SIS analytic results

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1 General definitions and properties

Following [Tam+22], we define the amplification factor as

$$F(w) = \frac{w}{2\pi\mathrm{i}}\int \mathrm{d}^2x\,\mathrm{e}^{\mathrm{i}w\phi(\boldsymbol{x})} \tag{1}$$

where the Fermat potential (or time delay) is

$$\phi(\boldsymbol{x}) = \frac{1}{2}|\boldsymbol{x}-\boldsymbol{y}|^2 - \psi(\boldsymbol{x}) \tag{2}$$

Fourier-transforming $F(w)/w$ we can define

$$\begin{aligned} I(\tau) &\equiv \frac{1}{2\pi}\int \mathrm{d}^2x \int_{-\infty}^{\infty}\mathrm{d}w\,\mathrm{e}^{\mathrm{i}w(\phi(\boldsymbol{x})-\tau-t_{\min})} && (3)\\ &= \int \mathrm{d}^2x\,\delta(\phi(\boldsymbol{x})-\tau-t_{\mathrm{tmin}}) && (4)\end{aligned}$$

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where we have also defined $t_{\rm min} \equiv \phi_{\rm min}$. For the SIS we have

$$\psi(\boldsymbol{x}) = \psi_0 |\boldsymbol{x}| \equiv \psi_0 x \tag{5}$$

The minimum of the Fermat potential is[1]

$$\begin{align} t_{\rm min} &= -\frac{1}{2}\psi_0(2y+\psi_0) \tag{6}\\ \mu_{\rm min} &= 1+\frac{\psi_0}{y} \tag{7}\\ x_{1,\,\rm min} &= y+\psi_0 \tag{8}\\ x_{2,\,\rm min} &= 0 \tag{9}\end{align}$$

If $y < \psi_0$, there is a second critical point (a saddle point)

$$\begin{align} t_{\rm saddle} &= \frac{1}{2}\psi_0(2y-\psi_0) \tag{10}\\ \mu_{\rm saddle} &= \left|1-\frac{\psi_0}{y}\right| \tag{11}\\ x_{1,\,\rm saddle} &= y-\psi_0 \tag{12}\\ x_{2,\,\rm saddle} &= 0 \tag{13}\end{align}$$

2 Time domain

2.1 Final result for a single SIS

The time-domain integral for the SIS is

$$\begin{align} I_{\rm SIS}(\tau) &= \int \mathrm{d}^2x\, \delta(\phi - t) \nonumber\\ &= \int \mathrm{d}x_1\mathrm{d}x_2\, \delta\left(\frac{1}{2}x_1^2 - x_1 y + \frac{1}{2}x_2^2 + \frac{1}{2}y^2 - \psi(x) - \tau - t_{\rm min}\right) \tag{14}\end{align}$$

where[2]

$$\psi(x) = \psi_0\, x\ , \qquad t_{\rm min} = \frac{1}{2}y^2 - \frac{1}{2}(\psi_0+y)^2 \tag{15}$$

Its solution can be written as

$$I_{\rm SIS}(\tau) = \mathcal{I}(u, R) \tag{16}$$

where u is the "time parameter" and r is just a constant[3]

$$\begin{align} u &\equiv \frac{\sqrt{2\tau}}{\psi_0+y} \tag{17a}\\ R &\equiv \frac{\psi_0-y}{\psi_0+y} \tag{17b}\end{align}$$

The integral can be compactly defined as

$$\begin{align} I_{\rm SIS}(\tau) &= 4\,\mathcal{I}(a,b,c,d)\ , \qquad (a>b>c>d) \nonumber\\ &= \frac{8}{\sqrt{(a-c)(b-d)}}\left\{(b-c)\,\Pi\left(\frac{a-b}{a-c}, r\right) + c\,K(r)\right\}\ , \qquad r\equiv\sqrt{\frac{(a-b)(c-d)}{(a-c)(b-d)}} \tag{18}\end{align}$$

[1] For axisymmetric lenses the magnification can be written as $\mu^{-1} = |1-\psi'/x|\,|1-\psi''|$.

[2] We will sometimes write $\tilde{\psi}_0 = \psi_0/y$.

[3] Notice that $R \sim -1+2\tilde{\psi}_0$ corresponds to the weak lensing regime while $R\sim 1-2/\tilde{\psi}_0$ is the extremely strong lensing regime. $R=1$ is the caustic.

where Π and K are the complete elliptic integrals of the third and first kind, respectively[4]. We must distinguish three regions

- *Region 1*: $(u > 1)$

$$\begin{aligned} a &= 1 + u & c &= 1 - u \\ b &= R + \sqrt{u^2 + R^2 - 1} & d &= R - \sqrt{u^2 + R^2 - 1} \end{aligned}$$

- *Region 2*: $(\sqrt{1 - R^2} < u < 1)$
  - *Case A:* $(R > 0)$

$$\begin{aligned} a &= 1 & c &= \sqrt{1 - u^2} \\ b &= R & d &= -\sqrt{1 - u^2} \end{aligned}$$

