UNIVERSITATEA DIN BUCUREȘTI
FACULTATEA DE MATEMATICĂ ȘI INFORMATICĂ

# TUTORIAT
# ALGEBRĂ ȘI GEOMETRIE
# V

STUDENȚI:

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# Proiecții. Simetrii.

*Exemplu 1:*

$(\mathbb{R}^3,+,\cdot)|_{\mathbb{R}}$

$V' = \{x \in \mathbb{R}^3 \mid -2x_1 + x_2 + 4x_3 = 0\};\ \mathbb{R}^3 = V' \oplus V''$

$Fie\ p\ proiectia\ pe\ V'si\ s\ simetria\ fata\ de\ V''.\ Sa\ se\ calculeze\ p(2\ ,-8\ ,2), s\left(\frac{1}{2}\ ,2\ ,-\frac{1}{2}\right).$

*Rezolvare:*

$x_2 = 2x_1 - 4x_3$

$V' = \{(x_1\ ,2x_1 - 4x_3\ ,x_3)|x_1\ ,x_3 \in \mathbb{R}\} = <\{(1\ ,2\ ,0),(0\ ,-4\ ,1)\}> (1)$

$\dim V' = 2 = |R'|\ (2)$

$Din\ (1),(2) \Longrightarrow R' = \{(1\ ,2\ ,0),(0\ ,-4\ ,1)\}\ reper$

$Extindem\ R'la\ un\ reper\ \mathbb{R}^3:$

$$rg\begin{pmatrix}1 & 0 & 0\\ 2 & -4 & 0\\ 0 & 1 & 1\end{pmatrix} = 3\ (maxim) \Longrightarrow R' \cup \{(0\ ,0\ ,1)\}\ reper\ in\ \mathbb{R}^3$$

$Consideram\ V'' =< \{(0\ ,1\ ,1)\}>$

$$(2\ ,-8\ ,2) = a(1\ ,2,0) + b(0\ ,-4\ ,1) + c(0\ ,0\ ,1), unde\ \begin{matrix} v' = a(1\ ,2\ ,0) + b(0\ ,-4\ ,1) \in V' \\ v'' = c(0\ ,0\ ,1) \in V''\end{matrix}$$

$$= \begin{cases} a = 2 \\ 2a - 4b = -8 \Longrightarrow \cdots \\ b + c = 2\end{cases} \Longrightarrow \begin{cases} a = 2 \\ b = 3 \\ c = -1\end{cases}$$

$(2\ ,3\ ,-1) coordonatele\ lui\ (2\ ,-8\ ,2)\ in\ raport\ cu\ reperul.$

$v' = a(1,2,0) + b(0,-4,1) = 2(1,2,0) + 3(0,-4,1) = (2,-8,3)$

$v'' = c(0,0,1) = -(0,0,1) = (0,0,-1)$

$\boldsymbol{p(2,-8,2) = v' = (2,-8,3)}$.

$$\left(\frac{1}{2},2,-\frac{1}{2}\right) = a(1,2,0) + b(0,-4,1) + c(0,0,1),$$

$$unde\ \begin{matrix} v_1' = a(1,2,0) + b(0,-4,1) \in V' \\ v_1'' = c(0,0,1) \in V'' \end{matrix} = \begin{cases} a = \frac{1}{2} \\ 2a - 4b = 2 \Longrightarrow \cdots \Longrightarrow \\ b + c = -\frac{1}{2} \end{cases} \begin{cases} a = \frac{1}{2} \\ b = -\frac{1}{4} \\ c = -\frac{1}{4} \end{cases}$$

$$\left(\frac{1}{2},2,-\frac{1}{4}\right) coordonatele\ lui\ \left(\frac{1}{2},2,-\frac{1}{2}\right)\ in\ raport\ cu\ reperul.$$

$v_1' = a(1,2,0) + b(0,-4,1) = \frac{1}{2}(1,2,0) - \frac{1}{4}(0,-4,1) = \left(\frac{1}{2},2,-\frac{1}{4}\right)$

$v_1'' = c(0,0,1) = -\frac{1}{4}(0,0,1) = \left(0,0,-\frac{1}{4}\right)$

$\boldsymbol{s(v) = 2p(v) - v = 2\left(0,0,-\frac{1}{4}\right) - \left(\frac{1}{2},2,-\frac{1}{2}\right) = \left(-\frac{1}{2},-2,0\right)}, unde\ p(v)$

$reprezinta$ $proiectia\ pe\ V''$ $\rightarrow\ p(v) = \left(0,0,-\frac{1}{4}\right)$.

***<u>Notă:</u>***

$$In\ general, tot\ ce\ trebuie\ sa\ stim\ pentru\ acest\ tip\ de\ exercitiu\ este\ ca:$$

$$p : V_1 \oplus V_2 \rightarrow V_1 \oplus V_2\ aplicatie\ liniara$$

$$p\ se\ numeste\ proiectie\ pe\ V_1 \Leftrightarrow p(v_1 + v_2) = v_1, unde\ \begin{matrix} v_1 \in V_1 \\ v_2 \in V_2 \end{matrix}$$

$$p : V_1 \oplus V_2 \rightarrow V_1 \oplus V_2\ aplicatie\ liniara$$

$$p(v) = p(v_1 + v_2) = v_1, unde \begin{matrix} v_1 \in V_1 \\ v_2 \in V_2 \end{matrix} proiectia\ pe\ V_1$$

$$s(v) = 2p(v) - v = 2v_1 - (v_1 + v_2) = v_1 - v_2\ simetria\ fata\ de\ V_1$$

‼ *Atentie fata de* "cine" *e simetria ori pe* "ce" *e proiectia* ‼

# Vectori proprii. Valori proprii. Diagonalizare.

*Exemplu 2:*

$f: \mathbb{R}^3 \to \mathbb{R}^3$

$f(x) = (-x_1 - 2x_3\ , x_1 + x_2 + x_3\ , x_1 + 2x_3)$

*Determinati valorile proprii si vectorii proprii corespunzatori. Se poate diagonaliza*

