Contents

1 Introduction

In this project a simulation of the control law algorithm of a satellite on orbit will be simulated. The dynamics are going to be modeled using classical physics and a disturbance created by the Earth's gravitational force will be added.

The development of this project has several parts. The theory of the control law will be explained and later a simulation will be done. This project is based on the development done in the book "Robust Autonomous Guidance" and has been presented as the bachelor's thesis of the writer in his studies of Industrial engineering.

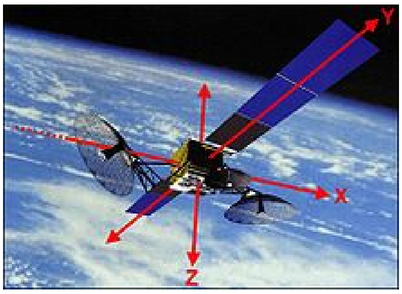


Figure 1: Satellite

2 Main ideas

2.1 Position and velocity

2.1.1 Matrix rotation

A reference system F is defined as $F = \{O, \overrightarrow{i}, \overrightarrow{j}, \overrightarrow{k}\}$ where it is defined a reference center and a canonical basis in R^3,

$$i = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad j = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad k = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

A rotation matrix is used to show the difference between two attitude positions expressed in two different coordinate systems. The first system is called $F_a = \{O_a, \overrightarrow{i_a}, \overrightarrow{j_a}, \overrightarrow{k_a}\}$ and the second one $F_b = \{O_b, \overrightarrow{i_b}, \overrightarrow{j_b}, \overrightarrow{k_b}\}$. The relative attitude between F_b and F_a is given by the rotation matrix:

$$R_{ab} = \begin{pmatrix} \overrightarrow{i_b} \cdot \overrightarrow{i_a} & \overrightarrow{j_b} \cdot \overrightarrow{i_a} & \overrightarrow{k_b} \cdot \overrightarrow{i_a} \\ \overrightarrow{i_b} \cdot \overrightarrow{j_a} & \overrightarrow{j_b} \cdot \overrightarrow{j_a} & \overrightarrow{k_b} \cdot \overrightarrow{j_a} \\ \overrightarrow{i_b} \cdot \overrightarrow{k_a} & \overrightarrow{j_b} \cdot \overrightarrow{k_a} & \overrightarrow{k_b} \cdot \overrightarrow{k_a} \end{pmatrix}$$

This rotation matrix is an element of the special orthogonal group $SO(3) \subset R^{3x3}$, this is, the set:

$$SO(3) = \{R \in R^{3x3} : RR^T = I, det(R) = 1\}$$

And gives a relationship between two unit vectors expressed in two different coordinate systems:

$$\begin{pmatrix} \overrightarrow{i_b} & \overrightarrow{j_b} & \overrightarrow{k_b} \end{pmatrix} = R_{ab} \begin{pmatrix} \overrightarrow{i_a} & \overrightarrow{j_a} & \overrightarrow{k_a} \end{pmatrix}$$

A generic vector $\overrightarrow{v}$ si given in F_a and F_b by:

$$\overrightarrow{v} = v_1^a \overrightarrow{i_a} + v_2^a \overrightarrow{j_a} + v_3^a \overrightarrow{k_a}$$

and

$$\overrightarrow{v} = v_1^b \overrightarrow{i_b} + v_2^b \overrightarrow{j_b} + v_3^b \overrightarrow{k_b}$$

The relationship betwwen these two vectors:

$$v^a = \begin{pmatrix} v_1^a \\ v_2^a \\ v_3^a \end{pmatrix}, \quad v^b = \begin{pmatrix} v_1^b \\ v_2^b \\ v_3^b \end{pmatrix}$$

is given by the rotation matrix:

$$v^a = R_{ab} v^b$$

2.1.2 Quaternions

Another way to express the attitude of two different coordinate systems is given by the use of quaternions. These are a quadruple of real numbers (q_0, q_1, q_2, q_3) which satisfies the following:

$$\sum q_i^2 = 1$$

And which is defined by the set:

$$S_4 = \{x \in R^4 : \left|\left|x\right|\right| = 1\}$$

Each quaternion is expressed by a scalar number and a vector with dimension 3:

$$\mathbf{q} = \begin{pmatrix} q_0 \\ q \end{pmatrix}$$

where q_0 is the scalar and

$$q = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}$$

the vector.

The rotation matrix defined previously is easy to visualize, however, quaternions are not easy to visualized since it is defined in the 4th dimension. The main idea is to realize that each set of 4 numbers shows a particular attitude, quaternions are easy to manipulate and compute.

For each $\mathbf{q}$ there is a matrix given by:

$$R(\mathbf{q}) = \begin{pmatrix} 1 - 2\mathrm{q}_2^2 - 2\mathrm{q}_3^2 & 2\mathrm{q}_1\mathrm{q}_2 - 2\mathrm{q}_0\mathrm{q}_3 & 2\mathrm{q}_1\mathrm{q}_3 + 2\mathrm{q}_0\mathrm{q}_2 \\ 2\mathrm{q}_1\mathrm{q}_2 + 2\mathrm{q}_0\mathrm{q}_3 & 1 - 2\mathrm{q}_1^2 - 2\mathrm{q}_3^2 & 2\mathrm{q}_2\mathrm{q}_3 - 2\mathrm{q}_0\mathrm{q}_1 \\ 2\mathrm{q}_1\mathrm{q}_3 - 2\mathrm{q}_0\mathrm{q}_2 & 2\mathrm{q}_2\mathrm{q}_3 + 2\mathrm{q}_0\mathrm{q}_1 & 1 - 2\mathrm{q}_1^2 - 2\mathrm{q}_2^2 \end{pmatrix}$$

which satisfies $R^T(\mathbf{q})R(\mathbf{q}) = I$ and $det(R(\mathbf{q})) = 1$. In the same way, it can be shown that for each rotation matrix R there is a quaternion $\mathbf{q}$which satisfies:

$$R = R(\mathbf{q})$$

2.1.3 Attitude dynamics: Rotation matrix and quaternions

If a coordinate system F_b changes in time in regard to F_a the rotation matrix R_{ab} will change in time. This relationship is given by:

$$\dot{R}_{ab} = R_{ab}Skew(w_{ab}^b) = R_{ab} \times w_{ab}^b$$

In the same way the dynamics can be expressed using quaternions as:

$$\dot{\mathbf{q}} = \frac{1}{2}E(\mathbf{q})w$$

where

$$E(q) = \begin{pmatrix} -q^T \\ q_0 I + Skew(q) \end{pmatrix} = \begin{pmatrix} -q_1 & -q_2 & -q_3 \\ q_0 & -q_3 & q_2 \\ q_3 & q_0 & -q_1 \\ -q_2 & q_1 & q_0 \end{pmatrix}$$

and where the lineal operator $Skew()$ is introduced for make things easier. This is:

$$Skew(v) = \begin{pmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{pmatrix}$$

And is equivalent to the cross product " $\times$ v"

2.2 Mathematical ideas

- A proposition is a statement which is either true or false. It is similar to a theorem although a theorem can be used as a principal statement.
- A proposition which is useful for showing a theorem is called lemma.

2.3 Lineal and nonlineal system

A lineal system is given by

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

A nonlineal system is given by:

$$\dot{x} = f(x, u)$$

$$y = k(x, u)$$

2.4 System inmersion

Definition: Given two systems with the same output y:

$$\begin{cases} \dot{x} = f(x) & , x \in \chi \\ y = h(x) & , y \in R^m \end{cases}$$

and

$$\begin{cases} \dot{X} = F(X) & , X \in \mathbf{X} \\ Y = H(X) & , Y \in R^m \end{cases}$$

the system $\{\chi, f, h\}$ is inmersed inside $\{\mathbf{X}, \mathrm{F}, \mathrm{H}\}$ if there is a function $\tau : \chi \to \mathbf{X}$ which satisfies $\tau(0) = 0$ and

$$\frac{\partial \tau}{\partial x} f(x) = F(\tau(x))$$

$$h(x) = H(\tau(x))$$

for every $x \in \chi$.

The idea of inmersion is highly relevent for the regulation problem which is stated in this project. This is because it gives the option to have an autonomous system:

$$\dot{w} = s(w)$$

$$u = c(w)$$

inmersed in a system:

$$\dot{\xi} = \varphi(\xi)$$

$$u = \gamma(\xi)$$

which could have properties that can be exploited to solve the problem.

2.5 Stability

2.5.1 Equilibria

A point x^* is called equilibria if it has the property that for any state of the system x, with initial conditon $x(0) = x^*$ belongs inside x^* for every time t.

2.5.2 Stability of equilibria

In addition to the idea of equilibria it is important to learn about the idea of equilibria stability. As a general form the stability of equilibria implies that if the state x is closed to the equilibria x^* , it will remain closed to it for every future time $t > 0$.

This is formally defined as:

" *The equilibria $x = 0$ is*

1. *stable if for any $\epsilon > 0$, it can be found a $\delta > 0$ that satisfies*

$$\|x(t_0)\| < \delta \textit{ means that } \quad \|x(t)\| < \epsilon \quad \textit{for every } t \geq t_0$$

2. *unstable if it is not stable*

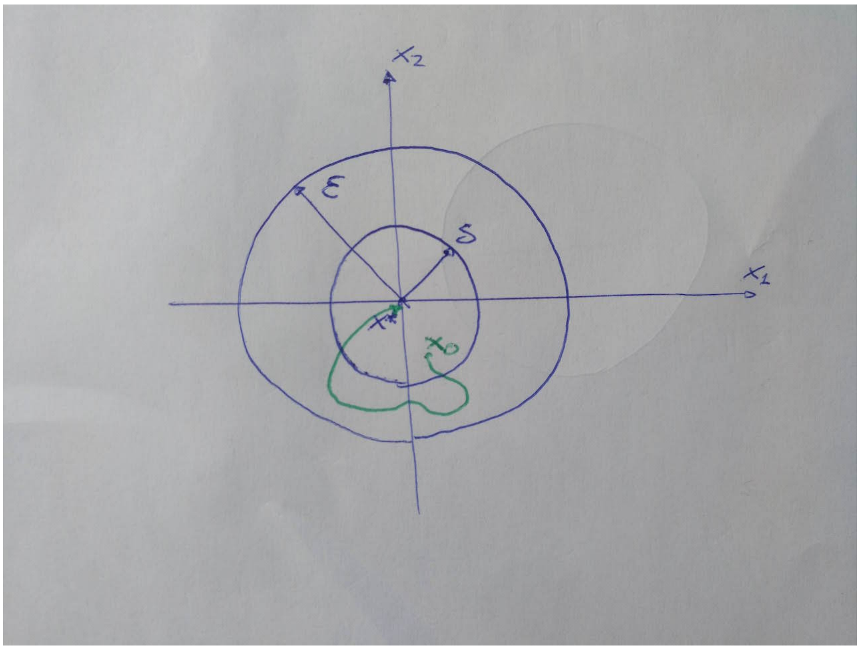


Figure 2: equilibria

3. *asymptotically stable if it is stable and it can be found a* $\delta > 0$ *that satisfies*
$$\|x(t_0)\| < \delta \quad \textit{means that} \quad \lim_{t \to \infty} x(t) = 0$$

4. *globally asymptotically stable if it is stable*
$$\lim_{t \to \infty} x(t) = 0, \; \textit{for every } x(t_0)$$

"

The definitions 3 y 4 shows how these ideas are applied to this project. If the attitude and angular velocity are defined by the state x and there are disturbances that causes that this state gets further from the equilibria by a distance δ, it is desired that the state gets closer to the equilbira. The definition number 3 is applied to the local scenario, this means applied to a point close to the equilibria. The definition number 4 is much more rigorous and is applied in a global scenario for any state.

