Advanced Mathematics Textbook
A Comprehensive Guide

Your Name

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Preface

This textbook provides a comprehensive introduction to advanced mathematical concepts. Each chapter builds upon previous knowledge while introducing new ideas and techniques essential for higher mathematics.

The book is organized into several major areas of mathematics, from foundational algebra and calculus through more advanced topics in analysis, linear algebra, and discrete mathematics. The approach follows classical treatments found in works such as Rudin [4] for analysis, Dummit and Foote [1] for abstract algebra, and Strang [6] for linear algebra.

Mathematical rigor is emphasized throughout, with careful attention to definitions, theorems, and proofs as advocated by Spivak [5]. The probability chapter draws from the foundational work of Ross [3], while discrete mathematics concepts follow the comprehensive treatment in Rosen [2].

Chapter 1

Mathematical Foundations

This chapter establishes the fundamental concepts and notation that will be used throughout the textbook. We begin with set theory, logic, and proof techniques.

1.1 Set Theory

Definition

Definition 1.1 (Set). A **set** is a well-defined collection of distinct objects, called elements or members of the set.

Example

Example 1.1. The set of natural numbers less than 5 can be written as:

$$N_5 = \{1, 2, 3, 4\}$$

1.1.1 Set Operations

Let A and B be sets. We define the following operations:

Definition

Definition 1.2 (Union). The **union** of sets A and B is:

$$A \cup B = \{x : x \in A \text{ or } x \in B\}$$

Definition

Definition 1.3 (Intersection). The **intersection** of sets A and B is:

$$A \cap B = \{x : x \in A \text{ and } x \in B\}$$

> **Definition**
>
> **Definition 1.4** (Complement)**.** The **complement** of set A with respect to universal set U is:
> $$A^c = \{x \in U : x \notin A\}$$

1.2 Logic and Proof Techniques

1.2.1 Logical Statements

> **Definition**
>
> **Definition 1.5** (Proposition)**.** A **proposition** is a statement that is either true or false, but not both.

> **Example**
>
> **Example 1.2.** The following are propositions:
>
> - $2 + 2 = 4$ (True)
>
> - All prime numbers are odd (False, since 2 is prime and even)
>
> - $\sqrt{2}$ is irrational (True)

1.2.2 Methods of Proof

> **Definition**
>
> **Definition 1.6** (Direct Proof)**.** A **direct proof** of a statement "If P, then Q" assumes P is true and uses logical steps to show that Q must be true.

> **Theorem**
>
> **Theorem 1.1.** If n is an even integer, then n^2 is even.

Proof. Let n be an even integer. Then $n = 2k$ for some integer k. We have:
$$n^2 = (2k)^2 = 4k^2 = 2(2k^2)$$
Since $2k^2$ is an integer, $n^2 = 2(2k^2)$ is even. $\square$

> **Definition**
>
> **Definition 1.7** (Proof by Contradiction)**.** A **proof by contradiction** assumes the negation of what we want to prove and shows this leads to a logical contradiction.

> **Theorem**
>
> **Theorem 1.2.** $\sqrt{2}$ is irrational.

Proof. Assume for contradiction that $\sqrt{2}$ is rational. Then $\sqrt{2} = \frac{p}{q}$ where p and q are integers with $\gcd(p, q) = 1$.

Squaring both sides: $2 = \frac{p^2}{q^2}$, so $2q^2 = p^2$.

This means p^2 is even, which implies p is even. Let $p = 2r$ for some integer r.

Substituting: $2q^2 = (2r)^2 = 4r^2$, so $q^2 = 2r^2$.

This means q^2 is even, which implies q is even.

But if both p and q are even, then $\gcd(p, q) \geq 2$, contradicting our assumption that $\gcd(p, q) = 1$.

Therefore, $\sqrt{2}$ is irrational. $\qquad\square$

1.3 Functions and Relations

> **Definition**
>
> **Definition 1.8** (Function). A **function** $f : A \to B$ is a relation that assigns to each element $a \in A$ exactly one element $f(a) \in B$.

> **Definition**
>
> **Definition 1.9** (Injective Function). A function $f : A \to B$ is **injective** (one-to-one) if for all $a_1, a_2 \in A$:
> $$f(a_1) = f(a_2) \implies a_1 = a_2$$

> **Definition**
>
> **Definition 1.10** (Surjective Function). A function $f : A \to B$ is **surjective** (onto) if for all $b \in B$, there exists $a \in A$ such that $f(a) = b$.

> **Definition**
>
> **Definition 1.11** (Bijective Function). A function is **bijective** if it is both injective and surjective.

1.4 Exercises

1. Prove that the intersection of two sets is commutative: $A \cap B = B \cap A$.

2. Show that if $f : A \to B$ and $g : B \to C$ are both injective, then $g \circ f : A \to C$ is injective.

3. Prove by contradiction that there are infinitely many prime numbers.

4. Let $A = \{1, 2, 3, 4\}$ and $B = \{2, 4, 6, 8\}$. Find $A \cup B$, $A \cap B$, and $A \setminus B$.

Chapter 2

Abstract Algebra

This chapter introduces the fundamental algebraic structures that form the foundation of modern algebra.

2.1 Groups

Definition

Definition 2.1 (Binary Operation). A **binary operation** on a set S is a function $* : S \times S \to S$ that assigns to each ordered pair (a, b) of elements in S an element $a * b$ in S.

Definition

Definition 2.2 (Group). A **group** is a set G together with a binary operation $*$ such that:

1. **Closure:** For all $a, b \in G$, we have $a * b \in G$.

2. **Associativity:** For all $a, b, c \in G$, we have $(a * b) * c = a * (b * c)$.

3. **Identity:** There exists an element $e \in G$ such that for all $a \in G$, $e * a = a * e = a$.

4. **Inverse:** For each $a \in G$, there exists an element $a^{-1} \in G$ such that $a * a^{-1} = a^{-1} * a = e$.

> **Example**
>
> **Example 2.1** (The Integers under Addition). The set $\mathbb{Z}$ with the operation of addition forms a group:
>
> - Closure: If $a, b \in \mathbb{Z}$, then $a + b \in \mathbb{Z}$
>
> - Associativity: $(a + b) + c = a + (b + c)$ for all $a, b, c \in \mathbb{Z}$
>
> - Identity: 0 is the identity element since $a + 0 = 0 + a = a$
>
> - Inverse: For each $a \in \mathbb{Z}$, the inverse is $-a$ since $a + (-a) = 0$

> **Theorem**
>
> **Theorem 2.1.** In any group G, the identity element is unique.

Proof. Suppose e and e' are both identity elements in G. Then:

$$e = e * e' = e'$$

where the first equality uses the fact that e' is an identity, and the second equality uses the fact that e is an identity. $\square$

2.1.1   Subgroups

> **Definition**
>
> **Definition 2.3** (Subgroup). Let $(G, *)$ be a group. A subset $H \subseteq G$ is a **subgroup** of G if $(H, *)$ is itself a group.

> **Theorem**
>
> **Theorem 2.2** (Subgroup Test). A non-empty subset H of a group G is a subgroup if and only if:
>
> 1. For all $a, b \in H$, we have $ab \in H$ (closure)
>
> 2. For all $a \in H$, we have $a^{-1} \in H$ (inverse)

2.2 Rings

> **Definition**
>
> **Definition 2.4** (Ring). A **ring** is a set R with two binary operations, addition $(+)$ and multiplication $(\cdot)$, such that:
>
> 1. $(R, +)$ is an abelian group
>
> 2. Multiplication is associative
>
> 3. The distributive laws hold: $a \cdot (b + c) = a \cdot b + a \cdot c$ and $(a + b) \cdot c = a \cdot c + b \cdot c$

> **Example**
>
> **Example 2.2** (The Integers). The set $\mathbb{Z}$ with ordinary addition and multiplication forms a ring.

> **Definition**
>
> **Definition 2.5** (Field). A **field** is a commutative ring with unity in which every non-zero element has a multiplicative inverse.

> **Example**
>
> **Example 2.3** (The Rational Numbers). The set $\mathbb{Q}$ of rational numbers forms a field under ordinary addition and multiplication.