  - *Case B:* $(R < 0)$

$$\begin{aligned} a &= 1 & c &= -\sqrt{1 - u^2} \\ b &= \sqrt{1 - u^2} & d &= R \end{aligned}$$

- *Region 3*: $(0 < u < \sqrt{1 - R^2})$

$$\begin{aligned} a &= 1 & c &= R \\ b &= \sqrt{1 - u^2} & d &= -\sqrt{1 - u^2} \end{aligned}$$

2.2 Derivation for region 1

In this region it is more convenient to solve the angular integral first

$$\begin{aligned} I(t) &= \int \mathrm{d}^2x\, \delta(\phi - t) = 2\int_{-1}^{1} \frac{\mathrm{d}z}{\sqrt{1 - z^2}} \int_0^\infty r\,\mathrm{d}r\, \delta\left(\frac{1}{2}r^2 - yrz + \frac{1}{2}y^2 - \psi_0 r - t\right) \\ &= 2\int_{-1}^{1} \frac{\mathrm{d}z}{\sqrt{1 - z^2}} \int_0^\infty \tilde{r}\,\mathrm{d}\tilde{r}\, \delta\left(\frac{1}{2}\tilde{r}^2 - \tilde{r}z + \frac{1}{2} - \tilde{\psi}_0\tilde{r} - \frac{\tau}{y^2} - \frac{t_{\min}}{y^2}\right) \end{aligned} \tag{19}$$

where we have defined $\psi_0 \equiv \psi_0/y$. Finally we get

$$\begin{aligned} I(\tau) &= 2\int_{-1}^{1} \frac{\mathrm{d}z}{\sqrt{1 - z^2}} \int_0^\infty \tilde{r}\,\mathrm{d}\tilde{r}\, \delta\left\{z - \overbrace{\frac{1}{\tilde{r}}\left(\frac{1}{2}\tilde{r}^2 - \tilde{\psi}_0\tilde{r} + \frac{1}{2} - \frac{\tau}{y^2} - \frac{t_{\min}}{y^2}\right)}^{z_*}\right\} \\ &= 2\int_0^\infty \frac{\mathrm{d}\tilde{r}}{\sqrt{1 - z_*^2}}\Theta(1 - z_*^2) \end{aligned} \tag{20}$$

We will introduce the following definitions

$$\begin{aligned} a^2 &\equiv \frac{2\tau}{y^2} & \delta &\equiv \tilde{\psi}_0 + 1 \\ z_*^2 - 1 &\equiv \frac{1}{4\tilde{r}^2}\phi_A\phi_B & \delta_{-1} &\equiv \tilde{\psi}_0 - 1 \end{aligned} \tag{21}$$

where (using also $t_{\min}/y^2 = (1 - \delta^2)/2$)

$$\phi_A \equiv 2\tilde{r}(z_* - 1) = \tilde{r}^2 - 2\delta\tilde{r} - (a^2 - \delta^2) \tag{22}$$

$$\phi_B \equiv 2\tilde{r}(z_* + 1) = \tilde{r}^2 - 2\delta_{-1}\tilde{r} - (a^2 - \delta^2) \tag{23}$$

[4]See the appendix for the definitions and the reference for the integral.

The integral can then be rewritten as

$$I(\tau) = 4\int_0^\infty \frac{\tilde{r}\mathrm{d}\tilde{r}}{\sqrt{-\phi_A\phi_B}}\Theta(-\phi_A\phi_B) \tag{24}$$

Equations (20) and (24) are general for any axisymmetric lense. The main simplification for the SIS comes from the fact that ϕ_A and ϕ_B are quadratic and can then be easily factorized, which will allow us to solve the Θ function. In particular, if $a > \delta$ (definition of region 1), we have

$$\phi_A = (r - r_+^A)(r - r_-^A)\ , \qquad r_\pm^A = \delta \pm a \tag{25}$$

$$\begin{aligned} \phi_B = (r - r_+^B)(r - r_-^B)\ , \qquad r_\pm^B &= \delta_{-1} \pm \sqrt{\delta_{-1}^2 + a^2 - \delta^2} \\ &= \delta - 2 \pm \sqrt{a^2 - 4\tilde{\psi}_0} \end{aligned} \tag{26}$$

In this regime, we can also prove that[5]

$$r_+^A > r_+^B > r_-^A > r_-^B \tag{27}$$

Since $r_-^A < 0$, inside the integral we can only have $r_+^A > r > r_+^B > r_-^A > r_-^B$. Then we get

$$\begin{aligned} I(\tau) &= 4\int_{r_+^B}^{r_+^A} \frac{r\mathrm{d}r}{\sqrt{(r_+^A - r)(r - r_+^B)(r - r_-^A)(r - r_-^B)}} \\ &= 4\mathcal{I}(r_+^A, r_+^B, r_-^A, r_-^B) \end{aligned} \tag{28}$$

