UNIVERSITY OF CAMBRIDGE

Assignment 1
The asynchronous & irregular state of cortical circuits

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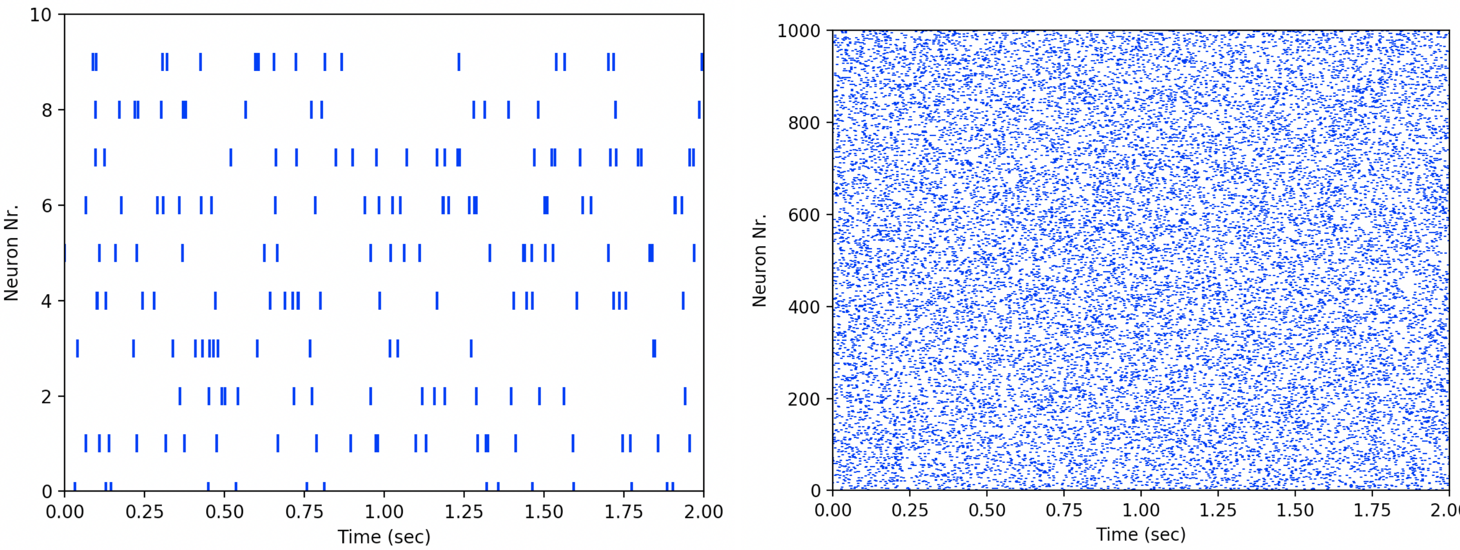
1 Generating Poisson Spike Trains

It is desired to simulate the activity of X-population neurons for period $T = 2$sec. This generation problem can be treated statistically by looking at the firing rates rather than looking at specific spike sequences. Therefore a Poisson process would be a suitable model to describe the activity of X-population. A rate $r_X = 10$Hz is chosen along with a population $N = 1000$. To be able to this in a computer simulation a sampling period $\delta_t = 0.1$ms is chosen. To simulate a Poisson process with rate r_X: $S_i[k]_{i=1,\dots,N}$ where k is the discrete-time variable, one could sample the value $S[k]$ in each bin independently from a Bernoulli distribution:

$$S[k] = \frac{1}{\delta_t} \times \begin{cases} 1 & \text{with probability } r_X \delta_t \\ 0 & \text{with probability } 1 - r_X \delta_t \end{cases} \tag{1}$$

The scalar multiplier $\frac{1}{\delta_t}$ is added so that the area over each bin will be equal to 1. Using this process the simulation of X-population is observed in a "raster plot" in *Figure 1b* below:



(a) X-population activity plot for $N = 10$ and $T = 2$ sec (b) X-population activity plot for $N = 1000$ and $T = 2$ sec

Figure 1: Raster Plot of X-population Activity

By simulating this process 30 times and each time counting the number of spikes fired by each neuron and then averaging over all $30 \times N$ neurons it is found that the average firing rate is equal to $r_X = 9.998 \pm 0.07$ Hz. As observed this value is very close to the expected value of r_X initially considered, $r_X = 10$Hz.

2 Single LIF neuron with one input spike train

In this part of the exercise it is required to simulate the dynamics and spiking activity of a single "Leaky Integrate and Fire" (LIF) neuron which receives input from a single Poisson neuron S_{in} with rate r_X defined as in **Section 1**. A synaptic weight w is given a value of $w = 0.9$. The equation to describe this process is:

$$\frac{dV(t)}{dt} = -\frac{V(t)}{\tau} + wS_{in}(t) \tag{2}$$

Where V(t) represents the membrane potential. *Equation 2* represents the continuous-time process. However, in order to simulate it is required to go into discrete time. Therefore using an "Euler Method" described in **Appendix A**, a discrete equation is produced:

$$V[k] = V[k-1] + \delta_t\left(-\frac{V[k-1]}{\tau} + wS_{in}[k-1]\right) \tag{3}$$

Where $V[0] = 0$. The voltage threshold required so that the neuron will fire is $V_{th} = 1$. The simulation of the LIF neuron is observed in *Figure 2* below:

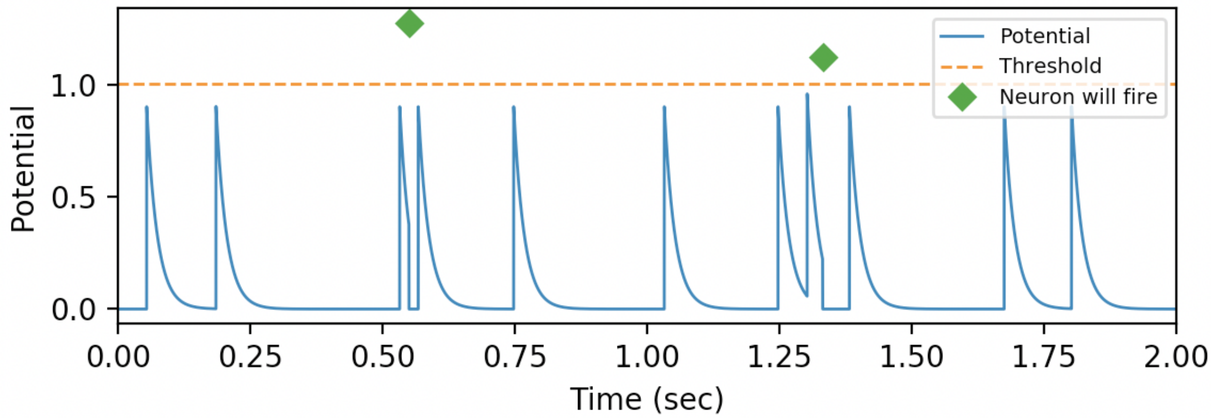


Figure 2: Membrane Potential of a LIF neuron when single neuron is used as input

As observed in the figure above a single spike input corresponds to a membrane potential value of 0.9. Therefore at least 2 spikes within a maximum time Δt are required to make the neuron fire. Δt can be calculated in the following way:

