UNIVERSITATEA DIN BUCUREȘTI
FACULTATEA DE MATEMATICĂ ȘI INFORMATICĂ

# TUTORIAT
# ALGEBRĂ ȘI GEOMETRIE
# VI

STUDENȚI:

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## Breviar – Forme biliniare (de văzut în curs)

$(V,+,\cdot)|_{\mathbb{K}}$

$g : V \times V \to \mathbb{K}$ *se numeste forma biliniara* $\Leftrightarrow$ *g este liniara in fiecare argument*

$$\begin{matrix} g(ax+by,z) = a\,g(x,z) + b\,g(y,z) \\ g(x,ay+bz) = a\,g(x,y) + b\,g(x,z) \end{matrix} \forall\, a,b \in \mathbb{K}\,; x,y,z \in V$$

*si notam cu* $L(V,V;\mathbb{K})$ *multimea formelor biliniare pe* $V$

$g$ *forma simetrica* $\Leftrightarrow g(x,y) = g(y,x)$ *cu* $L^s(V,V;\mathbb{K})$ *multimea f. bilin. simetrice*

$g$ *forma simetrica* $\Leftrightarrow g(x,y) = -g(y,x)$ *cu* $L^a(V,V;\mathbb{K})$ *multimea f. antisimetrice*

$Q : V \to \mathbb{K}$ *se numeste forma patratica* $\Leftrightarrow \exists\, g \in L^s(V,V;\mathbb{K})$ *astfel incat*

$Q(x) = g(x,x)\ \forall\, x \in V$

## Breviar – Matricea asociată unei forme biliniare (de văzut în curs).

$$G = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix};\ g: \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$$

$$g(x,y) = a_{11}(x_1,y_1) + a_{12}(x_1,y_2) + a_{13}(x_1,y_3) + a_{21}(x_2,y_1) + a_{22}(x_2,y_2) +$$

$$+a_{23}(x_2,y_3)a_{31}(x_3,y_1) + a_{32}(x_3,y_2) + a_{33}(x_3,y_3).$$

$g$ *nedegenarata* $\Leftrightarrow$ $G$ *nedegenerata (inversabila)*

$(V,+,\cdot)|_{\mathbb{R}}$

$Q : V \to \mathbb{R}$ *forma patratica reala*

$g$ *forma biliniara simetrica pozitiv definita* $\Leftrightarrow$ $Q$ *forma patratica asociata pozitiv*

$definita \Leftrightarrow \begin{cases} 1)\ Q(x) > 0\ \forall\ V \setminus \{0_V\} \\ 2)\ Q(x) = 0 \Leftrightarrow x = 0_V \end{cases} sau\ signatura$ ! $este\ (n, 0).$

# Metoda Jacobi. Metoda Gauss (de văzut în curs!).

*Metoda Jacobi:*

$Q : V \to \mathbb{R}\ forma\ patratica\ reala$

$Fie\ R\ reper\ in\ V\ astfel\ incat\ G\ asociata\ formei\ Q\ verifica:$

$$\Delta_1 = g_{11}\ ;\ \Delta_2 = \begin{vmatrix} g_{11} & g_{11} \\ g_{11} & g_{11} \end{vmatrix}\ ; \dots ;\ \Delta_n = \det(G)\ toti\ nenuli$$

$Atunci\ \exists\ un\ reper\ in\ V\ astefel\ incat$

$$Q(x) = \frac{1}{\Delta_1} {x_1}'^2 + \frac{\Delta_1}{\Delta_2} {x_2}'^2 + \cdots + \frac{\Delta_{n-1}}{\Delta_n} {x_n}'^2$$

$Mai\ mult,\ daca\ \Delta_i > 0\ \forall\ i = \overline{1, n} \implies Q\ este\ pozitiv\ definita.$

*Metoda Gauss:*

$Q : V \to \mathbb{K}\ forma\ patratica \Rightarrow \exists\ un\ reper\ in\ V\ astfel\ incat\ Q\ are\ o\ forma\ canonica$