2.5.3 Lyapunov's Theorem

The stability theorem of Lyapunov is defined as:

"Given a system

$$\dot{x} = f(x)$$

where $x \in R^n$ *and assuming that there exists a solution for* $x(t)$ *in an open set* $D \subset R^n$ *which has the origin (this means* $0 \in D$*). Given that* $x = 0$ *is an equilibria of the system. And considering a continous differentiable function* $V(x) : D \to R$ *such as* $V(0) = 0$ *and* $V(x) > 0$ *for* $x \in D$ *with* $x \neq 0$*. Assuming now that for the solution* $x(t)$*,*

$$\dot{V}(x) \leq 0$$

Then, $x = 0$ *is stable. In fact, if*

$$\dot{V}(x) \leq 0 \; for \; x \neq 0,$$

for the solutions $x(t)$*, then* $x = 0$ *is asymptotically stable."*

In addition it can be shown another similar result to show asymptotic stability:

" Assuming that the previous theorem holds and in addition that the Lyapunov function $V(x)$ *has the property that*

$$\|x\| \to \infty \quad \text{implies that } V(x) \to \infty$$

Then $x = 0$ *is globally asymptotically stable"*

2.5.4 LaSalle's Theorem

"Given a system

$$\dot{x} = f(x)$$

where $x \in R^n$*. It is said that a set* $M \subset R^n$ *is invariant in regard to this system when*

$$x(0) \in M \; implies \; x(t) \in M \; for \; every \; t \in R$$

"

This means that if $x(t)$ stays inside M for any time, it will keep staying inside M for every t.

It is interesting to know what idea holds the sentence " a point is getting closer to a set":

" It is said that $x(t)$ gets closer to a set M when $t \to \infty$ if for each $\epsilon > 0$ there exists $T > 0$ such as

$$dist(x(t),\, M) < \epsilon, \; for\ every\ t \geq T,$$

where $dist(p,\, M)$ means the distance from a point p to a set M, which is defined as

$$dist(p, M) = min\, \|p - q\|$$

"

LaSalle's Theorem is similar to Lyapunov's Theorem and allows to establish stability results in sets:

"Given a system

$$\dot{x} = f(x)$$

and assuming that there exists a solution $x(t)$ to the state equtions inside an open set $D \subset R^n$. Assuming now that for a solution $x(t)$ a closed and bounded set can be found $\Omega \subset D$ such as $x(t) \in \Omega$ for every $t \geq 0$. Given $V(x) : D \to R$ a continous differentiable function such as along the trajectories of the system $x(t)$ the first derivative is $\dot{V} \leq 0$ inside Ω. Given E the set of points inside Ω where $\dot{V}(x) = 0$, this is $E = \{x \in \Omega : \dot{V}(x) = 0\}$. Given M the largest invariant set inside E. Then $x(t) \to M$ when $t \to \infty$"

3 Theory of the problem

Given a system:

$$\dot{x} = f(x, w, u)$$

$$e = h(x, w)$$

The control problem which is solved in this project is a classical problem of the regulation of the output. This means, accomplish using feedback that the output of the system $y(t)$ tracks a desired known trajectory $y_{ref}(t)$ and in the same time rejects, if there is, a undesired disturbance $w(t)$. In any case, the problem is can be stated as accomplishing that the error

$$e(t) = y_{ref}(t) - y(t)$$

reaches 0.

The control law $u(t)$ that gets information from the plant has to be able to achieve this goal depending on which kind of information is receiving. There can be two kind of problems:

3.1 Case in which there is full information: knowledge about $x(t)$ and the disturbance $w(t)$

The most desirable situtation in which there are full information about the state $x$and disturbance w. In this case it is said that the controller gets full information, $u = \alpha(x, w)$. The system can be described in closed loop:

$$\begin{aligned} \dot{x} &= f(x, w, \alpha(x, w)) \\ \dot{w} &= s(w) \end{aligned}$$

This is problem is the easiest one and will not be resolve in this project

3.2 Case in which there is partial information: knowledge about the error $e(t)$

3.2.1 Control problem

The most realistic situation, in which there is only partial information about the error e. In this case it is said that the controller receives only information about the error. The control law will be implemented by a dynamic system which counteracts the disturbance's effect on the system:

$$\dot{\xi} = \phi(\xi, e)$$

$$u = \theta(\xi, e)$$

The closed loop system can be stated as:

$$\dot{x} = f(x, w, \theta(\xi))$$

$$\dot{\xi} = \phi(\xi, h(x, w))$$

$$\dot{w} = s(w)$$

This project is based on the $2^{\underline{o}}$ kind of problem, it is case in which given an error find a control law that taking into account the disturbance achieves to regulate the error. In addition, a stabilizer control law will be added u_{st}. The goal of this last one is to achieve stability.

3.2.2 Solution to the control problem

In the case that there is only information about the error, the solution to regulation problem is given by the following result:

Given the proposition 1.7.1 of [1]:

"*Assuming that a controller like the following one is found*

$$\dot{\xi} = \phi(\xi, e)$$

$$u = \theta(\xi, e)$$

satisfiying the equations,

$$\frac{\partial \pi}{\partial w} s(w) = f(\pi(w), c(w), w)$$

$$0 = h(\pi(w), w)$$

for three maps $\pi(w), \sigma(w), c(w)$. Assuming that all the trajectories of the system with initial conditions inside a set $\chi \times \Xi \times W$ are bounded and attracted by the manifold M_0. The mentioned controller solves the regulation problem."

Understanding the role of the internal model ξ is utterly imporant to understand how the problem is being approached. Taking into account that there is a disturbance w which can not be avoided, the internal model's goal is to "copy" the disturbance without having knowledge of this one. The main goal of this is to create a control law counteracting the effect of the disturbance.

3.2.3 Regulation equations

The first two equations:

$$\frac{\partial \pi}{\partial w} s(w) = f(\pi(w), c(w), w)$$

$$0 = h(\pi(w), w)$$

mean that the control law $u = c(w)$ makes that the regulation error is null, which is our goal making that the state x stays on the surface $\pi(w)$.

3.2.4 Internal Model property

The other two equations:

$$\frac{\partial \sigma}{\partial w} s(w) = \phi(\sigma(w), w))$$

$$c(w) = \theta(\sigma(w), w))$$

imply that the internal model is able to reproduce the control law $u = c(w)$. Without this last result the problem could not be solved since a priori there is no knowledge of the disturbance w. To sum up, this can be seen as the disturbance and control law $\dot{w} = s(w)$ and $u = c(w)$ being inmersed into the internal model:

$$\dot{\xi} = \phi(\xi, e)$$

$$u = \theta(\xi, e)$$

Inmersion implies that any control law $u = c(w)$ is the output of $u = \theta(\xi, e)$

However due to the internal model property, a dynamic system $\dot{\xi}$ can be created which counteracts the disturbance without having information of it.

3.2.5 Geometric meaning of the regulation equations:

The regulation equations and internal model can be seen from a geometric point view like the "manifold" M which is a mathematical object similar to the idea of lineal space but in a nonlineal scenario, in invariant under the control law defined. This means, and this is the principal idea of this project, if the state is inside M the control law of the internal model forces the state to remain inside the manifold. And since the error is null on this surface, it can be said that as long as the state stays inside the manifold, the tracking error will keep being null, which is the goal.

The geometric meaning of the regulation equations is the one shown below. In the space $x = \pi(w)$ the solution to the regulation equations of the error $e(t)$ is null, due to the fact that the control law $u = c(w)$ is a solution to the regulation equations. In addition $\xi = \sigma(w)$ is a solution to the internal model equations.

$$M = \{(x, \xi, w) : x = \pi(w), \xi = \sigma(w)\}$$

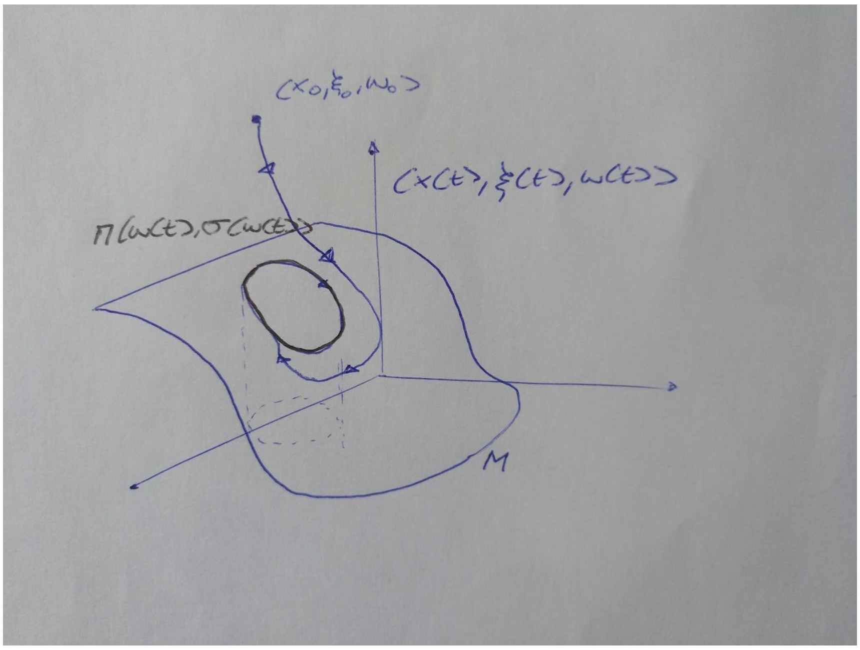


Figure 3: Manifold

4 Reference system

There are the following reference systems to take into account:

4.1 Earth centered

The coordinate system $F_e = \{O_e, \overrightarrow{i_e}, \overrightarrow{j}_e, \overrightarrow{k}_e\}$ centered in the Earth's center of mass. Where the unit vector $\overrightarrow{i}_e$ points to the direction of the March equinox, the unit vector $\overrightarrow{k}_e$ points to the direction of the Earth's rotation axis pointing towards the North Pole and the unit vector $\overrightarrow{j}_e$ in the equatorial plane. These coordinates are called v^e.

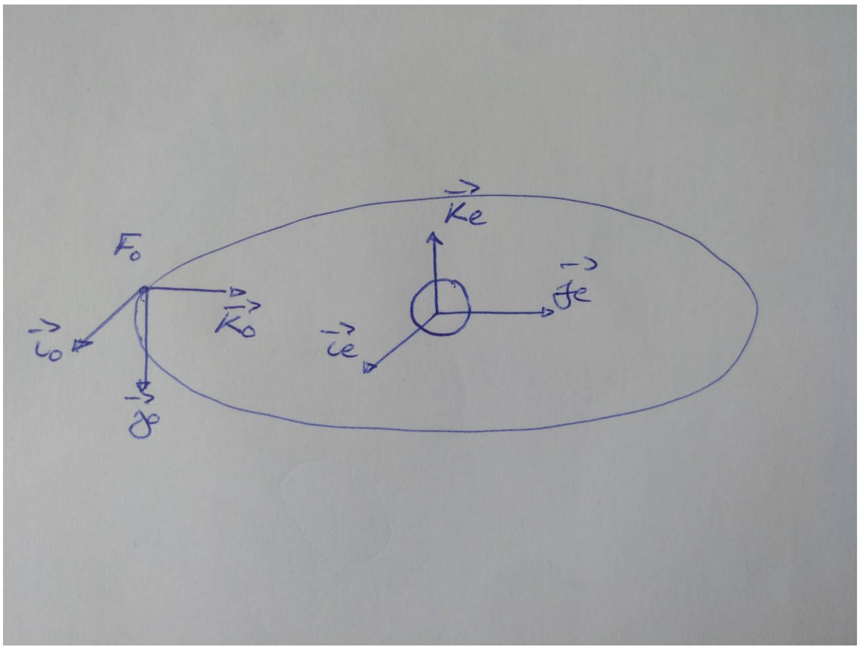


Figure 4: Local Orbit and Earth system

4.2 Centered in the satellite

The coordinate system $F_b = \{O_b, \overrightarrow{i_b}, \overrightarrow{j_b}, \overrightarrow{k}_b\}$ centered in the satellite, with origin in the center of mass of the vehicle. The alignment of the unit vectors are, not necessarily, aligned with the main inertia axis. These coordinates are called $v^b \in R^3$.