2.3 Vector Spaces

> **Definition**
>
> **Definition 2.6** (Vector Space). Let F be a field. A **vector space** over F is a set V together with operations of vector addition and scalar multiplication such that:
>
> 1. $(V, +)$ is an abelian group
>
> 2. For all $\alpha \in F$ and $\mathbf{v} \in V$, we have $\alpha \mathbf{v} \in V$
>
> 3. $\alpha(\mathbf{u} + \mathbf{v}) = \alpha \mathbf{u} + \alpha \mathbf{v}$
>
> 4. $(\alpha + \beta)\mathbf{v} = \alpha \mathbf{v} + \beta \mathbf{v}$
>
> 5. $(\alpha\beta)\mathbf{v} = \alpha(\beta\mathbf{v})$
>
> 6. $1\mathbf{v} = \mathbf{v}$ where 1 is the multiplicative identity in F

> **Example**
>
> **Example 2.4** (Euclidean Space)**.** The set $\mathbb{R}^n$ with componentwise addition and scalar multiplication forms a vector space over $\mathbb{R}$.

> **Definition**
>
> **Definition 2.7** (Linear Independence)**.** Vectors $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}$ in a vector space V are **linearly independent** if the only solution to
>
> $$\alpha_1 \mathbf{v_1} + \alpha_2 \mathbf{v_2} + \cdots + \alpha_n \mathbf{v_n} = \mathbf{0}$$
>
> is $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$.

> **Definition**
>
> **Definition 2.8** (Basis)**.** A **basis** for a vector space V is a set of vectors that is both linearly independent and spans V.

2.4   Exercises

1. Prove that in any group, each element has a unique inverse.

2. Show that the set of even integers forms a subgroup of $(\mathbb{Z}, +)$.

3. Verify that the set $\mathbb{Z}/n\mathbb{Z}$ forms a ring under addition and multiplication modulo n.

4. Prove that if $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}$ are linearly independent, then any subset is also linearly independent.

5. Find a basis for the vector space of 2×2 matrices over $\mathbb{R}$.

Chapter 3

Calculus

This chapter covers the fundamental concepts of differential and integral calculus, building upon the foundations established in previous chapters.

3.1 Limits and Continuity

Definition

Definition 3.1 (Limit). Let f be a function defined on some open interval containing a, except possibly at a itself. We say that

$$\lim_{x \to a} f(x) = L$$

if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that whenever $0 < |x - a| < \delta$, we have $|f(x) - L| < \varepsilon$.

Example

Example 3.1. To show that $\lim_{x \to 2}(3x - 1) = 5$:
Given $\varepsilon > 0$, we need $|(3x - 1) - 5| < \varepsilon$, which simplifies to $|3x - 6| = 3|x - 2| < \varepsilon$. Taking $\delta = \frac{\varepsilon}{3}$, whenever $0 < |x - 2| < \delta$, we have:

$$|(3x - 1) - 5| = 3|x - 2| < 3 \cdot \frac{\varepsilon}{3} = \varepsilon$$

Definition

Definition 3.2 (Continuity). A function f is **continuous** at a if:

$$\lim_{x \to a} f(x) = f(a)$$

Theorem

Theorem 3.1 (Intermediate Value Theorem). If f is continuous on $[a, b]$ and k is any number between $f(a)$ and $f(b)$, then there exists a number $c \in (a, b)$ such that $f(c) = k$.

3.2   Differentiation

Definition

Definition 3.3 (Derivative). The **derivative** of a function f at a is:

$$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

provided this limit exists.

Theorem

Theorem 3.2 (Chain Rule). If g is differentiable at a and f is differentiable at $g(a)$, then the composite function $f \circ g$ is differentiable at a and:

$$(f \circ g)'(a) = f'(g(a)) \cdot g'(a)$$

Example

Example 3.2 (Computing a Derivative). Let $f(x) = \sin(x^2)$. Using the chain rule:

$$f'(x) = \cos(x^2) \cdot \frac{d}{dx}[x^2] = \cos(x^2) \cdot 2x = 2x \cos(x^2)$$

3.2.1   Applications of Derivatives

Theorem

Theorem 3.3 (Mean Value Theorem). If f is continuous on $[a, b]$ and differentiable on (a, b), then there exists a number $c \in (a, b)$ such that:

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Definition

Definition 3.4 (Critical Point). A **critical point** of a function f is a number c in the domain of f where either $f'(c) = 0$ or $f'(c)$ does not exist.

3.3 Integration

Definition

Definition 3.5 (Riemann Sum)**.** For a function f defined on $[a, b]$, a **Riemann sum** is:

$$S = \sum_{i=1}^{n} f(x_i^*) \Delta x_i$$

where $\Delta x_i = x_i - x_{i-1}$ and $x_i^* \in [x_{i-1}, x_i]$.

Definition

Definition 3.6 (Definite Integral)**.** The **definite integral** of f from a to b is:

$$\int_a^b f(x)\, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*) \Delta x$$

where $\Delta x = \frac{b-a}{n}$ and $x_i^* = a + i\Delta x$.

Theorem

Theorem 3.4 (Fundamental Theorem of Calculus)**.** If f is continuous on $[a, b]$, then:

1. If $g(x) = \int_a^x f(t)\, dt$, then $g'(x) = f(x)$

2. $\int_a^b f(x)\, dx = F(b) - F(a)$ where F is any antiderivative of f

---

Example

Example 3.3 (Definite Integral)**.** Evaluate $\int_0^1 x^2\, dx$:
Since $\frac{d}{dx}\left[\frac{x^3}{3}\right] = x^2$, we have:

$$\int_0^1 x^2\, dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1^3}{3} - \frac{0^3}{3} = \frac{1}{3}$$

3.4   Techniques of Integration

3.4.1   Integration by Parts

> **Theorem**
>
> **Theorem 3.5** (Integration by Parts).
>
> $$\int u \, dv = uv - \int v \, du$$

> **Example**
>
> **Example 3.4.** Evaluate $\int xe^x \, dx$:
> Let $u = x$ and $dv = e^x dx$. Then $du = dx$ and $v = e^x$.
>
> $$\int xe^x \, dx = xe^x - \int e^x \, dx = xe^x - e^x + C = e^x(x-1) + C$$

3.4.2   Trigonometric Substitution

For integrals involving $\sqrt{a^2 - x^2}$, $\sqrt{a^2 + x^2}$, or $\sqrt{x^2 - a^2}$, we can use trigonometric substitutions:

- For $\sqrt{a^2 - x^2}$: use $x = a \sin \theta$

- For $\sqrt{a^2 + x^2}$: use $x = a \tan \theta$

- For $\sqrt{x^2 - a^2}$: use $x = a \sec \theta$

3.5   Series

> **Definition**
>
> **Definition 3.7** (Infinite Series). An **infinite series** is an expression of the form:
>
> $$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$$

> **Definition**
>
> **Definition 3.8** (Convergence). A series $\sum_{n=1}^{\infty} a_n$ **converges** to S if:
>
> $$\lim_{n \to \infty} S_n = S$$
>
> where $S_n = \sum_{k=1}^{n} a_k$ is the n-th partial sum.

> **Theorem**
>
> **Theorem 3.6** (Ratio Test). For a series $\sum a_n$ with positive terms, let $L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$.
>
> - If $L < 1$, the series converges
>
> - If $L > 1$, the series diverges
>
> - If $L = 1$, the test is inconclusive

3.6 Exercises

1. Use the definition of limit to prove that $\lim_{x\to3}(2x+1) = 7$.

2. Find the derivative of $f(x) = \ln(\cos(x^2))$.

3. Use the Mean Value Theorem to show that $|\sin a - \sin b| \le |a - b|$ for all real numbers a and b.

4. Evaluate $\int_0^{\pi/2} x \sin x \, dx$ using integration by parts.

5. Determine the convergence of the series $\sum_{n=1}^{\infty} \frac{n!}{n^n}$.

Chapter 4

Linear Algebra

Linear algebra studies vector spaces and linear transformations between them. This chapter covers the essential concepts and computational techniques.

4.1   Matrices and Systems of Linear Equations

Definition

Definition 4.1 (Matrix). An $m \times n$ **matrix** is a rectangular array of numbers arranged in m rows and n columns:

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

Definition

Definition 4.2 (Matrix Addition). If A and B are both $m \times n$ matrices, then $A + B$ is the $m \times n$ matrix whose (i, j)-entry is $a_{ij} + b_{ij}$.