$+\ metoda\ val.proprii\ (exemple\ la\ tutoriat)$

*Exemplu 1:*

$Q: \mathbb{R}^3 \to \mathbb{R} \qquad Q(x) = {x_1}^2 + 4x_1x_2 - 2x_2x_3 + {x_2}^2 - {x_3}^2$

$a)\ Q\ =?\ matricea\ asociata\ lui\ Q\ in\ raport\ cu\ R_0$

$b)\ Sa\ se\ aduca\ Q\ la\ o\ forma\ normala$

$c)\ Este\ Q\ pozitiv\ definita\ ?\ Verificati\ prin\ mai\ multe\ metode.$

-------------------------> $de\ facut$

# Procedeul Gram-Schmidt.

*Exemplu 1:*

$(\mathbb{R}^3, g_0)\ , \qquad V\ =<\{(1\ ,-1\ ,1)\ ,(2\ ,-1\ ,3)\ ,(1\ ,3\ ,5)\}>$

$a)\ V^{\perp} = ?$

$b)\ R\ = R_1 \cup R_2\ reper\ ortonormat = ?\ ;unde\ \ R_1\ reper\ ortonormat\ in\ V\ respectiv\ R_2$

$in\ V^{\perp}.$

$c)\ Fie\ p\ proiectia\ ortogonala\ pe\ V.\ p(0\ ,1\ ,0) = ?$

*Rezolvare:*

$a)$

$$V^{\perp} = \{x \in\ \mathbb{R}^3 |\ g_0(x\ ,f_1) = 0\ , g_0(x\ ,f_2) = 0\ , g_0(x\ ,f_3) = 0\} =$$

$$= \left\{x \in\ \mathbb{R}^3 \middle| \begin{cases} x_1 - x_2 + x_3 = 0 \\ 2x_1 - x_2 + 3x_3 = 0 \\ x_1 + 3x_2 + 5x_3 = 0 \end{cases} \right\}; \qquad \begin{pmatrix} 1 & -1 & 1 \\ 2 & -1 & 3 \\ 1 & 3 & 5 \end{pmatrix}$$

$$\begin{vmatrix} 1 & -1 & 1 \\ 2 & -1 & 3 \\ 1 & 3 & 5 \end{vmatrix} = -5 - 3 + 6 + 1 - 9 + 10 = 17 - 17 = 0$$

$$\begin{vmatrix} 1 & -1 \\ 2 & -1 \end{vmatrix} = -1 + 2 = 1 \neq 0$$

$$\begin{cases} x_1 - x_2 = -x_3 \\ 2x_1 - x_2 = -3x_3 \end{cases} \Rightarrow \cdots \Rightarrow x_1 = -2x_3;\ x_2 = -x_3.$$

$$V^{\perp} = \{(-2x_3 , -x_3 , x_3) | x_3 \in \mathbb{R}\} = \{x_3(-2 , -1 , 1) | x_3 \in \mathbb{R}\}$$

$$V^{\perp} =< \{(-2 , -1 , 1)\} >$$

$b)$

$$V =< \{f_1 = (1 , -1 , 1) , f_2 = (2 , -1 , 3)\} >$$

*Aplicam Gram − Schmidt*:

$$e_1 = f_1 = (1 , -1 , 1)$$

$$e_2 = f_2 - \frac{< f_2 , e_1 >}{< e_1 , e_1 >} f_1$$

*Calculam* $< f_2 , e_1 >$:

$$f_2 = (\mathbf{2} , \mathbf{-1} , \mathbf{3})$$

$$e_1 = (\mathbf{1} , \mathbf{-1} , \mathbf{1})$$

$$< f_2 , e_1 > = \mathbf{2} * \mathbf{1} + (\mathbf{-1}) * (\mathbf{-1}) + \mathbf{3} * \mathbf{1} = 2 + 1 + 3 = 6$$