$$
\begin{aligned}
V_{th} - V_{\text{spike}} &= V_{\text{spike}} \times (1 - \frac{\delta_t}{\tau})^{\frac{\Delta t}{\delta_t}} \\
\Delta t &= \delta_t \times \frac{\ln(\frac{V_{th} - V_{\text{spike}}}{V_{\text{spike}}})}{\ln(1 - \frac{\delta_t}{\tau})} \approx 0.0438 \,\text{seconds}
\end{aligned}
\tag{4}
$$

Now that a method of calculating the membrane potential from the input spikes has been established it is time to look at the spiking activity of this LIF neuron. Therefore the output of the neuron is:

$$
S_{out}[k] = \frac{1}{\delta_t} \times \begin{cases} 1 & \text{if } V[k] > V_t h \\ 0 & \text{otherwise} \end{cases}
\tag{5}
$$

Due to the fact that a spike is fired the neuron is depolarised resulting in a reset of the voltage V[k] to 0. A plot showing the output spike based on the input spikes is observed below:

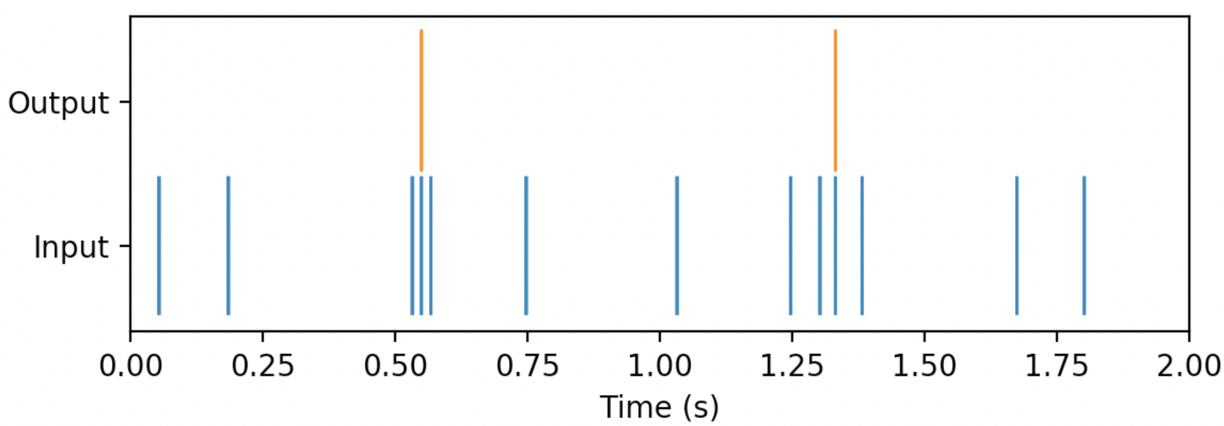


Figure 3: Output Spikes from the LIF neuron when a single neuron is used as a input

3 Single LIF neuron with many input spike trains

It is again looked at a single LIF neuron but this time receiving inputs from K independent Poisson neurons, each firing at constant rate r_X. The synaptic weight w, is chosen so that it is constant for all K neurons. The new dynamics are described by the following continuous-time equation:

$$\frac{dV(t)}{dt} = -\frac{V(t)}{\tau} + \underbrace{\frac{w}{K}\sum_{j=1}^{K} S_j(t)}_{h(t)} \tag{6}$$

By applying the "Euler" Method the discrete-time equation can be expressed as:

$$V[k] = V[k-1] + \delta_t\Big(-\frac{V[k-1]}{\tau} + \underbrace{\frac{w}{K}\sum_{j=1}^{K} S_j[k-1]\Big)}_{h[k-1]} \tag{7}$$

By disabling the spike reset mechanism, choosing a random K neurons out of X-population and setting $w = 1$ the membrane potential of a single LIF neuron at each δ_t is calculated and displayed in *Figure 4* below:

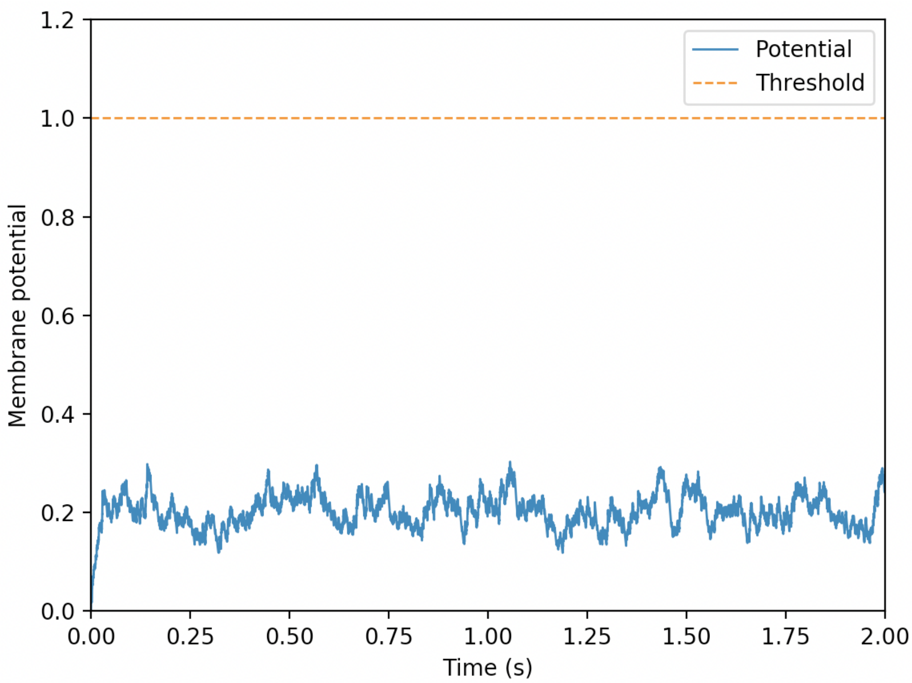


Figure 4: Membrane Potential of a LIF neuron when $K = 100$ neurons are used as input

As observed, after a transient response of ≈ 0.1 seconds, the membrane potential settles around a mean of 0.2 and it varies stochastically. These fluctuations can be approximated by a normal distribution with mean μ and variance σ^2 ($\mathcal{N} \sim (V; \mu_v, \sigma_v^2)$). The theoretical values of the mean and variance can be found analytically.

