Teorema Reziduurilor

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Iunie 2013

Rezumat

Aplicatii ale teoremei reziduurilor in calculul unor chestii interesante. In prima parte avem introducere apoi exemple din x urmate de aplicatii de tip y.

Cuprins

1 Integrala Riemann-Stieltjes a unei functii complexe de variabila reala

Definitie 1. *Fie $f = u + iv$ si $F = U + iV$, iar $[a; b]$ interval din $\mathbb{R}$. Spunem ca f este integrabila Riemann-Stieltjes in raport cu F pe intervalul $[a; b]$ daca u si v sunt integrabile Riemann-Stieltjes in raport cu U si V pe $[a; b]$.*

Notam :

$$\int_a^b f \, \mathrm{d}F := \int_a^b u \, \mathrm{d}U - \int_a^b v \, \mathrm{d}V + i \int_a^b u \, \mathrm{d}V + i \int_a^b v \, \mathrm{d}U \qquad (1)$$

Teorema 1. *Consideram $f = u + iv$, $F = U + iV$, iar $f_n : [a; b] \mapsto \mathbb{C}$, $F_n : [a; b] \mapsto \mathbb{C}$, si α , $\beta \in \mathbb{C}$*

Au loc urmatoarele proprietati:

1. *Daca f este integrabila R-S in raport cu F pe $[a; b]$, atunci F este integrabila R-S in raport cu f si*

$$\int_a^b f \, \mathrm{d}F + \int_a^b F \, \mathrm{d}f = f(b)F(b) - f(a)F(a)$$

2. *Daca f si g sunt integrabile R-S in raport cu F pe $[a; b]$, atunci αf si βg e integrabila dupa F si*

$$\int_a^b (\alpha f + \beta g) \, \mathrm{d}F = \alpha \int_a^b f \, \mathrm{d}F + \beta \int_a^b g \, \mathrm{d}F$$

3. *Daca f este continuua si F este cu variatie marginita pe $[a; b]$, atunci f este integrabila pe $[a; b]$ in raport cu F*

4. *Fie $(f_n)_{n \in \mathbb{N}}$ un sir de functii continuue ce converge uniform catre f pe $[a; b]$ si $(F_n)_{n \in \mathbb{N}}$ un sir de functii cu variatie marginita care converge punctual catre F, iar sirul $\mathrm{V}(F_n, [a; b])$ marginit Atunci avem ca:*

$$\lim_{\substack{n \to \infty \\ k \to \infty}} \int_a^b f_n \, \mathrm{d}F_k = \int_a^b f \, \mathrm{d}F$$

5. *Daca f e continuua , F derivabila si F' continuua, atunci :*

$$\int_a^b f \, \mathrm{d}F = \int_a^b f(t) F'(t) \, \mathrm{d}t$$

6. *Fie $c \in (a; b)$ si f integrabila in raport cu F pe $[a; b]$, atunci f este integrabila in raport cu F si pe $[a; c]$, si*

$$\int_a^b f \, \mathrm{d}F = \int_a^c f \, \mathrm{d}F + \int_c^b f \, \mathrm{d}F$$

7. *Daca f e integrabila in raport cu F pe $[a; b]$, si $h : [a'; b'] \mapsto [a; b]$ $h(a') = a$ si $h(b') = b$, h fiind omeomorfism, atunci $f \circ h$ e integrabila R-S pe $F \circ H$ si*

$$\int_a^b f \, \mathrm{d}F = \int_{a'}^{b'} (f \circ h) \, \mathrm{d}(F \circ H)$$

Definitie 2. *Consideram drumul rectificabil γ , iar $f : \{\gamma\} \mapsto \mathbb{C}$ continuua. Atunci $f \circ \gamma$ va fi continuua pe $[0; 1]$ si integrabila in raport cu γ Aceasta inegrala se numeste integrala complexa a drumului f de-a lungul lui γ*

$$\int_\gamma f := \int_\gamma f(\epsilon) \, \mathrm{d}\epsilon = \int_0^1 (f \circ \gamma) \, \mathrm{d}\gamma \tag{2}$$

Teorema 2. *Fie α , $\beta \in \mathbb{C}$, γ drum rectificabil din $\mathcal{D}(z_0; z_1)$ si f, g functii continuue din $\{\gamma\}$, atunci:*

1.

$$\int_\gamma \alpha f + \beta g = \alpha \int_\gamma f + \beta \int_\gamma g$$

2.

$$\int_\gamma -f = -\int_\gamma f$$

3. *Fie γ_1 drum rectificabil din $\mathcal{D}(z_1; z_2)$ si f continuua pe $\{\gamma_1\}$, atunci:*

$$\int_{\gamma \cup \gamma_1} f = \int_\gamma f + \int_{\gamma_1} -f$$

4. *Daca* $(\gamma_1, \gamma_2, \cdots, \gamma_n)$ *e o descompunere a lui* γ *atunci*

$$\int_\gamma f = \sum_{k=1}^\infty \int_{\gamma_k} f$$

5. *Daca pentru* $\forall t \in [0; 1]$ *avem ca* $|f(\gamma(t))| \le M$, *atunci*

$$\left| \int_\gamma f \right| \le M \cdot \mathrm{V}(\gamma)$$

6. *Fie* γ *un drum liniar atunci*

$$\int_\gamma f = (z_2 - z_1) \int_0^1 f[(1-t)z_1 + tz_2] \, \mathrm{d}t$$

7. *Fie* $f : G \mapsto \mathbb{C}$ *continuua,* G *multime deschisa din* $\mathbb{C}$ *, iar* $(\gamma_n)_{n \in \mathbb{N}} \in \mathcal{D}_G$
   rectificabile. $\{\gamma\} \subset G$ *si* $\{\gamma\}$ *converge uniform pe* $[0; 1]$ *catre* γ, *iar*
   $\mathrm{V}(\gamma_n)$ *e multime marginita. Atunci:*

$$\lim_{n \to \infty} \int_{\gamma_n} f = \int_\gamma f$$

8. *Fie* $(f_n)_{n \in \mathbb{N}}$ *sir de aplicatii continuue ,* $f_n : \{\gamma\} \mapsto \mathbb{C}$ *uniform conver-*
   gent pe $\{\gamma\}$ *catre* $\mathbb{C}$, *atunci*

$$\lim_{n \to \infty} \int_\gamma f_n = \int_\gamma f$$

Definitie 3. *Fie* $G \subset \mathbb{C}$ *multime deschisa,* $f : G \mapsto \mathbb{C}$ *si* $g \in \mathcal{H}(G)$*. Spunem
ca* g *este primitiva pentru* f *daca* $f = g'$.

