Teorema Reziduurilor

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Rezumat

Aplicatii ale teoremei reziduurilor in calulul unor chestii interesante. In prima parte avem introducere apoi exemple din x urmate de aplicatii de tip y.

Cuprins

1 Teorema Reziduurilor

Teorema 1 (Teorema Reziduurilor)**.** *Fie functia $f \in \mathcal{H}(G)$, unde $G \subset \mathbb{C}$ multime deschisa. Notam cu ρ mutimea tuturor punctelor singulare izolate ale lui f Fie $\widetilde{G} := G \cup S$, iar γ un contur in G omotop cu zero in $\widetilde{G}$*

$$\textit{Atunci suma: } \sum_{z \in \widetilde{G}} n(\gamma; z) \operatorname{Rez}(f; z) \textit{ este finita si}$$

$$\int_\gamma f(z)\,\mathrm{d}z = 2\pi i \sum_{z \in \widetilde{G}} n(\gamma; z) \operatorname{Rez}(f; z)$$

Demonstraţie. $\exists \varphi : [0;1]^2 \mapsto G$ deformare continuua, $k = \varphi([0;1]^2) \subset \widetilde{G}$ compact.

Fie

$$r := \frac{1}{2}\,\mathrm{d}\left(k,\ \mathbb{C} \setminus \widetilde{G}\right)$$
$$D := \bigcup_{z \in k} \mathcal{U}(z; r)$$

$k \subset D \subset \overline{D} \subset \widetilde{G}$
γ omotop cu 0 in D
$\overline{D} \cap \rho$ finita $\implies \exists \{b_1, \ldots, b_k\} = \overline{D} \cap \rho$
Fie $\Pi_k(z)$ partea principala a dezvoltarii lui f in b_k

Deci, functia $g := f - \sum_{k=1}^{n} \Pi_k$ olomorfa mai putin in b_k admite o prelungire olomorfa g_1 la D .

$$\int_\gamma g = \int_\gamma g_1 = 0$$
$$g = g_1|_{D = \{b_1, \ldots, b_k\}}$$
$$\implies \int_\gamma f = \sum_{k=1}^{n} \int_\gamma \Pi_k$$

Calculam

$$\int_\gamma \Pi_k \text{ , unde } \Pi_k(z) = \sum_{m=1}^{\infty} \frac{a^{(k)} - m}{(z - b_k)^m}$$

Seria este uniform convergenta pe $\forall$ parte compacta din $\mathbb{C} \setminus \{b_a\} \implies$ uniform convergenta pe $\{\gamma\} \implies$ putem integra termen cu termen si

$$\int_\gamma \frac{\mathrm{d}}{z - b_k} m = 0, \forall m > 1$$

Functia $\dfrac{1}{(z - b_n)^m}$ admite primitiva si $\displaystyle\int_\gamma \frac{\mathrm{d}z}{z - b_k} = 2\pi i \cdot n(\gamma; b_n) \cdot a_{-1}^{(k)}$ deci

$$\int_\gamma f = 2\pi i \sum_{k=1}^{n} n(\gamma; b_k) \operatorname{Rez}(f; b_n)$$

Trebuie sa mai aratam ca $\forall z_0 \in \widetilde{G} \setminus (D \cap \rho)\colon n(\gamma; z_0) \cdot \mathrm{Rez}(f; z_0) = 0$

Intr-adevar, daca pentru $z_0 \in \widetilde{G} \setminus (D \cap \rho)$ avem $\mathrm{Rez}(f; z_0) \neq 0 \implies z_0 \in \rho$, deci $z_0 \notin D$ si

$$n(\gamma; z_0) = \frac{1}{2\pi i} \int_\gamma \frac{\mathrm{d}\xi}{\xi - z_0} = 0$$

caci $h(\xi) = \frac{1}{\xi - z_0}$ olomorfa pe D si γ omotop cu zero

$$\implies \int_\gamma f = 2\pi i \sum_{z \in \widetilde{G}} n(\gamma; z) \cdot \mathrm{Rez}(f; z)$$

□

2 Puncte singulare izolate

Definitie 1. *Fie $G \subset \mathbb{C}$ multime deschisa si $f \in \mathcal{H}(G)$. Punctul $z_0 \in \mathbb{C}$ se numeste punct singular izolat pentru functia f daca $z_0 \notin G$, dar $\exists p > 0$ a.i $\dot{\mathcal{U}}(z_0; p) \subset G \implies f \in \mathcal{H}(\dot{\mathcal{U}}(z_0; p))$*

Observatie 1. *De exemplu functiile $\frac{\sin(z)}{z}$, $\frac{1}{z}$, $e^{\frac{1}{z}}$ au singularitati izolate in $z = 0$*

Observatie 2. *Daca z_0 este un punct singular izolat pentru $f \in \mathcal{H}(G)$, iar $p > 0$ a.i $\dot{\mathcal{U}}(z_0; p) \subset G$, atunci f admite o dezvoltare in serie Laurent de forma*