First, as it is known that the h varies independently at each k value, it can only be looked at $h[k]$ to gather all the information about the function h. The mean and variance for $h[k]$ is calculated (μ_h and σ_h^2). The $h[k]$ function is rewritten below:

$$h[k] = \frac{w}{K}\sum_{j=1}^{K} S_j[k] \tag{8}$$

Therefore the mean μ_h can be expressed as:

$$
\begin{aligned}
\mu_h = E\{h[k]\} &= E\{\frac{w}{K}\sum_{j=1}^{K} S_j[k]\} = \\
&= \frac{w}{K}E\{\sum_{j=1}^{K} S_j[k]\} \\
&\overset{(a)}{=} \frac{w}{K}\sum_{j=1}^{K} E\{S_j[k]\} \\
&\overset{(b)}{=} \frac{w}{K}\sum_{j=1}^{K} \frac{1}{\delta_t} r_X \delta_t \\
&= \frac{w}{K} K r_X \\
&\boxed{\mu_h = w r_X}
\end{aligned}
\tag{9}
$$

Where, step *(a)* comes from the fact that the expectations of a sum is the sum of the expectations. Step *(b)* comes from the fact that the expected value of the Poisson process $S_j[k]$ is equal to $E\{S_j[k]\} = \frac{1}{\delta_t} \times r_X \delta_t$. Next, σ_h^2 can be calculated as:

$$
\begin{aligned}
\sigma_h^2 &= E\{h[k]^2\} - E\{h[k]\}^2 \\
&= E\{\frac{w^2}{K^2}\sum_{i=1}^{K} S_i[k] \sum_{j=1}^{K} S_j[k]\} - \mu_h^2 \\
&= E\{\frac{w^2}{K^2}\sum_{i=1}^{K}\sum_{j=1}^{K} S_i[k] S_j[k]\} - \mu_h^2 \\
&\overset{(a)}{=} \frac{w^2}{K^2}\sum_{i=1}^{K}\sum_{j=1}^{K} E\{S_i[k] S_j[k]\} - \mu_h^2 \\
&= \frac{w^2}{K^2}(\underbrace{\sum_{j=1}^{K} E\{S_j[k]^2\}}_{j=i} + \underbrace{\sum_{j=1}^{K}\sum_{j\neq i, i=1}^{K} E\{S_j[k] S_i[k]\}}_{j\neq i}) - \mu_h^2 \\
&\overset{(b)}{=} \frac{w^2}{K^2}(K\frac{1}{\delta_t} r_X + K(K-1) r_X^2) - \mu_h^2 \\
&= \frac{w^2}{K}\frac{1}{\delta_t} r_X + \frac{w^2}{K}(K-1) r_X^2 - w^2 r_X^2 \\
&\boxed{\sigma_h^2 = \frac{w^2}{K}\frac{1}{\delta_t} r_X - \frac{w^2}{K} r_X}
\end{aligned}
\tag{10}
$$

Where step *(a)* comes from the same argument as in step *(a)* in Equation 9. Step *(b)* comes from the fact that the autocorrelation of the Poisson process simulated is equal to:

$$
E\{S_j[k] S_i[k]\} = \begin{cases} (\frac{1}{\delta_t})^2 r_X^2 \delta_t^2 & \text{if } i \neq j \\ (\frac{1}{\delta_t})^2 r_X \delta_t & \text{if } i = j \end{cases}
\tag{11}
$$

As the values of μ_h and σ_h^2 have been defined it is now time to look at Equation 7 and find the values for

μ_v and σ_v^2. First, it is started with the mean:

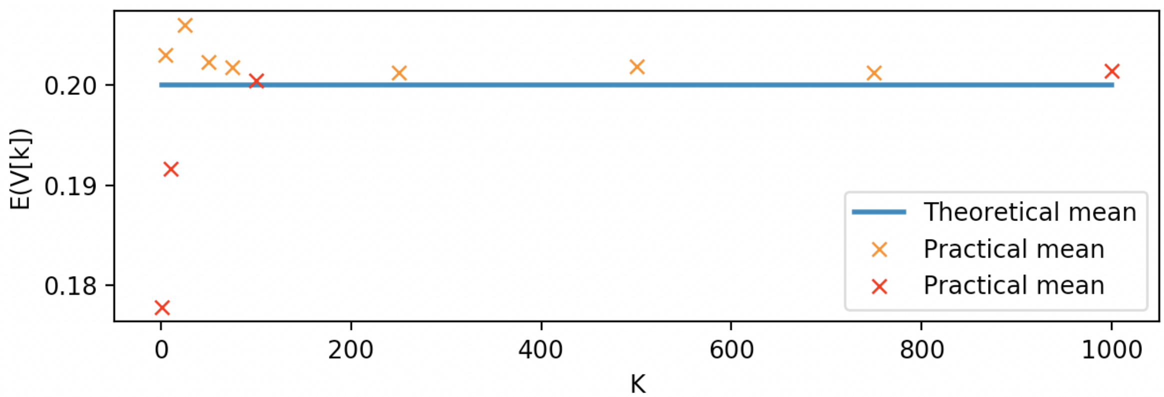
$$
\begin{aligned}
V[k] &= V[k-1] + \delta_t(-\frac{V[k-1]}{\tau} + h[k-1]) \quad |E\{\} \\
E\{V[k]\} &= E\{V[k-1]\} - \frac{\delta_t}{\tau}E\{V[k-1]\} + \delta_t E\{h[k-1]\} \\
\cancel{\mu_v} &= \cancel{\mu_v} - \frac{\delta_t}{\tau}\mu_v + \delta_t\mu_h \\
\frac{\cancel{\delta_t}}{\tau}\mu_v &= \cancel{\delta_t}\mu_h \\
\mu_v &= \tau\mu_h \\
&\boxed{\mu_v = \tau w r_X}
\end{aligned}
\tag{12}
$$

