}$$

2.3 Lattice stability

In the case of an evenly spaced equilibrium configuration, we know the position of every particle. We can define the spacing between particles to be $\Delta = \frac{L}{N}$, such that agents have equilibrium position $x_i^* = i\Delta$ for $i \in \{1, 2, 3, ..., N\}$. We obtain an expression for the Taylor expansion of the total energy function about $\vec{x}^*$ such that $\vec{x} = \vec{x}^* + \vec{\gamma}$ where $\vec{\gamma}$ is a vector of perturbations to each of the agent's position. The Taylor expansion is,

$$E(\vec{x}) \approx E(\vec{x}^*) + \vec{\gamma}^T \nabla E(\vec{x}^*) + \frac{1}{2}\vec{\gamma}^T \mathbf{H} \vec{\gamma} \tag{2.8}$$

$$\approx E(\vec{x}^*) + \sum_{j=1}^{N} \frac{\partial E(\vec{x}^*)}{\partial x_j}\gamma_j + \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} \frac{\partial^2 E(\vec{x}^*)}{\partial x_j \partial x_i}\gamma_j \gamma_i, \tag{2.9}$$

where the Hessian matrix is defined by,

$$\mathbf{H}_{ij} \equiv \frac{\partial^2 E(\vec{x}^*)}{\partial x_j \partial x_i}.$$

By the second partial derivative test, the eigenvalues of the Hessian matrix evaluated at equilibrium determine its linear stability. If the eigenvalues are positive definite, the equilibrium is stable and it is unstable otherwise. To gain some intuition, note that by the definition of an equilibrium point, $\frac{\partial E}{\partial x_j} = 0 \; \forall j \in \{1, 2, 3, ..., N\}$ and we can rearrange Equation 2.9 to find an expression for the change in energy caused by the perturbation,

$$E(\vec{x}) - E(\vec{x}^*) \approx \frac{1}{2}\vec{\gamma}^T \mathbf{H} \vec{\gamma}. \tag{2.10}$$

Now suppose the perturbation is the kth eigenvector $\vec{v}^{(k)}$ of $\mathbf{H}$ with corresponding eigenvalue $\lambda^{(k)}$, then the change in energy of the system is

$$E(\vec{x}) - E(\vec{x}^*) \approx \frac{1}{2}\lambda^{(k)} |\vec{v}^{(k)}|^2. \tag{2.11}$$

If the set of eigenvalues is positive definite, then the change in energy following any arbitrary perturbation is positive which suggests the equilibrium is a local minimum. This is because any arbitrary vector can be expressed in terms of the eigenbasis of the Hessian. If there is even one negative eigenvalue then there exists a perturbation that can lower the energy further by moving away from equilibrium suggesting that the equilibrium is unstable.

2.4 Analysis of Hessian

The Hessian is a matrix of second derivatives of our total energy function. We will begin by computing its first derivative,

$$\begin{aligned}
\frac{\partial E}{\partial x_k} &= \frac{\partial}{\partial x_k}\frac{1}{2}\sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N} m\tilde{Q}(x_i - x_j) \\
&= \frac{m}{2}\sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N}\frac{\partial}{\partial x_k}\tilde{Q}(x_i - x_j) \\
&= \frac{m}{2}\sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N}\left[\frac{\partial\tilde{Q}}{\partial x_k}\delta_{ik} - \frac{\partial\tilde{Q}}{\partial x_k}\delta_{jk}\right] \\
&= \frac{m}{2}\left[\sum_{\substack{j=1\\ j\neq k}}^{N}\frac{\partial\tilde{Q}(x_k - x_j)}{\partial x_k} - \sum_{\substack{i=1\\ i\neq k}}^{N}\frac{\partial\tilde{Q}(x_i - x_k)}{\partial x_k}\right]
\end{aligned}$$

Because the summations are independent of one another, we can use the property that $\tilde{Q}'(r)$ is an odd function so $-\tilde{Q}'(-r) = \tilde{Q}(r)$ and reindex the problem such that $i \to j$. Then,

$$\begin{aligned}
\frac{\partial E}{\partial x_k} &= \frac{m}{2}\left[\sum_{\substack{j=1\\ j\neq k}}^{N}\frac{\partial\tilde{Q}(x_k - x_j)}{\partial x_k} + \sum_{\substack{j=1\\ j\neq k}}^{N}\frac{\partial\tilde{Q}(x_k - x_j)}{\partial x_k}\right] \\
&= m\sum_{\substack{j=1\\ j\neq k}}^{N}\frac{\partial\tilde{Q}(x_k - x_j)}{\partial x_k} \qquad (2.12)
\end{aligned}$$

