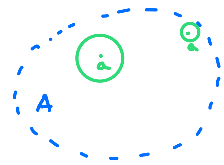
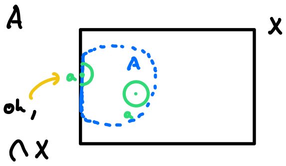
Množica v $\mathbb{R}^n$



T $A \subseteq X \subseteq \mathbb{R}^n$

- A je odprta v $\mathbb{R}^n$, če za $\forall a \in A$ $\exists \varepsilon > 0$ tako da $K(a,\varepsilon) \subseteq A$
- A je odprta v X, če za $\forall a \in A$ $\exists \varepsilon > 0$ tako da $K(a,\varepsilon) \cap X \subseteq A$



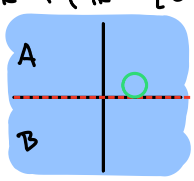
tudi ok, ker smo $\cap X$

A je odprta v X a ne v $\mathbb{R}^n$

- A je zaprta v X, če je $X \setminus A$ odprta v X
  ($\{\emptyset\}$ in X sta vedno odprti v X)
- X je povezana, če sta X in $\{\emptyset\}$ edini podmnožici mn. X ki sta hkrati odprti in zaprti v X.
- X je povezan s potmi, če za $\forall a, s \in X$ $\exists \gamma : [0,1] \to X$: $\gamma(0) = a$, $\gamma(1) = s$
  γ je zvezna. Vsaka s potmi povezana množica je povezana.
- X je kompaktna $\Leftrightarrow$ je zaprta in omejena.

1 Ali je množica povezana in odprta v $\mathbb{R}^2$?

a $\mathbb{R}^2 \setminus (\mathbb{R} \times \{0\})$



Množica je odprta, ni povezana, ni s potmi povezana

$A = \mathbb{R} \times \mathbb{R}^+ \subseteq X$

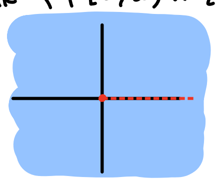
$B = \mathbb{R} \times \mathbb{R}^- \subseteq X$

$A \neq \emptyset$ $B \neq \emptyset$ $A \cap B = \emptyset$, $A \cup B = X$

A in B sta odprti v X in zato tudi zaprti v X

$\emptyset$ in X nista edini odprti in zaprti v X

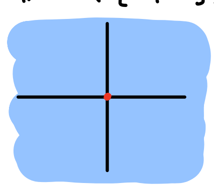
b $\mathbb{R}^2 \setminus ([0,\infty) \times \{0\})$



odprta: da

povezana: da, ker je povezana s potmi

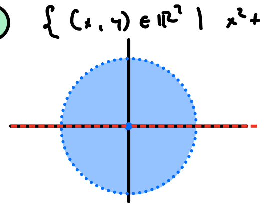
c $\mathbb{R}^2 \setminus (\{0,0\})$



odprta: da

povezana: da

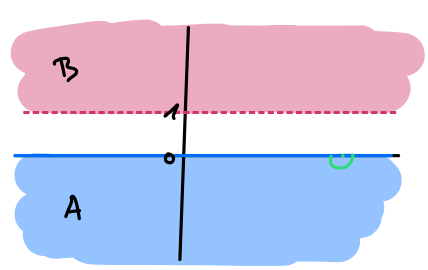
d $\{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1,\ y \neq 0\} \cup \{(0,0)\}$



odprta: ne, zaradi točke $(0,0)$

povezana: da

2 $A = \mathbb{R} \times (-\infty, 0]$, $B = \mathbb{R} \times (1,\infty)$, $X = A \cup B$



a Ali so A, B, X odprte ali zaprte v $\mathbb{R}^2$?

b Ali so A, B, X odprte ali zaprte v X? Ali je X povezan?

ⓐ A ni odprta, je zaprta v $\mathbb{R}^2$

B je odprta v $\mathbb{R}^2$

X je zaprta ali odprta v $\mathbb{R}^2$

ⓑ A je odprta v X

B je odprta v X

B je zaprta v X ker je $X \setminus B = A$ odprta

A je zaprta v X

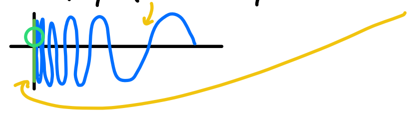
X je vedno odprta v X.

X je vedno zaprta v X.

X ni povezana.

Primer povezane množice, ki ni povezana s potmi

$X = \{(x,y);\ y = \sin \frac{1}{x};\ x > 0\} \cup (\{0\} \times [-1,1])$

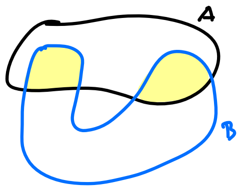


Lahko najdemo krogle (povezanost) ne moremo pa najti poti med in

③ Pokaži, da je presek s potmi povezanih množic v $\mathbb{R}$ s potmi povezan? Ali velja enako v $\mathbb{R}^2$?

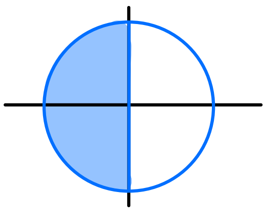
Presek intervalov je interval ali $\emptyset$ ker je spet povezano.