Definition

Definition 4.3 (Matrix Multiplication). If A is an $m \times p$ matrix and B is a $p \times n$ matrix, then AB is the $m \times n$ matrix whose (i, j)-entry is:

$$(AB)_{ij} = \sum_{k=1}^{p} a_{ik} b_{kj}$$

> **Example**
>
> **Example 4.1** (Matrix Multiplication). Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}$. Then:
>
> $$AB = \begin{pmatrix} 1 \cdot 5 + 2 \cdot 7 & 1 \cdot 6 + 2 \cdot 8 \\ 3 \cdot 5 + 4 \cdot 7 & 3 \cdot 6 + 4 \cdot 8 \end{pmatrix} = \begin{pmatrix} 19 & 22 \\ 43 & 50 \end{pmatrix}$$

4.1.1   Gaussian Elimination

> **Definition**
>
> **Definition 4.4** (Row Echelon Form). A matrix is in **row echelon form** if:
>
> 1. All nonzero rows are above any rows of all zeros
>
> 2. Each leading entry is in a column to the right of the leading entry in the row above it
>
> 3. All entries in a column below a leading entry are zeros

> **Definition**
>
> **Definition 4.5** (Reduced Row Echelon Form). A matrix is in **reduced row echelon form** if it is in row echelon form and:
>
> 1. The leading entry in each nonzero row is 1
>
> 2. Each leading 1 is the only nonzero entry in its column

4.2   Determinants

> **Definition**
>
> **Definition 4.6** (Determinant (2×2)). For a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$:
>
> $$\det(A) = ad - bc$$

Definition

Definition 4.7 (Determinant (n×n)). For an $n \times n$ matrix A, the determinant can be computed using cofactor expansion:

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} M_{ij}$$

where M_{ij} is the (i,j)-minor of A.

Theorem

Theorem 4.1 (Properties of Determinants). Let A and B be $n \times n$ matrices. Then:

1. $\det(AB) = \det(A) \det(B)$

2. $\det(A^T) = \det(A)$

3. If A is invertible, then $\det(A^{-1}) = \frac{1}{\det(A)}$

4. $\det(cA) = c^n \det(A)$ for scalar c

4.3   Vector Spaces and Subspaces

Definition

Definition 4.8 (Subspace). A subset W of a vector space V is a **subspace** if:

1. The zero vector is in W

2. W is closed under addition

3. W is closed under scalar multiplication

Example

Example 4.2 (Column Space). The **column space** of an $m \times n$ matrix A is the subspace of $\mathbb{R}^m$ spanned by the columns of A:

$$\text{Col}(A) = \{\mathbf{b} \in \mathbb{R}^m : A\mathbf{x} = \mathbf{b} \text{ has a solution}\}$$

Definition

Definition 4.9 (Null Space). The **null space** of an $m \times n$ matrix A is:

$$\text{Null}(A) = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{0}\}$$

4.4   Eigenvalues and Eigenvectors

Definition

Definition 4.10 (Eigenvalue and Eigenvector). Let A be an $n \times n$ matrix. A scalar λ is an **eigenvalue** of A if there exists a nonzero vector $\mathbf{v}$ such that:

$$A\mathbf{v} = \lambda\mathbf{v}$$

The vector $\mathbf{v}$ is called an **eigenvector** corresponding to λ.

Definition

Definition 4.11 (Characteristic Polynomial). The **characteristic polynomial** of an $n \times n$ matrix A is:

$$p(\lambda) = \det(A - \lambda I)$$

The eigenvalues of A are the roots of this polynomial.

Example

Example 4.3 (Finding Eigenvalues). For $A = \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}$:

$$A - \lambda I = \begin{pmatrix} 3 - \lambda & 1 \\ 0 & 2 - \lambda \end{pmatrix}$$

$$\det(A - \lambda I) = (3 - \lambda)(2 - \lambda) = \lambda^2 - 5\lambda + 6 = (\lambda - 2)(\lambda - 3)$$

So the eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = 3$.

Theorem

Theorem 4.2 (Diagonalization). An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors. In this case, $A = PDP^{-1}$ where D is diagonal and the columns of P are eigenvectors of A.

4.5 Inner Products and Orthogonality

> **Definition**
>
> **Definition 4.12** (Inner Product). An **inner product** on a real vector space V is a function $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ that satisfies:
>
> 1. $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$
>
> 2. $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$
>
> 3. $\langle c\mathbf{u}, \mathbf{v} \rangle = c\langle \mathbf{u}, \mathbf{v} \rangle$
>
> 4. $\langle \mathbf{v}, \mathbf{v} \rangle \geq 0$ with equality if and only if $\mathbf{v} = \mathbf{0}$

> **Definition**
>
> **Definition 4.13** (Orthogonal Vectors). Two vectors $\mathbf{u}$ and $\mathbf{v}$ are **orthogonal** if $\langle \mathbf{u}, \mathbf{v} \rangle = 0$.

> **Definition**
>
> **Definition 4.14** (Orthogonal Matrix). A square matrix Q is **orthogonal** if $Q^T Q = I$, or equivalently, $Q^{-1} = Q^T$.

> **Theorem**
>
> **Theorem 4.3** (Gram-Schmidt Process). Given linearly independent vectors $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_k}$, the Gram-Schmidt process produces orthogonal vectors $\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_k}$ such that span$\{\mathbf{u_1}, \ldots, \mathbf{u_j}\} = $ span$\{\mathbf{v_1}, \ldots, \mathbf{v_j}\}$ for each j.

4.6 Exercises

1. Compute AB and BA for $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$.

2. Find the determinant of $A = \begin{pmatrix} 2 & -1 & 3 \\ 1 & 0 & 4 \\ -2 & 1 & 1 \end{pmatrix}$.

3. Determine if the vectors $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $\begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$, and $\begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix}$ are linearly independent.

4. Find the eigenvalues and eigenvectors of $A = \begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix}$.

5. Use the Gram-Schmidt process to orthogonalize the vectors $\mathbf{v_1} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and $\mathbf{v_2} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$.

Chapter 5

Real Analysis

Real analysis provides the rigorous foundation for calculus and studies the properties of real numbers, sequences, series, and functions.

5.1 The Real Number System

> **Definition**
>
> **Definition 5.1** (Supremum)**.** Let $S \subseteq \mathbb{R}$ be bounded above. The **supremum** (or least upper bound) of S, denoted $\sup S$, is the smallest number that is greater than or equal to every element of S.

> **Theorem**
>
> **Theorem 5.1** (Completeness Axiom)**.** Every non-empty subset of $\mathbb{R}$ that is bounded above has a supremum in $\mathbb{R}$.

> **Definition**
>
> **Definition 5.2** (Archimedean Property)**.** For any real numbers x and y with $x > 0$, there exists a positive integer n such that $nx > y$.

5.2   Sequences and Series

> **Definition**
>
> **Definition 5.3** (Convergent Sequence)**.** A sequence (a_n) **converges** to L if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$:
>
> $$|a_n - L| < \varepsilon$$
>
> We write $\lim_{n \to \infty} a_n = L$.

> **Theorem**
>
> **Theorem 5.2** (Monotone Convergence Theorem)**.** Every bounded monotone sequence converges.

Proof. Let (a_n) be a bounded increasing sequence. Since (a_n) is bounded above, by the completeness axiom, $S = \{a_n : n \in \mathbb{N}\}$ has a supremum $L = \sup S$.

For any $\varepsilon > 0$, since L is the least upper bound, $L - \varepsilon$ is not an upper bound. Therefore, there exists N such that $a_N > L - \varepsilon$.

Since (a_n) is increasing, for all $n \geq N$:

$$L - \varepsilon < a_N \leq a_n \leq L < L + \varepsilon$$

Thus $|a_n - L| < \varepsilon$, proving convergence to L. $\square$

> **Definition**
>
> **Definition 5.4** (Cauchy Sequence)**.** A sequence (a_n) is **Cauchy** if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $m, n \geq N$:
>
> $$|a_m - a_n| < \varepsilon$$

> **Theorem**
>
> **Theorem 5.3** (Cauchy Criterion)**.** A sequence in $\mathbb{R}$ converges if and only if it is Cauchy.

