Universitatea Babes-Bolyai Cluj-Napoca
Facultatea de Matematica si Informatica
Specializarea Matematica

Lucrare de diploma

Teorema Reziduurilor si aplicatii

Autor :
Tapalaga Ecaterina Simona

Conducator stiintific:
Prof. Dr. Salagean Grigore

Iunie 2013

Cuprins

1 Introducere

Aceasta lucrare prezinta aplicatii ale teoremei reziduurilor atat la calulul unor integrale si serii, cat si la determinarea numarului de zerourilor a unor functii.

Lucrarea este alcatuita din trei capitole. Primul capitol reprezinta scheletul teoretic necesar pentru a putea introduce notiunea de reziduuri. Acesta cuprinde notiunile de integrala complexa, zerourile unei functii olomorfe si indexul unei curbe.

Urmatorul capitol enunta teorema reziduurilor impreuna cu demonstratia ei si calcularea reziduului intr-un pol. Acesta teorema are aplicatii variate si importante. Ultimul capitol, cel mai cuprinzator, prezinta unele dintre ele. Teorema reziduurilor ofera metode elegante de calcula unele integrale reale. Se poate utiliza deasemenea si in calcularea unor sume de serii, care in general sunt greu de determinat clasic. O alta aplicatie interesanta este demonstratia teoremei fundamentale a algebrei. Teorema lui Rouche este de asemenea un rezultat util al teoriei reziduurilor. Incheiem lucrarea cu aplicatii ale acesteia.

2 Notiuni introductive

2.1 Integrala Riemann-Stieltjes a unei functii complexe de variabila reala

Definitie 1. *Fie $f = u+iv$ si $F = U+iV$, iar $[a;b]$ interval din $\mathbb{R}$. Spunem ca f este integrabila Riemann-Stieltjes in raport cu F pe intervalul $[a;b]$ daca u si v sunt integrabile Riemann-Stieltjes in raport cu U si V pe $[a;b]$.*

Notam :

$$\int_a^b f\ \mathrm{d}F := \int_a^b u\ \mathrm{d}U - \int_a^b v\ \mathrm{d}V + i\int_a^b u\ \mathrm{d}V + i\int_a^b v\ \mathrm{d}U$$

Teorema 1. *Consideram $f = u + iv$, $F = U + iV$, iar $f_n : [a;b] \mapsto \mathbb{C}$, $F_n : [a;b] \mapsto \mathbb{C}$, si α , $\beta \in \mathbb{C}$.*

Au loc urmatoarele proprietati :

1. *Daca f este integrabila Riemann-Stieltjes in raport cu F pe $[a;b]$, atunci F este integrabila Riemann-Stieltjes in raport cu f si :*
$$\int_a^b f\ \mathrm{d}F + \int_a^b F\ \mathrm{d}f = f(b)F(b) - f(a)F(a)$$
2. *Daca f si g sunt integrabile Riemann-Stieltjes in raport cu F pe $[a;b]$, atunci $\alpha f + \beta g$ e integrabila dupa F si :*
$$\int_a^b (\alpha f + \beta g)\ \mathrm{d}F = \alpha\int_a^b f\ \mathrm{d}F + \beta\int_a^b g\ \mathrm{d}F$$
3. *Daca f este continua si F este cu variatie marginita pe $[a;b]$, atunci f este integrabila pe $[a;b]$ in raport cu F.*
4. *Fie $(f_n)_{n\in\mathbb{N}}$ un sir de functii continuue ce converge uniform catre f pe $[a;b]$ si $(F_n)_{n\in\mathbb{N}}$ un sir de functii cu variatie marginita care converge punctual catre F, iar sirul $\mathrm{V}(F_n,[a;b])$ marginit. Atunci avem ca:*
$$\lim_{\substack{n\to\infty \\ k\to\infty}} \int_a^b f_n\ \mathrm{d}F_k = \int_a^b f\ \mathrm{d}F$$

5. *Daca f e continua, F derivabila si F' continua, atunci :*

$$\int_a^b f \, \mathrm{d}F = \int_a^b f(t)F'(t) \, \mathrm{d}t$$

6. *Fie $c \in (a;b)$ si f integrabila in raport cu F pe $[a;b]$, atunci f este integrabila in raport cu F si pe $[a;c]$, si pe $[c;b]$, iar :*

$$\int_a^b f \, \mathrm{d}F = \int_a^c f \, \mathrm{d}F + \int_c^b f \, \mathrm{d}F$$

7. *Daca f e integrabila in raport cu F pe $[a;b]$, si $h : [a';b'] \mapsto [a;b]$ $h(a') = a$ si $h(b') = b$, h fiind omeomorfism, atunci $f \circ h$ e integrabila Riemann-Stieltjes pe $F \circ H$ si*

$$\int_a^b f \, \mathrm{d}F = \int_{a'}^{b'} (f \circ h) \, \mathrm{d}(F \circ H)$$

Definitie 2. *Consideram drumul rectificabil γ, iar $f : \{\gamma\} \mapsto \mathbb{C}$ continua. Atunci $f \circ \gamma$ va fi continua pe $[0;1]$ si integrabila in raport cu γ. Aceasta inegrala se numeste integrala complexa a drumului f de-a lungul lui γ :*

$$\int_\gamma f := \int_\gamma f(\zeta) \, \mathrm{d}\zeta = \int_0^1 (f \circ \gamma) \, \mathrm{d}\gamma$$

Teorema 2. *Fie γ drum rectificabil din $\mathcal{D}(z_0;z_1)$ si f o functie continua din $\{\gamma\}$. Atunci :*

1. *Fie g o alta functie continua din $\{\gamma\}$, α, $\beta \in \mathbb{C}$, atunci:*

$$\int_\gamma \alpha f + \beta g = \alpha \int_\gamma f + \beta \int_\gamma g$$

2.

$$\int_{\gamma^-} f = -\int_\gamma f$$

3. *Fie γ_1 un alt drum rectificabil din $\mathcal{D}(z_1;z_2)$, atunci :*

$$\int_{\gamma \cup \gamma_1} f = \int_\gamma f + \int_{\gamma_1} f$$

4. *Daca* $(\gamma_1, \gamma_2, \cdots, \gamma_n)$ *e o descompunere a lui* γ *atunci :*

$$\int_\gamma f = \sum_{k=1}^{n} \int_{\gamma_k} f$$

5. *Daca pentru* $\forall t \in [0;1]$ *avem ca* $|f(\gamma(t))| \leq M$, *atunci :*

$$\left| \int_\gamma f \right| \leq M \cdot \mathrm{V}(\gamma)$$

6. *Fie* γ *un drum liniar atunci* $\exists z_1, z_2 \in \mathbb{C}$ *a.i. :*

$$\int_\gamma f = (z_2 - z_1) \int_0^1 f[(1-t)z_1 + tz_2]\,\mathrm{d}t$$

7. *Fie* $f : G \mapsto \mathbb{C}$ *continua,* G *multime deschisa din* $\mathbb{C}$ *, iar* $(\gamma_n)_{n\in\mathbb{N}} \in \mathcal{D}_G$ *rectificabile.* $\{\gamma\} \subset G$ *si* $(\gamma_n)_{n\in\mathbb{N}}$ *converge uniform pe* $[0;1]$ *catre* γ, *iar* $\mathrm{V}(\gamma_n)$ *e multime marginita. Atunci:*

$$\lim_{n\to\infty} \int_{\gamma_n} f = \int_\gamma f$$

8. *Fie* $(f_n)_{n\in\mathbb{N}}$ *sir de aplicatii continue ,* $f_n : \{\gamma\} \mapsto \mathbb{C}$ *uniform convergent pe* $\{\gamma\}$ *catre* $\mathbb{C}$, *atunci*

$$\lim_{n\to\infty} \int_\gamma f_n = \int_\gamma f$$

Definitie 3. *Fie* $G \subset \mathbb{C}$ *multime deschisa,* $f : G \mapsto \mathbb{C}$ *si* $g \in \mathcal{H}(G)$. *Spunem ca* g *este primitiva pentru* f *daca* $f = g'$.