2.3 Derivation for regions 2 and 3

For these two regions[6], it is more convenient to solve the radial integral first. Defining $\delta_z \equiv \tilde{\psi}_0 + z$, we have

$$I(\tau) = 2\int_{-1}^1 \frac{\mathrm{d}z}{\sqrt{1-z^2}}\int_0^\infty r\,\mathrm{d}r\,\delta\left(\frac{1}{2}r^2 - \delta_z r + \frac{1}{2}(\delta^2 - a^2)\right) \tag{29}$$

Solving for r, we get

$$I(\tau) = 2\sum_*\int_{-1}^1 \frac{\mathrm{d}z}{\sqrt{1-z^2}}\frac{r_*\Theta(r_*)}{|r_* - \delta_z|} \tag{30}$$

where

$$\begin{aligned} r_* &= \delta_z \pm \sqrt{\delta_z^2 - \delta^2 + a^2} \\ &= \delta_z \pm \sqrt{(z - z_*^+)(z - z_*^-)}\ , \qquad z_*^\pm = -\tilde{\psi}_0 \pm \sqrt{\delta^2 - a^2} \end{aligned}$$

To compute the limits of the integral we must analyze when the solution exists and is positive. It is useful to study the behaviour of $z_*^\pm$ beforehand. We can prove that $\frac{\mathrm{d}z_*^\pm}{\mathrm{d}\tilde{\psi}_0} \neq 0$ for $0 < a < \delta$ so

$$\begin{aligned} z_*^+ &\in \Big(-\tilde{\psi}_0\ (\text{for } a = \delta),\ 1\ (\text{for } a = 0)\Big) \\ z_*^- &\in \Big(-1 - 2\tilde{\psi}_0\ (\text{for } a = 0),\ -\tilde{\psi}_0\ (\text{for } a = \delta)\Big) \end{aligned}$$

[5] It is easy to see that $r_+^A > r_+^B > (r_-^A, r_-^B)$. For the second inequality, we can write

$$r_-^A - r_-^B = 2 - a + \sqrt{(2-a)^2 + 4(a - \delta)} > 0 \quad \to \quad r_-^A > r_-^B$$

[6] $2\sqrt{\tilde{\psi}_0} < a < \delta$ and $0 < a < 2\sqrt{\tilde{\psi}_0}$

and for

$$
\begin{aligned}
&\text{if } \tilde{\psi}_0 > 1 \quad \to \quad z_*^+ < -1 \quad \text{for} \quad a > 2\sqrt{\tilde{\psi}_0} \\
&\text{if } \tilde{\psi}_0 < 1 \quad \to \quad z_*^- > -1 \quad \text{for} \quad a > 2\sqrt{\tilde{\psi}_0}
\end{aligned}
$$

We are ready to write the final result.

- *Region 2:* $\delta > a > 2\sqrt{\tilde{\psi}_0}$
  - *Case A:* $\tilde{\psi}_0 > 1, \quad (1 > z > -1 > z_*^+ > z_*^-)$

$$
\begin{aligned}
I(\tau) &= 4\int_{-1}^{1} \frac{(z+\tilde{\psi}_0)}{\sqrt{(1-z)(z+1)(z-z_*^+)(z-z_*^-)}} \\
&= 4\mathcal{I}(1,-1,z_*^+,z_*^-) + 4\tilde{\psi}_0\mathcal{J}(1,-1,z_*^+,z_*^-)
\end{aligned}
$$

  - *Case B:* $\tilde{\psi}_0 < 1, \quad (1 > z > z_*^+ > z_*^- > -1)$

$$
\begin{aligned}
I(\tau) &= 4\int_{z_*^+}^{1} \frac{(z+\tilde{\psi}_0)}{\sqrt{(1-z)(z-z_*^+)(z-z_*^-)(z+1)}} \\
&= 4\mathcal{I}(1,z_*^+,z_*^-,-1) + 4\tilde{\psi}_0\mathcal{J}(1,z_*^+,z_*^-,-1)
\end{aligned}
$$

- *Region 3:* $0 < a < 2\sqrt{\tilde{\psi}_0}, \quad (1 > z > z_*^+ > -1 > z_*^-)$

$$
\begin{aligned}
I(\tau) &= 4\int_{z_*^+}^{1} \frac{(z+\tilde{\psi}_0)}{\sqrt{(1-z)(z-z_*^+)(z+1)(z-z_*^-)}} \\
&= 4\mathcal{I}(1,z_*^+,-1,z_*^-) + 4\tilde{\psi}_0\mathcal{J}(1,z_*^+,-1,z_*^-)
\end{aligned}
$$

