Methods

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1.1 Standard

The time-domain integral is

$$I(t) = \int \mathrm{d}^2x \, \delta\left(\phi(\boldsymbol{x}) - t\right) \tag{1}$$

where the Fermat potential is

$$\phi(x_1, x_2) = \frac{1}{2}(x_1^2 + x_2^2) + x_1 \cos(\theta) + \frac{1}{2}y^2 - \psi(x_1, x_2) \tag{2}$$

and

$$t = \tau + t_{\text{min}} \tag{3}$$

We can change to polar coordinates

$$x_1 = x_1^0 + R\cos(\theta) \tag{4}$$

$$x_2 = x_2^0 + R\sin(\theta) \tag{5}$$

where (x_1^0, x_2^0) correspond to the position of the minimum

$$\phi(x_1^0, x_2^0) = t_{\text{min}} \tag{6}$$

With this change of coordinates, the integral can be expressed as

$$I(t) = \int R\mathrm{d}R\mathrm{d}\theta \, \delta\left(\phi(R, \theta) - t\right) \tag{7}$$

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If $\partial_R \phi \neq 0$, we can invert

$$\phi(R, \theta) = t \tag{8}$$

to obtain $R(\theta, t)$. We can then solve the integral over the δ function, plugging in this solution

$$I(t) = \int_0^{2\pi} \frac{R(\theta, t)}{|\partial_R \phi|} \mathrm{d}\theta \tag{9}$$

Finally, the system of differential equations that must be solved to find both the curve $R(\theta, t)$ and $I(t)$ is

$$\frac{\mathrm{d}I}{\mathrm{d}\theta} = \frac{R}{|\partial_R \phi|} \tag{10}$$

$$\frac{\mathrm{d}R}{\mathrm{d}\theta} = -\frac{\partial_\theta \phi}{\partial_R \phi} \tag{11}$$

Since we are just interested in $I(t)$, we can integrate this system from $\theta = 0$ to 2π, with initial conditions $I(\theta = 0, t) = 0$ and $R(\theta = 0, t)$ such that $\phi(R(0, t), 0) = t$. A couple of useful intermediate results:

$$\partial_\theta f = -\Delta x_2 \partial_{x_1} f + \Delta x_1 \partial_{x_2} f \tag{12}$$

$$\partial_R f = \frac{\Delta x_1}{R} \partial_{x_1} f + \frac{\Delta x_2}{R} \partial_{x_2} f \tag{13}$$

$$\tag{14}$$

where $\Delta x_i = x_i - x_i^0$.

1.2 Robust (parametric)

In general we cannot solve

$$\phi(R, \theta) = t \tag{15}$$

to obtain a relation $R(\theta, t)$, but instead we can find a parametric curve $R(\sigma, t)$ and $\theta(\sigma, t)$. We can think of it as a Hamiltonian system with a Hamiltonian ϕ, and the problem is to find the equations of motion for an energy t. The equations for these curves are

$$\frac{\mathrm{d}R}{\mathrm{d}\sigma} = -\partial_\theta \phi \tag{16}$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}\sigma} = \partial_R \phi \tag{17}$$

Now, making the coordinate transformation $(R, \theta) \to (\sigma, t')$, we end up with

$$I(t) = \int \frac{R(\sigma, t')}{|J|} \mathrm{d}\sigma \mathrm{d}t' \delta(t - t') \tag{18}$$

where

$$J = \begin{vmatrix} \partial_\sigma R & \partial_t R \\ \partial_\sigma \theta & \partial_t \theta \end{vmatrix} = 1 \tag{19}$$

so we end up with

$$I(t) = \oint R(\sigma, t) \mathrm{d}\sigma \tag{20}$$

The final system of ODEs that we must solve is

$$\frac{\mathrm{d}I}{\mathrm{d}\sigma} = R \tag{21}$$

$$\frac{\mathrm{d}R}{\mathrm{d}\sigma} = -\partial_\theta \phi \tag{22}$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}\sigma} = \partial_R \phi \tag{23}$$

This time we must integrate from $\sigma = 0$ until we close the curve[1], i.e. $\theta(\sigma_f, t) = 2\pi$ and $R(\sigma_f, t) = R(0, t)$. The initial conditions are chosen as before.

2 Single integral method

Let us start again with the general expression

$$I(t) = \int \mathrm{d}^2 x \, \delta(\phi(\boldsymbol{x}) - t) \tag{24}$$

This time we will assume that the lens is axisymmetric, so choosing coordinates centered in the lens position

$$x_1 = r\cos(\varphi) \tag{25}$$

$$x_2 = r\sin(\varphi) \tag{26}$$

we have

$$\psi(x_1, x_2) = \psi(r) \tag{27}$$

Using these coordinates, we can solve the angular integral (with $z \equiv \cos(\varphi)$)

$$I(t) = \int_{-1}^{1} \frac{\mathrm{d}z}{\sqrt{1-z^2}} \int_0^\infty r\mathrm{d}r \, \delta\left(\frac{1}{2}r^2 - yrz + \frac{1}{2}y^2 - \psi(r) - t\right) \tag{28}$$

$$= \int_0^\infty \frac{2\mathrm{d}r}{y\sqrt{1-z_*^2}} \Theta(1 - z_*^2) \ , \qquad z_* \equiv \frac{1}{2yr}\left(r^2 - 2\psi(r) - t\right) \tag{29}$$

We can finally express it as (using the notation in the code)

$$I(t) = \int_0^\infty \alpha(r)\mathrm{d}r$$

$$\alpha = \begin{cases} 2r/\sqrt{\beta} \ , & \beta > 0 \\ 0 \ , & \beta < 0 \end{cases}$$

$$\beta = -\phi_+\phi_-$$

$$\phi_\pm = \frac{1}{2}r^2 + \frac{1}{2}y^2 - \psi(r) - t \mp ry$$

The problem now reduces to 1) finding the zeros of ϕ_+ and ϕ_- and 2) performing the integral between them (where $\phi_+\phi_- < 0$).

3 Area/binning method

Finally, another way to solve the integral

$$I(t) = \int \mathrm{d}^2 x \, \delta(\phi(\boldsymbol{x}) - t) \tag{30}$$

is to compute it directly as a surface integral, representing the delta as

$$\delta_n(x) = \begin{cases} 0 \ , & x < -1/2n \\ n \ , & -1/2n < x < 1/2n \\ 0 \ , & x > 1/2n \end{cases} \tag{31}$$

when $n \to \infty$. In this way, we obtain a discrete representation

$$I(t) \simeq I_i \ , \quad \text{for } t \in [t_i, t_i + \Delta t_i] \tag{32}$$

that converges to the real result as we reduce the size of the boxes Δt_i.

[1]Notice that since we assume that (x_1^0, x_2^0) is *inside* the contour we know that the final θ that closes the contour is 2π, otherwise it could also be 0.