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Teorema Reziduurilor si aplicatii

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Cuprins

Introducere

1 Integrala Riemann-Stieltjes a unei functii complexe de variabila reala

Definitie 1. *Fie $f = u+iv$ si $F = U+iV$, iar $[a;b]$ interval din $\mathbb{R}$. Spunem ca f este integrabila Riemann-Stieltjes in raport cu F pe intervalul $[a;b]$ daca u si v sunt integrabile Riemann-Stieltjes in raport cu U si V pe $[a;b]$.*

Notam :

$$\int_a^b f \,\mathrm{d}F := \int_a^b u \,\mathrm{d}U - \int_a^b v \,\mathrm{d}V + i\int_a^b u \,\mathrm{d}V + i\int_a^b v \,\mathrm{d}U$$

Teorema 1. *Consideram $f = u + iv$, $F = U + iV$, iar $f_n : [a;b] \mapsto \mathbb{C}$, $F_n : [a;b] \mapsto \mathbb{C}$, si α , $\beta \in \mathbb{C}$.*

Au loc urmatoarele proprietati :

1. *Daca f este integrabila Riemann-Stieltjes in raport cu F pe $[a;b]$, atunci F este integrabila Riemann-Stieltjes in raport cu f si :*

$$\int_a^b f \,\mathrm{d}F + \int_a^b F \,\mathrm{d}f = f(b)F(b) - f(a)F(a)$$

2. *Daca f si g sunt integrabile Riemann-Stieltjes in raport cu F pe $[a;b]$, atunci $\alpha f + \beta g$ e integrabila dupa F si :*

$$\int_a^b (\alpha f + \beta g) \,\mathrm{d}F = \alpha\int_a^b f \,\mathrm{d}F + \beta\int_a^b g \,\mathrm{d}F$$

3. *Daca f este continua si F este cu variatie marginita pe $[a;b]$, atunci f este integrabila pe $[a;b]$ in raport cu F.*

4. *Fie $(f_n)_{n\in\mathbb{N}}$ un sir de functii continuue ce converge uniform catre f pe $[a;b]$ si $(F_n)_{n\in\mathbb{N}}$ un sir de functii cu variatie marginita care converge punctual catre F, iar sirul $\mathrm{V}(F_n, [a;b])$ marginit. Atunci avem ca:*

$$\lim_{\substack{n\to\infty \\ k\to\infty}} \int_a^b f_n \,\mathrm{d}F_k = \int_a^b f \,\mathrm{d}F$$

5. *Daca* f *e continua,* F *derivabila si* F' *continua, atunci :*

$$\int_a^b f \,\mathrm{d}F = \int_a^b f(t)F'(t)\,\mathrm{d}t$$

6. *Fie* $c \in (a;b)$ *si* f *integrabila in raport cu* F *pe* $[a;b]$*, atunci* f *este integrabila in raport cu* F *si pe* $[a;c]$*, si pe* $[c;b]$*, iar :*

$$\int_a^b f \,\mathrm{d}F = \int_a^c f \,\mathrm{d}F + \int_c^b f \,\mathrm{d}F$$

7. *Daca* f *e integrabila in raport cu* F *pe* $[a;b]$*, si* $h : [a';b'] \mapsto [a;b]$ $h(a') = a$ *si* $h(b') = b$*,* h *fiind omeomorfism, atunci* $f \circ h$ *e integrabila Riemann-Stieltjes pe* $F \circ H$ *si*

$$\int_a^b f \,\mathrm{d}F = \int_{a'}^{b'} (f \circ h)\,\mathrm{d}(F \circ H)$$

Definitie 2. *Consideram drumul rectificabil* γ*, iar* $f : \{\gamma\} \mapsto \mathbb{C}$ *continua. Atunci* $f \circ \gamma$ *va fi continua pe* $[0;1]$ *si integrabila in raport cu* γ*. Aceasta inegrala se numeste integrala complexa a drumului* f *de-a lungul lui* γ *:*

$$\int_\gamma f := \int_\gamma f(\zeta)\,\mathrm{d}\zeta = \int_0^1 (f \circ \gamma)\,\mathrm{d}\gamma$$

Teorema 2. *Fie* γ *drum rectificabil din* $\mathcal{D}(z_0;z_1)$ *si* f *o functie continua din* $\{\gamma\}$*. Atunci :*

1. *Fie* g *o alta functie continua din* $\{\gamma\}$*,* α*,* $\beta \in \mathbb{C}$ *, atunci:*

$$\int_\gamma \alpha f + \beta g = \alpha \int_\gamma f + \beta \int_\gamma g$$

2.

$$\int_{\gamma^-} f = -\int_\gamma f$$

3. *Fie* γ_1 *un alt drum rectificabil din* $\mathcal{D}(z_1;z_2)$*, atunci :*

$$\int_{\gamma\cup\gamma_1} f = \int_\gamma f + \int_{\gamma_1} f$$

4. *Daca* $(\gamma_1, \gamma_2, \cdots, \gamma_n)$ *e o descompunere a lui* γ *atunci :*

$$\int_\gamma f = \sum_{k=1}^{n} \int_{\gamma_k} f$$

5. *Daca pentru* $\forall t \in [0;1]$ *avem ca* $|f(\gamma(t))| \leq M$, *atunci :*

$$\left| \int_\gamma f \right| \leq M \cdot \mathrm{V}(\gamma)$$

6. *Fie* γ *un drum liniar atunci* $\exists z_1, z_2 \in \mathbb{C}$ *a.i. :*

$$\int_\gamma f = (z_2 - z_1) \int_0^1 f[(1-t)z_1 + tz_2] \,\mathrm{d}t$$

7. *Fie* $f : G \mapsto \mathbb{C}$ *continua,* G *multime deschisa din* $\mathbb{C}$ *, iar* $(\gamma_n)_{n\in\mathbb{N}} \in \mathcal{D}_G$ *rectificabile.* $\{\gamma\} \subset G$ *si* $(\gamma_n)_{n\in\mathbb{N}}$ *converge uniform pe* $[0;1]$ *catre* γ, *iar* $\mathrm{V}(\gamma_n)$ *e multime marginita. Atunci:*

$$\lim_{n\to\infty} \int_{\gamma_n} f = \int_\gamma f$$

8. *Fie* $(f_n)_{n\in\mathbb{N}}$ *sir de aplicatii continue ,* $f_n : \{\gamma\} \mapsto \mathbb{C}$ *uniform convergent pe* $\{\gamma\}$ *catre* $\mathbb{C}$, *atunci*

$$\lim_{n\to\infty} \int_\gamma f_n = \int_\gamma f$$

Definitie 3. *Fie* $G \subset \mathbb{C}$ *multime deschisa,* $f : G \mapsto \mathbb{C}$ *si* $g \in \mathcal{H}(G)$. *Spunem ca* g *este primitiva pentru* f *daca* $f = g'$.

Teorema 3 (Legatura dintre primitiva si integrala)**.** *Fie o functie* $f : D \mapsto \mathbb{C}$ *continua , unde* D *domeniu din* $\mathbb{C}$. *Atunci*

1. *Daca pentru orice contur* γ *din* D *avem ca* $\displaystyle\int_\gamma f = 0$, *atunci* f *admite primitiva pe* D *.*

2. *Daca g este o primitiva a lui f pe D, atunci pentru $\forall$ drum rectificabil γ din D are loc $\displaystyle\int_\gamma f = g(\gamma_1) - g(\gamma_0)$. Daca γ e contur (drum rectificabil inchis), atunci avem $\displaystyle\int_\gamma f = 0$*

Teorema 4 (Legatura dintre olomorfie si primitiva)**.** *Fie D un domeniu stelat in z_0, iar $d_1, \cdots, d_n$ drepte ce trec prin z_0, d reuniunea lor. Daca $f : D \mapsto \mathbb{C}$ e continua pe D si derivabila pe $D \setminus d$, atunci f admite primitiva pe D*

Teorema 5 (Cauchy)**.** *Fie G o multime deschisa. Daca functia $f \in \mathcal{H}(G)$, iar conturul γ e omotop cu zero in G, atunci*

$$\int_\gamma f = 0$$

2 Zerourile functiilor olomorfe

Definitie 4. *Fie $G \subset \mathbb{C}$ deschisa, iar $f \in \mathcal{H}(G)$. Daca $\exists$ un punct $z \in G$ a.i. $f(z) = 0$, atunci z se numeste zerou al functiei f. Daca $\exists$ un $k \in \mathbb{N}^*$ a.i. :*

$$f(z) = f'(z) = \cdots = f^{k-1}(z) = 0$$

si $f^k(z) \neq 0$, atunci z se numeste zerou multiplu de ordin k pentru f

Pentru $k = 1$ il numim pe z zerou simplu.