3 Frequency domain

The amplification factor for the SIS can be written as

$$
\begin{aligned}
F(w) &\equiv \frac{w}{2\pi\mathrm{i}}\int \mathrm{d}^2x\, \mathrm{e}^{\mathrm{i}w\phi(\boldsymbol{x})} &(31)\\
&= \frac{w}{2\pi\mathrm{i}}\int_0^{2\pi}\int_0^{\infty} r\mathrm{d}r \exp\left\{\mathrm{i}w\left(\frac{1}{2}r^2 - ry\cos\theta + \frac{1}{2}y^2 - \psi_0 r\right)\right\} &(32)\\
&= \mathrm{e}^{\mathrm{i}wy^2/2}\frac{w}{\mathrm{i}\pi}\int_0^{\pi}\mathrm{d}\theta\left\{\int_0^{\infty} r\mathrm{d}r\cos\left(\frac{w}{2}r^2 - wr(\psi_0 + y\cos\theta)\right)\right. \\
&\qquad \left. + \mathrm{i}\int_0^{\infty} r\mathrm{d}r\sin\left(\frac{w}{2}r^2 - wr(\psi_0 + y\cos\theta)\right)\right\} &(33)
\end{aligned}
$$

Defining now

$$\alpha(\theta) \equiv \alpha_0(1 + r\cos\theta) \tag{34}$$

$$\alpha_0 \equiv \psi_0\sqrt{w/\pi} \tag{35}$$

$$r \equiv y/\psi_0 \tag{36}$$

and using the results in the appendix, we have

$$F(w) = \mathrm{e}^{\mathrm{i}wy^2/2}\left\{1 + \int_0^{\pi}\mathrm{d}\theta\,\alpha f(-\alpha) - \mathrm{i}\int_0^{\pi}\mathrm{d}\theta\,\alpha g(-\alpha)\right\} \tag{37}$$

Using the symmetry properties of the Fresnel auxiliary functions we can rewrite the two integrals in a more convenient form

$$\int_0^\pi \mathrm{d}\theta\, \alpha g(-\alpha) \overset{(r\leq 1)}{=} -I_g(1-r, 1+r, \alpha_0) + \frac{\alpha_0}{2} I_s(1-r, 1+r, \alpha_0^2) + \frac{\alpha_0}{2} I_c(1-r, 1+r, \alpha_0^2) \quad (38)$$

$$\overset{(r\geq 1)}{=} -J_g(r-1, r+1, \alpha_0) - J_g(r+1, r-1, \alpha_0) + \frac{\alpha_0}{2} J_s(r-1, r+1, \alpha_0^2) + \frac{\alpha_0}{2} J_c(r-1, r+1, \alpha_0^2) \quad (39)$$

$$\int_0^\pi \mathrm{d}\theta\, \alpha f(-\alpha) \overset{(r\leq 1)}{=} -I_f(1-r, 1+r, \alpha_0) - \frac{\alpha_0}{2} I_s(1-r, 1+r, \alpha_0^2) + \frac{\alpha_0}{2} I_c(1-r, 1+r, \alpha_0^2) \quad (40)$$

$$\overset{(r\geq 1)}{=} -J_f(r-1, r+1, \alpha_0) - J_f(r+1, r-1, \alpha_0) - \frac{\alpha_0}{2} J_s(r-1, r+1, \alpha_0^2) + \frac{\alpha_0}{2} J_c(r-1, r+1, \alpha_0^2) \quad (41)$$

Each of the integrals is defined as

$$I_s(a, b, \omega) \equiv \int_{a^2}^{b^2} \frac{\sin(\omega u)\mathrm{d}u}{\sqrt{(b-\sqrt{u})(\sqrt{u}-a)}} = \int_a^b \frac{2\beta \sin(\omega\beta^2)\mathrm{d}\beta}{\sqrt{(b-\beta)(\beta-a)}} \quad (42)$$

$$J_s(a, b, \omega) \equiv \int_0^{b^2} \frac{\sin(\omega u)\mathrm{d}u}{\sqrt{(b-\sqrt{u})(\sqrt{u}+a)}} = \int_0^b \frac{2\beta \sin(\omega\beta^2)\mathrm{d}\beta}{\sqrt{(b-\beta)(\beta+a)}} \quad (43)$$

$$I_g(a, b, \alpha) \equiv \int_a^b \frac{\mathrm{d}\beta}{\sqrt{(b-\beta)(\beta-a)}} \tilde{g}(\alpha\beta) \quad (44)$$

$$J_g(a, b, \alpha) \equiv \int_0^b \frac{\mathrm{d}\beta}{\sqrt{(b-\beta)(\beta+a)}} \tilde{g}(\alpha\beta) \quad (45)$$

with $(a, b, \alpha, \omega > 0)$ and the obvious substitutions $\sin \to \cos$ and $g \to f$ for the other half. Using

$$\beta = \frac{b+a}{2} + \frac{b-a}{2}\cos\theta \quad \to \quad \frac{\mathrm{d}\beta}{\sqrt{(b-\beta)(\beta-a)}} = -\mathrm{sign}(\theta)\mathrm{d}\theta \quad (46)$$

with $\beta = b \to \theta = 0$ and $\beta = a \to \theta = \pi$, we can recover the 'angular' integrals. So far, it seems that the best strategy to integrate these expressions at high frequencies is to leave I_f and I_g into ther 'angular' forms and to rewrite the other two as