5.3   Continuity and Uniform Continuity

> **Definition**
>
> **Definition 5.5** (Continuity at a Point)**.** A function $f : D \to \mathbb{R}$ is **continuous at** $c \in D$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x \in D$:
>
> $$|x - c| < \delta \implies |f(x) - f(c)| < \varepsilon$$

> **Definition**
>
> **Definition 5.6** (Uniform Continuity)**.** A function $f : D \to \mathbb{R}$ is **uniformly continuous** on D if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in D$:
>
> $$|x - y| < \delta \implies |f(x) - f(y)| < \varepsilon$$

> **Theorem**
>
> **Theorem 5.4** (Uniform Continuity on Compact Sets)**.** If f is continuous on a compact set K, then f is uniformly continuous on K.

5.4 Differentiation

> **Theorem**
>
> **Theorem 5.5** (Rolle's Theorem)**.** If f is continuous on $[a, b]$, differentiable on (a, b), and $f(a) = f(b)$, then there exists $c \in (a, b)$ such that $f'(c) = 0$.

> **Theorem**
>
> **Theorem 5.6** (Mean Value Theorem)**.** If f is continuous on $[a, b]$ and differentiable on (a, b), then there exists $c \in (a, b)$ such that:
>
> $$f'(c) = \frac{f(b) - f(a)}{b - a}$$

> **Definition**
>
> **Definition 5.7** (Uniform Convergence)**.** A sequence of functions (f_n) **converges uniformly** to f on a set E if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$ and all $x \in E$:
> $$|f_n(x) - f(x)| < \varepsilon$$

> **Theorem**
>
> **Theorem 5.7** (Uniform Convergence and Continuity)**.** If (f_n) is a sequence of continuous functions that converges uniformly to f on a set E, then f is continuous on E.

5.5   Integration

Definition

Definition 5.8 (Riemann Integrable)**.** A bounded function f on $[a, b]$ is **Riemann integrable** if:

$$\overline{\int_a^b} f = \underline{\int_a^b} f$$

where $\overline{\int_a^b} f$ and $\underline{\int_a^b} f$ are the upper and lower Darboux integrals, respectively.

Theorem

Theorem 5.8 (Riemann Integrability Criterion)**.** A bounded function f on $[a, b]$ is Riemann integrable if and only if the set of discontinuities of f has measure zero.

Theorem

Theorem 5.9 (Fundamental Theorem of Calculus (Rigorous))**.** Let f be Riemann integrable on $[a, b]$.

1. If $F(x) = \int_a^x f(t)\, dt$, then F is continuous on $[a, b]$.

2. If f is continuous at $c \in [a, b]$, then $F'(c) = f(c)$.

3. If g is differentiable on $[a, b]$ and $g' = f$, then $\int_a^b f(x)\, dx = g(b) - g(a)$.

5.6   Metric Spaces

Definition

Definition 5.9 (Metric Space)**.** A **metric space** is a set X together with a function $d : X \times X \to [0, \infty)$ such that for all $x, y, z \in X$:

1. $d(x, y) = 0$ if and only if $x = y$

2. $d(x, y) = d(y, x)$ (symmetry)

3. $d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality)

> **Definition**
>
> **Definition 5.10** (Open Ball). In a metric space (X, d), the **open ball** of radius $r > 0$ centered at $x \in X$ is:
> $$B_r(x) = \{y \in X : d(x, y) < r\}$$

> **Definition**
>
> **Definition 5.11** (Compact Set). A subset K of a metric space is **compact** if every open cover of K has a finite subcover.

> **Theorem**
>
> **Theorem 5.10** (Heine-Borel Theorem). A subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.

5.7 Exercises

1. Prove that $\lim_{n\to\infty} \frac{1}{n} = 0$ using the definition of convergence.

2. Show that the sequence $a_n = \frac{n}{n+1}$ is Cauchy.

3. Prove that $f(x) = x^2$ is uniformly continuous on any bounded interval $[a, b]$.

4. Use the Mean Value Theorem to prove that $|\sin x - \sin y| \leq |x - y|$ for all real x, y.

5. Show that the function $f(x) = \frac{1}{x}$ is not uniformly continuous on $(0, 1)$.

Chapter 6

Discrete Mathematics

Discrete mathematics studies mathematical structures that are fundamentally discrete rather than continuous, forming the foundation for computer science and combinatorics.

6.1 Graph Theory

> **Definition**
>
> **Definition 6.1** (Graph). A **graph** $G = (V, E)$ consists of a finite set V of vertices (or nodes) and a set E of edges, where each edge connects two vertices.

> **Definition**
>
> **Definition 6.2** (Degree). The **degree** of a vertex v, denoted $\deg(v)$, is the number of edges incident to v.

> **Theorem**
>
> **Theorem 6.1** (Handshaking Lemma). For any graph $G = (V, E)$:
>
> $$\sum_{v \in V} \deg(v) = 2|E|$$

Proof. Each edge contributes 1 to the degree of each of its two endpoints, so the sum of all degrees counts each edge exactly twice. $\square$

> **Definition**
>
> **Definition 6.3** (Path). A **path** in a graph is a sequence of vertices $v_1, v_2, \ldots, v_k$ such that for each i, there is an edge between v_i and v_{i+1}.

> **Definition**
>
> **Definition 6.4** (Connected Graph). A graph is **connected** if there is a path between every pair of vertices.

> **Definition**
>
> **Definition 6.5** (Tree). A **tree** is a connected graph with no cycles.

> **Theorem**
>
> **Theorem 6.2** (Tree Characterization). For a graph G with n vertices, the following are equivalent:
>
> 1. G is a tree
>
> 2. G is connected and has $n - 1$ edges
>
> 3. G is acyclic and has $n - 1$ edges
>
> 4. There is exactly one path between any two vertices in G

6.2   Combinatorics

> **Definition**
>
> **Definition 6.6** (Permutation). A **permutation** of n distinct objects is an arrangement of these objects in a specific order. The number of permutations of n objects is $n!$.

> **Definition**
>
> **Definition 6.7** (Combination). A **combination** is a selection of objects where order does not matter. The number of ways to choose k objects from n objects is:
>
> $$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

> **Theorem**
>
> **Theorem 6.3** (Binomial Theorem). For any real numbers x and y and positive integer n:
>
> $$(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

Example

Example 6.1 (Counting Problem). How many ways can we arrange the letters in "MATHEMATICS"?

The word has 11 letters with repetitions: M(2), A(2), T(2), H(1), E(1), I(1), C(1), S(1).

The number of arrangements is:

$$\frac{11!}{2! \cdot 2! \cdot 2! \cdot 1! \cdot 1! \cdot 1! \cdot 1! \cdot 1!} = \frac{11!}{2^3} = \frac{39,916,800}{8} = 4,989,600$$

6.2.1 Inclusion-Exclusion Principle

Theorem

Theorem 6.4 (Inclusion-Exclusion Principle). For finite sets $A_1, A_2, \ldots, A_n$:

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = \sum_i |A_i| - \sum_{i<j} |A_i \cap A_j| \tag{6.1}$$

$$+ \sum_{i<j<k} |A_i \cap A_j \cap A_k| - \cdots + (-1)^{n+1} |A_1 \cap A_2 \cap \cdots \cap A_n| \tag{6.2}$$

6.3 Number Theory

Definition

Definition 6.8 (Divisibility). An integer a **divides** an integer b (written $a \mid b$) if there exists an integer k such that $b = ak$.

Theorem

Theorem 6.5 (Division Algorithm). For any integers a and b with $b > 0$, there exist unique integers q and r such that:

$$a = bq + r \quad \text{where } 0 \leq r < b$$

Definition

Definition 6.9 (Greatest Common Divisor). The **greatest common divisor** of integers a and b, denoted $\gcd(a, b)$, is the largest positive integer that divides both a and b.

> **Theorem**
>
> **Theorem 6.6** (Euclidean Algorithm)**.** The Euclidean algorithm computes $\gcd(a, b)$ by repeatedly applying:
> $$\gcd(a, b) = \gcd(b, a \bmod b)$$
> until one of the numbers becomes 0.