4.3 Local orbit

The coordinate system $F_o = \{O_o, \overrightarrow{i_o}, \overrightarrow{j_o}, \overrightarrow{k_o}\}$. This system is usually called "Local Vertical Local Horizontal". It has its origin in O_o at the center of mass of the satellite, with the vector $\overrightarrow{k_o}$, pointing towards the direction of the vector $O_o - O_e$ pointing to the center of the Earth. The vector $\overrightarrow{i_o}$ is found on the orbital plane, in the direction of the velocity vector of the satellite which is tangent to the circular orbit. The vector $\overrightarrow{j_o}$ is orthogonal to the orbital plane and with an opposite direction to the angular momentum of the satellite. These coordinates are called v^o.

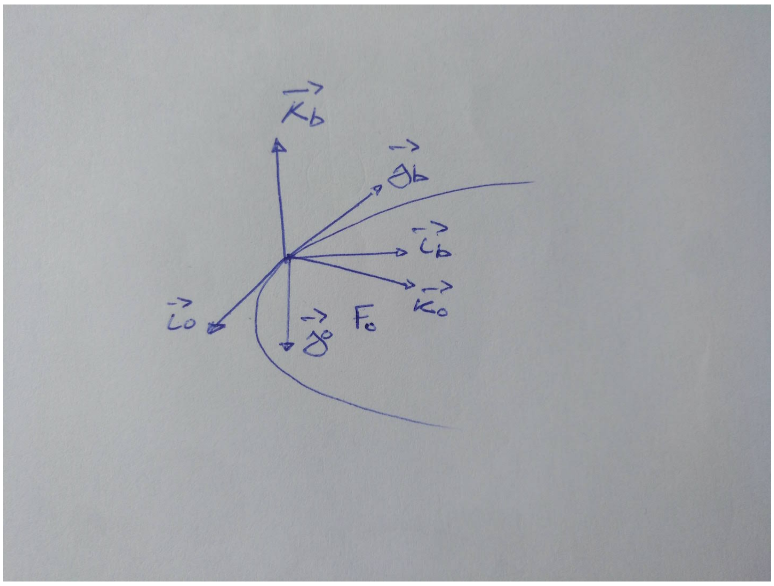


Figure 5: Satellite centered system

4.4 Position and angular velocity in regard to the coordinate system

A vector with respect to another one can be expressed using the rotation matrix:

$$v^b = R_{be} v^e$$

The rotation matrix satisfies the following:

$$R_{be} = R_{eb}^T$$

The angular velocity vector of F_b with respect to F_e is called $\overrightarrow{w}_{eb}$ and its components can be expressed with respect to another and its components can be expressed with respect to one of the coordinate systems: if it is expressed respect to the satellite centered system is called w_{eb}^b . And if it expressed respect to the earth centered system is called w_{eb}^e .

5 Disturbances on the system

In the following lines the quantification of w is shown.

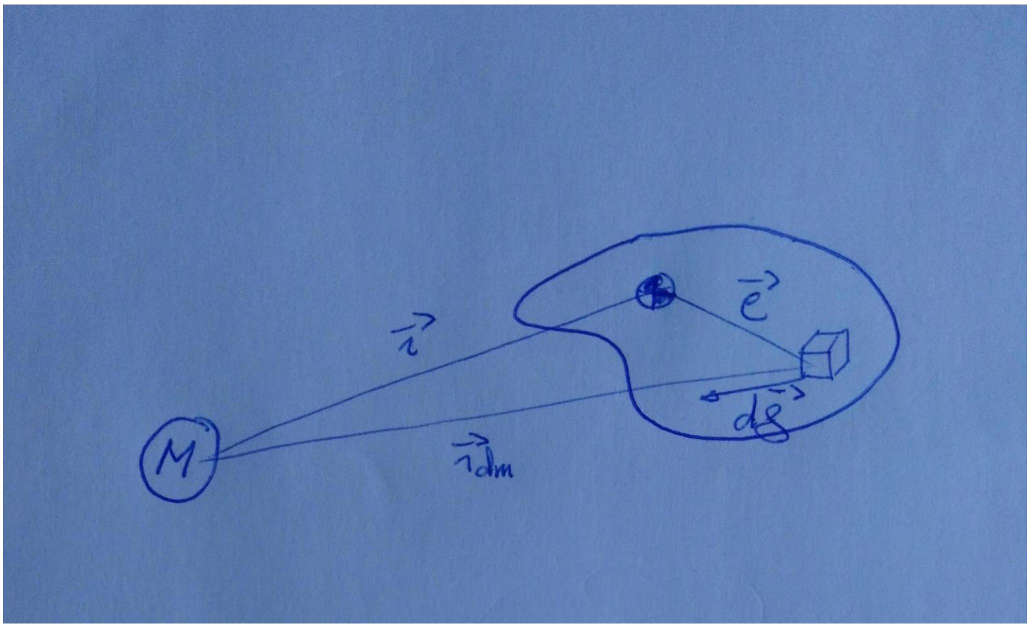


Figure 6: Gravity attraction force

5.1 Gravitational gradient

5.1.1 Gravitational force

The torque due to the gravitational gradient is due to the fact that the gravity force of the Earth mass acting on the satellite is not constant with the distance but grows quadratically. As a consecuence, the section of the satellite further from the Earth is less affected by the Earth than the section closer to it. This difference of forces create a momentum respect to the center of mass.

The gravitational force on a differential element of mass dm located to a distance $\overrightarrow{\rho}$ from the center of mass of the satellite is expressed by

$$d\overrightarrow{\mathbf{f}} = -\frac{\mu(\overrightarrow{\mathbf{r}} + \overrightarrow{\rho})}{|\overrightarrow{\mathbf{r}} + \overrightarrow{\rho}|^3} dm$$

where $\overrightarrow{\mathbf{r}}$ is the orbital position of the center of mass of the satellite. Let's assume that the distance is ρ is much smaller than the radius of the Earth r this is $\rho \ll r$. And where $\mu = GM$.

Then using Taylor series it can be approximated:

$$|\overrightarrow{\mathbf{r}} + \overrightarrow{\rho}|^3 \approx \frac{1}{r^3}(1 - \frac{3\overrightarrow{\mathbf{r}} \cdot \overrightarrow{\rho}}{r^2})$$

Replacing this force equation previously mentioned:

$$d\overrightarrow{\mathbf{f}} = \frac{\mu(\overrightarrow{\mathbf{r}} + \overrightarrow{\rho})}{r^3}(1 - \frac{3\overrightarrow{\mathbf{r}} \cdot \overrightarrow{\rho}}{r^2})dm$$

5.1.2 Gravitational gradient torque

The total torque over the center of mass is given by:

$$\overrightarrow{\tau}_{grav} = \frac{3\mu}{r^5}\int_V \overrightarrow{\rho} \times d\overrightarrow{\mathrm{f}} =$$

$$= \int_V \overrightarrow{\rho} \times (-\frac{\mu\,(\overrightarrow{r} + \overrightarrow{\rho})}{r^3})dm + \int_V \overrightarrow{\rho} \times (\frac{3\,\mu\,\overrightarrow{r} \cdot \overrightarrow{\rho}}{r^5}(\overrightarrow{r} + \overrightarrow{\rho}))dm$$

where the integral is calculated over the entire volume of the satellite. Taking into account that $\int_V \overrightarrow{\rho}dm = 0$ since the origin is located in the center of mass. And that $\overrightarrow{\rho} \times \overrightarrow{\rho} = \mathbf{0}$, the expression can be given by:

$$\overrightarrow{\tau}_{grav} = \frac{3\mu}{r^5}\int_V \overrightarrow{\rho} \times \overrightarrow{r}(\overrightarrow{r} \cdot \overrightarrow{\rho})dm$$

Due to the distribution of mass of the satellite the force of gravity over the satellite is stronger in the section closer to it, this creates a torque which affects the desired position of the satellite.

It can be concluded that the torque is given by:

$$\overrightarrow{\tau}_{grav} = \frac{3\mu}{r^3}(\overrightarrow{k}_0 \times \overrightarrow{J} \cdot \overrightarrow{k}_0)$$

where $\overrightarrow{k}_0$ is the unit vector of the local orbit basis.

If this torque is expressed respect to the coordinate system centered on the center of mass of the satellite F_b, this unit vector can be given by:

$$\overrightarrow{k}_0 = R_{bo}\overrightarrow{k}_b = R_{ob}^T\overrightarrow{k}_b$$

where R_{ob} is the rotation matrix which links the basis of F_0 with F_b.

Therefore, this torque respect with the coordinate system before mentioned is given by:

$$\overrightarrow{\tau}^b_{grav} = \frac{3\mu}{r^3}(R_{ob}^T\overrightarrow{k}) \times \overrightarrow{J} \cdot R_{ob}^T\overrightarrow{k}$$

6 System dynamics

6.1 Satellite physics

The satellite considered in this project is moving according a circular orbit affected by the force of gravity of the Earth and its dynamics is described using Newton laws which links the movement of a particle.

The 1º Law states that if there is no force acting over a body, the resting body will keep motionless and the body in motion will keep in straight line movement.

The 2º Law states that if a force is applied there will be a change of velocity, this means, an acceleration proportional to the force and in the direction in which is the force is applied. This law is expressed as:

$$F = m\,a$$

where F is the force, m is the mass particle, and a the acceleration.

The 3º Law states that if a body causes a force over another body, this body will create a force of the same magnitude but with opposite direction over the first body.

The Newton's law of universal gravitations states that two particles with mass m_1 and m_2 and separated by a distance r are mutually attracted atraídas with the same force and in the direction with goes from one to other. This magnitude of the force is:

$$F = G\frac{m_1 m_2}{r^2}$$

where $G = 6.67259 \times 10^{-11} N \frac{m^2}{kg^2}$ is the gravitational constant.

In a circular orbit the satellite finds itself in a situation of free fall. The satellite accelerates towards the center of the Earth while its moving in a straight line. The velocity changes continuously in direction but not in magnitude. From the Newton Laws it can be seen that the velocity's direction is changing, there exists acceleration. This acceleration, called centripetal acceleration has a direction pointed to the Earth's center and is expressed as

$$a = \frac{v^2}{r}$$

where v is the satellite's velocity and r is the radius of the orbit. This acceleration in accordance with the Newton's laws has to be caused by a force, the centripetal force which magnitude is expressed as:

$$F = \frac{m\,v^2}{r}$$

The direction of F in any moment has the same direction as a.

Now after a balance of forces acting on the satellite in accordance with the 2º Newton Law: centripetal and gravitational forces, the velocity of the satellite can be calculated:

$$\frac{v^2}{r} = \frac{GM}{r^2}$$

or

$$v = \sqrt{\frac{G\,M}{r}}$$

And the angular velocity of the satellite as:

$$w = \frac{v}{r} \rightarrow w^2 = \frac{v^2}{r^2}$$

Taking into account the last equation of the centripetal acceleration:

$$w^2 = \frac{a}{r}$$

Therefore,

$$w^2 = \frac{GM}{r^3}$$

This is the module of $\overrightarrow{w}_{eb}$ which is the velocity the satellite has rotating around the Earth. The value depends on the gravitational constant, the Earth's mass and the radius of the orbit.

