Regularization

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Contents

1 Old regularization scheme

Minimum (type I)

$$I^{\mathrm{m}}_{\mathrm{sing}}(\tau) \equiv 2\pi\sqrt{|\mu_j|}\Theta(\tau-\tau_j) \tag{1}$$

$$F^{\mathrm{m}}_{\mathrm{sing}}(w) = \sqrt{|\mu_j|}\mathrm{e}^{\mathrm{i}w\tau_j} \tag{2}$$

Maximum (type II)

$$I^{\mathrm{M}}_{\mathrm{sing}}(\tau) \equiv 2\pi\sqrt{|\mu_j|}\Theta(\tau_j-\tau) \tag{3}$$

$$F^{\mathrm{M}}_{\mathrm{sing}}(w) = -\sqrt{|\mu_j|}\mathrm{e}^{\mathrm{i}w\tau_j} \tag{4}$$

Saddle (type III)

$$I^{\mathrm{s}}_{\mathrm{sing}}(\tau) \equiv -2\sqrt{|\mu_j|}\mathrm{e}^{-|\tau-\tau_j|/T}\log|\tau-\tau_j| \tag{5}$$

$$F^{\mathrm{s}}_{\mathrm{sing}}(w) = \frac{2\mathrm{i}w}{\pi}\sqrt{|\mu_j|}\mathrm{e}^{\mathrm{i}w\tau_j}\ \Re(\mathcal{I}) \tag{6}$$

$$\mathcal{I} \equiv \int_0^\infty \mathrm{d}t\,\log(t)\mathrm{e}^{-t/T+\mathrm{i}wt} = -\frac{\gamma_E+\log(T^{-1}-\mathrm{i}w)}{T^{-1}-\mathrm{i}w} \tag{7}$$

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2 New regularization scheme

2.1 Regularizing functions

We can use the following set of functions to regularize the initial step function, power-law tails (both at $\tau \to 0$ and $\tau \to \infty$) as well as the saddle points.

$$R_0(\alpha,\beta,\sigma;x) \equiv \frac{\beta\,\Theta(x)}{\left(x^2+(\beta/\alpha)^{2/\sigma}\right)^{\sigma/2}} \qquad \begin{cases} R_0(0)=\alpha \\ R_0'(0)=0 \\ R_0(x\to\infty)\sim \beta/x^\sigma \end{cases} \tag{8}$$

$$R_1(\alpha,\beta,\sigma;x) \equiv xR_0(\alpha,\beta,\sigma+1;x) \qquad \begin{cases} R_1(0)=0 \\ R_1'(0)=\alpha \\ R_1(x\to\infty)\sim \beta/x^\sigma \end{cases} \tag{9}$$

$$R_L(\alpha,\beta;x) \equiv R_1(\alpha,\beta,1;x) = \frac{\beta x\,\Theta(x)}{x^2+\beta/\alpha} \qquad \begin{cases} R_L(0)=0 \\ R_L'(0)=\alpha \\ R_L(x\to\infty)\sim \beta/x \end{cases} \tag{10}$$

$$S(A,B;x) \equiv \frac{AB}{2}\Theta(x)\log\left|\frac{B+x}{B-x}\right| \qquad \begin{cases} S(0)=0 \\ S'(0)=A \\ S(x\to\infty)\sim AB^2/x \end{cases} \tag{11}$$

$$S_{\text{full}}(A,B;x) \equiv S(A,B;x) - R_L(A,AB^2;x) \qquad \begin{cases} S_{\text{full}}(0)=0 \\ S_{\text{full}}'(0)=0 \\ S_{\text{full}}(x\to\infty)\sim \dfrac{4AB^4}{3x^3} \end{cases} \tag{12}$$

2.2 Fourier transform

We can define the direct and inverse Fourier transform as

$$\mathcal{F}\Big(f(x)\Big) \equiv \int_{-\infty}^{\infty} \mathrm{d}x\,\mathrm{e}^{-\mathrm{i}\omega x} f(x) \tag{13}$$

$$\mathcal{F}^{-1}\Big(f(\omega)\Big) \equiv \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi}\,\mathrm{e}^{\mathrm{i}\omega x} f(\omega) \tag{14}$$