Teorema 6. *Daca z este un zerou multiplu de ordin k al functiei $f \in \mathcal{H}(G)$, atunci $\exists g \in \mathcal{H}(G)$ a.i.*

$$g(x) \neq 0 \text{ si } f(x) = (x - z)^k g(x) \forall x \in G$$

Teorema 7. *Fie $D \subset \mathbb{C}$ domeniu si $f, g : D \mapsto \mathbb{C}$ functii olomorfe pe D. Urmatoarele afirmatii sunt echivalente:*

1. $f \equiv g$;

2. $\exists$ un punct $a \in D$ a.i. $f^{(k)}(a) = g^{(k)}(a)\ \forall k \in \mathbb{N}$;

3. $\{z \in D \colon f(z) = g(z)\} \neq \emptyset$.

Teorema 8 (Zerourile unei functii olomorfe). *Fie $D \subset \mathbb{C}$ domeniu si $f \in \mathcal{H}(G)$ nu este identic nula pe D, iar $z_0 \in D$ este un zerou al lui f, atunci $\exists r = r(z_0) > 0$ a.i. $\mathcal{U}(z_0; r) \subset D$ si $f(z) \neq 0, z \in \dot{\mathcal{U}}(z_0; r)$.*

Teorema 9 (Maximul modulului). *Fie $D \subset \mathbb{C}$ domeniu si $f : D \mapsto \mathbb{C}$ o functie olomorfa. Daca $\exists$ un punct $z_0 \in D$ a.i. $|f(z)| \leq |f(z_0)|$, $\forall z \in D$, atunci f este constanta.*

Teorema 10 (Lema lui Schwarz). *Fie functia f olomorfa pe $\mathcal{U}(0; 1)$ a.i. $f(0) = 0$ si $|f(z)| < M$, $z \in \mathcal{U}$, $M > 0$. Atunci:*

$$|f(z)| \leq M|z| \ , \ z \in \mathcal{U} \ \textit{si} \ |f'(0)| \leq M$$

Daca $\exists z_0 \in \dot{\mathcal{U}}(z_0; r)$ a.i. $|f(z_0)| = M|z_0|$ sau daca $|f'(0)| = M$, atunci $\exists \alpha \in \mathbb{C}$ a.i. $|\alpha| = M$ si $f(z) = \alpha z$, $z \in \mathcal{U}$

3 Serii Laurent

Definitie 5. *Se numeste seria Laurent in jurul lui $z_0 \in \mathbb{C}$:*

$$\sum_{n=-\infty}^{\infty} a_n (z - z_0)^n = \cdots + \frac{a_{-n}}{(z - z_0)^n} + \cdots + \frac{a_{-1}}{z - z_0} + a_0 + \cdots a_n (z - z_0)^n + \cdots$$

unde $a_n \in \mathbb{C}$ si se numesc coeficientii seriei.

Daca $\forall n < 0$ avem $a_n = 0$ spunem ca seria Laurent se reduce la o serie de puteri.

$\sum_{n=1}^{\infty} a_{-n}(z-z_0)^{-n}$ *se numeste partea principala, iar*

$\sum_{n=0}^{\infty} a_n(z-z_0)^n$ *se numeste partea tayloreana.*

Teorema 11 (Coroanei de convergenta)**.** *Fie* $\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$ *serie Laurent si folosim notatiile :*

$$r = \overline{\lim_{n\to\infty}} \sqrt[n]{|a_{-n}|}$$
$$\frac{1}{R} = \overline{\lim_{n\to\infty}} \sqrt[n]{|a_n|}$$

In conditiile in care $r < R$, avem:

1. *$\mathcal{U}(z_0; r; R) = \{z\colon r < |z-z_0| < R\}$ coroana de convergenta a seriei Laurent converge absolut si uniform pe compacte.*

2. $$\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n \text{ diverge in } \mathbb{C} \setminus \overline{\mathcal{U}}(z_0; r; R) \ .$$

3. $$f(z) = \sum_{n=-\infty}^{\infty} a_n(z-z_0)^n \in \mathcal{H}(\mathcal{U}(z_0; r; R)) \ .$$

4 Index unei curbe

Definitie 6. *Fie γ un drum rectificabil din $\mathbb{C}$ si $z_0 \in \mathbb{C}\setminus\{\gamma\}$. Numim indexul lui γ in raport cu z_0 :*

$$n(\gamma; z_0) := \frac{1}{2\pi i}\int_{\gamma} \frac{\mathrm{d}\zeta}{\zeta - z_0}$$

Teorema 12. *1. Fie γ_1 si γ_2 drumuri rectificabile din $\mathbb{C}$ si $z_0 \in \mathbb{C}\setminus\{\gamma_j\}$, $j = \overline{1,2}$. Daca $\gamma_1 \sim \gamma_2$ in $\mathbb{C}\setminus\{z_0\} \implies n(\gamma_1; z_0) = n(\gamma_2; z_0)$.*

2. Daca γ_1 si γ_2 drumuri rectificabile din $\mathbb{C}$ a.i. $\gamma_1(1) = \gamma_2(0)$, $z_0 \notin \{\gamma_j\}$, $j = \overline{1,2}$ atunci $n(\gamma_1 \cup \gamma_2; z_0) = n(\gamma_1; z_0) + n(\gamma_2; z_0)$

$$\gamma_1 \cup \gamma_2 : [0;1] \mapsto \mathbb{C}$$

$$(\gamma_1 \cup \gamma_2)(t) = \begin{cases} \gamma_1(2t) \ , \ t \in \left[0; \frac{1}{2}\right] \\ \gamma_2(2t-1) \ , \ t \in \left[\frac{1}{2}; 1\right] \end{cases}$$

3. $n(\gamma^-; z_0) = -n(\gamma; z_0)$, unde γ drum rectificabil pe $\mathbb{C}$ $z_0 \notin \{\gamma\}$, unde $\gamma^-(t) = \gamma(1-t)$, $t \in [0;1]$.

Teorema 13 (Teorema indexului)**.** *Fie γ un contur din $\mathbb{C}$. Atunci*

$$n(\gamma; z) \in \mathbb{Z} \ , \ \forall z \in \mathbb{C} \setminus \{\gamma\} \ .$$

Definitie 7. *Fie γ contur din $\mathbb{C}$. γ se numeste* contur Jordan *daca γ contur simplu ($\gamma|_{(0;1)}$ - functie injectiva) si $n(\gamma; z) = 1$, $\forall z \in (\gamma)$, unde (γ) e domeniul marginit cu frontiera γ.*

Teorema 14 (Formulele lui Cauchy pentru contururi)**.** *Fie $G \subset \mathbb{C}$ deschisa, $f \in \mathcal{H}(G)$, γ contur din G, $\gamma \underset{G}{\sim} 0$. Atunci :*

$$n(\gamma; z) f^{(k)}(z) = \frac{k!}{2\pi i} \int_\gamma \frac{f(\xi)}{(\xi - z)^{k+1}} \ , \ \forall z \in G \setminus \{\gamma\} \ , \ k \in \mathbb{N} \qquad (1)$$

5 Functii meromorfe

Definitie 8. *Fie $f : \widetilde{G} \mapsto \mathbb{C}$, unde $\widetilde{G}$ multime deschisa din G. Spunem ca f este meromorfa pe $\widetilde{G}$ si notam $f \in \mathcal{M}(\widetilde{G})$ daca $\exists$ o multime E care sa fie alcatuita numai din punctele eliminabile, respectiv poli ai functiei f si f sa fie olomorfa pe $\widetilde{G} \setminus E$.*

Definitie 9. *Fie functia* $f \in \mathcal{M}(\widetilde{G})$, *unde* $\widetilde{G}$ *multime deschisa din* $\mathbb{C}$, $z_0 \in \widetilde{G}$, $n \in \mathbb{Z}$. *Spunem ca* $f(z)$ *este divizibila cu* $(z-z_0)^n$ *daca* $\exists k > 0$ *si o functie* h *olomorfa pe* $\mathcal{U}(z_0;k)$ *a.i.* $h(z_0) \neq 0$, $\mathcal{U}(z_0;k) \subset \widetilde{G}$ *si* $f(z) = (z-z_0)^n h(z)$, $\forall z \in \dot{\mathcal{U}}(z_0;k)$.

