Teorema Reziduurilor

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Rezumat

Aplicatii ale teoremei reziduurilor in calulul unor chestii interesante. In prima parte avem introducere apoi exemple din x urmate de aplicatii de tip y.

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1 Teorema Reziduurilor

Teorema 1. *Fie functia $f \in \mathcal{H}(G)$, unde $G \subset \mathbb{C}$ multime deschisa. Notam cu ρ mutimea tuturor punctelor singulare izolate ale lui f Fie $\widetilde{G} := G \cup S$, iar γ un contur in G omotop cu zero in $\widetilde{G}$*

$$\text{Atunci suma: } \sum_{z \in \widetilde{G}} n(\gamma; z) Rez(f; z) \text{ este finita si}$$

$$\int_\gamma f(z)\mathrm{d}z = 2\pi i \sum_{z \in \widetilde{G}} n(\gamma; z) Rez(f; z)$$

Demonstraţie. $\exists \varphi : [0; 1]^2 \mapsto G$ deformare continuua, $k = \varphi([0; 1]^2) \subset \widetilde{G}$ compact.

Fie

$$r := \frac{1}{2}\mathrm{d}\left(k, \mathbb{C} \setminus \widetilde{G}\right)$$
$$D := \bigcup_{z \in k} \mathcal{U}(z; r)$$

$k \subset D \subset \overline{D} \subset \widetilde{G}$
γ omotop cu 0 in D
$\overline{D} \cap \rho$ finita $\implies \exists \{b_1, \dots, b_k\} = \overline{D} \cap \rho$
Fie $\Pi_k(z)$ partea principala a dezvoltarii lui f in b_k

Deci, functia $g := f - \sum_{k=1}^{n} \Pi_k$ olomorfa mai putin in b_k admite o prelungire olomorfa g_1 la D .

$$\int_\gamma g = \int_\gamma g_1 = 0$$
$$g = g_1|_{D = \{b_1, \dots, b_k\}}$$
$$\implies \int_\gamma f = \sum_{k=1}^{n} \int_\gamma \Pi_k$$

Calculam

$$\int_\gamma \Pi_k \text{ , unde } \Pi_k(z) = \sum_{m=1}^{\infty} \frac{a^{(k)} - m}{(z - b_k)^m}$$

Seria este uniform convergenta pe $\forall$ parte compacta din $\mathbb{C} \setminus \{b_a\} \implies$ uniform convergenta pe $\{\gamma\} \implies$ putem integra termen cu termen si

$$\int_\gamma \frac{\mathrm{d}}{z - b_k} m = 0, \forall m > 1$$

Functia $\dfrac{1}{(z - b_n)^m}$ admite primitiva si $\displaystyle\int_\gamma \frac{\mathrm{d}z}{z - b_k} = 2\pi i \cdot n(\gamma; b_n) \cdot a_{-1}^{(k)}$ deci

$$\int_\gamma f = 2\pi i \sum_{k=1} nn(\gamma; b_k) Rez(f; b_n)$$

Trebuie sa mai aratam ca $\forall z_0 \in \widetilde{G} \setminus (D \cap \rho)\colon n(\gamma; z_0) \cdot Rez(f; z_0) = 0$

Intr-adevar, daca pentru $z_0 \in \widetilde{G} \setminus (D \cap \rho)$ avem $Rez(f; z_0) \neq 0 \implies z_0 \in \rho$, deci $z_0 \notin D$ si

$$n(\gamma; z_0) = \frac{1}{2\pi i} \int_\gamma \frac{\mathrm{d}\xi}{\xi - z_0} = 0$$

caci $h(\xi) = \frac{1}{\xi - z_0}$ olomorfa pe D si γ omotop cu zero

$$\implies \int_\gamma f = 2\pi i \sum_{z \in \widetilde{G}} n(\gamma; z) \cdot Rez(f; z)$$

□

2 Puncte singulare izolate

Definitie 1. *Fie $G \subset \mathbb{C}$ multime deschisa si $f \in \mathcal{H}(G)$. Punctul $z_0 \in \mathbb{C}$ se numeste punct singular izolat pentru functia f daca $z_0 \notin G$, dar $\exists p > 0$ a.i $\dot{\mathcal{U}}(z_0; p) \subset G \implies f \in \mathcal{H}(\dot{\mathcal{U}}(z_0; p))$*