

UNIVERSITATEA DIN BUCUREŞTI
FACULTATEA DE MATEMATICĂ ŞI INFORMATICĂ

**TUTORIAT
ALGEBRĂ ŞI GEOMETRIE
VI**

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Breviar – Forme biliniare (de văzut în curs).

$(V, +, \cdot)|_{\mathbb{K}}$

$g : V \times V \rightarrow \mathbb{K}$ se numește forma biliniara $\Leftrightarrow g$ este liniara în fiecare argument

$$\begin{aligned} g(ax + by, z) &= a g(x, z) + b g(y, z) \\ g(x, ay + bz) &= a g(x, y) + b g(x, z) \end{aligned} \quad \forall a, b \in \mathbb{K}; x, y, z \in V$$

si notam cu $L(V, V; \mathbb{K})$ multimea formelor biliniare pe V

g forma simetrică $\Leftrightarrow g(x, y) = g(y, x)$ cu $L^s(V, V; \mathbb{K})$ multimea f. bilin. simetrice

g forma simetrică $\Leftrightarrow g(x, y) = -g(y, x)$ cu $L^a(V, V; \mathbb{K})$ multimea f. antisimetrice

$Q : V \rightarrow \mathbb{K}$ se numește forma patratica $\Leftrightarrow \exists g \in L^s(V, V; \mathbb{K})$ astfel incat

$Q(x) = g(x, x) \quad \forall x \in V$

Breviar – Matricea asociată unei forme biliniare (de văzut în curs).

$$G = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}; g : \mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathbb{R}$$

$$\begin{aligned} g(x, y) &= a_{11}(x_1, y_1) + a_{12}(x_1, y_2) + a_{13}(x_1, y_3) + a_{21}(x_2, y_1) + a_{22}(x_2, y_2) + \\ &+ a_{23}(x_2, y_3) + a_{31}(x_3, y_1) + a_{32}(x_3, y_2) + a_{33}(x_3, y_3). \end{aligned}$$

g nedegenerata $\Leftrightarrow G$ nedegenerata (inversabilă)

$(V, +, \cdot)|_{\mathbb{R}}$

$Q : V \rightarrow \mathbb{R}$ forma patratica reală

g forma biliniara simetrică pozitiv definită $\Leftrightarrow Q$ forma patratica asociată pozitiv

definita $\Leftrightarrow \begin{cases} 1) Q(x) > 0 \forall V \setminus \{0_V\} \\ 2) Q(x) = 0 \Leftrightarrow x = 0_V \end{cases}$ sau *signatura* este $(n, 0)$.

Metoda Jacobi. Metoda Gauss (de văzut în curs).

Metoda Jacobi:

$Q : V \rightarrow \mathbb{R}$ forma patratica reala

Fie R reper in V astfel incat G asociata formei Q verifica:

$$\Delta_1 = g_{11}; \Delta_2 = \begin{vmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{vmatrix}; \dots; \Delta_n = \det(G) \text{ toti nenuli}$$

Atunci \exists un reper in V astfel incat

$$Q(x) = \frac{1}{\Delta_1} {x_1}'^2 + \frac{\Delta_1}{\Delta_2} {x_2}'^2 + \dots + \frac{\Delta_{n-1}}{\Delta_n} {x_n}'^2$$

Mai mult, daca $\Delta_i > 0 \forall i = \overline{1, n} \Rightarrow Q$ este pozitiv definita.

Metoda Gauss:

$Q : V \rightarrow \mathbb{K}$ forma patratica $\Rightarrow \exists$ un reper in V astfel incat Q are o forma canonica

[+ exemple la tutoriat](#)

Procedeu Gram-Schmidt.

Exemplu 1:

$$(\mathbb{R}^3, g_0), \quad V = \langle \{(1, -1, 1), (2, -1, 3), (1, 3, 5)\} \rangle$$

a) $V^\perp = ?$

b) $R = R_1 \cup R_2$ reper ortonormat = ?; unde R_1 reper ortonormat in V respectiv R_2 in V^\perp .

c) Fie p proiectia ortogonală pe V . $p(0, 1, 0) = ?$

Rezolvare:

a)

$$V^\perp = \{x \in \mathbb{R}^3 \mid g_0(x, f_1) = 0, g_0(x, f_2) = 0, g_0(x, f_3) = 0\} =$$

$$= \left\{ x \in \mathbb{R}^3 \mid \begin{cases} x_1 - x_2 + x_3 = 0 \\ 2x_1 - x_2 + 3x_3 = 0 \\ x_1 + 3x_2 + 5x_3 = 0 \end{cases}; \quad \begin{pmatrix} 1 & -1 & 1 \\ 2 & -1 & 3 \\ 1 & 3 & 5 \end{pmatrix} \right.$$

$$\begin{vmatrix} 1 & -1 & 1 \\ 2 & -1 & 3 \\ 1 & 3 & 5 \end{vmatrix} = -5 - 3 + 6 + 1 - 9 + 10 = 17 - 17 = 0$$