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k, \quad z \in \dot{\mathcal{U}}(z_0; p)$$

Coeficientul a_{-1} al termenului $(z - z_0)^{-1}$ se numeste reziduul functiei f in z_0 si se noteaza cu $a_{-1} = \mathrm{Rez}(f; z_0)$

Definitie 2. *Fie $G \subset \mathbb{C}$ multime deschisa, $f \in \mathcal{H}(G)$, iar z_0 punct singular izolat al functiei f. Spunem ca:*

1. *z_0 este punct eliminabil daca f se extinde olomorf la $\Omega \cup \{z_0\}$*
2. *z_0 este pol daca $lim_{z \to z_0} f(z) = \infty$*
3. *z_0 este punct esential izolat daca $\nexists$ limita a lui f in z_0*
4. *Un punct z este regular pentru f daca z este eliminabil pentru f sau f este derivabila in z*

3 Calcularea reziduului intr-un pol

1. Daca z_0 este un pol de ordin k pentru f atunci

$$\mathrm{Rez}(f; z_0) = \frac{1}{(k-1)!} \lim_{z \to z_0} \left[(z - z_0)^k f(z)\right]^{(k-1)}$$

2. In cazul unui punct singular esential reziduul se calculeaza cu ajutoril dezvoltarii in serie Laurent
3. Intr-un punct regular reziduul este 0

4 Aplicatii ale teoriei reziduurilor la calculul unor integrale definite reale

Tipul 1 (1). *Fie integrala* $I = \displaystyle\int_0^{2\pi} R(\sin x, \cos x)\ \mathrm{d}x$, *unde* $R(u, v)$ *este o functie rationala reala ce nu are poli pe cercul* $u^2 + v^2 = 1$

$$\text{Atunci } \int_0^{2\pi} R(\sin x, \cos x)\ \mathrm{d}x = 2\pi i \sum_{z \in \mathcal{U}(0;1)} \mathrm{Rez}(f; z)$$

$$\text{unde } f(z) = \frac{1}{z} R\left(\frac{z - \frac{1}{z}}{2i}, \ \frac{z + \frac{1}{z}}{2}\right)$$

Demonstraţie. Utilizand formulele lui Euler:

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}\ , \quad \sin x = \frac{e^{ix} - e^{-ix}}{2i} x \in \mathbb{R}$$

si substitutia $e^{ix} = z$, avem ca

$$\int_0^{2\pi} R(\sin x, \cos x)\ \mathrm{d}x = \int_{\partial\mathcal{U}(0;1)} R\left(\frac{z - \frac{1}{z}}{2i}, \frac{z + \frac{1}{z}}{2}\right) \frac{\mathrm{d}z}{iz} \implies$$

$$\int_0^{2\pi} R(\sin x, \cos x)\ \mathrm{d}x = -i \int_{\partial\mathcal{U}(0;1)} f(z)\ \mathrm{d}z \overset{T.Rez}{\implies}$$

$$\int_{\partial\mathcal{U}(0;1)} f(z)\ \mathrm{d}z = 2\pi i \sum_{|z|<1} \mathrm{Rez}(f; z) \implies$$

$$\int_0^{2\pi} R(\sin x, \cos x)\ \mathrm{d}x = 2\pi \sum_{|z|<1} \mathrm{Rez}(f; z)$$

□

Tipul 2. *Fie* R *o functie rationala reala ,* $R = P/Q$ *unde* P *si* Q *polinoame de grad* n *, respectiv* m, $Q(x) \neq 0 \quad \forall x \in \mathbb{R}$, $\lim\limits_{z\to\infty} zf(z) = 0, (n \leq m - 2)$

Atunci

$$\int_{-\infty}^{\infty} R(x)\ \mathrm{d}x = 2\pi i \sum_{\mathrm{Im}\, z = 0} \mathrm{Rez}(f; z)$$

Demonstraţie. $\exists M, r_1 > 0$ a.i

$$\left|\frac{P(x)}{Q(x)}\right| \leq \frac{M}{|x|^2}, \quad |x| \geq r_1$$

$$\int_{r_1}^{\infty} \frac{1}{x^2}\ \mathrm{d}x \text{ converge } \implies \int_{r_1}^{\infty} \frac{P(x)}{Q(x)}\ \mathrm{d}x \text{ converge}$$

Analog

$$\int_{-\infty}^{-r_1} \frac{P(x)}{Q(x)}\ \mathrm{d}x \text{ converge}$$

$$\text{Dar } \frac{P}{Q} \text{ continuua pe } [-r_1, r_1] \implies \exists \int_{-r_1}^{r_1} \frac{P(x)}{Q(x)}\,\mathrm{d}x$$

$$\int_{-\infty}^{0} \frac{P(x)}{Q(x)}\,\mathrm{d}x \text{ si } \int_{0}^{\infty} \frac{P(x)}{Q(x)}\,\mathrm{d}x \text{ converg } \implies \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\,\mathrm{d}x \text{ converge}$$

Fie $r > 0$ suficient de mare astfel incat toti polii lui f din semiplanul superior sa fie continuti in Ω_r, unde $\Omega_r = \{z \in \mathbb{C} \colon |z| < r, \quad \operatorname{Im} z > 0\}$.