$A = [f]_{R_0,R_0} = ?$

*Rezolvare:*

$$\begin{cases} -x_1 - 2x_3 = 0 \\ x_1 + x_2 + x_3 = 0 \\ x_1 + 2x_3 = 0 \end{cases} \Longrightarrow \qquad A = \begin{pmatrix} -1 & 0 & -2 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{pmatrix}$$

$p(\lambda) = \det(A - \lambda I_3) = 0$

$$\begin{vmatrix} -1-\lambda & 0 & -2 \\ 1 & 1-\lambda & 1 \\ 1 & 0 & 2-\lambda \end{vmatrix} = (-1-\lambda)(1-\lambda)(2-\lambda) + 0 + 0 + 2(1-\lambda) - 0 - 0 =$$

$$= (1-\lambda)\big((-1-\lambda)(2-\lambda) + 2\big) = (1-\lambda)(-2 + \lambda - 2\lambda + \lambda^2 + 2) =$$

$$= (1-\lambda)(\lambda^2 - \lambda) = \lambda(1-\lambda)(\lambda - 1) = -\lambda(1-\lambda)^2$$

$$\det(A - \lambda I_3) = 0 \Leftrightarrow \begin{cases} \lambda_1 = 0\ ;\ m_1 = 1 \\ \lambda_2 = 1\ ;\ m_1 = 2 \end{cases}$$

$$V_{\lambda_1} = \{x \in \mathbb{R}^3 | f(x) = \lambda_1 X\}$$

$$AX = \lambda_1 I_3 X \implies (A - \lambda_1 I_3)X = 0$$

$$\left(\begin{pmatrix} -1 & 0 & -2 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \implies \begin{pmatrix} -1 & 0 & -2 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\dim V_{\lambda_1} = 3 - rg\begin{pmatrix} -1 & 0 & -2 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{pmatrix} = 3 - 2 = 1$$

$$\dim V_{\lambda_1} = 1 = m_{\lambda_1}$$

$\begin{cases} -x_1 - 2x_3 = 0 \\ x_1 + x_2 + x_3 = 0 \end{cases} \implies x_2 - x_3 = 0 \implies x_2 = x_3$

$-x_1 - 2x_2 = 0$

$x_1 = -2x_2$

$$V_{\lambda_1} = \{(-2x_2\,, x_2\,, x_2) | x_2 \in \mathbb{R}\} =< \{(-2\,,1\,,1)\} >$$

$$R_1 = \{(-2\,,1\,,1)\}\, reper\ in\ \ V_{\lambda_1}$$

$$V_{\lambda_2} = \{x \in \mathbb{R}^3 | f(x) = \lambda_2 X\}$$

$$AX = \lambda_2 I_3 X \implies (A - \lambda_2 I_3)X = 0$$

$$\left(\begin{pmatrix} -1 & 0 & -2 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \implies \begin{pmatrix} -2 & 0 & -2 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\dim V_{\lambda_2} = 3 - rg\begin{pmatrix} -2 & 0 & -2 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix} = 3 - 1 = 2$$

$$\dim V_{\lambda_2} = 2 = m_{\lambda_2}$$

$x_1 + x_3 = 0 \Leftrightarrow x_3 = -x_1\ \forall\ x_2 \in \mathbb{R}$

$$V_{\lambda_2} = \{(x_1\,, x_2\,, -x_1) | x_1\,, x_2 \in \mathbb{R}\} =< \{(1\,,0\,,-1), (0\,,1\,,0)\} >$$

$$R_2 = \{(1\,,0\,,-1), (0\,,1\,,0)\}\, reper\ in\ \ V_{\lambda_2}$$

$Verificam\ diagonalizarea$:

$p(\lambda) = 0$

$\lambda_1 = 0, m_{\lambda_1} = 1; \dim V_{\lambda_1} = 1 = m_{\lambda_1}, R_1\ reper\ in\ V_{\lambda_1}$

$\lambda_2 = 1, m_{\lambda_2} = 2; \dim V_{\lambda_2} = 2 = m_{\lambda_2}, R_2\ reper\ in\ V_{\lambda_2}$

$\Longrightarrow \boldsymbol{A\ se\ poate\ diagonaliza}$

$$[f]_{R,R} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix};\ R_1 \cup R_2 = R = \{(-2,1,1),(1,0,-1),(0,1,0)\}$$

*Exemplu 3:*

$f: \mathbb{R}^3 \to \mathbb{R}^3$

$f(x) = (2x_1 + x_3, x_1 + x_2 + 2x_3, 6x_1 + 3x_3)$

$Aflati\ valorile\ proprii\ si\ vectorii\ proprii\ corespunzatori.\ Se\ poate\ diagonaliza\ ?$

--------------------------> $de\ facut$

‼*Reminder:Teorema de diagonalizare*

$$f \in V$$

$$(V,+,\cdot)|_{\mathbb{K}}\ sp.vect.cu\ dim_{\mathbb{K}}V = n$$

$$\exists\ R\ reper\ in\ V\ astfel\ incat\ [f]_{R,R}\ diagonala\ \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} (1) toate\ radacinile\ p(\lambda) se\ afla\ in\ \mathbb{K} \\ (2) dimensiunile\ subspatiilor\ proprii\ sunt\ egale\ cu\ multiplicitatile\ val.proprii\ coresp. \end{cases}$$