

Generating Connected Random Graphs

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We present an algorithm to produce connected random graphs using a Metropolis-Hastings framework. The algorithm extends to a general framework for sampling from a known distribution of graphs, conditioned on a desired property. We demonstrate the method to generate connected spatially embedded random graphs, specifically the well known Waxman network, and illustrate the convergence and practicalities of the algorithm.

I. INTRODUCTION

Random graphs are commonly used as underlying models in many fields: *e.g.*, computer networking, biology, social sciences and physics [1–5]. The ability to generate random graphs with desired properties is crucial, as they may be used in conjunction with complex models, for instance a routing protocol in computer networking.

Real-world networks come with countless properties that one may consider modelling, *e.g.*, degree distributions, clustering levels etc. Most random graph models focus on one (or sometimes more) of these properties to represent an observed network. However, many current methods for generating random graphs result in networks with some undesirable properties for a particular applications. For instance,

- the graphs may not be connected, *e.g.*, the Gilbert-Erdős-Rényi model; or
- the graphs might have multi-edges or self-loops, *e.g.*, the configuration model.

While one might argue that this is a modelling problem, there are nevertheless very many instances in the literature where a model matches enough properties of the real networks in question that it is useful, except for one deficiency such as noted above.

Examples include:

- using the Waxman graph [6] to model physical networks that are inherently connected, *e.g.*, router networks; and
- using the configuration model that generates graphs with self-loops and multi-edges to model simple networks.

We present an algorithm to produce random graphs from a known ensemble conditioned on an extra desired property of the network. Our algorithm uses Markov chain Monte Carlo (MCMC) methods to sample from

the ensemble of interest. In particular we focus here on generating connected networks. We show the algorithm samples graphs from the desired distribution and demonstrate the algorithm on spatially embedded random networks (SERNs), in particular the Waxman random graph. We show that the algorithm is $\mathcal{O}(K)$ for K iterations, and show convergence scales like $\mathcal{O}(N^2)$ in the number of nodes in the graph.

The algorithm not only has practical applications in that one can generate connected graphs as needed in various applications, but also, such a simulation algorithm could be used to estimate the probability of such graphs in an ensemble.

II. BACKGROUND

A. Mathematical formalities

A graph (or network) $G = (V, E)$ consists of a set of N nodes, which, without loss of generality, we label $V = \{1, 2, \dots, N\}$. The graph has edges (or links) $E \subset V \times V$, where $|E| = \ell$. We are primarily concerned here with undirected graphs (though much work on random graphs is easy to generalise to directed graphs).

We say that a link exists between two nodes i and j if $(i, j) \in E$. We say that they are *connected* if a path $i = n_1 - \dots - n_k = j$ exists, where $(n_l, n_{l+1}) \in E$. The graph is connected if all pairs of nodes (i, j) are connected.

The well-known Gilbert-Erdős-Rényi (GER) random graph [7], $G_{n,p}$ of n nodes is constructed by assigning each edge (i, j) to be in E independently, with fixed probability p .

Spatially embedded random networks (SERNs) stem from the notion that longer links are more expensive. The probability of an edge in a SERN is dependent on the distance between the two nodes. Formally, we create a SERN by placing N nodes uniformly at random within some defined region R of a metric space Ω with distance metric $d(x, y)$. Each pair of nodes is made adjacent independently, with probability p_{ij} , which is a function of $d(x_i, x_j)$. In the Waxman case,

$$p_{ij} = qe^{-sd_{ij}}. \quad (1)$$

for $q \in (0, 1]$, $s \geq 0$, and the Euclidean distance d_{ij} . The

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parameter s controls the extent to which spatial structure is incorporated into the graph. Note that when $s = 0$ we recover the GER random graph, with edge probability q . In general, the q value controls the overall edge density in the graph. Note that the parametrisation in (1) differs from much of the literature on Waxman graphs. We chose to do this as unfortunately, the parameters (α, β) used traditionally have become confused by frequent reversal.