V $\mathbb{R}^2$ ne velja npr:



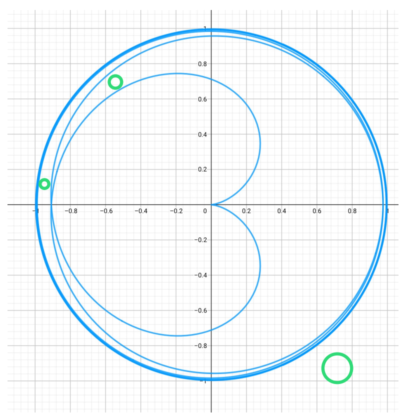
⑤ Katere množice so kompaktne?

ⓐ $\{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1,\ x \leq 0\} \cup S^1$ krožnica



je zaprta in omejena zato je kompaktna

ⓒ $\left\{ \frac{\varphi^2}{1+\varphi^2} (\cos\varphi, \sin\varphi) \mid \varphi \in \mathbb{R} \right\} \cup S^1$



Omejena: da

Zaprta: da

Kompaktna: da

⑦ $S^2 = \{(x,y,z);\ x^2 + y^2 + z^2 = 1\}$

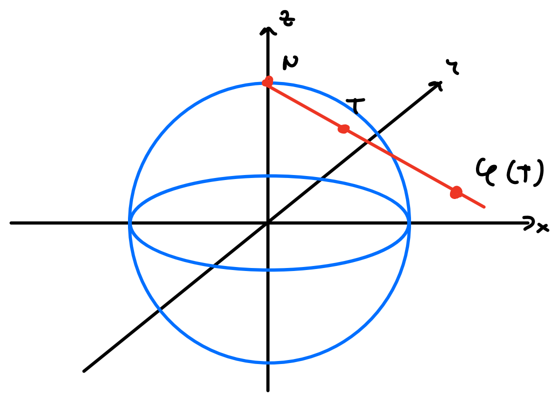
$\varphi: S^2 \setminus \{(0,0,1)\} \longrightarrow \mathbb{R}^2$ Stereografska projekcija

$\varphi(x,y,z)$ je presečišče premice skozi $(0,0,1)$ in (x,y,z) z ravnino $z=0$

ⓐ predpis za φ

ⓑ dokaži bijektivnost φ in zveznost φ in φ^{-1}

ⓒ Kam φ^{-1} preslika premico v ravnini?



ⓐ $\vec{r}(t) = (0,0,1) + t(x, y, z-1) = (*, *, 0)$

$$1 + t(z-1) = 0 \qquad t = \frac{1}{1-z}$$

$$\Rightarrow \vec{r}(t) = (0,0,1) + \frac{1}{1-z}(x, y, z-1) = \left(\frac{x}{1-z}, \frac{y}{1-z}, 0\right)$$

ⓑ $\varphi(x,y,z) = \left(\underbrace{\frac{x}{1-z}}_{u}, \underbrace{\frac{y}{1-z}}_{v}\right)$

Izrazimo x, y, z z u, v

$$u = \frac{x}{1-z} \qquad v = \frac{y}{1-z} \qquad x^2 + y^2 + z^2 = 1$$

$$x = u(1-z) \qquad y = v(1-z) \qquad (1-z)^2(u^2+v^2) = (1-z)(1+z)$$

$$u^2 + v^2 - zu^2 - zv^2 = 1+z$$

$$z = \frac{u^2+v^2-1}{u^2+v^2+1}$$

$$x = \frac{2u}{u^2+v^2+1} \qquad y = \frac{2v}{u^2+v^2+1} \qquad 1-z = \frac{2}{u^2+v^2+1}$$

$$\varphi^{-1}(u,v) = \left(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1}\right)$$

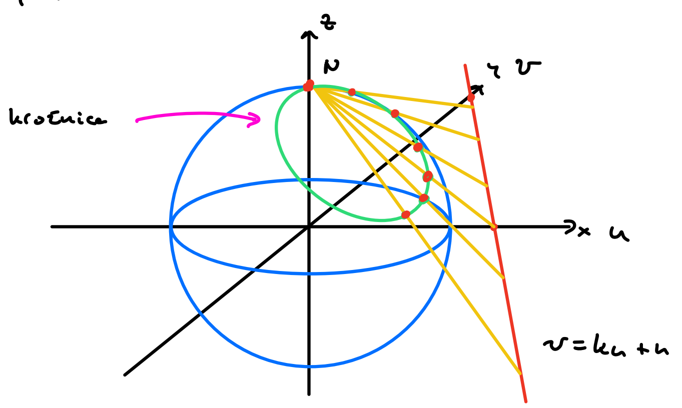
Da preverimo bijektivnost $\varphi \circ \varphi^{-1} = id \qquad \varphi^{-1} \circ \varphi = id$

$$\varphi(\varphi^{-1}(u,v)) = \overset{DN}{\dots} = (u,v)$$
$$\varphi^{-1}(\varphi(x,y,z)) = \dots = (x,y,z)$$