$$I_s(a, b, w) = \int_{(a+\Delta)^2}^{(b-\Delta)^2} \frac{\sin(\omega u)\mathrm{d}u}{\sqrt{(b-\sqrt{u})(\sqrt{u}-a)}} + 2\int_0^{\theta_b} \beta(\theta) \sin\left(\omega\beta^2(\theta)\right) \mathrm{d}\theta + 2\int_{\theta_a}^{\pi} \beta(\theta) \sin\left(\omega\beta^2(\theta)\right) \mathrm{d}\theta \quad (47)$$

where $\beta(\theta)$ is defined above and

$$\theta_b = \arccos\left(1 - \frac{2}{b-a}\Delta\right) \quad (48)$$

$$\theta_a = \arccos\left(-1 + \frac{2}{b-a}\Delta\right) \quad (49)$$

$$(50)$$

We can choose Δ as

$$\Delta = \sqrt{\frac{\pi n}{\omega}}\,, \qquad n < \frac{(b-a)^2\omega}{4\pi} \quad (51)$$

In our case $b = 1 + r$ and $a = 1 - r$ so we can write

$$\beta = 1 + r\cos(\theta) \tag{52}$$
$$\theta_b = \arccos(1 - \Delta/r) \tag{53}$$
$$\theta_a = \arccos(-1 + \Delta/r) \tag{54}$$
$$n < r^2 \frac{\omega}{\pi} \tag{55}$$

For the weak lensing case we can play the same game

$$J_s(a,b,\omega) = \int_0^{(b-\Delta)^2} \frac{\sin(\omega u)\mathrm{d}u}{\sqrt{(b-\sqrt{u})(\sqrt{u}+a)}} + 2\int_0^{\theta_b} \beta(\theta)\sin\left(\omega\beta^2\right)\mathrm{d}\theta \tag{56}$$

where

$$\beta(\theta) = \frac{b-a}{2} + \frac{b+a}{2}\cos(\theta) \tag{57}$$
$$\theta_b = \arccos\left(1 - \frac{2}{b+a}\Delta\right) \tag{58}$$
$$\theta_0 = \arccos\left(-\frac{b-a}{b+a}\right) \tag{59}$$
$$\Delta = \sqrt{\frac{\pi n}{\omega}}\,, \qquad n < \frac{(b-a)^2\omega}{4\pi} \tag{60}$$

Now we must distinguish two cases (for J_f and J_g). First, if $b = r - 1$ and $a = r + 1$ we have

$$\beta(\theta) = -1 + r\cos(\theta) \tag{61}$$
$$\theta_b = \arccos(1 - \Delta/r) \tag{62}$$
$$\theta_0 = \arccos(1/r) \tag{63}$$

and n is the same as before. The second case is $b = r + 1$ and $a = r - 1$

$$\beta(\theta) = 1 + r\cos(\theta) \tag{64}$$
$$\theta_b = \arccos(1 - \Delta/r) \tag{65}$$
$$\theta_0 = \arccos(-1/r) \tag{66}$$

A Special functions

A.1 Elliptic integrals

From [GR14] we have

$$\begin{aligned}
\mathcal{I}(a,b,c,d) &\equiv \int_b^a \frac{x\mathrm{d}x}{\sqrt{(a-x)(b-x)(x-c)(x-d)}}\,, \qquad &&\text{[GR, 3.148.6, p.276]}\\
&= \frac{2}{\sqrt{(a-c)(b-d)}}\left\{(b-c)\,\Pi\left(\frac{a-b}{a-c}, r\right) + c\,K(r)\right\}\\
\mathcal{J}(a,b,c,d) &\equiv \int_b^a \frac{\mathrm{d}x}{\sqrt{(a-x)(b-x)(x-c)(x-d)}}\,, \qquad &&\text{[GR, 3.147.6, p.275]}\\
&= \frac{2}{\sqrt{(a-c)(b-d)}}\,K(r)
\end{aligned}$$

both for $(a > b > c > d)$ and where

$$r = \sqrt{\frac{(a-b)(c-d)}{(a-c)(b-d)}}$$

The complete elliptic integrals are defined as

$$\begin{aligned}
K(k) &\equiv \int_0^{\pi/2} \frac{\mathrm{d}\alpha}{\sqrt{1-k^2\sin^2\alpha}} \\
E(k) &\equiv \int_0^{\pi/2} \sqrt{1-k^2\sin^2\alpha}\,\mathrm{d}\alpha \\
\Pi(n,k) &\equiv \int_0^{\pi/2} \frac{\mathrm{d}\alpha}{(1-n\sin^2\alpha)\sqrt{1-k^2\sin^2\alpha}}
\end{aligned}$$

All three are equal to $\pi/2$ at the origin $n = k = 0$.