Now, we can compute the Hessian as follows,

$$\begin{aligned}
\mathbf{H}_{ik} &= \frac{\partial}{\partial x_i}\frac{\partial E}{\partial x_k} \\
&= m\frac{\partial}{\partial x_i}\sum_{\substack{j=1\\ j\neq k}}^{N}\tilde{Q}'(x_k - x_j) \\
&= m\sum_{\substack{j=1\\ j\neq k}}^{N}\tilde{Q}''(x_k - x_j)\frac{\partial}{\partial x_i}\big[x_k - x_j\big] \\
&= m\sum_{\substack{j=1\\ j\neq k}}^{N}\tilde{Q}''(x_k - x_j)\big[\delta_{ki} - \delta_{ji}\big] \\
&= m\left[\delta_{ki}\sum_{\substack{j=1\\ j\neq k}}^{N}\tilde{Q}''(x_k - x_j)\right] - m\tilde{Q}''(x_k - x_i).
\end{aligned}$$

This can be rewritten such that one term is a constant that defines the values of the matrix along the diagonal and the other is matrix of off-diagonal terms. Evaluated at equilibrium with positions $x_i^* = i\Delta$, the Hessian becomes,

$$\mathbf{H}_{ij} = m\delta_{ij}\sum_{\substack{k=1\\ k\neq j}}^{N}\tilde{Q}''\Big(\Delta(j-k)\Big) - m(1-\delta_{ij})\tilde{Q}''\Big(\Delta(j-i)\Big). \tag{2.13}$$

So, we define

$$P \equiv \sum_{k=1}^{N-1}\tilde{Q}''\Big(k\Delta\Big), \qquad S_{ij} \equiv -\tilde{Q}''\Big(\Delta(i-j)\Big)$$

such that,

$$\mathbf{H}_{ij} = \delta_{ij}P + (1-\delta_{ij})S_{ij}.$$

The Hessian is symmetric by the symmetry of second derivatives and its circulant. The diagonal matrix P is trivially circulant and S_{ij} is circulant because $\tilde{Q}''(r)$ is an even and periodic function. The advantage of working with a circulant matrix is that the eigenvalues can be found directly regardless of eigenvectors.

2.5 On circulant matrices

A circulant matrix is defined as a square $N \times N$ matrix that is fully specified by the first row vector, $\vec{h}$, where each subsequent row is a cyclic permutation of this vector. Colloquially, this means that every row is the same except every element is shifted to the right by one. Consider a circulant matrix $\mathbf{H}$,

$$\mathbf{H} = \begin{bmatrix} h_1 & h_2 & h_3 & \cdots & h_N \\ h_N & h_1 & h_2 & \cdots & h_{N-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h_2 & h_3 & h_4 & \cdots & h_1 \end{bmatrix}.$$

The first eigenvector and eigenvalue pair, $\vec{v}^{(1)}$ and $\lambda^{(1)}$, are attainable by inspection. Suppose $\vec{v}^{(1)}$ is an $N \times 1$ vector of one's such that,

$$\mathbf{H}\vec{v}^{(1)} = \begin{bmatrix} h_1 & h_2 & h_3 & \cdots & h_N \\ h_N & h_1 & h_2 & \cdots & h_{N-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h_2 & h_3 & h_4 & \cdots & h_1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = (h_1 + \cdots + h_N)\vec{v}^{(1)} = \lambda^{(1)}\vec{v}^{(1)}. \quad (2.14)$$

So, we know $\lambda^{(1)} = (h_1 + h_2 + \cdots + h_N)$. All the eigenvectors are expressible in terms of roots of unity. The N^{th} root of unity is defined by $\omega_N = e^{\frac{2\pi i}{N}}$. The exponents of ω_N are periodic such that $\omega_N^{j+N} = \omega_N^j$. The eigenvectors are

$$\vec{v}^{(k)} = \begin{bmatrix} \omega_N^k \\ \omega_N^{2k} \\ \omega_N^{3k} \\ \vdots \\ \omega_N^{Nk} \end{bmatrix}.$$

Now we will derive an expression for the eigenvalues of a general circulant matrix. Consider a set of vectors $\vec{y}^{(k)}$ such that,

$$\vec{y}^{(k)} = \mathbf{H}\vec{v}^{(k)} = \begin{bmatrix} h_1 & h_2 & h_3 & \cdots & h_N \\ h_N & h_1 & h_2 & \cdots & h_{N-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h_2 & h_3 & h_4 & \cdots & h_1 \end{bmatrix} \begin{bmatrix} \omega_N^k \\ \omega_N^{2k} \\ \vdots \\ \omega_N^{Nk} \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^N h_j \omega_N^{jk} \\ \sum_{j=1}^N h_{(j-1)} \omega_N^{jk} \\ \sum_{j=1}^N h_{(j-2)} \omega_N^{jk} \\ \vdots \\ \sum_{j=1}^N h_{(j-N+1)} \omega_N^{jk} \end{bmatrix}.$$

To make things clearer, we will work with the ℓ^{th} element of the vector $\vec{y}^{(k)}$ which is given by,

$$\begin{aligned}\vec{y}_\ell^{(k)} &= \sum_{j=1}^{N} h_{(j-\ell)}\omega_N^{jk} = \sum_{j=1}^{N} h_{(j-\ell)}\left(\omega_N^{\ell k}\cdot\omega_N^{(j-\ell)k}\right)\\ &= \omega_N^{\ell k}\left(\sum_{j=1}^{N} h_{(j-\ell)}\omega_N^{(j-\ell)k}\right).\end{aligned}$$

