# Teorema Reziduurilor

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**Rezumat**

Aplicatii ale teoremei reziduurilor in calulul unor chestii interesante. In prima parte avem introducere apoi exemple din x urmate de aplicatii de tip y.

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# 1 Teorema Reziduurilor

**Teorema 1.** *Fie functia $f \in \mathcal{H}(G)$, unde $G \subset \mathbb{C}$ multime deschisa. Notam cu $\rho$ mutimea tuturor punctelor singulare izolate ale lui $f$ Fie $\widetilde{G} := G \cup S$, iar $\gamma$ un contur in $G$ omotop cu zero in $\widetilde{G}$*

$$\textit{Atunci suma: } \sum_{z \in \widetilde{G}} n(\gamma; z) Rez(f; z) \textit{ este finita si}$$

$$\int_\gamma f(z)\,\mathrm{d}z = 2\pi i \sum_{z \in \widetilde{G}} n(\gamma; z) Rez(f; z)$$

*Demonstraţie.* $\exists \varphi : [0;1]^2 \mapsto G$ deformare continuua, $k = \varphi([0;1]^2) \subset \widetilde{G}$ compact.

Fie

$$r := \frac{1}{2}\,\mathrm{d}\left(k, \mathbb{C} \setminus \widetilde{G}\right)$$
$$D := \bigcup_{z \in k} \mathcal{U}(z; r)$$

$k \subset D \subset \overline{D} \subset \widetilde{G}$

$\gamma$ omotop cu 0 in $D$

$\overline{D} \cap \rho$ finita $\implies \exists \{b_1, \ldots, b_k\} = \overline{D} \cap \rho$

Fie $\Pi_k(z)$ partea principala a dezvoltarii lui $f$ in $b_k$

Deci, functia $g := f - \sum_{k=1}^{n} \Pi_k$ olomorfa mai putin in $b_k$ admite o prelungire olomorfa $g_1$ la $D$ .

$$\int_\gamma g = \int_\gamma g_1 = 0$$
$$g = g_1|_{D = \{b_1, \ldots, b_k\}}$$
$$\implies \int_\gamma f = \sum_{k=1}^{n} \int_\gamma \Pi_k$$

Calculam

$$\int_\gamma \Pi_k \text{ , unde } \Pi_k(z) = \sum_{m=1}^{\infty} \frac{a^{(k)} - m}{(z - b_k)^m}$$

Seria este uniform convergenta pe $\forall$ parte compacta din $\mathbb{C} \setminus \{b_a\} \implies$ uniform convergenta pe $\{\gamma\} \implies$ putem integra termen cu termen si

$$\int_\gamma \frac{\mathrm{d}}{z - b_k} m = 0, \forall m > 1$$

Functia $\dfrac{1}{(z - b_n)^m}$ admite primitiva si $\displaystyle\int_\gamma \frac{\mathrm{d}z}{z - b_k} = 2\pi i \cdot n(\gamma; b_n) \cdot a_{-1}^{(k)}$ deci

$$\int_\gamma f = 2\pi i \sum_{k=1} nn(\gamma; b_k) Rez(f; b_n)$$

Trebuie sa mai aratam ca $\forall z_0 \in \widetilde{G} \setminus (D \cap \rho)\colon n(\gamma; z_0) \cdot Rez(f; z_0) = 0$

Intr-adevar, daca pentru $z_0 \in \widetilde{G} \setminus (D \cap \rho)$ avem $Rez(f; z_0) \neq 0 \implies z_0 \in \rho$, deci $z_0 \notin D$ si

$$n(\gamma; z_0) = \frac{1}{2\pi i} \int_\gamma \frac{\mathrm{d}\xi}{\xi - z_0} = 0$$

caci $h(\xi) = \frac{1}{\xi - z_0}$ olomorfa pe $D$ si $\gamma$ omotop cu zero

$$\implies \int_\gamma f = 2\pi i \sum_{z \in \widetilde{G}} n(\gamma; z) \cdot Rez(f; z)$$

□

## 2 Puncte singulare izolate

**Definitie 1.** *Fie $G \subset \mathbb{C}$ multime deschisa si $f \in \mathcal{H}(G)$. Punctul $z_0 \in \mathbb{C}$ se numeste punct singular izolat pentru functia $f$ daca $z_0 \notin G$, dar $\exists p > 0$ a.i $\dot{\mathcal{U}}(z_0; p) \subset G \implies f \in \mathcal{H}(\dot{\mathcal{U}}(z_0; p))$*

**Observatie 1.** *De exemplu functiile $\frac{\sin(z)}{z}$ , $\frac{1}{z}$, $e^{\frac{1}{z}}$ au singularitati izolate in $z = 0$*