> **Theorem**
>
> **Theorem 6.7** (Bézout's Identity)**.** For any integers a and b, there exist integers x and y such that:
> $$\gcd(a, b) = ax + by$$

6.3.1   Prime Numbers

> **Definition**
>
> **Definition 6.10** (Prime Number)**.** A positive integer $p > 1$ is **prime** if its only positive divisors are 1 and p.

> **Theorem**
>
> **Theorem 6.8** (Fundamental Theorem of Arithmetic)**.** Every integer greater than 1 can be expressed uniquely (up to order) as a product of prime numbers.

> **Theorem**
>
> **Theorem 6.9** (Infinitude of Primes)**.** There are infinitely many prime numbers.

Euclid's Proof. Suppose there are only finitely many primes $p_1, p_2, \ldots, p_k$. Consider:

$$N = p_1 p_2 \cdots p_k + 1$$

Since $N > 1$, it has a prime divisor p. But p cannot be any of $p_1, \ldots, p_k$ since $N \equiv 1 \pmod{p_i}$ for each i. This contradicts our assumption that we listed all primes.   □

6.4   Recurrence Relations

> **Definition**
>
> **Definition 6.11** (Recurrence Relation)**.** A **recurrence relation** is an equation that defines a sequence recursively, expressing each term in terms of previous terms.

Example

Example 6.2 (Fibonacci Sequence). The Fibonacci sequence is defined by:

$$F_0 = 0, \quad F_1 = 1, \quad F_n = F_{n-1} + F_{n-2} \text{ for } n \geq 2$$

Theorem

Theorem 6.10 (Linear Homogeneous Recurrence Relations). The recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ with characteristic equation $r^2 - c_1 r - c_2 = 0$ has:

- If roots $r_1 \neq r_2$: general solution $a_n = A r_1^n + B r_2^n$

- If repeated root r: general solution $a_n = (A + Bn) r^n$

6.5 Boolean Algebra

Definition

Definition 6.12 (Boolean Algebra). A **Boolean algebra** is a set B with two binary operations $\wedge$ (AND) and $\vee$ (OR), a unary operation $\neg$ (NOT), and constants 0 and 1, satisfying certain axioms.

Theorem

Theorem 6.11 (De Morgan's Laws). For any Boolean expressions A and B:

$$\neg(A \wedge B) = \neg A \vee \neg B \tag{6.3}$$
$$\neg(A \vee B) = \neg A \wedge \neg B \tag{6.4}$$

Definition

Definition 6.13 (Boolean Function). A **Boolean function** is a function $f : \{0,1\}^n \to \{0,1\}$ that maps n-tuples of Boolean values to a Boolean value.

6.6 Exercises

1. Prove that in any graph, the number of vertices with odd degree is even.

2. Find the number of ways to distribute 10 identical balls into 4 distinct boxes.

3. Use the inclusion-exclusion principle to find the number of integers from 1 to 100 that are divisible by 2, 3, or 5.

4. Find $\gcd(252, 198)$ using the Euclidean algorithm and express it in the form $252x + 198y$.

5. Solve the recurrence relation $a_n = 5a_{n-1} - 6a_{n-2}$ with initial conditions $a_0 = 1, a_1 = 0$.

Chapter 7

Probability Theory

Probability theory provides the mathematical foundation for analyzing random phenomena and uncertainty.

7.1 Sample Spaces and Events

> **Definition**
>
> **Definition 7.1** (Sample Space). A **sample space** Ω is the set of all possible outcomes of a random experiment.

> **Definition**
>
> **Definition 7.2** (Event). An **event** is a subset of the sample space Ω.

> **Example**
>
> **Example 7.1** (Coin Flipping). For the experiment of flipping a coin twice:
>
> - Sample space: $\Omega = \{HH, HT, TH, TT\}$
> - Event "at least one head": $A = \{HH, HT, TH\}$
> - Event "exactly one tail": $B = \{HT, TH\}$

7.2 Probability Measures

> **Definition**
>
> **Definition 7.3** (Probability Measure). A **probability measure** P on a sample space Ω is a function that assigns to each event A a number $P(A)$ satisfying:
>
> 1. $P(A) \geq 0$ for all events A
>
> 2. $P(\Omega) = 1$
>
> 3. If $A_1, A_2, \ldots$ are pairwise disjoint events, then:
>
> $$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

> **Theorem**
>
> **Theorem 7.1** (Basic Properties of Probability). For any events A and B:
>
> 1. $P(\emptyset) = 0$
>
> 2. $P(A^c) = 1 - P(A)$
>
> 3. If $A \subseteq B$, then $P(A) \leq P(B)$
>
> 4. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

7.3 Conditional Probability

> **Definition**
>
> **Definition 7.4** (Conditional Probability). The **conditional probability** of event A given event B with $P(B) > 0$ is:
>
> $$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

> **Theorem**
>
> **Theorem 7.2** (Law of Total Probability). If $B_1, B_2, \ldots, B_n$ form a partition of Ω with $P(B_i) > 0$ for all i, then:
>
> $$P(A) = \sum_{i=1}^{n} P(A|B_i)P(B_i)$$

Theorem

Theorem 7.3 (Bayes' Theorem). If $B_1, B_2, \ldots, B_n$ form a partition of Ω with $P(B_i) > 0$ for all i, then:
$$P(B_j|A) = \frac{P(A|B_j)P(B_j)}{\sum_{i=1}^{n} P(A|B_i)P(B_i)}$$

Example

Example 7.2 (Medical Testing). A disease affects 1% of the population. A test for the disease is 95% accurate (both sensitivity and specificity). If someone tests positive, what's the probability they have the disease?
Let $D =$ "has disease" and $T =$ "tests positive".

$$P(D) = 0.01, \quad P(D^c) = 0.99 \tag{7.1}$$
$$P(T|D) = 0.95, \quad P(T|D^c) = 0.05 \tag{7.2}$$

By Bayes' theorem:

$$P(D|T) = \frac{P(T|D)P(D)}{P(T|D)P(D) + P(T|D^c)P(D^c)} = \frac{0.95 \times 0.01}{0.95 \times 0.01 + 0.05 \times 0.99} \approx 0.161$$

7.4 Random Variables

Definition

Definition 7.5 (Random Variable). A **random variable** is a function $X : \Omega \to \mathbb{R}$ that assigns a real number to each outcome in the sample space.

Definition

Definition 7.6 (Probability Mass Function). For a discrete random variable X, the **probability mass function** (PMF) is:

$$p_X(x) = P(X = x)$$

Definition

Definition 7.7 (Cumulative Distribution Function). The **cumulative distribution function** (CDF) of a random variable X is:

$$F_X(x) = P(X \leq x)$$

7.5   Expected Value and Variance

> **Definition**
>
> **Definition 7.8** (Expected Value)**.** The **expected value** of a discrete random variable X is:
> $$E[X] = \sum_x x \cdot P(X = x)$$

> **Definition**
>
> **Definition 7.9** (Variance)**.** The **variance** of a random variable X is:
> $$Var(X) = E[(X - E[X])^2] = E[X^2] - (E[X])^2$$

> **Theorem**
>
> **Theorem 7.4** (Linearity of Expectation)**.** For random variables X and Y and constants a and b:
> $$E[aX + bY] = aE[X] + bE[Y]$$

7.6   Common Discrete Distributions

7.6.1   Binomial Distribution

> **Definition**
>
> **Definition 7.10** (Binomial Distribution)**.** A random variable X follows a **binomial distribution** with parameters n and p, denoted $X \sim \text{Binomial}(n, p)$, if:
> $$P(X = k) = \binom{n}{k} p^k (1 - p)^{n-k} \quad \text{for } k = 0, 1, \ldots, n$$

> **Theorem**
>
> **Theorem 7.5.** If $X \sim \text{Binomial}(n, p)$, then:
> $$E[X] = np \tag{7.3}$$
> $$Var(X) = np(1 - p) \tag{7.4}$$

7.6.2 Poisson Distribution

Definition

Definition 7.11 (Poisson Distribution)**.** A random variable X follows a **Poisson distribution** with parameter $\lambda > 0$, denoted $X \sim \text{Poisson}(\lambda)$, if:

$$P(X = k) = \frac{\lambda^k e^{-\lambda}}{k!} \quad \text{for } k = 0, 1, 2, \dots$$

Theorem

Theorem 7.6. If $X \sim \text{Poisson}(\lambda)$, then:

$$E[X] = \lambda \tag{7.5}$$
$$Var(X) = \lambda \tag{7.6}$$

7.7 Continuous Distributions

7.7.1 Normal Distribution

Definition

Definition 7.12 (Normal Distribution)**.** A random variable X follows a **normal distribution** with parameters μ and σ^2, denoted $X \sim N(\mu, \sigma^2)$, if it has probability density function:

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

Theorem

Theorem 7.7. If $X \sim N(\mu, \sigma^2)$, then:

$$E[X] = \mu \tag{7.7}$$
$$Var(X) = \sigma^2 \tag{7.8}$$

> **Theorem**
>
> **Theorem 7.8** (Central Limit Theorem). Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed random variables with mean μ and variance σ^2. Then:
>
> $$\frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} \xrightarrow{d} N(0, 1) \quad \text{as } n \to \infty$$
>
> where $\bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i$.