Fie $\gamma_r(t) = re^{\pi i t}, t \in [0;1], \gamma = [-r;r] \cup \gamma_r$.

Atunci $\gamma = \partial\Omega_r$, iar $(\gamma) = \Omega_r \overset{T.Rez}{\Longrightarrow}$

$$\int_\gamma f(z)\,\mathrm{d}z = 2\pi i \sum_{z \in \Omega_r} \operatorname{Rez}(f;z) = 2\pi i \sum_{\operatorname{Im} z > 0} \operatorname{Rez}(f;z) \qquad (*)$$

Pe de alta parte

$$\int_\gamma f(z)\,\mathrm{d}z = \int_{\gamma_r} f(z)\,\mathrm{d}z + \int_{-r}^{r} f(x)\,\mathrm{d}x \qquad (**)$$

Din $(*)$ si $(**)$ trecand la limita $\implies$

$$2\pi i \sum_{\operatorname{Im} z > 0} \operatorname{Rez}(f;z) = \lim_{r \to \infty} \int_{\gamma_r} f(z)\,\mathrm{d}z + \int_{-\infty}^{\infty} f(x)\,\mathrm{d}x$$

$$\text{Dar, } \lim_{z \to \infty} z f(z) = 0 \implies \lim_{r \to \infty} \int_{\gamma_r} f(z)\,\mathrm{d}z = 0$$

$$\implies \int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = 2\pi i \sum_{\operatorname{Im} z > 0} \operatorname{Rez}(f;z)$$

□

Tipul 3. *Fie R o functie rationala reala de forma $R = \frac{P}{Q}$, $Q(x) \neq 0$, $x \in \mathbb{R}$, grad $Q >$ grad $P + 1$ si $\lim_{|z| \to \infty} R(z) = 0$*

Atunci

$$\int_{-\infty}^{\infty} R(x)e^{ix}\,\mathrm{d}x \textit{ converge si } \int_{-\infty}^{\infty} R(x)e^{ix}\,\mathrm{d}x = 2\pi i \sum_{\operatorname{Im} z > 0} \operatorname{Rez}(f;z)$$

unde $f(z) = R(z)e^{iz}$.

Demonstrație. Fie $r > 0$ suficient de mare a.i. toti polii functiei f din semiplanul superior sa fie continuti in D, unde $D = \{z \in \mathbb{C} \colon |z| < r;\ \operatorname{Im}\ z > 0\}$

Fie $C = \partial D \implies C = [-r;r] \cup \gamma_r$

$$\overset{\text{T. Rez}}{\Longrightarrow} \int_C f(z)\,\mathrm{d}z = 2\pi i \sum_{\operatorname{Im}\ z > 0} \operatorname{Rez}(f;z)$$

$$\left.\begin{array}{r} \text{Dar } \displaystyle\int_C f(z)\,\mathrm{d}z = \int_{-r} r f(x)\,\mathrm{d}x + \int_{\gamma_r} f(z)\,\mathrm{d}z \\ r \to \infty \end{array}\right\} \implies$$

$$\implies 2\pi i \sum_{\text{Im } z>0} \text{Rez}(f;z) = \int_{-\infty} \infty f(x)dx + \lim_{r\to\infty} \int_{\gamma_r} f(z)\,\mathrm{d}z$$
$$= \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} e^{ix}\,\mathrm{d}x + \lim_{r\to\infty} \int_{\gamma_r} \frac{P(z)}{Q(z)} e^{iz}\,\mathrm{d}z$$

$$g(z) = \frac{P(z)}{Q(z)}$$

deci,

$$\lim_{z\to\infty} g(z) = 0 \overset{L.Jordan}{\Longrightarrow} \lim_{r\to\infty} \int_{\gamma_r} g(z) e^{iz}\,\mathrm{d}z = 0$$

Asadar,

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} e^{ix}\,\mathrm{d}x = 2\pi i \sum_{\text{Im } z>0} \text{Rez}(f;z)$$

□

Tipul 4. *Fie integrala*

$$I = \int_{-\infty}^{\infty} f(x) e^{i\alpha x}\,\mathrm{d}x$$

unde $f = P/Q$, $Q(x) \neq 0$, $x \in \mathbb{R}$, *grad* $P = k$, *grad* $Q = p$, *iar* $p \geq k+1$.

Daca $\alpha > 0$, *atunci:*

$$I = \int_{-\infty}^{\infty} f(x) e^{i\alpha x}\,\mathrm{d}x = 2\pi i \sum_{\text{Im } z>0} \text{Rez}(g;z)$$

, *unde* $g(z) = f(z)e^{i\alpha z}$.