φ, φ^{-1} zvezni? Ni: polov na D_f, rac. funk. zvezni

ⓒ $v = ku + n$ premica v ravnini

Kot presek ravnin in sfere



Holomorfne funkcije

Ⓣ

- Naj bo $D \subseteq \mathbb{C}$ območje (tj. odprta povezana množica). Funkcija $f : D \to \mathbb{C}$ je *holomorfna* v točki $z \in D$, če obstaja limita $\lim_{h\to 0} \frac{f(z+h)-f(z)}{h}$.
- Vsaka potenčna vrsta $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ je holomorfna na svojem konv. območju (konv. radij $R = \frac{1}{\limsup \sqrt[n]{a_n}}$).
- Funkcije cos, sin, ch, sh so definirane kot potenčne vrste in so holomorfne povsod. Velja $\cos z = \operatorname{ch}(iz)$, $i\sin z = \operatorname{sh}(iz)$, $e^{iz} = \cos z + i\sin z$. $= \operatorname{ch} iz + \operatorname{sh} iz$
- Naj bo $f : D \to \mathbb{C}$ funkcija, definirana na območju $D \subseteq \mathbb{C}$. Pišimo $f(x+iy) = u(x,y) + iv(x,y)$, $u, v : D \to \mathbb{R}$. Potem je f holomorfna na D natanko tedaj, ko sta u, v zvezno parcialno odvedljivi in veljata Cauchy-Riemannovi enakosti: $u_x = v_y$ in $u_y = -v_x$. Tedaj je $f' = u_x + iv_x$, u in v pa sta harmonični funkciji.

① Kompleksne ničle sin, cos, sh, ch

Vse holomorfne zveze, ki veljajo v $\mathbb{R}$, veljajo tudi v $\mathbb{C}$

ⓐ $\sin z = 0$

$\sin a+ib = 0$

$\sin a \cos ib + \cos a \sin ib = 0$

$\sin a \, \underset{\neq 0}{\operatorname{ch} b} + i \cos a \operatorname{sh} b = 0$

$\sin a \operatorname{ch} b = 0 \qquad \cos a \operatorname{sh} b = 0$

$a = k\pi \quad k \in \mathbb{Z} \qquad \underset{\neq 0}{\cos a} \quad b = 0$

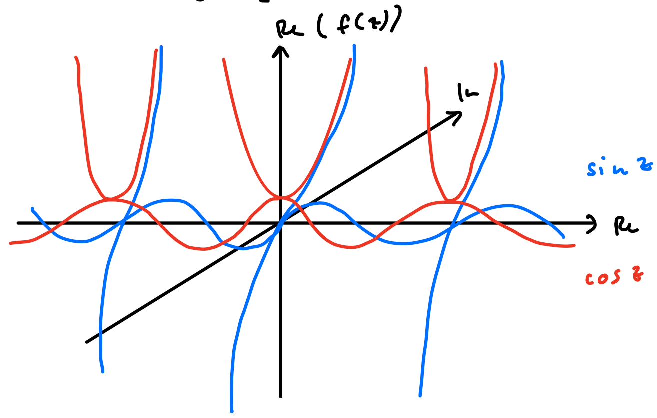
sin ima le realne ničle $z = k\pi + i0$

ⓑ $\cos z = 0$

$\sin\left(\frac{\pi}{2} - z\right) = 0$

$\frac{\pi}{2} - z = k\pi$

$z = \frac{\pi}{2} + k\pi$



ⓒ $\operatorname{sh} z = 0$

$i \sin iz = 0$

$\sin iz = 0$

$iz = k\pi$

$z = k\pi i$

ⓓ $\operatorname{ch} z = 0$

$\cos iz = 0$

$iz = \frac{\pi}{2} + k\pi$

$z = \frac{\pi}{2} i + k\pi i$

② $|\sin(x+iy)|^2 = |\sin x \operatorname{ch} y + i \operatorname{sh} x \cos y|^2 = \sin^2 x \operatorname{ch}^2 y + \operatorname{sh}^2 x \cos^2 y = (1 - \cos^2 x)\operatorname{ch}^2 y + (1 + \operatorname{ch}^2 x)\cos^2 y$

$= \operatorname{ch}^2 y + \cos^2 y$

④ $\Omega = \{ z ; \operatorname{Im} z > 0 \}$

$f : \Omega \to \mathbb{C}$

$$f(x+iy) = \underbrace{\sqrt{\sqrt{x^2+y^2} + x}}_{u} + i \underbrace{\sqrt{\sqrt{x^2+y^2} - x}}_{v}$$

ⓐ f holomorfna ?

u, v parc. zv. odv. ✓

$$u_x = \frac{1}{2} \frac{\frac{x}{\sqrt{x^2+y^2}} + 1}{\sqrt{\sqrt{x^2+y^2} + x}} \qquad v_x = \frac{1}{2} \frac{\frac{x}{\sqrt{x^2+y^2}} - 1}{\sqrt{\sqrt{x^2+y^2} - x}}$$

$$u_y = \frac{1}{2} \frac{\frac{y}{\sqrt{x^2+y^2}}}{\sqrt{\sqrt{x^2+y^2} + x}} \qquad v_y = \frac{1}{2} \frac{\frac{y}{\sqrt{x^2+y^2}}}{\sqrt{\sqrt{x^2+y^2} - x}}$$