From the definition of $I(\tau)$ we have

$$I(\tau) \equiv \frac{1}{2\pi}\int \mathrm{d}^2x \int_{-\infty}^{\infty} \mathrm{d}w\,\mathrm{e}^{\mathrm{i}w(\phi-\tau-t_{\min})} \tag{15}$$

$$= \mathcal{F}\left(\mathrm{e}^{-\mathrm{i}wt_{\min}}\frac{\mathrm{i}\,F(w)}{w}\right) \tag{16}$$

and then

$$F(w) = \frac{w}{\mathrm{i}}\mathrm{e}^{\mathrm{i}wt_{\min}}\mathcal{F}^{-1}\Big(I(\tau)\Big) \tag{17}$$

$$= \frac{w}{2\pi\,\mathrm{i}}\mathrm{e}^{\mathrm{i}wt_{\min}}\int_{-\infty}^{\infty}\mathrm{d}\tau\,\mathrm{e}^{\mathrm{i}w\tau} I(\tau) \tag{18}$$

We can define the Fourier counterpart of the regularizing functions as

$$R(w) \equiv w\,\mathrm{e}^{\mathrm{i}wt_{\min}}\left\{\int_0^\infty \sin(w\tau)R(\tau)\mathrm{d}\tau - \mathrm{i}\int_0^\infty \cos(w\tau)R(\tau)\mathrm{d}\tau\right\} \tag{19}$$

We can use the equations in the appendices to write

$$\int_0^\infty \mathrm{d}\tau \frac{\sin(w\tau)}{(\tau^2+C^2)^{\sigma/2}} = \frac{1}{\sqrt{\pi}}\left(\frac{2C}{w}\right)^{\frac{1-\sigma}{2}} \cos\left(\frac{\pi\sigma}{2}\right)\Gamma(1-\sigma/2)K_{\frac{1-\sigma}{2}}(wC) - \frac{\sqrt{\pi}}{2}\Gamma(1-\sigma/2)C^{(1-\sigma)}\tilde{\mathbb{M}}_{\frac{1-\sigma}{2}}(wC) \tag{20}$$

$$\int_0^\infty \mathrm{d}\tau \frac{\cos(w\tau)}{(\tau^2+C^2)^{\sigma/2}} = \frac{1}{\sqrt{\pi}}\left(\frac{2C}{w}\right)^{\frac{1-\sigma}{2}} \sin\left(\frac{\pi\sigma}{2}\right)\Gamma(1-\sigma/2)K_{\frac{1-\sigma}{2}}(wC) \tag{21}$$

where $C \equiv (\beta/\alpha)^{1/\sigma}$ for R_0 and $C \equiv (\beta/\alpha)^{1/(\sigma+1)}$ for R_1.

$$\int_0^\infty \mathrm{d}\tau \frac{\tau\sin(w\tau)}{(\tau^2+C^2)^{(\sigma+1)/2}} = \frac{C}{\sqrt{\pi}}\left(\frac{2C}{w}\right)^{-\sigma/2} \cos\left(\frac{\pi\sigma}{2}\right)\Gamma\left(\frac{1-\sigma}{2}\right)K_{1-\sigma/2}(wC) \tag{22}$$

$$\int_0^\infty \mathrm{d}\tau \frac{\tau\cos(w\tau)}{(\tau^2+C^2)^{(\sigma+1)/2}} = \frac{C}{\sqrt{\pi}}\left(\frac{2C}{w}\right)^{-\sigma/2} \sin\left(\frac{\pi\sigma}{2}\right)\Gamma\left(\frac{1-\sigma}{2}\right)K_{1-\sigma/2}(wC) - \frac{C^{(1-\sigma)}}{1-\sigma}\left\{1+\sqrt{\pi}\Gamma\left(\frac{3-\sigma}{2}\right)\left(\frac{wC}{2}\right)\tilde{\mathbb{M}}_{1-\sigma/2}(wC)\right\} \tag{23}$$

$$\int_0^\infty \mathrm{d}\tau \frac{\tau\sin(w\tau)}{\tau^2+\beta/\alpha} = \frac{\pi}{2}\mathrm{e}^{-w\sqrt{\beta/\alpha}} \tag{24}$$