The basic properties of the Waxman graph can be derived: For instance it is shown [8] that the average node degree is given by

$$\bar{z} = (n - 1)q\tilde{G}(s), \quad (2)$$

where $\tilde{G}(s)$ is the Laplace transform of $g(t)$, the probability density function between an arbitrary pair of random points (the Line-Picking Problem) [8, 9]. The Waxman is just one example of a SERN, and we use it here to provide a simple and clear example. All results generalise to other SERNs.

B. Markov chain Monte Carlo

Markov chain Monte Carlo (MCMC) methods are widely used to sample from complex probability distributions that are difficult to generate directly. These approaches generate Markov chains that converge to the distribution of interest.

Specifically, we use the Metropolis-Hastings (M-H) algorithm [10, 11], given in Algorithm 1 to draw samples from our distribution of interest, namely, the distribution of networks with our desired property.

Consider the target distribution $\pi(\theta)$ we wish to sample from. We use the M-H algorithm to create a Markov chain $\theta^{(1)}, \theta^{(2)}, \dots$. To do so, we choose a proposal distribution $Q(\theta'|\theta)$ to propose the next candidate θ' from the current state θ . The proposal distribution must be able to explore the entire space in a finite number of steps [12].

The proposed parameter value θ' is accepted with some probability given by, in the case of M-H, the acceptance probability,

$$\alpha = \min \left(1, \frac{\pi(\theta')Q(\theta|\theta')}{\pi(\theta)Q(\theta'|\theta)} \right).$$

If the proposal distribution is symmetric (as originally proposed by Metropolis) then

$$\alpha = \min \left(1, \frac{\pi(\theta')}{\pi(\theta)} \right).$$

The chain is generated from the proposed parameter θ' as follows

$$\theta^{(t+1)} = \begin{cases} \theta', & \text{if accepted,} \\ \theta^{(t)}, & \text{otherwise,} \end{cases}$$

where θ' is generated from $Q(\theta'|\theta^{(t)})$.

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1: for k=1...K do
2:   Given  $\theta^{(t)}$ ,
3:   Generate  $\theta' \sim Q(\theta'|\theta^{(t)})$ 
4:   Take  $\theta^{(t+1)} = \begin{cases} \theta', & \text{with probability } \alpha \\ \theta^{(t)}, & \text{with probability } 1 - \alpha. \end{cases}$ 
       where  $\alpha = \min \left( 1, \frac{\pi(\theta')Q(\theta|\theta')}{\pi(\theta)Q(\theta'|\theta)} \right)$ 
5: end for

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Algorithm 1. General Metropolis-Hastings algorithm [12].

C. Connectedness

We present our algorithm in the context of generating connected random networks. The property of connectedness is often observed in physical networks, such as a telecommunications network, where there is the requirement that a path exist between all nodes. Other physical examples include the Internet routing network. It is also important in the application of social networks. In general each individual may not be connected to all others through some path. However, in the application of information diffusion we are particularly interested in the network over which information propagates. To participate in a cascade the individual must have observed the information; therefore, there is necessarily a path between all individuals in the network over which the cascade is observed.

Many random graph generators do not consider connectivity and prove properties like the distribution of size of connected components in the asymptotic limit. However, in many applications we are interested in generating connected networks of finite size.

Often rejection sampling is used to simulate connected networks by generating networks from the ensemble and rejecting disconnected networks. While appropriate in some cases, there are many situations in which this method is extremely slow, as the probability of all nodes being connected can be very low even for quite reasonable parameter values.

III. GENERATING CONNECTED GRAPHS

We assume a random graph model that generates an ensemble of sometimes unconnected graphs, and that the model provides a probability distribution across the ensemble, *i.e.*, the probability $P(G)$ for each graph G . Even if we assume that this probability is calculable, direct simulation from the distribution is usually intractable because the size of these ensembles is very large. Usually, there is an algorithm to generate the graphs from this ensemble.