Teorema 3 (Legatura dintre primitiva si integrala)**.** *Fie o functie* $f : D \mapsto \mathbb{C}$ *continua , unde* D *domeniu din* $\mathbb{C}$. *Atunci*

1. *Daca pentru orice contur* γ *din* D *avem ca* $\displaystyle\int_\gamma f = 0$, *atunci* f *admite primitiva pe* D *.*

2. *Daca g este o primitiva a lui f pe D, atunci pentru $\forall$ drum rectificabil γ din D are loc $\displaystyle\int_\gamma f = g(\gamma_1) - g(\gamma_0)$. Daca γ e contur (drum rectificabil inchis), atunci avem $\displaystyle\int_\gamma f = 0$*

Teorema 4 (Legatura dintre olomorfie si primitiva)**.** *Fie D un domeniu stelat in z_0, iar $d_1, \cdots, d_n$ drepte ce trec prin z_0, d reuniunea lor. Daca $f : D \mapsto \mathbb{C}$ e continua pe D si derivabila pe $D \setminus d$, atunci f admite primitiva pe D*

Teorema 5 (Cauchy)**.** *Fie G o multime deschisa. Daca functia $f \in \mathcal{H}(G)$, iar conturul γ e omotop cu zero in G, atunci*

$$\int_\gamma f = 0$$

2.2 Zerourile functiilor olomorfe

Definitie 4. *Fie $G \subset \mathbb{C}$ deschisa, iar $f \in \mathcal{H}(G)$. Daca $\exists$ un punct $z \in G$ a.i. $f(z) = 0$, atunci z se numeste zerou al functiei f. Daca $\exists$ un $k \in \mathbb{N}^*$ a.i. :*

$$f(z) = f'(z) = \cdots = f^{k-1}(z) = 0$$

si $f^k(z) \neq 0$, atunci z se numeste zerou multiplu de ordin k pentru f

Pentru $k = 1$ il numim pe z zerou simplu.

Teorema 6. *Daca z este un zerou multiplu de ordin k al functiei $f \in \mathcal{H}(G)$, atunci $\exists g \in \mathcal{H}(G)$ a.i.*

$$g(x) \neq 0 \text{ si } f(x) = (x - z)^k g(x) \forall x \in G$$

Teorema 7. *Fie $D \subset \mathbb{C}$ domeniu si $f, g : D \mapsto \mathbb{C}$ functii olomorfe pe D. Urmatoarele afirmatii sunt echivalente:*

1. $f \equiv g$;
2. $\exists$ *un punct* $a \in D$ *a.i.* $f^{(k)}(a) = g^{(k)}(a)$ $\forall k \in \mathbb{N}$;
3. $\{z \in D\colon f(z) = g(z)\} \neq \emptyset$.

Teorema 8 (Zerourile unei functii olomorfe)**.** *Fie $D \subset \mathbb{C}$ domeniu si $f \in \mathcal{H}(G)$ nu este identic nula pe D, iar $z_0 \in D$ este un zerou al lui f, atunci $\exists r = r(z_0) > 0$ a.i. $\mathcal{U}(z_0; r) \subset D$ si $f(z) \neq 0, z \in \dot{\mathcal{U}}(z_0; r)$.*

Teorema 9 (Maximul modulului)**.** *Fie $D \subset \mathbb{C}$ domeniu si $f : D \mapsto \mathbb{C}$ o functie olomorfa. Daca $\exists$ un punct $z_0 \in D$ a.i. $|f(z)| \leq |f(z_0)|$, $\forall z \in D$, atunci f este constanta.*

Teorema 10 (Lema lui Schwarz)**.** *Fie functia f olomorfa pe $\mathcal{U}(0; 1)$ a.i. $f(0) = 0$ si $|f(z)| < M$, $z \in \mathcal{U}$, $M > 0$. Atunci:*

$$|f(z)| \leq M|z| \ , \ z \in \mathcal{U} \text{ si } |f'(0)| \leq M$$

Daca $\exists z_0 \in \dot{\mathcal{U}}(z_0; r)$ *a.i.* $|f(z_0)| = M|z_0|$ *sau daca* $|f'(0)| = M$, *atunci* $\exists \alpha \in \mathbb{C}$ *a.i.* $|\alpha| = M$ *si* $f(z) = \alpha z,\ z \in \mathcal{U}$

2.3 Serii Laurent

Definitie 5. *Se numeste seria Laurent in jurul lui* $z_0 \in \mathbb{C}$ *:*

$$\sum_{n=-\infty}^{\infty} a_n (z-z_0)^n = \cdots + \frac{a_{-n}}{(z-z_0)^n} + \cdots + \frac{a_{-1}}{z-z_0} + a_0 + \cdots a_n(z-z_0)^n + \cdots$$

unde $a_n \in \mathbb{C}$ *si se numesc coeficientii seriei.*

Daca $\forall n < 0$ *avem* $a_n = 0$ *spunem ca seria Laurent se reduce la o serie de puteri.*

$$\sum_{n=1}^{\infty} a_{-n}(z-z_0)^{-n} \text{ se numeste partea principala, iar}$$
$$\sum_{n=0}^{\infty} a_n(z-z_0)^{n} \text{ se numeste partea tayloreana.}$$

Teorema 11 (Coroanei de convergenta). *Fie* $\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$ *serie Laurent si folosim notatiile :*

$$r = \overline{\lim_{n\to\infty}} \sqrt[n]{|a_{-n}|}$$
$$\frac{1}{R} = \overline{\lim_{n\to\infty}} \sqrt[n]{|a_n|}$$

In conditiile in care $r < R$, *avem:*

1. $\mathcal{U}(z_0; r; R) = \{z \colon r < |z-z_0| < R\}$ *coroana de convergenta a seriei Laurent converge absolut si uniform pe compacte.*

2. $$\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n \text{ diverge in } \mathbb{C} \setminus \overline{\mathcal{U}}(z_0; r; R)\ .$$

3.

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n \in \mathcal{H}(\mathcal{U}(z_0; r; R)) \ .$$

2.4 Index unei curbe

Definitie 6. *Fie γ un drum rectificabil din $\mathbb{C}$ si $z_0 \in \mathbb{C} \setminus \{\gamma\}$. Numim indexul lui γ in raport cu z_0 :*

$$n(\gamma; z_0) := \frac{1}{2\pi i} \int_\gamma \frac{\mathrm{d}\zeta}{\zeta - z_0}$$

Teorema 12. *1. Fie γ_1 si γ_2 drumuri rectificabile din $\mathbb{C}$ si $z_0 \in \mathbb{C} \setminus \{\gamma_j\}$, $j = \overline{1,2}$. Daca $\gamma_1 \sim \gamma_2$ in $\mathbb{C} \setminus \{z_0\} \implies n(\gamma_1; z_0) = n(\gamma_2; z_0)$.*

2. Daca γ_1 si γ_2 drumuri rectificabile din $\mathbb{C}$ a.i. $\gamma_1(1) = \gamma_2(0)$, $z_0 \notin \{\gamma_j\}$, $j = \overline{1,2}$ atunci $n(\gamma_1 \cup \gamma_2; z_0) = n(\gamma_1; z_0) + n(\gamma_2; z_0)$

$$\gamma_1 \cup \gamma_2 : [0;1] \mapsto \mathbb{C}$$

$$(\gamma_1 \cup \gamma_2)(t) = \begin{cases} \gamma_1(2t) \ , \ t \in \left[0; \dfrac{1}{2}\right] \\ \gamma_2(2t-1) \ , \ t \in \left[\dfrac{1}{2}; 1\right] \end{cases}$$

3. $n(\gamma^-; z_0) = -n(\gamma; z_0)$, unde γ drum rectificabil pe $\mathbb{C}$ $z_0 \notin \{\gamma\}$, unde $\gamma^-(t) = \gamma(1-t)$, $t \in [0;1]$.

Teorema 13 (Teorema indexului)**.** *Fie γ un contur din $\mathbb{C}$. Atunci*

$$n(\gamma; z) \in \mathbb{Z} \ , \ \forall z \in \mathbb{C} \setminus \{\gamma\} \ .$$

Definitie 7. *Fie γ contur din $\mathbb{C}$. γ se numeste* contur Jordan *daca γ contur simplu ($\gamma|_{(0;1)}$ - functie injectiva) si $n(\gamma; z) = 1$, $\forall z \in (\gamma)$, unde (γ) e domeniul marginit cu frontiera γ.*

Teorema 14 (Formulele lui Cauchy pentru contururi)**.** *Fie $G \subset \mathbb{C}$ deschisa, $f \in \mathcal{H}(G)$, γ contur din G, $\gamma \underset{G}{\sim} 0$. Atunci :*

$$n(\gamma; z) f^{(k)}(z) = \frac{k!}{2\pi i} \int_{\gamma} \frac{f(\xi)}{(\xi - z)^{k+1}} \ , \ \forall z \in G \setminus \{\gamma\} \ , \ k \in \mathbb{N} \tag{1}$$

2.5 Functii meromorfe

Definitie 8. *Fie $f : \widetilde{G} \mapsto \mathbb{C}$, unde $\widetilde{G}$ multime deschisa din G. Spunem ca f este meromorfa pe $\widetilde{G}$ si notam $f \in \mathcal{M}(\widetilde{G})$ daca $\exists$ o multime E care sa fie alcatuita numai din punctele eliminabile, respectiv poli ai functiei f si f sa fie olomorfa pe $\widetilde{G} \setminus E$.*