$$\begin{vmatrix} 1 & -1 \\ 2 & -1 \end{vmatrix} = -1 + 2 = 1 \neq 0$$

$$\begin{cases} x_1 - x_2 = -x_3 \\ 2x_1 - x_2 = -3x_3 \end{cases} \Rightarrow \dots \Rightarrow x_1 = -2x_3; x_2 = -x_3.$$

$$V^\perp = \{(-2x_3, -x_3, x_3) \mid x_3 \in \mathbb{R}\} = \{x_3(-2, -1, 1) \mid x_3 \in \mathbb{R}\}$$

$$V^\perp = \langle \{(-2, -1, 1)\} \rangle$$

b)

$$V = \langle \{f_1 = (1, -1, 1), f_2 = (2, -1, 3)\} \rangle$$

Aplicam Gram – Schmidt:

$$e_1 = f_1 = (1, -1, 1)$$

$$e_2 = f_2 - \frac{\langle f_2, e_1 \rangle}{\langle e_1, e_1 \rangle} e_1$$

Calculam $\langle f_2, e_1 \rangle$:

$$f_2 = (\textcolor{red}{2}, \textcolor{green}{-1}, \textcolor{violet}{3})$$

$$e_1 = (\textcolor{blue}{1}, \textcolor{red}{-1}, \textcolor{blue}{1})$$

$$\langle f_2, e_1 \rangle = \textcolor{red}{2} * \textcolor{blue}{1} + (\textcolor{green}{-1}) * (\textcolor{red}{-1}) + \textcolor{violet}{3} * \textcolor{blue}{1} = 2 + 1 + 3 = 6$$

Calculam $\langle e_1, e_1 \rangle$:

$$e_1 = (1, -1, 1)$$

$$\langle e_1, e_1 \rangle = 1 * 1 + (-1) * (-1) + 1 * 1 = 1 + 1 + 1 = 3$$

$$e_2 = (2, -1, 3) - \frac{6}{3}(1, -1, 1) = (2, -1, 3) - (2, -2, 2) = (0, 1, 1)$$

$R'_1 = \{e_1 = (\textcolor{red}{1}, \textcolor{green}{-1}, \textcolor{blue}{1}), e_2 = (0, 1, 1)\}$ reper ortogonal in V

$$R_1 = \left\{ \frac{e_1}{\|e_1\|}, \frac{e_2}{\|e_2\|} \right\} \text{ reper ortonormal in } V$$

$$\|e_1\| = \sqrt{\textcolor{red}{1}^2 + (\textcolor{green}{-1})^2 + \textcolor{blue}{1}^2} = \sqrt{3}$$

$$\|e_2\| = \sqrt{0^2 + 1^2 + 1^2} = \sqrt{2}$$

$$R_1 = \left\{ \frac{1}{\sqrt{3}}(1, -1, 1), \frac{1}{\sqrt{2}}(0, 1, 1) \right\} \text{ reper ortonormal in } V (1)$$

$$V^\perp = \langle \{f_3 = (-2, -1, 1)\} \rangle$$

$R'_2 = \{e_3 = (-2, -1, 1)\}$ reper ortogonal in V^\perp

$R_2 = \left\{ \frac{e_3}{\|e_3\|} \right\}$ reper ortonormat in V^\perp

$$\|e_3\| = \sqrt{(-2)^2 + (-1)^2 + 1^2} = \sqrt{6}$$

$R_2 = \left\{ \frac{1}{\sqrt{6}}(-2, -1, 1) \right\}$ reper ortonormat in $V^\perp(2)$

(1), (2) $\Rightarrow R = R_1 \cup R_2$ reper ortonormat in \mathbb{R}^3

c)

$$(0, 1, 0) = a \left(\frac{1}{\sqrt{3}}(1, -1, 1) \right) + b \left(\frac{1}{\sqrt{2}}(0, 1, 1) \right) + c \left(\frac{1}{\sqrt{6}}(-2, -1, 1) \right)$$

$$a = \dots; b = \dots; c = \dots$$

$$\Rightarrow p(0, 1, 0) = a \left(\frac{1}{\sqrt{3}}(1, -1, 1) \right) + b \left(\frac{1}{\sqrt{2}}(0, 1, 1) \right) = \dots \text{ (de vazut in Tutoriatul 5 –)}$$

– Proiectii si simetrii \ominus)

Exemplu 2:

$$(\mathbb{R}^3, g_0), \quad U = \left\{ x \in \mathbb{R}^3 \middle| \begin{cases} x_1 + x_2 - x_3 = 0 \\ 3x_1 - 2x_2 + 2x_3 = 0 \\ 6x_1 + x_2 - x_3 = 0 \end{cases} \right\}$$

a) $U^\perp = ?$

b) $R = R_1 \cup R_2$ reper ortonormat = ?; unde R_1 reper ortonormat in U respectiv R_2 in U^\perp .

c) Fie s simetria ortogonală fata de U^\perp . $s(-1, 0, 1) = ?$

-----> de facut

Exemplu 3:

$$(\mathbb{R}^3, g_0), v = (1, 5, -1)$$

a) $\langle \{v\}^\perp \rangle = ?$

b) Determinati un reper ortonormat in $\langle \{v\}^\perp \rangle$.

-----> de facut