Given the model, we would like to generate connected graphs with the same conditional probability distribution as the model of interest, *i.e.*, we would like to generate

connected graphs G with probabilities

$$P\{G|G \text{ is connected}\} = \frac{P\{G \text{ and } G \text{ is connected}\}}{P\{G \text{ is connected}\}},$$

where the numerator is given by:

$$P\{G \text{ and } G \text{ is connected}\} = \begin{cases} P(G), & \text{for } G \text{ connected} \\ 0, & \text{otherwise.} \end{cases}$$

However, in general, we do not know $P\{G \text{ is connected}\}$ across the ensemble, and for many models it is intractable to calculate directly. This prevents direct calculation of the conditional probability distribution. Moreover, graphs are not often generated from the distribution $P(G)$, so knowing $P\{G \text{ is connected}\}$ would not allow simulation of connected graphs. Therefore, we utilise properties of the distributions and known generators to construct random graphs with the desired conditional probability distribution.

The required connected random graphs are samples from the unknown conditional probability distribution $P\{G|G \text{ is connected}\}$. This leads naturally to the use of well known MCMC methods as the basis for the sampling algorithm.

We implement the Metropolis-Hastings method to generate a Markov chain that will result in samples from the desired distribution. The algorithm produces a new graph $G' = (V, E')$ based on the old graph G . The two main components are a symmetric proposal distribution that can explore the entire space and a tractable acceptance ratio.

We initialise the algorithm using the underlying model to create a random graph, $G^{(-1)}$. This network is connected by adding arbitrary links. We require a connected graph, not necessarily one chosen correctly from our ensemble, so almost any procedure to obtain connectivity is adequate. This results in a connected random graph $G^{(0)}$ used as the input to the M-H algorithm.

The algorithm is given in Algorithm 2 and the process described in detail below.

Step 1 - Proposal: The probability density $Q(G'|G)$, is the proposal distribution that gives the next candidate for the algorithm. An important feature of Q for the M-H algorithm is that it be symmetric, *i.e.*, $Q(G'|G) = Q(G|G')$, as this simplifies the acceptance ratio.

Here we perform the algorithm link by link. At each step, we select a link (i, j) at random, and consider adding or removing the link to obtain the new network. In practice we choose two distinct nodes at random and consider the possible link between them. If the link is already in the graph, we consider it for deletion, and if the link is not in the graph, we consider it for inclusion.

Mathematically,

1. if $(i, j) \in E$ then $E' = E \setminus (i, j)$
2. if $(i, j) \notin E$ then $E' = E \cup (i, j)$

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1: Generate  $G^{(-1)}$  from the model
2: Connect  $G^{(-1)}$  to get  $G^{(0)}$ 
3: for k=1..K do
4:   Generate a random edge  $(i, j)$ 
5:   if  $(i, j) \in E$  then
6:     Remove the edge:  $E' = E \setminus (i, j)$ 
7:     if  $G'$  is connected then
8:       accept  $G'$  with probability  $P(G')/P(G)$ 
9:     else
10:      reject  $G'$ 
11:    end if
12:  else
13:    Add edge:  $E' = E \cup (i, j)$ 
14:    accept  $G'$  with probability  $P(G')/P(G)$ 
15:  end if
16: end for

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Algorithm 2. Metropolis-Hastings method for generating connected graphs.

All links are chosen with equal probability, so $Q(G'|G) = 1/\ell$ for all G and G' that differ by one link. Therefore, the transition is symmetric.

Note that this proposal has been used in graph sampling previously, *e.g.*, [13], and there is no consideration of connectivity in this step.

Step 2 - Acceptance: The Metropolis-Hastings acceptance ratio (the probability of accepting the proposed transition) given that the proposal is symmetric is given by

$$\alpha = \min \left\{ 1, \frac{P\{G'|G' \text{ is connected}\}}{P\{G|G \text{ is connected}\}} \right\}. \quad (3)$$

If the proposed graph has a higher probability than the previous graph we accept the move. If the proposed graph is less likely than the old graph, we accept with some probability dependent on the ratio of the two graph probabilities. However, the ratio is intractable in this form, as we cannot calculate $P\{G|G \text{ is connected}\}$.

