UNIVERSITATEA DIN BUCUREȘTI
FACULTATEA DE MATEMATICĂ ȘI INFORMATICĂ

TUTORIAT
ALGEBRĂ ȘI GEOMETRIE
IV

STUDENȚI:

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Aplicații liniare. Repere în Ker f respectiv Im f.

Exemplu 1:

$$f \in End(\mathbb{R}^3)$$

$$f(x) = (\ x_1 - 2x_2 + x_3, 3x_1 + 3x_3, 7x_1 + 3x_2 + 7x_3\ )$$

$a)\ Aratati\ ca\ f\ este\ liniara.$

$b)\ Ker\ f\ = ? \qquad Im\ f\ = ?$

$Sa\ se\ gaseasca\ cate\ un\ reper\ in\ Ker\ f\ respectiv\ Im\ f.$

Rezolvare:

$a)$

$$f\ liniara\ \Leftrightarrow f(ax + by) = a\,f(x) + b\,f(y) \begin{matrix} \forall\, x, y\ \in\ \mathbb{R}^3 \\ \forall\, a, b\ \in\ \mathbb{R} \end{matrix}.$$

$f(ax + by) = f(ax_1 + by_1, ax_2 + by_2, ax_3 + by_3) =$

$= (ax_1 + by_1 - 2ax_2 - 2by_2 + ax_3 + by_3\ , 3ax_1 + 3by_1 + 3ax_3 + 3by_3, 7ax_1 + 7by_1 +$

$+3ax_2 + 3by_2 + 7ax_3 + 7by_3\ ) = (a(x_1 - 2x_2 + x_3) + b(y_1 - 2y_2 + y_3), a(3x_1 + 3x_3) +$

$+b(3y_1 + 3y_3), a(7x_1 + 3x_2 + 7x_3) + b(7y_1 + 3y_2 + 7y_3)) =$

$= a(\ x_1 - 2x_2 + x_3, 3x_1 + 3x_3, 7x_1 + 3x_2 + 7x_3\ ) + b(\ y_1 - 2y_2 + y_3, 3y_1 + 3y_3, 7y_1 +$

$+3y_2 + 7y_3\ ) = a\,f(x) + b\,f(y) \begin{matrix} \forall\, x, y\ \in\ \mathbb{R}^3 \\ \forall\, a, b\ \in\ \mathbb{R} \end{matrix}.$

$b)$

$$Ker\ f\ = \{x \in\ \mathbb{R}^3 \mid f(x) = 0_{\mathbb{R}^3}\} = S(A)$$

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ 3x_1 \qquad\quad + 3x_3 = 0 \\ 7x_1 + 3x_2 + 7x_3 = 0 \end{cases} \qquad A = \begin{pmatrix} 1 & -2 & 1 \\ 3 & 0 & 3 \\ 7 & 3 & 7 \end{pmatrix}$$

$Observam\ ca\ C_1 = C_3 \implies \det A = 0.$

$$rg\begin{pmatrix}1 & -2 & 1\\ 3 & 0 & 3\\ 7 & 3 & 7\end{pmatrix} = 2\ \left(d_p = \begin{vmatrix}3 & 0\\ 7 & 3\end{vmatrix} \neq 0\right) \implies \begin{cases}3x_1 & = -3x_3\\ 7x_1 + 3x_2 & = -7x_3\end{cases} \implies \begin{cases}x_1 & = -x_3\\ 7x_1 + 3x_2 & = -7x_3\end{cases}$$

$\implies 3x_2 = 0 \implies x_2 = 0$

$$S(A) = \{(-x_3, 0, x_3) \mid x_3 \in \mathbb{R}\}$$

$$Ker\ f = <\{(-1, 0, 1)\}>$$

$$R_1 = \{(-1, 0, 1)\}\ reper\ in\ Ker\ f$$

$\underline{Metoda\ 1}$:

$$Im\ f = \{y \in \mathbb{R}^3 \mid \exists\ x \in \mathbb{R}^3\ astfel\ incat\ f(x) = y\}$$

$$B = \left\{\begin{array}{ccc|c}1 & -2 & 1 & y_1\\ 3 & 0 & 3 & y_2\\ 7 & 3 & 7 & y_3\end{array}\right\}$$