6.2 Satellite dynamics equations

The satellite's dynamics are described using Euler's equations for a rigid solid in rotation, which can be found in any classical book.

$$\dot{R}_{eb} = R_{eb}\,Skew(w^b_{eb})$$

$$J\dot{w}^b_{eb} = -w^b_{eb} \times Jw^b_{eb} + \tau^b$$

in which the inertia matrix is described in the satellite's centered coordinate system, is symmetric and positive. The term R_{eb} is the relative position of the body in respect to the Earth's coordinate system. It is expressed using matrix notation and is called matrix rotation. This matrix rotation has useful properties which can be used to design the controller. The second term w_{eb}^b is the angular velocity of the satellite respect to the Earth's coordinate system. The operator $Skew$ is the equivalent operator to $R_{eb} \times w_{eb}^b$.

The second equation describes the dynamics of the solid taking into account forces acting on it. The term J in this case is the inertial matrix of the satellite with dimension 3 and depends on the geometry and mass distribution of it. It is expressed in regard to the satellite's axis, which are defined as the principal inertia axis. This matrix J will be one of the design parameters of the satellite.

The dynamic equation depends as well on the angular velocity and the torque applied to the body which can be expressed by τ. This term $\overrightarrow{\tau} = \overrightarrow{\tau}_{dist} + \overrightarrow{u}$ is compound of a term which contains the torque which causes disturbances on the dynamics of the satellite.

6.3 Dynamics equations of the error

Forward up the dynamics equations of the position error will be deduced. In order to do this, the attitude and angular velocity error will be taken into account.

Below it is first defined the attitude error as the difference between the real attitude of the satellite in respect to the the local orbit and the desired attitude of the satellite in respect to the local orbit. Secondly segundo, the angular velocity error is defined as the difference between the real angular velocity of the satellite in respect to the Earth and the desired angular velocity of the satellite in respect to the Earth.

$$\widetilde{R}(t) = R_{db}(t) = R_{od}^T R_{ob} = R_{ed}^T R_{eb}$$

$$\widetilde{w} = w_{eb}^b - w_{ed}^b$$

As said before, we make the assumption that the control law has knowledge about the error $(\widetilde{R}, \widetilde{w})$. Developing the dynamics equations the error dynamics equations can be obtained as:

$$\dot{\widetilde{R}} = \widetilde{R}\, Skew(\widetilde{w})$$

$$J\dot{\widetilde{w}} = \tau_0^b + \tau_{grav}^b + u^b$$

where it is replaced

$$w_{eb}^b = \widetilde{w} + w_{ed}^b$$

so that:

$$\tau_0^b = -\widetilde{w} \times J\widetilde{w} - w_{ed}^b \times J\widetilde{w} - \widetilde{w} \times Jw_{ed}^b - w_{ed}^b \times Jw_{ed}^b - J\dot{w}_{ed}^b$$

The desired angular velocity of the satellite in respect to the Earth in the coordinate system of the satellite's body is:

$$w_{ed}^b = \widetilde{R}^T w_{ed}^d$$

And in where the desired angular velocity of the satellite in respect to the Earth in the coordinate system of the desired attitude is the sum of: the desired angular velocity in respect to the local orbit and the angular velocity of the local orbit in respect to the Earth, both expressed in the coordinate system of the desired attitude:

$$w_{ed}^d = w_{od}^d + w_{eo}^d = w_{od}^d + R_{od}^T w_{eo}^o$$

and taking into account that the angular velocity of the local orbit in respect to the Earth in the coordinate system of the local orbit is $w_{eo}^o = -w_0 j$. Then:

$$w_{ed}^d = w_{od}^d + w_{eo}^d = w_{od}^d + R_{od}^T w_{eo}^o = w_{od}^d - w_0 R_{od}^T j$$

So:

$$w_{ed}^b = \widetilde{R}^T w_{ed}^d = \widetilde{R}^T [w_{od}^d - w_0 R_{od}^T j]$$

And the first derivative w_{ed}^b is calculated using the properties of the product derivative:

$$\dot{w}_{ed}^b = \frac{d}{dt}(\widetilde{R}^T [w_{od}^d - w_0 R_{od}^T j])$$

$$= -\widetilde{w} \times \widetilde{R}^T [w_{od}^d - w_0 R_{od}^T j] + \widetilde{R}^T [\dot{w}_{od}^d - w_0 \dot{R}_{od}^T j]$$

Given $\dot{w}_{od}^d = 0$ since the satellite turns at constant speed::

$$= -\widetilde{w} \times \widetilde{R}^T [w_{od}^d - w_0 R_{od}^T j] + w_0 \widetilde{R}^T [w_{od}^d \times R_{od}^T j]$$

Therefore the dynamics equations of the error are given by:

$$\dot{\widetilde{R}} = \widetilde{R}\, Skew(\widetilde{w})$$

$$J\dot{\widetilde{w}} = \tau_0^b(\widetilde{R}, \widetilde{w}, R_{od}, w_{od}, \mu) + \tau_{grav}^b(\widetilde{R}, R_{od}, \mu) + u^b$$

which depend on the error variables and exogenous inputs. Therefore if a control law u^b can be obtained equal to:

$$u^b = -\tau_0^b(I, 0, R_{od}, w_{od}^d, \mu) - \tau_{grav}^b(I, R_{od}, \mu)$$

At the state $(\widetilde{R}, \widetilde{w}) = (I, 0)$ there will be a point of equilibria of the error dynamics:

$$\dot{\widetilde{R}} = \widetilde{R}\, Skew(\widetilde{w})$$

$$J\dot{\widetilde{w}} = 0$$

At this point it is satisfied $\widetilde{R} = I$ and $\widetilde{w} = 0$ for every $t > 0$. For this an internal model able to copy the behavior of this control law without knowledge of the state of the disturbances R_{od}, w_{od}, μ has to be designed.

7 Nonlineal control law

7.1 Internal Model control law u_{im}

The system which has to be regulated is given by:

$$\dot{\widetilde{R}} = \widetilde{R}\, Skew(\widetilde{w})$$

$$J\dot{\widetilde{w}} = \tau_0^b + \tau_{grav}^b + u^b$$

where the attitude and angular velocity error are:

$$\widetilde{R}(t) = R_{db}(t) = R_{od}^T R_{ob}$$

$$\widetilde{w} = w_{ob} - w_{od}$$

- The control law which satisfies the regulation equations:

$$\frac{\partial \pi}{\partial w} s(w) = f(\pi(w), c(w), w)$$

$$0 = h(\pi(w), w)$$

is given by

$$u^b = c(R_{od}, w_{od}^d, \mu) = -\tau_0^b(I, 0, R_{od}, w_{od}^d, \mu) - \tau_{grav}^b(I, R_{od}, \mu)$$

$$c(\, R_{od},\, w_{od}^d, \mu) = \, (w_{od}^d - w_0 R_{od}^T j) \times J(w_{od}^d - w_0 R_{od}^T j) + w_0 J(w_{od}^d \times R_{od}^T j) - 3w_0^2 (R_{od}^T k) \times J R_{od}^T k$$

causing the point $(\widetilde{R}, \widetilde{w}) = (I, 0)$ to be a point of equilibria of the system.

At this equilibria a map is defined $(\widetilde{R}, \widetilde{w}) = \pi(R_{od}, w_{od}, \mu)$ in where the state $(\widetilde{R}, \widetilde{w})$ remains over time when this control law is defined:

$$u^b = c(R_{od}, w_{od}^d, \mu)$$

and in which the orbital position is equivalent to the desired one $R_{ob} = R_{od}$ like the angular velocity $w_{ob} = w_{od}$

- The internal model able to reproduce this control law and satisfying the internal model equations: :

$$\frac{\partial \sigma}{\partial w} s(w) = \phi(\sigma(w), w))$$

$$c(w) = \theta(\sigma(w), w))$$

is a dynamical system and will being shown later.

This dynamical system is able to generate the control law $c(R_{od}, w_{od}^d, \mu)$ which satisfies the regulation equations without knowledge of the exogenous inputs (R_{od}, w_{od}^d, μ)

which has a specific form depending of the control law $u^b = c(R_{od}, w_{od}^d, \mu)$ aiming to reproduce, this means depending of the mission.

7.2 Stabilizer control law u_{st}

The stabilizer control law u_{st} is designed using the principles of Lyapunov stability and LaSalle's theorem. First, change of variables is done:

$$\widetilde{\xi} = \xi - \sigma(w)$$

$$z = \widetilde{w} + k_1 \widetilde{q}$$

where k_1 is a parameter to be chosen.

Then the function of Lyapunov is given in accordance with [1] by :

$$V(\widetilde{\xi},\, \widetilde{q},\, \widetilde{w}+k_1\widetilde{q}) = \frac{\gamma}{2}\widetilde{\xi}^T P\widetilde{\xi} + (1-\widetilde{q}_0)^2 + \widetilde{q}^T\widetilde{q} + \frac{1}{2}(\widetilde{w}+k_1\widetilde{q})^T J(\widetilde{w}+k_1\widetilde{q})$$

And after several mathematical manipulations and picking the stabilizer control law as:

$$u^b_{st} = -k_2(1+\left|\left|(\widetilde{w}+k_1\widetilde{q})\right|\right|)(\widetilde{w}+k_1\widetilde{q})$$

It is concluded that the derivative of the Lyapunov's function is upper bounded:

$$\dot{V}(\widetilde{\xi},\widetilde{q},z) \leq -\frac{k_1}{2}\left|\left|\widetilde{q}\right|\right|^2 - \epsilon(1+\left|\left|z\right|\right|)\left|\left|z\right|\right|^2 < 0$$

where $\epsilon > 0$.

This control law by arguments of Lyapunov and LaSalle assures that the closed loop system is stable. .

7.2.1 Stability of the interconnection of the error dynamics and the internal model

After designing the internal model ξ which counteracts the effect of the exogenous system and adding it as a state variable it has to be ensured the stability of the connection $(x,\, \xi)$. In order to do this a term like g_{st} iis added which is designed using Lyapunov's arguments. The term assures that the interconnection betweenthe state x and internal model ξ is stable.

$$\dot{\xi} = \Phi\xi + g_{st}$$

$$u = \Gamma\xi$$

The value of this term is calculated manipulating the Lyapunov's function previously shown in order to guarantee that the first deritave is smaller than 0. In the end we get that:

$$g_{st} = -\frac{1}{\gamma}P^{-1}\Gamma^T z$$

where $\gamma > 0$ is a paramater to be picked and$P$ is a positive definite matrix which is the solution of the matrix inequality:

$$P\Phi + \Phi^T P \leq 0$$

7.3 Control law: internal model control law u_{im} and stabilizer control law u_{st}

The control law implemented is the sum of the internal model control law and the stabilizer control law:

$$u^b = u_{im} + u_{st}$$

LIke previously shown the internal model control law u_{im} ensures that the regulation error is 0 as long as there exists this control law and internal model satisfying the regulator and internal model equations. The role of the stabilizer control law u_{st} is to guarantee convergence to the desired mainfold from any initial condition.

8 Simulation using Simulink

Below is shown the simulation for two different missions. First a mission in which the satellite has to remained facing the Earth along the orbit is going to be simulated. In order to do this the internal model has to be designed. In the second mission a satellite turning around along the orbit is going to be simulated. Equally, in this case an internal model has to be designed which will be more difficult than in the first mission.