Demonstraţie. Observam ca $\exists \int_{-\infty}^{\infty} f(x)e^{i\alpha x}\,\mathrm{d}x$ si este convergenta. Intr-adevar, pentru ca $p \geq k+1 \implies \lim_{z\to\infty} f(z) = 0$. Dar $f'(z) = \dfrac{h(z)}{Q^2(z)}$, unde h este un polinom de grad cel mult $k+p-1$.

Fie x_0 zeroul lui h de modul maxim $\implies f'(x)$ are semn constant pentru $x > |x_0| \implies f(x)$ monotona pentru $x > |x_0|$.

Fie $x_1,\ x_2 \in \mathbb{R}$ cu $x_2 > x_1 > |x_0|$

$$\text{Cum } \lim_{z\to\infty} f(z) = 0 \implies \text{fie } f > 0 \text{ si } \lim_{x\to\infty} f(x) = 0^+, x > |x_0|$$
$$\text{fie } f < 0 \text{ si } \lim_{x\to\infty} f(x) = 0^-, x > |x_0|$$

Aplicand a doua teorema de medie din calculul integral $\implies \exists \xi \in (x_1; x_2)$ a.i.

$$\int_{x_1}^{x_2} f(x)\cos\alpha x\ \ \mathrm{d}x = f(x_1)\int_{x_1}^{\xi} \cos\alpha t\ \ \mathrm{d}t + f(x_2)\int_{\xi}^{x_2} \cos\alpha t\ \ \mathrm{d}t$$

$$\implies \left| \int_{x_1}^{x_2} f(x)\cos\alpha x\ \ \mathrm{d}x \right| \leq \frac{2}{\alpha}|f(x_1)| + \frac{2}{\alpha}|f(x_2)|$$

Stiind ca $\lim\limits_{z\to\infty} f(z) = 0 \implies \forall \epsilon > 0 \quad \exists \delta(\epsilon) > 0$ a.i. $|f(x)| < \dfrac{\epsilon\alpha}{4} x > \delta(\epsilon)$

Deci,

$$\left| \int_{x_1}^{x_2} f(x) \cos \alpha x \ \mathrm{d}x \right| \leq \frac{2}{\alpha} \left[|f(x_1)| + |f(x_2)| \right] < \epsilon,$$

$$x_2 > x_1 > \max\{|x_0|,\ \delta(\epsilon)\} \implies \int_0^\infty f(x) \cos \alpha x \ \mathrm{d}x \text{ converge}$$

Analog $\exists$ si converge

$$\int_0^\infty f(x) \sin \alpha x \, \mathrm{d}x$$

$$\implies \int_0^\infty f(x) e^{i\alpha x} \ \mathrm{d}x$$

este deasemenea convergenta.

Fie $\Omega_r = \{z \in \mathbb{C} \colon |z| < r; \mathrm{Im}\ \ z > 0\}$ ce contine toti polii functiei g din semiplanul superior

$$\overset{T.Rez}{\implies} \int_{\partial\Omega_r} g(z) \ \mathrm{d}z = 2\pi i \sum_{z \in \Omega_r} \mathrm{Rez}(g; z) = 2\pi i \sum_{\mathrm{Im}\ \ z>0} Rez(g; z)$$

Dar

$$\int_{\partial\Omega_r} g(z) \ \mathrm{d}z = \int_{-r}^{r} f(x) e^{i\alpha x} \ \mathrm{d}x + \int_{\gamma_r} g(z) \ \mathrm{d}z$$

$$\overset{L.Jordan}{\implies} \lim_{r\to\infty} \int_{\gamma_r} g(z) \ \mathrm{d}z = 0$$

$$\overset{r\to\infty}{\implies} \int_{-\infty}^{\infty} f(x) e^{i\alpha x} \ \mathrm{d}x = 2\pi i \sum_{\mathrm{Im}\ \ z>0} \mathrm{Rez}(g; z)$$

□

Aplicatie (1). *Sa se calculeze integrala*

$$I = \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{a^4 + x^4}$$

Demonstraţie. Este o integrala de tipul II

$$\left.\begin{array}{l} P(x) = 1 \\ Q(x) = a^4 + x^4 \end{array}\right\} grad\ Q > grad\ P + 2$$

$$f(z) = \frac{a}{a^4 + x^4}$$

$$a^4 + x^4 = 0 \implies z^4 = -a^4 = a^4(\cos \pi + i \sin \pi)$$

$$\implies z_k = a \left(\cos \frac{\pi + 2k\pi}{4} + i \sin \frac{\pi + 2k\pi}{4} \right), k = \overline{0, 3}$$

$$z_0 = a\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) = \frac{a}{\sqrt{2}}(1+i)$$
$$z_1 = \frac{a}{\sqrt{2}}(-1+i)$$
$$z_2 = \frac{a}{\sqrt{2}}(-1-i)$$
$$z_3 = \frac{a}{\sqrt{2}}(1-i)$$

$$I = 2\pi i \sum_{\mathrm{Im}\ z_k > 0} \mathrm{Rez}(f; z_k)$$

$$\Longrightarrow I = 2\pi i[\mathrm{Rez}(f; z_0) + \mathrm{Rez}(f; z_1)]$$

$$\mathrm{Rez}(f; z_k) = \lim_{z \to z_k} (z - z_k)\frac{1}{z^4 + a^4} \overset{\frac{0}{0}}{\underset{\mathrm{UH}}{=}} \lim_{z \to z_k} \frac{1}{4z^3} = -\frac{z_k}{4a^4}$$