Because the indices are periodic and the sum independent of ℓ, we can re-index ℓ out of the sum $(j-\ell)\rightarrow j$ and the expression becomes,

$$\vec{y}_\ell^{(k)} = \omega_N^{\ell k}\left(\sum_{j=1}^{N} h_j\omega_N^{jk}\right) = \vec{v}_\ell^{(k)}\left(\sum_{j=1}^{N} h_j\omega_N^{jk}\right) = \vec{v}_\ell^{(k)}\lambda^{(k)}. \tag{2.15}$$

Finally we see that,

$$\lambda^{(k)} = \sum_{j=1}^{N} h_j\omega_N^{jk}. \tag{2.16}$$

This result is significant because the eigenvalues are computable using the Discrete Fourier Transform (DFT) matrix,

$$\mathbf{F} = \left[\vec{v}^{(1)}, \vec{v}^{(2)}, \vec{v}^{(3)}, \cdots, \vec{v}^{(N)}\right].$$

Multiplying a vector of first row elements, $\vec{h}$, by the DFT matrix we obtain a vector of eigenvalue elements such that,

$$\mathbf{F}\vec{h} = \begin{bmatrix} \omega_N^{11} & \omega_N^{12} & \omega_N^{13} & \cdots & \omega_N^{1N}\\ \omega_N^{21} & \omega_N^{22} & \omega_N^{23} & \cdots & \omega_N^{2N}\\ \omega_N^{31} & \omega_N^{32} & \omega_N^{33} & \cdots & \omega_N^{3N}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ \omega_N^{N1} & \omega_N^{N2} & \omega_N^{N3} & \cdots & \omega_N^{NN}\end{bmatrix}\begin{bmatrix} h_1\\ h_2\\ h_3\\ \vdots\\ h_N\end{bmatrix} = \begin{bmatrix}\lambda^{(1)}\\ \lambda^{(2)}\\ \lambda^{(3)}\\ \vdots\\ \lambda^{(N)}\end{bmatrix} = \vec{\lambda}.$$

Much of this theory can be found in the work Karner et al. (2003).

2.6 Eigenvalues of Hessian matrix

Now knowing how to find the eigenvalues of a circulant matrix, we come back to the Hessian defined in Equation 2.13. To make things clear, the

Hessian is

$$\mathbf{H} = \begin{bmatrix} P & -\tilde{Q}''\Big(-\Delta\Big) & -\tilde{Q}''\Big(-2\Delta\Big) & \cdots & -\tilde{Q}''\Big((1-N)\Delta\Big) \\ -\tilde{Q}''\Big(\Delta\Big) & P & -\tilde{Q}''\Big(-\Delta\Big) & \cdots & -\tilde{Q}''\Big((2-N)\Delta\Big) \\ -\tilde{Q}''\Big(2\Delta\Big) & -\tilde{Q}''\Big(\Delta\Big) & P & \cdots & -\tilde{Q}''\Big((3-N)\Delta\Big) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\tilde{Q}''\Big((N-1)\Delta\Big) & -\tilde{Q}''\Big((N-2)\Delta\Big) & -\tilde{Q}''\Big((N-3)\Delta\Big) & \cdots & P \end{bmatrix}.$$

Because $\tilde{Q}''(r)$ is an even L-periodic function and the particles are evenly distributed throughout the period at equilibrium, we know that $\tilde{Q}''((j-N)\Delta) = \tilde{Q}''(j\Delta) = \tilde{Q}''(-j\Delta)$ making it a circulant matrix. Now, we can use the DFT and the first row of our Hessian to find its eigenvalues. To make the eigenvalue calculation simpler, we must reindex the exponents of the roots of unity. Because they are N-periodic, we can rewrite the DFT matrix such that,

$$\mathbf{F} = \begin{bmatrix} \omega_N^{00} & \omega_N^{01} & \omega_N^{02} & \cdots & \omega_N^{0(N-1)} \\ \omega_N^{10} & \omega_N^{11} & \omega_N^{12} & \cdots & \omega_N^{1(N-1)} \\ \omega_N^{20} & \omega_N^{21} & \omega_N^{22} & \cdots & \omega_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \omega_N^{(N-1)0} & \omega_N^{(N-1)1} & \omega_N^{(N-1)2} & \cdots & \omega_N^{(N-1)(N-1)} \end{bmatrix}.$$

Now, multiplying $\mathbf{F}$ by the first row $\vec{h}$ of $\mathbf{H}$, the k^{th} eigenvalue is

$$\begin{aligned} \lambda^{(k)} &= P\omega_N^{0k} - \tilde{Q}''(-\Delta)\omega_N^{1k} - \cdots - \tilde{Q}''((2-N)\Delta)\omega_N^{(N-2)k} - \tilde{Q}''((1-N)\Delta)\omega_N^{(N-1)k} \\ &= P\omega_N^{0k} - \tilde{Q}''(\Delta)\omega_N^{1k} - \cdots - \tilde{Q}''(2\Delta)\omega_N^{(N-2)k} - \tilde{Q}''(\Delta)\omega_N^{(N-1)k}. \end{aligned}$$