In both missions the stabilizer control law u_{st} will be the one previously shown.

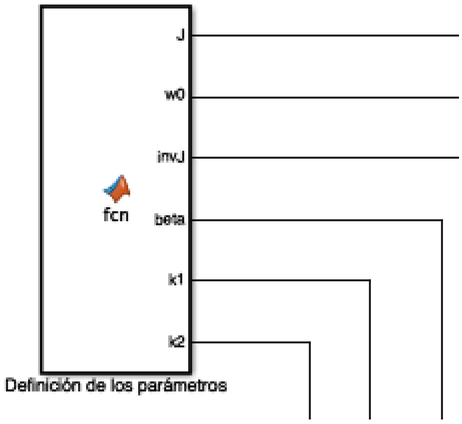
The plots and code will be shown with the aim of showing how the simulation has been done. The satellite parameters are chosen as:

$J_x = 1.3$	$J_{xy} = 0.2$
$J_y = 0.9$	$J_{yz} = 0.09$
$J_z = 1.8$	$J_{zx} = 0.08$
$w_0 = 6.2 * 10^{-3}$	

The gains are picked as:

$k_1 = 0.25$	$k_2 = 25$	$\gamma = 5 * 10^{-3}$

These parameters are designed inside a matlab block:



In this block the parameters of the system are defined:

```
function [J, w0, invJ, beta, k1, k2] = fcn
%Definition of parameters
J=[1.3 0.2 0.08;
     0.2 0.9  0.09;
     0.08 0.09 1.8];

beta=5*10^-3;
k1=0.25;
k2=25;
w0= 6.2*10^-3; invJ= inv(J);
```

Figure 7: System's parameters

8.1 Mission: Inertial flight

8.1.1 Mission goal

As shown in the picture below, the goal of the mission is to make that the satellite remains in the desired attitude, in order to do this it has to be going turning around the j axis along as time passes.

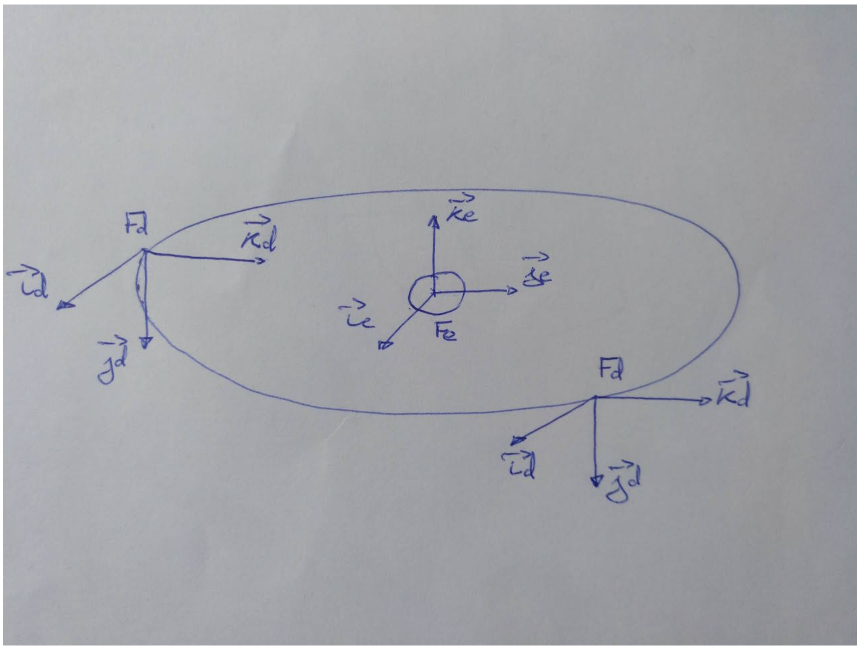


Figure 8: Mission: Inertial flight

The initial conditions of the reference state are:

$$w_{od}^{d}(0) = \begin{pmatrix} 0 \\ w_0 \\ 0 \end{pmatrix} \qquad R_{od}(0) = I$$

And the reference to be tracked by the satellite has the following form:

$$w_{od}^{d}(t) = \begin{pmatrix} 0 \\ w_0 \\ 0 \end{pmatrix}$$

and

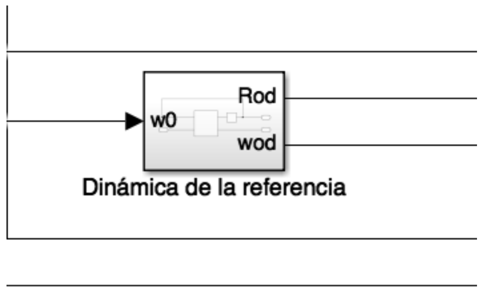
$$R_{od}(t) = \begin{pmatrix} cos(w_0 t) & 0 & sin(w_0 t) \\ 0 & 1 & 0 \\ -sin(w_0 t) & 0 & cos(w_0 t) \end{pmatrix}$$

which can be viewed as a disturbance generatedl by the dynamical system:

$$\begin{cases} \dot{R}_{od} = R_{od}\, Skew(w_{od}^d) \\ \dot{w}_{od}^d = 0 \\ \dot{\mu} = 0 \end{cases}$$

The uncertainty of the parameters $\mu = \begin{pmatrix} J \\ w_0 \end{pmatrix}$ can be as well viewed as an exogenous input into the system:

This has been implemented in Simulink as:



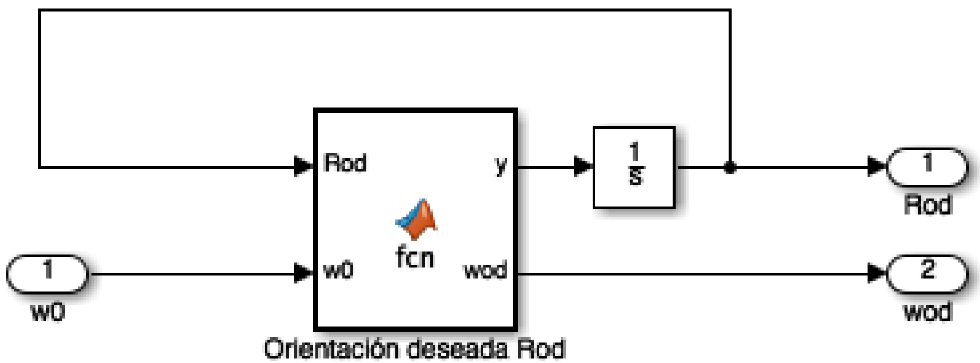


Figure 9: Desired attitude R_{od} y w_{od}

The block "Desired attitude R_{od}" contains the following code which expresses the dynamics before mentioned:

This function takes as input the desired attitude R_{od} at t and the desired angular velocity w_0 , with these data the variation of the desired attitude is calculated $\frac{dR_{od}}{dt}$ and then an integration to find this R_{od} is done.

```
function [y, wod] = fcn(Rod,w0)
wod=[0;w0;0];

y = Rod*[0 -wod(3) wod(2);
          wod(3) 0 -wod(1);
          -wod(2) wod(1)  0;];
```

8.1.2 Design of the control law by an internal model approach u_{im}

Like previously mentioned the control law to be implemented is:

$$u^b = c(R_{od}, w_{od}^d, \mu) = -\tau_0^b(I, 0, R_{od}, w_{od}^d, \mu) - \tau_{grav}^b(I, R_{od}, \mu)$$

An internal model able to copy the control law satisfying the regulation equations will be designed. The state variables of the exogenous input, which must be able to generate w_{od}^d, μ are constant along time. The variable R_{od} is not constant and changes along time. Therefore the internal model has to have dynamics.

$$u^b(t) = c(R_{od}, w_{od}^d, \mu) = c(R_{od}(t), w_{od}^d(0), \mu(0))$$

Given the fact that the desired angular velocity and the parameter's uncertainty do not change along time, they can be expressed as constant. The only term that changes along time is the desired attitude R_{od}.

And the term $R_{od}(t)$ can be been in accordance with [1] as:

$$R_{od}(t) = R_{od}(0)[I + sin(\Omega t)\, Skew(\lambda) + (1 - cos(\Omega t))[Skew(\lambda)]^2]$$

With the term $R_{od}(0)$ being the attitude at $t = 0$ and $Skew()$ the operator previously mentioned, and where Ω is an upper bound to the angular velocity at $t = 0$. This makes sense, due to the fact that the satellite has a maximum speed by design:

$$\Omega = \left|\left| w_{od}^d(0) \right|\right|$$

the term Ω is the norm of the angular velocity at $t = 0$, and by convenience the term λ is created as:

$$\lambda = \frac{1}{\Omega} w_{od}^d(0)$$

It can been shown that each component of the matrix $R_{od_{ij}}$ with dimension 3 can be expressed in respect to time as as sum of sinusoids:

$$R_{od_{ij}}(t) = c_{0_{ij}} + c_{1_{ij}} cos(\Omega t) + c_{2_{ij}} sin(\Omega t)$$

what is more each component of the vector $c(R_{od}, w_{od}^d, \mu)$ is a polynomial with grade 2 and depends on the terms of the rotation matrix. Therefore the desired control law $u_i^b(t)$ where $i = 1, 2, 3$ for each axis is:

$$u_i^b(t) = d_{0i} + d_{1i} cos(\Omega t) + d_{2i} sin(\Omega t) + d_{11_i} cos^2(\Omega t) + d_{12_i} cos(\Omega t) sin(\Omega t) + d_{22_i} sin^2(\Omega t)$$

where the parameters depend on the initial state of the exosystem $(R_{od}(0), w_{od}^d(0), \mu(0))$.

This input $u_i^b(t)$ can be shown to be the solution of the following differential equation:

$$\frac{du^5}{dt^5} + 5\Omega^2 \frac{du^3}{dt^3} + 4\Omega^4 \frac{du}{dt} = 0$$

This homogeneous solution allows to express the control law as:

$$u_i^b(t) = Q\xi_i(t) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \end{pmatrix} \xi_i(t)$$

where

$$\dot{\xi}_i = S\xi_i$$

being

$$\xi_i = \begin{pmatrix} u_i^b(t) \\ \frac{du_i^b(t)}{dt} \\ \frac{du_i^b(t)^2}{dt^2} \\ \frac{du_i^b(t)^3}{dt^3} \\ \frac{du_i^b(t)^4}{dt^4} \end{pmatrix}$$

and

$$S = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & -4\Omega^4 & 0 & -5\Omega^2 & 0 \end{pmatrix}$$

It can be shown now that the control law:

$$\dot{\xi} = S\xi$$

$$u = Q\xi$$

satisfies the regulation equations.

A map τ_i is now defined:

$$\tau_i : SO(3) \times R^3 \times P \to R^5$$

which is designed equivalent to the defined system ξ_i:

$$\xi_i(t) = \tau_i(R_{od}(t), w_{od}^d(t), \mu(t))$$

satisfying the internal model equations defined before. This map τ assures that there exists a control law $c(w)$ which makes the tracking error to be zero:

$$\frac{d}{dt}\tau(R_{od}(t), w_{od}^d(t), \mu(t)) = \mathit{\Phi}\,\tau(R_{od}(t), w_{od}^d(t), \mu(t))$$

$$c(R_{od}(t), w_{od}^d(t), \mu(t)) = \Gamma\,\tau(R_{od}(t), w_{od}^d(t), \mu(t))$$

Therefore, it can be concluded that if this model ξ is implemented the control law u will replicate the behaviour of the control law $c(w)$.