Next, σ_v is calculated:

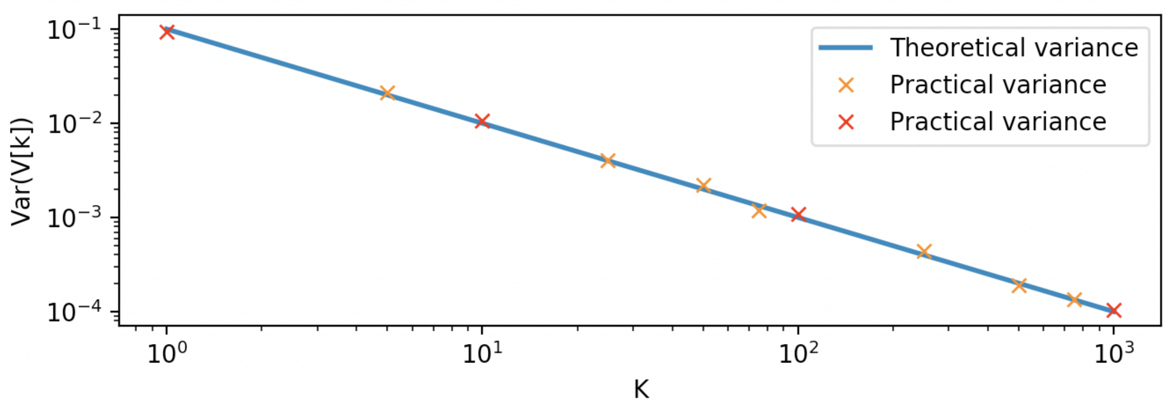
$$
\begin{aligned}
V[k] &= V[k-1] + \delta_t(-\frac{V[k-1]}{\tau} + h[k-1]) \quad | - \mu_v, ()^2, E\{\} \\
E\{(V[k]-\mu_v)^2\} &= E\{[(V[k-1]-\mu_v) + \delta_t(h[k-1] - \frac{1}{\tau}V[k-1])]^2\} \\
\sigma_v^2 &= E\{(V[k-1]-\mu_v)^2\} + E\{2\delta_t(V[k-1]-\mu_v)(h[k-1] - \frac{1}{\tau}V[k-1])\} + \\
&+ E\{\delta_t^2(h[k-1] - \frac{1}{\tau}V[k-1])^2\} \\
\cancel{\sigma_v^2} &= \cancel{\sigma_v^2} + 2\delta_t E\{V[k-1]h[k-1]\} - \frac{2\delta_t}{\tau}E\{V[k-1]^2\} - 2\delta_t\mu_v E\{h[k-1]\} + \\
&+ \frac{2\delta_t}{\tau}\mu_v E\{V[k-1]\} + \delta_t^2 E\{h[k-1]^2\} - \frac{2\delta_t^2}{\tau}E\{h[k-1]V[k-1]\} + \\
&+ \delta_t^2 E\{V[k-1]^2\} \\
0 &= \cancel{2\delta_t\mu_v\mu_h} - \frac{2\delta_t}{\tau}E\{V[k-1]^2\} - \cancel{2\delta_t\mu_v\mu_h} + \frac{2\delta_t}{\tau}\mu_v^2 + \delta_t^2 E\{h[k-1]^2\} - \\
&- \frac{2\delta_t^2}{\tau}\mu_h\mu_v + \frac{\delta_t^2}{\tau^2}E\{V[k-1]^2\} \\
0 &= \frac{2\delta_t}{\tau}(\underbrace{E\{V[k-1]^2\} - \mu_v^2}_{\sigma_v^2}) + \delta_t^2 E\{h[k-1]^2\} - \frac{\delta_t^2}{\tau}\mu_h\mu_v - \frac{\delta_t^2}{\tau}\mu_h\mu_v + \frac{\delta_t^2}{\tau^2}E\{V[k-1]^2\} \\
0 &= -\frac{2\delta_t}{\tau}\sigma_v^2 + \delta_t^2 E\{h[k-1]^2\} - \frac{\delta_t^2}{\tau}\mu_h \underbrace{\tau\mu_h}_{\mu_v} - \frac{\delta_t^2}{\tau}\underbrace{\frac{\mu_v}{\tau}}_{\mu_h}\mu_v + \frac{\delta_t^2}{\tau^2}E\{V[k-1]^2\} \\
0 &= -\frac{2\delta_t}{\tau}\sigma_v^2 + \delta_t^2(\underbrace{E\{h[k-1]^2\} - \mu_h^2}_{\sigma_h^2}) + \frac{\delta_t^2}{\tau^2}(\underbrace{E\{V[k-1]^2\} - \mu_v^2}_{\sigma_v^2}) \\
0 &= -\frac{2}{\tau}\sigma_v^2 + \delta_t\sigma_h^2 + \frac{\delta_t}{\tau^2}\sigma_v^2 \\
\sigma_v^2 &= \frac{\tau^2}{2\tau - \underbrace{\delta_t}_{\text{negligible}}}\delta_t\sigma_h^2
\end{aligned}
\tag{13}
$$

$$\sigma_v^2 = \frac{\tau}{2}\delta_t\sigma_h^2$$
$$\sigma_v^2 = \frac{\tau}{2}\delta_t(\frac{w^2}{K}\frac{1}{\delta_t}r_X - \frac{w^2}{K}r_X^2)$$
$$\sigma_v^2 = \frac{\tau}{2}\frac{w^2}{K}r_X - \underbrace{\frac{\tau}{2}\delta_t\frac{w^2}{K}r_X^2}_{\text{negligible}}$$
$$\boxed{\sigma_v^2 = \frac{\tau}{2}\frac{w^2}{K}r_X}$$

Therefore, V[k] can be approximated by a Normal Distribution : $\mathbf{N} \sim (V[k], \tau w r_X, \frac{\tau w^2}{2K}r_X)$. It is observed that the mean of $V[k]$ is independent of K while the variance of $V[k]$ is inversely proportional to K. In the figures below the theoretical values are compared against the practical values obtained from the simulation by neglecting the initial transient response.