To determine a tractable acceptance ratio, we consider the connectivity of each proposed graph. Recall, we start with a valid connected graph $G^{(0)}$. If G' is unconnected, then $P\{G'|G' \text{ is connected}\} = 0$, so unconnected graphs will never be accepted, and this ensures that we remain in the space of connected graphs.

We use this to establish a tractable ratio. When G and G' are connected, the conditionals can be dropped from the probabilities, as $P\{G \text{ is connected}\}$ is constant over the ensemble.

This gives

$$\alpha = \min \left\{ 1, \frac{P(G')}{P(G)} \right\}, \quad (4)$$

for connected graphs G and G' . The ratio is tractable in many cases where we can calculate the ratio of the probability distributions.

The process is iterated a number of times until the Markov chain converges and the networks are being sampled from the stationary distribution of interest.

IV. THEORETICAL CONVERGENCE

Theorem 1 Algorithm 2 generates samples from the random graph ensemble with probability distribution $P\{G|G \text{ is connected}\}$.

Proof 1 Theorem (7.4) of Robert & Casella [12] states that the chain produced by the Metropolis-Hastings algorithm (Algorithm 1) converges to the stationary distribution π if:

1. it is irreducible, and
2. it is aperiodic.

Consider the Markov chain produced by Algorithm 2. We show there exists a sequence of a finite number of steps with positive probability from any connected graph H to any connected H' .

Consider adding all edges that not in H to create a clique. Then remove the edges in subsequent steps to reach H' . Each step, and hence the sequence, has positive probability.

$$P(H \rightarrow H_{\text{clique}} \rightarrow H') = \prod_{e \in \mathcal{E} \setminus H} p_e \prod_{e \in \mathcal{E} \setminus H'} (1 - p_e) > 0.$$

Therefore, the chain is **irreducible**.

A sufficient condition for the Markov chain to be aperiodic is to choose Q such that the probability of the event $\{X^{(t+1)} = X^{(t)}\}$ is non-zero for some state. From the acceptance probability in (4), the event of remaining in the same state occurs with positive probability.

Therefore, the chain is **aperiodic**.

Note that the acceptance probability construction ensures $\pi = P\{G|G \text{ is connected}\}$. Hence, by Theorem (7.4) of Robert & Casella, Algorithm 2, with acceptance probability α converges to the distribution of interest.

Unfortunately this result only assures us that after infinite time the process will be sampling from the distribution of interest. We show evidence for convergence in finite time in Section VII.

V. COMPLEXITY

Theorem 2 Algorithm 2 with K iterations has computational complexity $\mathcal{O}(K)$, independent of the size of the graph, for sparse graphs.

Proof 2 We use a dictionary data structure to describe the edges in the graph. This results in $\mathcal{O}(1)$ operations to check edges for existence and add/remove edges at each iteration.

We check for connectivity when edge removal is proposed. The breadth first search algorithm is $\mathcal{O}(N)$ for a network with N nodes.

For a sparse graph the number of edges is $\mathcal{O}(N)$, and so the probability of selecting an edge to delete is $\mathcal{O}(1/N)$. That is, for large N

$$P\{\text{edge } (i, j) \text{ exists}\} \sim \frac{1}{N}.$$

So, the probability there exists an edge between the two chosen nodes, requiring the $\mathcal{O}(N)$ connectedness routine, decreases like $1/N$. Therefore, each iteration is on average $\mathcal{O}(1)$, and overall the algorithm is $\mathcal{O}(K)$ in the number of iterations.

VI. SERN EXAMPLE

Here we present the example of the spatially embedded networks to demonstrate the algorithm.

Edges in a SERN are independent (conditional on distance), and hence the probability distribution of a spatially embedded random network is given by

$$P(G) = \prod_{(i,j) \in E} p_{ij} \prod_{(i,j) \notin E} (1 - p_{ij}), \quad (5)$$

where p_{ij} is the probability of an edge for the specific SERN of interest. For example, in the case of a Waxman network the edge (i, j) is given by

$$p_{ij} = qe^{-sd_{ij}},$$

for nodes separated by distance d . In the Waxman formulation, d is calculated by the Euclidean distance.

