UNIVERSITATEA DIN BUCUREȘTI
FACULTATEA DE MATEMATICĂ ȘI INFORMATICĂ

TUTORIAT
ALGEBRĂ ȘI GEOMETRIE
III

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Operații cu subspații vectoriale.

Exemplu 1:

$$(\mathbb{R}^3,+,\cdot)|_{\mathbb{R}}$$

$$V_1 = \{(x\,,y\,,z) \in \mathbb{R}^3\ |x = 0\}$$

$$V_2 = \{(x\,,y\,,z) \in \mathbb{R}^3\ |\begin{cases} y = 0 \\ z = 0 \end{cases}\}$$

$$V_2' = \{(x\,,y\,,z) \in \mathbb{R}^3|\begin{cases} x = z \in \mathbb{R} \\ y = 0 \end{cases}\}$$

$a)\ V = V_1 \oplus V_2$

$b)\ V = V_1 \oplus V_2{}'$

$c)\ V = V_1 \oplus W\ ; W = ?\,; W \neq V_2 \neq V_2'.$

Rezolvare:

$a)$

$$V_1 = \{(x\,,y\,,z) \in \mathbb{R}^3\ |x = 0\} = \{y * (0\,,1\,,0) + z * (0\,,0\,,1)\ |\ y\,,z \in \mathbb{R}\} =$$

$$= < \{(0\,,1,0),(0\,,0,1)\} > SG\ (1)$$

$$rg\begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} = 2(maxim) \Longrightarrow SLI\ (2)$$

$$Din\ (1)\ și\ (2) \Longrightarrow\ R_1 = \{(0\,,1,0),(0\,,0,1)\}\ reper\ in\ V_1$$

$$V_2 = \{(x\,,y\,,z) \in \mathbb{R}^3\ |\begin{cases} y = 0 \\ z = 0 \end{cases}\} = \{x * (1\,,0\,,0)\ |\ x \in \mathbb{R}\} =< \{(1\,,0,0)\} >$$

$$Analog, R_2 = \{(1\,,0,0)\}\ reper\ in\ V_2$$

$$R_0 = R_2 \cup R_1 \Longrightarrow \mathbb{R}^3 = V_1 \oplus V_2.$$

$b) \dashrightarrow$ *ca la a)*

<u>*Metoda* 2. ***Grassmann***</u>:

$$(V,+,\cdot)|_{\mathbb{K}};\ V_1, V_2 \subseteq V \textit{ subspatii vectoriale}$$

$$\dim(V_1 + V_2) = \dim V_1 + \dim V_2 - \dim(V_1 \cap V_2)$$

In acest caz:

$$V_1 = \{y * (0,1,0) + z * (0,0,1) \mid y, z \in \mathbb{R}\}$$

$$V_2' = \{x * (1,0,1) \mid x \in \mathbb{R}\}$$

$$V_1 \cap V_2' = \{0_{\mathbb{R}^3}\} \Longrightarrow \dim(V_1 \cap V_2') = 0$$

$$\dim(V_1 \oplus V_2') = \dim V_1 + \dim V_2{}' - \dim(V_1 \cap V_2') = 2 + 1 - 0 = 3 =$$

$$= \dim \mathbb{R}^3 \Longrightarrow V_1 \oplus V_2' = \mathbb{R}^3.$$

$c)\ R_1 = \{(0,1,0), (0,0,1)\}$ *reper in* V_1

Extindem R_1 *la un reper in* $\mathbb{R}^3$ *(a se vedea Tutoriatul II − Aflarea SLI maximal si extinderea la o baza a acestuia)*

In plus, conditia $W \neq V_2 \neq V_2'$!!!