$u_x \overset{?}{=} v_y$

$$\left(\frac{x}{\sqrt{x^2+y^2}} + 1\right) \sqrt{\sqrt{x^2+y^2} - x} = \frac{y}{\sqrt{x^2+y^2}} \sqrt{\sqrt{x^2+y^2} + x} \quad /^2$$

$$(x^2 + 2x\sqrt{} + x^2 + y^2)\left(\sqrt{x^2+y^2} - x\right) = y^2 \left(\sqrt{x^2+y^2} + x\right)$$

⋮ ✓

$y = y$

✓ $u_y = -v_x$

Cauchy–Riemannove enačbe

f je holomorfna $\Leftrightarrow$

u, v parc. zv. odv.

in veljajo CR enačbe

$u_x = v_y \qquad u_y = -v_x$

(b) Izračunaj kaj si predpis za f

Namig f^2

$$\left(\sqrt{\sqrt{x^2+y^2}+x} + i\sqrt{\sqrt{x^2+y^2}-x}\right)^2 = \sqrt{x^2+y^2}+x - \sqrt{x^2+y^2}+x + 2i\sqrt{x^2+y^2-x^2} =$$

$$= 2iy + 2x$$

$$\Rightarrow \quad f(x,y) = \sqrt{2(x+iy)} = \sqrt{2z}$$

(c) Odvod f

$$f' = \frac{1}{2\sqrt{2z}}\, 2 = \frac{1}{\sqrt{2z}}$$

(5) Poišči vse cele funkcije, katerih realni del je $x^3 - 3xy^2$.

Cela = holomorfna na celi ravnini ($\mathbb{C}$)

$$f(x+iy) = \underset{\substack{\| \\ x^3-3xy^2}}{u(x,y)} + i\, v(x,y)$$

$$u_x = v_y \qquad v_x = -u_y$$

$$3x^2 - 3y^2 = v_y \qquad v_x = 6xy$$

$$v = 3x^2y - y^3 + C(x) \qquad v = 3x^2y + C(y)$$

$$v = 3x^2y - y^3 + c$$

$$f(x,y) = (x^3 - 3xy^2) + i(3x^2y - y^3 + c) = (x+iy)^3 + Ci = z^3 + Ci \qquad C \in \mathbb{R}$$

(7) Katere od spodnjih funkcij so holomorfne za vsako celo funkcijo f?

(a) $f_1(z) = f(\bar{z})$

$f(z)$ holom., $f(\bar{z})$ pa ne

$f(z) = z \qquad f_1(z) = \bar{z}$

konjugiranje ni holomorfno

(b) $f_2(z) = \overline{f(z)}$

$f(z) = z \qquad f_2(z) = \bar{z}$ ni holomorfna

(c) $f_3(z) = \overline{f(\bar{z})}$

$$f_3(z) = \sin z = \overline{\bar{z} - \frac{\bar{z}^3}{3!} \dots} = z - \frac{z^3}{3!} \dots = \sin(z)$$

Vsako holomorfno funkcijo se da razviti v vrsto $\Rightarrow$ dvojno konjugiranje se pokrajša

Rešimo brez razvoja

f ... holomorfna $\Rightarrow f(x+iy) = u(x,y) + i v(x,y)$

$$u_x = v_y \qquad u_y = -v_x$$

$$f_3(z) = \overline{f(\bar{z})} = \overline{f(x-iy)} = \overline{(u(x,-y) + i v(x,-y))} = \underbrace{u(x,-y)}_{\alpha} - i \underbrace{v(x,-y)}_{\beta}$$

CR enačbe

$\alpha_x = \beta_y$ $\quad\quad$ $\alpha_y = -\beta_x$

$u_x = -v_y \ (-1)$ $\quad\quad$ $-u_y = -(-1)\, v_x$

$u_x = v_y$ $\quad\quad$ $u_y = -v_x$

8. $f: D \to \mathbb{C}$ holomorfna

D območje (odprta in povezana množica)

$|f(z)| = C$ za $\forall z \in D$

Dokaži, da je f konstantna

$f = u + iv$, $\quad u_x = v_y \quad u_y = -v_x$

$\sqrt{u^2+v^2} = c \quad u^2+v^2 = c^2 \quad \Big/ \frac{d}{dx}, \frac{d}{dy}$

$2u\, u_x + 2v\, v_x = 0 \qquad 2u\, u_y + 2v v_y = 0$

$\downarrow \qquad \swarrow \quad -u V_x + v u_x = 0$

$u u_x + v v_x = 0 \quad /u$

$v u_x - u v_x = 0 \quad /v$

$\Downarrow$

...

$(u^2+v^2)\, u_x = 0$

$c^2 \ u_x = 0$

BŠS $\quad c \neq 0$

...