Using (5) above, the acceptance probability when adding a link (i, j) becomes

$$\frac{P(G')}{P(G)} = \frac{p_{ij}}{1 - p_{ij}},$$

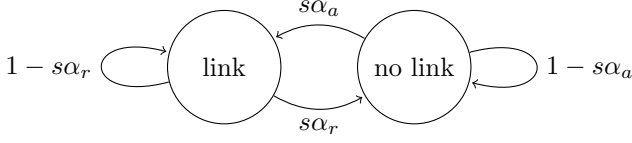
and for removing a link

$$\frac{P(G')}{P(G)} = \frac{1 - p_{ij}}{p_{ij}}.$$

While the probability distribution of the ensemble is known, it is often difficult in practice to determine the value of $P(G)$ explicitly. However, we only require the ratio of the probabilities between each pair of graphs, a much easier calculation. This also prevents issues with floating point precision caused if calculation of $P(G)$ explicitly is required.

Often, we assume that we are dealing with sparse graphs. Dense graphs are more likely to be connected, and so would not require this algorithm. Additionally, in physical networks there exists a cost constraint of constructing links, resulting in many sparse real-world networks. While definitions of network sparsity differs, all sparse networks have $p_{ij} < 1/2$. Therefore, the probability of accepting the removal of an edge is 1 if it leaves the graph connected.

A. Single link Markov chain: Waxman



The algorithm is $\mathcal{O}(K)$ in the number of iterations K , but we need to choose a reasonable value of K to use in practice. K must be large enough to ensure that the Markov chain has sufficiently converged to the stationary distribution. Here we use a single link analysis to estimate the scaling of K .

Recall that our method is performed link by link; therefore, let us consider a single link (i, j) in the graph G . The probability of accepting a change is

$$\alpha_a = \min \left(1, \frac{p_{ij}}{1 - p_{ij}} \right) \text{ if adding,}$$

$$\alpha_r = \min \left(1, \frac{1 - p_{ij}}{p_{ij}} \right) \text{ if removing.}$$

Additionally, consider that we choose the link (i, j) with some probability s . For the single link the transition probabilities of the Markov chain become

$$P = \begin{pmatrix} \text{link} & \text{no link} \\ 1 - s\alpha_a & s\alpha_a \\ s\alpha_r & 1 - s\alpha_r \end{pmatrix} \begin{matrix} \text{link} \\ \text{no link} \end{matrix} \quad (6)$$

In the limit this converges to the stationary probability of a link between nodes i and j

$$\begin{aligned} p(\text{link}) &= \frac{\alpha_a}{\alpha_a + \alpha_r}, \\ &= \frac{\min \left(1, \frac{p_{ij}}{1 - p_{ij}} \right)}{\min \left(1, \frac{p_{ij}}{1 - p_{ij}} \right) + \min \left(1, \frac{1 - p_{ij}}{p_{ij}} \right)}, \\ &= \begin{cases} \frac{\frac{p_{ij}}{1 - p_{ij}}}{\frac{p_{ij}}{1 - p_{ij}} + 1}, & \text{if } p < 0.5, \\ \frac{1}{1 + \frac{1 - p_{ij}}{p_{ij}}}, & \text{if } p \geq 0.5, \end{cases} \\ &= p_{ij}. \end{aligned}$$

Hence, the MCMC process will produce a network with the required link probability.

To extend this to the connected case of sampling from $P\{G|G \text{ is connected}\}$ we note that the probability of removing a link and moving into a ‘no link’ state where the network is disconnected is zero. As we never start in this absorbing state (i.e. the initial network is always connected), the connected system of interest is equivalent to the simplified case presented above.

The mixing of this Markov chain is important in the application of the algorithm in finite time. The spectral gap controls the rate of exponential decay to equilibrium

and the relaxation time gives an indication of how fast the chain converges. The two eigenvalues of the transition matrix (6) are $\lambda_1 = 1$ and $\lambda_2 = 1 - s\alpha_r - s\alpha_a$, giving a spectral gap of $\gamma^* = s(\alpha_r + \alpha_a)$. Note that we select edge (i, j) with probability $s \sim 1/N^2$ and $\alpha_r + \alpha_a$ is constant for any given link.

