UNIVERSITATEA DIN BUCUREȘTI
FACULTATEA DE MATEMATICĂ ȘI INFORMATICĂ

TUTORIAT
ALGEBRĂ ȘI GEOMETRIE
VII-VIII

STUDENȚI:

Radu Dimitrie Octavian

Țigănilă Ștefania

Bălan Mihai

BUCUREȘTI

2025

Tutoriat VII

- Recapitulare din tutoriatele anterioare, exerciții ca la testul de seminar + exerciții ca la examen cu noțiunile din urmă.

Transformări ortogonale.

Exemplu 1:

$g_0: \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$; $g_0(x,y) = x_1y_1 + x_2y_2 + x_3y_3$

$f \in End(\mathbb{R}^3)$

$$f(x_1, x_2, x_3) = \frac{1}{9}(x_1 - 8x_2 + 4x_3, 4x_1 + 4x_2 + 7x_3, -8x_1 + x_2 + 4x_3)$$

$a)\ f \in O(\mathbb{R}^3) = ?$

$b)\ \varphi = ?$

$$c)\ R\ reper\ ortonormat\ astfel\ incat\ [f]_{R,R} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{pmatrix}$$

Rezolvare:

$a)$

$$A = [f]_{R_0,R_0} = \frac{1}{9}\begin{pmatrix} 1 & -8 & 4 \\ 4 & 4 & 7 \\ -8 & 1 & 4 \end{pmatrix}$$

$$A * A^t = \frac{1}{9^2}\begin{pmatrix} 1 & -8 & 4 \\ 4 & 4 & 7 \\ -8 & 1 & 4 \end{pmatrix}\begin{pmatrix} 1 & 4 & -8 \\ -8 & 4 & 1 \\ 4 & 7 & 4 \end{pmatrix} = \frac{1}{9^2}\begin{pmatrix} 81 & 0 & 0 \\ 0 & 81 & 0 \\ 0 & 0 & 81 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I_3 \Rightarrow$$

$\Rightarrow A \in O(3) \Rightarrow f \in O(\mathbb{R}^3)$

$b)$

$\det A = \cdots = 1 \ \Rightarrow f \ de \ speta \ I$

$$\left.\begin{array}{r} Tr\ (A) = 1 + 2\cos\varphi \\ Tr\ (A) = \frac{1}{9}(1+4+4) = \frac{9}{9} = 1 \end{array}\right| \Rightarrow 1 + 2\cos\varphi = 1 \Rightarrow 2\cos\varphi = 0 \Rightarrow \cos\varphi = 0 \Rightarrow$$

$$\Rightarrow \varphi = \frac{\pi}{2}$$

$c)$

$Axa \ de \ rotatie \ f(x) = x.$

$$\begin{cases} x_1 - 8x_2 + 4x_3 = 9x_1 \\ 4x_1 + 4x_2 + 7x_3 = 9x_2 \\ -8x_1 + x_2 + 4x_3 = 9x_3 \end{cases} \Rightarrow \begin{cases} -8x_1 - 8x_2 + 4x_3 = 0 \\ 4x_1 - 5x_2 + 7x_3 = 0 \\ -8x_1 + x_2 - 5x_3 = 9x_3 \end{cases}$$

$$B = \begin{pmatrix} -8 & -8 & 4 \\ 4 & -5 & 7 \\ -8 & 1 & -5 \end{pmatrix}$$

$$Aflam \ rg(B) = 2 \ \Rightarrow \begin{cases} -8x_1 - 8x_2 = -4x_3 \\ 4x_1 - 5x_2 = -7x_3 \end{cases} \Rightarrow \cdots \ \Rightarrow x_2 = x_3 \ \Rightarrow \ x_1 = -\frac{1}{2}x_2$$

$$\left\{\left(-\frac{1}{2}x_2\,, x_2\,, x_2\right) \mid x_2 \in \ \mathbb{R}\right\} = < \left\{f_1 = \left(-\frac{1}{2}\,, 1\,, 1\right)\right\} >$$

$Aplicam \ Gram - Schmidt(nu \ facem \ nimic, un \ singur \ vector)$

$$e_1 = f_1 = \left(-\frac{1}{2}\,, 1\,, 1\right)$$

$$\|e_1\| = \sqrt{\left(-\frac{1}{2}\right)^2 + 1^2 + 1^2} = \sqrt{\frac{9}{4}} = \frac{3}{2}$$

$$e_1{}' = \frac{e_1}{\|e_1\|} = \left(-\frac{1}{3}\,, \frac{2}{3}\,, \frac{2}{3}\right) \ versorul \ axei$$

$$e_1{}'^{\perp} = \{x \in \mathbb{R}^3 \mid g_0(x, e_1{}') = 0\} = \{x \in \mathbb{R}^3 \mid -\frac{1}{3}x_1 + \frac{2}{3}x_2 + \frac{2}{3}x_3 = 0\} \ \Rightarrow 2x_2 + 2x_3 =$$

$$= x_1$$

$e_1'^{\perp} = \{(2x_2 + 2x_3, x_2, x_3) | x_2, x_3 \in \mathbb{R}\} =< \{(2,1,0), (2,0,1)\} >$