$\Rightarrow \ u_x = 0 \quad v_x = 0 \qquad\qquad v_y = 0 \quad u_y = 0$

Ker je D povezano območje

$\Rightarrow$ u, v sta konstanti

10. Razvij $\sin z$ okrog točke i

$\sin(z-i) = (z-i) - \frac{(z-i)^3}{3!} + \dots$ $\quad$ razvoj okrog i

$\sin((z-i)+i) = \sin(z-i)\cos i + \cos(z-i)\sin i =$

$= \left((z-i) - \frac{(z-i)^3}{3!} + \dots\right)\cos i + \left(1 - \frac{(z-i)^2}{2!} + \dots\right)\sin i$

$\cos i = \operatorname{ch} 1$

$\sin i = i \operatorname{sh} 1$

11. Določi konvergenčno območje

$\sum_{n=0}^{\infty} z^{2^n}$ $\qquad R = \frac{1}{\limsup \sqrt[n]{|a_n|}} = 1$

$a_n = \begin{cases} 0 \ ; & n = 2k \\ 1 \ ; & \text{sicer} \end{cases}$ $\quad = \limsup \sqrt[n]{|a_n|} = 1$

Kompleksni integrali

T

- $\int_\gamma f(z)dz = \int_{t_1}^{t_2} f(z(t))\dot{z}(t)dt$
- Naj bo f holomorfna na enostavno povezanem območju D, $\alpha \in D$ in γ sklenjena pot v D, ki ne gre skozi α. Potem je:

brez lukenj

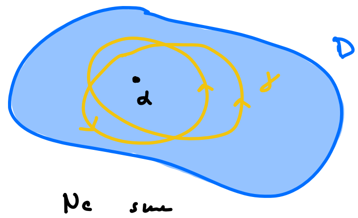
(1) $\oint_\gamma f(z)dz = 0$

(2) $\frac{1}{2\pi i}\oint_\gamma \frac{f(z)}{z-\alpha}dz = I_\gamma(\alpha)f(\alpha)$

(3) $\frac{1}{2\pi i}\oint_\gamma \frac{f(z)}{(z-\alpha)^{n+1}}dz = I_\gamma(\alpha)\cdot\frac{f^{(n)}(\alpha)}{n!}$, $n \in \mathbb{N}_0$

index povezanosti / ovojno število

Cauchyjeva formula



Ne sme biti lukenj

1 Izračunaj integrale

a) $\int_{|z|=1} z^n dz$ $\quad n \in \mathbb{Z}$

$\oint z^n dz = 0$

ker nima polov in z^n je holomorf. za $z \geq 0$

$z = \cos t + i\sin t = e^{it}$ $\quad t \in [0, 2\pi)$ $\quad dz = ie^{it}dt$

$$\int_0^{2\pi} e^{nit}\, ie^{it}\, dt =$$

$$= i\int_0^{2\pi} e^{(n+1)it}dt = \frac{i}{(n+1)i} e^{(n+1)it}\Big|_0^{2\pi} =$$

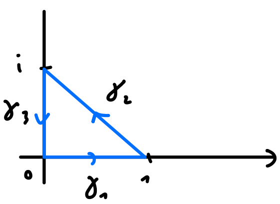
$$= \frac{1}{n+1}\left(e^{2\pi i(n+1)} - e^0\right) = \frac{1}{n+1}(1-1) = 0$$

$n \neq -1$

$n = -1$ $\quad i\int_0^{2\pi} e^0 dt = 2\pi i$

b) $\int_\gamma \bar{z}\, dz$ $\quad \gamma$ rob $\Delta(0, 1, i)$

Ker $\bar{z}$ ni hol. ne moremo uporabiti Cauchyjeve formule



$\gamma_1: \; z(t) = t \quad t \in [0,1] \quad dz = dt$

$\gamma_2: \; z(t) = 1 + t(i-1) \quad t \in [0,1] \quad dz = (i-1)dt$

$\gamma_3: \; z(t) = i(1-t) \quad t \in [0,1] \quad dz = -i\,dt$

$$\int_0^1 t\, dt + \int_0^1 (1 + it - t)(i-1)dt + \int_0^1 (i - it)(-i)dt =$$

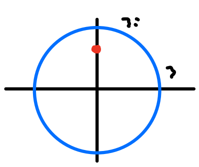
$$= \frac{1}{2} + (i-1)\left(1 + \frac{i}{2} - \frac{1}{2}\right) + 1 - \frac{1}{2}$$

$$= i$$

2 Izračunaj s Cauchyjevo formulo

a) $\oint_{|z|=2} \frac{\sin z}{z+i}dz = 2\pi i \cdot 1 \cdot \sin(-i) = 2\pi i\,(-1)\, i \sinh 1 = 2\pi \sinh 1$

b) $\int_{|z|=3} \frac{z^2}{z-2i}dz = 2\pi i\,(2i)^2 = -8\pi i$



Pol mora biti znotraj γ

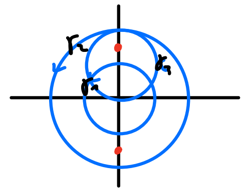
(c) $\int_{|z|=1} \bar z \, dz = \int_{|z|=1} \frac{1}{z}\, dz = 2\pi i \cdot 1 \cdot 1 = 2\pi i$

na krožnici velja $z \bar z = 1 \Rightarrow \bar z = \frac{1}{z}$

se ujemata na območju integracije

(d) $\oint \frac{dz}{9+z^2} = \oint \frac{dz}{(z+3i)(z-3i)} = 0$ $\quad \gamma = |z| = 2$ ni polov znotraj