**Observatie 2.** *Daca $z_0$ este un punct singular izolat pentru $f \in \mathcal{H}(G)$, iar $p > 0$ a.i $\dot{\mathcal{U}}(z_0; p) \subset G$ , atunci $f$ admite o dezvoltare in serie Laurent de forma*

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k, \quad z \in \dot{\mathcal{U}}(z_0; p)$$

*Coeficientul $a_{-1}$ al termenului $(z - z_0)^{-1}$ se numeste reziduul functiei $f$ in $z_0$ si se noteaza cu $a_{-1} = Rez(f; z_0)$*

**Definitie 2.** *Fie $G \subset \mathbb{C}$ multime deschisa, $f \in \mathcal{H}(G)$ , iar $z_0$ punct singular izolat al functiei $f$. Spunem ca:*

1. *$z_0$ este punct eliminabil daca $f$ se extinde olomorf la $\Omega \cup \{z_0\}$*
2. *$z_0$ este pol daca $lim_{z \to z_0} f(z) = \infty$*
3. *$z_0$ este punct esential izolat daca $\nexists$ limita a lui $f$ in $z_0$*
4. *Un punct $z$ este regular pentru $f$ daca $z$ este eliminabil pentru $f$ sau $f$ este derivabila in $z$*

## 3 Calcularea reziduului intr-un pol

1. Daca $z_0$ este un pol de ordin $k$ pentru $f$ atunci

$$Rez(f; z_0) = \frac{1}{(k-1)!} \lim_{z \to z_0} \left[ (z - z_0)^k f(z) \right]^{(k-1)}$$

2. In cazul unui punct singular esential reziduul se calculeaza cu ajutoril dezvoltarii in serie Laurent
3. Intr-un punct regular reziduul este 0

# 4 Aplicatii ale teoriei reziduurilor la calculul unor integrale definite reale

**Tipul 1** (1). *Fie integrala* $I = \displaystyle\int_0^{2\pi} R(\sin x, \cos x)\ \mathrm{d}x$, *unde* $R(u, v)$ *este o functie rationala reala ce nu are poli pe cercul* $u^2 + v^2 = 1$

$$\text{Atunci} \int_0^{2\pi} R(\sin x, \cos x)\ \mathrm{d}x = 2\pi i \sum_{z \in \mathcal{U}(0;1)} Rez(f; z)$$

$$\text{unde } f(z) = \frac{1}{z} R\left(\frac{z - \frac{1}{z}}{2i}, \frac{z + \frac{1}{z}}{2}\right)$$

*Demonstraţie.* Utilizand formulele lui Euler:

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}, \sin x = \frac{e^{ix} - e^{-ix}}{2i} x \in \mathbb{R}$$

si substitutia $e^{ix} = z$, avem ca

$$\int_0^{2\pi} R(\sin x, \cos x)\ \mathrm{d}x = \int_{\partial\mathcal{U}(0;1)} R\left(\frac{z - \frac{1}{z}}{2i}, \frac{z + \frac{1}{z}}{2}\right) \frac{\mathrm{d}z}{iz} \implies$$

$$\int_0^{2\pi} R(\sin x, \cos x)\ \mathrm{d}x = -i \int_{\partial\mathcal{U}(0;1)} f(z)\ \mathrm{d}z \overset{T.Rez}{\implies}$$

$$\int_{\partial\mathcal{U}(0;1)} f(z)\ \mathrm{d}z = 2\pi i \sum_{|z|<1} Rez(f; z) \implies$$

$$\int_0^{2\pi} R(\sin x, \cos x)\ \mathrm{d}x = 2\pi \sum_{|z|<1} Rez(f; z)$$

□

**Tipul 2.** *Fie* $R$ *o functie rationala reala ,* $R = P/Q$ *unde* $P$ *si* $Q$ *polinoame de grad* $n$ *, respectiv* $m$, $Q(x) \neq 0 \quad \forall x \in \mathbb{R}$, $\lim\limits_{z\to\infty} zf(z) = 0, (n \leq m - 2)$

*Atunci*

$$\int_{-\infty}^{\infty} R(x)\ \mathrm{d}x = 2\pi i \sum_{Imz=0} Rez(f; z)$$

*Demonstraţie.* $\exists M, r_1 > 0$ a.i

$$\left|\frac{P(x)}{Q(x)}\right| \leq \frac{M}{|x|^2}, \quad |x| \geq r_1$$

$$\int_{r_1}^{\infty} \frac{1}{x^2}\ \mathrm{d}x \text{ converge } \implies \int_{r_1}^{\infty} \frac{P(x)}{Q(x)}\ \mathrm{d}x \text{ converge}$$