Deci,

$$I = 2\pi i\left[\frac{a}{\sqrt{2}}(1 + i - 1 + i)\right] = \frac{2\pi i \cdot a \cdot 2i}{\sqrt{2}} \Longrightarrow I = -\frac{4\pi a}{\sqrt{2}}$$

□

Aplicatie. *Sa se calculeze integrala*

$$I = \int_{-\infty}^{\infty} \frac{\cos x}{x^2 + a^2}\,\mathrm{d}x,\ \textit{unde } a > 0$$

Demonstraţie. Este o integrala de tip III:

$$\text{Fie } I_1 = \int_{-\infty}^{\infty} \frac{\cos x}{x^2 + a^2}\,\mathrm{d}x$$
$$\text{si } I_2 = \int_{-\infty}^{\infty} \frac{\sin x}{x^2 + a^2}\,\mathrm{d}x (= 0 \text{ pe ca e impara})$$
$$\text{si fie } I = I_1 + I_2$$
$$\Longrightarrow I = \int_{-\infty}^{\infty} \frac{e^{ix}}{x^2 + a^2}\,\mathrm{d}x$$

$$\left.\begin{array}{l} P(x) = 1 \\ Q(x) = a^2 + x^2 \end{array}\right\} \begin{array}{c} \textit{grad } Q \geq \textit{grad } P + 1 \\ 2 \geq 1 \end{array}$$
$$f(z) = \frac{e^{iz}}{a^2 + x^2}$$

$a^2 + x^2 = 0 \Longrightarrow z_{1,2} = \pm ia$, dar doar $z_1 = ia$ pol de gradul I $\in$ semiplanul superior

$$\Longrightarrow I = 2\pi i\ \mathrm{Rez}(f; z_1) = 2\pi i\ \mathrm{Rez}(f; ia)$$

$$\mathrm{Rez}(f; ia) = \lim_{z \to ia} (z - ia)\frac{e^{iz}}{z^2 + a^2} = \frac{e^{-ia}}{z + ia} = \frac{e^{-ia}}{2ia}$$

$$\Longrightarrow I = 2\pi i \frac{e^{-ia}}{2ia} = \frac{e^{-a}\pi}{a}$$

$$I_1 = \text{Re } I \quad I_2 = \text{Im } I \implies I_1 = \frac{e^{-a}\pi}{a}; \quad I_2 = 0$$

□

Teorema 2. *Fie $f \in \mathcal{M}(\mathbb{C})$ si $z_1, \ldots, z_k$ poli ai functiei f cu reziduurile $u_1, \ldots, u_k$. Daca $f(z) \neq 0$, $z \in \mathbb{Z}$, $z_j \notin \mathbb{Z}$, $j = 1, \ldots, k$, iar $f(z) = O(z^{-2}), z \to \infty$, atunci*

$$\sum_{-\infty}^{\infty} f(\varphi) = -\pi \sum_{j=1}^{k} \text{Rez}(\operatorname{ctg} \pi z \cdot f(z); z_j)$$

Demonstraţie. pag 95-98 carte portocala □

Aplicatie. *Sa se calculeze*

$$\sum_{n=1}^{\infty} \frac{1}{n^4 + 1}$$

Demonstraţie. Se vede ca

$$\sum_{-\infty}^{\infty} \frac{1}{n^4 + 1} = 1 + 2 \sum_{n=1}^{\infty} \frac{1}{n^4 + 1}$$

Fie $f(z) = \dfrac{1}{z^4 + 1}$, atunci $f \in \mathbb{C}$, cu polii simplii $\pm 1, \pm i$

$$\text{Rez}(f; z_n) = \lim z \to z_n \frac{1}{z^4 + 1} \underset{\text{UH}}{\overset{\frac{0}{0}}{=}} \lim_{z \to z_n} \frac{1}{4z^3} = \frac{z_n}{-4}$$

$$\begin{aligned}
\sum_{-\infty}^{\infty} \frac{1}{n^4 + 1} &= -\pi \left[-\frac{1}{4} \operatorname{ctg} \pi + \frac{1}{4} \operatorname{ctg}(-\pi) - \frac{i}{4} \operatorname{ctg} i\pi + \frac{i}{4} \operatorname{ctg}(-i\pi) \right] \\
&= \frac{\pi}{4} [\operatorname{ctg} \pi + \operatorname{ctg} \pi + i \operatorname{ctg} i\pi + i \operatorname{ctg} i\pi] \\
&= \frac{\pi}{2} \operatorname{ctg} \pi + \frac{\pi}{2} \text{cth } \pi
\end{aligned}$$

$$\text{Deci, } 1 + 2 \sum_{n=1}^{\infty} \frac{1}{n^4 + 1} = \frac{\pi}{2} \operatorname{ctg} \pi + \frac{\pi}{2} \text{cth } \pi$$

$$\implies \sum_{n=1}^{\infty} \frac{1}{n^4 + 1} = \frac{\pi}{4} [\operatorname{ctg} \pi + \frac{\pi}{2} \text{cth } \pi] - \frac{1}{2}$$