Teorema 3 (Legatura dintre primitiva si integrala)**.** *Fie o functie* $f : D \mapsto \mathbb{C}$
continuua , unde D *domeniu din* $\mathbb{C}$. *Atunci*

1. *Daca pentru orice contur* γ *din* D *avem ca* $\int_\gamma f = 0$, *atunci* f *admite
   primitiva pe* D

2. *Daca g este o primitiva a lui f pe D, atunci pentru $\forall$ drum rectificabil γ din D are loc $\int_\gamma f = g(\gamma_1) - g(\gamma_0)$. Daca γ e contur (drum rectificabil inchis), atunci avem $\int_\gamma f = 0$*

Teorema 4 (Legatura dintre olomorfie si primitiva). *Fie D un domeniu stelat in z_0, iar $d_1, \cdots, d_n$ drepte ce trec prin z_0, d reuniunea lor. Daca $f : D \mapsto \mathbb{C}$ e continuua pe D si derivabila pe $D \setminus d$, atunci f admite primitiva pe D*

Teorema 5 (Cauchy). *Fie G o multime deschisa. Daca functia $f \in \mathcal{H}(G)$, iar conturul γ e omotop cu zero in G, atunci*

$$\int_\gamma f = 0$$

2 Zerourile functiilor olomorfe

Definitie 4. *Fie $G \subset \mathbb{C}$ deschisa, iar $f \in \mathcal{H}(G)$. Daca $\exists$ un punct $z \in G$ a.i. $f(z) = 0$, atunci z se numeste zerou al functiei f. Daca $\exists$ un $k \in \mathbb{N}^*$ a.i.*

$$f(z) = f'(z) = \cdots = f^{k-1}(z) = 0$$

si $f^k(z) \neq 0$, atunci z se numeste zerou multiplu de ordin k pentru f

Pentru $k = 1$ il numim pe z zerou simplu

Teorema 6. *Daca z este un zerou multiplu de ordin k al functiei $f \in \mathcal{H}(G)$, atunci $\exists g \in \mathcal{H}(G)$ a.i.*

$$g(x) \neq 0 \ si \ f(x) = (x - z)^k g(x) \forall x \in G$$

Teorema 7. *Fie $D \subset \mathbb{C}$ domeniu si $f, g : D \mapsto \mathbb{C}$ functii olomorfe pe D. Urmatoarele afirmatii sunt echivalente:*

1. $f \equiv g$

2. $\exists$ un punct $a \in D$ a.i. $f^k(a) = g^k(a) \forall k \in \mathbb{N}$

3. $\{z \in D \colon f(z) = g(z)\} \neq \emptyset$

Teorema 8 (Zerourile unei functii olomorfe)**.** *Fie $D \subset \mathbb{C}$ domeniu si $f \in \mathcal{H}(G)$ nu este identic nula pe D, iar $z_0 \in D$ este un zerou al lui f, atunci $\exists r = r(z_0) > 0$ a.i. $\mathcal{U}(z_0; r) \subset D$ si $f(z) \neq 0, z \in \dot{\mathcal{U}}(z_0; r)$*

Teorema 9 (Maximul modulului)**.** *Fie $D \subset \mathbb{C}$ domeniu si $f : D \mapsto \mathbb{C}$ o functie olomorfa. Daca $\exists$ un punct $z_0 \in D$ a.i. $|f(z)| \leq |f(z_0)|$, $\forall z \in D$, atunci f este constanta*

Teorema 10 (Lemma lui Schwarz)**.** *Fie functia f olomorfa pe $\mathcal{U}(0; 1)$ a.i. $f(0) = 0$ si $|f(z)| < M$, $z \in \mathcal{U}$, $M > 0$. Atunci:*

$$|f(z)| \leq M|z| \ , \ z \in \mathcal{U} \ si \ |f'(0)| \leq M$$

Daca $\exists z_0 \in \dot{\mathcal{U}}(z_0; r)$ a.i. $|f(z_0)| = M|z_0|$ sau daca $|f'(0)| = M$, atunci $\exists \alpha \in \mathbb{C}$ a.i. $|\alpha| = M$ si $f(z) = \alpha z$, $z \in \mathcal{U}$

3 Serii Laurent

Definitie 5. *Se numeste seria Laurent in jurul lui $z_0 \in \mathbb{C}$:*

$$\sum_{n=-\infty}^{\infty} a_n (z-z_0)^n = \cdots + \frac{a_{-n}}{(z-z_0)^n} + \cdots + \frac{a_{-1}}{z-z_0} + a_0 + \cdots a_n (z-z_0)^n + \cdots \quad (3)$$

unde $a_n \in \mathbb{C}$ si se numesc coeficientii seriei.

Daca $\forall n < 0$ avem $a_n = 0$ spunem ca seria Laurent se reduce la o serie de puteri.

$$\sum_{n=1}^{\infty} a_{-n}(z - z_0)^{-n} \text{ se numeste partea principala, iar}$$

$$\sum_{n=0}^{\infty} a_n(z - z_0)^n \text{ se numeste partea tayloreana}$$

Teorema 11 (Coroanei de convergenta). *Fie* $\displaystyle\sum_{n=-\infty}^{\infty} a_n(z - z_0)^n$ *serie Laurent si folosim notatiile*

$$r = \varlimsup_{n \to \infty} \sqrt[n]{|a_{-n}|}$$

$$\frac{1}{R} = \varlimsup_{n \to \infty} \sqrt[n]{|a_n|}$$

In conditiile in care $r < R$, avem:

1. $\mathcal{U}(z_0; r; R) = \{z \colon r < |z - z_0| < R\}$ *coroana de convergenta a seriei Laurent converge absolut si uniform pe compacte*

2.
$$\sum_{n=-\infty}^{\infty} a_n(z - z_0)^n \text{ diverge in } \mathbb{C} \setminus \overline{\mathcal{U}}(z_0; r; R)$$

3.
$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z - z_0)^n \in \mathcal{H}(\mathcal{U}(z_0; r; R))$$

4 Functii meromorfe

Definitie 6. *Fie $f : \widetilde{G} \mapsto \mathbb{C}$, unde $\widetilde{G}$ multime deschisa din G. Spunem ca f este meromorfa pe $\widetilde{G}$ si notam $f \in \mathcal{M}(\widetilde{G})$ daca $\exists$ o multime E care sa fie alcatuita numai din punctele eliminabile, respectiv poli ai functiei f si sa fie olomorfa pe $\widetilde{G} \setminus E$*

