Math Guide

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Preface

This is a math guide for The University of Queensland's MATH1051, MATH1052, MATH2001, MATH1061, and MATH2302 courses. It is intended to be a quick reference for students studying these subjects.

Contents

Part I

MATH1051 Calculus & Linear Algebra I

Chapter 1

Eigenvalues and Eigenvectors

Consider a linear transformation represented by a square matrix, denoted A. When a vector is transformed by the matrix, its direction and magnitude can change. Consider the unique case of special vectors, called **eigenvectors**, denoted $\underline{x}$, that only change in magnitude (not direction) under this transformation. The magnitude by which this eigenvector is scaled is called the **eigenvalue**, denoted λ.

The relationship between a matrix A, its eigenvalues λ, and eigenvectors $\underline{x}$ is given by the equation:

$$A\underline{x} = \lambda\underline{x}$$

This is known as the **definition of eigenvalues and eigenvectors**.

1.1 Eigenvalue Equation

To find the eigenvalues λ of a matrix A, we solve the equation:

$$|A - \lambda I| = 0$$

1.2 Eigenvector Equation

To find the eigenvectors $\underline{x}$ corresponding to a specific eigenvalue λ, we solve the equation:

$$(A - \lambda I)\underline{x} = \underline{0}$$

1.3 Derivation of Eigenvalue and Eigenvector Equations

To find the eigenvector equation, we can rearrange the definition of eigenvalues and eigenvectors.

$$\begin{aligned} A\underline{x} &= \lambda\underline{x} \\ \iff A\underline{x} &= \lambda I\underline{x} \\ \iff A\underline{x} - \lambda\underline{x} &= \underline{0} \\ \iff (A - \lambda I)\underline{x} &= \underline{0} \end{aligned}$$

To find the eigenvalue equation, note that eigenvectors are non-zero vectors $\underline{x} \neq \underline{0}$. Therefore, the equation $(A - \lambda I)\underline{x} = \underline{0}$ must have non-trivial solutions. This occurs when the matrix $(A - \lambda I)$ is singular, when its determinant is zero:

$$|A - \lambda I| = 0$$

Therefore, we have derived both the eigenvalue and eigenvector equations from the definition of eigenvalues and eigenvectors.

1.4 Problems

Question 1 *Determine the eigenvalues and corresponding eigenvectors of each matrix given below. For every case, verify that the obtained eigenvalues and eigenvectors satisfy the equation* $A\underline{x} = \lambda\underline{x}$.

a) $A = \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}$

b) $A = \begin{pmatrix} 0 & -2 \\ 1 & -3 \end{pmatrix}$

c) $A = \begin{pmatrix} -6 & -5 \\ 4 & -2 \end{pmatrix}$

d) $A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 4 \\ 0 & 0 & 5 \end{pmatrix}$

e) $A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

f) $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$

Question 2 *Determine whether the vector* $\mathbf{v}$ *is a linear combination of the given vectors.*

a) $\mathbf{v} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \mathbf{u}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{u}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

b) $\mathbf{v} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}, \mathbf{u}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \mathbf{u}_2 = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$

c) $\mathbf{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \mathbf{u}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{u}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

d) $\mathbf{v} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, \mathbf{u}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \mathbf{u}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \mathbf{u}_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

e) $\mathbf{v} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{u}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \mathbf{u}_2 = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}, \mathbf{u}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

f) $\mathbf{v} = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \end{pmatrix}, \mathbf{u}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{u}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \mathbf{u}_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

Question 3 *For each of the following sets of vectors, determine if the set is linearly independent or linearly dependent.*

a) $\left\{ \mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \ \mathbf{v}_2 = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \right\}$

b) $\left\{ \mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \ \mathbf{v}_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$

c) $\left\{ \mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix}, \ \mathbf{v}_2 = \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix}, \ \mathbf{v}_3 = \begin{pmatrix} 4 \\ 1 \\ 11 \end{pmatrix} \right\}$

d) $\left\{ \mathbf{v}_1 = \begin{pmatrix} 2 \\ 2 \\ 1 \\ -3 \end{pmatrix}, \ \mathbf{v}_2 = \begin{pmatrix} 2 \\ -2 \\ 2 \\ -5 \end{pmatrix}, \ \mathbf{v}_3 = \begin{pmatrix} 0 \\ -2 \\ -2 \\ 1 \end{pmatrix}, \ \mathbf{v}_4 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ -1 \end{pmatrix} \right\}$

e) $\left\{ \mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \ \mathbf{v}_2 = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, \ \mathbf{v}_3 = \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix} \right\}$

f) $\left\{ \mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \ \mathbf{v}_2 = \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}, \ \mathbf{v}_3 = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} \right\}$

Question 4 *The vectors* $\mathbf{v}, \mathbf{u}, \mathbf{w}$ *are linearly independent. Determine whether the following sets of vectors are linearly independent or linearly dependent.*

a) $\{\mathbf{u}+\mathbf{v}, \mathbf{v}+\mathbf{w}, \mathbf{u}+\mathbf{w}\}$

b) $\{\mathbf{u}-\mathbf{v}, \mathbf{v}-\mathbf{w}, \mathbf{u}-\mathbf{w}\}$

Question 5 *Determine whether the following sets of vectors form a vector space.*

a) $U = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; 3x - y + z = 0 \right\}$

b) $V = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; x^2 + y^2 \leq z^2 \right\}$

c) $W = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; 2y - z = 4 \right\}$

d) $X = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; x - y = 0 \right\}$

e) $Y = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; x + 2y + 3z = 0 \right\}$

f) $Z = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; xy = 1 \right\}$

Question 6 *Determine whether the following sets of vectors form a vector space. If they do, find a basis and the dimension of the vector space.*

a) $A = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; x - y \leq z \right\}$

b) $B = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; 3x - y + 2z = 0 \right\}$

c) $C = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; 2x + y - z = 0 \text{ and } x - y + z = 0 \right\}$

d) $D = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; 2x + y - z = 0 \text{ or } x + 2y + z = 0 \right\}$

e) $E = \left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \in \mathbb{R}^4 \;\middle|\; x + y + z + w = 0 \right\}$

f) $F = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; x^2 + y^2 = z^2 \right\}$

1.5 Solutions

Chapter 2

Vector Spaces

2.1 Linear Combination

2.2 Linear Independence

2.3 Invertibility Equivalence

2.4 Vector Space

2.5 Subspace

2.6 Eigenspace

2.7 Null Space

2.8 Span

2.9 Basis

2.10 Dimension

2.11 Properties of Basis

2.12 Problems

Part II

MATH1052 Multivariable Calculus & Linear Algebra I

Chapter 3

Ordinary Differential Equations

3.1 Introduction

Part III

MATH2001 Calculus & Linear Algebra II

Chapter 4

Solutions to First-Order Ordinary Differential Equations

Part IV

MATH1061 Discrete Mathematics I

Chapter 5

Logical Forms

Part V

MATH2302 Discrete Mathematics II

Chapter 6

Selections