γ krožnica $|z|=2$, $|z|=4$ oz. $|z-2i|=2$



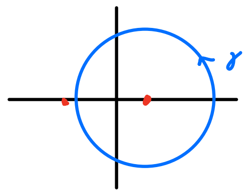
$$\oint_{\gamma_2} \frac{dz}{(z+3i)(z-3i)} = \oint_{\gamma_2} \frac{-1/6i}{z+3i} + \frac{1/6i}{z-3i}\, dz =$$

$$= -\frac{1}{6i}\, 2\pi i + \frac{1}{6i}\, 2\pi i = 0$$

Drug način: razdelimo na dve poti

$$\oint_{\gamma_3} \frac{dz}{(z+3i)(z-3i)} = \oint_{\gamma_3} \frac{\frac{1}{z+3i}}{z-3i}\, dz = 2\pi i \, \frac{1}{3i+3i} = \frac{\pi}{3}$$

(e) $\oint_{|z-1|=\frac{3}{2}} \frac{dz}{(z+1)(z-1)^3} = \oint_\gamma \frac{\frac{1}{z+1}}{(z-1)^3}\, dz =$



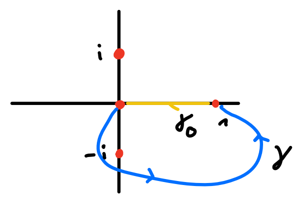
$$= 2\pi i \, (-1)(-2) \frac{1}{(z+1)^3} \, \frac{1}{2!} =$$

$$\oint \frac{f(z)}{(z-\alpha)^{n+1}}\, dz = 2\pi i \, \mathrm{Ind} \cdot \frac{f^{(n)}(\alpha)}{n!}$$

$$= 2\pi i \, \frac{1}{2^3} = \frac{\pi}{4} i$$

(3) Katere vrednosti lahko zavzame $\int_0^1 \frac{dz}{z^2+1}$ če int. po vseh poteh od 0 do 1

- γ ... daljica $\quad \int \ldots = \arctan \Big|_0^1 = \frac{\pi}{4}$



- γ ni daljica $\quad \sigma = \gamma \cup \gamma_0$

$$\oint_\sigma \frac{dz}{z^2+1} = \int_\gamma \frac{dz}{z^2+1} + \int_{\gamma_0} \frac{dz}{z^2+1} = 2\pi i \, \frac{1}{i+i} I_\sigma(i) + 2\pi i \, \frac{1}{-i-i} I_\sigma(-i)$$

$$= \pi \left(I_\sigma(i) - I_\sigma(-i) \right) = \int_\gamma \frac{dz}{z^2+1} - \frac{\pi}{4}$$

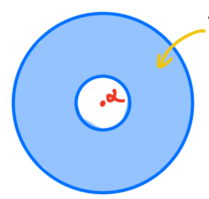
$$\Rightarrow \int_\gamma \frac{dz}{z^2+1} = \frac{\pi}{4} + k\pi$$

Laurentove vrsta

- *Laurentov razvoj* holomorfne funkcije f na kolobarju $r < |z-\alpha| < R$ je

$$f(z) = \sum_{n=0}^{\infty} c_n (z-\alpha)^n + \sum_{n=1}^{\infty} c_{-n} (z-\alpha)^{-n} = \sum_{n=-\infty}^{\infty} c_n (z-\alpha)^n, \qquad c_n = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{(z-\alpha)^{n+1}}\, dz,$$

kjer je γ poljubna sklenjena pot na danem kolobarju z ovojnim številom $I_\gamma(\alpha) = 1$. Prva vrsta v formuli konvergira vsaj na območju $|z-\alpha| < R$, druga pa vsaj na območju $|z-\alpha| > r$.

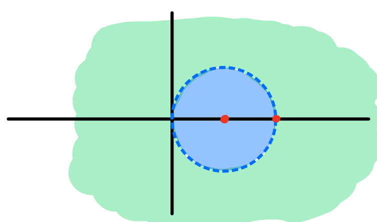


tu razvijemo in tu je holom.

(1) Razvij v L. vrsto $f(z) = \frac{1}{(z-1)(z-2)}$, da bo vrsta konvergirala na:

(a) $0 < |z-1| < 1$

(b) $|z-1| > 1$



ⓐ $\frac{1}{(z-1)(z-2)} = \frac{1}{z-1} \frac{1}{(z-1)-1} \overset{\text{geo.}}{=} -\frac{1}{z-1}\left(1+(z-1)+(z-1)^2+(z-1)^3+\dots\right) =$ Radij konvergen.