The relaxation rate is given by,

$$\begin{aligned} t_{\text{rel}} &= \frac{1}{\gamma^*}, \\ &= \frac{1}{s(\alpha_r + \alpha_a)}, \end{aligned}$$

where

$$\alpha_r + \alpha_a = \begin{cases} \frac{1}{1 - p_{ij}}, & \text{if } p < 0.5, \\ \frac{1}{p_{ij}}, & \text{if } p \geq 0.5, \end{cases} \in [1, 2].$$

In the case of sparse graphs (p_{ij} small), $\alpha_r + \alpha_a \approx 1$. Therefore, in general, $t_{\text{rel}} \sim N^2$, and we expect that $K \sim \mathcal{O}(N^2)$ for the algorithm to converge.

We next turn to the practicalities of implementing in large networks, and investigate the convergence of the algorithm on real networks.

VII. IMPLEMENTATION

Section IV showed that Algorithm 2 will converge to the distribution of connected Waxman graphs in infinite time, but we expect approximate convergence in $K \sim \mathcal{O}(N^2)$ steps. The critical question becomes, how long is required in practice?

We implement the algorithm described above using the NetworkX package in Python 2.7.13 [14] to produce connected SERNs. In order to simulate networks in finite time we must provide evidence for the convergence of the chain. Many applications of MCMC use visual means to determine when the chain seems to have converged. Here we use a heuristic that uses statistics of the graph.

Summary statistics are often used to describe network ensembles. Here we utilise the distribution of a summary statistic over the ensemble to determine convergence. In particular, the distribution of average degrees. When we condition on connectedness, we expect a slight increase in average degree due to the extra links used to connect the graph. This results in a shift in the distribution of the average degree over the ensemble. After convergence we expect no change in the distribution of summary statistics of the network as they are being drawn from the same underlying distribution. Other summary statistics, such as average edge length, could also be used.

Figure 1 shows the confidence intervals of average degree in 200 chains of the MCMC process, i.e., values at intervals along the process in 200 runs of the algorithm. This demonstrates a steady increase in average degree as the algorithm progresses. We conservatively suggest that there is no significant change in average degree after

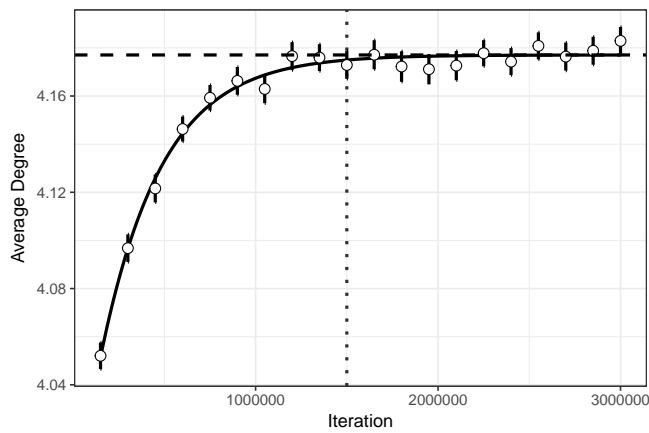


FIG. 1. Average degree over the MCMC process for a Waxman network with $N = 1000$ nodes. The mean of the average degree over 200 chains are shown with 95% confidence intervals. The solid fitted regression curve is shown, and the dashed line represents the fitted carrying capacity. Note that there is evidence for convergence at approximately 1.5 million iterations.

1.5 million iterations. Therefore, we have evidence that the system has converged and we are sampling from the posterior distribution of connected Waxman networks.

A. Iterations until convergence

To determine K , the number of steps required until convergence, we must investigate how the number of iterations to convergence scales with the number of nodes in the network. Therefore, determining convergence by eye is insufficient. We develop a framework to automate the process and give estimates of the required iterations to convergence.

