Lenses

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1 Rescaling

$$w \equiv 8\pi G M_{Lz} f \tag{1}$$

$$M_{Lz} \equiv \frac{\xi_0^2}{4Gd_{\text{eff}}} \tag{2}$$

2 Magnification

From [2] (ch.5, p.162)

$$\phi(\boldsymbol{x}, \boldsymbol{y}) = \frac{1}{2}|\boldsymbol{x} - \boldsymbol{y}|^2 - \psi(\boldsymbol{x}) \tag{3}$$

$$A_{ij} = \frac{\partial y_i}{\partial x_j} = \phi_{ij} = \delta_{ij} - \psi_{ij} \quad \rightarrow \quad \mu = \frac{1}{\det A} \tag{4}$$

Defining

$$\gamma_1 = \frac{1}{2}(\psi_{11} - \psi_{22}) \tag{5}$$

$$\gamma_2 = \psi_{12} \tag{6}$$

$$\kappa = \frac{1}{2}(\psi_{11} + \psi_{22}) \tag{7}$$

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And then we can write

$$\mathrm{eig}(A) = 1 - \kappa \mp \gamma \tag{8}$$
$$\det A = (1-\kappa)^2 - \gamma^2 \tag{9}$$
$$\mathrm{tr}\, A = 2(1-\kappa) \tag{10}$$

where $\gamma = \sqrt{\gamma_1^2 + \gamma_2^2}$. The different types of images are

$$\text{Type I (minimum)} \quad \rightarrow \quad \det A > 0,\ \mathrm{tr}\, A > 0 \tag{11}$$
$$\text{Type II (saddle)} \quad \rightarrow \quad \det A < 0 \tag{12}$$
$$\text{Type III (maximum)} \quad \rightarrow \quad \det A > 0,\ \mathrm{tr}\, A < 0 \tag{13}$$
$$\tag{14}$$

3 Singular Isothermal Sphere (SIS)

4 Cored Isothermal Sphere (CIS)

5 Point lens

6 Uniform density sphere

7 Navarro-Frenk-White (NFW)

Following [1], we start with

$$\rho(r) = \frac{\rho_s}{(r/r_s)(1+r/r_s)^2} \tag{15}$$

and we have the lensing potential

$$\psi(r) = \frac{1}{2}\psi_0 \left(\log^2(u/2) + (u^2-1)\mathcal{F}(u)\right) , \qquad \psi_0 = \frac{4\kappa_s r_s^2}{\xi_0^2} \tag{16}$$

where $u \equiv r/r_s = x/x_s$ and $x \equiv r/\xi_0$. We also have

$$\kappa_s = \frac{\rho_s r_s}{\Sigma_{\mathrm{crit}}} \tag{17}$$
$$\Sigma_{\mathrm{crit}} = \frac{1}{4\pi G d_{\mathrm{eff}}(1+z_L)} \tag{18}$$
$$d_{\mathrm{eff}} = \frac{D_L D_{SL}}{(1+z_L)D_S} \tag{19}$$

Then

$$\psi_0 = \frac{1}{\xi_0^2} 4\kappa_s r_s^2 = \frac{4G d_{\mathrm{eff}}(1+z_L)M_{\mathrm{NFW}}}{\xi_0^2} \tag{20}$$

where $M_{\mathrm{NFW}} \equiv 4\pi\rho_s r_s^3$. We can then choose

$$\xi_0^2 = 4G d_{\mathrm{eff}}(1+z_L)M_{\mathrm{NFW}} \tag{21}$$
$$M_{Lz} = (1+z_L)M_{\mathrm{NFW}} \tag{22}$$

We can relate the NFW parameters to the concentration parameter defined as [Wikipedia]

$$r_{\mathrm{vir}} = c\, r_s \tag{23}$$
$$M_{\mathrm{vir}} = \int_0^{r_{\mathrm{vir}}} 4\pi r^2 \rho(r) \mathrm{d}r = M_{\mathrm{NFW}} \left(\log(1+c) - \frac{c}{1+c}\right) \tag{24}$$

Ref.	ξ_0	ψ_0	x_s	M_{Lz}
Fairbairn+	$R_E(M_{\rm vir})$	$M_{\rm NFW}/M_{\rm vir}$	$r_s/R_E(M_{\rm vir})$	$(1+z_L)M_{\rm vir}$
Guo+	r_s	$R_E(M_{\rm NFW})/r_s$	1	$(1+z_L)M_{\rm NFW}\frac{r_s^2}{R_E^2(M_{\rm NFW})}$
Ours (?)	$R_E(M_{\rm NFW})$	1	$r_s/R_E(M_{\rm NFW})$	$(1+z_L)M_{\rm NFW}$

Table 1: Possible choices of normalization. Here $R_E(M) \equiv \sqrt{4Gd_{\rm eff}(1+z_L)M}$.

with typical values ranging from 10 to 15 for the Milky Way and 4 to 40 for halos. With the additional definition of the virial radius

$$M_{\rm vir} = \frac{4}{3}\pi r_{\rm vir}^3\ 200\rho_{\rm crit} \tag{25}$$

we can obtain $r_s(M_{\rm vir}, c)$ and $M_{\rm NFW}(M_{\rm vir}, c)$. So finally we have

$$M_{Lz} = \frac{(1+z_L)}{\log(1+c) - \frac{c}{1+c}} M_{\rm vir} \tag{26a}$$

$$\xi_0^2 = 4Gd_{\rm eff}M_{Lz} \tag{26b}$$

$$x_s \equiv \frac{r_s}{\xi_0} = \frac{1}{\xi_0}\left(\frac{3M_{\rm vir}}{800\pi c^3\rho_{\rm crit}}\right)^{1/3} \tag{26c}$$

References

[1] Charles R. Keeton. A catalog of mass models for gravitational lensing. 2 2001.

[2] Peter Schneider, Jürgen Ehlers, and Emilio E. Falco. *Gravitational Lenses*. 1992.