Definitie 9. *Fie functia $f \in \mathcal{M}(\widetilde{G})$, unde $\widetilde{G}$ multime deschisa din $\mathbb{C}$, $z_0 \in \widetilde{G}$, $n \in \mathbb{Z}$. Spunem ca $f(z)$ este divizibila cu $(z - z_0)^n$ daca $\exists k > 0$ si o functie h olomorfa pe $\mathcal{U}(z_0; k)$ a.i. $h(z_0) \neq 0$, $\mathcal{U}(z_0; k) \subset \widetilde{G}$ si $f(z) = (z - z_0)^n h(z)$, $\forall z \in \dot{\mathcal{U}}(z_0; k)$.*

Definitie 10. *Numim* ordinul *lui f in z_0:*

$$o(f; z_0) := \max\{n \in \mathbb{Z} \colon f(z) \text{ divizibila cu } (z - z_0)^n\} \tag{2}$$

Teorema 15 (Proprietatii ale ordinului)**.** *Daca $f_1, \ f_2 \in \mathcal{M}(\widetilde{G})$, $z_0 \in \widetilde{G}$, atunci :*

1. $o(f_1 f_2; z_0) = o(f_1; z_0) + o(f_2; z_0)$;
2. $o\left(\frac{f_1}{f_2}; z_0\right) = o(f_1; z_0) - o(f_2; z_0)$;
3. *Daca $D \subset \widetilde{G}$ si $\sum\limits_{z \in D} o(f; z)$ finita , atunci $o(f; D) := \sum\limits_{z \in D} o(f; z)$ si se numeste ordinul functiei f pe D .*

Daca functia $f \in \mathcal{M}(\widetilde{G})$, $\widetilde{G}$ *- multime dechisa din* $\mathbb{C}$ $z_0 \in \widetilde{G}$ *, atunci:*

$$o(f; z_0) = \begin{cases} n, & \textit{daca } z_0 \textit{ este un zerou de ordin n pentru } f \\ 0, & \textit{daca } z_0 \textit{ punct regular pentru } f \textit{ dar nu se anuleaza} \\ -n, & \textit{daca } z_0 \textit{ pol de ordin n pentru } f \end{cases}$$

Definitie 11. $o(f; z) := \infty$*, cand* $f \equiv 0$*, iar* $z \in \widetilde{G}$ *.*

Teorema 16 (Teorema lui Cauchy relativa la zerouri si poli)**.** *Fie* $\widetilde{G}$ *multime deschisa,* $f \in \mathcal{M}(\widetilde{G})$*,* $f \neq 0$ $g \in \mathcal{M}(\widetilde{G})$*,* γ *contur din* $\widetilde{G}$ *care nu trece prin niciun zerou, respectiv pol al functiei* f *a.i.* $\gamma \underset{G}{\sim} 0$*. Atunci :*

$$\sum_{z \in \widetilde{G}} n(\gamma; z) \cdot o(f; z) \cdot g(z) \textit{ este finita si}$$

$$\int_\gamma \frac{f'(z)}{f(z)} g(z) \, \mathrm{d}z = 2\pi i \sum_{z \in G} n(\gamma; z) \cdot o(f; z) \cdot g(z) \ .$$

3 Teorema reziduurilor

3.1 Teorema Reziduurilor

Teorema 17 (Teorema Reziduurilor)**.** *Fie functia $f \in \mathcal{H}(G)$, unde $G \subset \mathbb{C}$ multime deschisa. Notam cu S mutimea tuturor punctelor singulare izolate ale lui f. Fie $\widetilde{G} := G \cup S$, iar γ un contur in G omotop cu zero in $\widetilde{G}$.*

$$\text{Atunci suma: } \sum_{z \in \widetilde{G}} n(\gamma; z) \operatorname{Rez}(f; z) \text{ este finita si}$$

$$\int_\gamma f(z)\, \mathrm{d}z = 2\pi i \sum_{z \in \widetilde{G}} n(\gamma; z) \operatorname{Rez}(f; z)\ .$$

Demonstraţie. $\exists \varphi : [0;1]^2 \mapsto G$ deformare continuua, $k = \varphi([0;1]^2) \subset \widetilde{G}$ compact.

Fie

$$r := \frac{1}{2}\, \mathrm{d}\left(k,\ \mathbb{C} \setminus \widetilde{G}\right)$$

$$D := \bigcup_{z \in k} \mathcal{U}(z; r)$$

$k \subset D \subset \overline{D} \subset \widetilde{G}$

γ omotop cu 0 in D

$\overline{D} \cap S$ finita $\implies \exists \{b_1, \ldots, b_k\} = \overline{D} \cap S$

Fie $\Pi_k(z)$ partea principala a dezvoltarii lui f in b_k

Deci, functia $g := f - \sum_{k=1}^{n} \Pi_k$ olomorfa mai putin in b_k, admite o prelungire olomorfa g_1 la D :

$$\int_\gamma g = \int_\gamma g_1 = 0$$

$$g = g_1|_{D = \{b_1, \ldots, b_k\}}$$

$$\implies \int_\gamma f = \sum_{k=1}^{n} \int_\gamma \Pi_k$$

Calculam :

$$\int_\gamma \Pi_k \text{ , unde } \Pi_k(z) = \sum_{m=1}^{\infty} \frac{a_{-m}^{(k)}}{(z-b_k)^m} \text{ .}$$

Seria este uniform convergenta pe $\forall$ parte compacta din $\mathbb{C} \setminus \{b_k\} \implies$ uniform convergenta pe $\{\gamma\} \implies$ putem integra termen cu termen si

$$\int_\gamma \frac{\mathrm{d}z}{(z-b_k)^m} = 0, \forall m > 1 \text{ .}$$

Functia $\dfrac{1}{(z-b_n)^m}$ admite primitiva si $\displaystyle\int_\gamma \frac{\mathrm{d}z}{z-b_k} = 2\pi i \cdot n(\gamma; b_n) \cdot a_{-1}^{(k)}$ deci

$$\int_\gamma f = 2\pi i \sum_{k=1}^{n} n(\gamma; b_k) \operatorname{Rez}(f; b_n) \text{ .}$$

Trebuie sa mai aratam ca $\forall z_0 \in \widetilde{G} \setminus (D \cap S)$: $n(\gamma; z_0) \cdot \operatorname{Rez}(f; z_0) = 0$.

Intr-adevar, daca pentru $z_0 \in \widetilde{G} \setminus (D \cap S)$ avem $\operatorname{Rez}(f; z_0) \neq 0 \implies z_0 \in S$, deci $z_0 \notin D$ si

$$n(\gamma; z_0) = \frac{1}{2\pi i} \int_\gamma \frac{\mathrm{d}\xi}{\xi - z_0} = 0$$

caci $h(\xi) = \frac{1}{\xi - z_0}$ olomorfa pe D si γ omotop cu zero

$$\implies \int_\gamma f = 2\pi i \sum_{z \in \widetilde{G}} n(\gamma; z) \cdot \operatorname{Rez}(f; z).$$

□

3.2 Puncte singulare izolate

Definitie 12. *Fie $G \subset \mathbb{C}$ multime deschisa si $f \in \mathcal{H}(G)$. Punctul $z_0 \in \mathbb{C}$ se numeste punct singular izolat pentru functia f daca $z_0 \notin G$, dar $\exists p > 0$ a.i $\dot{\mathcal{U}}(z_0; p) \subset G \implies f \in \mathcal{H}(\dot{\mathcal{U}}(z_0; p))$.*

Observatie 1. *De exemplu functiile $\frac{\sin(z)}{z}$, $\frac{1}{z}$, $e^{\frac{1}{z}}$ au singularitati izolate in $z = 0$.*

Observatie 2. *Daca z_0 este un punct singular izolat pentru $f \in \mathcal{H}(G)$, iar $p > 0$ a.i $\dot{\mathcal{U}}(z_0;p) \subset G$, atunci f admite o dezvoltare in serie Laurent de forma :*

$$f(z) = \sum_{k=-\infty}^{\infty} a_k(z - z_0)^k, \quad z \in \dot{\mathcal{U}}(z_0;p).$$

Coeficientul a_{-1} al termenului $(z - z_0)^{-1}$ se numeste reziduul functiei f in z_0 si se noteaza cu $a_{-1} = \mathrm{Rez}(f; z_0)$.