$z-1 \neq 0$ $|z-1|<1$

$= -\frac{1}{z-1} - 1 - (z-1) - (z-1)^2 - \dots$

ⓑ $\frac{1}{(z-1)(z-2)} = \frac{1}{z-1}\cdot\frac{1}{(z-1)-1} = \frac{1}{z-1}\cdot\frac{\frac{1}{z-1}}{1-\frac{1}{z-1}} = \frac{1}{(z-1)^2}\frac{1}{1-\frac{1}{z-1}} =$ $\left|\frac{1}{z-1}\right|<1$

$1<|z-1|$

$= \frac{1}{(z-1)^2}\left(1+\frac{1}{z-1}+\left(\frac{1}{z-1}\right)^2+\dots\right) = \frac{1}{(z-1)^2}+\frac{1}{(z-1)^3}+\frac{1}{(z-1)^4}+\dots$

② $f(z) = \frac{1}{z(z-1)(z-2)}$

ⓐ $0<|z|<1$

$f(z) = \frac{1}{z}\frac{1}{(z-1)(z-2)} = \frac{1}{z}\left(-\frac{1}{z-1}+\frac{1}{z-2}\right) = \frac{1}{z}\left(1+z+z^2+z^3+\dots -\frac{1}{2}\left(1+\frac{z}{2}+\left(\frac{z}{2}\right)^2+\dots\right)\right) =$

Območje konv. $|z|<1$ $\left|\frac{z}{2}\right|<1$

$z\neq 0$ $|z|<2$

$= \frac{1}{z}\left(\frac{1}{2}+\frac{3z}{4}+\frac{7}{8}z^2+\dots\right) = \sum_{n\geq -1}\left(1-\frac{1}{2^{n+2}}\right)z^n$

ⓑ $1<|z|<2$

$f(z) = \frac{1}{z}\left(-\frac{1}{z-1}+\frac{1}{z-2}\right) = \frac{1}{z}\left(-\frac{1/z}{1-1/z}+\frac{1}{z-2}\right)$

$\left|\frac{1}{z}\right|<1$ $|z|>1$ $|z|<2$

$= \frac{1}{z}\left(-\frac{1}{z}\left(1+\frac{1}{z}+\left(\frac{1}{z}\right)^2+\dots\right)-\frac{1}{2}\left(1+\frac{z}{2}+\left(\frac{z}{2}\right)^2+\dots\right)\right) =$

$= \frac{1}{z}\sum_{n=0}^{\infty} -\frac{1}{z}\frac{1}{z^n} - \frac{1}{2}\left(\frac{z}{2}\right)^n = \frac{1}{z}\sum_{n=0}^{\infty} -\frac{1}{z^{n+1}} - \frac{z^n}{2^{n+1}} = \sum_{n=0}^{\infty} -\frac{1}{z^{n+2}} - \frac{z^{n-1}}{2^{n+1}}$

ⓒ $|z|>2$

$f(z) = \frac{1}{z}\left(-\frac{1}{z-1}+\frac{1}{z-2}\right) = \frac{1}{z}\left(-\frac{1/z}{1-1/z}+\frac{1/z}{1-2/z}\right) =$

$|z|>1$ $|z|>2$

$= \frac{1}{z}\left(-\frac{1}{z}\sum_{n=0}^{\infty}\frac{1}{z^n}+\frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n\right) = \frac{1}{z^2}\sum_{n=0}^{\infty}\frac{2^n}{z^n} - \frac{1}{z^2} = \sum_{n=0}^{\infty}\frac{2^n-1}{z^{n+2}}$

③ $f(z) = \frac{z}{z^3-3z+2}$ okoli $a=1$ (okoli pomeni najmanjši kolobar do prve nesingu.). Konv. območje?

$f(z) = \frac{z}{(z-1)(z^2+z-2)} = \frac{1}{(z-1)^2}\frac{z}{z+2} = \frac{1}{(z-1)^2}\, z\, \frac{1}{(z-1)+3} = \frac{1}{(z-1)^2}\frac{(z-1)+1}{3\left(\frac{z-1}{3}+1\right)} =$

$= \frac{(z-1)+1}{(z-1)^2}\frac{1}{3}\left(1-\frac{z-1}{3}+\left(\frac{z-1}{3}\right)^2-\dots\right) =$ Območje konv. $\left|\frac{z-1}{3}\right|<1$

$|z-1|<3$

$= \frac{1}{3}\left(1-\frac{z-1}{3}+\left(\frac{z-1}{3}\right)^2\dots+(z-1)-\frac{(z-1)^2}{3}+\frac{(z-1)^3}{3^2}-\dots\right)\frac{1}{(z-1)^2} =$

$= \frac{1}{3}\sum_{n=0}^{\infty}\frac{(z-1)+1}{(z-1)^2}\left(\frac{z-1}{3}\right)^n(-1)^n = \sum_{n=0}^{\infty}3^{-n-1}(-1)^n(z-1)^{n-2}\left((z-1)+1\right) =$

$= \sum_{n=0}^{\infty}3^{-n-1}(-1)^n\left((z-1)^{n-1}+(z-1)^{n-2}\right)$

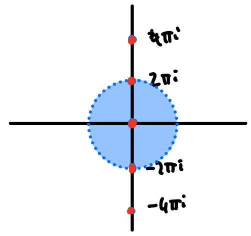
4. $f(z)=\frac{1}{e^z-1}$ okoli $a=0$, vsaj 4 členi, območje konv.?