As stated in Section 2.5, the exponents of ω_N are N-periodic such that $\omega_N^{jk} = \omega_N^{(N-j)k}$ and we can combine many of the terms of $\lambda^{(k)}$. For example, the jth term after P and the jth term from the end can be reduced as follows:

$$\begin{aligned} -\tilde{Q}''(j\Delta)\omega_N^{jk} - \tilde{Q}''((j-N)\Delta)\omega_N^{(N-j)k} &= -\tilde{Q}''(j\Delta)\Big[\omega_N^{jk} + \omega_N^{(N-j)k}\Big] \\ &= -\tilde{Q}''(j\Delta)\Big[e^{\frac{2\pi ijk}{N}} + e^{\frac{2\pi i(N-j)k}{N}}\Big] \\ &= -\tilde{Q}''(j\Delta)\Big[e^{\frac{2\pi ijk}{N}} + e^{-\frac{2\pi ijk}{N}}\Big] \\ &= -\tilde{Q}''(j\Delta)\Big[2\cos\Big(\frac{2\pi jk}{N}\Big)\Big] \end{aligned}$$

As long as N is an odd integer, there will be an even number of terms to pair and every term after P will be simplified in this way. For this reason, we assume from here onward that N is an odd integer to obtain a concise expression of eigenvalues that can be numerically evaluated. The resulting expression assuming an odd integer N is,

$$\begin{aligned}\lambda^{(k)} &= P - 2\tilde{Q}''(\Delta)\cos\left(\frac{2\pi k}{N}\right) - \cdots - 2\tilde{Q}''\left(\frac{N-1}{2}\Delta\right)\cos\left(\frac{2\pi\frac{N-1}{2}k}{N}\right) \\ &= P - 2\sum_{\ell=1}^{(N-1)/2}\tilde{Q}''(\ell\Delta)\cos\left(\frac{2\pi\ell k}{N}\right) \end{aligned} \tag{2.17}$$

Now we re-index the problem such that, $N = 2M + 1$ and instead the agents' positions are $x_i^* = i\Delta$ where $i \in \{-M, -M+1, .., M\}$. Now the eigenvalues for an odd number of individuals are given by,

$$\lambda^{(k)} = P - 2\sum_{\ell=1}^{M}\tilde{Q}''(\ell\Delta)\cos\left(\frac{2\pi\ell k}{N}\right) \quad k \in \{-M, -M+1, ..., M\}. \tag{2.18}$$

2.7 Symmetries of the system

A symmetry of a system is defined by a transformations that leaves the action of the system unchanged. An action in physics is a functional that is the time integral of the Lagrangian of the system. Given a mechanical system with N generalized coordinates, x_i for $i \in \{1, 2, 3, ..., N\}$ or $\mathbf{x}$, the Lagrangian and the corresponding action respectively are:

$$\begin{aligned} L &= T - U \\ S[\mathbf{x}(t)] &= \int_{t_a}^{t_b} L(t, \mathbf{x}, \dot{\mathbf{x}})dt \end{aligned}$$

where T is the total kinetic energy of the particles and U is the total potential energy due to the system's social forces. In classical mechanics, the dynamics of the system is governed by the function $\mathbf{x}(t)$ that makes the action stationary. Consider a transformation to each coordinate such that $x_i' = x_i + C$. Kinetic energy depends on $\dot{\mathbf{x}}$ and the potential energy depends on the distance between particles $|x_i - x_j|$, both of which are invariant under the transformation. Because the Lagrangian is unchanged by the transformation so is the action and our system of N pairwise interacting

individuals is transitionally invariant. As a consequence, if the system is shifted along the real line by a constant, the energy of the system remains unchanged as well. As was demonstrated in Section 2.5, a vector $\vec{v}^{(1)}$ of one's is always an eigenvector of a circulant matrix. By Equation 2.11, the symmetry means that the corresponding eigenvalue $\lambda^{(1)}$ is equal to zero,

$$\lambda^{(1)} = h_1 + h_2 + \cdots + h_N = P - 2\sum_{\ell=1}^{M} \tilde{Q}''(\ell\Delta) = 0.$$

This result confirms our original definition of P in Section 2.4 and gives us a constraint on our numerical results. Every row and column of our Hessian matrix individually sums to zero. The expression for the k^{th} eigenvalue becomes,

$$\lambda^{(k)} = 2\sum_{\ell=1}^{M} \tilde{Q}''(\ell\Delta)\Big(1 - \cos\Big(\frac{2\pi}{N}\ell k\Big)\Big). \tag{2.19}$$

2.8 Final eigenvalues

By calculating the second derivative of the pairwise energy function $\tilde{Q}(r)$, we can finalize our derivation of the eigenvalues of the system in terms of its parameters.

$$\tilde{Q}''(r) = \frac{\cosh\left(L/2 - \mid r \mid\right)}{\sinh(L/2)} - \frac{F}{A}\frac{\cosh\left(\frac{L/2-|r|}{A}\right)}{\sinh\left(\frac{L}{2A}\right)} \quad r \in [0, L] \tag{2.20}$$

Now, insert Equation 2.20 into Equation 2.19 to obtain,

$$\lambda^{(k)} = 2\sum_{\ell=1}^{M}\left[\frac{\cosh\left(L/2 - \ell\Delta\right)}{\sinh(L/2)} - \frac{F}{A}\frac{\cosh\left(\frac{L/2-\ell\Delta}{A}\right)}{\sinh\left(\frac{L}{2A}\right)}\right]\Big(1 - \cos\Big(\frac{2\pi}{N}\ell k\Big)\Big) \quad \text{for} \quad -M \le k \le M. \tag{2.21}$$

Note that $\lambda^{(0)} = 0$ and $\lambda^{(-k)} = \lambda^{(k)}$.