Definitie 13. *Fie $G \subset \mathbb{C}$ multime deschisa, $f \in \mathcal{H}(G)$, iar z_0 punct singular izolat al functiei f. Spunem ca:*

1. *z_0 este punct eliminabil daca f se extinde olomorf la $\Omega \cup \{z_0\}$;*
2. *z_0 este pol daca $lim_{z \to z_0} f(z) = \infty$;*
3. *z_0 este punct esential izolat daca $\nexists$ limita a lui f in z_0;*
4. *Un punct z este regular pentru f daca z este eliminabil pentru f sau f este derivabila in z.*

3.3 Calcularea reziduului intr-un pol

1. Daca z_0 este un pol de ordin k pentru f atunci

$$\mathrm{Rez}(f; z_0) = \frac{1}{(k-1)!} \lim_{z \to z_0} \left[(z - z_0)^k f(z)\right]^{(k-1)}.$$

2. In cazul unui punct singular esential reziduul se calculeaza cu ajutorul dezvoltarii in serie Laurent.

3. Intr-un punct regular reziduul este 0.

4 Aplicatii ale teoremei reziduurilor

4.1 Aplicatii ale teoriei reziduurilor la calculul unor integrale definite reale

Tipul 1 (1). *Fie integrala* $I = \displaystyle\int_0^{2\pi} R(\sin x, \cos x)\,\mathrm{d}x$*, unde* $R(u,v)$ *este o functie rationala reala ce nu are poli pe cercul* $u^2+v^2=1$ *.*

$$\textit{Atunci :} \int_0^{2\pi} R(\sin x, \cos x)\,\mathrm{d}x = 2\pi \sum_{z\in\mathcal{U}(0;1)} \mathrm{Rez}(f;z)$$

$$\textit{unde } f(z) = \frac{1}{z} R\left(\frac{z-\frac{1}{z}}{2i}, \frac{z+\frac{1}{z}}{2}\right)$$

Demonstraţie. Utilizand formulele lui Euler:

$$\cos x = \frac{e^{ix}+e^{-ix}}{2}\ , \quad \sin x = \frac{e^{ix}-e^{-ix}}{2i}\ , \quad x\in\mathbb{R}$$

si substitutia $e^{ix}=z$, avem ca :

$$\int_0^{2\pi} R(\sin x, \cos x)\,\mathrm{d}x = \int_{\partial\mathcal{U}(0;1)} R\left(\frac{z-\frac{1}{z}}{2i}, \frac{z+\frac{1}{z}}{2}\right)\frac{\mathrm{d}z}{iz} \implies$$

$$\int_0^{2\pi} R(\sin x, \cos x)\,\mathrm{d}x = -i\int_{\partial\mathcal{U}(0;1)} f(z)\,\mathrm{d}z \overset{T.Rez}{\Longrightarrow}$$

$$\int_{\partial\mathcal{U}(0;1)} f(z)\,\mathrm{d}z = 2\pi i \sum_{|z|<1} \mathrm{Rez}(f;z) \implies$$

$$\int_0^{2\pi} R(\sin x, \cos x)\,\mathrm{d}x = 2\pi \sum_{|z|<1} \mathrm{Rez}(f;z).$$

□

Tipul 2. *Fie* R *o functie rationala reala ,* $R=P/Q$ *unde* P *si* Q *polinoame de grad* n *, respectiv* m*,* $Q(x)\neq 0 \quad \forall x\in\mathbb{R}$*,* $\lim\limits_{z\to\infty} zf(z)=0, (n\leq m-2)$ *.*

Atunci :

$$\int_{-\infty}^{\infty} R(x)\,\mathrm{d}x = 2\pi i \sum_{\operatorname{Im} z>0} \operatorname{Rez}(f;z).$$

Demonstraţie. $\exists M, r_1 > 0$ a.i :

$$\left|\frac{P(x)}{Q(x)}\right| \leq \frac{M}{|x|^2}, \quad |x| \geq r_1$$

$$\int_{r_1}^{\infty} \frac{1}{x^2}\,\mathrm{d}x \text{ converge } \implies \int_{r_1}^{\infty} \frac{P(x)}{Q(x)}\,\mathrm{d}x \text{ converge .}$$

Analog :

$$\int_{-\infty}^{-r_1} \frac{P(x)}{Q(x)}\,\mathrm{d}x \text{ converge .}$$

$$\text{Dar } \frac{P}{Q} \text{ continua pe } [-r_1, r_1] \implies \exists \int_{-r_1}^{r_1} \frac{P(x)}{Q(x)}\,\mathrm{d}x.$$

$$\int_{-\infty}^{0} \frac{P(x)}{Q(x)}\,\mathrm{d}x \text{ si } \int_{0}^{\infty} \frac{P(x)}{Q(x)}\,\mathrm{d}x \text{ converg } \implies \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\,\mathrm{d}x \text{ converge}$$

Fie $r > 0$ suficient de mare astfel incat toti polii lui f din semiplanul superior sa fie continuti in Ω_r, unde $\Omega_r = \{z \in \mathbb{C} \colon |z| < r, \quad \operatorname{Im} z > 0\}$.

Fie $\gamma_r(t) = re^{\pi i t}, t \in [0;1], \gamma = [-r;r] \cup \gamma_r$.

Atunci $\gamma = \partial\Omega_r$, iar $(\gamma) = \Omega_r \overset{T.Rez}{\Longrightarrow}$

$$\int_{\gamma} f(z)\,\mathrm{d}z = 2\pi i \sum_{z\in\Omega_r} \operatorname{Rez}(f;z) = 2\pi i \sum_{\operatorname{Im} z>0} \operatorname{Rez}(f;z) \qquad (*)$$

Pe de alta parte :

$$\int_{\gamma} f(z)\,\mathrm{d}z = \int_{\gamma_r} f(z)\,\mathrm{d}z + \int_{-r}^{r} f(x)\,\mathrm{d}x \qquad (**)$$

Din $(*)$ si $(**)$ trecand la limita $\implies$

$$2\pi i \sum_{\operatorname{Im} z>0} \operatorname{Rez}(f;z) = \lim_{r\to\infty} \int_{\gamma_r} f(z)\,\mathrm{d}z + \int_{-\infty}^{\infty} f(x)\,\mathrm{d}x$$

$$\text{Dar, } \lim_{z\to\infty} zf(z) = 0 \implies \lim_{r\to\infty}\int_{\gamma_r} f(z)\,\mathrm{d}z = 0$$

$$\implies \int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = 2\pi i \sum_{\mathrm{Im}\, z>0} \mathrm{Rez}(f;z).$$

$\square$

Tipul 3. *Fie R o functie rationala reala de forma $R = \dfrac{P}{Q}$, $Q(x) \neq 0$, $x \in \mathbb{R}$, grad $Q \geq$ grad $P+1$ si $\lim\limits_{|z|\to\infty} R(z) = 0$*

Atunci

$$\int_{-\infty}^{\infty} R(x)e^{ix}\,\mathrm{d}x \textit{ converge si} \int_{-\infty}^{\infty} R(x)e^{ix}\,\mathrm{d}x = 2\pi i \sum_{\mathrm{Im}\, z>0} \mathrm{Rez}(f;z)$$

unde $f(z) = R(z)e^{iz}$.

Demonstraţie. Fie $r>0$ suficient de mare a.i. toti polii functiei f din semiplanul superior sa fie continuti in D, unde $D = \{z \in \mathbb{C}\colon |z| < r;\ \mathrm{Im}\ z > 0\}$.

Fie $C = \partial D \implies C = [-r;r] \cup \gamma_r$

$$\overset{\text{T. Rez}}{\implies} \int_C f(z)\,\mathrm{d}z = 2\pi i \sum_{\mathrm{Im}\ z>0} \mathrm{Rez}(f;z)$$

$$\left.\begin{array}{r} \text{Dar } \displaystyle\int_C f(z)\,\mathrm{d}z = \int_{-r}^{r} f(x)\,\mathrm{d}x + \int_{\gamma_r} f(z)\,\mathrm{d}z \\ r \to \infty \end{array}\right\} \implies$$

$$\implies 2\pi i \sum_{\mathrm{Im}\ z>0} \mathrm{Rez}(f;z) = \int_{-\infty}^{\infty} f(x)dx + \lim_{r\to\infty}\int_{\gamma_r} f(z)\,\mathrm{d}z$$

$$= \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}e^{ix}\,\mathrm{d}x + \lim_{r\to\infty}\int_{\gamma_r} \frac{P(z)}{Q(z)}e^{iz}\,\mathrm{d}z$$

$$g(z) = \frac{P(z)}{Q(z)}$$

deci,

$$\lim_{z\to\infty} g(z) = 0 \overset{L.Jordan}{\Longrightarrow} \lim_{r\to\infty} \int_{\gamma_r} g(z)e^{iz}\ \mathrm{d}z = 0$$

Asadar,

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} e^{ix}\ \mathrm{d}x = 2\pi i \sum_{\mathrm{Im}\ z>0} \mathrm{Rez}(f;z)$$

$\square$

Tipul 4. *Fie integrala*

$$I = \int_{-\infty}^{\infty} f(x)e^{i\alpha x}\ \mathrm{d}x$$

unde $f = P/Q$, $Q(x) \neq 0$, $x \in \mathbb{R}$, *grad* $P = k$, *grad* $Q = p$, *iar* $p \geq k+1$.

Daca $\alpha > 0$, *atunci:*

$$I = \int_{-\infty}^{\infty} f(x)e^{i\alpha x}\ \mathrm{d}x = 2\pi i \sum_{\mathrm{Im}\ z>0} \mathrm{Rez}(g;z)$$

, *unde* $g(z) = f(z)e^{i\alpha z}$.