For every axis $i = 1, 2, 3$:

$$\dot{\xi} = \Phi\xi$$

$$u = \Gamma\xi$$

where,

$$\Phi = \begin{pmatrix} S & 0 & 0 \\ 0 & S & 0 \\ 0 & 0 & S \end{pmatrix}$$

$$\mathit{\Gamma} = \begin{pmatrix} Q & 0 & 0 \\ 0 & Q & 0 \\ 0 & 0 & Q \end{pmatrix}$$

And it has been implemented in Simulink as:

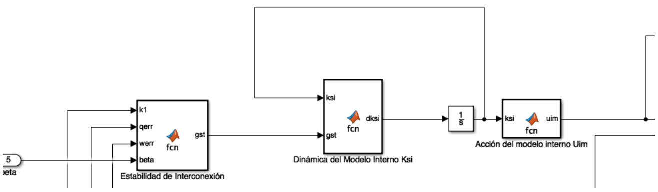


Figure 10: Internal model control law

Like shown, an integration of the dynamics of the internal model ξ has been implemented. And where the block "Dinámica del Modelo Interno Ksi" has been implemented using the following code:

```
function dksi = fcn(ksi, gst)

W= 6.2*10^-3; %||w(0)|| desired velocity norm

S=[0 1 0 0 0;
        0 0 1 0 0;
        0 0 0 1 0;
        0 0 0 0 1;
        0 -4*W^4 0 -5*W^2 0];

phi=[S zeros(5,5) zeros(5,5);
        zeros(5,5) S zeros(5,5);
        zeros(5,5) zeros(5,5) S];

dksi = phi*ksi + gst;
```

8.1.3 Stabiliy of the interconnection

After designing the form of the internal model ξ the stability of the connection $(x,\, \xi)$ has to be assured. For this the term g_{st} is added to the internal model and is designed using Lyapunov's arguments.

The term g_{st} has been implemented as :

$$g_{st} = -\frac{1}{\gamma} P^{-1} \Gamma^T (\widetilde{w} + k_1 \widetilde{q})$$

```
function gst = fcn(k1, qerr, werr)

        beta= 5*10^-3;

        Q=[1 0 0 0 0];
        gamma=[Q zeros(1,5) zeros(1,5);
                        zeros(1,5) Q zeros(1,5);
                        zeros(1,5) zeros(1,5) Q];

        P= [7.95,-96.09,694.77,-2150.92,-4867.73,0,0,0,0,0,0,0,0,0,0;
         -96.09,1817.74,-17243.04,74694.01,112765.08,0,0,0,0,0,0,0,0,0,0;
694.77,-17243.04,204788.29,-1199232.40,-104586.91,0,0,0,0,0,0,0,0,0,0;
-2150.92,74694.01,-1199232.40,10430890.31,-27944049.88,0,0,0,0,0,0,0,0,0,0;
-4867.73,112765.08,-104586.91,-27944049.88,536545695.41,0,0,0,0,0,0,0,0,0,0;
0,0,0,0,0,7.95,-96.09,694.77,-2150.92,-4867.73,0,0,0,0,0;
         0,0,0,0,0,-96.09,1817.74,-17243.04,74694.01,112765.08,0,0,0,0,0;
0,0,0,0,0,694.77,-17243.04,204788.29,-1199232.40,-104586.91,0,0,0,0,0;
```

```
0,0,0,0,0,-2150.92,74694.01,-1199232.40,10430890.31,-27944049.88,0,0,0,0,0;
0,0,0,0,0,-4867.73,112765.08,-104586.91,-27944049.88,536545695.42,0,0,0,0,0;
0,0,0,0,0,0,0,0,0,0,7.95,-96.09,694.77,-2150.92,-4867.73;
        0,0,0,0,0,0,0,0,0,0,-96.09,1817.74,-17243.04,74694.01,112765.08;
0,0,0,0,0,0,0,0,0,0,694.77,-17243.04,204788.29,-1199232.40,-104586.91;
0,0,0,0,0,0,0,0,0,0,-2150.92,74694.01,-1199232.40,10430890.31,-27944049.88;
0,0,0,0,0,0,0,0,0,0,-4867.73,112765.08,-104586.91,-27944049.88,536545695.41]

        gst= -(1/beta)*inv(P)*gamma'*(werr+k1*[qerr(2);qerr(3);qerr(4)]);
```

Where P is a matrix with dimension 15 which is the solution of the matrix inequality solved with the code shown in the appendix II.

8.1.4 Simulation of the control algorithm using Simulink

The attitude error dynamics, angular velocity error dynamics, internal model and control law designed simulated are:

$$\dot{\widetilde{q}} = \frac{1}{2}E(\widetilde{q})\,\widetilde{w}$$

$$J\dot{\widetilde{w}} = f(R(\widetilde{q}), \widetilde{w}, R_{od}, w_{od}^{d}, \mu) + u^{b}$$

$$\dot{\xi} = \Phi\xi - \frac{1}{\gamma}P^{-1}\Gamma^{T}(\widetilde{w} + k_1\widetilde{q})$$

$$u^{b} = \Gamma\xi - k_2(1 + \left|\left|\widetilde{w} + k_1\widetilde{q}\right|\right|)(\widetilde{w} + k_1\widetilde{q})$$

Now it can be seen how it has been implemented in Simulink:

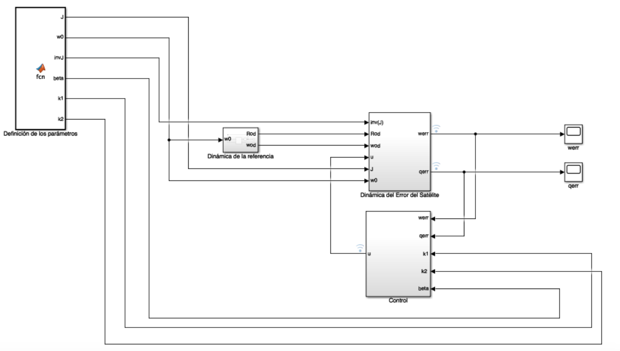


Figure 11: Overview of the controller and system

Now the controller sum of the internal model and stabilizer can be seen:

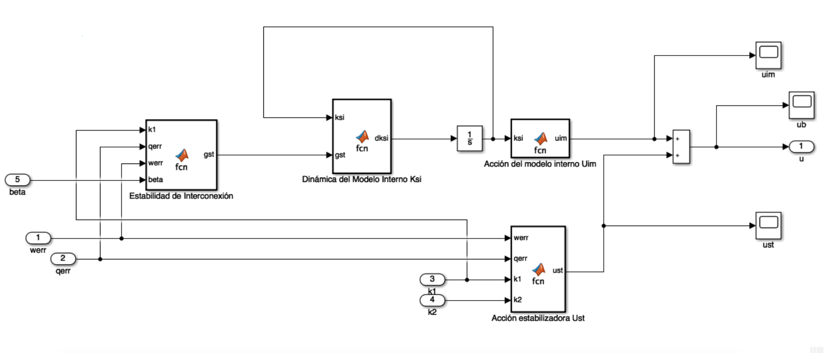


Figure 12: Overview of the controller

In addition the dynamics block is showed below. As it can be seen there are two blocks "tau0" and "taugrav", these are the torques created by the satellite's inertia and by the gravitational force.

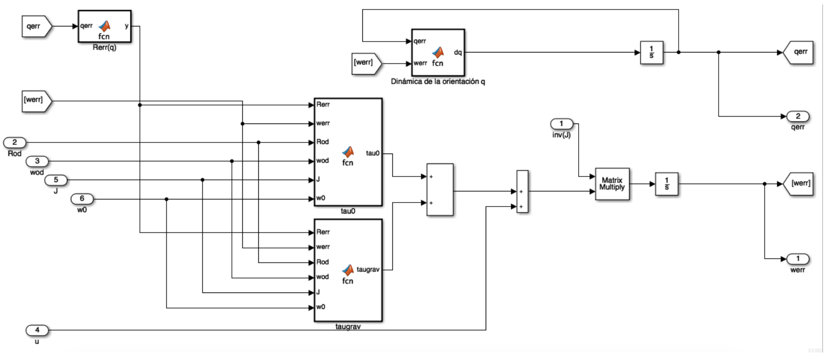


Figure 13: Overview of the block "Error dynamics of the satellite" "

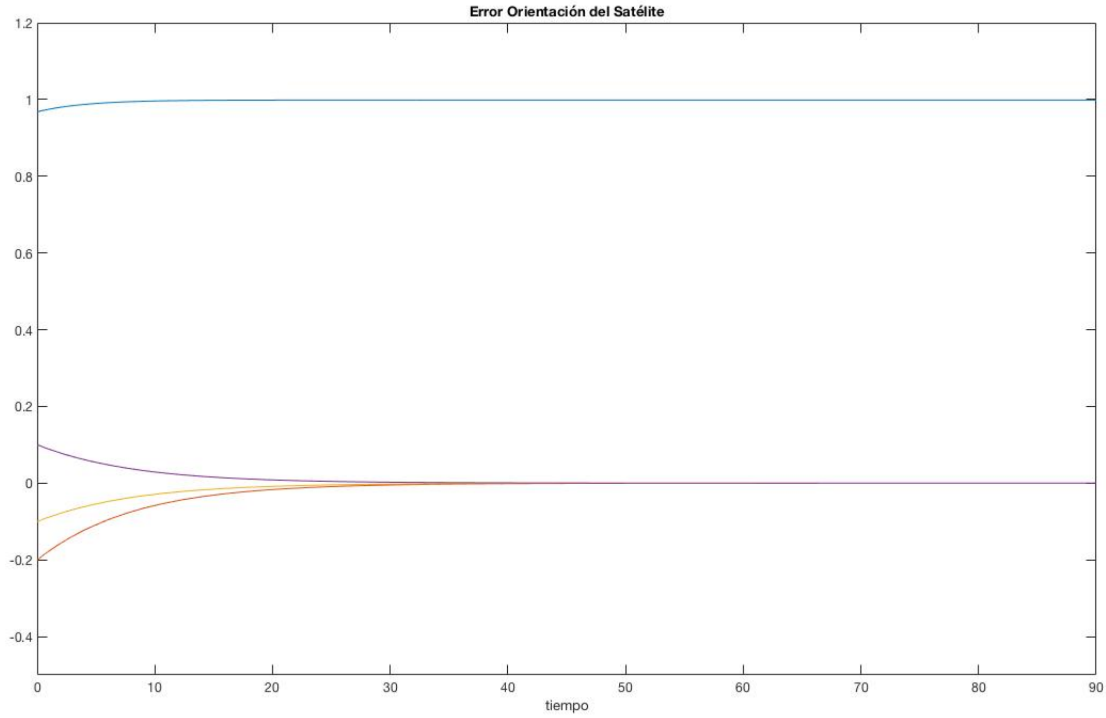


Figure 14: Attitude error of the satellite

The results of the simulation can be shown below:

The error of the angular velocity can be seen below:

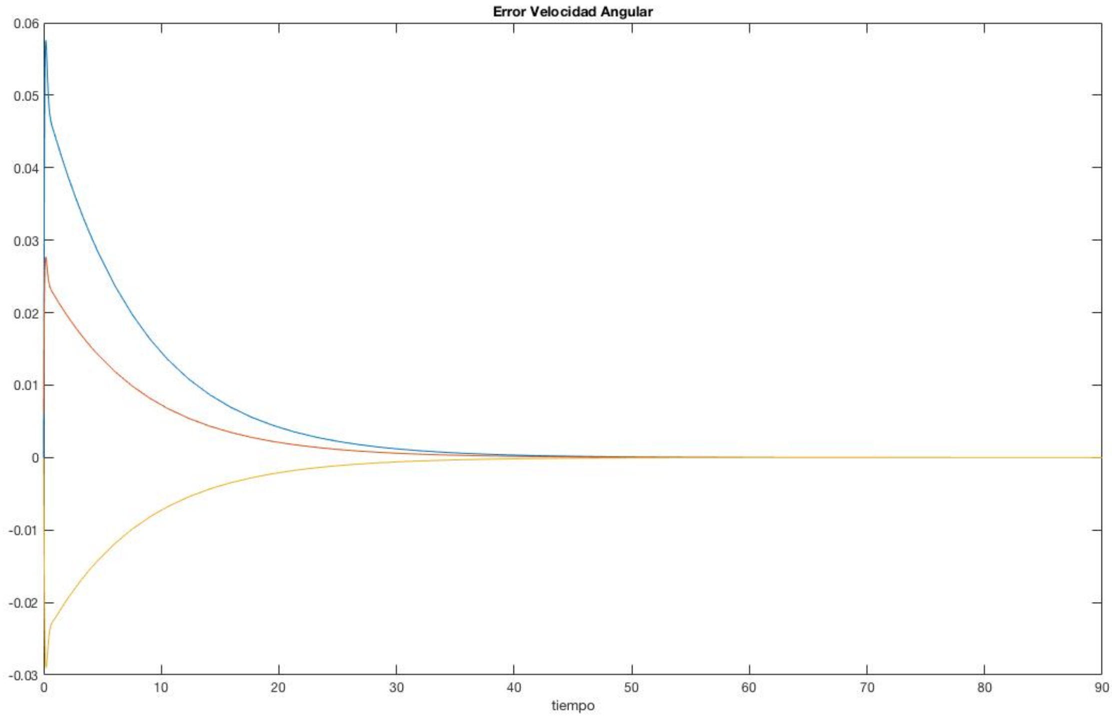


Figure 15: Angular velocity error $\widetilde{w}$

And the control law:

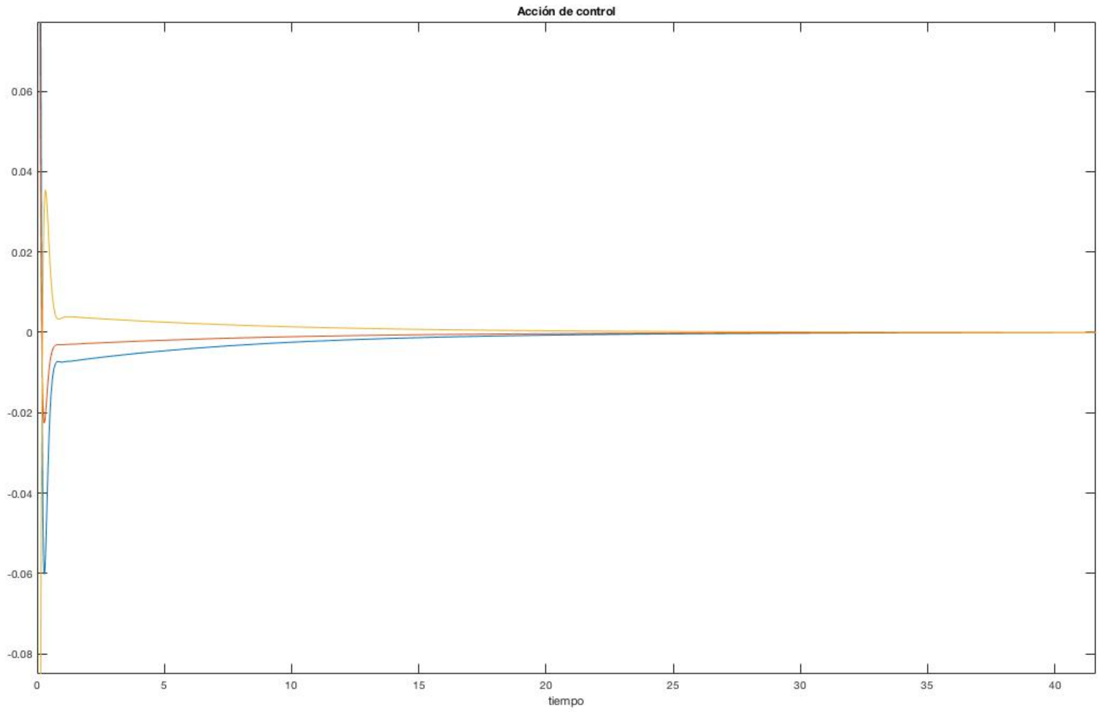


Figure 16: Control law $u^b(t)$

8.2 Mission: Flight oriented towards the Earth

8.2.1 Mission goal

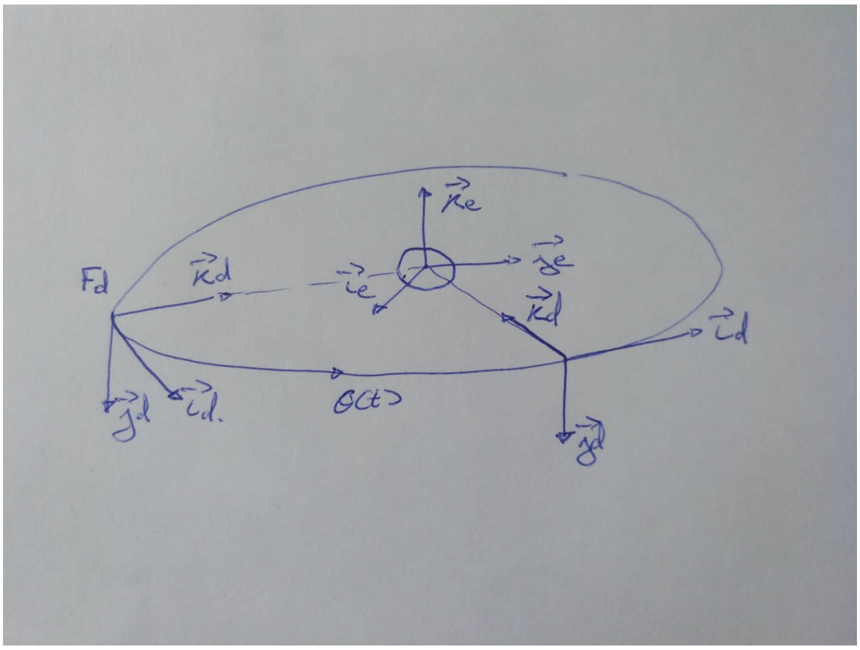


Figure 17: Mission goal: Earth oriented

Like the previous mission, the initial condition of the reference system are:

$$w_{od}^{d}(0) = 0 \qquad R_{od}(0) = I$$

And the reference to track by the satellite has the following form:

$$w_{od}^{d}(t) = 0$$

and

$$R_{od}(t) = I$$

which can be seen as a disturbance created by the dynamic system:

$$\begin{cases} \dot{R}_{od} = R_{od}\, Skew(w_{od}^{d}) \\ \dot{w}_{od}^{d} = 0 \\ \dot{\mu} = 0 \end{cases}$$

The uncertainty of the state parameters $\mu = \begin{pmatrix} J \\ w_0 \end{pmatrix}$ can be seen as weell as an exogenous input of the system.

8.2.2 Design of the control law by an internal model approach u_{im}

The other mission aims to keep the satellite in orbit facing the Earth at any moment. In this case, the satellite's attitude needs to stay in the same reference system as the orbital's one. This means keeping the angular velocity of the satellite equal to zero and the attitude of the satellite equal to the orbital's system.

In this case the dynamics equations are similar to the previously mentioned with

$$R_{od} = I \qquad w^d_{od} = 0$$

Where the desired angular acceleration and velocity in respect to the Earth are:

$$w^b_{ed} = \widetilde{R}^T w^d_{ed} = \widetilde{R}^T[w^d_{od} - w_0 R^T_{od} j] \Longrightarrow w^b_{ed} = -w_0 \widetilde{R}^T \mathrm{j}$$

$$\dot{w}^b_{ed} = \frac{d}{dt}(\widetilde{R}^T[w^d_{od} - w_0 R^T_{od}\mathrm{j}]) \Longrightarrow \dot{w}^b_{ed} = w_0 \widetilde{w} \times (\widetilde{R}^T \mathrm{j})$$

The control law to be replicated by the internal model would be:

$$u^b = c(R_{od}, w^d_{od}, \mu) = -\tau^b_0(I, 0, R_{od}, w^d_{od}, \mu) - \tau^b_{grav}(I, R_{od}, \mu)$$

Where the inertial torque is:

$$\tau^b_0(I, 0, R_{od}, w^d_{od}, \mu) = -w_0^2 j \times (Jj) = -\mathrm{w}_0^2 \begin{pmatrix} J_{yz} \\ 0 \\ J_{xy} \end{pmatrix}$$

And the gravitational torque:

$$\tau^b_{grav}(I, R_{od}, \mu) = 3w_0^2 \begin{pmatrix} J_{yz} \\ -J_{xz} \\ 0 \end{pmatrix}$$

Due to the fact that every terms are constant, it can be concluded that the feedback control that makes $R_{ob}(t) = R_{od}(t)$ is constant. It has to be remarked that the uncertainty's parameters μ are constant. Therefore the control law

$$u^b(t) = c(R_{od}(t), w^d_{od}(0), \mu(0)) = c(I, 0, \mu)$$

can be generated by an internal model:

$$\dot{\xi} = 0$$

$$u = \xi$$

with initial condition:

$$\xi(0) = c(I, 0, \mu) = -\tau^b_0(I, 0, R_{od}, w^d_{od}, \mu) - \tau^b_{grav}(I, R_{od}, \mu)$$

$$= \mathrm{w}_0^2 \begin{pmatrix} J_{yz} \\ 0 \\ J_{xy} \end{pmatrix} - 3\mathrm{w}_0^2 \begin{pmatrix} J_{yz} \\ -J_{xz} \\ 0 \end{pmatrix}$$

8.2.3 Control algorith simulation

In the picture below the law control is implemented in simulink. In this case it can be shown two main blocks, the internal model control law and the stabilizer control law. In this case the internal model does not have dynamics and is constant, this is due to the fact that the disturbance introduced by gravity is constant because the desired position of the satellite is constant. Therefore, the internal model has just to counteract the initial value.