Chapter 3

Continuum Formulation

3.1 Local repulsion

So far, we have considered one potential, $\tilde{Q}(r)$, where the repulsive and attractive contributions apply to each pair of individuals. For biological swarms, repulsion functions as a collision avoidance mechanism and thus only acts locally. To account for this, we rewrite our discrete model in such a way that repulsion only acts on an individual's nearest neighbors and attraction acts globally. We separate $\tilde{Q}(r)$ into two potentials such that,

$$\tilde{Q}(r) = R(r) + A(r), \tag{3.1}$$

where $R(r)$ is a repulsive potential energy and $A(r)$ is an attractive potential energy. In this chapter, we relax the requirement that the pairwise interaction energy is a Morse Potential such that we can evaluate the effects of different repulsion potentials in simulation.

3.2 Discrete energy

We define the energy contribution of particle j by,

$$e_j(\vec{x}) = \frac{1}{2}\left[R\left(\frac{\Delta_j^+}{h}\right) + R\left(\frac{\Delta_j^-}{h}\right)\right] + \frac{1}{2}\sum_{i=1}^{N} mA(x_j - x_i), \tag{3.2}$$

where the constant $h = \frac{L}{N}$ is the average distance between particles, $\Delta_j^+ = (x_{j+1} - x_j)$, and $\Delta_j^- = (x_j - x_{j-1})$. The factors h in $R(r)$ and m multiplied to $A(r)$ ensure that repulsion and attraction remain bounded as N approaches

infinity. We find the total energy by summing over all individual energies such that,

$$E(\vec{x}) = \sum_{j=1}^{N} me_j(\vec{x}) = \sum_{j=1}^{N} \frac{m}{2}\left[R\left(\frac{\Delta_j^+}{h}\right) + R\left(\frac{\Delta_j^-}{h}\right)\right] + \sum_{i=1}^{N}\sum_{\substack{j=1 \\ j\neq i}}^{N} \frac{m^2}{2} A(x_i - x_j) \quad (3.3)$$

3.3 Discrete to continuum

To derive the analogue continuum model, we define a discrete mass density function,

$$\rho_d(x) = \sum_{i=1}^{N} m\delta(x - x_i)\, dx,$$

where each particle of mass m has position x_i. With $\rho_d(x)$, we find the cumulative density function to be,

$$\Psi_d(x) = \begin{cases} 0 & x < x_1 \\ m[1/2 + (i-1)] & x = x_i, i = 1, ..., N \\ im & x_i < x < x_{i+1}, i = 1, ..., N \\ M & x > x_N, \end{cases}$$

where we have used the convention that integrating up to a δ-function yields half the mass of integrating through it. To establish correspondence between discrete and continuum models, we constrain $\Psi_d(x_i) = \Psi_c(x_i)$ where $\Psi_c(x)$ is a continuous cumulative density function,

$$\Psi_c(x) = \int_0^x \rho(x)\, dx,$$

and $\rho(x)$ is the continuous density of the swarm. As $N \to \infty$ for a fixed value of M, $\Psi_d(x)$ converges to $\Psi_c(x)$. Now, we will go through the derivation of the continuum energy term by term. First consider the attractive term,

$$E^A(\vec{x}) = \sum_{i=1}^{N}\sum_{\substack{j=1 \\ j\neq i}}^{N} \frac{m^2}{2} A(x_i - x_j).$$

By the cumulative density we know,

$$m = \Psi_c(x_{i+1}) - \Psi_c(x_i) = \int_{x_i}^{x_{i+1}} \rho(x)\,dx \approx \rho(x_i)\Delta_i^+, \tag{3.4}$$

where the approximation comes from a Riemann sum. Using the approximation we can rewrite the attractive energy,

$$E^A(\vec{x}) = \frac{1}{2}\sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N} \rho(x_i)\rho(x_j)A(x_i - x_j)\Delta_i^+\Delta_j^+$$

such that,

$$\lim_{N\to\infty} E^A(\vec{x}) = W^A[\rho(x)] = \frac{1}{2}\int_0^L\int_0^L \rho(x)\rho(y)A(x-y)\,dxdy. \tag{3.5}$$