Definitie 10. *Numim* ordinul *lui* f *in* z_0*:*

$$o(f;z_0) := \max\{n \in \mathbb{Z} \colon f(z) \text{ divizibila cu } (z-z_0)^n\} \tag{2}$$

Teorema 15 (Proprietatii ale ordinului)**.** *Daca* $f_1,\ f_2 \in \mathcal{M}(\widetilde{G})$, $z_0 \in \widetilde{G}$, *atunci :*

1. $o(f_1 f_2; z_0) = o(f_1;z_0) + o(f_2;z_0)$;
2. $o\left(\dfrac{f_1}{f_2}; z_0\right) = o(f_1;z_0) - o(f_2;z_0)$;
3. *Daca* $D \subset \widetilde{G}$ *si* $\sum\limits_{z\in D} o(f;z)$ *finita , atunci* $o(f;D) := \sum\limits_{z\in D} o(f;z)$ *si se numeste ordinul functiei* f *pe* D .

Daca functia $f \in \mathcal{M}(\widetilde{G})$, $\widetilde{G}$ *- multime dechisa din* $\mathbb{C}$ $z_0 \in \widetilde{G}$ *, atunci:*

$$o(f;z_0) = \begin{cases} n, & \text{daca } z_0 \text{ este un zerou de ordin n pentru } f \\ 0, & \text{daca } z_0 \text{ punct regular pentru } f \text{ dar nu se anuleaza} \\ -n, & \text{daca } z_0 \text{ pol de ordin n pentru } f \end{cases}$$

Definitie 11. $o(f;z) := \infty$, *cand* $f \equiv 0$, *iar* $z \in \widetilde{G}$.

Teorema 16 (Teorema lui Cauchy relativa la zerouri si poli)**.** *Fie* $\widetilde{G}$ *multime deschisa,* $f \in \mathcal{M}(\widetilde{G})$, $f \neq 0$ $g \in \mathcal{M}(\widetilde{G})$, γ *contur din* $\widetilde{G}$ *care nu trece prin niciun zerou, respectiv pol al functiei* f *a.i.* $\gamma \underset{G}{\sim} 0$. *Atunci :*

$$\sum_{z\in\widetilde{G}} n(\gamma;z)\cdot o(f;z)\cdot g(z) \text{ este finita si}$$

$$\int_\gamma \frac{f'(z)}{f(z)} g(z)\,\mathrm{d}z = 2\pi i \sum_{z\in G} n(\gamma;z)\cdot o(f;z)\cdot g(z)\ .$$

6 Teorema Reziduurilor

Teorema 17 (Teorema Reziduurilor)**.** *Fie functia $f \in \mathcal{H}(G)$, unde $G \subset \mathbb{C}$ multime deschisa. Notam cu S mutimea tuturor punctelor singulare izolate ale lui f. Fie $\widetilde{G} := G \cup S$, iar γ un contur in G omotop cu zero in $\widetilde{G}$.*

$$\textit{Atunci suma: } \sum_{z \in \widetilde{G}} n(\gamma; z) \operatorname{Rez}(f; z) \textit{ este finita si}$$

$$\int_\gamma f(z) \, \mathrm{d}z = 2\pi i \sum_{z \in \widetilde{G}} n(\gamma; z) \operatorname{Rez}(f; z) \ .$$

Demonstrație. $\exists \varphi : [0;1]^2 \mapsto G$ deformare continuua, $k = \varphi([0;1]^2) \subset \widetilde{G}$ compact.

Fie

$$r := \frac{1}{2} \, \mathrm{d}\left(k, \ \mathbb{C} \setminus \widetilde{G}\right)$$
$$D := \bigcup_{z \in k} \mathcal{U}(z; r)$$

$k \subset D \subset \overline{D} \subset \widetilde{G}$

γ omotop cu 0 in D

$\overline{D} \cap S$ finita $\implies \exists \{b_1, \ldots, b_k\} = \overline{D} \cap S$

Fie $\Pi_k(z)$ partea principala a dezvoltarii lui f in b_k

Deci, functia $g := f - \sum_{k=1}^{n} \Pi_k$ olomorfa mai putin in b_k, admite o prelungire olomorfa g_1 la D :

$$\int_\gamma g = \int_\gamma g_1 = 0$$

$$g = g_1|_{D = \{b_1, \ldots, b_k\}}$$

$$\implies \int_\gamma f = \sum_{k=1}^{n} \int_\gamma \Pi_k$$

Calculam :

$$\int_\gamma \Pi_k \text{ , unde } \Pi_k(z) = \sum_{m=1}^{\infty} \frac{a_{-m}^{(k)}}{(z-b_k)^m} \ .$$

Seria este uniform convergenta pe $\forall$ parte compacta din $\mathbb{C} \setminus \{b_k\} \implies$ uniform convergenta pe $\{\gamma\} \implies$ putem integra termen cu termen si

$$\int_\gamma \frac{\mathrm{d}z}{(z-b_k)^m} = 0, \forall m > 1 \ .$$

Functia $\dfrac{1}{(z-b_n)^m}$ admite primitiva si $\displaystyle\int_\gamma \frac{\mathrm{d}z}{z-b_k} = 2\pi i \cdot n(\gamma; b_n) \cdot a_{-1}^{(k)}$ deci

$$\int_\gamma f = 2\pi i \sum_{k=1}^{n} n(\gamma; b_k) \operatorname{Rez}(f; b_n) \ .$$

Trebuie sa mai aratam ca $\forall z_0 \in \widetilde{G} \setminus (D \cap S)$: $n(\gamma; z_0) \cdot \operatorname{Rez}(f; z_0) = 0$.

Intr-adevar, daca pentru $z_0 \in \widetilde{G} \setminus (D \cap S)$ avem $\operatorname{Rez}(f; z_0) \neq 0 \implies z_0 \in S$, deci $z_0 \notin D$ si

$$n(\gamma; z_0) = \frac{1}{2\pi i} \int_\gamma \frac{\mathrm{d}\xi}{\xi - z_0} = 0$$

caci $h(\xi) = \frac{1}{\xi - z_0}$ olomorfa pe D si γ omotop cu zero

$$\implies \int_\gamma f = 2\pi i \sum_{z \in \widetilde{G}} n(\gamma; z) \cdot \operatorname{Rez}(f; z).$$

$\square$

7 Puncte singulare izolate

Definitie 12. *Fie $G \subset \mathbb{C}$ multime deschisa si $f \in \mathcal{H}(G)$. Punctul $z_0 \in \mathbb{C}$ se numeste punct singular izolat pentru functia f daca $z_0 \notin G$, dar $\exists p > 0$ a.i $\dot{\mathcal{U}}(z_0; p) \subset G \implies f \in \mathcal{H}(\dot{\mathcal{U}}(z_0; p))$.*

Observatie 1. *De exemplu functiile $\frac{\sin(z)}{z}$, $\frac{1}{z}$, $e^{\frac{1}{z}}$ au singularitati izolate in $z = 0$.*

Observatie 2. *Daca z_0 este un punct singular izolat pentru $f \in \mathcal{H}(G)$, iar $p > 0$ a.i $\dot{\mathcal{U}}(z_0; p) \subset G$, atunci f admite o dezvoltare in serie Laurent de forma :*

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k, \quad z \in \dot{\mathcal{U}}(z_0; p).$$

Coeficientul a_{-1} al termenului $(z - z_0)^{-1}$ se numeste reziduul functiei f in z_0 si se noteaza cu $a_{-1} = \mathrm{Rez}(f; z_0)$.

Definitie 13. *Fie $G \subset \mathbb{C}$ multime deschisa, $f \in \mathcal{H}(G)$, iar z_0 punct singular izolat al functiei f. Spunem ca:*

1. *z_0 este punct eliminabil daca f se extinde olomorf la $\Omega \cup \{z_0\}$;*
2. *z_0 este pol daca $lim_{z \to z_0} f(z) = \infty$;*
3. *z_0 este punct esential izolat daca $\nexists$ limita a lui f in z_0;*
4. *Un punct z este regular pentru f daca z este eliminabil pentru f sau f este derivabila in z.*

8 Calcularea reziduului intr-un pol

1. Daca z_0 este un pol de ordin k pentru f atunci
$$\mathrm{Rez}(f; z_0) = \frac{1}{(k-1)!} \lim_{z \to z_0} \left[(z - z_0)^k f(z) \right]^{(k-1)}.$$
2. In cazul unui punct singular esential reziduul se calculeaza cu ajutorul dezvoltarii in serie Laurent.
3. Intr-un punct regular reziduul este 0.