$$\int_0^\infty \mathrm{d}\tau \frac{\tau\cos(w\tau)}{\tau^2+\beta/\alpha} = -\frac{1}{2}\left\{\mathrm{e}^{-w\sqrt{\beta/\alpha}}\mathrm{Ei}\left(w\sqrt{\frac{\beta}{\alpha}}\right) - \mathrm{e}^{w\sqrt{\beta/\alpha}}E_1\left(w\sqrt{\frac{\beta}{\alpha}}\right)\right\} \tag{25}$$

$$\int_0^\infty \mathrm{d}\tau \sin(w\tau) B\log\left|\frac{\tau+B}{\tau-B}\right| = \frac{\pi B}{w}\sin(wB) \tag{26}$$

$$\int_0^\infty \mathrm{d}\tau \cos(w\tau) B\log\left|\frac{\tau+B}{\tau-B}\right| = \frac{\pi B}{w}\cos(wB) + \frac{2B}{w}\Big(\cos(wB)\mathrm{si}(wB) - \sin(wB)\mathrm{ci}(wB)\Big) \tag{27}$$

We can combine everything to write (omitting the $\mathrm{e}^{\mathrm{i}wt_{\min}}$ factor)

$$R_0(\alpha,\beta,\sigma;\ w) = \sqrt{\pi}\alpha\Gamma\left(1-\frac{\sigma}{2}\right)\left\{\frac{2}{\pi}\left(\frac{wC}{2}\right)^{\frac{1+\sigma}{2}}\mathrm{e}^{-\mathrm{i}\frac{\pi\sigma}{2}}K_{\frac{1-\sigma}{2}}(wC) - \left(\frac{wC}{2}\right)\tilde{\mathbb{M}}_{\frac{1-\sigma}{2}}(wC)\right\} \tag{28}$$

$$R_1(\alpha,\beta,\sigma;\ w) = \frac{2\alpha C}{\sqrt{\pi}}\left(\frac{wC}{2}\right)^{1+\frac{\sigma}{2}}\mathrm{e}^{-\mathrm{i}\frac{\pi\sigma}{2}}\Gamma\left(\frac{1-\sigma}{2}\right)K_{1-\frac{\sigma}{2}}(wC) + \frac{\mathrm{i}\alpha C}{1-\sigma}wC\left\{1+\sqrt{\pi}\Gamma\left(\frac{3-\sigma}{2}\right)\left(\frac{wC}{2}\right)\tilde{\mathbb{M}}_{1-\frac{\sigma}{2}}(wC)\right\} \tag{29}$$

$$R_L(\alpha,\beta;\ w) = \frac{\pi}{2}\beta w\,\mathrm{e}^{-w\sqrt{\beta/\alpha}} + \mathrm{i}\beta\frac{w}{2}\left\{\mathrm{e}^{-w\sqrt{\beta/\alpha}}\mathrm{Ei}\left(w\sqrt{\frac{\beta}{\alpha}}\right) - \mathrm{e}^{w\sqrt{\beta/\alpha}}E_1\left(w\sqrt{\frac{\beta}{\alpha}}\right)\right\} \tag{30}$$

$$S(A,B;\ w) = -\mathrm{i}\frac{\pi}{2}AB\,\mathrm{e}^{\mathrm{i}wB} - \mathrm{i}AB\Big(\cos(wB)\mathrm{si}(wB) - \sin(wB)\mathrm{ci}(wB)\Big) \tag{31}$$

where, again, $C \equiv (\beta/\alpha)^{1/\sigma}$ for R_0 and $C \equiv (\beta/\alpha)^{1/(\sigma+1)}$ for R_1.

2.3 Low and high frequency limits

High-frequency behaviour

$$R_0(\alpha,\beta,\sigma;\ w) \sim \alpha + \mathcal{O}\Big(w^{-1}\Big) \tag{32}$$

$$R_1(\alpha,\beta,\sigma;\ w) \sim \mathcal{O}\Big(w^{-1}\Big) \tag{33}$$

$$R_L(\alpha,\beta;\ w) \sim \mathcal{O}\Big(w^{-1}\Big) \tag{34}$$

$$S(A,B;\ w) \sim -\mathrm{i}\frac{\pi}{2}AB\mathrm{e}^{\mathrm{i}wB} + \mathcal{O}\Big(w^{-1}\Big) \tag{35}$$