5   Index unei curbe

Definitie 7. *Fie γ un drum rectificabil din $\mathbb{C}$ si $z_0 \in \mathbb{C} \backslash \{\gamma\}$. Numim indexul lui γ in raport cu z_0 :*

$$n(\gamma; z_0) := \frac{1}{2\pi i} \int_\gamma \frac{\mathrm{d}\zeta}{\zeta - z_0} \tag{4}$$

Teorema 12.   *1. Fie γ_1 si γ_2 drumuri rectificabile din $\mathbb{C}$ si $z_0 \in \mathbb{C} \setminus \{\gamma_j\}$, $j = \overline{1,2}$. Daca $\gamma_1 \sim \gamma_2$ in $\mathbb{C} \setminus \{z_0\} \implies n(\gamma_1; z_0) = n(\gamma_2; z_0)$*

   2. TODO

   3. TODO

6   Teorema Reziduurilor

Teorema 13 (Teorema Reziduurilor)**.** *Fie functia $f \in \mathcal{H}(G)$, unde $G \subset \mathbb{C}$ multime deschisa. Notam cu ρ mutimea tuturor punctelor singulare izolate ale lui f Fie $\widetilde{G} := G \cup S$, iar γ un contur in G omotop cu zero in $\widetilde{G}$*

$$\text{Atunci suma: } \sum_{z \in \widetilde{G}} n(\gamma; z) \, \mathrm{Rez}(f; z) \text{ este finita si}$$

$$\int_\gamma f(z) \, \mathrm{d}z = 2\pi i \sum_{z \in \widetilde{G}} n(\gamma; z) \, \mathrm{Rez}(f; z)$$

Demonstraţie. $\exists \varphi : [0;1]^2 \mapsto G$ deformare continuua, $k = \varphi([0;1]^2) \subset \widetilde{G}$ compact.

   Fie

$$r := \frac{1}{2} \, \mathrm{d}\left(k, \, \mathbb{C} \setminus \widetilde{G}\right)$$

$$D := \bigcup_{z \in k} \mathcal{U}(z; r)$$

$k \subset D \subset \overline{D} \subset \widetilde{G}$

γ omotop cu 0 in D

$\overline{D} \cap \rho$ finita $\implies \exists \{b_1, \ldots, b_k\} = \overline{D} \cap \rho$

Fie $\Pi_k(z)$ partea principala a dezvoltarii lui f in b_k

Deci, functia $g := f - \sum_{k=1}^{n} \Pi_k$ olomorfa mai putin in b_k admite o prelungire olomorfa g_1 la D .

$$\int_{\gamma} g = \int_{\gamma} g_1 = 0$$

$$g = g_1|_{D = \{b_1, \ldots, b_k\}}$$

$$\implies \int_{\gamma} f = \sum_{k=1}^{n} \int_{\gamma} \Pi_k$$

Calculam

$$\int_{\gamma} \Pi_k \text{ , unde } \Pi_k(z) = \sum_{m=1}^{\infty} \frac{a^{(k)} - m}{(z - b_k)^m}$$

Seria este uniform convergenta pe $\forall$ parte compacta din $\mathbb{C} \setminus \{b_a\} \implies$ uniform convergenta pe $\{\gamma\} \implies$ putem integra termen cu termen si

$$\int_{\gamma} \frac{\mathrm{d}}{z - b_k} m = 0, \forall m > 1$$

Functia $\dfrac{1}{(z - b_n)^m}$ admite primitiva si $\displaystyle\int_{\gamma} \frac{\mathrm{d}z}{z - b_k} = 2\pi i \cdot n(\gamma; b_n) \cdot a_{-1}^{(k)}$ deci

$$\int_{\gamma} f = 2\pi i \sum_{k=1}^{n} n(\gamma; b_k) \operatorname{Rez}(f; b_n)$$

Trebuie sa mai aratam ca $\forall z_0 \in \widetilde{G} \setminus (D \cap \rho) \colon n(\gamma; z_0) \cdot \operatorname{Rez}(f; z_0) = 0$

Intr-adevar, daca pentru $z_0 \in \widetilde{G} \setminus (D \cap \rho)$ avem $\operatorname{Rez}(f; z_0) \neq 0 \implies z_0 \in \rho$, deci $z_0 \notin D$ si

$$n(\gamma; z_0) = \frac{1}{2\pi i} \int_{\gamma} \frac{\mathrm{d}\xi}{\xi - z_0} = 0$$

caci $h(\xi) = \frac{1}{\xi - z_0}$ olomorfa pe D si γ omotop cu zero

$$\Longrightarrow \int_\gamma f = 2\pi i \sum_{z \in \widetilde{G}} n(\gamma; z) \cdot \text{Rez}(f; z)$$

$\square$

7   Puncte singulare izolate

Definitie 8. *Fie $G \subset \mathbb{C}$ multime deschisa si $f \in \mathcal{H}(G)$. Punctul $z_0 \in \mathbb{C}$ se numeste punct singular izolat pentru functia f daca $z_0 \notin G$, dar $\exists p > 0$ a.i $\dot{\mathcal{U}}(z_0; p) \subset G \implies f \in \mathcal{H}(\dot{\mathcal{U}}(z_0; p))$*

Observatie 1. *De exemplu functiile $\frac{\sin(z)}{z}$, $\frac{1}{z}$, $e^{\frac{1}{z}}$ au singularitati izolate in $z = 0$*

Observatie 2. *Daca z_0 este un punct singular izolat pentru $f \in \mathcal{H}(G)$, iar $p > 0$ a.i $\dot{\mathcal{U}}(z_0; p) \subset G$, atunci f admite o dezvoltare in serie Laurent de forma*

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k, \quad z \in \dot{\mathcal{U}}(z_0; p)$$

Coeficientul a_{-1} al termenului $(z - z_0)^{-1}$ se numeste reziduul functiei f in z_0 si se noteaza cu $a_{-1} = \text{Rez}(f; z_0)$

Definitie 9. *Fie $G \subset \mathbb{C}$ multime deschisa, $f \in \mathcal{H}(G)$, iar z_0 punct singular izolat al functiei f. Spunem ca:*

1. *z_0 este punct eliminabil daca f se extinde olomorf la $\Omega \cup \{z_0\}$*

2. *z_0 este pol daca $\lim_{z \to z_0} f(z) = \infty$*

3. *z_0 este punct esential izolat daca $\nexists$ limita a lui f in z_0*

4. *Un punct z este regular pentru f daca z este eliminabil pentru f sau f este derivabila in z*

8 Calcularea reziduului intr-un pol

1. Daca z_0 este un pol de ordin k pentru f atunci

$$\text{Rez}(f; z_0) = \frac{1}{(k-1)!} \lim_{z \to z_0} \left[(z - z_0)^k f(z) \right]^{(k-1)}$$