9 Aplicatii ale teoriei reziduurilor la calculul unor integrale definite reale

Tipul 1 (1). *Fie integrala* $I = \displaystyle\int_0^{2\pi} R(\sin x, \cos x)\ \mathrm{d}x$, *unde* $R(u,v)$ *este o functie rationala reala ce nu are poli pe cercul* $u^2 + v^2 = 1$

$$\textit{Atunci} \int_0^{2\pi} R(\sin x, \cos x)\ \mathrm{d}x = 2\pi i \sum_{z\in\mathcal{U}(0;1)} \mathrm{Rez}(f;z)$$

$$\textit{unde } f(z) = \frac{1}{z} R\left(\frac{z-\frac{1}{z}}{2i}, \frac{z+\frac{1}{z}}{2}\right)$$

Demonstraţie. Utilizand formulele lui Euler:

$$\cos x = \frac{e^{ix}+e^{-ix}}{2}\ , \quad \sin x = \frac{e^{ix}-e^{-ix}}{2i} x \in \mathbb{R}$$

si substitutia $e^{ix} = z$, avem ca

$$\int_0^{2\pi} R(\sin x, \cos x)\ \mathrm{d}x = \int_{\partial\mathcal{U}(0;1)} R\left(\frac{z-\frac{1}{z}}{2i}, \frac{z+\frac{1}{z}}{2}\right) \frac{\mathrm{d}z}{iz} \implies$$

$$\int_0^{2\pi} R(\sin x, \cos x)\ \mathrm{d}x = -i \int_{\partial\mathcal{U}(0;1)} f(z)\ \mathrm{d}z \overset{T.Rez}{\implies}$$

$$\int_{\partial\mathcal{U}(0;1)} f(z)\ \mathrm{d}z = 2\pi i \sum_{|z|<1} \mathrm{Rez}(f;z) \implies$$

$$\int_0^{2\pi} R(\sin x, \cos x)\ \mathrm{d}x = 2\pi \sum_{|z|<1} \mathrm{Rez}(f;z)$$

□

Tipul 2. *Fie* R *o functie rationala reala ,* $R = P/Q$ *unde* P *si* Q *polinoame de grad* n *, respectiv* m, $Q(x) \neq 0 \quad \forall x \in \mathbb{R}$, $\lim\limits_{z\to\infty} zf(z) = 0, (n \leq m-2)$

Atunci

$$\int_{-\infty}^{\infty} R(x)\ \mathrm{d}x = 2\pi i \sum_{\mathrm{Im}\, z=0} \mathrm{Rez}(f;z)$$

Demonstraţie. $\exists M, r_1 > 0$ a.i

$$\left|\frac{P(x)}{Q(x)}\right| \leq \frac{M}{|x|^2}, \quad |x| \geq r_1$$

$$\int_{r_1}^{\infty} \frac{1}{x^2}\,\mathrm{d}x \text{ converge } \implies \int_{r_1}^{\infty} \frac{P(x)}{Q(x)}\,\mathrm{d}x \text{ converge}$$

Analog

$$\int_{-\infty}^{-r_1} \frac{P(x)}{Q(x)}\,\mathrm{d}x \text{ converge}$$

$$\text{Dar } \frac{P}{Q} \text{ continuua pe } [-r_1, r_1] \implies \exists \int_{-r_1}^{r_1} \frac{P(x)}{Q(x)}\,\mathrm{d}x$$

$$\int_{-\infty}^{0} \frac{P(x)}{Q(x)}\,\mathrm{d}x \text{ si } \int_{0}^{\infty} \frac{P(x)}{Q(x)}\,\mathrm{d}x \text{ converg } \implies \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\,\mathrm{d}x \text{ converge}$$

Fie $r > 0$ suficient de mare astfel incat toti polii lui f din semiplanul superior sa fie continuti in Ω_r, unde $\Omega_r = \{z \in \mathbb{C} \colon |z| < r, \quad \operatorname{Im} z > 0\}$.

Fie $\gamma_r(t) = re^{\pi i t}, t \in [0; 1], \gamma = [-r; r] \cup \gamma_r$.

Atunci $\gamma = \partial\Omega_r$, iar $(\gamma) = \Omega_r \overset{T.Rez}{\implies}$

$$\int_{\gamma} f(z)\,\mathrm{d}z = 2\pi i \sum_{z \in \Omega_r} \mathrm{Rez}(f; z) = 2\pi i \sum_{\operatorname{Im} z > 0} \mathrm{Rez}(f; z) \qquad (*)$$

Pe de alta parte

$$\int_{\gamma} f(z)\,\mathrm{d}z = \int_{\gamma_r} f(z)\,\mathrm{d}z + \int_{-r}^{r} f(x)\,\mathrm{d}x \qquad (**)$$

Din $(*)$ si $(**)$ trecand la limita $\implies$

$$2\pi i \sum_{\operatorname{Im} z > 0} \mathrm{Rez}(f; z) = \lim_{r \to \infty} \int_{\gamma_r} f(z)\,\mathrm{d}z + \int_{-\infty}^{\infty} f(x)\,\mathrm{d}x$$

$$\text{Dar, } \lim_{z \to \infty} z f(z) = 0 \implies \lim_{r \to \infty} \int_{\gamma_r} f(z)\,\mathrm{d}z = 0$$

$$\Longrightarrow \int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = 2\pi i \sum_{\operatorname{Im} z>0} \operatorname{Rez}(f;z)$$

$\square$

Tipul 3. *Fie R o functie rationala reala de forma $R = \dfrac{P}{Q}$, $Q(x) \neq 0$, $x \in \mathbb{R}$, grad $Q >$ grad $P + 1$ si $\lim\limits_{|z|\to\infty} R(z) = 0$*

Atunci

$$\int_{-\infty}^{\infty} R(x)e^{ix}\,\mathrm{d}x \textit{ converge si } \int_{-\infty}^{\infty} R(x)e^{ix}\,\mathrm{d}x = 2\pi i \sum_{\operatorname{Im} z>0} \operatorname{Rez}(f;z)$$

unde $f(z) = R(z)e^{iz}$.

Demonstraţie. Fie $r > 0$ suficient de mare a.i. toti polii functiei f din semiplanul superior sa fie continuti in D, unde $D = \{z \in \mathbb{C}\colon |z| < r;\ \operatorname{Im}\ z > 0\}$

Fie $C = \partial D \implies C = [-r;r] \cup \gamma_r$

$$\overset{\text{T. Rez}}{\Longrightarrow} \int_C f(z)\,\mathrm{d}z = 2\pi i \sum_{\operatorname{Im}\ z>0} \operatorname{Rez}(f;z)$$

$$\left.\begin{array}{r} \text{Dar } \displaystyle\int_C f(z)\,\mathrm{d}z = \int_{-r} r f(x)\,\mathrm{d}x + \int_{\gamma_r} f(z)\,\mathrm{d}z \\ r \to \infty \end{array}\right\} \Longrightarrow$$

$$\begin{aligned} \Longrightarrow\ 2\pi i \sum_{\operatorname{Im}\ z>0} \operatorname{Rez}(f;z) &= \int_{-\infty} \infty f(x)dx + \lim_{r\to\infty} \int_{\gamma_r} f(z)\,\mathrm{d}z \\ &= \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} e^{ix}\,\mathrm{d}x + \lim_{r\to\infty} \int_{\gamma_r} \frac{P(z)}{Q(z)} e^{iz}\,\mathrm{d}z \end{aligned}$$

$$g(z) = \frac{P(z)}{Q(z)}$$

deci,

$$\lim_{z\to\infty} g(z) = 0 \overset{L.Jordan}{\Longrightarrow} \lim_{r\to\infty} \int_{\gamma_r} g(z)e^{iz}\,\mathrm{d}z = 0$$

Asadar,

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} e^{ix} \, \mathrm{d}x = 2\pi i \sum_{\mathrm{Im} \ z>0} \mathrm{Rez}(f;z)$$

$\square$

Tipul 4. *Fie integrala*

$$I = \int_{-\infty}^{\infty} f(x) e^{i\alpha x} \, \mathrm{d}x$$

unde $f = P/Q$, $Q(x) \neq 0$, $x \in \mathbb{R}$, *grad* $P = k$, *grad* $Q = p$, *iar* $p \geq k+1$.

Daca $\alpha > 0$, *atunci:*

$$I = \int_{-\infty}^{\infty} f(x) e^{i\alpha x} \, \mathrm{d}x = 2\pi i \sum_{\mathrm{Im} \ z>0} \mathrm{Rez}(g;z)$$

, *unde* $g(z) = f(z) e^{i\alpha z}$.

Demonstrație. Observam ca $\exists \displaystyle\int_{-\infty}^{\infty} f(x) e^{i\alpha x} \, \mathrm{d}x$ si este convergenta. Intr-adevar, pentru ca $p \geq k+1 \implies \lim\limits_{z\to\infty} f(z) = 0$. Dar $f'(z) = \dfrac{h(z)}{Q^2(z)}$, unde h este un polinom de grad cel mult $k+p-1$.