2.4 Scheme

0) Starting point, $I_{\text{reg}}(\tau) = I(\tau)$.

$$I_{\text{reg}}(0) = \sqrt{\mu_{\text{min}}}\ , \qquad I_{\text{reg}}(-\tau) = 0\ , \tag{36}$$

$$I'_{\text{reg}}(0) = G_0\ , \qquad I_{\text{reg}}(x\to\infty) \sim 1 + \frac{I_{\text{asymp}}}{\tau^\sigma}\ . \tag{37}$$

1) Subtract all maxima ($\mathcal{C}_{\text{max}} \equiv \sum_{\text{max}}\sqrt{|\mu_j|}$)

$$I_{\text{reg}}(0) = \sqrt{\mu_{\text{min}}} - \mathcal{C}_{\text{max}}\ , \qquad I_{\text{reg}}(-\tau) = -\mathcal{C}_{\text{max}}\ , \tag{38}$$

$$I'_{\text{reg}}(0) = G_0\ , \qquad I_{\text{reg}}(x\to\infty) \sim 1 + \frac{I_{\text{asymp}}}{\tau^\sigma}\ . \tag{39}$$

2) Subtract all minima, except the global minimum ($\mathcal{C}_{\text{min}} \equiv \sum_{\text{loc min}}\sqrt{|\mu_j|}$)

$$I_{\text{reg}}(0) = \sqrt{\mu_{\text{min}}} - \mathcal{C}_{\text{max}}\ , \qquad I_{\text{reg}}(-\tau) = -\mathcal{C}_{\text{max}}\ , \tag{40}$$

$$I'_{\text{reg}}(0) = G_0\ , \qquad I_{\text{reg}}(x\to\infty) \sim 1 - \mathcal{C}_{\text{min}} + \frac{I_{\text{asymp}}}{\tau^\sigma}\ . \tag{41}$$

3) (Intermediate step) Add constant $\mathcal{C}_{\text{max}}$ and subtract step $\Theta(\tau)(1-\mathcal{C}_{\text{min}}+\mathcal{C}_{\text{max}})$

$$I_{\text{reg}}(0) = \sqrt{\mu_{\text{min}}} - 1 - \mathcal{C}_{\text{max}} - \mathcal{C}_{\text{min}}\ , \qquad I_{\text{reg}}(-\tau) = 0\ , \tag{42}$$

$$I'_{\text{reg}}(0) = G_0\ , \qquad I_{\text{reg}}(x\to\infty) \sim \frac{I_{\text{asymp}}}{\tau^\sigma}\ . \tag{43}$$

4) Subtract the global minimum with R_0

$$I_{\text{reg}}(0) = 0\ , \qquad I_{\text{reg}}(-\tau) = 0\ , \tag{44}$$

$$I'_{\text{reg}}(0) = G_0\ , \qquad I_{\text{reg}}(x\to\infty) \sim 0 \qquad (\text{faster than } 1/\tau^\sigma)\ . \tag{45}$$

5) Subtract the saddles with S_{full}

$$I_{\text{reg}}(0) = 0\ , \qquad I_{\text{reg}}(-\tau) = 0\ , \tag{46}$$

$$I'_{\text{reg}}(0) = G_0\ , \qquad I_{\text{reg}}(x\to\infty) \sim 0 \qquad (<\ 1/\tau^\sigma, \geq\ 1/\tau^3)\ . \tag{47}$$

The overall operation can be written as

$$\begin{aligned}\frac{I_{\text{reg}}(\tau)}{2\pi} = \frac{I(\tau)}{2\pi} &- \sum_{\text{max}}\sqrt{|\mu_j|}\Theta(\tau_j-\tau) - \sum_{\text{loc min}}\sqrt{|\mu_j|}\Theta(\tau-\tau_j)\\ &+ \mathcal{C}_{\text{max}} - (1-\mathcal{C}_{\text{min}}+\mathcal{C}_{\text{max}})\,\Theta(\tau)\\ &- R_0\left(\sqrt{|\mu_{\text{min}}|} - 1 - \mathcal{C}_{\text{max}} + \mathcal{C}_{\text{min}}, I_{\text{asymp}}, \sigma;\ \tau\right)\\ &- \sum_{\text{saddle}} S_{\text{full}}\left(\frac{2\sqrt{|\mu_j|}}{\pi\tau_j}, \tau_j;\ \tau\right)\end{aligned} \tag{48}$$

where

$$\mathcal{C}_{\max} \equiv \sum_{\max} \sqrt{|\mu_j|}\ , \qquad \mathcal{C}_{\min} \equiv \sum_{\text{loc min}} \sqrt{|\mu_j|} \tag{49}$$