2. In cazul unui punct singular esential reziduul se calculeaza cu ajutoril dezvoltarii in serie Laurent

3. Intr-un punct regular reziduul este 0

9 Aplicatii ale teoriei reziduurilor la calculul unor integrale definite reale

Tipul 1 (1). *Fie integrala* $I = \displaystyle\int_0^{2\pi} R(\sin x, \cos x) \, \mathrm{d}x$, *unde* $R(u, v)$ *este o functie rationala reala ce nu are poli pe cercul* $u^2 + v^2 = 1$

$$Atunci \int_0^{2\pi} R(\sin x, \cos x) \, \mathrm{d}x = 2\pi i \sum_{z \in \mathcal{U}(0;1)} \text{Rez}(f; z)$$

$$unde \ f(z) = \frac{1}{z} R \left(\frac{z - \frac{1}{z}}{2i}, \ \frac{z + \frac{1}{z}}{2} \right)$$

Demonstraţie. Utilizand formulele lui Euler:

$$\cos x = \frac{e^{ix} + e^{-ix}}{2} \ , \quad \sin x = \frac{e^{ix} - e^{-ix}}{2i} x \in \mathbb{R}$$

si substitutia $e^{ix} = z$, avem ca

$$\int_0^{2\pi} R(\sin x, \cos x) \,\mathrm{d}x = \int_{\partial \mathcal{U}(0;1)} R\left(\frac{z - \frac{1}{z}}{2i}, \frac{z + \frac{1}{z}}{2}\right) \frac{\mathrm{d}z}{iz} \implies$$

$$\int_0^{2\pi} R(\sin x, \cos x) \,\mathrm{d}x = -i \int_{\partial \mathcal{U}(0;1)} f(z) \,\mathrm{d}z \overset{T.Rez}{\implies}$$

$$\int_{\partial \mathcal{U}(0;1)} f(z) \,\mathrm{d}z = 2\pi i \sum_{|z|<1} \mathrm{Rez}(f; z) \implies$$

$$\int_0^{2\pi} R(\sin x, \cos x) \,\mathrm{d}x = 2\pi \sum_{|z|<1} \mathrm{Rez}(f; z)$$

$\square$

Tipul 2. *Fie R o functie rationala reala , $R = P/Q$ unde P si Q polinoame de grad n , respectiv m, $Q(x) \neq 0 \quad \forall x \in \mathbb{R}$, $\lim_{z \to \infty} zf(z) = 0, (n \leq m - 2)$*

Atunci

$$\int_{-\infty}^{\infty} R(x) \,\mathrm{d}x = 2\pi i \sum_{\mathrm{Im}\, z = 0} \mathrm{Rez}(f; z)$$

Demonstraţie. $\exists M, r_1 > 0$ a.i

$$\left|\frac{P(x)}{Q(x)}\right| \leq \frac{M}{|x|^2}, \quad |x| \geq r_1$$

$$\int_{r_1}^{\infty} \frac{1}{x^2} \,\mathrm{d}x \text{ converge} \implies \int_{r_1}^{\infty} \frac{P(x)}{Q(x)} \,\mathrm{d}x \text{ converge}$$

Analog

$$\int_{-\infty}^{-r_1} \frac{P(x)}{Q(x)} \,\mathrm{d}x \text{ converge}$$

$$\text{Dar } \frac{P}{Q} \text{ continuua pe } [-r_1, r_1] \implies \exists \int_{-r_1}^{r_1} \frac{P(x)}{Q(x)} \,\mathrm{d}x$$

$$\int_{-\infty}^{0} \frac{P(x)}{Q(x)} \,\mathrm{d}x \text{ si } \int_0^{\infty} \frac{P(x)}{Q(x)} \,\mathrm{d}x \text{ converg} \implies \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \,\mathrm{d}x \text{ converge}$$

Fie $r > 0$ suficient de mare astfel incat toti polii lui f din semiplanul superior sa fie continuti in Ω_r, unde $\Omega_r = \{z \in \mathbb{C} \colon |z| < r, \quad \mathrm{Im}\, z > 0\}$.

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Fie $\gamma_r(t) = re^{\pi it}, t \in [0; 1], \gamma = [-r; r] \cup \gamma_r$.

Atunci $\gamma = \partial\Omega_r$, iar $(\gamma) = \Omega_r \overset{T.Rez}{\Longrightarrow}$

$$\int_\gamma f(z)\,\mathrm{d}z = 2\pi i \sum_{z \in \Omega_r} \mathrm{Rez}(f; z) = 2\pi i \sum_{\mathrm{Im}\, z > 0} \mathrm{Rez}(f; z) \qquad (*)$$

Pe de alta parte

$$\int_\gamma f(z)\,\mathrm{d}z = \int_{\gamma_r} f(z)\,\mathrm{d}z + \int_{-r}^{r} f(x)\,\mathrm{d}x \qquad (**)$$

Din $(*)$ si $(**)$ trecand la limita $\Longrightarrow$

$$2\pi i \sum_{\mathrm{Im}\, z > 0} \mathrm{Rez}(f; z) = \lim_{r \to \infty} \int_{\gamma_r} f(z)\,\mathrm{d}z + \int_{-\infty}^{\infty} f(x)\,\mathrm{d}x$$

$$\text{Dar, } \lim_{z \to \infty} z f(z) = 0 \implies \lim_{r \to \infty} \int_{\gamma_r} f(z)\,\mathrm{d}z = 0$$

$$\implies \int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = 2\pi i \sum_{\mathrm{Im}\, z > 0} \mathrm{Rez}(f; z)$$

$$\square$$

Tipul 3. *Fie R o functie rationala reala de forma $R = \dfrac{P}{Q}$, $Q(x) \neq 0$, $x \in \mathbb{R}$, grad $Q >$ grad $P + 1$ si $\lim\limits_{|z| \to \infty} R(z) = 0$*

Atunci

$$\int_{-\infty}^{\infty} R(x) e^{ix}\,\mathrm{d}x \text{ converge si } \int_{-\infty}^{\infty} R(x) e^{ix}\,\mathrm{d}x = 2\pi i \sum_{\mathrm{Im}\, z > 0} \mathrm{Rez}(f; z)$$

unde $f(z) = R(z) e^{iz}$.