□

5 Calcularea unei integrale pe un arc de curba simplu si rectificabil, dar nu inchis

In acest caz putem incerca sa formam o curba inchisa $\gamma_0 \cup \gamma_1$ a.i. sa poata sa se aplice teorema reziduurilor , iar integrala pe noua curba $\gamma = \gamma_0 \cup \gamma_1$ sa se poata calcula cu reziduuri direct sau sa aiba o relatie simpla cu integrala cautata.

Daca integrala este improprie , fiind limita unei alte integrale

$$\int_{\gamma_0} = \lim_{\gamma \to \gamma_0} \int_{\gamma}$$

atunci si arcul adaugat va varia si vom putea calcula integrala improprie cunoscand limita $\int_{\gamma_1}$ si daca suma reziduurilor din domeniu G variabil are limita cunoscuta:

$$\int_{\gamma_0} f \, \mathrm{d}z = -\lim \int_{\gamma_1} f \, \mathrm{d}z + 2\pi i \lim \sum \operatorname{Rez}(f; z)$$

Aplicatie. *Sa se calculeze*

$$I = \int_0^{\infty} \frac{\cos ax}{\operatorname{ch} \pi x} \, \mathrm{d}x, a \in \mathbb{R}$$

Demonstraţie.

$$f(z) = \frac{\cos az}{\operatorname{ch} \pi z}$$

Polii acestei functii sunt simpli , $z = (zk+1)\frac{i}{2}$, $k \in \mathbb{Z}$ Pentru a evita seria de reziduuri care este divergenta alegem conturul

IMAGINE

Pe latura $z = R + iy \quad (0 \leq y \leq 1)$

$$\left| i \int_0^1 \frac{\cos a(R+iy)}{\operatorname{ch} \pi(R+iy)} \, \mathrm{d}y \right| = \left| i \int_0^1 \frac{e^{ia(R+iy)} + e^{-ia(R+iy)}}{e^{\pi(R+iy)} + e^{-\pi(R+iy)}} \, \mathrm{d}y \right|$$
$$< \frac{\int_0^1 (e^{-ay} + e^{ay}) \, \mathrm{d}y}{e^{\pi R} - e^{-\pi R}} \to 0$$

Ramane : $2 \int_0^R \frac{\cos ax}{\operatorname{ch} \pi x} \, \mathrm{d}x + \int_0^R \left[-\frac{\cos a(i+x)}{\operatorname{ch} \pi(i+x)} - \frac{\cos a(i-x)}{\operatorname{ch} \pi(i-x)} \right] \, \mathrm{d}x \longrightarrow 2\pi i \operatorname{Rez}\left(f; \frac{i}{2}\right)$

Stiind ca

$$\operatorname{ch} \pi(x \pm i) = -\operatorname{ch} \pi x$$
$$\cos a(x \pm i) = \cos ax \cdot \operatorname{ch} a \mp \sin ax \cdot \operatorname{ch} a$$

obtinem ca

$$2(1 + \operatorname{ch} a) \int_0^R \frac{\cos ax}{\operatorname{ch} \pi x} \, \mathrm{d}x \longrightarrow 2\pi i \operatorname{Rez}\left(f; \frac{i}{2}\right)$$

$$\operatorname{Rez}\left(f; \frac{i}{2}\right) = \lim_{z \to \frac{i}{2}} \left(z - \frac{i}{2}\right) \frac{\cos az}{\operatorname{ch} \pi z} = \frac{\cos \frac{ai}{2}}{\pi \operatorname{ch} \frac{\pi}{2}} = \frac{\operatorname{ch} \frac{a}{2}}{\pi i \operatorname{ch} \frac{\pi}{2}} = \frac{\operatorname{ch} \frac{a}{2}}{\pi i}$$

$$\Longrightarrow I = 2\pi i \frac{\operatorname{ch} \frac{a}{2}}{\pi} = 2 \operatorname{ch} \frac{a}{2}$$

□

6 Aplicatii la dezvoltari in serie

Teorema 3. *Fie $f(z)$ o functie mereomorfa ai carei poli formeaza un sir infinit $z_k \to \infty$ si D_n un domeniu marginit de o curba rectificabila γ_n si care nu trece prin nici un pol z_n*

$$\text{Atunci} \int_{\gamma_n} f(z)\,\mathrm{d}z = 2\pi i \sum_{k=1}^{n} \mathrm{Rez}(f; z_n)$$

Observatie 3.