In the frequency domain this reads

$$F(w) = F_{\text{reg}}(w) + F_{\text{sing}}(w) \tag{50}$$

where

$$\begin{aligned} F_{\text{sing}}(w) = 1 - \mathcal{C}_{\min} + \mathcal{C}_{\max} + \sum_{\text{loc min}} \sqrt{|\mu_j|} \mathrm{e}^{\mathrm{i} w \tau_j} - \sum_{\max} \sqrt{|\mu_j|} \mathrm{e}^{\mathrm{i} w \tau_j} \\ + R_0 \left(\sqrt{|\mu_{\min}|} - 1 - \mathcal{C}_{\max} + \mathcal{C}_{\min}, I_{\text{asymp}}, \sigma;\ w \right) \\ + \sum_{\text{saddle}} S_{\text{full}} \left(\frac{2\sqrt{|\mu_j|}}{\pi \tau_j}, \tau_j;\ w \right) \end{aligned} \tag{51}$$

A Special functions

A.1 Gamma function

Reflection property [DLM23]

$$\Gamma(1-z) = \frac{\pi}{\Gamma(z)\sin(\pi z)} \tag{52}$$

Recurrence

$$\Gamma(z+1) = z\Gamma(z) \tag{53}$$

Particular values

$$\Gamma(1/2) = \sqrt{\pi} \tag{54}$$

A.2 Modified Bessel functions

The derivative of the modified Bessel function is [DLM23]

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(z^{-\nu} I_\nu\right) = z^{-\nu} I_{\nu+1} \tag{55}$$

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(z^{\nu} I_\nu\right) = z^{\nu} I_{\nu+1} \tag{56}$$

For K_ν we have similar relations

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(z^{-\nu} K_\nu\right) = -z^{-\nu} K_{\nu+1} \tag{57}$$

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(z^{\nu} K_\nu\right) = -z^{\nu} K_{\nu+1} \tag{58}$$

Another useful property

$$I_{-\nu}(z) = \frac{2}{\pi}\sin(\pi\nu) K_\nu(z) + I_\nu(z) \tag{59}$$

For the other modified Bessel function we have $K_{-\nu} = K_\nu$. Power series expansions

$$I_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{(z/2)^{2k}}{k!\Gamma(\nu+k+1)} \tag{60}$$

Asymptotic expansions

$$I_\nu(z) \sim \frac{\mathrm{e}^z}{\sqrt{2\pi z}} \sum_{k=0}^{\infty} (-1)^k \frac{a_k(\nu)}{z^k} \tag{61}$$

$$K_\nu(z) \sim \sqrt{\frac{\pi}{2z}} \mathrm{e}^{-z} \sum_{k=0}^{\infty} \frac{a_k(\nu)}{z^k} \tag{62}$$

where the coefficients a_k can be found in [DLM23]. Both functions can be related as

$$K_\nu(z) = \left(\frac{\pi}{2}\right) \frac{I_{-\nu}(z) - I_\nu(z)}{\sin(\nu\pi)} \tag{63}$$

Recurrence relation

$$I_{\nu-1}(z) - I_{\nu+1}(z) = \frac{2\nu}{z} I_\nu(z) \tag{64}$$

A.3 Modified Struve functions

The modified Struve function $\mathbb{M}_\nu$ is defined as [DLM23]

$$\mathbb{M}_\nu(z) \equiv \mathbb{L}_\nu(z) - I_\nu(z) \tag{65}$$

where $\mathbb{L}_\nu$ is another modified Struve function and I_ν is a modified Bessel function. For our purposes, we will define

$$\tilde{\mathbb{M}}_\nu(z) \equiv \left(\frac{2}{z}\right)^\nu \mathbb{M}_\nu(z) \tag{66}$$