Demonstrație. Observam ca $\exists \displaystyle\int_{-\infty}^{\infty} f(x)e^{i\alpha x}\ \mathrm{d}x$ si este convergenta. Intradevar, pentru ca $p \geq k+1 \implies \lim_{z\to\infty} f(z) = 0$. Dar $f'(z) = \dfrac{h(z)}{Q^2(z)}$, unde h este un polinom de grad cel mult $k+p-1$.

Fie x_0 zeroul lui h de modul maxim $\implies f'(x)$ are semn constant pentru $x > |x_0| \implies f(x)$ monotona pentru $x > |x_0|$.

Fie $x_1,\ x_2 \in \mathbb{R}$ cu $x_2 > x_1 > |x_0|$

$$\text{Cum } \lim_{z\to\infty} f(z) = 0 \implies \text{fie } f > 0 \text{ si } \lim_{x\to\infty} f(x) = 0^+, x > |x_0|$$

$$\text{fie } f < 0 \text{ si } \lim_{x\to\infty} f(x) = 0^-, x > |x_0|$$

Aplicand a doua teorema de medie din calculul integral $\implies \exists \xi \in (x_1;x_2)$ a.i.

$$\int_{x_1}^{x_2} f(x)\cos\alpha x\ \ \mathrm{d}x = f(x_1)\int_{x_1}^{\xi} \cos\alpha t\ \ \mathrm{d}t + f(x_2)\int_{\xi}^{x_2} \cos\alpha t\ \ \mathrm{d}t$$

$$\Longrightarrow \left| \int_{x_1}^{x_2} f(x) \cos \alpha x \ \ \mathrm{d}x \right| \leq \frac{2}{\alpha} |f(x_1)| + \frac{2}{\alpha} |f(x_2)|$$

Stiind ca $\lim_{z \to \infty} f(z) = 0 \implies \forall \varepsilon > 0 \quad \exists \delta(\varepsilon) > 0$ a.i. $|f(x)| < \dfrac{\varepsilon \alpha}{4}$, $x > \delta(\varepsilon)$

Deci,

$$\left| \int_{x_1}^{x_2} f(x) \cos \alpha x \ \ \mathrm{d}x \right| \leq \frac{2}{\alpha} \left[|f(x_1)| + |f(x_2)| \right] < \varepsilon,$$

$$x_2 > x_1 > \max\{|x_0|, \ \delta(\varepsilon)\} \implies \int_0^\infty f(x) \cos \alpha x \ \ \mathrm{d}x \text{ converge }.$$

Analog $\exists$ si converge :

$$\int_0^\infty f(x) \sin \alpha x \, \mathrm{d}x$$

$$\Longrightarrow \int_0^\infty f(x) e^{i\alpha x} \, \mathrm{d}x$$

este deasemenea convergenta.

Fie $\Omega_r = \{z \in \mathbb{C} \colon |z| < r ; \mathrm{Im} \ \ z > 0\}$ ce contine toti polii functiei g din semiplanul superior.

$$\overset{T.Rez}{\Longrightarrow} \int_{\partial \Omega_r} g(z) \ \mathrm{d}z = 2\pi i \sum_{z \in \Omega_r} \mathrm{Rez}(g; z) = 2\pi i \sum_{\mathrm{Im} \ \ z > 0} Rez(g; z)$$

Dar :

$$\int_{\partial \Omega_r} g(z) \ \mathrm{d}z = \int_{-r}^{r} f(x) e^{i\alpha x} \ \mathrm{d}x + \int_{\gamma_r} g(z) \ \mathrm{d}z$$

$$\overset{L.Jordan}{\Longrightarrow} \lim_{r \to \infty} \int_{\gamma_r} g(z) \ \mathrm{d}z = 0$$

$$\overset{r \to \infty}{\Longrightarrow} \int_{-\infty}^{\infty} f(x) e^{i\alpha x} \ \mathrm{d}x = 2\pi i \sum_{\mathrm{Im} \ \ z > 0} \mathrm{Rez}(g; z).$$

$\square$

Aplicatia 1. *Sa se calculeze integrala :*

$$I = \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{a^4 + x^4} \quad , \ unde \ a > 0.$$

Demonstrație. Este o integrala de tipul II

$$\left.\begin{aligned} P(x) &= 1 \\ Q(x) &= a^4 + x^4 \end{aligned}\right\} grad\ Q > grad\ P + 2$$

$$f(z) = \frac{1}{a^4 + z^4}$$

$$a^4 + z^4 = 0 \implies z^4 = -a^4 = a^4(\cos\pi + i\sin\pi)$$

$$\implies z_k = a\left(\cos\frac{\pi + 2k\pi}{4} + i\sin\frac{\pi + 2k\pi}{4}\right), k = \overline{0,3}$$

unde $z_k, k = \overline{0,3}$ sunt poli simpli pentru f.

$$z_0 = a\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) = \frac{a}{\sqrt{2}}(1+i)$$

$$z_1 = \frac{a}{\sqrt{2}}(-1+i)$$

$$z_2 = \frac{a}{\sqrt{2}}(-1-i)$$

$$z_3 = \frac{a}{\sqrt{2}}(1-i)$$

$$I = 2\pi i \sum_{\mathrm{Im}\ z_k > 0} \mathrm{Rez}(f; z_k)$$

$$\implies I = 2\pi i[\mathrm{Rez}(f; z_0) + \mathrm{Rez}(f; z_1)]$$

$$\mathrm{Rez}(f; z_k) = \lim_{z \to z_k}(z - z_k)\frac{1}{z^4 + a^4} \overset{\frac{0}{0}}{\underset{\mathrm{L'H}}{=}} \lim_{z \to z_k}\frac{1}{4z^3} = -\frac{z_k}{4a^4}$$

Deci,

$$I = 2\pi i\left[\frac{a}{\sqrt{2}}(1 + i - 1 + i)\right] = \frac{2\pi i \cdot a \cdot 2i}{\sqrt{2}} \implies I = -\frac{4\pi a}{\sqrt{2}}$$

□

Aplicatia 2. *Sa se calculeze integrala*

$$I = \int_{-\infty}^{\infty} \frac{\cos x}{x^2 + a^2}\,\mathrm{d}x,\ \textit{unde } a > 0.$$

Demonstraţie. Este o integrala de tip III :

$$\begin{aligned}
\text{Fie } I_1 &= \int_{-\infty}^{\infty} \frac{\cos x}{x^2+a^2}\,\mathrm{d}x \\
\text{si } I_2 &= \int_{-\infty}^{\infty} \frac{\sin x}{x^2+a^2}\,\mathrm{d}x (= 0 \text{ pe ca e impara})
\end{aligned}$$

si fie $I = I_1 + iI_2$

$$\implies I = \int_{-\infty}^{\infty} \frac{e^{ix}}{x^2+a^2}\,\mathrm{d}x$$

$$\left.\begin{aligned} P(x) &= 1 \\ Q(x) &= a^2 + x^2 \end{aligned}\right\} \begin{aligned} &grad\ Q \geq grad\ P + 1 \\ &\qquad 2 \geq 1 \end{aligned}$$

$$f(z) = \frac{e^{iz}}{a^2+z^2}$$

$a^2 + z^2 = 0 \implies z_{1,2} = \pm ia$, dar doar $z_1 = ia$ pol de gradul I $\in$ semiplanul superior

$$\implies I = 2\pi i\ \mathrm{Rez}(f; z_1) = 2\pi i\ \mathrm{Rez}(f; ia)$$

$$\mathrm{Rez}(f; ia) = \lim_{z \to ia} (z - ia) \frac{e^{iz}}{z^2+a^2} = \frac{e^{-a}}{z+ia} = \frac{e^{-a}}{2ia}$$

$$\implies I = 2\pi i \frac{e^{-a}}{2ia} = \frac{e^{-a}\pi}{a}$$

$$I_1 = \mathrm{Re}\ I \quad I_2 = \mathrm{Im}\ I \implies I_1 = \frac{e^{-a}\pi}{a}; \quad I_2 = 0$$

□

Teorema 18. *Fie $f \in \mathcal{M}(\mathbb{C})$ si $z_1, \ldots, z_k$ poli ai functiei f cu reziduurile $u_1, \ldots, u_k$. Daca $f(z) \neq 0$, $z \in \mathbb{Z}$, $z_j \notin \mathbb{Z}$, $j = 1, \ldots, k$, iar $f(z) = O(z^{-2}), z \to \infty$, atunci*

$$\sum_{-\infty}^{\infty} f(\varphi) = -\pi \sum_{j=1}^{k} \mathrm{Rez}(\operatorname{ctg} \pi z \cdot f(z); z_j)$$

Aplicatia 3. *Sa se calculeze*

$$\sum_{n=1}^{\infty} \frac{1}{n^4+1}$$

Demonstraţie. Se vede ca

$$\sum_{-\infty}^{\infty} \frac{1}{n^4+1} = 1 + 2\sum_{n=1}^{\infty} \frac{1}{n^4+1}$$