Demonstraţie. Fie $r > 0$ suficient de mare a.i. toti polii functiei f din semi-planul superior sa fie continuti in D, unde $D = \{z \in \mathbb{C} : |z| < r;\ \mathrm{Im}\ z > 0\}$

Fie $C = \partial D \implies C = [-r; r] \cup \gamma_r$

$$\overset{T.Rez}{\Longrightarrow} \int_C f(z)\,\mathrm{d}z = 2\pi i \sum_{\mathrm{Im}\ z > 0} \mathrm{Rez}(f; z)$$

$$\text{Dar } \int_C f(z) \, \mathrm{d}z = \int_{-r} r f(x) \, \mathrm{d}x + \int_{\gamma_r} f(z) \, \mathrm{d}z \Bigg\} \implies$$
$$r \to \infty$$

$$\implies 2\pi i \sum_{\text{Im } z>0} \text{Rez}(f; z) = \int_{-\infty} \infty f(x)dx + \lim_{r\to\infty} \int_{\gamma_r} f(z) \, \mathrm{d}z$$
$$= \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} e^{ix} \, \mathrm{d}x + \lim_{r\to\infty} \int_{\gamma_r} \frac{P(z)}{Q(z)} e^{iz} \, \mathrm{d}z$$

$$g(z) = \frac{P(z)}{Q(z)}$$

deci,

$$\lim_{z\to\infty} g(z) = 0 \overset{L.Jordan}{\implies} \lim_{r\to\infty} \int_{\gamma_r} g(z) e^{iz} \, \mathrm{d}z = 0$$

Asadar,

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} e^{ix} \, \mathrm{d}x = 2\pi i \sum_{\text{Im } z>0} \text{Rez}(f; z)$$

$\square$

Tipul 4. *Fie integrala*
$$I = \int_{-\infty}^{\infty} f(x) e^{i\alpha x} \, \mathrm{d}x$$
unde $f = P/Q$, $Q(x) \neq 0$, $x \in \mathbb{R}$, grad $P = k$, grad $Q = p$, iar $p \geq k+1$
.

Daca $\alpha > 0$, atunci:
$$I = \int_{-\infty}^{\infty} f(x) e^{i\alpha x} \, \mathrm{d}x = 2\pi i \sum_{\text{Im } z>0} \text{Rez}(g; z)$$
, unde $g(z) = f(z) e^{i\alpha z}$.

Demonstraţie. Observam ca $\exists \int_{-\infty}^{\infty} f(x) e^{i\alpha x} \, \mathrm{d}x$ si este convergenta. Intr-adevar, pentru ca $p \geq k+1 \implies \lim_{z\to\infty} f(z) = 0$. Dar $f'(z) = \frac{h(z)}{Q^2(z)}$, unde h este un polinom de grad cel mult $k + p - 1$.

Fie x_0 zeroul lui h de modul maxim $\implies f'(x)$ are semn constant pentru $x > |x_0| \implies f(x)$ monotona pentru $x > |x_0|$.

Fie $x_1,\ x_2 \in \mathbb{R}$ cu $x_2 > x_1 > |x_0|$

Cum $\lim\limits_{z \to \infty} f(z) = 0 \implies$ fie $f > 0$ si $\lim\limits_{x \to \infty} f(x) = 0^+, x > |x_0|$

fie $f < 0$ si $\lim\limits_{x \to \infty} f(x) = 0^-, x > |x_0|$

Aplicand a doua teorema de medie din calculul integral $\implies \exists \xi \in (x_1; x_2)$ a.i.

$$\int_{x_1}^{x_2} f(x) \cos \alpha x \ \mathrm{d}x = f(x_1) \int_{x_1}^{\xi} \cos \alpha t \ \mathrm{d}t + f(x_2) \int_{\xi}^{x_2} \cos \alpha t \ \mathrm{d}t$$

$$\implies \left| \int_{x_1}^{x_2} f(x) \cos \alpha x \ \mathrm{d}x \right| \le \frac{2}{\alpha} |f(x_1)| + \frac{2}{\alpha} |f(x_2)|$$

Stiind ca $\lim\limits_{z \to \infty} f(z) = 0 \implies \forall \epsilon > 0 \quad \exists \delta(\epsilon) > 0$ a.i. $|f(x)| < \dfrac{\epsilon \alpha}{4} x > \delta(\epsilon)$

Deci,

$$\left| \int_{x_1}^{x_2} f(x) \cos \alpha x \ \mathrm{d}x \right| \le \frac{2}{\alpha} \big[|f(x_1)| + |f(x_2)| \big] < \epsilon,$$

$$x_2 > x_1 > \max\{ |x_0|,\ \delta(\epsilon)\} \implies \int_0^{\infty} f(x) \cos \alpha x \ \mathrm{d}x \text{ converge}$$

Analog $\exists$ si converge

$$\int_0^{\infty} f(x) \sin \alpha x \ \mathrm{d}x$$

$$\implies \int_0^{\infty} f(x) e^{i\alpha x} \ \mathrm{d}x$$

este deasemenea convergenta.

Fie $\Omega_r = \{ z \in \mathbb{C} : |z| < r; \operatorname{Im} z > 0 \}$ ce contine toti polii functiei g din semiplanul superior

$$\overset{T.Rez}{\implies} \int_{\partial \Omega_r} g(z) \ \mathrm{d}z = 2\pi i \sum_{z \in \Omega_r} \operatorname{Rez}(g; z) = 2\pi i \sum_{\operatorname{Im} z > 0} Rez(g; z)$$

Dar

$$\int_{\partial \Omega_r} g(z) \ \mathrm{d}z = \int_{-r}^{r} f(x) e^{i\alpha x} \ \mathrm{d}x + \int_{\gamma_r} g(z) \ \mathrm{d}z$$

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$$\overset{L.Jordan}{\Longrightarrow} \lim_{r \to \infty} \int_{\gamma_r} g(z) \, \mathrm{d}z = 0$$

$$\overset{r \to \infty}{\Longrightarrow} \int_{-\infty}^{\infty} f(x)e^{i\alpha x} \, \mathrm{d}x = 2\pi i \sum_{\mathrm{Im}\ z>0} \mathrm{Rez}(g; z)$$