7.8 Limit Theorems

> **Theorem**
>
> **Theorem 7.9** (Law of Large Numbers). Let $X_1, X_2, \ldots$ be independent and identically distributed random variables with finite mean μ. Then:
>
> $$\bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i \to \mu \quad \text{as } n \to \infty$$

7.9 Exercises

1. A fair six-sided die is rolled twice. Find the probability that the sum is 7 given that at least one roll shows a 3.

2. If $X \sim \text{Binomial}(10, 0.3)$, compute $P(X = 3)$ and $E[X]$.

3. Customers arrive at a store according to a Poisson process with rate 2 per hour. What is the probability that exactly 3 customers arrive in a 2-hour period?

4. If $X \sim N(50, 100)$, find $P(40 < X < 60)$.

5. Use Bayes' theorem to solve: A bag contains 3 red balls and 2 blue balls. A ball is drawn and replaced 3 times, with 2 reds and 1 blue observed. What's the probability the bag actually contains 4 red balls and 1 blue ball?

Chapter 8

Geometry

This chapter explores the fundamental concepts of Euclidean geometry, coordinate geometry, and introduces elements of non-Euclidean geometry.

8.1 Euclidean Geometry

> **Definition**
>
> **Definition 8.1** (Point, Line, Plane). The fundamental objects of Euclidean geometry are:
>
> - A **point** has no dimension (position only)
>
> - A **line** is one-dimensional and extends infinitely in both directions
>
> - A **plane** is two-dimensional and extends infinitely in all directions

8.1.1 Euclid's Axioms

> **Theorem**
>
> **Theorem 8.1** (Euclid's Five Postulates).   1. A straight line can be drawn between any two points
>
> 2. Any finite straight line can be extended indefinitely
>
> 3. A circle can be drawn with any center and any radius
>
> 4. All right angles are equal to each other
>
> 5. If a line intersects two other lines such that the sum of interior angles on one side is less than two right angles, then the two lines will intersect on that side when extended

> **Theorem**
>
> **Theorem 8.2** (Parallel Postulate Equivalent). Through a point not on a given line, there exists exactly one line parallel to the given line.

8.2   Triangles and Congruence

> **Definition**
>
> **Definition 8.2** (Triangle). A **triangle** is a polygon with three vertices and three sides.

> **Theorem**
>
> **Theorem 8.3** (Triangle Congruence Criteria). Two triangles are congruent if any of the following conditions hold:
>
> 1. **SSS:** Three sides are equal
>
> 2. **SAS:** Two sides and the included angle are equal
>
> 3. **ASA:** Two angles and the included side are equal
>
> 4. **AAS:** Two angles and a non-included side are equal
>
> 5. **RHS:** Right angle, hypotenuse, and one side are equal

> **Theorem**
>
> **Theorem 8.4** (Pythagorean Theorem). In a right triangle with legs of length a and b and hypotenuse of length c:
> $$a^2 + b^2 = c^2$$

Proof. Consider a square with side length $(a+b)$ containing four copies of the right triangle. The area can be computed in two ways:

$$(a+b)^2 = 4 \cdot \frac{1}{2}ab + c^2 \tag{8.1}$$
$$a^2 + 2ab + b^2 = 2ab + c^2 \tag{8.2}$$
$$a^2 + b^2 = c^2 \tag{8.3}$$

$\square$

8.3 Circles

> **Definition**
>
> **Definition 8.3** (Circle). A **circle** is the set of all points in a plane that are equidistant from a fixed point called the center.

> **Theorem**
>
> **Theorem 8.5** (Inscribed Angle Theorem). An inscribed angle is half the central angle that subtends the same arc.

> **Theorem**
>
> **Theorem 8.6** (Power of a Point). For a point P and a circle with center O and radius r, the power of P is $|PO|^2 - r^2$. For any line through P intersecting the circle at points A and B:
> $$PA \cdot PB = ||PO|^2 - r^2|$$

8.4 Coordinate Geometry

> **Definition**
>
> **Definition 8.4** (Distance Formula). The distance between points (x_1, y_1) and (x_2, y_2) is:
> $$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

> **Definition**
>
> **Definition 8.5** (Equation of a Line). A line can be represented by:
>
> - **Slope-intercept form:** $y = mx + b$
> - **Point-slope form:** $y - y_1 = m(x - x_1)$
> - **General form:** $Ax + By + C = 0$

> **Definition**
>
> **Definition 8.6** (Equation of a Circle). A circle with center (h, k) and radius r has equation:
> $$(x - h)^2 + (y - k)^2 = r^2$$

8.5   Transformations

Definition

Definition 8.7 (Rigid Transformations). **Rigid transformations** preserve distances and angles:

- **Translation:** $(x, y) \mapsto (x + a, y + b)$

- **Rotation:** $(x, y) \mapsto (x \cos \theta - y \sin \theta, x \sin \theta + y \cos \theta)$

- **Reflection:** $(x, y) \mapsto (-x, y)$ (across y-axis)

Definition

Definition 8.8 (Similarity Transformations). **Similarity transformations** preserve angles but may change distances by a constant factor:

- **Scaling:** $(x, y) \mapsto (kx, ky)$ for some $k > 0$

- **Homothety:** Combination of scaling and translation

8.6   Area and Volume

Theorem

Theorem 8.7 (Area Formulas). • **Triangle:** $A = \frac{1}{2}bh$ or $A = \sqrt{s(s-a)(s-b)(s-c)}$ (Heron's formula)

- **Rectangle:** $A = lw$

- **Circle:** $A = \pi r^2$

- **Ellipse:** $A = \pi ab$ where a and b are the semi-axes

Theorem

Theorem 8.8 (Volume Formulas). • **Rectangular prism:** $V = lwh$

- **Cylinder:** $V = \pi r^2 h$

- **Sphere:** $V = \frac{4}{3}\pi r^3$

- **Cone:** $V = \frac{1}{3}\pi r^2 h$

8.7 Vectors in Geometry

> **Definition**
>
> **Definition 8.9** (Vector). A **vector** is a quantity with both magnitude and direction, often represented as $\overrightarrow{AB}$ or $\mathbf{v} = \langle a, b \rangle$.

> **Definition**
>
> **Definition 8.10** (Dot Product). For vectors $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$:
>
> $$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 = |\mathbf{u}||\mathbf{v}| \cos \theta$$
>
> where θ is the angle between the vectors.

> **Definition**
>
> **Definition 8.11** (Cross Product). For vectors $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$:
>
> $$\mathbf{u} \times \mathbf{v} = \langle u_2 v_3 - u_3 v_2, u_3 v_1 - u_1 v_3, u_1 v_2 - u_2 v_1 \rangle$$

8.8 Introduction to Non-Euclidean Geometry

> **Definition**
>
> **Definition 8.12** (Hyperbolic Geometry). **Hyperbolic geometry** is a non-Euclidean geometry where the parallel postulate is replaced with: Through a point not on a given line, there exist infinitely many lines parallel to the given line.

> **Definition**
>
> **Definition 8.13** (Spherical Geometry). **Spherical geometry** is the geometry on the surface of a sphere, where "lines" are great circles and there are no parallel lines.