Fie $f(z) = \dfrac{1}{z^4+1}$, atunci $f \in \mathcal{M}(\mathbb{C})$, cu polii simpli $\pm 1, \pm i$.

$$\mathrm{Rez}(f; z_k) = \lim_{z \to z_k} \frac{z - z_k}{z^4+1} \underset{\mathrm{L'H}}{\overset{\frac{0}{0}}{=}} \lim_{z \to z_k} \frac{1}{4z^3} = \frac{z_k}{-4}$$

$$\begin{aligned}
\sum_{-\infty}^{\infty} \frac{1}{n^4+1} &= -\pi\left[-\frac{1}{4}\operatorname{ctg}\pi + \frac{1}{4}\operatorname{ctg}(-\pi) - \frac{i}{4}\operatorname{ctg} i\pi + \frac{i}{4}\operatorname{ctg}(-i\pi)\right] \\
&= \frac{\pi}{4}[\operatorname{ctg}\pi + \operatorname{ctg}\pi + i\operatorname{ctg} i\pi + i\operatorname{ctg} i\pi] \\
&= \frac{\pi}{2}\operatorname{ctg}\pi + \frac{\pi}{2}\operatorname{cth}\pi
\end{aligned}$$

$$\text{Deci, } 1 + 2\sum_{n=1}^{\infty} \frac{1}{n^4+1} = \frac{\pi}{2}\operatorname{ctg}\pi + \frac{\pi}{2}\operatorname{cth}\pi$$

$$\Longrightarrow \sum_{n=1}^{\infty} \frac{1}{n^4+1} = \frac{\pi}{4}[\operatorname{ctg}\pi + \operatorname{cth}\pi] - \frac{1}{2}$$

□

4.2 Calcularea unei integrale pe un arc de curba simplu si rectificabil, dar nu inchis

In acest caz putem incerca sa formam o curba inchisa $\gamma_0 \cup \gamma_1$ a.i. sa poata sa se aplice teorema reziduurilor , iar integrala pe noua curba $\gamma = \gamma_0 \cup \gamma_1$ sa se poata calcula cu reziduuri direct sau sa aiba o relatie simpla cu integrala cautata.

Daca integrala este improprie , fiind limita unei alte integrale

$$\int_{\gamma_0} = \lim_{\gamma \to \gamma_0} \int_{\gamma}$$

atunci si arcul adaugat va varia si vom putea calcula integrala improprie cunoscand limita $\int_{\gamma_1}$ si daca suma reziduurilor din domeniu G variabil are limita cunoscuta:

$$\int_{\gamma_0} f \, \mathrm{d}z = -\lim \int_{\gamma_1} f \, \mathrm{d}z + 2\pi i \lim \sum \mathrm{Rez}(f; z)$$

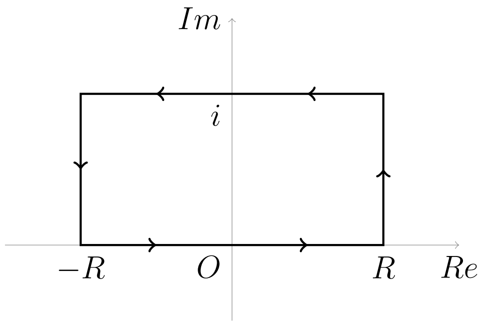
Aplicatia 4. *Sa se calculeze*

$$I = \int_0^{\infty} \frac{\cos ax}{\operatorname{ch} \pi x} \, \mathrm{d}x, a \in \mathbb{R}$$

Demonstraţie.

$$f(z) = \frac{\cos az}{\operatorname{ch} \pi z}$$

Polii acestei functii sunt simpli , $z = (2k+1)\frac{i}{2}$, $k \in \mathbb{Z}$ Pentru a evita seria de reziduuri care este divergenta alegem conturul



Pe latura $z = R + iy \quad (0 \le y \le 1)$

$$\left| i \int_0^1 \frac{\cos a(R+iy)}{\operatorname{ch} \pi(R+iy)} \, \mathrm{d}y \right| = \left| \int_0^1 \frac{e^{ia(R+iy)} + e^{-ia(R+iy)}}{e^{\pi(R+iy)} + e^{-\pi(R+iy)}} \, \mathrm{d}y \right|$$
$$< \frac{\int_0^1 (e^{-ay} + e^{ay}) \, \mathrm{d}y}{e^{\pi R} - e^{-\pi R}} \to 0$$

Ramane : $2\int_0^R \frac{\cos ax}{\operatorname{ch}\pi x}\,\mathrm{d}x + \int_0^R \left[-\frac{\cos a(i+x)}{\operatorname{ch}\pi(i+x)} - \frac{\cos a(i-x)}{\operatorname{ch}\pi(i-x)}\right]\,\mathrm{d}x \longrightarrow 2\pi i\,\mathrm{Rez}\left(f;\frac{i}{2}\right)$

Stiind ca :

$$\operatorname{ch}\pi(x\pm i) = -\operatorname{ch}\pi x$$
$$\cos a(x\pm i) = \cos ax\cdot\operatorname{ch}a \mp \sin ax\cdot\operatorname{sh}a$$

obtinem ca :

$$2(1+\operatorname{ch}a)\int_0^R \frac{\cos ax}{\operatorname{ch}\pi x}\,\mathrm{d}x \longrightarrow 2\pi i\,\mathrm{Rez}\left(f;\frac{i}{2}\right)$$
$$\mathrm{Rez}\left(f;\frac{i}{2}\right) = \lim_{z\to\frac{i}{2}}\left(z-\frac{i}{2}\right)\frac{\cos az}{\operatorname{ch}\pi z} = \frac{\cos\frac{ai}{2}}{\pi\operatorname{sh}\frac{\pi i}{2}} = \frac{\operatorname{ch}\frac{a}{2}}{\pi i\operatorname{sh}\frac{\pi}{2}} = \frac{\operatorname{ch}\frac{a}{2}}{\pi i}$$
$$\Longrightarrow\ I = 2\pi i\frac{\operatorname{ch}\frac{a}{2}}{\pi i} = 2\operatorname{ch}\frac{a}{2}$$

□

4.3 Aplicatii la dezvoltari in serie

Teorema 19. *Fie $f(z)$ o functie meromorfa ai carei poli formeaza un sir infinit $z_k\to\infty$ si D_n un domeniu marginit de o curba rectificabila γ_n si care nu trece prin nici un pol z_k.*

$$\textit{Atunci : }\int_{\gamma_n} f(z)\,\mathrm{d}z = 2\pi i\sum_{k=1}^{n}\mathrm{Rez}(f;z_k).$$

Observatie 3.

1. *Daca $n\to\infty, \gamma_n$ variaza a.i. D_n tinde catre un domeniu ce cuprinde toti polii a_n. Daca integrala din membrul* I *are o limita finita, atunci obtinem suma seriei de reziduuri $\sum_{k=1}^{\infty}\mathrm{Rez}(f;z_k)$ insumata dupa domeniul D_n.*

2. Daca indicele k ia valorile $1, 2, \cdots$ si $|z_k|$ sunt strict crescatoare $|z_1| < |z_2| < \cdots$, a.i. intre 2 curbe consecutive sa se afle un singur pol, vom obtine suma seriei convergente $\sum_{k=1}^{\infty} \mathrm{Rez}(f; z_k)$.

3. Daca $|z_k|$ si $|z_{-k}|$ sunt crescatori vom putea obtine suma seriei convergente $\mathrm{Rez}(f; z_0) + \sum_{k=1}^{n} [\mathrm{Rez}(f; z_k) + \mathrm{Rez}(f; z_{-k})]$ adica :

$$\sum_{-\infty}^{\infty} \mathrm{Rez}(f; z_k) = \frac{1}{2\pi i} \lim_{k \to \infty} \int_{\gamma_k} f(z)\, \mathrm{d}z$$

4. Fie $f(z)$ o functie mereomorfa avand polii de gradul I , $z_k \to \infty$ si $g(z)$ o functie uniforma cu un numar finit de puncte singulare a_h, diferite de z_k . Fie γ_n cu $n > n_0$ ce contine punctele a_h in interiorul sau. Atunci pentru functia $f(z) \cdot g(z)$ avem ca

$$\mathrm{Rez}(f \cdot g; z_k) = g(z_k)\, \mathrm{Rez}(f; z_k)$$

Formula din Obs 3 se transforma astfel

$$\sum_{-\infty}^{\infty} \mathrm{Rez}(f; z_k) g(z_k) = \frac{1}{2\pi i} \lim_{k \to \infty} \int_{\gamma_k} f(z) g(z)\, \mathrm{d}z - \sum_{k \in \mathbb{C}} \mathrm{Rez}(f \cdot g; a_h)$$