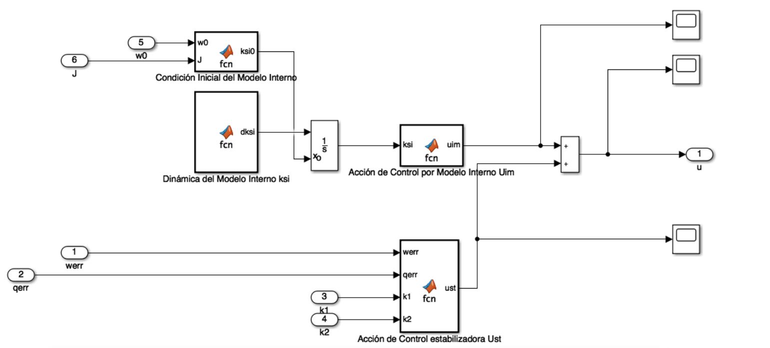


Figure 18: Internal model and stabilizer

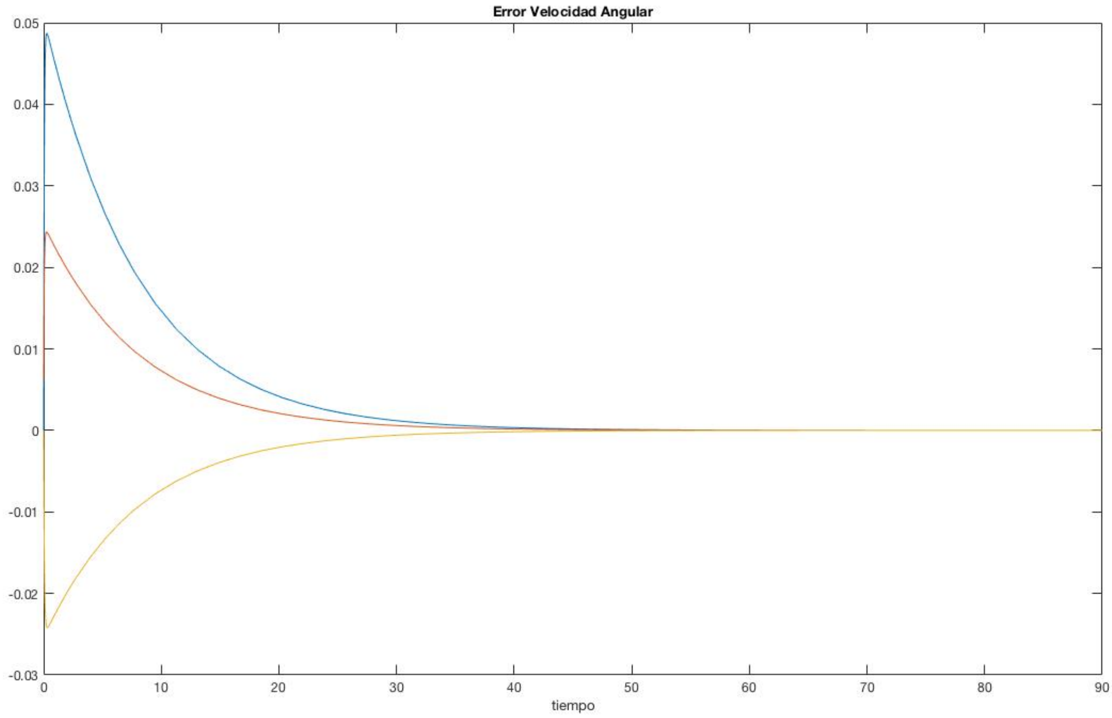


Figure 19: Angular velocity error $\widetilde{w}$

Below, a simulation of the angular error can be seen:

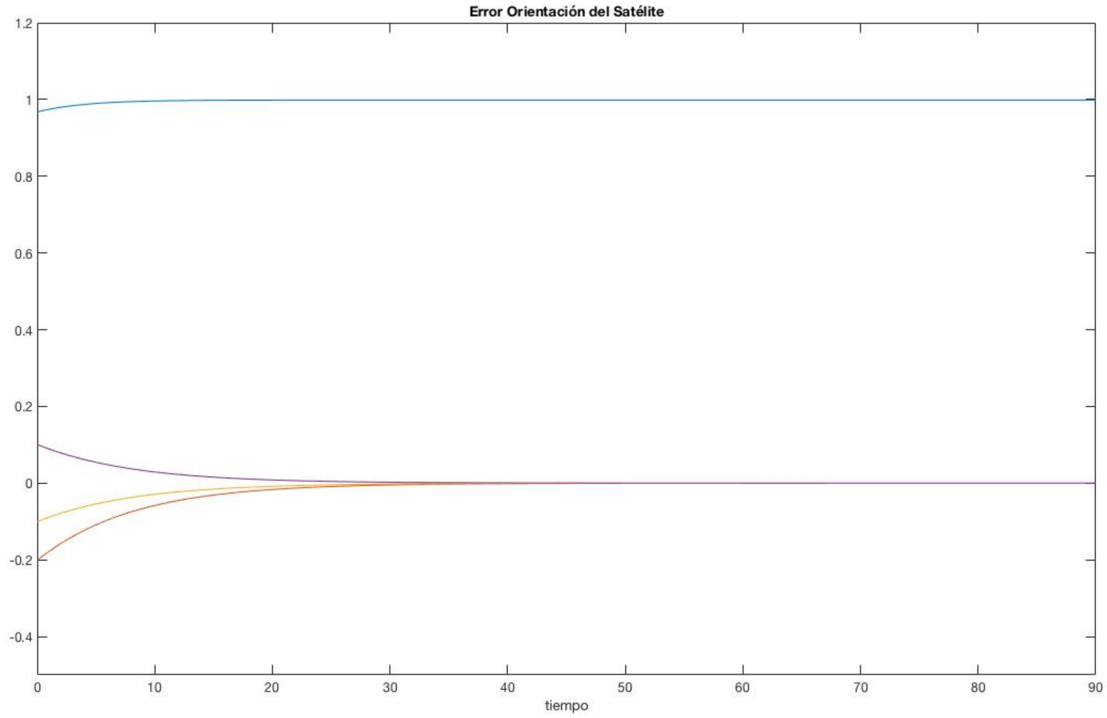


Figure 20: Attitude error $\widetilde{q}$

The attitude error is:

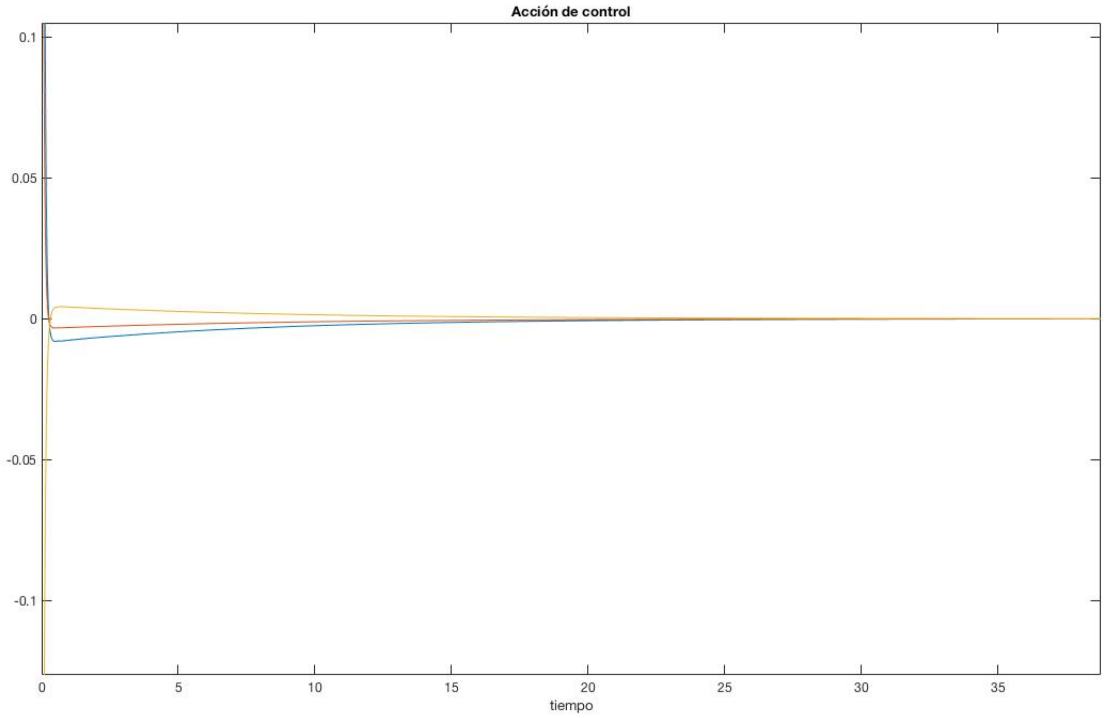


Figure 21: Control law u^b

In addition the control law representation is:

8.3 Conclusión

Different simulations have been shown for two different settings. As it can be seen the regulation error for the attitude and angular velocity are successfully regulated. In addition, the control law, both the internal model law and stabilizer law are shown.

Therefore, it has been shown that the statements discussed at the beginning about the regulation equations and internal model equations are emperically satisfied by a numeric simulation. In addition, given an initial condition, it has been shown that the control law derived by Lyapunov arguments achieves to stabilize the system. Future developments about this project could include more sources of disturbances like solar radiation or further explanations of the mathematical ideas and developments.

APPENDIX I

A quaternion can be given by a scalar part q_0 and a vector $(q_1\ q_2\ q_3)$

$$q = \begin{pmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix}$$

Each quaternion can be represented using a rotation matrix

$$R(\mathbf{q}) = \begin{pmatrix} 1 - 2\mathrm{q}_2^2 - 2\mathrm{q}_3^2 & 2\mathrm{q}_1\mathrm{q}_2 - 2\mathrm{q}_0\mathrm{q}_3 & 2\mathrm{q}_1\mathrm{q}_3 + 2\mathrm{q}_0\mathrm{q}_2 \\ 2\mathrm{q}_1\mathrm{q}_2 + 2\mathrm{q}_0\mathrm{q}_3 & 1 - 2\mathrm{q}_1^2 - 2\mathrm{q}_3^2 & 2\mathrm{q}_2\mathrm{q}_3 - 2\mathrm{q}_0\mathrm{q}_1 \\ 2\mathrm{q}_1\mathrm{q}_3 - 2\mathrm{q}_0\mathrm{q}_2 & 2\mathrm{q}_2\mathrm{q}_3 + 2\mathrm{q}_0\mathrm{q}_1 & 1 - 2\mathrm{q}_1^2 - 2\mathrm{q}_2^2 \end{pmatrix}$$

Indeed it can be shown that any rotation matrix R can be represented by a quaternion $\mathbf{q}$

Given two quaternions q y p, the following is true:

1. $\mathbf{q} + \mathbf{p} = \begin{pmatrix} \mathrm{q}_0 + \mathrm{p}_0 \\ \mathrm{q}+\mathrm{p} \end{pmatrix}$

2. $\mathbf{q} * \mathbf{p} = \begin{pmatrix} \mathrm{q}_0 & -\mathrm{q}^{\mathrm{T}} \\ \mathrm{q} & \mathrm{q}_0\mathrm{I} + \mathrm{Skew}(\mathrm{q}) \end{pmatrix} \begin{pmatrix} p_0 \\ p \end{pmatrix}$

3. A sequence of rotations using rotation matrix

$$R_{ac} = R_{ab}R_{bc}$$

, can be calculated using quaternions

$$\mathbf{q_{ac}} = \mathbf{q_{ab}} * \mathbf{q_{bc}}$$

4. The solution to the differential equation for rotation matrix:

$$\dot{R} = RSkew(w)$$

, can be seen using quaternions as

$$\dot{\mathbf{q}} = \frac{1}{2}\mathrm{E}(\mathbf{q})\mathrm{w}$$

, where

$$E(\mathbf{q}) = \begin{pmatrix} -\mathrm{q}^{\mathrm{T}} \\ \mathrm{q}_0\mathrm{I} + \mathrm{Skew}(\mathrm{q}) \end{pmatrix}$$

APPENDIX II

In this part a calculation of the positive P matrix is made. This is useful to proof stability using Lyapunov's Theorem:

$$PA + A^T P \leq 0$$

Using matlab a linear matrix inequality will be solved for the solution P

$$\begin{bmatrix} PA + A^T P & 0 \\ 0 & -P \end{bmatrix} < 0$$

The code in matlab is the following:

```
>> %It starts here, the information available is the matrix A
>> %Setting of the LMIs solver
>> setlmis([])
>> %Specification of the structure of P
>>
>> P= lmivar(1, [size(A,1) 1])
>>% Definition of the LMI
>>
>> lmiterm([1 1 1 P], 1, A, 's');
>> lmiterm([1 1 2 0], 1);
>> lmiterm([1 2 2 P], -1, 1);
>>
>>%Solution
>>
>> LMISYS= getlmis;
>> [tmin, Psol]= feasp(LMISYS);
>> P= dec2mat(LMISYS, Psol, P)
```

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