$\{f_2 = (2,1,0), f_3 = (2,0,1)\}$ *reper in* $e_1'^{\perp}$

Aplicam Gram – Schmidt:

$$e_2 = f_2 = (2,1,0)$$

$$e_3 = f_3 - \frac{< f_3, e_2 >}{< e_2, e_2 >} f_2$$

Calculam $< f_3, e_2 >$:

$$f_3 = (\mathbf{2}, \mathbf{0}, \mathbf{1})$$

$$e_2 = (\mathbf{2}, \mathbf{1}, \mathbf{0})$$

$$< f_2, e_1 > = \mathbf{2} * \mathbf{2} + \mathbf{0} * \mathbf{1} + \mathbf{1} * \mathbf{0} = 4$$

Calculam $< e_2, e_2 >$:

$$e_2 = (2,1,0)$$

$$< e_2, e_2 > = 2 * 2 + 1 * 1 + 0 * 0 = 4 + 1 + 0 = 5$$

$$e_3 = (2,0,1) - \frac{4}{5}(2,1,0) = (2,0,1) - \left(\frac{8}{5}, \frac{4}{5}, 0\right) = \left(\frac{2}{5}, -\frac{4}{5}, 1\right)$$

$\left\{e_2 = (2,1,0), e_3 = \left(\frac{2}{5}, -\frac{4}{5}, 1\right)\right\}$ *reper ortogonal in* $e_1'^{\perp}$

$\|e_2\| = \sqrt{2^2 + 1^2 + 0^2} = \sqrt{5}$

$\|e_3\| = \sqrt{\left(\frac{2}{5}\right)^2 + \left(-\frac{4}{5}\right)^2 + 1^2} = \cdots = \frac{3\sqrt{5}}{5}$

$\left\{e_2' = \frac{e_2}{\|e_2\|}, e_3' = \frac{e_3}{\|e_3\|}\right\}$ *reper optonormat in* $e_1'^{\perp}$

$R = \{e_1{}', e_2{}', e_3{}'\}$ *reperul ortonormat in raport cu care*

$$[f]_{R,R} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} / \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

Exemplu 2:

$g_0 : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$; $g_0(x, y) = x_1 y_1 + x_2 y_2 + x_3 y_3$

$f \in End(\mathbb{R}^3)$

$f(x_1, x_2, x_3) = (x_3 , x_1 , x_2)$

$a)\ f \in O(\mathbb{R}^3) = ?$

$b)\ \varphi = ?$

$c)\ R$ *reper ortonormat astfel incat* $[f]_{R,R} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{pmatrix}$

-------------------------> *de facut*

Endomorfisme simetrice.

Exemplu 1:

$(\mathbb{R}^3, g_0)$, $f \in End(\mathbb{R}^3)$; $[f]_{R_0,R_0} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

$a)\ f \in Sim(\mathbb{R}^3)$; $f(x) = ?$

$b)\ Q = ?$ *forma patratica asociata*

$c)$ *Sa se aduca* Q *la o forma canonica printr* $-$ *o transformare ortogonala*

Rezolvare:

$a)$

$Verificam\ daca\ A = A^t$

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \Rightarrow f \in Sim(\mathbb{R}^3)$$

$$f: \mathbb{R}^3 \to \mathbb{R}^3; \qquad f(x_1, x_2, x_3) = (x_2 + x_3, x_1 + x_3, x_1 + x_2)$$

$b)$

$Q(x) = < x, f(x) >$

$Q(x) = 2x_1x_2 + 2x_2x_3 + 2x_1x_3$

$c)$

$Aplicam\ metoda\ valorilor\ proprii$:

$$\det(A - \lambda I_3) = 0 \Rightarrow \begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = 0 \Rightarrow \cdots \Rightarrow (\lambda + 1)^2(\lambda - 2) = 0 \Rightarrow$$

$$\Rightarrow \begin{matrix} \lambda_1 = -1 & m_1 = 2 \\ \lambda_2 = 2 & m_2 = 1 \end{matrix}$$