$\square$

Aplicatia 1 (1). *Sa se calculeze integrala*

$$I = \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{a^4 + x^4}$$

Demonstraţie. Este o integrala de tipul II

$$\left. \begin{array}{l} P(x) = 1 \\[2mm] Q(x) = a^4 + x^4 \end{array} \right\} \mathit{grad}\ Q > \mathit{grad}\ P + 2$$

$$f(z) = \frac{a}{a^4 + x^4}$$

$$a^4 + x^4 = 0 \implies z^4 = -a^4 = a^4(\cos\pi + i\sin\pi)$$

$$\implies z_k = a\left(\cos\frac{\pi + 2k\pi}{4} + i\sin\frac{\pi + 2k\pi}{4}\right), k = \overline{0,3}$$

$$z_0 = a\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) = \frac{a}{\sqrt{2}}(1 + i)$$

$$z_1 = \frac{a}{\sqrt{2}}(-1 + i)$$

$$z_2 = \frac{a}{\sqrt{2}}(-1 - i)$$

$$z_3 = \frac{a}{\sqrt{2}}(1 - i)$$

$$I = 2\pi i \sum_{\mathrm{Im}\ z_k>0} \mathrm{Rez}(f; z_k)$$

$$\implies I = 2\pi i[\mathrm{Rez}(f; z_0) + \mathrm{Rez}(f; z_1)]$$

$$\mathrm{Rez}(f; z_k) = \lim_{z \to z_k}(z - z_k)\frac{1}{z^4 + a^4} \overset{\frac{0}{0}}{\underset{\mathrm{UH}}{=\!=}} \lim_{z \to z_k}\frac{1}{4z^3} = -\frac{z_k}{4a^4}$$

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Deci,

$$I = 2\pi i \left[\frac{a}{\sqrt{2}}(1 + i - 1 + i) \right] = \frac{2\pi i \cdot a \cdot 2i}{\sqrt{2}} \implies I = -\frac{4\pi a}{\sqrt{2}}$$

$\square$

Aplicatia 2. *Sa se calculeze integrala*

$$I = \int_{-\infty}^{\infty} \frac{\cos x}{x^2 + a^2} \, dx, \ \ unde \ a > 0$$

Demonstraţie. Este o integrala de tip III:

$$\text{Fie } I_1 = \int_{-\infty}^{\infty} \frac{\cos x}{x^2 + a^2} \, dx$$

$$\text{si } I_2 = \int_{-\infty}^{\infty} \frac{\sin x}{x^2 + a^2} \, dx (= 0 \text{ pe ca e impara})$$

$$\text{si fie } I = I_1 + I_2$$

$$\implies I = \int_{-\infty}^{\infty} \frac{e^{ix}}{x^2 + a^2} \, dx$$

$$\left. \begin{array}{l} P(x) = 1 \\ Q(x) = a^2 + x^2 \end{array} \right\} \begin{array}{c} grad \ Q \geq grad \ P + 1 \\ 2 \geq 1 \end{array}$$

$$f(z) = \frac{e^{iz}}{a^2 + x^2}$$

$a^2 + x^2 = 0 \implies z_{1,2} = \pm ia$, dar doar $z_1 = ia$ pol de gradul I $\in$ semiplanul superior

$$\implies I = 2\pi i \ \text{Rez}(f; z_1) = 2\pi i \ \text{Rez}(f; ia)$$

$$\text{Rez}(f; ia) = \lim_{z \to ia} (z - ia) \frac{e^{iz}}{z^2 + a^2} = \frac{e^{-ia}}{z + ia} = \frac{e^{-ia}}{2ia}$$

$$\implies I = 2\pi i \frac{e^{-ia}}{2ia} = \frac{e^{-a}\pi}{a}$$

$$I_1 = \text{Re } I \quad I_2 = \text{Im } I \implies I_1 = \frac{e^{-a}\pi}{a}; \quad I_2 = 0$$

$\square$

Teorema 14. *Fie $f \in \mathcal{M}(\mathbb{C})$ si $z_1, \ldots, z_k$ poli ai functiei f cu reziduurile $u_1, \ldots, u_k$. Daca $f(z) \neq 0$, $z \in \mathbb{Z}$, $z_j \notin \mathbb{Z}$, $j = 1, \ldots, k$, iar $f(z) = O(z^{-2}), z \to \infty$, atunci*

$$\sum_{-\infty}^{\infty} f(\varphi) = -\pi \sum_{j=1}^{k} \operatorname{Rez}(\operatorname{ctg} \pi z \cdot f(z); z_j)$$

Demonstraţie. pag 95-98 carte portocala ▢

Aplicatia 3. *Sa se calculeze*

$$\sum_{n=1}^{\infty} \frac{1}{n^4 + 1}$$

Demonstraţie. Se vede ca

$$\sum_{-\infty}^{\infty} \frac{1}{n^4 + 1} = 1 + 2 \sum_{n=1}^{\infty} \frac{1}{n^4 + 1}$$

Fie $f(z) = \dfrac{1}{z^4 + 1}$, atunci $f \in \mathbb{C}$, cu polii simplii $\pm 1, \pm i$

$$\operatorname{Rez}(f; z_n) = \lim z \to z_n \frac{1}{z^4 + 1} \overset{\frac{0}{0}}{\underset{\text{UH}}{=}} \lim_{z \to z_n} \frac{1}{4z^3} = \frac{z_n}{-4}$$

$$\sum_{-\infty}^{\infty} \frac{1}{n^4 + 1} = -\pi \left[-\frac{1}{4} \operatorname{ctg} \pi + \frac{1}{4} \operatorname{ctg}(-\pi) - \frac{i}{4} \operatorname{ctg} i\pi + \frac{i}{4} \operatorname{ctg}(-i\pi) \right]$$

$$= \frac{\pi}{4}[\operatorname{ctg} \pi + \operatorname{ctg} \pi + i \operatorname{ctg} i\pi + i \operatorname{ctg} i\pi]$$

$$= \frac{\pi}{2} \operatorname{ctg} \pi + \frac{\pi}{2} \operatorname{cth} \pi$$

Deci, $1 + 2 \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4 + 1} = \frac{\pi}{2} \operatorname{ctg} \pi + \frac{\pi}{2} \operatorname{cth} \pi$

$$\implies \sum_{n=1}^{\infty} \frac{1}{n^4 + 1} = \frac{\pi}{4}[\operatorname{ctg} \pi + \frac{\pi}{2} \operatorname{cth} \pi] - \frac{1}{2}$$

▢

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10 Calcularea unei integrale pe un arc de curba simplu si rectificabil, dar nu inchis

In acest caz putem incerca sa formam o curba inchisa $\gamma_0 \cup \gamma_1$ a.i. sa poata sa se aplice teorema reziduurilor , iar integrala pe noua curba $\gamma = \gamma_0 \cup \gamma_1$ sa se poata calcula cu reziduuri direct sau sa aiba o relatie simpla cu integrala cautata.