Using the power series expansion

$$\left(\frac{2}{z}\right)^\nu \mathbb{L}_\nu(z) = \frac{z}{2} \sum_{n=0}^{\infty} \frac{(z/2)^{2n}}{\Gamma(n+3/2)\Gamma(n+\nu+3/2)} \tag{67}$$

and combining it with the one for I_ν, we can write

$$\tilde{\mathbb{M}}_\nu(z) = \frac{z}{2} \sum_{n=0}^{\infty} \frac{(z/2)^{2n}}{\Gamma(n+3/2)\Gamma(n+\nu+3/2)} - \sum_{n=0}^{\infty} \frac{(z/2)^{2n}}{n!\Gamma(\nu+n+1)} \tag{68}$$

On the other hand, the asymptotic expansion is

$$\tilde{\mathbb{M}}_\nu(z) \sim \frac{1}{\pi} \sum_{k=0}^{\infty} (-1)^{k+1} \frac{\Gamma(k+1/2)}{\Gamma(\nu+1/2-k)} \left(\frac{z}{2}\right)^{-2k-1} \tag{69}$$

For its convergence properties see [DLM23]. It is also possible to derive an expansion in Bessel functions

$$\left(\frac{2}{z}\right)^\nu \mathbb{L}_\nu(z) = \frac{\sqrt{z/2}}{\Gamma(\nu+1/2)} \sum_{n=0}^{\infty} \frac{(-1)^n (z/2)^n}{n!(n+\nu+1/2)} I_{n+1/2}(z) \tag{70}$$

Another property that will prove useful is

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(z^{-\nu}\mathbb{L}_\nu\right) = \frac{2^{-\nu}}{\sqrt{\pi}\Gamma(\nu+3/2)} + z^{-\nu}\mathbb{L}_{\nu+1} \tag{71}$$

Using the similar properties of the modified Bessel function, we have

$$\frac{\mathrm{d}}{\mathrm{d}z}\tilde{\mathbb{M}}_\nu = \frac{1}{\sqrt{\pi}\Gamma(\nu+3/2)} + \frac{z}{2}\tilde{\mathbb{M}}_{\nu+1} \tag{72}$$

Recurrence relation (from [AS64])

$$\mathbb{L}_{\nu-1} - \mathbb{L}_{\nu+1} = \frac{2\nu}{z}\mathbb{L}_\nu + \frac{(z/2)^\nu}{\sqrt{\pi}\Gamma(\nu+3/2)} \tag{73}$$

With this we can write

$$\tilde{\mathbb{M}}_{\nu-1} = \left(\frac{z}{2}\right)^2 \tilde{\mathbb{M}}_{\nu+1} + \nu\tilde{\mathbb{M}}_\nu + \frac{z}{2\sqrt{\pi}\Gamma(\nu+3/2)} \tag{74}$$

A.4 Sine and cosine integrals

The sine and cosine integrals are defined as [DLM23]

$$\mathrm{Si}(z) \equiv \int_0^z \frac{\sin(t)}{t} \mathrm{d}t \tag{75}$$

$$\mathrm{si}(z) \equiv -\int_z^\infty \frac{\sin(t)}{t} \mathrm{d}t = \mathrm{Si}(z) - \frac{\pi}{2} \tag{76}$$

$$\mathrm{Ci}(z) \equiv \mathrm{ci}(z) = -\int_z^\infty \frac{\cos(t)}{t} \mathrm{d}t \tag{77}$$

The auxiliary functions can be defined as

$$f(z) \equiv \mathrm{ci}(z) \sin(z) - \mathrm{si}(z) \cos(z) \tag{78}$$

$$g(z) \equiv -\mathrm{ci}(z) \cos(z) - \mathrm{si}(z) \sin(z) \tag{79}$$

with derivatives

$$\frac{\mathrm{d}f(z)}{\mathrm{d}z} = -g(z) \tag{80}$$

$$\frac{\mathrm{d}g(z)}{\mathrm{d}z} = f(z) - \frac{1}{z} \tag{81}$$

The asymptotic expansions are

$$f(z) \sim \frac{1}{z} \sum_{m=0} (-1)^m \frac{(2m)!}{z^{2m}} \tag{82}$$

$$g(z) \sim \frac{1}{z^2} \sum_{m=0} (-1)^m \frac{(2m+1)!}{z^{2m}} \tag{83}$$

The exponential integral is defined as

$$\mathrm{Ei}(x) = \int_{-\infty}^x \frac{\mathrm{e}^t}{t} \mathrm{d}t \tag{84}$$