(a) Mean Theoretical vs Mean Practical against multiple values of K



(b) Variance Theoretical vs Variance Practical in a log plot against multiple values of K

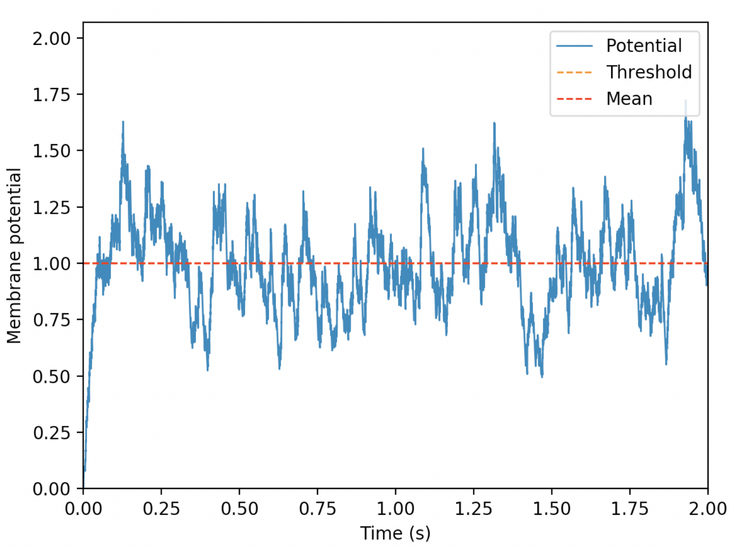
Figure 5: Mean and Variance comparison

From the figures above it is observed that at low values of K the practical value of the mean fails to match the theoretical value due to higher variance in membrane potential but it improves as K increases. On the other hand, it can be observed that the practical value of the variance matched very well the theoretical value at all values of K.

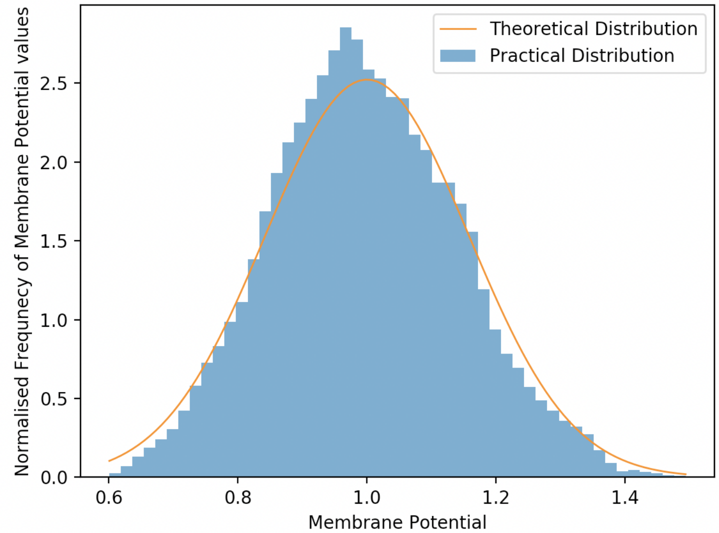
In order to set the mean value of $V[k]$ equal to V_{th} for $K = 100$ it is required to adjust the synaptic weight such that it equals:

$$w^* = \frac{V_{th}}{\tau r_X} = 5 \tag{14}$$

Therefore calculating the new membrane potential for $T = 2$ seconds using w^* as synaptic weight results in the figure below:



(a) Membrane Potential of a LIF neuron with K excitatory neurons and the synaptic weight is set to $w = 5$

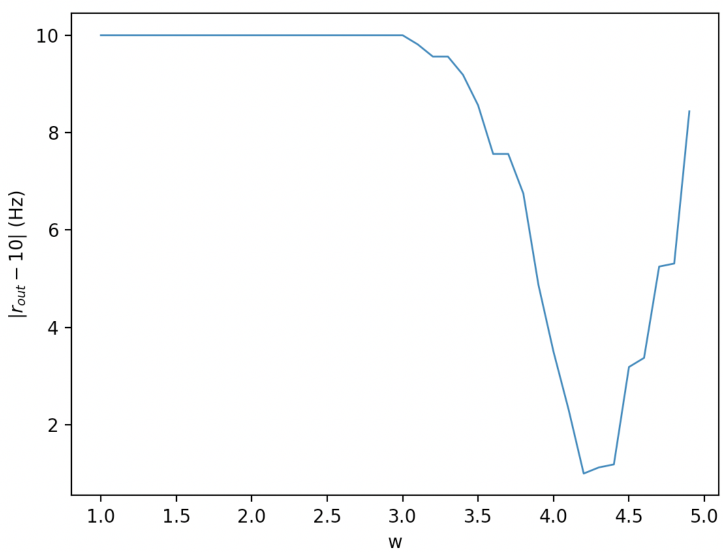


(b) The distribution of the Membrane potential compared to a true an ideal normal distribution with mean 1 and variance $\frac{\tau}{2}\frac{w^2}{K}r_X$

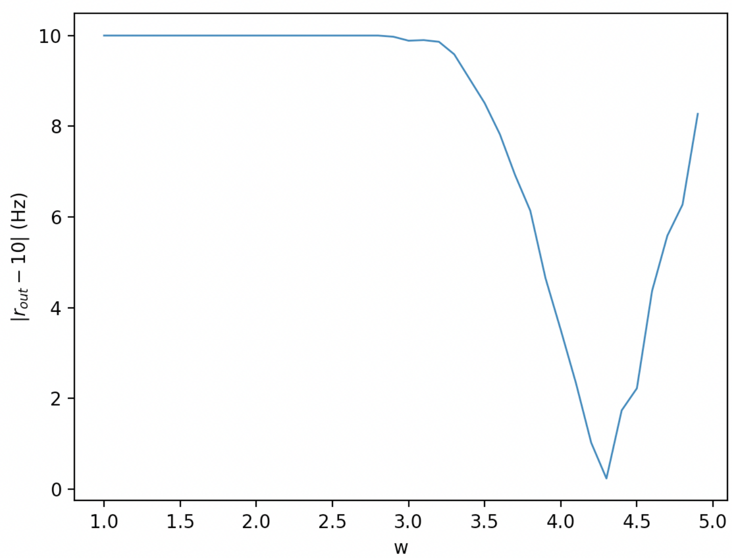
Figure 6: Raster Plot of X-population Activity

As observed in *Figure 6a* the mean μ_v and V_th completely overlap showing that our theory matches the practical simulation. From *Figure 6b* it is observed that indeed the simulation of the membrane potential corresponds to sampling from a normal distribution.