Daca integrala este improprie , fiind limita unei alte integrale

$$\int_{\gamma_0} = \lim_{\gamma \to \gamma_0} \int_{\gamma}$$

atunci si arcul adaugat va varia si vom putea calcula integrala improprie cunoscand limita $\int_{\gamma_1}$ si daca suma reziduurilor din domeniu G variabil are limita cunoscuta:

$$\int_{\gamma_0} f \, \mathrm{d}z = -\lim \int_{\gamma_1} f \, \mathrm{d}z + 2\pi i \lim \sum \operatorname{Rez}(f; z)$$

Aplicatia 4. *Sa se calculeze*

$$I = \int_0^{\infty} \frac{\cos ax}{\operatorname{ch} \pi x} \, \mathrm{d}x, a \in \mathbb{R}$$

Demonstraţie.

$$f(z) = \frac{\cos az}{\operatorname{ch} \pi z}$$

Polii acestei functii sunt simpli , $z = (zk + 1)\frac{i}{2}$, $k \in \mathbb{Z}$ Pentru a evita seria de reziduuri care este divergenta alegem conturul

IMAGINE

Pe latura $z = R + iy$ $(0 \le y \le 1)$

$$\left| i \int_0^1 \frac{\cos a(R + iy)}{\operatorname{ch} \pi (R + iy)} \, \mathrm{d}y \right| = \left| i \int_0^1 \frac{e^{ia(R+iy)} + e^{-ia(R+iy)}}{e^{\pi(R+iy)} + e^{-\pi(R+iy)}} \, \mathrm{d}y \right|$$
$$< \frac{\int_0^1 (e^{-ay} + e^{ay}) \, \mathrm{d}y}{e^{\pi R} - e^{-\pi R}} \to 0$$

Ramane : $2 \int_0^R \frac{\cos ax}{\operatorname{ch} \pi x} \, \mathrm{d}x + \int_0^R \left[-\frac{\cos a(i+x)}{\operatorname{ch} \pi(i+x)} - \frac{\cos a(i-x)}{\operatorname{ch} \pi(i-x)} \right] \mathrm{d}x \longrightarrow 2\pi i \operatorname{Rez} \left(f; \frac{i}{2} \right)$

Stiind ca

$$\operatorname{ch} \pi(x \pm i) = -\operatorname{ch} \pi x$$

$$\cos a(x \pm i) = \cos ax \cdot \operatorname{ch} a \mp \sin ax \cdot \operatorname{ch} a$$

obtinem ca

$$2(1 + \operatorname{ch} a) \int_0^R \frac{\cos ax}{\operatorname{ch} \pi x} \, \mathrm{d}x \longrightarrow 2\pi i \operatorname{Rez} \left(f; \frac{i}{2} \right)$$

$$\operatorname{Rez} \left(f; \frac{i}{2} \right) = \lim_{z \to \frac{i}{2}} \left(z - \frac{i}{2} \right) \frac{\cos az}{\operatorname{ch} \pi z} = \frac{\cos \frac{ai}{2}}{\pi \operatorname{ch} \frac{\pi}{2}} = \frac{\operatorname{ch} \frac{a}{2}}{\pi i \operatorname{ch} \frac{\pi}{2}} = \frac{\operatorname{ch} \frac{a}{2}}{\pi i}$$

$$\implies I = 2\pi i \frac{\operatorname{ch} \frac{a}{2}}{\pi} = 2 \operatorname{ch} \frac{a}{2}$$

$\square$

11 Aplicatii la dezvoltari in serie

Teorema 15. *Fie $f(z)$ o functie mereomorfa ai carei poli formeaza un sir infinit $z_k \to \infty$ si D_n un domeniu marginit de o curba rectificabila γ_n si care nu trece prin nici un pol z_n*

$$Atunci \int_{\gamma_n} f(z) \, \mathrm{d}z = 2\pi i \sum_{k=1}^n \operatorname{Rez}(f; z_n)$$

Observatie 3.

1. *Daca $n \to \infty, \gamma_n$ variaza a.i. D_n tinde catre un domeniu ce cuprinde toti polii a_n. Daca integrala din membrul I are o limita finita, atunci obtinem suma seriei de Reziduuri $\sum_{n=1}^\infty \operatorname{Rez}(f; z_n)$ insumata dupa domeniul D_n*

2. *Daca indicele k ia valorile $1, 2, \cdots$ si $|z_k|$ sunt strict crescatoare $|z_1| < |z_2| < \cdots$, a.i. intre 2 curbe consecutive sa se afle un singur pol, vom obtine suma seriei convergente $\sum_{n=1}^{\infty} \text{Rez}(f; z_k)$*

3. *Daca $|z_n|$ si $|z_{-n}|$ sunt crescatori vom putea obtine suma seriei convergente $\text{Rez}(f; z_0) + \sum_{k=1}^{n}[\text{Rez}(f; z_k) + \text{Rez}(f; z_{-k})]$ adica*

$$\sum_{-\infty}^{\infty} \text{Rez}(f; z_n) = \frac{1}{2\pi i} \lim_{k \to \infty} \int_{\gamma_k} f(z) \, \mathrm{d}z$$

4. *Fie $f(z)$ o functie mereomorfa avand polii de gradul I , $z_k \to \infty$ si $g(z)$ o functie uniforma cu un numar finit de puncte singulare a_h, diferite de z_n . Fie γ_n cu $n > n_0$ ce contine punctele a_h in interiorul sau. Atunci pentru functia $f(z) \cdot g(z)$ avem ca*

$$\text{Rez}(f \cdot g; z_n) = g(z_k) \, \text{Rez}(f; z_n)$$

Formula din Obs 3 se transforma astfel

$$\sum_{-\infty}^{\infty} \text{Rez}(f; z_n)g(z_n) = \frac{1}{2\pi i} \lim_{k \to \infty} \int_{\gamma_k} f(z)g(z) \, \mathrm{d}z - \sum_{k \in \mathbb{C}} \text{Rez}(f \cdot g; a_h)$$

A doua suma este nula pentru $g(z)$ functie intreaga

Aplicatia 5. *Sa se calculeza integrala*

$$I = \int_0^{2\pi} \frac{\mathrm{d}x}{a + \cos x}, a > 1$$

Demonstraţie. Se observa ca I este o integrala de tipul I. Din formulele lui Euler stim ca:

$$\cos z = \frac{e^{ix} + e^{-ix}}{2}, \quad e^{ix} = z \implies \mathrm{d}x = \frac{\mathrm{d}z}{iz} \implies \cos x = \frac{z + \frac{1}{z}}{2}$$

$$f(z) = \frac{1}{z} \frac{1}{a + \frac{z+1/2}{2}} \implies f(z) = \frac{1}{z^2 + 2az + 1}$$

$$I = \int_{\partial \mathcal{U}(0;1)} \frac{\frac{\mathrm{d}z}{iz}}{a + \frac{z+1/2}{2}} = -2i \int_{\partial \mathcal{U}(0;1)} \frac{\mathrm{d}z}{z^2 + 2az + 1}$$

$$z^2 + 2az + 1 = 0 \implies \Delta = 4a^2 - 4$$

$$\implies \begin{cases} z_1 = \dfrac{-2a + \sqrt{4a^2 - 4}}{2} = -a + \sqrt{a^2 - 1} \\ z_2 = \dfrac{-2a - \sqrt{4a^2 - 4}}{2} = -a - \sqrt{a^2 - 1} \end{cases}$$

$$|z_1| < 1 \iff |-a + \sqrt{a^2 - 1}| = a - \sqrt{a^2 - 1} < 1$$

$$\iff a - 1 < \sqrt{a^2 - 1} \Big|^2 \iff a^2 - 2a + 1 < a^2 - 1$$

$$\iff 2a > 0 \text{ Adevarat}$$

$$|z_2| < 1 \iff |-a - \sqrt{a^2 - 1}| < 1 \text{ Fals} \implies z_2 \notin \mathcal{U}(0;1)$$