Now we will consider the repulsive energy and its continuum limit. Using Equation 3.4, the repulsive energy can be reformulated,

$$\begin{aligned} E^R(\vec{x}) &= \frac{1}{2}\sum_{j=1}^{N} m\left[R\Big(\frac{\Delta_j^+}{h}\Big) + R\Big(\frac{\Delta_j^-}{h}\Big)\right] \\ &= \frac{1}{2}\sum_{j=1}^{N} \frac{M}{N}\left[R\Big(\frac{M}{L\rho(x_j)}\Big) + R\Big(\frac{M}{L\rho(x_j)}\Big)\right] \\ &= \sum_{j=1}^{N} \frac{M}{L} R\Big(\frac{M}{L\rho(x_j)}\Big)h. \end{aligned}$$

In the limit of infinite N, h becomes an infinitesimal such that,

$$\lim_{N\to\infty} E^R(\vec{x}) = W^R[\rho(x)] = \sigma\int_0^L R\Big(\frac{\sigma}{\rho(x)}\Big)\,dx, \tag{3.6}$$

where $\sigma = \frac{M}{L}$ is the average density of the swarm. Finally, the complete continuum energy in functional form is given by,

$$W[\rho(x)] = \sigma\int_0^L R\Big(\frac{\sigma}{\rho(x)}\Big)\,dx + \frac{1}{2}\int_0^L\int_0^L \rho(x)\rho(y)A(x-y)\,dxdy. \tag{3.7}$$

3.4 Linear stability of homogeneous density

We use variational calculus to determine whether a constant density solution is a minimizer of our energy functional, $W[\rho(x)]$. Let

$$\rho(x) = \bar{\rho} + \epsilon\tilde{\rho}(x), \tag{3.8}$$

where $\bar{\rho}$ is a constant density solution of total mass M and $\epsilon\tilde{\rho}(x)$ is a zero mass perturbation. We can expand the functional such that,

$$W[\rho(x)] = W[\bar{\rho}] + \epsilon W_1[\bar{\rho}, \tilde{\rho}] + \epsilon^2 W_2[\bar{\rho}, \tilde{\rho}], \tag{3.9}$$

where W_1 is the first variation and W_2 the second variation. For $\bar{\rho}$ to be an equilibrium solution the first variation must equal zero and, for that equilibrium to be a stable minimizer, the second variation must be greater than zero.

The first variation is given by,

$$\begin{aligned} W_1[\bar{\rho}, \tilde{\rho}] &= -\int_0^L \frac{\tilde{\rho}(y)\sigma^2}{\bar{\rho}(y)^2} R'\Big(\frac{\sigma}{\bar{\rho}(y)}\Big)\, dy + \int_0^L \tilde{\rho}(y) \int_0^L \bar{\rho}(x) A(x-y)\, dx dy \\ &= \int_0^L \tilde{\rho}(y)\Big[\int_0^L \bar{\rho}(x) A(x-y)\, dx - \frac{\sigma^2}{\bar{\rho}(y)^2} R'\Big(\frac{\sigma}{\bar{\rho}(y)}\Big)\Big]\, dy. \end{aligned} \tag{3.10}$$

Because $\tilde{\rho}(y)$ is a zero mass perturbation, it's integral over the period $[0, L]$ is zero. As long as the bracketed portion of Equation 3.10 is a constant, then the first variation is equal to zero. Moreover, if the bracketed expression is not constant then one can find a perturbation such that the first variation is nonzero indicating that the solution is not an equilibrium of the system. For $W_1[\bar{\rho}, \tilde{\rho}] = 0$,

$$\int_0^L \bar{\rho}(x) A(x-y)\, dx - \frac{\sigma^2}{\bar{\rho}(y)^2} R'\Big(\frac{\sigma}{\bar{\rho}(y)}\Big) = \Lambda(y), \tag{3.11}$$

where $\Lambda(y)$ is a constant function of y. The repulsive term in Equation 3.10 is contant in y because $\bar{\rho}(y)$ is constant. The integral of attraction can be shown also to be constant in y. Consider a substitution such that $x - y = z$ and $dx = dz$. Then the integral becomes,

$$\int_0^L \bar{\rho}(y+z) A(z)\, dz, \tag{3.12}$$

where $\bar{\rho}(y+z)$ is still a constant and $A(z)$ is a periodic function for $0 < z < L$ so that the integral results in a constant in y. Thus, we have shown that if $\bar{\rho}(x)$ is a constant function, then it is an equilibrium of the energy functional defined in Equation 3.7.

For the constant density equilibrium to be stable, W_2 must be greater than zero. The second variation is given by,

$$\begin{aligned} W_2[\bar{\rho},\tilde{\rho}] &= \int_0^L \tilde{\rho}(x)^2\left[\frac{\sigma^2}{\bar{\rho}^3}R'\Big(\frac{\sigma}{\bar{\rho}}\Big)+\frac{\sigma^3}{2\bar{\rho}^4}R''\Big(\frac{\sigma}{\bar{\rho}}\Big)\right]dx+\frac{1}{2}\int_0^L\int_0^L \tilde{\rho}(x)\tilde{\rho}(y)A(x-y)\,dxdy \\ &= \int_0^L \tilde{\rho}(x)^2\big[B\big]\,dx+\frac{1}{2}\int_0^L\int_0^L \tilde{\rho}(x)\tilde{\rho}(y)A(x-y)\,dxdy, \end{aligned} \tag{3.13}$$

where B is a constant. To arrive at stability conditions, we must use a Fourier series approximation of the perturbation $\tilde{\rho}(x)$. Suppose,

$$\tilde{\rho}(x) = \sum_{n=1}^{\infty} a_n \cos\Big(\frac{2n\pi}{L}x\Big)+b_n\sin\Big(\frac{2n\pi}{L}x\Big). \tag{3.14}$$