Exemplu 2:

$$(\mathbb{R}^3,+,\cdot)|_{\mathbb{R}}$$

$$V_1 = \{(2x, 3y, z) \in \mathbb{R}^3 \mid z = 0\}$$

$$V_2 = \{(x, y, 5z) \in \mathbb{R}^3 \mid \begin{cases} x = 0 \\ y = 0 \end{cases}\}$$

$$V_2' = \{(x, y, z) \in \mathbb{R}^3 \mid x = y = z \in \mathbb{R}\}$$

$a)\ V = V_1 \oplus V_2$

$b)\ V = V_1 \oplus V_2{}'$

$c)\ V = V_1 \oplus W;\ W = ?;\ W \neq V_2 \neq V_2'.$

-------------------------> *de facut*

Exemplu 3:

$$(\mathbb{R}^3, +,\cdot)|_{\mathbb{R}}$$

$$V_1 = \{(x\,,y\,,z) \in \mathbb{R}^3 \mid 3x - y + 2z = 0\} = S(A_1)$$

$a)\ R_1 =?\ astfel\ incat\ R_1\ reper\ in\ V_1$

$b)\ \ V_1 \oplus\ V_2 = \mathbb{R}^3;\ V_2 =?$

$c)\ Sa\ se\ descompuna\ x = \left(\frac{2}{3}\,,8\,,\frac{1}{2}\right) in\ raport\ cu\ \mathbb{R}^3 =\ V_1 \oplus\ V_2.$

Rezolvare:

$a)$

$$A_1 = (3\ \ -1\ \ 2)$$

$\dim V_1 = 3 - rg(A_1) = 3 - 1 = 2$

$3x - y + 2z = 0 \Longrightarrow y = 3x + 2z$

$$V_1 = \{(x\,,3x + 2z\,,z) \mid x,z \in \mathbb{R}\} = \{x(1\,,3\,,0) + z(0\,,2\,,1) \mid x,z \in \mathbb{R}\} =$$

$$= < \{(1\,,3\,,0),(0\,,2\,,1)\} > SG\ (1)$$

$$R_1 = \{(1\,,3\,,0),(0\,,2\,,1)\}$$

$$\dim V_1 = |R_1| = 2\ SLI\ (2)$$

$$Din\ (1) si\ (2) \Longrightarrow R_1\ reper\ in\ V_1$$

$b)\ Extindem\ R_1\ la\ un\ reper\ in\ \ \mathbb{R}^3$

$$rg\begin{pmatrix} 1 & 0 & 0 \\ 3 & 2 & 1 \\ 0 & 1 & 0 \end{pmatrix} = 3\ (maxim)$$

$$R_2 = \{(0\,,1,0)\} reper\ in\ V_2$$

$$V_2 = < R_2 >$$

$c)\ \left(\frac{2}{3}, 8, \frac{1}{2}\right) = a(1,3,0) + b(0,2,1) + c(0,1,0). - - - - \rightarrow e\ de\ forma\ x =$

$= x_1 + x_2\ cu\ x_1 \in V_1\ respectiv\ x_2 \in V_2.$

$Mai\ departe \ldots . a = ?\ b = ?\ c = ?\ (rezolvare - tip\ in\ tutoriatele\ anterioare)$

Exemplu 4:

$$(\mathbb{R}^3, +, \cdot)|_{\mathbb{R}}$$

$$V_1 = \{(x, y, z) \in \mathbb{R}^3 \mid x + 2y + 7z = 0\} = S(A_1)$$

$a)\ R_1 = ?\ astfel\ incat\ R_1\ reper\ in\ V_1$

$b)\ Sa\ se\ descompuna\ x = (-23, 5, 2) in\ raport\ cu\ \mathbb{R}^3 = V_1 \oplus V_2.\ V_2 = ?$

-------------------------> $de\ facut$

Descrierea unui subspațiu printr-un sistem de ecuații liniare.

Exemplu 5:

$$(\mathbb{R}^4, +, \cdot)|_{\mathbb{R}}$$

$$V' = < (1, 2, 0, 0), (3, 0, -2, 1) >$$

$a)\ Sa\ se\ descrie\ V' printr - un\ sistem\ de\ ecuatii\ liniare;$

$b)\ V' \oplus V'' = \mathbb{R}^4;\ V'' = ?$

$c)\ Sa\ se\ descompuna\ x = (0, 10, 7, -2) in\ raport\ cu\ aceasta\ suma\ directa.$

Rezolvare:

$a)$

$$A = \begin{pmatrix} 1 & 3 \\ 2 & 0 \\ 0 & -2 \\ 0 & 1 \end{pmatrix}$$

$$\begin{vmatrix} 1 & 3 \\ 2 & 0 \end{vmatrix} = 0 - 6 = -6 \neq 0 \Longrightarrow rg(A) = 2(maxim) \Longrightarrow$$

$$\Longrightarrow R' = \{(1\ ,2\ ,0\ ,0), (3\ ,0\ ,-2\ ,1)\}\ SLI\ (1)$$

$$V' =< R' > SG\ (2)$$

$$Din\ (1)\ si\ (2) \Longrightarrow R'reper\ in\ V'$$

$$\forall\ x \in V' \Longrightarrow \exists\ a\ ,b \in \mathbb{R}\ astfel\ incat\ x = a(1\ ,2\ ,0\ ,0) + b(3\ ,0\ ,-2\ ,1)$$

$$\left(x_{1\ ,}x_{2\ ,}x_{3\ ,}x_4\right) = a(1\ ,2\ ,0\ ,0) + b(3\ ,0\ ,-2\ ,1) = (a + 3b\ ,2a\ ,-2a\ ,b\ )$$

$$\begin{cases} a + 3b = x_1 \\ 2a = x_2 \\ -2b = x_3 \\ b = x_4 \end{cases} \qquad A' = \left(\begin{array}{cc|c} 1 & 3 & x_1 \\ 2 & 0 & x_2 \\ 0 & -2 & x_3 \\ 0 & 1 & x_4 \end{array}\right)$$

$$d_p = \begin{vmatrix} 1 & 3 \\ 2 & 0 \end{vmatrix} \neq 0 \Longrightarrow SCD\ \Leftrightarrow \begin{cases} d_{c_1} = 0 \\ d_{c_2} = 0 \end{cases}$$

$$d_{c_1} = \begin{vmatrix} 1 & 3 & x_1 \\ 2 & 0 & x_2 \\ 0 & -2 & x_3 \end{vmatrix} = 0 \overset{L_2 - 2L_1}{\Longleftrightarrow} \begin{vmatrix} 1 & 3 & x_1 \\ 0 & -6 & x_2 - 2x_1 \\ 0 & -2 & x_3 \end{vmatrix} = 0 \Leftrightarrow -6x_3 + 2x_2 - 4x_1 = 0$$

$$d_{c_2} = \begin{vmatrix} 1 & 3 & x_1 \\ 2 & 0 & x_2 \\ 0 & 1 & x_4 \end{vmatrix} = 0 \overset{L_2 - 2L_1}{\Longleftrightarrow} \begin{vmatrix} 1 & 3 & x_1 \\ 0 & -6 & x_2 - 2x_1 \\ 0 & 1 & x_4 \end{vmatrix} = 0 \Leftrightarrow -6x_4 - x_2 + 2x_1 = 0$$

$$V' = \left\{x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \middle| \begin{matrix} 4x_1 - 2x_2 + 6x_3 = 0 \\ 2x_1 - x_2 + 6x_4 = 0 \end{matrix} \right\}$$

$b)$--------------------------> $de\ facut$

$c)$--------------------------> $de\ facut$

Exemplu 6:

$$(\mathbb{R}^4, +, \cdot)|_{\mathbb{R}}$$

$$V' =< (1\ ,2\ ,0,0)\ ,(3\ ,0\ ,2\ ,2) >$$

$$V'' =< (1\ ,1\ ,0,0)\ ,(5\ ,0\ ,3\ ,7) >$$

$$Sa\ se\ descrie\ V' respectiv\ V'' prin\ sisteme\ de\ ecuatii\ liniare.$$

Exemplu 7 (test seminar 2025):

$$Fie\ x_1 = (2\ ,1\ ,2), x_2 = (-1\ ,0\ ,2), y_1 = (3\ ,1\ ,2), y_2 = (0\ ,1\ ,-1)$$

$$L_1 =< (x_1, x_2) >$$

$$L_2 =< (y_1, y_2) >$$

$a)\ baza\ in\ L_1 \cap L_2$

$b)\ L_3 =?\ astfel\ incat\ L_1 \oplus\ L_3 = \mathbb{R}^3.$