1. *Daca $n \to \infty, \gamma_n$ variaza a.i. D_n tinde catre un domeniu ce cuprinde toti polii a_n. Daca integrala din membrul* I *are o limita finita, atunci obtinem suma seriei de Reziduuri $\sum_{n=1}^{\infty} \mathrm{Rez}(f; z_n)$ insumata dupa domeniul D_n*

2. *Daca indicele k ia valorile $1, 2, \cdots$ si $|z_k|$ sunt strict crescatoare $|z_1| < |z_2| < \cdots$, a.i. intre 2 curbe consecutive sa se afle un singur pol, vom obtine suma seriei convergente $\sum_{n=1}^{\infty} \mathrm{Rez}(f; z_k)$*

3. *Daca $|z_n|$ si $|z_{-n}|$ sunt crescatori vom putea obtine suma seriei convergente $\mathrm{Rez}(f; z_0) + \sum_{k=1}^{n} [\mathrm{Rez}(f; z_k) + \mathrm{Rez}(f; z_{-k})]$ adica*

$$\sum_{-\infty}^{\infty} \mathrm{Rez}(f; z_n) = \frac{1}{2\pi i} \lim_{k \to \infty} \int_{\gamma_k} f(z)\,\mathrm{d}z$$

4. *Fie $f(z)$ o functie mereomorfa avand polii de gradul* I *, $z_k \to \infty$ si $g(z)$ o functie uniforma cu un numar finit de puncte singulare a_h, diferite de z_n . Fie γ_n cu $n > n_0$ ce contine punctele a_h in interiorul sau. Atunci pentru functia $f(z) \cdot g(z)$ avem ca*

$$\mathrm{Rez}(f \cdot g; z_n) = g(z_k)\,\mathrm{Rez}(f; z_n)$$

Formula din Obs 3 se transforma astfel

$$\sum_{-\infty}^{\infty} \mathrm{Rez}(f; z_n) g(z_n) = \frac{1}{2\pi i} \lim_{k \to \infty} \int_{\gamma_k} f(z) g(z)\,\mathrm{d}z - \sum_{k \in \mathbb{C}} \mathrm{Rez}(f \cdot g; a_h)$$

A doua suma este nula pentru $g(z)$ functie intreaga

Aplicatie. *Sa se calculeza integrala*

$$I = \int_0^{2\pi} \frac{\mathrm{d}x}{a + \cos x}, a > 1$$

Demonstraţie. Se observa ca I este o integrala de tipul I. Din formulele lui Euler stim ca:

$$\cos z = \frac{e^{ix} + e^{-ix}}{2}, \quad e^{ix} = z \implies \mathrm{d}x = \frac{\mathrm{d}z}{iz} \implies \cos x = \frac{z + \frac{1}{z}}{2}$$

$$f(z) = \frac{1}{z} \frac{1}{a + \frac{z + 1/2}{2}} \implies f(z) = \frac{1}{z^2 + 2az + 1}$$

$$I=\int_{\partial\mathcal{U}(0;1)}\frac{\frac{\mathrm{d}z}{iz}}{a+\frac{z+1/2}{2}}=-2i\int_{\partial\mathcal{U}(0;1)}\frac{\mathrm{d}z}{z^2+2az+1}$$

$$z^2+2az+1=0 \implies \Delta=4a^2-4$$

$$\implies \begin{cases} z_1=\dfrac{-2a+\sqrt{4a^2-4}}{2}=-a+\sqrt{a^2-1} \\ z_2=\dfrac{-2a-\sqrt{4a^2-4}}{2}=-a-\sqrt{a^2-1} \end{cases}$$

$$\begin{aligned} |z_1|<1 &\iff |-a+\sqrt{a^2-1}|=a-\sqrt{a^2-1}<1 \\ &\iff a-1<\sqrt{a^2-1}\,\Big|^2 \iff a^2-2a+1<a^2-1 \\ &\iff 2a>0 \text{ Adevarat} \\ |z_2|<1 &\iff |-a-\sqrt{a^2-1}|<1 \text{ Fals } \implies z_2\notin\mathcal{U}(0;1) \end{aligned}$$

Deci, $I=2\pi\,\mathrm{Rez}(f;z_1)$ cu z_1 pol simplu

$$\begin{aligned} \mathrm{Rez}(f;z_1) &= \lim_{z\to z_1}(z-z_1)\frac{1}{z^2+2az+1} \overset{\frac{0}{0}}{\underset{\text{UH}}{=}} \lim_{z\to z_1}\frac{1}{2z+2a} \\ &= \frac{1}{2(a+\sqrt{a^2-1})+2a}=-\frac{1}{2\sqrt{a^2-1}} \end{aligned}$$