Fie x_0 zeroul lui h de modul maxim $\implies f'(x)$ are semn constant pentru $x > |x_0| \implies f(x)$ monotona pentru $x > |x_0|$.

Fie $x_1, \ x_2 \in \mathbb{R}$ cu $x_2 > x_1 > |x_0|$

$$\text{Cum } \lim_{z\to\infty} f(z) = 0 \implies \text{fie } f > 0 \text{ si } \lim_{x\to\infty} f(x) = 0^+, x > |x_0|$$
$$\text{fie } f < 0 \text{ si } \lim_{x\to\infty} f(x) = 0^-, x > |x_0|$$

Aplicand a doua teorema de medie din calculul integral $\implies \exists \xi \in (x_1; x_2)$ a.i.

$$\int_{x_1}^{x_2} f(x) \cos\alpha x \ \ \mathrm{d}x = f(x_1) \int_{x_1}^{\xi} \cos\alpha t \ \ \mathrm{d}t + f(x_2) \int_{\xi}^{x_2} \cos\alpha t \ \ \mathrm{d}t$$

$$\implies \left| \int_{x_1}^{x_2} f(x) \cos\alpha x \ \ \mathrm{d}x \right| \leq \frac{2}{\alpha} |f(x_1)| + \frac{2}{\alpha} |f(x_2)|$$

Stiind ca $\lim_{z\to\infty} f(z) = 0 \implies \forall\epsilon > 0 \quad \exists\delta(\epsilon) > 0$ a.i. $|f(x)| < \frac{\epsilon\alpha}{4} x > \delta(\epsilon)$

Deci,

$$\left|\int_{x_1}^{x_2} f(x)\cos\alpha x \ \ \mathrm{d}x\right| \leq \frac{2}{\alpha}\left[|f(x_1)| + |f(x_2)|\right] < \epsilon,$$

$$x_2 > x_1 > \max\{|x_0|,\ \delta(\epsilon)\} \implies \int_0^\infty f(x)\cos\alpha x \ \ \mathrm{d}x \text{ converge}$$

Analog $\exists$ si converge

$$\int_0^\infty f(x)\sin\alpha x \ \mathrm{d}x$$

$$\implies \int_0^\infty f(x)e^{i\alpha x} \ \mathrm{d}x$$

este deasemenea convergenta.

Fie $\Omega_r = \{z \in \mathbb{C}\colon |z| < r; \mathrm{Im}\ \ z > 0\}$ ce contine toti polii functiei g din semiplanul superior

$$\overset{T.Rez}{\Longrightarrow} \int_{\partial\Omega_r} g(z) \ \mathrm{d}z = 2\pi i \sum_{z\in\Omega_r} \mathrm{Rez}(g; z) = 2\pi i \sum_{\mathrm{Im}\ \ z>0} Rez(g; z)$$

Dar

$$\int_{\partial\Omega_r} g(z) \ \mathrm{d}z = \int_{-r}^{r} f(x)e^{i\alpha x} \ \mathrm{d}x + \int_{\gamma_r} g(z) \ \mathrm{d}z$$

$$\overset{L.Jordan}{\Longrightarrow} \lim_{r\to\infty} \int_{\gamma_r} g(z) \ \mathrm{d}z = 0$$

$$\overset{r\to\infty}{\Longrightarrow} \int_{-\infty}^{\infty} f(x)e^{i\alpha x} \ \mathrm{d}x = 2\pi i \sum_{\mathrm{Im}\ \ z>0} \mathrm{Rez}(g; z)$$

$\square$

Aplicatia 1 (1)**.** *Sa se calculeze integrala*

$$I = \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{a^4 + x^4}$$

Demonstrație. Este o integrala de tipul II

$$\left.\begin{aligned} P(x) &= 1 \\ Q(x) &= a^4 + x^4 \end{aligned}\right\} grad\ Q > grad\ P + 2$$

$$f(z) = \frac{a}{a^4 + x^4}$$

$$a^4 + x^4 = 0 \implies z^4 = -a^4 = a^4(\cos\pi + i\sin\pi)$$

$$\implies z_k = a\left(\cos\frac{\pi + 2k\pi}{4} + i\sin\frac{\pi + 2k\pi}{4}\right), k = \overline{0,3}$$

$$\begin{aligned} z_0 &= a\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) = \frac{a}{\sqrt{2}}(1+i) \\ z_1 &= \frac{a}{\sqrt{2}}(-1+i) \\ z_2 &= \frac{a}{\sqrt{2}}(-1-i) \\ z_3 &= \frac{a}{\sqrt{2}}(1-i) \end{aligned}$$

$$I = 2\pi i \sum_{\text{Im } z_k > 0} \text{Rez}(f; z_k)$$

$$\implies I = 2\pi i[\text{Rez}(f; z_0) + \text{Rez}(f; z_1)]$$

$$\text{Rez}(f; z_k) = \lim_{z \to z_k}(z - z_k)\frac{1}{z^4 + a^4} \overset{\frac{0}{0}}{\underset{\text{UH}}{=}} \lim_{z \to z_k}\frac{1}{4z^3} = -\frac{z_k}{4a^4}$$

Deci,

$$I = 2\pi i\left[\frac{a}{\sqrt{2}}(1 + i - 1 + i)\right] = \frac{2\pi i \cdot a \cdot 2i}{\sqrt{2}} \implies I = -\frac{4\pi a}{\sqrt{2}}$$

□

Aplicatia 2. *Sa se calculeze integrala*

$$I = \int_{-\infty}^{\infty} \frac{\cos x}{x^2 + a^2}\,\mathrm{d}x,\ \ unde\ a > 0$$

Demonstraţie. Este o integrala de tip III:

$$\text{Fie } I_1 = \int_{-\infty}^{\infty} \frac{\cos x}{x^2 + a^2}\,\mathrm{d}x$$
$$\text{si } I_2 = \int_{-\infty}^{\infty} \frac{\sin x}{x^2 + a^2}\,\mathrm{d}x (= 0 \text{ pe ca e impara})$$

si fie $I = I_1 + I_2$

$$\Longrightarrow I = \int_{-\infty}^{\infty} \frac{e^{ix}}{x^2 + a^2}\,\mathrm{d}x$$

$$\left.\begin{array}{l} P(x) = 1 \\ Q(x) = a^2 + x^2 \end{array}\right\} \begin{array}{c} grad\ Q \geq grad\ P + 1 \\ 2 \geq 1 \end{array}$$

$$f(z) = \frac{e^{iz}}{a^2 + x^2}$$

$a^2 + x^2 = 0 \implies z_{1,2} = \pm ia$, dar doar $z_1 = ia$ pol de gradul I $\in$ semiplanul superior

$$\Longrightarrow I = 2\pi i\ \mathrm{Rez}(f; z_1) = 2\pi i\ \mathrm{Rez}(f; ia)$$

$$\mathrm{Rez}(f; ia) = \lim_{z \to ia} (z - ia) \frac{e^{iz}}{z^2 + a^2} = \frac{e^{-ia}}{z + ia} = \frac{e^{-ia}}{2ia}$$

$$\Longrightarrow I = 2\pi i \frac{e^{-ia}}{2ia} = \frac{e^{-a}\pi}{a}$$

$$I_1 = \mathrm{Re}\ I \quad I_2 = \mathrm{Im}\ I \implies I_1 = \frac{e^{-a}\pi}{a}; \quad I_2 = 0$$

□

Teorema 18. *Fie $f \in \mathcal{M}(\mathbb{C})$ si $z_1, \ldots, z_k$ poli ai functiei f cu reziduurile $u_1, \ldots, u_k$. Daca $f(z) \neq 0$, $z \in \mathbb{Z}$, $z_j \notin \mathbb{Z}$, $j = 1, \ldots, k$, iar $f(z) = O(z^{-2}), z \to \infty$, atunci*

$$\sum_{-\infty}^{\infty} f(\varphi) = -\pi \sum_{j=1}^{k} \mathrm{Rez}(\mathrm{ctg}\, \pi z \cdot f(z); z_j)$$

Demonstrație. pag 95-98 carte portocala $\square$

Aplicatia 3. *Sa se calculeze*

$$\sum_{n=1}^{\infty} \frac{1}{n^4+1}$$

Demonstrație. Se vede ca

$$\sum_{-\infty}^{\infty} \frac{1}{n^4+1} = 1 + 2\sum_{n=1}^{\infty} \frac{1}{n^4+1}$$