A doua suma este nula pentru $g(z)$ functie intreaga

Aplicatia 5. *Sa se calculeza integrala*

$$I = \int_0^{2\pi} \frac{\mathrm{d}x}{a + \cos x}, a > 1$$

Demonstraţie. Se observa ca I este o integrala de tipul I. Din formulele lui Euler stim ca:

$$\cos z = \frac{e^{ix} + e^{-ix}}{2}, \quad e^{ix} = z \implies \mathrm{d}x = \frac{\mathrm{d}z}{iz} \implies \cos x = \frac{z + \frac{1}{z}}{2}$$

$$f(z) = \frac{1}{z} \frac{1}{a + \frac{z + 1/z}{2}} \implies f(z) = \frac{1}{z^2 + 2az + 1}$$

$$I=\int_{\partial\mathcal{U}(0;1)}\frac{\frac{\mathrm{d}z}{iz}}{a+\frac{z+1/z}{2}}=-2i\int_{\partial\mathcal{U}(0;1)}\frac{\mathrm{d}z}{z^2+2az+1}$$

$$z^2+2az+1=0 \implies \Delta=4a^2-4$$

$$\implies \begin{cases} z_1=\dfrac{-2a+\sqrt{4a^2-4}}{2}=-a+\sqrt{a^2-1}\\ z_2=\dfrac{-2a-\sqrt{4a^2-4}}{2}=-a-\sqrt{a^2-1}\end{cases}$$

$$\begin{aligned}|z_1|<1 &\iff |-a+\sqrt{a^2-1}|=a-\sqrt{a^2-1}<1\\ &\iff a-1<\sqrt{a^2-1}\,\Big|^2 \iff a^2-2a+1<a^2-1\\ &\iff 2a>0 \text{ Adevarat.}\\ |z_2|<1 &\iff |-a-\sqrt{a^2-1}|<1 \text{ Fals . } \implies z_2\notin\mathcal{U}(0;1)\end{aligned}$$

Deci, $I=2\pi\,\mathrm{Rez}(f;z_1)$ cu z_1 pol simplu :

$$\begin{aligned}\mathrm{Rez}(f;z_1)&=\lim_{z\to z_1}(z-z_1)\frac{1}{z^2+2az+1}\overset{\frac{0}{0}}{\underset{\mathrm{L'H}}{=}}\lim_{z\to z_1}\frac{1}{2z+2a}\\ &=\frac{1}{2(a+\sqrt{a^2-1})+2a}=-\frac{1}{2\sqrt{a^2-1}}\end{aligned}$$

Asadar ,

$$I=2\pi\frac{1}{2\sqrt{a^2-1}}=\frac{\pi}{\sqrt{a^2-1}}$$

□

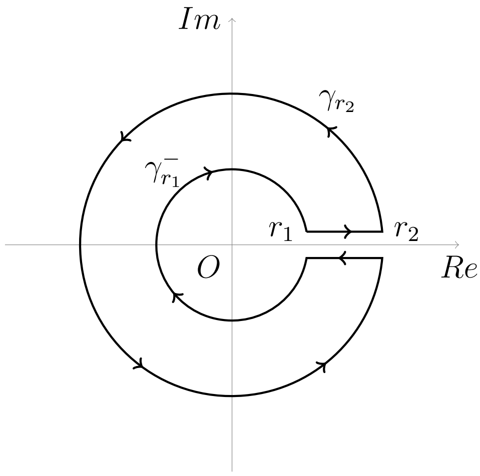
Teorema 20. *Fie functia reala rationala $f=P/Q$, neavand poli pe $[0;1]$, fie $0<\alpha<1$ si $\lim\limits_{z\to\infty}f(z)=0$. Atunci avem ca :*

$$\int_0^\infty\frac{f(x)}{x^\alpha}\,\mathrm{d}x=\frac{\pi e^{\alpha\pi i}}{\sin\alpha\pi}\sum_{z\in\mathbb{C}^*}\mathrm{Rez}(h;z),$$

$$\textit{unde } h(z)=\frac{f(z)}{z^\alpha}\ ,\ \textit{iar } z^\alpha=e^{\alpha\log z},$$

cu $\log z$ ramura uniforma a aplicatiei multivoce Logaritm.

Demonstrație. Fie Γ conturul din imagine:



Atunci :

$$\int_\Gamma g(z)\,\mathrm{d}z = 2\pi i \sum_{z\in\mathbb{C}^*} \mathrm{Rez}(g;z)$$

si :

$$\int_\Gamma g(z)\,\mathrm{d}z = \int_{r_1}^{r_2} \frac{f(x)}{x^\alpha}\,\mathrm{d}x + \int_{\gamma_{r_2}} \frac{f(z)}{z^\alpha}\,\mathrm{d}z - \int_{r_1}^{r_2} \frac{f(x)}{e^{\alpha[\ln x + 2\pi i]}}\,\mathrm{d}x - \int_{\gamma_{r_1}} \frac{f(z)}{z^\alpha}\,\mathrm{d}z$$

Deci :

$$(*)\quad 2\pi i \sum_{z\in\mathbb{C}^*} \mathrm{Rez}(g;z) = \int_{\gamma_{r_2}} \frac{f(z)}{z^\alpha}\,\mathrm{d}z - \int_{\gamma_{r_1}} \frac{f(z)}{z^\alpha}\,\mathrm{d}z + (1-e^{-2\pi i\alpha}) \int_{r_1}^{r_2} \frac{f(x)}{x^\alpha}\,\mathrm{d}x$$

Cum $\lim\limits_{z\to\infty} f(z) = 0$ urmeaza ca $p \le k+1$, unde k si p sunt gradele polinoamelor P respectiv Q. Deoarece $\alpha \in (0;1)$, obtinem imediat ca :

$$\lim_{z\to\infty} zg(z) = \lim_{z\to\infty} z^{1-\alpha}\frac{P(z)}{Q(z)} = 0 \text{ si } \lim_{z\to 0} zg(z) = \lim_{z\to 0} z^{1-\alpha}\frac{P(z)}{Q(z)} = 0.$$

Trecand la limita in $(*)$ pentru $r_1 \to 0$ si $r_2 \to \infty$ deducem concluzia teoremei. □

Aplicatia 6. *Sa se calculeze integrala :*

$$I = \int_0^\infty \frac{\mathrm{d}x}{\sqrt[3]{x}(x^5+a^5)}, a > 0.$$

Demonstrație. Aceasta integrala este de tip V.

$$f(x) = \frac{1}{x^5 + a^5}$$
$$h(z) = \frac{1}{z^{1/3}(z^5 + a^5)}$$

$$z^5 + a^5 = 0 \implies z^5 = -a^5$$
$$z_k = a\left(\cos\frac{\pi + 2k\pi}{5} + i\sin\frac{\pi + 2k\pi}{5}\right), k = \overline{0,4}$$

unde z_k sunt poli simpli.

$$I = \frac{\pi e^{\frac{\pi i}{3}}}{\sin\frac{\pi}{3}} \sum_{k=0}^{4} \mathrm{Rez}(h; z_k)$$

$$\begin{aligned}
\mathrm{Rez}(h; z_k) &= \lim_{z\to z_k}(z - z_k)\frac{1}{z^{1/3}(z^5 + a^5)} \overset{\frac{0}{0}}{\underset{\text{L'H}}{=}} \frac{1}{z_k^{1/3}} \lim_{z\to z_k}\frac{1}{5z^4} = \frac{1}{z_k^{1/3}} \lim_{z\to z_k}\frac{z}{5z^4} \\
&= \frac{1}{z_k^{1/3}}\frac{z_k}{5a^5} = -\frac{z_k^{2/3}}{5a^5} = -\frac{1}{5a^5}e^{\frac{2}{3}\log z_k} = -\frac{1}{5a^5}e^{\frac{2}{3}[\ln a + i\theta(z)]} \\
&= -\frac{1}{5a^5}e^{\frac{2}{3}\left[\ln a + i\frac{\pi+2k\pi}{5}\right]} = -\frac{a^{2/3}}{5a^5}e^{\frac{2}{3}\frac{i\pi+2k\pi}{5}} \\
I &= -\frac{\pi e^{\frac{\pi i}{3}}}{\frac{\sqrt{3}}{2}}\frac{a^{2/3}}{5a^5}e^{i\frac{2}{3}}\left[e^{\frac{\pi}{5}} + e^{\frac{3\pi}{5}} + e^{\frac{5\pi}{5}} + e^{\frac{7\pi}{5}} + e^{\frac{9\pi}{5}}\right]
\end{aligned}$$

□

4.4 Aplicatii in teoria functiilor

Urmatorul rezultat face legatura intre numarul de zerouri si numarul de poli ai unei functii analitice.