Analog

$$\int_{-\infty}^{-r_1} \frac{P(x)}{Q(x)}\ \mathrm{d}x \text{ converge}$$

$$\text{Dar } \frac{P}{Q} \text{ continuua pe } [-r_1, r_1] \implies \exists \int_{-r_1}^{r_1} \frac{P(x)}{Q(x)}\, \mathrm{d}x$$

$$\int_{-\infty}^{0} \frac{P(x)}{Q(x)}\, \mathrm{d}x \text{ si } \int_{0}^{\infty} \frac{P(x)}{Q(x)}\, \mathrm{d}x \text{ converg } \implies \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\, \mathrm{d}x \text{ converge}$$

Fie $r > 0$ suficient de mare astfel incat toti polii lui $f$ din semiplanul superior sa fie continuti in $\Omega_r$, unde $\Omega_r = \{z \in \mathbb{C} \colon |z| < r, \quad Imz > 0\}$ .

Fie $\gamma_r(t) = re^{\pi i t}, t \in [0; 1], \gamma = [-r; r] \cup \gamma_r$.

Atunci $\gamma = \partial\Omega_r$, iar $(\gamma) = \Omega_r \overset{T.Rez}{\implies}$

$$\int_{\gamma} f(z)\, \mathrm{d}z = 2\pi i \sum_{z \in \Omega_r} Rez(f; z) = 2\pi i \sum_{Imz > 0} Rez(f; z) \qquad (*)$$

Pe de alta parte

$$\int_{\gamma} f(z)\, \mathrm{d}z = \int_{\gamma_r} f(z)\, \mathrm{d}z + \int_{-r}^{r} f(x)\, \mathrm{d}x \qquad (**)$$

Din $(*)$ si $(**)$ trecand la limita $\implies$

$$2\pi i \sum_{Imz>0} Rez(f; z) = \lim_{r \to \infty} \int_{\gamma_r} f(z)\, \mathrm{d}z + \int_{-\infty}^{\infty} f(x)\, \mathrm{d}x$$

$$\text{Dar, } \lim_{z \to \infty} zf(z) = 0 \implies \lim_{r \to \infty} \int_{\gamma_r} f(z)\, \mathrm{d}z = 0$$

$$\implies \int_{-\infty}^{\infty} f(x)\, \mathrm{d}x = 2\pi i \sum_{Imz>0} Rez(f; z)$$

□

**Tipul 3.** *Fie $R$ o functie rationala reala de forma $R = \frac{P}{Q}$, $Q(x) \neq 0$ , $x \in \mathbb{R}$ , grad $Q >$ grad $P + 1$ si $\lim_{|z| \to \infty} R(z) = 0$*

*Atunci*

$$\int_{-\infty}^{\infty} R(x)e^{ix}\, \mathrm{d}x \textit{ converge si } \int_{-\infty}^{\infty} R(x)e^{ix}\, \mathrm{d}x = 2\pi i \sum_{Imz>0} Rez(f; z)$$

*unde $f(z) = R(z)e^{iz}$ .*

*Demonstraţie.* Fie $r > 0$ suficient de mare a.i. toti polii functiei $f$ din semiplanul superior sa fie continuti in $D$, unde $D = \{z \in \mathbb{C} \colon |z| < r;\ Im\ z > 0\}$

Fie $C = \partial D \implies C = [-r; r] \cup \gamma_r$

$$\overset{T.Rez}{\implies} \int_{C} f(z)\, \mathrm{d}z = 2\pi i \sum_{Im\ z>0} Rez(f; z)$$

$$\left.\begin{array}{r} \text{Dar } \displaystyle\int_{C} f(z)\, \mathrm{d}z = \int_{-r} rf(x)\, \mathrm{d}x + \int_{\gamma_r} f(z)\, \mathrm{d}z \\ r \to \infty \end{array}\right\} \implies$$

$$\Longrightarrow 2\pi i \sum_{Im\ z>0} Rez(f;z) = \int_{-\infty} \infty f(x)dx + \lim_{r\to\infty}\int_{\gamma_r} f(z)\ \mathrm{d}z$$

$$= \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} e^{ix}\ \mathrm{d}x + \lim_{r\to\infty}\int_{\gamma_r} \frac{P(z)}{Q(z)} e^{iz}\ \mathrm{d}z$$

$$g(z) = \frac{P(z)}{Q(z)}$$

deci,

$$\lim_{z\to\infty} g(z) = 0 \overset{L.Jordan}{\Longrightarrow} \lim_{r\to\infty}\int_{\gamma_r} g(z)e^{iz}\ \mathrm{d}z = 0$$