A.2 Fresnel integrals

We will follow the DLMF notation [DLM23] for the Fresnel integrals

$$C(z) \equiv \int_0^z \cos\left(\frac{1}{2}\pi t^2\right) \tag{67}$$

$$S(z) \equiv \int_0^z \sin\left(\frac{1}{2}\pi t^2\right) \tag{68}$$

Other commonly used definitions are

$$C_1(z) \equiv C(\sqrt{2/\pi z})\ , \qquad C_2(z) \equiv C(\sqrt{2z/\pi}) \tag{69}$$

$$S_1(z) \equiv S(\sqrt{2/\pi z})\ , \qquad S_2(z) \equiv S(\sqrt{2z/\pi}) \tag{70}$$

The auxiliary functions are defined as

$$f(z) \equiv \left(\frac{1}{2} - S(z)\right)\cos\left(\frac{\pi}{2}z^2\right) - \left(\frac{1}{2} - C(z)\right)\sin\left(\frac{\pi}{2}z^2\right) \tag{71}$$

$$g(z) \equiv \left(\frac{1}{2} - C(z)\right)\cos\left(\frac{\pi}{2}z^2\right) + \left(\frac{1}{2} - S(z)\right)\sin\left(\frac{\pi}{2}z^2\right) \tag{72}$$

with the symmetries

$$f(-z) = \cos\left(\frac{\pi}{2}z^2\right) - \sin\left(\frac{\pi}{2}z^2\right) - f(z) \tag{73}$$

$$g(-z) = \cos\left(\frac{\pi}{2}z^2\right) + \sin\left(\frac{\pi}{2}z^2\right) - g(z) \tag{74}$$

and derivatives

$$f'(z) = -\pi z g(z) \tag{75}$$

$$g'(z) = \pi z f(z) - 1 \tag{76}$$

From [GR14]

$$\int_0^\infty \sin(ax^2 + 2bx)\mathrm{d}x = \sqrt{\frac{\pi}{2a}} f\left(\sqrt{\frac{2}{\pi}}\frac{b}{\sqrt{a}}\right) \qquad \text{[GR, 3.693.1, p.417]} \tag{77}$$

$$\int_0^\infty \cos(ax^2 + 2bx)\mathrm{d}x = \sqrt{\frac{\pi}{2a}} g\left(\sqrt{\frac{2}{\pi}}\frac{b}{\sqrt{a}}\right) \qquad \text{[GR, 3.693.2, p.417]} \tag{78}$$

We can derive both with respect to b to obtain

$$\int_0^\infty x\sin(ax^2+2bx)\mathrm{d}x = \frac{1}{2a} - \frac{\pi}{2a}\left(\sqrt{\frac{2}{\pi}}\frac{b}{\sqrt{a}}\right) f\left(\sqrt{\frac{2}{\pi}}\frac{b}{\sqrt{a}}\right) \tag{79}$$

$$\int_0^\infty x\cos(ax^2+2bx)\mathrm{d}x = -\frac{\pi}{2a}\left(\sqrt{\frac{2}{\pi}}\frac{b}{\sqrt{a}}\right) g\left(\sqrt{\frac{2}{\pi}}\frac{b}{\sqrt{a}}\right) \tag{80}$$

We will also use the notation

$$\tilde{g}(z) \equiv zg(z)\,, \qquad \tilde{f}(z) \equiv zf(z) \tag{81}$$

We can evaluate the Fresnel integrals using the approximate formulas derived in [Boe60]. In this reference, an approximate expression is found to evaluate

$$f_B(x) = C_2(x) - \mathrm{i}S_2(x) \tag{82}$$

that we can also write as

$$\mathrm{e}^{\mathrm{i}x} f_B(x) = \cos(x)C_2(x) + \sin(x)S_2(x) - \mathrm{i}\left(\cos(x)S_2(x) - \sin(x)C_2(x)\right) \tag{83}$$

The auxiliary functions can be expressed in terms of this combination as

$$f\left(\sqrt{\frac{2x}{\pi}}\right) = \frac{1}{2}\left(\cos(x) - \sin(x)\right) + \Im(f_b\mathrm{e}^{\mathrm{i}x}) \tag{84}$$

$$g\left(\sqrt{\frac{2x}{\pi}}\right) = \frac{1}{2}\left(\cos(x) + \sin(x)\right) - \Re(f_B\mathrm{e}^{\mathrm{i}x}) \tag{85}$$

So, we can use the results from this reference to approximate f and g as

$$g(x) = \frac{1}{\sqrt{2}}\sin\left(\frac{\pi x^2}{2} + \frac{\pi}{4}\right) - \sqrt{\frac{\pi x^2}{8}}\sum_{n=0}^{11} a_n \left(\frac{\pi x^2}{8}\right)^n \tag{86}$$

$$f(x) = \frac{1}{\sqrt{2}}\cos\left(\frac{\pi x^2}{2} + \frac{\pi}{4}\right) + \sqrt{\frac{\pi x^2}{8}}\sum_{n=0}^{11} b_n \left(\frac{\pi x^2}{8}\right)^n \tag{87}$$

for $0 \leq x \leq 2\sqrt{2/\pi}$, and

$$g(x) = -\sqrt{\frac{8}{\pi x^2}}\sum_{n=0}^{11} c_n \left(\frac{\pi x^2}{8}\right)^n \tag{88}$$

$$f(x) = \sqrt{\frac{8}{\pi x^2}}\sum_{n=0}^{11} d_n \left(\frac{\pi x^2}{8}\right)^n \tag{89}$$

for $x \geq 2\sqrt{2/\pi}$. The coefficients can be found in the original reference. From my experiments, these yield a maximum absolute error of 10^{-8} and a maximum relative error 10^{-6}.