Now, the spike reset is enabled again. An optimisation is performed to find the value of synaptic weight w that would correspond to an output firing rate of 10Hz. In the plot below the firing rate dependence on synaptic weight is observed:



(a) Output Firing Rate vs Synaptic Weight for $T = 2\,\mathrm{sec}$



(b) Output Firing Rate vs Synaptic Weight for $T = 10\,\mathrm{sec}$

Figure 7: Synaptic weight optimisation

It is observed that a value of synaptic weight $w \approx 4.3$ is determining a firing rate close to 10Hz. By further optimisation by trial and error, it is found that a value of $w = 4.29$ optimum. With this value of w the process is run for 30 times and by introducing a counting window of 100ms the Fano factor is computed to be 0.4665 ± 0.1267. This value of Fano factor is lower than 1 which suggest that there is not enough spike variability in this model compared to the cortex. The reason why the Fano factor is small is because of not having any inhibitory neurons in the model to bring the mean to 0 so that more variability in the spiking would occur.

4 Single LIF neuron with many E and I Poisson Inputs

It is again considered a single LIF neuron but this time it is receiving excitatory input from K independent Poisson neurons and respectively inhibitory input from K independent Poisson neurons. The same firing rate r_X is kept for both of these inputs and the synaptic weight are adjusted to $\frac{w}{\sqrt{K}}$ for excitatory input and $-\frac{w}{\sqrt{K}}$ for inhibitory input. Therefore the continuous-time equation of the LIF model is:

$$\frac{dV(t)}{dt} = -\frac{V(t)}{\tau} + \sqrt{K}(\underbrace{\frac{w}{K}\sum_{j=1}^{K} S_j(t)}_{h_e(t)} - \underbrace{\frac{w}{K}\sum_{i=1}^{K} S_i(t)}_{h_i(t)}) \tag{15}$$

By applying "Euler Method" the discrete-time equation can be expressed as:

$$V[k] = V[k-1] + \delta_t[-\frac{V[k-1]}{\tau} + \sqrt{K}(\underbrace{\frac{w}{K}\sum_{j=1}^{K} S_j[k-1]}_{h_e[k-1]} - \underbrace{\frac{w}{K}\sum_{i=1}^{K} S_i[k-1]}_{h_i[k-1]})] \tag{16}$$

For the equation described above it is observed that a again V can be approximated by a normal distribution with mean $\mu_{\hat{v}}$ and variance $\sigma_{\hat{v}^2}$. Firstly, it is observed that h_e and h_i are equivalent to h used in **Section 3** and $\hat{h}$ will be used to express:

$$\hat{h}[k] = h_e[k] - h_i[k] \tag{17}$$

Therefore the new $\hat{h}$ has a new mean $\mu_{\hat{h}}$ and a new variance $\sigma^2_{\hat{h}}$ which can be expressed analytically. Firstly the expression for the mean is found.

$$\begin{aligned}
\mu_{\hat{h}} &= E\{\hat{h}[k]\} \\
&= E\{h_e[k] - h_i[k]\} \\
&= E\{h_e[k]\} - E\{h_i[k]\} \\
&= E\{h[k]\} - E\{h[k]\} \\
&\boxed{\mu_{\hat{h}} = 0}
\end{aligned} \tag{18}$$

Next, the variance $\sigma^2_{\hat{h}}$ is calculated:

$$\begin{aligned}
\sigma^2_{\hat{h}} &= Var\{\hat{h}[k]\} \\
&= Var\{h_e[k] - h_i[k]\} \\
&\overset{(a)}{=} Var\{h_e[k]\} + Var\{h_i[k]\} \\
&= Var\{h[k]\} + Var\{h[k]\} \\
&= 2\sigma^2_h \\
&\boxed{\sigma^2_{\hat{h}} = 2(\frac{w^2}{K}\frac{1}{\delta_t}r_X - \frac{w^2}{K}r_X^2)}
\end{aligned} \tag{19}$$

Where step *(a)* comes from the fact that the variance of the sum of two independently random variables is the sum of the variance. This is proven in **Appendix A**. Now using Equations 12 and 13 but with $\sqrt{K}\mu_{\hat{h}}$

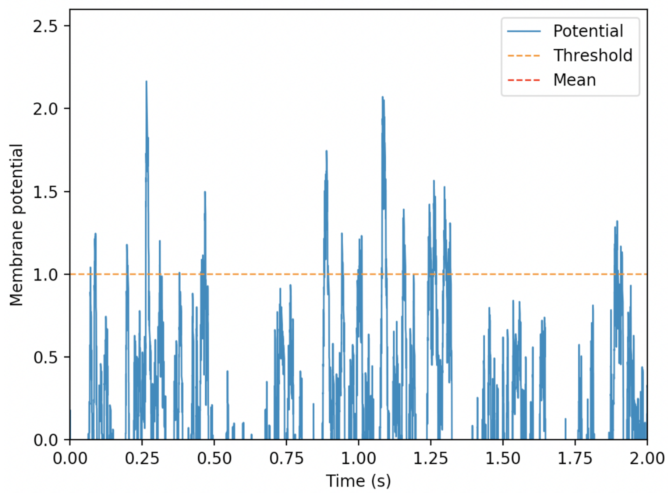
instead of μ_h and $K\sigma_{\hat{h}}^2$ instead of σ_h^2 it is found that:

$$\begin{cases} \mu_{\hat{v}} = \tau\sqrt{K}\mu_{\hat{h}} \\ \sigma_{\hat{v}}^2 = \frac{\tau}{2}\delta_t K\sigma_{\hat{h}}^2 \end{cases} = \begin{cases} \mu_{\hat{v}} = \tau\sqrt{K} \times 0 \\ \sigma_{\hat{v}}^2 = \frac{\tau}{2}\delta_t K \times 2(\frac{w^2}{K}\frac{1}{\delta_t}r_X - \frac{w^2}{K}r_X^2) \end{cases} = \begin{cases} \mu_{\hat{v}} = 0 \\ \sigma_{\hat{v}}^2 = \tau w^2 r_X - \underbrace{\delta_t \tau w^2 r_X^2}_{\text{negligible}} \end{cases} \tag{20}$$