Fie $f(z) = \dfrac{1}{z^4+1}$, atunci $f \in \mathbb{C}$, cu polii simplii $\pm 1, \pm i$

$$\text{Rez}(f; z_n) = \lim z \to z_n \frac{1}{z^4+1} \stackrel{\frac{0}{0}}{\underset{\text{UH}}{=}} \lim_{z \to z_n} \frac{1}{4z^3} = \frac{z_n}{-4}$$

$$\begin{aligned}\sum_{-\infty}^{\infty} \frac{1}{n^4+1} &= -\pi \left[-\frac{1}{4} \operatorname{ctg} \pi + \frac{1}{4} \operatorname{ctg}(-\pi) - \frac{i}{4} \operatorname{ctg} i\pi + \frac{i}{4} \operatorname{ctg}(-i\pi) \right] \\ &= \frac{\pi}{4}[\operatorname{ctg} \pi + \operatorname{ctg} \pi + i \operatorname{ctg} i\pi + i \operatorname{ctg} i\pi] \\ &= \frac{\pi}{2} \operatorname{ctg} \pi + \frac{\pi}{2} \operatorname{cth} \pi\end{aligned}$$

$$\text{Deci, } 1 + 2\sum_{n=1}^{\infty} \frac{1}{n^4+1} = \frac{\pi}{2} \operatorname{ctg} \pi + \frac{\pi}{2} \operatorname{cth} \pi$$

$$\Longrightarrow \sum_{n=1}^{\infty} \frac{1}{n^4+1} = \frac{\pi}{4}[\operatorname{ctg} \pi + \frac{\pi}{2} \operatorname{cth} \pi] - \frac{1}{2}$$

$\square$

10 Calcularea unei integrale pe un arc de curba simplu si rectificabil, dar nu inchis

In acest caz putem incerca sa formam o curba inchisa $\gamma_0 \cup \gamma_1$ a.i. sa poata sa se aplice teorema reziduurilor , iar integrala pe noua curba $\gamma = \gamma_0 \cup \gamma_1$ sa se poata calcula cu reziduuri direct sau sa aiba o relatie simpla cu integrala cautata.

Daca integrala este improprie , fiind limita unei alte integrale

$$\int_{\gamma_0} = \lim_{\gamma \to \gamma_0} \int_{\gamma}$$

atunci si arcul adaugat va varia si vom putea calcula integrala improprie cunoscand limita $\int_{\gamma_1}$ si daca suma reziduurilor din domeniu G variabil are limita cunoscuta:

$$\int_{\gamma_0} f \, \mathrm{d}z = -\lim \int_{\gamma_1} f \, \mathrm{d}z + 2\pi i \lim \sum \mathrm{Rez}(f; z)$$

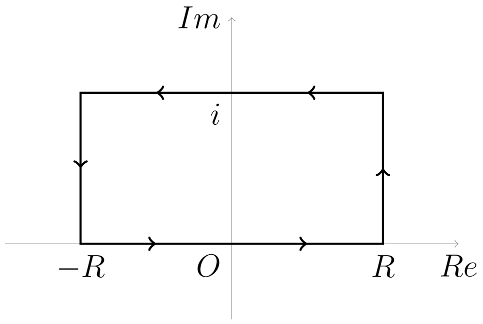
Aplicatia 4. *Sa se calculeze*

$$I = \int_0^{\infty} \frac{\cos ax}{\operatorname{ch} \pi x} \, \mathrm{d}x, a \in \mathbb{R}$$

Demonstraţie.

$$f(z) = \frac{\cos az}{\operatorname{ch} \pi z}$$

Polii acestei functii sunt simpli , $z = (zk+1)\frac{i}{2}$, $k \in \mathbb{Z}$ Pentru a evita seria de reziduuri care este divergenta alegem conturul



Pe latura $z = R + iy \quad (0 \le y \le 1)$

$$\left| i \int_0^1 \frac{\cos a(R+iy)}{\operatorname{ch} \pi(R+iy)} \, \mathrm{d}y \right| = \left| i \int_0^1 \frac{e^{ia(R+iy)} + e^{-ia(R+iy)}}{e^{\pi(R+iy)} + e^{-\pi(R+iy)}} \, \mathrm{d}y \right|$$
$$< \frac{\int_0^1 (e^{-ay} + e^{ay}) \, \mathrm{d}y}{e^{\pi R} - e^{-\pi R}} \to 0$$

Ramane : $2\int_0^R \frac{\cos ax}{\operatorname{ch}\pi x}\,\mathrm{d}x + \int_0^R \left[-\frac{\cos a(i+x)}{\operatorname{ch}\pi(i+x)} - \frac{\cos a(i-x)}{\operatorname{ch}\pi(i-x)}\right]\,\mathrm{d}x \longrightarrow 2\pi i\,\operatorname{Rez}\left(f;\frac{i}{2}\right)$

Stiind ca

$$\operatorname{ch}\pi(x\pm i) = -\operatorname{ch}\pi x$$

$$\cos a(x\pm i) = \cos ax\cdot\operatorname{ch}a \mp \sin ax\cdot\operatorname{ch}a$$

obtinem ca

$$2(1+\operatorname{ch}a)\int_0^R \frac{\cos ax}{\operatorname{ch}\pi x}\,\mathrm{d}x \longrightarrow 2\pi i\,\operatorname{Rez}\left(f;\frac{i}{2}\right)$$

$$\operatorname{Rez}\left(f;\frac{i}{2}\right) = \lim_{z\to\frac{i}{2}}\left(z-\frac{i}{2}\right)\frac{\cos az}{\operatorname{ch}\pi z} = \frac{\cos\frac{ai}{2}}{\pi\operatorname{ch}\frac{\pi}{2}} = \frac{\operatorname{ch}\frac{a}{2}}{\pi i\operatorname{ch}\frac{\pi}{2}} = \frac{\operatorname{ch}\frac{a}{2}}{\pi i}$$

$$\Longrightarrow\ I = 2\pi i\frac{\operatorname{ch}\frac{a}{2}}{\pi} = 2\operatorname{ch}\frac{a}{2}$$

□

11 Aplicatii la dezvoltari in serie

Teorema 19. *Fie $f(z)$ o functie mereomorfa ai carei poli formeaza un sir infinit $z_k \to \infty$ si D_n un domeniu marginit de o curba rectificabila γ_n si care nu trece prin nici un pol z_n*

$$\textit{Atunci}\ \int_{\gamma_n} f(z)\,\mathrm{d}z = 2\pi i\sum_{k=1}^{n}\operatorname{Rez}(f;z_n)$$

Observatie 3.

1. *Daca $n\to\infty, \gamma_n$ variaza a.i. D_n tinde catre un domeniu ce cuprinde toti polii a_n. Daca integrala din membrul* I *are o limita finita, atunci obtinem suma seriei de Reziduuri $\sum_{n=1}^{\infty}\operatorname{Rez}(f;z_n)$ insumata dupa domeniul D_n*

2. *Daca indicele k ia valorile $1, 2, \cdots$ si $|z_k|$ sunt strict crescatoare $|z_1| < |z_2| < \cdots$, a.i. intre 2 curbe consecutive sa se afle un singur pol, vom obtine suma seriei convergente $\sum_{n=1}^{\infty} \operatorname{Rez}(f; z_k)$*

3. *Daca $|z_n|$ si $|z_{-n}|$ sunt crescatori vom putea obtine suma seriei convergente $\operatorname{Rez}(f; z_0) + \sum_{k=1}^{n} [\operatorname{Rez}(f; z_k) + \operatorname{Rez}(f; z_{-k})]$ adica*

$$\sum_{-\infty}^{\infty} \operatorname{Rez}(f; z_n) = \frac{1}{2\pi i} \lim_{k \to \infty} \int_{\gamma_k} f(z)\,\mathrm{d}z$$

4. *Fie $f(z)$ o functie mereomorfa avand polii de gradul I , $z_k \to \infty$ si $g(z)$ o functie uniforma cu un numar finit de puncte singulare a_h, diferite de z_n . Fie γ_n cu $n > n_0$ ce contine punctele a_h in interiorul sau. Atunci pentru functia $f(z) \cdot g(z)$ avem ca*

$$\operatorname{Rez}(f \cdot g; z_n) = g(z_k) \operatorname{Rez}(f; z_n)$$

Formula din Obs 3 se transforma astfel

$$\sum_{-\infty}^{\infty} \operatorname{Rez}(f; z_n) g(z_n) = \frac{1}{2\pi i} \lim_{k \to \infty} \int_{\gamma_k} f(z) g(z)\,\mathrm{d}z - \sum_{k \in \mathbb{C}} \operatorname{Rez}(f \cdot g; a_h)$$