Asadar,

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} e^{ix}\ \mathrm{d}x = 2\pi i \sum_{Im\ z>0} Rez(f;z)$$

□

**Tipul 4.** *Fie integrala*

$$I = \int_{-\infty}^{\infty} f(x)e^{i\alpha x}\ \mathrm{d}x$$

*unde* $f = P/Q$ , $Q(x) \neq 0$ , $x \in \mathbb{R}$ , *grad* $P = k$ , *grad* $Q = p$, *iar* $p \geq k+1$ .
*Daca* $\alpha > 0$, *atunci:*

$$I = \int_{-\infty}^{\infty} f(x)e^{i\alpha x}\ \mathrm{d}x = 2\pi i \sum_{Im\ z>0} Rez(g;z)$$

, *unde* $g(z) = f(z)e^{i\alpha z}$.

*Demonstraţie.* Observam ca $\exists \displaystyle\int_{-\infty}^{\infty} f(x)e^{i\alpha x}\ \mathrm{d}x$ si este convergenta. Intr-adevar, pentru ca $p \geq k+1 \implies \displaystyle\lim_{z\to\infty} f(z) = 0$. Dar $f'(z) = \dfrac{h(z)}{Q^2(z)}$, unde $h$ este un polinom de grad cel mult $k+p-1$.

Fie $x_0$ zeroul lui $h$ de modul maxim $\implies f'(x)$ are semn constant pentru $x > |x_0| \implies f(x)$ monotona pentru $x > |x_0|$.

Fie $x_1,\ x_2 \in \mathbb{R}$ cu $x_2 > x_1 > |x_0|$

$$\text{Cum } \lim_{z\to\infty} f(z) = 0 \implies \text{fie } f > 0 \text{ si } \lim_{x\to\infty} f(x) = 0^+, x > |x_0|$$
$$\text{fie } f < 0 \text{ si } \lim_{x\to\infty} f(x) = 0^-, x > |x_0|$$

Aplicand a doua teorema de medie din calculul integral $\implies \exists \xi \in (x_1;x_2)$ a.i.

$$\int_{x_1}^{x_2} f(x)\cos\alpha x\ \ \mathrm{d}x = f(x_1)\int_{x_1}^{\xi} \cos\alpha t\ \ \mathrm{d}t + f(x_2)\int_{\xi}^{x_2} \cos\alpha t\ \ \mathrm{d}t$$

$$\implies \left|\int_{x_1}^{x_2} f(x)\cos\alpha x\ \ \mathrm{d}x\right| \leq \frac{2}{\alpha}|f(x_1)| + \frac{2}{\alpha}|f(x_2)|$$

Stiind ca $\lim_{z\to\infty} f(z) = 0 \implies \forall\epsilon > 0 \quad \exists\delta(\epsilon) > 0$ a.i. $|f(x)| < \dfrac{\epsilon\alpha}{4} x > \delta(\epsilon)$

Deci,

$$\left|\int_{x_1}^{x_2} f(x)\cos\alpha x \ \mathrm{d}x\right| \leq \frac{2}{\alpha}\left[|f(x_1)| + |f(x_2)|\right] < \epsilon,$$

$$x_2 > x_1 > \max\{|x_0|,\ \delta(\epsilon)\} \implies \int_0^\infty f(x)\cos\alpha x \ \mathrm{d}x \text{ converge}$$

Analog $\exists$ si converge

$$\int_0^\infty f(x)\sin\alpha x\,\mathrm{d}x$$

$$\implies \int_0^\infty f(x)e^{i\alpha x}\,\mathrm{d}x$$

este deasemenea convergenta.

Fie $\Omega_r = \{z \in \mathbb{C}\colon |z| < r; Im\ z > 0\}$ ce contine toti polii functiei $g$ din semiplanul superior

$$\stackrel{T.Rez}{\implies} \int_{\partial\Omega_r} g(z)\,\mathrm{d}z = 2\pi i \sum_{z\in\Omega_r} Rez(g;z) = 2\pi i \sum_{Im\ z>0} Rez(g;z)$$

Dar

$$\int_{\partial\Omega_r} g(z)\,\mathrm{d}z = \int_{-r}^{r} f(x)e^{i\alpha x}\,\mathrm{d}x + \int_{\gamma_r} g(z)\,\mathrm{d}z$$

$$\stackrel{L.Jordan}{\implies} \lim_{r\to\infty}\int_{\gamma_r} g(z)\,\mathrm{d}z = 0$$

$$\stackrel{r\to\infty}{\implies} \int_{-\infty}^{\infty} f(x)e^{i\alpha x}\,\mathrm{d}x = 2\pi i \sum_{Im\ z>0} Rez(g;z)$$

$\square$