*Calculam* $< e_1 , e_1 >$:

$$e_1 = (1 , -1 , 1)$$

$$< e_1 , e_1 > = 1 * 1 + (-1) * (-1) + 1 * 1 = 1 + 1 + 1 = 3$$

$$e_2 = (2 , -1 , 3) - \frac{6}{3}(1 , -1 , 1) = (2 , -1 , 3) - (2 , -2 , 2) = (0 , 1 , 1)$$

$$R_1' = \{e_1 = (\mathbf{1} , \mathbf{-1} , \mathbf{1}) , e_2 = (0 , 1 , 1)\}\ reper\ ortogonal\ in\ V$$

$$R_1 = \left\{ \frac{e_1}{\|e_1\|}, \frac{e_2}{\|e_2\|} \right\} \textit{ reper ortonormat in } V$$

$$\|e_1\| = \sqrt{1^2 + (-1)^2 + 1^2} = \sqrt{3}$$

$$\|e_2\| = \sqrt{0^2 + 1^2 + 1^2} = \sqrt{2}$$

$$R_1 = \left\{ \frac{1}{\sqrt{3}}(1,-1,1), \frac{1}{\sqrt{2}}(0,1,1) \right\} \textit{ reper ortonormat in } V \ (1)$$

$$V^{\perp} = < \{f_3 = (-2,-1,1)\} >$$

$$R_2' = \{e_3 = (-2,-1,1)\} \textit{ reper ortogonal in } V^{\perp}$$

$$R_2 = \left\{ \frac{e_3}{\|e_3\|} \right\} \textit{ reper ortonormat in } V^{\perp}$$

$$\|e_3\| = \sqrt{(-2)^2 + (-1)^2 + 1^2} = \sqrt{6}$$

$$R_2 = \left\{ \frac{1}{\sqrt{6}}(-2,-1,1) \right\} \textit{ reper ortonormat in } V^{\perp}(2)$$

$(1),(2) \Rightarrow R = R_1 \cup R_2$ *reper ortonormat in* $\mathbb{R}^3$

$c)$

$$(0,1,0) = a\left(\frac{1}{\sqrt{3}}(1,-1,1)\right) + b\left(\frac{1}{\sqrt{2}}(0,1,1)\right) + c\left(\frac{1}{\sqrt{6}}(-2,-1,1)\right)$$

$$a = \cdots; b = \cdots; c = \cdots$$

$\Rightarrow p(0,1,0) = a\left(\frac{1}{\sqrt{3}}(1,-1,1)\right) + b\left(\frac{1}{\sqrt{2}}(0,1,1)\right) = \cdots$ (*de vazut in Tutoriatul* 5 –

– *Proiectii si simetrii* 😊)

*Exemplu 2:*

$(\mathbb{R}^3, g_0)$ , $\quad U = \left\{ x \in \mathbb{R}^3 \middle| \begin{cases} x_1 + x_2 - x_3 = 0 \\ 3x_1 - 2x_2 + 2x_3 = 0 \\ 6x_1 + x_2 - x_3 = 0 \end{cases} \right\}$

$a)\ U^{\perp} = ?$

$b)\ R = R_1 \cup R_2\ reper\ ortonormat = ?\,;unde\ \ R_1\ reper\ ortonormat\ in\ U\ respectiv\ R_2$

$in\ U^{\perp}.$

$c)\ Fie\ s\ simetria\ ortogonala\ fata\ de\ U^{\perp}.\ s(-1\ ,0\ ,1) = ?$

--------------------------> $de\ facut$

*Exemplu 3:*

$(\mathbb{R}^3, g_0), v = (1\ ,5\ ,-1)$

$a) < \{v\}^{\perp} > = ?$

$b)\ Determinati\ un\ reper\ ortonormat\ in\ < \{v\}^{\perp} >.$

--------------------------> $de\ facut$