We use non-linear least squares in R to fit a logistic function to determine when the average degree distribution is no longer changing. The function, of the form

$$f(x) = \frac{C}{1 + e^{-k(x-x_0)}}, \quad (7)$$

is fitted to the full data (not just the means) to determine the carrying capacity. The carrying capacity is used as the ‘converged’ value of the mean average degree over the ensemble, see Figure 1. We define strong evidence for convergence conservatively to be when the fitted values are within 99.9% of the carrying capacity.

We apply this framework to the MCMC process for varying N to determine the scaling of convergence. From the results in Figure 2 we note that the line of best fit is a power-law with an exponent of 1.97 ± 0.04 . We conclude that the mixing time of this algorithm (number of iterations to convergence) is approximately $\mathcal{O}(N^2)$. This agrees with the theoretical analysis in Section VI A.

Combining with results from Section V, the overall complexity of the algorithm $\mathcal{O}(N^2)$.

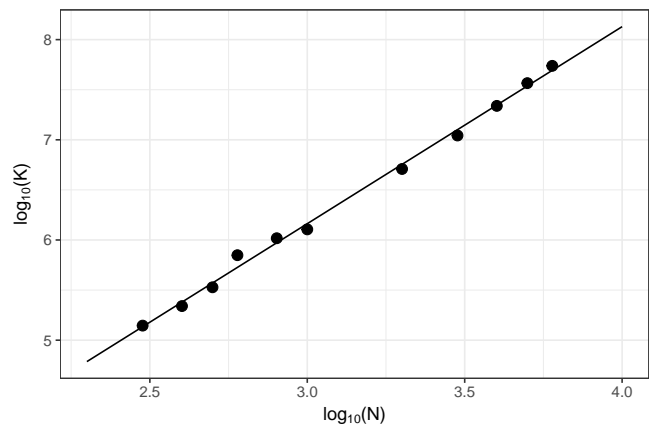


FIG. 2. Log-log plot of the time to convergence of the algorithm for varying size networks. We note that the equation of the fitted line is $\text{Iterations} = N^{1.97 \pm 0.04}$. This supports the $\mathcal{O}(N^2)$ mixing time expected over the edges in a network.

VIII. DISCUSSION

We have introduced the algorithm in the context of generating connected networks. However, this method generalises to generate networks from the probability distribution given by

$$P(G|G \text{ has some properties}),$$

assuming the properties can be tested. For example, generating a network without self loops or multi-edges would be easily implemented as above. Note although we only condition on connectedness here, the process is not restricted to a single property, a set of properties can be used.

The above algorithm assumes that the probability distribution of the network has the form in (5). However, other probability distributions, for example that of exponential random graphs [13] can easily be used. Note that we must be able to calculate the ratio of densities of graphs that differ by one link.

We initialise the algorithm by simulating a graph the model of interest; *e.g.*, the Waxman network, and connecting arbitrarily. However, any connected network can be used in this step as the MCMC process by design forgets the initial point of the Markov chain. This is particularly useful where the generation of the graph of interest is computationally expensive. However, starting ‘further’ from the distribution of interest may increase time to convergence.

IX. CONCLUSION & FUTURE WORK

This paper describes an algorithm to create random networks from a known ensemble conditioned on an extra desired property. We use a Bayesian framework, implemented with MCMC, to generate connected random net-

works. This implementation can be extended to include other desired properties of a network. We demonstrate the time complexity is $\mathcal{O}(N^2)$ with strong evidence of convergence to the desired ensemble.

Future work includes improving the efficiency of the connected components algorithm by exploiting the similarity between the connected network and the network of interest. There are extensions of the Metropolis-Hastings method, such as importance sampling, that aim to improve mixing and complexity of convergence that could be implemented in this algorithm. Additionally, when simulating a network the average degree (or some other property) is often an input. A study into the quantitative change in graph metrics over the process would be beneficial.

X. ACKNOWLEDGEMENTS

The authors acknowledge the Data to Decisions CRC (D2D CRC), the Cooperative Research Centres Programme and the ARC Center of Excellence for Mathematical and Statistical Frontiers (ACEMS) for funding this research. This research is supported by an Australian Government Research Training Program (RTP) Scholarship.

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