B Sketch of derivation for two SIS with additional symmetries

The main trick for solving the SIS was to convert the integral into a factorization problem for quadratic equations. The same trick can be applied to other systems, in particular to a lense composed of two SIS

$$\phi(\boldsymbol{x},\boldsymbol{y}) = \frac{1}{2}|\boldsymbol{x}-\boldsymbol{y}|^2 - \psi_1(|\boldsymbol{x}-\boldsymbol{L}/2|) - \psi_2(|\boldsymbol{x}+\boldsymbol{L}/2|) \tag{90}$$

in two particular configurations:

- *Configuration x_1:* Both lenses aligned with $\boldsymbol{y}$, i.e. in the $\hat{x}_1$ axis.
- *Configuration x_2:* When $\boldsymbol{y}$ and the vector $\boldsymbol{L}$ joining the center of the two lenses are perpendicular, i.e. both lenses at the same x_1 and situated at symmetric positions $x_2^{\rm obs} + L/2$ and $x_2^{\rm obs} - L/2$ with respect to the observer (only for equal lenses).

B.1 Alternative coordinate systems

Two-center bipolar coordinates. r_1 is the distance with respect to a point situated in $(L/2, 0)$ and r_2 with respect to a point in $(-L/2, 0)$

$$x_1 = \frac{r_2^2 - r_1^2}{2L} \tag{91}$$

$$x_2 = \pm\frac{1}{2L}\sqrt{4L^2 r_2^2 - (r_2^2 - r_1^2 + L^2)^2} \tag{92}$$

The variable x_2 can be rewritten in a number of ways

$$\begin{aligned} x_2 &= \pm\frac{1}{2L}\sqrt{(r_2^2 - r_1^2 + L^2 + 2Lr_2)(r_1^2 - r_2^2 + 2Lr_2 - L^2)} && (93)\\ &= \pm\frac{1}{2L}\sqrt{(r_2 + r_1 + L)(r_2 - r_1 + L)(r_1 + r_2 - L)(r_1 - r_2 + L)} && (94)\\ &= \pm\sqrt{\frac{1}{2}(r_1^2 + r_2^2) - \frac{1}{4}L^2 - \frac{1}{4L^2}(r_2^2 - r_1^2)} && (95) \end{aligned}$$

The limits are transformed as (check)

$$r_1^{\rm min} = |r_2 - L| \tag{96}$$

$$r_1^{\rm max} = r_2 + L \tag{97}$$

Jacobian

$$\frac{\partial x_1}{\partial r_1} = -\frac{r_1}{L} \qquad \frac{\partial x_2}{\partial r_1} = \frac{2r_1}{x_2}(r_2^2 - r_1^2 + L^2) \tag{98}$$

$$\frac{\partial x_1}{\partial r_2} = \frac{r_2}{L} \qquad \frac{\partial x_2}{\partial r_2} = \frac{4L^2 r_2}{x_2} - \frac{2r_2}{x_2}(r_2^2 - r_1^2 + L^2) \tag{99}$$

so that

$$J = \frac{4Lr_1r_2}{|x_2|} \tag{100}$$

We also have

$$\frac{1}{2}(x_1^2 + x_2^2) = \frac{1}{4}(r_1^2 + r_2^2) - \frac{1}{8}L^2 \tag{101}$$

Rotated version

$$u \equiv \frac{1}{\sqrt{2}}(r_1 + r_2)\ , \qquad r_2 = \frac{1}{\sqrt{2}}(v + u) \tag{102}$$

$$v \equiv \frac{1}{\sqrt{2}}(r_2 - r_1)\ , \qquad r_1 = \frac{1}{\sqrt{2}}(u - v) \tag{103}$$

The Cartesian coordinates in this system are

$$x_1 = \frac{uv}{L} \tag{104}$$

$$\begin{aligned} x_2 &= \pm\sqrt{\frac{1}{2}(u^2 + v^2) - \frac{1}{4}L^2 - \frac{1}{L^2}u^2v^2} && (105)\\ &= \pm\sqrt{(u + L)(v + L)(u - L)(L - v)} && (106) \end{aligned}$$

and some other useful combinations are

$$r_2^2 - r_1^2 = 2uv \tag{107}$$

$$r_1 r_2 = -\frac{1}{2}(v^2 - u^2) \tag{108}$$

$$= -\frac{1}{2}\Big((v+L)(v-L) - (u+L)(u-L)\Big) \tag{109}$$

B.2 Configuration $\boldsymbol{x_1}$

In this case, it is more convenient to use the two-center bipolar coordinates.