> **Theorem**
>
> **Theorem 8.9** (Gauss-Bonnet Theorem (Simple Form)). For a triangle on a sphere with angles α, β, and γ:
>
> $$\alpha + \beta + \gamma = \pi + \frac{A}{R^2}$$
>
> where A is the area of the triangle and R is the radius of the sphere.

8.9   Exercises

1. Prove that the diagonals of a rhombus are perpendicular bisectors of each other.

2. Find the equation of the circle passing through points $(1, 2)$, $(3, 4)$, and $(5, 2)$.

3. Use vectors to prove that the diagonals of a parallelogram bisect each other.

4. A triangle has sides of length 3, 4, and 5. Find its area using both the base-height formula and Heron's formula.

5. In spherical geometry, what is the sum of angles in a triangle whose area is $\frac{1}{4}$ the area of a hemisphere of radius 1?

Appendix A

Mathematical Notation

This appendix provides a comprehensive reference for the mathematical notation used throughout this textbook.

A.1 Set Theory Notation

Symbol	Meaning
$\in$	Element of (belongs to)
$\notin$	Not an element of
$\subset$	Subset of
$\subseteq$	Subset of or equal to
$\supset$	Superset of
$\supseteq$	Superset of or equal to
$\cup$	Union
$\cap$	Intersection
$\setminus$	Set difference
A^c	Complement of set A
$\emptyset$	Empty set
$\mathbb{U}$	Universal set
$|A|$	Cardinality of set A
$\mathcal{P}(A)$	Power set of A

A.2 Number Systems

Symbol	Meaning
$\mathbb{N}$	Natural numbers $\{1, 2, 3, \ldots\}$
$\mathbb{N}_0$	Natural numbers including zero $\{0, 1, 2, 3, \ldots\}$
$\mathbb{Z}$	Integers $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$
$\mathbb{Q}$	Rational numbers
$\mathbb{R}$	Real numbers
$\mathbb{C}$	Complex numbers
$\mathbb{Z}^+$	Positive integers
$\mathbb{R}^+$	Positive real numbers

A.3 Logic and Proof Notation

Symbol	Meaning
$\wedge$	Logical AND
$\vee$	Logical OR
$\neg$	Logical NOT
$\implies$	Implies
$\iff$	If and only if
$\forall$	For all (universal quantifier)
$\exists$	There exists (existential quantifier)
$\exists!$	There exists a unique
$\therefore$	Therefore
$\because$	Because
$\square$	End of proof

A.4 Functions and Relations

Symbol	Meaning
$f : A \to B$	Function from set A to set B
$f(x)$	Value of function f at x
f^{-1}	Inverse function of f
$f \circ g$	Composition of functions f and g
$\operatorname{dom}(f)$	Domain of function f
$\operatorname{ran}(f)$	Range of function f
$f\vert_A$	Restriction of f to set A

A.5 Calculus Notation

Symbol	Meaning
$\lim_{x \to a} f(x)$	Limit of $f(x)$ as x approaches a
$\lim_{x \to a^+} f(x)$	Right-hand limit
$\lim_{x \to a^-} f(x)$	Left-hand limit
$f'(x)$	Derivative of f with respect to x
$\frac{df}{dx}$	Derivative of f with respect to x
$\frac{d^n f}{dx^n}$	n-th derivative of f
$\int f(x)\, dx$	Indefinite integral of f
$\int_a^b f(x)\, dx$	Definite integral from a to b
$\sum_{i=1}^{n} a_i$	Sum from $i = 1$ to n
$\prod_{i=1}^{n} a_i$	Product from $i = 1$ to n

A.6 Linear Algebra Notation

Symbol	Meaning
$\mathbf{v}$	Vector $\mathbf{v}$
$|\mathbf{v}|$ or $\|\mathbf{v}\|$	Magnitude (norm) of vector $\mathbf{v}$
$\mathbf{u} \cdot \mathbf{v}$	Dot product of vectors $\mathbf{u}$ and $\mathbf{v}$
$\mathbf{u} \times \mathbf{v}$	Cross product of vectors $\mathbf{u}$ and $\mathbf{v}$
A^T	Transpose of matrix A
A^{-1}	Inverse of matrix A
$\det(A)$	Determinant of matrix A
$\operatorname{tr}(A)$	Trace of matrix A
$\operatorname{rank}(A)$	Rank of matrix A
$\operatorname{null}(A)$	Null space of matrix A
$\operatorname{col}(A)$	Column space of matrix A
I	Identity matrix
$\mathbf{0}$	Zero vector or zero matrix

A.7 Probability Notation

Symbol	Meaning
$P(A)$	Probability of event A
$P(A|B)$	Conditional probability of A given B
$A \cap B$	Intersection of events A and B
$A \cup B$	Union of events A and B
A^c	Complement of event A
Ω	Sample space
X	Random variable
$E[X]$	Expected value of random variable X
$\text{Var}(X)$	Variance of random variable X
σ_X	Standard deviation of random variable X
$X \sim D$	Random variable X follows distribution D
$F_X(x)$	Cumulative distribution function
$f_X(x)$	Probability density function
$p_X(x)$	Probability mass function

A.8 Number Theory Notation

Symbol	Meaning
$a|b$	a divides b
$a \nmid b$	a does not divide b
$\gcd(a, b)$	Greatest common divisor of a and b
$\text{lcm}(a, b)$	Least common multiple of a and b
$a \equiv b \pmod{n}$	a is congruent to b modulo n
$a \bmod n$	Remainder when a is divided by n
$\phi(n)$	Euler's totient function
$\mathbb{Z}_n$	Integers modulo n

A.9 Graph Theory Notation

Symbol	Meaning
$G = (V, E)$	Graph with vertex set V and edge set E
$|V|$	Number of vertices
$|E|$	Number of edges
$\deg(v)$	Degree of vertex v
$d(u, v)$	Distance between vertices u and v
K_n	Complete graph on n vertices
C_n	Cycle graph on n vertices
P_n	Path graph on n vertices

A.10   Common Mathematical Constants

Symbol	Name	Approximate Value
π	Pi	3.14159...
e	Euler's number	2.71828...
ϕ	Golden ratio	1.61803...
γ	Euler-Mascheroni constant	0.57721...
$\sqrt{2}$	Square root of 2	1.41421...

Appendix B

Selected Solutions

This appendix provides detailed solutions to selected exercises from each chapter.

B.1   Chapter 1: Mathematical Foundations

B.1.1   Exercise 1

Problem: Prove that the intersection of two sets is commutative: $A \cap B = B \cap A$.

   Solution: We need to show that $A \cap B = B \cap A$ by proving two inclusions.

   First, we show $A \cap B \subseteq B \cap A$: Let $x \in A \cap B$. By definition of intersection, $x \in A$ and $x \in B$. Since $x \in B$ and $x \in A$, we have $x \in B \cap A$. Therefore, $A \cap B \subseteq B \cap A$.

   Next, we show $B \cap A \subseteq A \cap B$: Let $x \in B \cap A$. By definition of intersection, $x \in B$ and $x \in A$. Since $x \in A$ and $x \in B$, we have $x \in A \cap B$. Therefore, $B \cap A \subseteq A \cap B$.

   Since both inclusions hold, $A \cap B = B \cap A$.

B.1.2   Exercise 3

Problem: Prove by contradiction that there are infinitely many prime numbers.

   Solution: Assume for contradiction that there are only finitely many prime numbers. Let these primes be $p_1, p_2, \ldots, p_k$.

   Consider the number $N = p_1 \cdot p_2 \cdot \ldots \cdot p_k + 1$.

   Since $N > 1$, by the fundamental theorem of arithmetic, N must have at least one prime divisor. Let p be a prime divisor of N.

   If p is one of $p_1, p_2, \ldots, p_k$, then p divides the product $p_1 \cdot p_2 \cdot \ldots \cdot p_k$. Since p also divides N, it must divide their difference: $N - p_1 \cdot p_2 \cdot \ldots \cdot p_k = 1$

   But no prime can divide 1, which is a contradiction.

   Therefore, p cannot be any of $p_1, p_2, \ldots, p_k$, meaning we have found a new prime not in our original list. This contradicts our assumption that we had listed all primes.