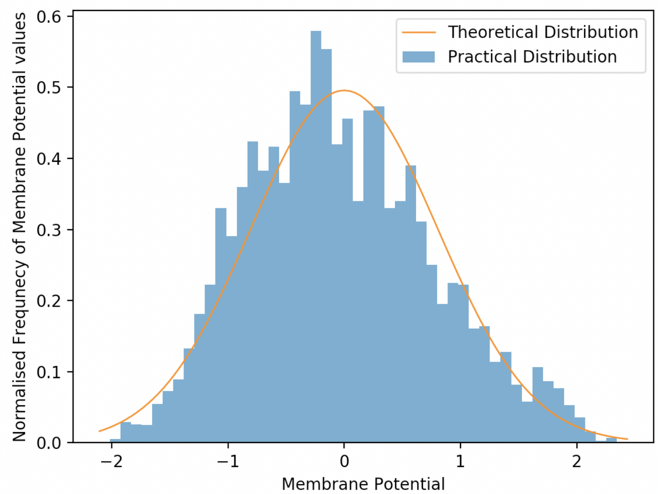
$$\boxed{\mu_{\hat{v}} = 0}$$

$$\boxed{\sigma_{\hat{v}}^2 = \tau w^2 r_X}$$

Therefore the membrane potential V can be approximated by a Normal Distribution with a mean 0 and variance $\tau w^2 r_X$. The simulation for finding membrane potential can be observed in the figures below:



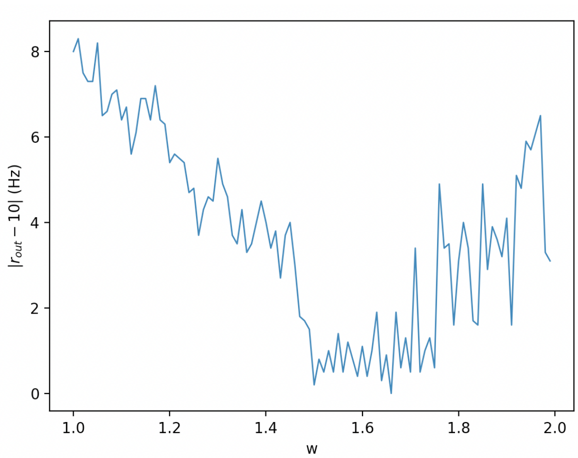
(a) Membrane Potential of a LIF neuron with K=100 inhibitory and excitatory neurons and the synaptic weight is set to $w = 1.8$



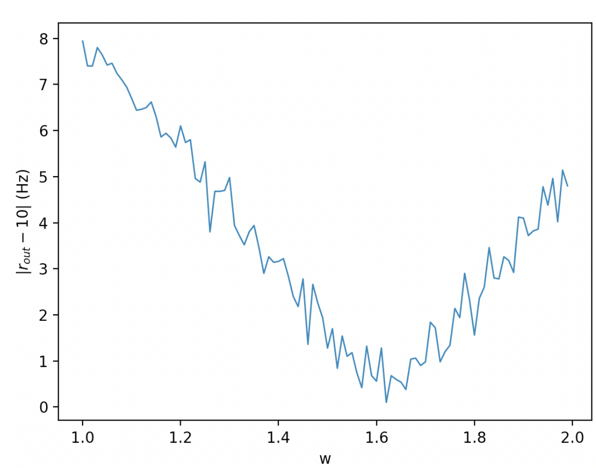
(b) The distribution of the Membrane potential compared to a true an ideal normal distribution with mean 0 and variance $\tau w^2 r_X$

Figure 8: Membrane Potential simulation

Again it is observed from *Figure 8b* that the assumption of sampling from a normal distribution still holds for this model. In order to set the value of the output rate to be $\approx 10\,\text{Hz}$, the synaptic weight is adjusted. In the figures below the output firing rate dependency on w:



(a) Output Firing Rate vs Synaptic Weight for $T = 2\,\text{sec}$



(b) Output Firing Rate vs Synaptic Weight for $T = 10\,\text{sec}$

Figure 9: Synaptic weight optimisation

From the above figures, it is observed that by simulating for short timescale $T = 2\,\text{sec}$ the output rate will oscillate. Increasing the timescale reduces the oscillation because the membrane potential will start looking more and more like a normal distribution and therefore will become smoother. From the figures, it can be seen that an optimum value of $w = 1.62$ yields the closest approximation. By further optimisation around this value, it is found that an optimum synaptic weight of $w = 1.63$ performs the best. For this value, the output firing rate is found to be 10.2267 ± 0.8794 and the Fano factor 0.9993 ± 0.0978. The Fano factor is ≈ 1 which is much closer to the values found in a real cortex. This is because there is no transient after the spike reset mechanism since it is now reset to the mean of the voltage of 0 and continue to be at the stationary distribution, which allows there to be more variability in the spiking.

5 Full Network

In this section, the network will be composed of three populations of N neurons each: one excitatory ('E') population, one inhibitory ('I') population, and a third, "external" population ('X') which will provide baseline input to the network. The diagram of this network is observed in *Figure 10* below:

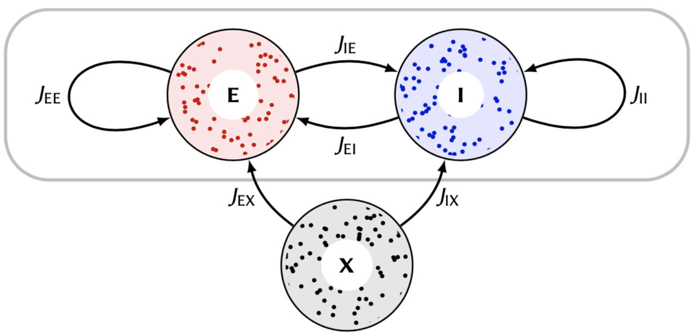


Figure 10: Diagram of the Full Network

The membrane potential V_i of the ith neuron in population α obeys first-order dynamics given by the following continuous-time function:

$$\frac{dV_i^\alpha(t)}{dt} = -\frac{V_i^\alpha(t)}{\tau} + \sum_{\beta \in \{\text{E,I,X}\}} \frac{J_{\alpha\beta}}{\sqrt{K}} \sum_{j \in \mathcal{C}_i^{\alpha\beta}} S_j^\beta(t) \tag{21}$$