$$\begin{aligned}\phi(x_1, x_2, y) &= \frac{1}{2}(x_1^2 + x_2^2) + \frac{1}{2}y^2 - x_1 y - \psi_1 - \psi_2 \\ &= \frac{1}{4}(r_1^2 + r_2^2) - \frac{1}{8}L^2 + \frac{1}{2}y^2 - \frac{y}{2L}(r_2^2 - r_1^2) - \psi_1(r_1) - \psi_2(r_2) \\ &= \frac{1}{4}(A_+ r_1^2 + A_- r_2^2) - \psi_1(r_1) - \psi_2(r_2) - \frac{L^2}{8}A_+ A_- \end{aligned} \tag{110}$$

where $A_\pm \equiv 1 \pm 2y/L$. The complete integral is

$$I(t) = \int \mathrm{d}^2 x\, \delta(\phi - t) \tag{111}$$

$$= 8L \int_0^\infty \mathrm{d}r_2 \int_{|r_2 - L|}^{r_2 + L} \mathrm{d}r_1 \frac{r_1 r_2}{|x_2|} \delta(\phi - t) \tag{112}$$

If the first lense is a SIS ($\psi_1 = \psi_0^1 r_1$), we can solve the δ function and reduce the problem to a single integral for any axisymmetric lense ψ_2. Furthermore, if both are SIS, we only have to deal with quadratic equations to obtain the limits, so the problem can be completely solved.

B.3 Configuration $\boldsymbol{x_2}$

This problem can be reduced to a (bi-)quadratic form using the rotated version of the two-center bipolar coordinates.

$$\begin{aligned}\phi(x_1, x_2, y) &= \frac{1}{2}(x_1^2 + x_2^2) + \frac{1}{2}y^2 - x_2 y - \psi_1 - \psi_2 \\ &= \frac{1}{4}(r_1^2 + r_2^2) - \frac{1}{8}L^2 + \frac{1}{2}y^2 - y x_2 - \psi_1(r_1) - \psi_2(r_2) \\ &= \frac{1}{4}(u^2 + v^2) - \frac{1}{8}L^2 + \frac{1}{2}y^2 \mp \frac{y}{4}\sqrt{\left(\frac{2u^2}{L^2} - 1\right)\left(1 - \frac{2v^2}{L^2}\right)} - \psi_1(r_1) - \psi_2(r_2) \end{aligned} \tag{113}$$

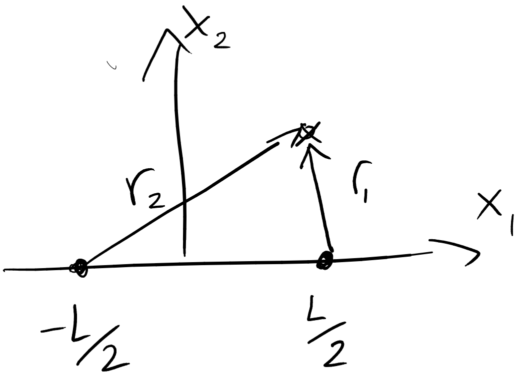
The key simplification for two equal-mass SIS is

$$\psi_1(r_1) + \psi_2(r_2) = \psi_0 r_1 + \psi_0 r_2 = \sqrt{2}\psi_0 u \tag{114}$$

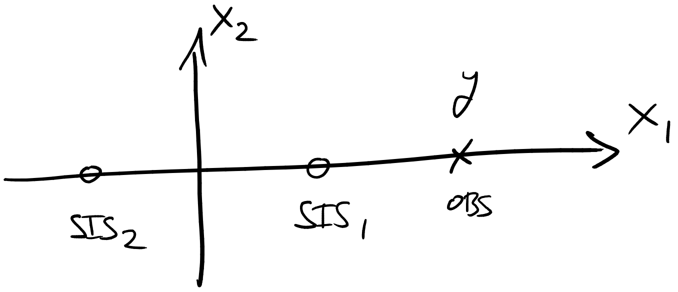
so

$$\phi = \frac{1}{4}(u^2 + v^2) - \frac{1}{8}L^2 + \frac{1}{2}y^2 \mp \frac{y}{4}\sqrt{\left(\frac{2u^2}{L^2} - 1\right)\left(1 - \frac{2v^2}{L^2}\right)} - \sqrt{2}\psi_0 u \tag{115}$$

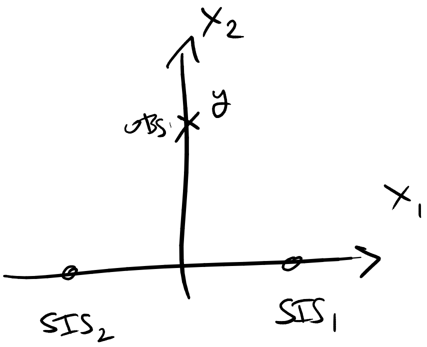
Written in this way, when solving for $\phi = t$ we will get a biquadratic equation for v.



Two-center bipolar coordinates



Configuration x_1



Configuration x_2

($SIS_1 = SIS_2$)

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