   Therefore, there must be infinitely many prime numbers.

B.2　Chapter 2: Abstract Algebra

B.2.1　Exercise 1

Problem: Prove that in any group, each element has a unique inverse.

　　Solution: Let G be a group with operation $*$ and identity element e. Let $a \in G$.

　　Suppose b and c are both inverses of a. Then: - $a * b = b * a = e$ - $a * c = c * a = e$

　　We need to show $b = c$.

　　Starting with b:

$$b = b * e \quad \text{(identity property)} \tag{B.1}$$
$$= b * (a * c) \quad \text{(since } a * c = e) \tag{B.2}$$
$$= (b * a) * c \quad \text{(associativity)} \tag{B.3}$$
$$= e * c \quad \text{(since } b * a = e) \tag{B.4}$$
$$= c \quad \text{(identity property)} \tag{B.5}$$

　　Therefore, $b = c$, proving uniqueness of the inverse.

B.3　Chapter 3: Calculus

B.3.1　Exercise 1

Problem: Use the definition of limit to prove that $\lim_{x \to 3}(2x + 1) = 7$.

　　Solution: We need to show that for every $\varepsilon > 0$, there exists $\delta > 0$ such that whenever $0 < |x - 3| < \delta$, we have $|(2x + 1) - 7| < \varepsilon$.

　　First, let's simplify the expression we need to bound: $|(2x + 1) - 7| = |2x - 6| = 2|x - 3|$

　　For this to be less than ε, we need: $2|x - 3| < \varepsilon$ $|x - 3| < \frac{\varepsilon}{2}$

　　Therefore, we can choose $\delta = \frac{\varepsilon}{2}$.

　　Verification: If $0 < |x - 3| < \delta = \frac{\varepsilon}{2}$, then: $|(2x + 1) - 7| = 2|x - 3| < 2 \cdot \frac{\varepsilon}{2} = \varepsilon$

　　This proves that $\lim_{x \to 3}(2x + 1) = 7$.

B.3.2　Exercise 4

Problem: Evaluate $\int_0^{\pi/2} x \sin x \, dx$ using integration by parts.

　　Solution: Using integration by parts with $u = x$ and $dv = \sin x \, dx$: - $du = dx$ - $v = -\cos x$

Applying the integration by parts formula:

$$\int_0^{\pi/2} x \sin x \, dx = [x(-\cos x)]_0^{\pi/2} - \int_0^{\pi/2} (-\cos x) \, dx \tag{B.6}$$

$$= [-x \cos x]_0^{\pi/2} + \int_0^{\pi/2} \cos x \, dx \tag{B.7}$$

$$= [-x \cos x]_0^{\pi/2} + [\sin x]_0^{\pi/2} \tag{B.8}$$

$$= \left[-\frac{\pi}{2} \cos \frac{\pi}{2} - (-0 \cos 0) \right] + \left[\sin \frac{\pi}{2} - \sin 0 \right] \tag{B.9}$$

$$= [0 - 0] + [1 - 0] \tag{B.10}$$

$$= 1 \tag{B.11}$$

Therefore, $\int_0^{\pi/2} x \sin x \, dx = 1$.

B.4 Chapter 4: Linear Algebra

B.4.1 Exercise 2

Problem: Find the determinant of $A = \begin{pmatrix} 2 & -1 & 3 \\ 1 & 0 & 4 \\ -2 & 1 & 1 \end{pmatrix}$.

Solution: Using cofactor expansion along the second row (which has a zero):

$\det(A) = 1 \cdot \det \begin{pmatrix} -1 & 3 \\ 1 & 1 \end{pmatrix} - 0 \cdot (\text{something}) + 4 \cdot \det \begin{pmatrix} 2 & -1 \\ -2 & 1 \end{pmatrix}$

Computing the 2×2 determinants: - $\det \begin{pmatrix} -1 & 3 \\ 1 & 1 \end{pmatrix} = (-1)(1) - (3)(1) = -1 - 3 = -4$ -

$\det \begin{pmatrix} 2 & -1 \\ -2 & 1 \end{pmatrix} = (2)(1) - (-1)(-2) = 2 - 2 = 0$

Therefore: $\det(A) = 1 \cdot (-4) - 0 + 4 \cdot 0 = -4$

B.5 Chapter 5: Real Analysis

B.5.1 Exercise 1

Problem: Prove that $\lim_{n \to \infty} \frac{1}{n} = 0$ using the definition of convergence.

Solution: We need to show that for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$: $\left| \frac{1}{n} - 0 \right| < \varepsilon$

This simplifies to showing $\frac{1}{n} < \varepsilon$.

Given $\varepsilon > 0$, by the Archimedean property, there exists a positive integer N such that $N > \frac{1}{\varepsilon}$, which means $\frac{1}{N} < \varepsilon$.

For any $n \geq N$, we have $n \geq N > \frac{1}{\varepsilon}$, so $\frac{1}{n} \leq \frac{1}{N} < \varepsilon$.

Therefore, $\left| \frac{1}{n} - 0 \right| = \frac{1}{n} < \varepsilon$ for all $n \geq N$.

This proves that $\lim_{n \to \infty} \frac{1}{n} = 0$.

B.6   Chapter 6: Discrete Mathematics

B.6.1   Exercise 1

Problem: Prove that in any graph, the number of vertices with odd degree is even.

Solution: Let $G = (V, E)$ be a graph. Let V_{odd} be the set of vertices with odd degree and V_{even} be the set of vertices with even degree.

By the handshaking lemma: $\sum_{v \in V} \deg(v) = 2|E|$

Since $2|E|$ is even, the sum of all degrees is even.

We can partition this sum: $\sum_{v \in V} \deg(v) = \sum_{v \in V_{\text{even}}} \deg(v) + \sum_{v \in V_{\text{odd}}} \deg(v)$

The first sum, $\sum_{v \in V_{\text{even}}} \deg(v)$, is a sum of even numbers, so it's even.

Since the total sum is even and the first part is even, the second part $\sum_{v \in V_{\text{odd}}} \deg(v)$ must also be even.

But this is a sum of odd numbers. For a sum of odd numbers to be even, there must be an even number of terms.

Therefore, $|V_{\text{odd}}|$ is even, meaning the number of vertices with odd degree is even.

B.7   Chapter 7: Probability Theory

B.7.1   Exercise 2

Problem: If $X \sim \text{Binomial}(10, 0.3)$, compute $P(X = 3)$ and $E[X]$.

Solution: For a binomial distribution with parameters $n = 10$ and $p = 0.3$:

$P(X = 3) = \binom{10}{3}(0.3)^3(0.7)^7$

Computing each part: - $\binom{10}{3} = \frac{10!}{3! \cdot 7!} = \frac{10 \cdot 9 \cdot 8}{3 \cdot 2 \cdot 1} = 120$ - $(0.3)^3 = 0.027$ - $(0.7)^7 = 0.0823543$

Therefore: $P(X = 3) = 120 \times 0.027 \times 0.0823543 \approx 0.2668$

For the expected value: $E[X] = np = 10 \times 0.3 = 3$

B.8   Chapter 8: Geometry

B.8.1   Exercise 4

Problem: A triangle has sides of length 3, 4, and 5. Find its area using both the base-height formula and Heron's formula.

Solution: First, note that $3^2 + 4^2 = 9 + 16 = 25 = 5^2$, so this is a right triangle with legs of length 3 and 4, and hypotenuse of length 5.

Method 1: Base-height formula Using the legs as base and height: $A = \frac{1}{2} \times \text{base} \times \text{height} = \frac{1}{2} \times 3 \times 4 = 6$

Method 2: Heron's formula First, find the semi-perimeter: $s = \frac{3+4+5}{2} = 6$

Then apply Heron's formula:

$$A = \sqrt{s(s-a)(s-b)(s-c)} \tag{B.12}$$
$$= \sqrt{6(6-3)(6-4)(6-5)} \tag{B.13}$$
$$= \sqrt{6 \times 3 \times 2 \times 1} \tag{B.14}$$
$$= \sqrt{36} \tag{B.15}$$
$$= 6 \tag{B.16}$$

Both methods give the same result: the area is 6 square units.

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