Where $C_i^{\alpha\beta}$ denote the set that contains the indices of i those K neurons in population β that have been randomly chosen to connect onto the ith neuron of population α and $\alpha \in$ E,I,X. The equivalent discrete-time equation can be expressed as:

$$V_i^\alpha[k] = V_i^\alpha[k-1] + \delta_t(-\frac{V_i^\alpha[k-1]}{\tau} + \sum_{\beta \in \{\text{E,I,X}\}} \frac{J_{\alpha\beta}}{\sqrt{K}} \sum_{j \in \mathcal{C}_i^{\alpha\beta}} S_j^\beta(t)) \tag{22}$$

If the two cases $\alpha = E$ and $\alpha = I$ are considered, setting $w = 1$ from previous sections and taking expectations of the Equation 22 then the firing rates should satisfy the following equations:

$$
\begin{aligned}
&\begin{cases} \mu_v^E = \mu_v^E + \delta_t[-\frac{1}{\tau}\mu_v^E + \sqrt(K)(J_{EE}E\{h_E[k-1]\} + J_{EI}E\{h_I[k-1]\} + J_{EX}E\{h_X[k-1]\})] \\ \mu_v^I = \mu_v^I + \delta_t[-\frac{1}{\tau}\mu_v^I + \sqrt(K)(J_{IE}E\{h_E[k-1]\} + J_{II}E\{h_I[k-1]\} + J_{IX}E\{h_X[k-1]\})] \end{cases} \\
(a) &\begin{cases} \frac{1}{\tau\sqrt{K}}\mu_v^E = J_{EE}r_E + J_{EI}r_I + J_{EX}r_X \\ \frac{1}{\tau\sqrt{K}}\mu_v^I = J_{IE}r_E + J_{II}r_I + J_{IX}r_X \end{cases} \\
(b) &\begin{cases} 0 = J_{EE}r_E + J_{EI}r_I + J_{EX}r_X \\ 0 = J_{IE}r_E + J_{II}r_I + J_{IX}r_X \end{cases} \\
&\begin{cases} 0 = r_E - 2r_I + r_X \\ 0 = r_E - 1.8r_I + 0.8r_X \end{cases}
\end{aligned}
\tag{23}
$$

Where step *(a)* comes from the fact that provided $w = 1$, then $E\{h_E[k]\} = r_E$ and the same principle is applied for X and I. Step *(b)* comes from the fact that as $K \to \infty$ the left hand side will tend towards 0. By solving the simultaneous equation above with r_X fixed the result shows that $r_E = r_I = r_X$. For multiple values of r_X, $r_X = [5, 10, 15, 20]$ the practical firing rates from the simulation are found and compared to the theoretical ones in the table below:

Theoretical Values			**Simulation Values**		
r_X (Hz)	r_E (Hz)	r_I (Hz)	r_X (Hz)	r_E (Hz)	r_I (Hz)
5.00	5.00	5.00	5.00	7.07	5.86
10.00	10.00	10.00	10.00	12.94	11.60
15.00	15.00	15.00	15.00	18.64	17.01
20.00	20.00	20.00	20.00	23.49	22.07

Table 1: Theoretical firing rates compared to Simulation firing rates

From *Table 1* above it is clear that there is a mismatch between theoretical and simulation values. This is because we have a finite value for K instead of the infinite value required to make the left hand side of Equation 23 equal to 0.

By setting $N = 100$ and $K = 100$ a "Raster" plot is produced showing the activity of the E-population. The figure is displayed in the figure below:

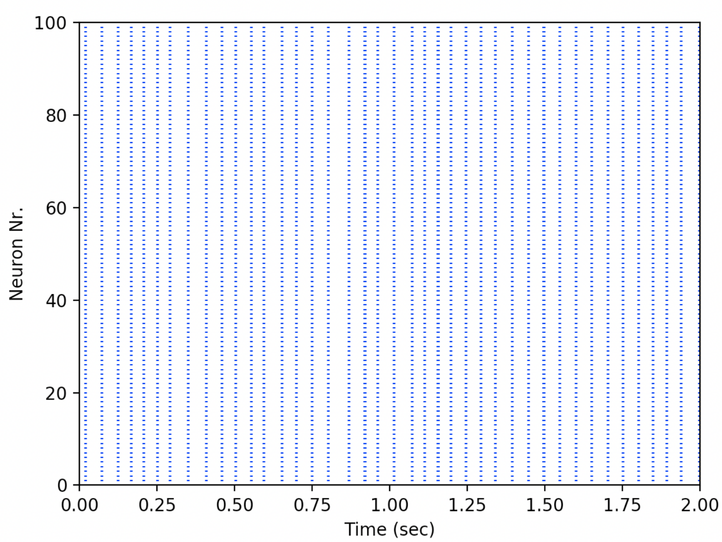


Figure 11: E-population activity plot with $K = 100$ and $N = 100$

From *Figure 11* it is observed that all neurons fire synchronously. This is because E and I population neurons are receiving the exact input from the full X population. The inter-neuron independence is broken because for this model of the network E and I population neurons are highly correlated.

6 Appendix

6.1 Appendix A: Mathematical Identities

6.1.1 Euler Method

A method of approximating a first-order numerical procedure for solving ordinary differential equations (ODEs):

$$\begin{aligned} V(t+\delta_t) &= V(t) + \delta_t \times \frac{dV(t)}{dt} \\ \frac{dV(t)}{dt} &= \frac{V(t+\delta_t) - V(t)}{\delta_t} \end{aligned} \tag{24}$$

6.1.2 Variance of a difference of two independent random variables

Let X and Y be two independent random variables:

$$\begin{aligned} Var\{X-Y\} &= E\{(X-Y)^2\} - E\{X-Y\}^2 \\ &= E\{X^2 - 2XY + Y^2\} - (E\{X\} - E\{Y\})^2 \\ &= E\{X^2\} - 2E\{X\}E\{Y\} + E\{Y^2\} - E\{X\}^2 + 2E\{X\}E\{Y\} - E\{Y\}^2 \\ &= E\{X^2\} - E\{X\}^2 + E\{Y^2\} - E\{Y\}^2 \\ &= Var\{X\} + Var\{Y\} \end{aligned} \tag{25}$$