Deci, $I = 2\pi \operatorname{Rez}(f; z_1)$ cu z_1 pol simplu

$$\operatorname{Rez}(f; z_1) = \lim_{z \to z_1} (z - z_1) \frac{1}{z^2 + 2az + 1} \overset{\frac{0}{0}}{\underset{\mathrm{UH}}{=}} \lim_{z \to z_1} \frac{1}{2z + 2a}$$

$$= \frac{1}{2(a + \sqrt{a^2 - 1}) + 2a} = -\frac{1}{2\sqrt{a^2 - 1}}$$

Asadar ,

$$I = 2\pi \frac{1}{2\sqrt{a^2 - 1}} = \frac{\pi}{\sqrt{a^2 - 1}}$$

$\square$

Teorema 16. *Fie functia reala rationala $f = P/Q$, avand polii pe $[0;1]$, fie $0 < \alpha < 1$ si $\lim\limits_{z \to \infty} f(z) = 0$. Atunci avem ca :*

$$\int_0^\infty \frac{f(x)}{x^\alpha} \, \mathrm{d}x = \frac{\pi e^{\alpha \pi i}}{\sin \alpha \pi} \sum_{z \in \mathbb{C}^*} \operatorname{Rez}(h; z)$$

Unde $h(z) = \dfrac{f(z)}{z^\alpha}$, iar $z^\alpha = e^{\alpha \log z}$

cu $\log z$ ramura uniforma a aplicatiei multivoce Logaritm

Demonstraţie. Fie Γ conturul din imagine

IMAGINE

Atunci

$$\int_\Gamma g(z)\,\mathrm{d}z = 2\pi i \sum_{z\in\mathbb{C}^*}\mathrm{Rez}(g;z)$$

si

$$\int_\Gamma g(z)\,\mathrm{d}z = \int_{r_1}^{r_2}\frac{f(x)}{x^\alpha}\,\mathrm{d}x + \int_{\gamma_{r_2}}\frac{f(z)}{z^\alpha}\,\mathrm{d}z - \int_{r_1}^{r_2}\frac{f(x)}{e^\alpha[\ln x + 2\pi i]}\,\mathrm{d}x - \int_{\gamma_{r_1}}\frac{f(z)}{z^\alpha}\,\mathrm{d}z$$

Deci

$$(*)\ 2\pi i\sum_{z\in\mathbb{C}^*}\mathrm{Rez}(g;z) = \int_{\gamma_{r_2}}\frac{f(z)}{z^\alpha}\,\mathrm{d}z - \int_{\gamma_{r_1}}\frac{f(z)}{z^\alpha}\,\mathrm{d}z + (1-e^{-2\pi i\alpha})\int_{r_1}^{r_2}\frac{f(x)}{x^\alpha}\,\mathrm{d}x$$

Cum $lim_{z\to\infty}f(z)=0$ urmeaza ca $p \le k+1$, unde k si p sunt gradele polinoamelor P respectiv Q. Deoarece $a\in(0;1)$, obtinem imediat ca

$$\lim_{z\to\infty}zg(z) = \lim_{z\to\infty}z^{1-\alpha}\frac{P(z)}{Q(z)} = 0 \text{ si } \lim_{z\to0}zg(z) = \lim_{z\to0}z^{1-\alpha}\frac{P(z)}{Q(z)} = 0$$

Trecand la limita in (*) pentru $r_1\to 0$ si $r_2\to\infty$ deducem concluzia teoremei. $\qquad\square$

Aplicatia 6. *Sa se calculeze integrala*

$$I = \int_0^\infty \frac{\mathrm{d}x}{\sqrt[3]{x}(x^5+a^5)}, a>0$$

Demonstraţie. Aceasta integrala este de tip V

$$f(x) = \frac{1}{x^5+a^5}$$
$$h(z) = \frac{1}{z^{1/3}(z^5+a^5)}$$
$$z^5+a^5 = 0 \implies z^5 = -a^5$$
$$z_k = a\left(\cos\frac{\pi+2k\pi}{5} + i\sin\frac{\pi+2k\pi}{5}\right), k=\overline{0,4}$$

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unde z_k sunt poli simpli

$$I = \frac{\pi e^{\frac{\pi i}{3}}}{\sin \frac{\pi}{3}} \sum_{k=0}^{n} \text{Rez}(h; z_n)$$

$$\text{Rez}(h; z_n) = \lim_{z \to z_n} (z - z_n) \frac{1}{z^{1/3}(z^5 + a^5)} \overset{\frac{0}{0}}{\underset{\text{UH}}{=}} \frac{1}{z_n^{1/3}} \lim_{z \to z_n} \frac{1}{5z^4} = \frac{1}{z_n^{1/3}} \lim_{z \to z_n} \frac{z}{5z^4}$$

$$= \frac{1}{z_n^{1/3}} \frac{z_n}{5a^5} = -\frac{z_k^{2/3}}{5a^5} = -\frac{1}{5a^5} e^{\frac{2}{3} \log z_n} = -\frac{1}{5a^5} e^{\frac{2}{3}[\ln a + io(z)]}$$

$$= -\frac{1}{5a^5} e^{\frac{2}{3}\left[\ln a + i\frac{\pi + 2k\pi}{5}\right]} = -\frac{a^{2/3}}{5a^5} e^{\frac{2}{3} \frac{i\pi + 2k\pi}{5}}$$

$$I = -\frac{\pi e^{\frac{\pi i}{3}}}{\frac{\sqrt{3}}{2}} \frac{a^{2/3}}{5a^5} e^{i\frac{2}{3}} \left[e^{\frac{\pi}{5}} + e^{\frac{3\pi}{5}} + e^{\frac{5\pi}{5}} + e^{\frac{7\pi}{5}} + e^{\frac{9\pi}{5}}\right]$$

$\square$