Asadar ,

$$I=2\pi\frac{1}{2\sqrt{a^2-1}}=\frac{\pi}{\sqrt{a^2-1}}$$

□

Teorema 4. *Fie functia reala rationala $f=P/Q$, avand polii pe $[0;1]$, fie $0<\alpha<1$ si $\lim_{z\to\infty} f(z)=0$. Atunci avem ca :*

$$\int_0^\infty \frac{f(x)}{x^\alpha}\,\mathrm{d}x=\frac{\pi e^{\alpha\pi i}}{\sin\alpha\pi}\sum_{z\in\mathbb{C}^*}\mathrm{Rez}(h;z)$$

$$\textit{Unde } h(z)=\frac{f(z)}{z^\alpha}\ ,\ \textit{iar} z^\alpha=e^{\alpha\log z}$$

cu $\log z$ *ramura uniforma a aplicatiei multivoce Logaritm*

Demonstraţie. Fie Γ conturul din imagine

IMAGINE

Atunci

$$\int_\Gamma g(z)\,\mathrm{d}z=2\pi i\sum_{z\in\mathbb{C}^*}\mathrm{Rez}(g;z)$$

si

$$\int_\Gamma g(z)\,\mathrm{d}z=\int_{r_1}^{r_2}\frac{f(x)}{x^\alpha}\,\mathrm{d}x+\int_{\gamma_{r_2}}\frac{f(z)}{z^\alpha}\,\mathrm{d}z-\int_{r_1}^{r_2}\frac{f(x)}{e^\alpha[\ln x+2\pi i]}\,\mathrm{d}x-\int_{\gamma_{r_1}}\frac{f(z)}{z^\alpha}\,\mathrm{d}z$$

Deci

$$(*)\ 2\pi i \sum_{z\in\mathbb{C}^*} \mathrm{Rez}(g;z) = \int_{\gamma_{r_2}} \frac{f(z)}{z^\alpha}\,\mathrm{d}z - \int_{\gamma_{r_1}} \frac{f(z)}{z^\alpha}\,\mathrm{d}z + (1-e^{-2\pi i\alpha})\int_{r_1}^{r_2} \frac{f(x)}{x^\alpha}\,\mathrm{d}x$$

Cum $lim_{z\to\infty} f(z) = 0$ urmeaza ca $p \leq k+1$, unde k si p sunt gradele polinoamelor P respectiv Q. Deoarece $a \in (0;1)$, obtinem imediat ca

$$\lim_{z\to\infty} zg(z) = \lim_{z\to\infty} z^{1-\alpha}\frac{P(z)}{Q(z)} = 0 \text{ si } \lim_{z\to 0} zg(z) = \lim_{z\to 0} z^{1-\alpha}\frac{P(z)}{Q(z)} = 0$$

Trecand la limita in (*) pentru $r_1 \to 0$ si $r_2 \to \infty$ deducem concluzia teoremei.

□

Aplicatie. *Sa se calculeze integrala*

$$I = \int_0^\infty \frac{\mathrm{d}x}{\sqrt[3]{x}(x^5+a^5)}, a > 0$$

Demonstraţie. Aceasta integrala este de tip V

$$f(x) = \frac{1}{x^5+a^5}$$
$$h(z) = \frac{1}{z^{1/3}(z^5+a^5)}$$

$$z^5 + a^5 = 0 \implies z^5 = -a^5$$
$$z_k = a\left(\cos\frac{\pi+2k\pi}{5} + i\sin\frac{\pi+2k\pi}{5}\right), k = \overline{0,4}$$

unde z_k sunt poli simpli

$$I = \frac{\pi e^{\frac{\pi i}{3}}}{\sin\frac{\pi}{3}} \sum_{k=0}^{n} \mathrm{Rez}(h; z_n)$$

$$\begin{aligned}
\mathrm{Rez}(h;z_n) &= \lim_{z\to z_n}(z-z_n)\frac{1}{z^{1/3}(z^5+a^5)} \overset{\frac{0}{0}}{\underset{\text{UH}}{=}} \frac{1}{z_n^{1/3}} \lim_{z\to z_n}\frac{1}{5z^4} = \frac{1}{z_n^{1/3}}\lim_{z\to z_n}\frac{z}{5z^4} \\
&= \frac{1}{z_n^{1/3}}\frac{z_n}{5a^5} = -\frac{z_k^{2/3}}{5a^5} = -\frac{1}{5a^5}e^{\frac{2}{3}\log z_n} = -\frac{1}{5a^5}e^{\frac{2}{3}[\ln a + io(z)]} \\
&= -\frac{1}{5a^5}e^{\frac{2}{3}\left[\ln a + i\frac{\pi+2k\pi}{5}\right]} = -\frac{a^{2/3}}{5a^5}e^{\frac{2}{3}\frac{i\pi+2k\pi}{5}} \\
I &= -\frac{\pi e^{\frac{\pi i}{3}}}{\frac{\sqrt{3}}{2}}\frac{a^{2/3}}{5a^5}e^{i\frac{2}{3}}\left[e^{\frac{\pi}{5}} + e^{\frac{3\pi}{5}} + e^{\frac{5\pi}{5}} + e^{\frac{7\pi}{5}} + e^{\frac{9\pi}{5}}\right]
\end{aligned}$$

□