$$E_1(x) = \int_x^\infty \frac{\mathrm{e}^{-t}}{t} \mathrm{d}t \tag{85}$$

with $x > 0$. The first expression must be understood as a principal value. We also have

$$\mathrm{Ei}(-x) = -E_1(x) \tag{86}$$

Power series

$$\mathrm{Ei}(x) = \gamma + \log(x) + \sum_{n=1}^\infty \frac{x^n}{n!n} \tag{87}$$

$$E_1(x) = -\gamma - \log(x) - \sum_{n=1}^\infty \frac{(-1)^n x^n}{n!n} \tag{88}$$

$$\mathrm{si}(x) = -\frac{\pi}{2} + \sum_{n=0}^\infty \frac{(-1)^n z^{2n+1}}{(2n+1)!(2n+1)} \tag{89}$$

$$\mathrm{ci}(x) = \gamma + \log(x) + \sum_{n=1}^\infty \frac{(-1)^n z^{2n}}{(2n)!(2n)} \tag{90}$$

Asymptotic expansions

$$\mathrm{Ei}(x) \sim \frac{\mathrm{e}^x}{x} \left(1 + \frac{1!}{x} + \frac{2!}{x^2} + \frac{3!}{x^3} + \dots\right) \tag{91}$$

$$E_1(x) \sim \frac{\mathrm{e}^{-x}}{x} \left(1 - \frac{1!}{x} + \frac{2!}{x^2} - \frac{3!}{x^3} + \dots\right) \tag{92}$$

B Integrals

Some useful integrals (from [GR14], unless otherwise stated).

[GR, p.441, 3.771]

$$\begin{aligned}
&1)\quad \int_0^\infty (\beta^2+x^2)^{\nu-1/2}\sin(ax)\mathrm{d}x = \frac{\sqrt{\pi}}{2}\left(\frac{2\beta}{a}\right)^\nu \Gamma(\nu+1/2)\left[I_{-\nu}(a\beta)-\mathbb{L}_\nu(a\beta)\right]\\
&\qquad \text{with}\quad a>0,\ \Re(\beta)>0,\ \Re(\nu)<1/2,\ \nu\neq -n/2,\ n=1,2,\dots
\end{aligned}\tag{93}$$

$$\begin{aligned}
&2)\quad \int_0^\infty (\beta^2+x^2)^{\nu-1/2}\cos(ax)\mathrm{d}x = \frac{1}{\sqrt{\pi}}\left(\frac{2\beta}{a}\right)^\nu \cos(\pi\nu)\Gamma(\nu+1/2)K_\nu(a\beta)\\
&\qquad \text{with}\quad a,\Re(\beta)>0,\ \Re(\nu)<1/2
\end{aligned}\tag{94}$$

$$\begin{aligned}
&5)\quad \int_0^\infty x(\beta^2+x^2)^{\nu-1/2}\sin(ax)\mathrm{d}x = \frac{\beta}{\sqrt{\pi}}\left(\frac{2\beta}{a}\right)^\nu \cos(\pi\nu)\Gamma(\nu+1/2)K_{\nu+1}(a\beta)\\
&\qquad \text{with}\quad a,\Re(\beta)>0,\ \Re(\nu)<0
\end{aligned}\tag{95}$$

[GR, p.424, 3.723] Both for $a>0$ and $\Re(\beta)>0$

$$3)\quad \int_0^\infty \frac{x\sin(ax)}{\beta^2+x^2}\mathrm{d}x = \frac{\pi}{2}\mathrm{e}^{-a\beta}\tag{96}$$

$$4)\quad \int_0^\infty \frac{x\cos(ax)}{\beta^2+x^2}\mathrm{d}x = -\frac{1}{2}\left[\mathrm{e}^{-a\beta}\overline{\mathrm{Ei}}(a\beta)+\mathrm{e}^{a\beta}\mathrm{Ei}(-a\beta)\right]\tag{97}$$