$Sistemul\ este\ compatibil \implies rg(B) = rg(B^*)$

$d_p = \begin{vmatrix}1 & -2\\ 3 & 0\end{vmatrix} \neq 0. In\ plus, toti\ d_c = 0.$

$$d_c = \begin{vmatrix}1 & -2 & y_1\\ 3 & 0 & y_2\\ 7 & 3 & y_3\end{vmatrix} = 0 \Leftrightarrow \cdots \Leftrightarrow 9y_1 - 17y_2 + 6y_3 = 0$$

$$Im\ f = \{y \in \mathbb{R}^3 \mid 9y_1 - 17y_2 + 6y_3 = 0\}$$

$$B' = (\ 9 - 17\ \ 6\ \mid\ 0)$$

$\dim Im\ f = 3 - rg(B') = 3 - 1 = 2$

$$y_2 = \frac{9y_1 + 6y_3}{17}$$

$$Im\ f = \left\{\left(y_1, \frac{9}{17}y_1 + \frac{6}{17}y_3, y_3\right) \middle|\ y_1, y_3 \in \mathbb{R}\right\} = \{y_1\left(1, \frac{9}{17}, 0\right) + y_2(0, \frac{6}{17}, 1) \mid y_1, y_3 \in \mathbb{R}\}$$

$$Im\ f = <\left\{\left(1, \frac{9}{17}, 0\right), \left(0, \frac{6}{17}, 1\right)\right\}>$$

$$R_2 = \left\{\left(1, \frac{9}{17}, 0\right), \left(0, \frac{6}{17}, 1\right)\right\} reper\ in\ Im\ f$$

$\underline{Metoda\ 2:}$

$Extindem\ R_1\ la\ un\ reper\ in\ \mathbb{R}^3$

$R_1\ \cup \{\ vect\_ales1\ , vect\_ales2\}\ \ reper\ in\ \mathbb{R}^3$

$R_2\ \cup \{f(vect\_ales1), f(vect\_ales2)\} reper\ in\ Im\ f$

Exemplu 2:

$$f \in End(\mathbb{R}^3)$$

$$f(x) = (\ x_1, 3x_1 - 2x_2 - 3x_3, -x_1 + 4x_2 + 6x_3\ )$$

$a)\ Aratati\ ca\ f\ este\ liniara.$

$b)\ Ker\ f\ =?\qquad Im\ f\ = ?$

$Sa\ se\ gaseasca\ cate\ un\ reper\ in\ Ker\ f\ respectiv\ Im\ f.$

-------------------------> $de\ facut$

Matricea asociată unei aplicații liniare.

Exemplu 3:

$$f \in End(\mathbb{R}^3)$$

$$f(x_1, x_2, x_3) = (-2x_1 - 2x_2 - x_3, 2x_1 + 2x_2 + x_3, x_1 + 3x_2 + x_3)$$

$a)\ [f]_{R_0, R_0}\ = ?$

$b)\ Ker\ f\ = ?\qquad Im\ f\ = ?$

Rezolvare:

$a)$

$R_0 = \{e_1 = (1\,,0\,,0), e_2 = (0\,,1\,,0), e_3 = (0\,,0\,,1)\}\ reperul\ canonic$

$f(e_1) = f(1\,,0\,,0) = (-2 * 1 - 2 * 0 - 0\,, 2 * 1 + 2 * 0 + 0\,, 1 + 3 * 0 + 0) = (-2\,,2\,,1) =$

$= -2e_1 + 2e_2 + 1e_3$

$f(e_2) = f(0\,,1\,,0) = (-2 * 0 - 2 * 1 - 0\,, 2 * 0 + 2 * 1 + 0\,, 0 + 3 * 1 + 0) = (-2\,,2\,,3) =$

$= -2e_1 + 2e_2 + 3e_3$

$f(e_1) = f(0\,,0\,,1) = (-2 * 0 - 2 * 0 - 1\,, 2 * 0 + 2 * 0 + 1\,, 0 + 3 * 0 + 1) = (-1\,,1\,,1) =$

$= -1e_1 + 1e_2 + 1e_3$

$$A = \begin{pmatrix} -2 & -2 & -1 \\ 2 & 2 & 1 \\ 1 & 3 & 1 \end{pmatrix} = [f]_{R_0,R_0}$$

$b)\ Analog\ exemplului\ 1.$

Exemplu 4:

$$f \in End(\mathbb{R}^3)$$

$$f(x) = (\,-x_1 - 2x_3, x_1 + x_2 + x_3, x_1 + 2x_3\,)$$

$a)\ Verificati\ daca\ f\ este\ liniara.$

$b)\ Ker\ f\ = ? \qquad Im\ f\ = ?$

$Sa\ se\ gaseasca\ cate\ un\ reper\ in\ Ker\ f\ respectiv\ Im\ f.$

$c)\ Este\ R = \{(1\,,3\,,-11), (1\,,4\,,0), (1\,,2\,,-1)\}\ reper\ ?$

$[f]_{R_0,R_0} = ?$

$[f]_{R,R} = ?$

-------------------------> $de\ facut$

Exemplu 5:

$$f \in End(\mathbb{R}^2)$$

$$f(x) = (\ x_1 + x_2\ , 3x_1 - x_2\ )$$

$a)$ $[f]_{R_0,R_0} = ?$

$b)$ $[f]_{R,R} = ?\ ; R = \{(1\ ,1)\ , (-1\ ,3)\}\ baza.$

-------------------------> $de\ facut$

Idee: de folosit formula de schimbare a matricei transformării liniare (... $C^{-1}AC$...)😊.