The first term of Equation 3.13 can be expressed in terms of an L^2-inner product such that,

$$\begin{aligned} W_2^R[\bar{\rho},\tilde{\rho}] &= B\int_0^L \tilde{\rho}(x)^2\,dx = B\big\langle\tilde{\rho}(x),\tilde{\rho}(x)\big\rangle \\ &= B\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left[a_na_m\Big\langle\cos\Big(\frac{2n\pi}{L}x\Big),\cos\Big(\frac{2m\pi}{L}x\Big)\Big\rangle+b_nb_m\Big\langle\sin\Big(\frac{2n\pi}{L}x\Big),\sin\Big(\frac{2m\pi}{L}x\Big)\Big\rangle\right] \\ &= B\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{L}{2}\Big[a_na_m\delta_{nm}+b_nb_m\delta_{nm}\Big] \\ &= B\sum_{n=1}^{\infty}\frac{L}{2}\Big(a_n^2+b_n^2\Big) \end{aligned}$$

Now consider the second term of Equation 3.13 where $A(r)$ can also be re-expressed as a Fourier series. Because the attractive potential is an even periodic function, it has a cosine series approximation, such that

$$A(x-y) = \frac{c_0}{2}+\sum_{j=1}^{\infty} c_j\cos\Big(\frac{2j\pi}{L}(x-y)\Big). \tag{3.15}$$

Using an angle-sum trigonometric identity, the series can be written as,

$$A(x-y) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_j \Big[\cos\Big(\frac{2j\pi}{L}x\Big)\cos\Big(\frac{2j\pi}{L}y\Big) + \sin\Big(\frac{2j\pi}{L}x\Big)\sin\Big(\frac{2j\pi}{L}y\Big)\Big].$$

The attractive term of $W_2[\bar{\rho},\tilde{\rho}]$ becomes a large expression,

$$\begin{aligned} W_2^A[\bar{\rho},\tilde{\rho}] = \frac{1}{2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{j=1}^{\infty}\Bigg[&\int_0^L\int_0^L \frac{c_0}{2}\Big(a_n\cos\Big(\frac{2n\pi}{L}x\Big) + b_n\sin\Big(\frac{2n\pi}{L}x\Big)\Big)\,dxdy \\ &+ \int_0^L\int_0^L c_j\Big(a_n\cos\Big(\frac{2n\pi}{L}x\Big) + b_n\sin\Big(\frac{2n\pi}{L}x\Big)\Big)\cos\Big(\frac{2j\pi}{L}x\Big)\cos\Big(\frac{2j\pi}{L}y\Big)\,dxdy \\ &+ \int_0^L\int_0^L c_j\Big(a_n\cos\Big(\frac{2n\pi}{L}x\Big) + b_n\sin\Big(\frac{2n\pi}{L}x\Big)\Big)\sin\Big(\frac{2j\pi}{L}x\Big)\sin\Big(\frac{2j\pi}{L}y\Big)\,dxdy\Bigg], \end{aligned}$$

that can be simplified massively by evaluating the terms as inner products. The first set of integrals vanishes because each trigonometric function is integrated over a full period. So, the attractive term of second variation is,

$$\begin{aligned} W_2^A[\bar{\rho},\tilde{\rho}] &= \frac{L^2}{8}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{j=1}^{\infty} c_j\Big(a_n a_m \delta_{jn}\delta_{jm} + b_n b_m \delta_{jn}\delta_{jm}\Big) \\ &= \sum_{n=1}^{\infty}\frac{c_n L^2}{8}\Big(a_n^2 + b_n^2\Big). \end{aligned}$$

Finally, the complete second variation is,

$$W_2[\bar{\rho},\tilde{\rho}] = \frac{L}{2}\sum_{n=1}^{\infty}\Big(B + \frac{c_n L}{4}\Big)\Big(a_n^2 + b_n^2\Big). \tag{3.16}$$

With this equation, we can evaluate the stability of the constant density equilibrium solution, $\bar{\rho}$. The stability of constant density to the k^{th} Fourier mode of the perturbation is given by the sign of,

$$\Big(B + \frac{c_n L}{4}\Big).$$

When the expression is positive, it is a stable mode and, when negative, an unstable mode.