A doua suma este nula pentru $g(z)$ functie intreaga

Aplicatia 5. *Sa se calculeza integrala*

$$I = \int_0^{2\pi} \frac{\mathrm{d}x}{a + \cos x}, a > 1$$

Demonstrație. Se observa ca I este o integrala de tipul I. Din formulele lui Euler stim ca:

$$\cos z = \frac{e^{ix} + e^{-ix}}{2}, \quad e^{ix} = z \implies \mathrm{d}x = \frac{\mathrm{d}z}{iz} \implies \cos x = \frac{z + \frac{1}{z}}{2}$$

$$f(z) = \frac{1}{z} \frac{1}{a + \frac{z+1/2}{2}} \implies f(z) = \frac{1}{z^2 + 2az + 1}$$

$$I = \int_{\partial\mathcal{U}(0;1)} \frac{\frac{\mathrm{d}z}{iz}}{a + \frac{z+1/2}{2}} = -2i \int_{\partial\mathcal{U}(0;1)} \frac{\mathrm{d}z}{z^2 + 2az + 1}$$

$$z^2 + 2az + 1 = 0 \implies \Delta = 4a^2 - 4$$

$$\implies \begin{cases} z_1 = \dfrac{-2a + \sqrt{4a^2 - 4}}{2} = -a + \sqrt{a^2 - 1} \\ z_2 = \dfrac{-2a - \sqrt{4a^2 - 4}}{2} = -a - \sqrt{a^2 - 1} \end{cases}$$

$$\begin{aligned} |z_1| < 1 &\iff |-a + \sqrt{a^2 - 1}| = a - \sqrt{a^2 - 1} < 1 \\ &\iff a - 1 < \sqrt{a^2 - 1} \,\Big|^2 \iff a^2 - 2a + 1 < a^2 - 1 \\ &\iff 2a > 0 \text{ Adevarat} \\ |z_2| < 1 &\iff |-a - \sqrt{a^2 - 1}| < 1 \text{ Fals } \implies z_2 \notin \mathcal{U}(0;1) \end{aligned}$$

Deci, $I = 2\pi \operatorname{Rez}(f; z_1)$ cu z_1 pol simplu

$$\begin{aligned} \operatorname{Rez}(f; z_1) &= \lim_{z \to z_1} (z - z_1) \frac{1}{z^2 + 2az + 1} \overset{\frac{0}{0}}{\underset{\text{UH}}{=}} \lim_{z \to z_1} \frac{1}{2z + 2a} \\ &= \frac{1}{2(a + \sqrt{a^2 - 1}) + 2a} = -\frac{1}{2\sqrt{a^2 - 1}} \end{aligned}$$

Asadar ,

$$I = 2\pi \frac{1}{2\sqrt{a^2 - 1}} = \frac{\pi}{\sqrt{a^2 - 1}}$$

□

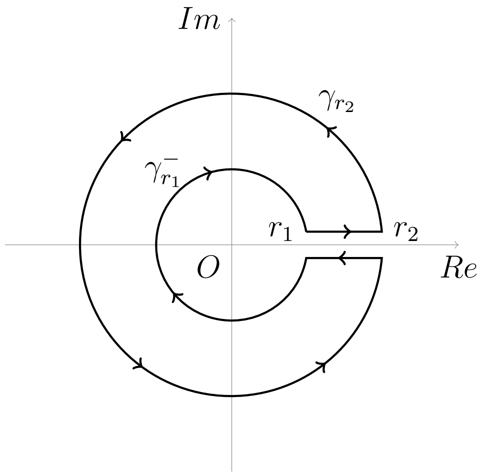
Teorema 20. *Fie functia reala rationala $f = P/Q$, avand polii pe $[0;1]$, fie $0 < \alpha < 1$ si $\lim_{z \to \infty} f(z) = 0$. Atunci avem ca :*

$$\int_0^\infty \frac{f(x)}{x^\alpha} \, \mathrm{d}x = \frac{\pi e^{\alpha\pi i}}{\sin \alpha\pi} \sum_{z \in \mathbb{C}^*} \operatorname{Rez}(h; z)$$

$$\textit{Unde } h(z) = \frac{f(z)}{z^\alpha} \text{ , } \textit{iar} z^\alpha = e^{\alpha \log z}$$

cu $\log z$ ramura uniforma a aplicatiei multivoce Logaritm

Demonstraţie. Fie Γ conturul din imagine



Atunci

$$\int_\Gamma g(z)\,\mathrm{d}z = 2\pi i \sum_{z\in\mathbb{C}^*} \mathrm{Rez}(g;z)$$

si

$$\int_\Gamma g(z)\,\mathrm{d}z = \int_{r_1}^{r_2} \frac{f(x)}{x^\alpha}\,\mathrm{d}x + \int_{\gamma_{r_2}} \frac{f(z)}{z^\alpha}\,\mathrm{d}z - \int_{r_1}^{r_2} \frac{f(x)}{e^\alpha[\ln x + 2\pi i]}\,\mathrm{d}x - \int_{\gamma_{r_1}} \frac{f(z)}{z^\alpha}\,\mathrm{d}z$$

Deci

$$(*)\ 2\pi i \sum_{z\in\mathbb{C}^*} \mathrm{Rez}(g;z) = \int_{\gamma_{r_2}} \frac{f(z)}{z^\alpha}\,\mathrm{d}z - \int_{\gamma_{r_1}} \frac{f(z)}{z^\alpha}\,\mathrm{d}z + (1-e^{-2\pi i\alpha}) \int_{r_1}^{r_2} \frac{f(x)}{x^\alpha}\,\mathrm{d}x$$

Cum $lim_{z\to\infty} f(z) = 0$ urmeaza ca $p \leq k+1$, unde k si p sunt gradele polinoamelor P respectiv Q. Deoarece $a \in (0;1)$, obtinem imediat ca

$$\lim_{z\to\infty} zg(z) = \lim_{z\to\infty} z^{1-\alpha}\frac{P(z)}{Q(z)} = 0 \text{ si } \lim_{z\to 0} zg(z) = \lim_{z\to 0} z^{1-\alpha}\frac{P(z)}{Q(z)} = 0$$

Trecand la limita in (*) pentru $r_1 \to 0$ si $r_2 \to \infty$ deducem concluzia teoremei. $\square$

Aplicatia 6. *Sa se calculeze integrala*

$$I = \int_0^\infty \frac{\mathrm{d}x}{\sqrt[3]{x}(x^5+a^5)}, a > 0$$

Demonstraţie. Aceasta integrala este de tip V

$$f(x) = \frac{1}{x^5 + a^5}$$
$$h(z) = \frac{1}{z^{1/3}(z^5 + a^5)}$$
$$z^5 + a^5 = 0 \implies z^5 = -a^5$$
$$z_k = a\left(\cos\frac{\pi + 2k\pi}{5} + i\sin\frac{\pi + 2k\pi}{5}\right), k = \overline{0,4}$$

unde z_k sunt poli simpli

$$I = \frac{\pi e^{\frac{\pi i}{3}}}{\sin\frac{\pi}{3}} \sum_{k=0}^{n} \text{Rez}(h; z_n)$$

$$\begin{aligned}
\text{Rez}(h; z_n) &= \lim_{z \to z_n} (z - z_n) \frac{1}{z^{1/3}(z^5 + a^5)} \overset{\frac{0}{0}}{\underset{\text{UH}}{=}} \frac{1}{z_n^{1/3}} \lim_{z \to z_n} \frac{1}{5z^4} = \frac{1}{z_n^{1/3}} \lim_{z \to z_n} \frac{z}{5z^4} \\
&= \frac{1}{z_n^{1/3}} \frac{z_n}{5a^5} = -\frac{z_k^{2/3}}{5a^5} = -\frac{1}{5a^5} e^{\frac{2}{3}\log z_n} = -\frac{1}{5a^5} e^{\frac{2}{3}[\ln a + io(z)]} \\
&= -\frac{1}{5a^5} e^{\frac{2}{3}\left[\ln a + i\frac{\pi + 2k\pi}{5}\right]} = -\frac{a^{2/3}}{5a^5} e^{\frac{2}{3}\frac{i\pi + 2k\pi}{5}} \\
I &= -\frac{\pi e^{\frac{\pi i}{3}}}{\frac{\sqrt{3}}{2}} \frac{a^{2/3}}{5a^5} e^{i\frac{2}{3}} \left[e^{\frac{\pi}{5}} + e^{\frac{3\pi}{5}} + e^{\frac{5\pi}{5}} + e^{\frac{7\pi}{5}} + e^{\frac{9\pi}{5}}\right]
\end{aligned}$$

□

12 Aplicatii in teoria functiilor

Urmatorul rezultat face legatura intre numarul de zerouri si numarul de poli ai unei functii analitice.