Teorema 21. *Fie $D \subset \mathbb{C}$ domeniu stelat si $f \in \mathcal{M}(D)$ cu zerouurile : $a_1, \cdots, a_n \in D$, si polii $b_1, \cdots, b_m \in D$. Atunci pentru orice contur γ din D ce evita toate zerourile si toti polii lui f avem:*

$$\frac{1}{2\pi i}\int_\gamma \frac{f'}{f}(\zeta)\,\mathrm{d}\zeta = \sum_{k=1}^{n} o(f; a_k)\, n(\gamma; a_k) + \sum_{l=1}^{m} o(f; b_l)\, n(\gamma; b_l)$$

O aplicatie a teoremei anterioare e teorema:

Teorema 22 (Hurwitz)**.** *Fie $f_1, f_2, \cdots : D \mapsto \mathbb{C}$ un sir de functii ce converge local uniform la functia analitica $f : D \mapsto \mathbb{C}$. Daca $\forall i$, f_i nu e identic nula pe D, atunci f fie e identic nula, fie nu are nici un zerou in D.*

Teorema 23. *Fie $D \in \mathbb{C}$ domeniu stelat si $f \in \mathcal{M}(D)$ cu zerouurile : $a_1, \cdots, a_n \in D$, si polii $b_1, \cdots, b_m \in D$. Notam:*

$$N(0) := \sum_{k=1}^{n} o(f; a_k) \qquad \textit{numarul tuturor zerourilor lui } f \ ;$$

$$N(\infty) := -\sum_{l=1}^{m} o(f; b_l) \qquad \textit{numarul tuturor polilor lui } f \ ;$$

numarand multiplicitatile. Fie γ din D ce inconjoara cu index 1 toate zerourile si toti polii. Atunci avem:

$$\frac{1}{2\pi i}\int_\gamma \frac{f'}{f}(\zeta)\, \mathrm{d}\zeta = N(0) - N(\infty).$$

Daca f nu are poli obtinem o formula pentru numarul de zerouri intr-un domeniu.

Aplicatia 7 (Teorema fundamentala a algebrei)**.** *Orice polinom $P(z)$ de grad n cu coeficienti complecsi, are exact n radacini complexe.*

Demonstraţie. Deoarece $\lim_{|z|\to\infty} P(z) = \infty$, $\exists R > 0$ a.i. P nu are radacini z cu $|z| \geq R$. Numarul de zerouri a lui P e :

$$N(0) = \frac{1}{2\pi i}\int_{|\zeta|=R} \frac{P'(\zeta)}{P(\zeta)}\, \mathrm{d}\zeta.$$

Functia P'/P are in ∞ un zerou simplu. Seria Laurent in ∞ e de forma :

$$\frac{n}{z} + \frac{c_2}{z^2} + \frac{c_3}{z^3} + \cdots (n = grad(P)).$$

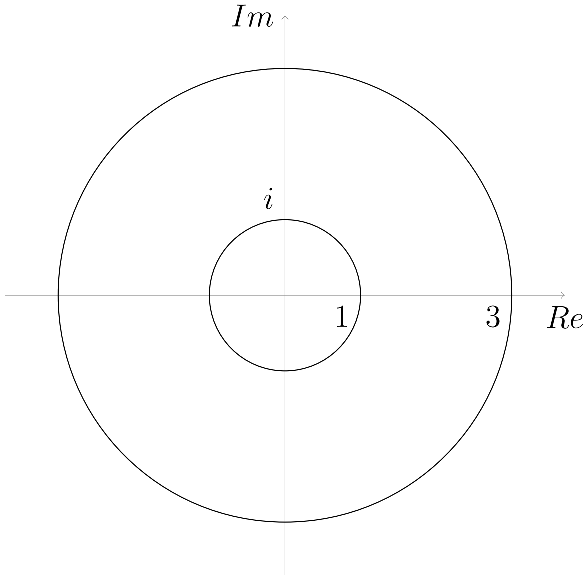
Deci:

$$N(0) = n = grad(P).$$

□

Teorema 24 (Rouche)**.** *Fie f, $g : D \mapsto \mathbb{C}$ analitice si γ un contur din D ce inconjoara orice punct din interiorul sau exact o data. Daca $|g(\zeta)| < |f(\zeta)| \forall \zeta \in \{\gamma\}$ atunci f, $f+g$ nu au zerouri in $\{\gamma\}$ si au in interiorul lui γ acelasi numar de zerouri considerand multiplicitatile.*

Aplicatia 8. *Sa se determine numarul solutiilor ecuatiei $z^4 - 8z + 10 = 0$ in $\mathcal{U}(0;1;3)$.*



Demonstraţie.

$$\mathcal{U}(0;1;3) = \mathcal{U}(0;3) \setminus \big(\mathcal{U}(0;1) \cup \partial\mathcal{U}(0;1)\big)$$

$$\begin{aligned} N_1 &:= \text{ numarul solutiilor ecuatiei in } \mathcal{U}(0;3) \\ N_2 &:= \text{ numarul solutiilor ecuatiei in } \overline{\mathcal{U}}(0;1) \\ N &:= \text{ numarul solutiilor ecuatiei in } \mathcal{U}(0;1;3) \\ N &= N_1 - N_2 \end{aligned}$$

Determinam N_1 : Avem $|z| = 3$.

Alegem $f(z) = z^4$ si $g(z) = -8z + 10$.

$$\begin{aligned}
|z^4| &= 3^4 = 81 \\
|-8z+10| &\leq 8|z| + 10 = 24 + 10 = 34 < 81 \implies \\
|g(z)| &< |f(z)| \text{ pentru } |z| = 3 \overset{T.Rouche}{\implies} \\
f(z) = 0 &\text{ si } f(z) + g(z) = 0 \text{ au acelasi numar de solutii in } \mathcal{U}(0;3) \\
&\implies N_1 = 4
\end{aligned}$$

Determinam N_2 :

$$\begin{aligned}
N_2' &:= \text{ numarul solutiilor ecuatiei in } \mathcal{U}(0;1) \\
N_2'' &:= \text{ numarul solutiilor ecuatiei pe } \partial\mathcal{U}(0;1) \\
N_2 &= N_2' + N_2''
\end{aligned}$$

$$\begin{aligned}
|z| &= 1 \\
f(z) &= -8z + 10 \implies g(z) = z^4 \\
|f(z)| &\leq 8|z| + 10 = 18 \\
|g(z)| &= |z^4| = 1 < 18 \overset{T.Rouche}{\implies} \\
f(z) &= 0 \text{ si } f(z) + g(z) = 0 \text{ au acelasi numar de solutii} \\
-8z + 10 = 0 &\implies z = 2 > 1 \implies N_2 = 0 \text{ numar de solutii in } \mathcal{U}(0;1). \\
|f(z) + g(z)| \geq |f(z)| + |g(z)| &> 0, |z| = 1 \implies N_2'' = 0.
\end{aligned}$$

Deci $N = 4 - 0 = 4$. □

Aplicatia 9. *Fie* $P_n(z) = a_0 + a_1 z + \cdots + a_n z^n$, $z \in \mathbb{C}$, *unde* $a_n \neq 0$.

$$\textit{Fie } \alpha_n = \frac{\sum_{k=0}^{n-1} |a_k|}{|a_n|} \text{ , } \textit{si} \quad r > \max\{\alpha_n, 1\}.$$

Sa se arate ca toate solitiile polinomului $P_n \in \mathcal{U}(0;r)$.

Demonstraţie. Fie :

$$f(z) := a_n z^n$$
$$g(z) := a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} = P_n(z) - f(z)$$

Pentru $|z| = r$ avem:

$$|f(z)| = a_n |r|^n = |a_n| r^n$$
$$|g(z)| = |a_0 + a_1 z + \cdots + a_{n-1} z^{n-1}|$$
$$\leq |a_0| + |a_1||z| + \cdots + |a_{n-1}||z|^{n-1}$$
$$= |a_0| + |a_1| r + \cdots + |a_{n-1}| r^{n-1}$$

Deoarece $r^k \leq r^{n-1}\ \forall k = \overline{0, n-1}$

$$|g(z)| \leq r^{n-1}(|a_0| + |a_1| + \cdots + |a_{n-1}|) = r^{n-1} \sum_{k=0}^{n-1} |a_k| \implies$$
$$|g(z)| \leq r^{n-1} \alpha_n |a_n| = \frac{\alpha_n r^n |a_n|}{r} = \frac{\alpha_n}{r} |f(z)|$$

Cum $\dfrac{\alpha_n}{r} < 1$ avem ca :

$$|g(z)| < |f(z)|\ , \quad |z| = r \overset{T.Rouche}{\Longrightarrow}$$

$f(z) = 0$ si $f(z) + g(z) = 0$ au acelasi numar de solutii in $\mathcal{U}(0; r)$.

$$\begin{cases} f(z) = 0 \\ P_n(z) = 0 \end{cases} \text{au acelasi numar de solutii} \implies N = n.$$

$\square$

Bibliografie

[1] Octavian Mayer. *Functii de o variabila complexa.*

[2] P. Hamburg si P. Mocanu si N. Negoescu. *Analiza Matematica (functii complexe).* Editura didactica si pedagogica, 1982.

[3] Gabriela Kohr si Petru T. Mocanu. *Capitole speciale de analiza complexa.* Cluj University Press, 2005.

[4] Eberhart Freitag si Rolf Busam. *Complex Analysis.* Springer, 2005.