Chapter 4

Numerical Results

4.1 Stability in discrete model

In Chapter 2, we derived an analytic expression for the eigenvalues of the Hessian of our system given by Equation 2.21,

$$\lambda^{(k)} = 2\sum_{\ell=1}^{M}\left[\frac{\cosh\left(L/2-\ell\Delta\right)}{\sinh(L/2)} - \frac{F}{A}\frac{\cosh\left(\frac{L/2-\ell\Delta}{A}\right)}{\sinh\left(\frac{L}{2A}\right)}\right]\left(1-\cos\left(\frac{2\pi}{N}\ell k\right)\right) \quad \text{for} \quad -M \leq k \leq M.$$

We numerically evaluate this result in MATLAB in two ways. Firstly, we build our Hessian and call an in-built function for the eigenvalues of a matrix. Then, we use the Fast Fourier Transform on the first row vector of the Hessian. We compare these two sets of eigenvalues against the results of our analytic expression to find that they agree.

We also find the transition to instability as the strength of attraction increases. In Figure 4.1, we see that the set of eigenvalues is positive definite for $F < 1.1$ and contains at least one negative eigenvalue for greater attraction.

4.2 Stability in continuum model

The most important insight into our continuum model is whether there is a transition to instability as the average density, $\frac{M}{L}$, of the model increases. Figure 4.2 shows that there is in fact a transition to instability for a square power law in repulsion. The smallest wavenumber perturbations are the first to become unstable and as the average density increases more wavenumbers are unstable. In Figure 4.3, we can have a closer look at the phase transition

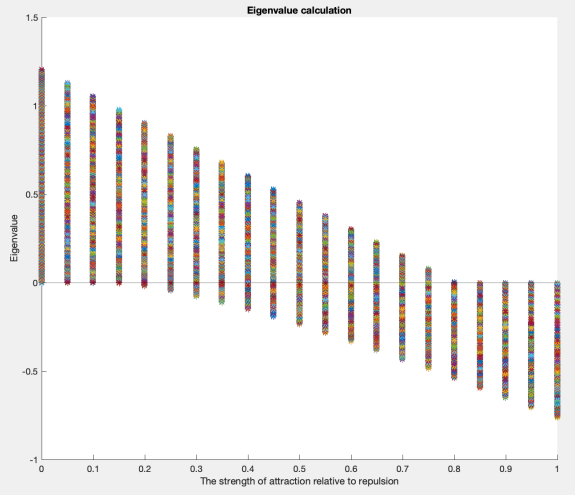

Figure 4.1 The strength of attraction, F, is on the x-axis in discrete steps and the resulting eigenvalues from Equation 2.21 are on the y-axis.

to instability. At an average density of about 0.2, the first Fourier mode becomes unstable.

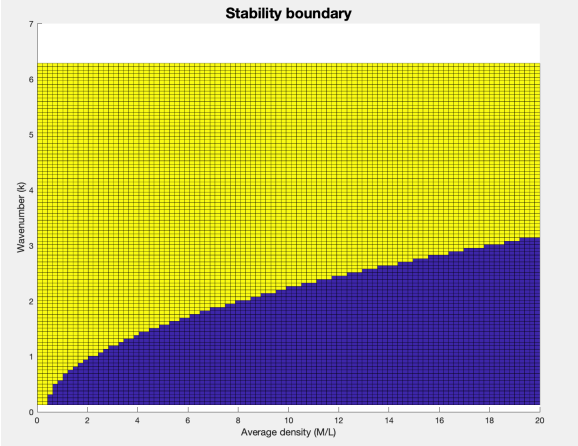

Figure 4.2 Stability of constant density as a function of wavenumber and average density. The yellow indicates a stable mode at that average density, while purple indicates instability. In this figure we use $R(r) = 1/r^2$.

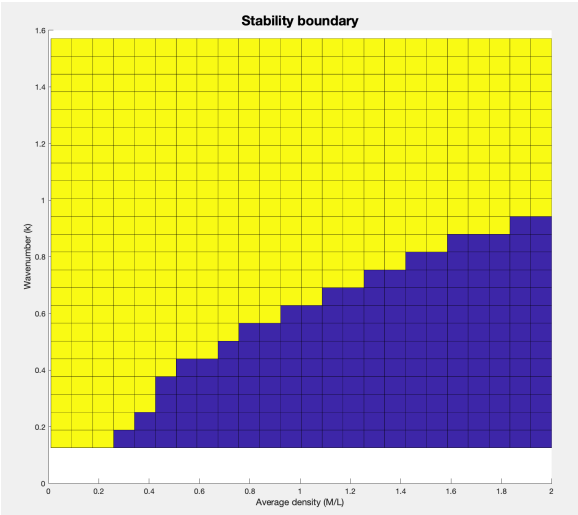

Figure 4.3 The transition to instability occurs just after $\sigma = 0.2$ because the first mode has negative second variation. In this figure we use $R(r) = 1/r^2$.

Chapter 5

Conclusion

We show that analytic expression of eigenvalues exists for a first order model of global attraction and repulsion forces on a periodic boundary. Our expression shows that the stability of the system is sensitive to the strength of attraction. We also show that if repulsion is made local, the stability of a constant density solution depends on the total mass of the swarm. This is an important result because much of the literature on aggregation models has treated global attraction with results that are not empirically supported for many swarms in nature. Our continuum model with local repulsion shows that there is a density threshold after which the constant density solution is unstable. Looking to the future, we plan to generalize the nearest neighbor model to two dimensions and apply our energy framework to evaluate stability.

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