Teorema 21. *Fie $D \in \mathbb{C}$ domeniu stelat si $f \in \mathcal{M}(D)$ cu zerouurile : $a_1, \cdots, a_n \in D$, si polii $b_1, \cdots, b_m \in D$. Atunci pentru orice contur γ din D ce evita toate zerourile si toti polii lui f avem:*

$$\frac{1}{2\pi i} \int_\gamma \frac{f'}{f}(\zeta)\, \mathrm{d}\zeta = \sum_{k=1}^{n} o(f; a_k)\, n(\gamma; a_k) + \sum_{l=1}^{m} o(f; b_l)\, n(\gamma; b_l)$$

O aplicatie a teoremei anterioare e teorema:

Teorema 22 (Hurwitz)**.** *Fie $f_1, f_2, \cdots : D \mapsto \mathbb{C}$ un sir de functii ce converge local uniform la functia analitica $f : D \mapsto \mathbb{C}$. Daca $\forall i f_i$ nu e identic nula pe D atunci f fie e identic nula ori nu are nici un zerou in D*

Teorema 23. *Fie $D \in \mathbb{C}$ domeniu stelat si $f \in \mathcal{M}(D)$ cu zerouurile : $a_1, \cdots, a_n \in D$, si polii $b_1, \cdots, b_m \in D$. Notam:*

$$N(0) := \sum_{k=1}^{n} o(f; a_k) \qquad \textit{numarul tuturor zerourilor lui } f\ ;$$

$$N(\infty) := -\sum_{l=1}^{m} o(f; b_l) \qquad \textit{numarul tuturor polilor lui } f\ ;$$

numarand multiplicitatile. Fie γ din D ce inconjoara cu index 1 toate zerourile si toti polii. Atunci avem:

$$\frac{1}{2\pi i} \int_{\gamma} \frac{f'}{f}(zeta)\, \mathrm{d}\zeta = N(0) - N(\infty).$$

Daca f nu are poli obtinem o formula pentru numarul de zerouri intr-un domeniu.

Aplicatia 7 (Teorema fundamentala a algebrei)**.** *Orice polinom $P(z)$ de grad n cu coeficienti complecsi, are exact n radacini complexe.*

Demonstraţie. Deoarece $\lim_{|z|\to\infty} P(z) = \infty$, $\exists R > 0$ a.i. P nu are radacini z cu $|z| \geq R$. Numarul de zerouri a lui P e :

$$N(0) = \frac{1}{2\pi i} \int_{|\zeta|=R} \frac{P'(\zeta)}{P(\zeta)}\, \mathrm{d}\zeta.$$

Functia P'/P are in ∞ un zerou simplu. Seria Laurent in ∞ e de forma :

$$\frac{n}{z} + \frac{c_2}{z^2} + \frac{c_3}{z^3} + \cdots (n = grad(P)).$$

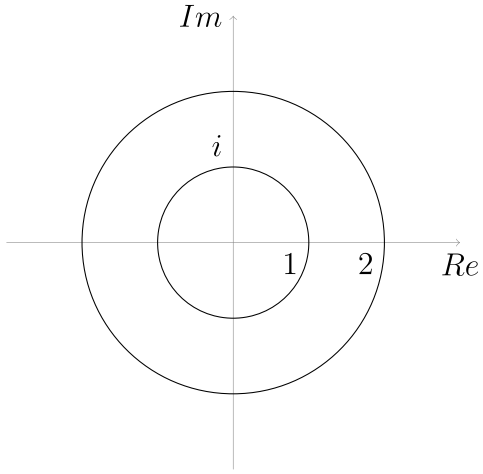
Deci:

$$N(0) = n = grad(P).$$

$\square$

Teorema 24 (Rouche)**.** *Fie $f,\ g : D \mapsto \mathbb{C}$ analitice si γ un contur din D ce inconjoara orice punct din interiorul sau exact o data. Daca $|g(\zeta)| < |f(\zeta)| \forall \zeta \in \{\gamma\}$ atunci $f,\ f+g$ nu au zerouri in $\{\gamma\}$ si au in interiorul lui γ acelasi numar de zerouri considerand multiplicitatile.*

Aplicatia 8. *Sa se determine numarul solutiilor ecuatiei $z^4 - 8z + 10 = 0$ in $\mathcal{U}(0;1;3)$*



Demonstraţie.

$$\mathcal{U}(0;1;3) = \mathcal{U}(0;3) \setminus \big(\mathcal{U}(0;1) \cup \partial\mathcal{U}(0;1)\big)$$

$$N_1 := \text{ numarul solutiilor ecuatiei in } \mathcal{U}(0;3)$$

$$N_2 := \text{ numarul solutiilor ecuatiei in } \mathcal{U}(0;1)$$

$$N := \text{ numarul solutiilor ecuatiei in } \mathcal{U}(0;1;3)$$

$$N = N_1 - N_2$$

Determinam N_1. Avem $|z| < 3$.

Alegem $f(z) = z^4$ si $g(z) = -8z + 10$.

$$|z^4| = 3^4 = 81$$
$$|-8z + 10| \leq 8|z| + 10 = 24 + 10 = 34 < 81 \implies$$
$$|g(z)| < |f(z)| \text{ pentru } |z| = 2 \overset{T.Rouche}{\implies}$$
$$f(z) = 0 \text{ si } f(z) + g(z) = 0 \text{ au acelasi numar de solutii in } \mathcal{U}(0;3)$$
$$\implies N_1 = 4$$

Determinam N_2.

$$N_2' := \text{ numarul solutiilor ecuatiei in } \mathcal{U}(0;1)$$
$$N_2'' := \text{ numarul solutiilor ecuatiei pe } \partial\mathcal{U}(0;1)$$
$$N_2 = N_2' + N_2''$$

$$|z| = 1$$
$$f(z) = -8z + 10 \implies g(z) = z^4$$
$$|f(z)| \leq -8|z| + 10 = 18$$
$$|g(z)| = |z^4| = 1 < 18 \overset{T.Rouche}{\implies}$$
$$f(z) = 0 \text{ si } f(z) + g(z) = 0 \text{ au acelasi numar de solutii}$$
$$-8z + 10 = 0 \implies z = 2 > 1 \implies N_2 = 0 \text{ numar de solutii in } \mathcal{U}(0;1)$$
$$|f(z) + g(z)| \geq |f(z)| + |g(z)| > 0, |z| = 1 \implies N_2'' = 0$$

Deci $N = 4 - 0 = 4$ $\square$

Aplicatia 9. *Fie $P_n(z) = a_0 + a_1 z + \cdots + a_n z^n$, $z \in \mathbb{C}$, unde $a_n \neq 0$.*

$$\textit{Fie } \alpha_n = \frac{\sum_{k=0}^{n-1} |a_k|}{|a_n|}, \textit{ si } \quad r > \max\{\alpha_n, 1\}.$$

Sa se arate ca toate solitiile polinomului $P_n \in \mathcal{U}(0;r)$.

Demonstrație. Fie :

$$f(z) := a_n z^n$$
$$g(z) := a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} = P_n(z) - f(z)$$

Avem :

$$|z| = r$$
$$|f(z)| = a_n |r|^n = |a_n| r^n$$
$$|g(z)| = |a_0 + a_1 z + \cdots + a_{n-1} z^{n-1}|$$
$$\le |a_0| + |a_1||z| + \cdots + |a_{n-1}||z|^{n-1}$$
$$= |a_0| + |a_1| r + \cdots + |a_{n-1}| r^{n-1}$$

Deoarece $r^k \le r^{n-1} \forall k = \overline{0, n-1}$

$$|g(z)| \le r^{n-1}(|a_0| + |a_1| + \cdots + |a_{n-1}|) = r^{n-1} \sum_{k=0}^{n-1} |a_k| \implies$$
$$|g(z)| \le r^{n-1} \alpha_n |a_n| = \frac{\alpha_n r^n |a_n|}{r} = \frac{\alpha_n}{r} |f(z)|$$

Cum $\dfrac{\alpha_n}{r} < 1$ avem ca :

$$|g(z)| < |f(z)|\,, \quad |z| = r \overset{T.Rouche}{\Longrightarrow}$$

$f(z) = 0$ si $f(z) + g(z) = 0$ au acelasi numar de solutii in $\mathcal{U}(0; r)$

$$\begin{cases} f(z) = 0 \\ P_n(z) = 0 \end{cases} \text{au acelasi numar de solutii} \implies N = n$$

□

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