[GR, p.581, 4.381] Both for $a>0, b>0$ (there seems to be a typo in [GR14], saying $a<0$ for the first one, check the original reference [MBE55])

$$1)\quad \int_0^\infty \log\left|\frac{x+a}{x-a}\right|\sin(bx)\mathrm{d}x = \frac{\pi}{b}\sin(ab)\tag{98}$$

$$2)\quad \int_0^\infty \log\left|\frac{x+a}{x-a}\right|\cos(bx)\mathrm{d}x = \frac{2}{b}\left[\cos(ab)\left\{\mathrm{si}(ab)+\frac{\pi}{2}\right\}-\sin(ab)\mathrm{ci}(ab)\right]\tag{99}$$

These integrals are the base for the regularization of the saddle points and the power-law tails. We need to derive one more result. Starting with the first one, we can rewrite it as

$$\begin{aligned}
\int_0^\infty (\beta^2+x^2)^{\nu-1/2}\sin(ax)\mathrm{d}x = {}&\frac{1}{\sqrt{\pi}}\left(\frac{2\beta}{a}\right)^\nu \sin(\pi\nu)\Gamma(\nu+1/2)K_\nu(a\beta)\\
&-\frac{\sqrt{\pi}}{2}\Gamma(\nu+1/2)\beta^{2\nu}\tilde{\mathbb{M}}_\nu(a\beta)
\end{aligned}\tag{100}$$

from this we can obtain

$$\begin{aligned}
\int_0^\infty x(\beta^2+x^2)^{\nu-1/2}\cos(ax)\mathrm{d}x = {}&-\frac{\beta}{\sqrt{\pi}}\left(\frac{2\beta}{a}\right)^\nu \sin(\pi\nu)\Gamma(\nu+1/2)K_{\nu+1}(a\beta)\\
&-\frac{\beta^{2\nu+1}}{2\nu+1}\left\{1+\sqrt{\pi}\Gamma(\nu+3/2)\left(\frac{a\beta}{2}\right)\tilde{\mathbb{M}}_{\nu+1}(a\beta)\right\}
\end{aligned}\tag{101}$$

More useful integrals, useful for the log-tail regularization, are (this time from [MBE55], vol. I)

$$\begin{aligned}
&*)\quad \int_0^\infty x^{\nu-1}\log(x)\cos(xy) = \Gamma(\nu)y^{-\nu}\cos\left(\frac{\pi\nu}{2}\right)\left[\psi(\nu)-\frac{\pi}{2}\tan\left(\frac{\pi\nu}{2}\right)-\log(y)\right]\\
&\qquad \text{with}\quad 0<\Re(\nu)<1
\end{aligned}\tag{102}$$

$$\begin{aligned}
&*)\quad \int_0^\infty x^{\nu-1}\log(x)\sin(xy) = \Gamma(\nu)y^{-\nu}\sin\left(\frac{\pi\nu}{2}\right)\left[\psi(\nu)+\frac{\pi}{2}\mathrm{cotan}\left(\frac{\pi\nu}{2}\right)-\log(y)\right]\\
&\qquad \text{with}\quad |\Re(\nu)|<1
\end{aligned}\tag{103}$$

$$*) \quad \int_0^\infty x^{\nu-1} \mathrm{e}^{-ax} \log(x) \cos(xy) = \Gamma(\nu)(a^2+y^2)^{-\nu/2} \cos(\nu\alpha) \left\{ \psi(\nu) - \frac{1}{2}\log(a^2+y^2) - \alpha \tan(\nu\alpha) \right\}$$

$$\text{with} \quad \Re(a), \Re(\nu) > 0 \quad \text{and} \quad \alpha \equiv \tan^{-1}(y/a) \tag{104}$$

$$*) \quad \int_0^\infty x^{\nu-1} \mathrm{e}^{-ax} \log(x) \sin(xy) = \Gamma(\nu)(a^2+y^2)^{-\nu/2} \sin(\nu\alpha) \left\{ \psi(\nu) - \frac{1}{2}\log(a^2+y^2) + \alpha \mathrm{cotan}(\nu\alpha) \right\}$$

$$\text{with} \quad \Re(a) > 0,\ \Re(\nu) > -1 \quad \text{and} \quad \alpha \equiv \tan^{-1}(y/a) \tag{105}$$

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