$V_{\lambda_1} = \{x \in \mathbb{R}^3 | f(x) = -x\}$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad rg(\dots) = 1 \Rightarrow$$

$$\Rightarrow V_{\lambda_1} = \{x \in \mathbb{R}^3 | x_1 + x_2 + x_3 = 0\} = \{(-x_2 - x_3, x_2, x_3) | x_2, x_3 \in \mathbb{R}\} =$$

$$= < \{f_1 = (-1, 1, 0), f_2 = (-1, 0, 1)\} >$$

$Aplicam\ Gram-Schmidt:$

$$e_1 = f_1 = (-1,1,0)$$

$$e_2 = f_2 - \frac{<f_2, e_1>}{<e_1, e_1>} f_1$$

$Calculam\ <f_2, e_1>:$

$$f_2 = (\mathbf{-1}, \mathbf{0}, \mathbf{1})$$

$$e_1 = (\mathbf{-1}, \mathbf{1}, \mathbf{0})$$

$$<f_2, e_1> = (\mathbf{-1}) * (\mathbf{-1}) + \mathbf{0} * \mathbf{1} + \mathbf{1} * \mathbf{0} = 1$$

$Calculam\ <e_1, e_1>:$

$$e_1 = (-1,1,0)$$

$$<e_1, e_1> = (-1) * (-1) + 1 * 1 + 0 * 0 = 1 + 1 = 2$$

$$e_2 = (-1,0,1) - \frac{1}{2}(-1,1,0) = (-1,0,1) - \left(\frac{1}{2}, -\frac{1}{2}, 0\right) = \left(-\frac{1}{2}, -\frac{1}{2}, 1\right)$$

$\left\{e_1 = (-1,1,0), e_2 = \left(-\frac{1}{2}, -\frac{1}{2}, 1\right)\right\}\ reper\ ortogonal\ in\ V_{\lambda_1}$

$$\|e_1\| = \sqrt{(\mathbf{-1})^2 + (\mathbf{1})^2 + \mathbf{0}^2} = \sqrt{2}$$

$$\|e_2\| = \sqrt{\left(-\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2 + 1^2} = \frac{3}{2}$$

$R_1 = \left\{\frac{1}{\sqrt{2}}(-1,1,0), \frac{2}{3}\left(-\frac{1}{2}, -\frac{1}{2}, 1\right)\right\}\ reper\ ortonormat\ in\ V_{\lambda_1}$

$V_{\lambda_2} = \{x \in \mathbb{R}^3 | f(x) = 2x\}$

$\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad rg(\ldots) = 2 \Rightarrow$

$\Rightarrow V_{\lambda_2} = \{(x_3, x_3, x_3) | x_3 \in \mathbb{R}\} =< \{f_3 = (1\,,1\,,1)\} >$

$Aplicam\ Gram - Schmidt$:

$$e_3 = f_3 = (1\,,1\,,1)$$

$\{e_3 = (1\,,1\,,1)\}\ reper\ ortogonal\ in\ V_{\lambda_2}$

$$\|e_2\| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$$

$R_2 = \left\{\frac{1}{\sqrt{3}}(1\,,1\,,1)\right\} reper\ ortonormat\ in\ V_{\lambda_2}$

$R = R_1 \cup R_2 = \left\{\frac{1}{\sqrt{2}}(-1\,,1\,,0)\,,\frac{2}{3}\left(-\frac{1}{2}\,,-\frac{1}{2}\,,1\right),\frac{1}{\sqrt{3}}(1\,,1\,,1)\right\} reper\ ortonormat\ in\ \mathbb{R}^3$

$A' = [f]_{R,R} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

$Q(x) = -{x_1}'^2 - {x_2}'^2 + 2{x_3}'^2\ forma\ canonica\ la\ care\ am\ adus\ forma\ initiala$

$R_0 \xrightarrow{C} R$

$C = \begin{pmatrix} -\frac{1}{\sqrt{2}} & -3 & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -3 & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{3} & \frac{1}{\sqrt{3}} \end{pmatrix}$

$h \in O(\mathbb{R}^3) \quad h(x) = \left(-\frac{1}{\sqrt{2}}x_1 - 3x_2 + \frac{1}{\sqrt{3}}x_3, \frac{1}{\sqrt{2}}x_1 - 3x_2 + \frac{1}{\sqrt{3}}x_3, \frac{2}{3}x_2 + \frac{1}{\sqrt{3}}x_3\right)$

Exemplu 2:

$(\mathbb{R}^3, g_0), \ \ f \in End(\mathbb{R}^3); \quad [f]_{R_0,R_0} = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}$

$a)\ f \in Sim(\mathbb{R}^3); \ \ f(x) = ?$

$b)\ Q = ?\ forma\ patratica\ asociata$

$c)\ Sa\ se\ aduca\ Q\ la\ o\ forma\ canonica\ printr-o\ transformare\ ortogonala$

--------------------------> $de\ facut$