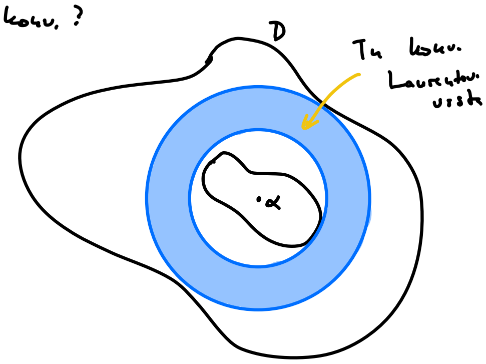
Singularnosti:

$e^z-1=0$

$e^z=1$

$z=2k\pi i$ $\quad k\in\mathbb{Z}$





Konv. območje $\quad 0<|z|<2\pi$

$$f(z)=\frac{1}{1+z+\frac{z^2}{2!}+\dots-1}=\frac{1}{z}\cdot\frac{1}{\underbrace{1+\frac{z}{2!}+\frac{z^2}{3!}+\dots}_{\varepsilon}}=\frac{1}{z}\left(1-\left(\frac{z}{2!}+\frac{z^2}{3!}+\dots\right)+\left(\frac{z}{2!}+\frac{z^2}{3!}+\dots\right)^2\mp\dots\right)=\dots$$

Območje konv. $\quad \left|\frac{z}{2!}+\frac{z^2}{3!}+\dots\right|<1$

$$\dots=\frac{1}{z}\left(1-\frac{z}{2}+\left(-\frac{z^2}{6}+\frac{z^2}{4}\right)+\left(-\frac{z^3}{4!}+2\frac{z^3}{2!\,3!}-\frac{z^3}{2^3}\right)\mp\dots\right)=$$

$$=\frac{1}{z}\left(1-\frac{1}{2}z+\frac{1}{12}z^2+0z^3+\dots\right)=\frac{1}{z}-\frac{1}{2}+\frac{1}{12}z+\mathcal{O}(z^3)$$

5. $f(z)=\frac{1}{z^2(1-z)^2}$ okoli $a=0$

$g(z)=\frac{1}{(1-z)^2}$ $\qquad \int g=\frac{1}{1-z}=1+z+z^2+\dots \;\Big|\; \frac{d}{dz}$

$g=1+2z+3z^2+\dots$ $\qquad |z|<1$

$$f(z)=\frac{1}{z^2}\left(1+2z+3z^2+\dots\right)=\frac{1}{z^2}+\frac{2}{z}+3+4z+\dots$$

Logaritem, Liouvillov izrek, princip identičnosti

- $\log z=\log|z|+i\arg z$, $-\pi<\arg z<\pi$
- Če je f *cela* funkcija (tj. holomorfna na $\mathbb{C}$) in je $|f|$ omejena, potem je f konstantna (Liouvillov izrek).
- Naj bosta f,g holomorfni na območju D. Če se f in g ujemata na množici $A\subseteq D$ s stekališčem v D, potem je $f=g$ na D (princip identičnosti).

$z^r=e^{r\ln z}$

1. $f(z)=\log(\sin z)$

a) $f(i\log 2)=?$

b) $D_f=?$

c) Dokaži ali ovrzi: $z\in D_f$, $f(z)\in\mathbb{R}\Rightarrow f(z)<0$

a) $$f(i\ln 2)=\ln(\sin(i\ln 2))=\ln(i\,\mathrm{sh}(\ln 2))=\ln i\,\frac{e^{\ln 2}-e^{-\ln 2}}{2}=\ln i\,\frac{2-\frac{1}{2}}{2}=\ln\frac{3i}{4}=$$

$$=\ln\frac{3}{4}+i\arg\left(\frac{3i}{4}\right)=\ln\frac{3}{4}+i\frac{\pi}{2}$$

b) D_f vse razen: $\quad \sin z\in\mathbb{R}$, $\sin z\le 0$

$z=a+bi$

$\sin(a+bi)=\sin a\,\mathrm{ch}\,b+i\cos a\,\mathrm{sh}\,b \quad \in\mathbb{R}, \;\le 0$

$\operatorname{sh} b \cos a = 0$

$b = 0$ ali $\cos a = 0$

$a = \frac{\pi}{2} + k\pi$

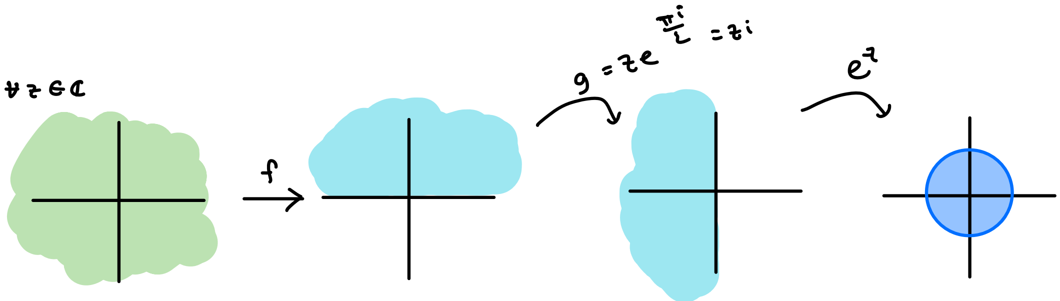
$\Downarrow$ $\Downarrow$ $\sin a \operatorname{ch} b \le 0$

$\sin a \le 0$ $\quad a = \frac{3\pi}{2} + 2k\pi$

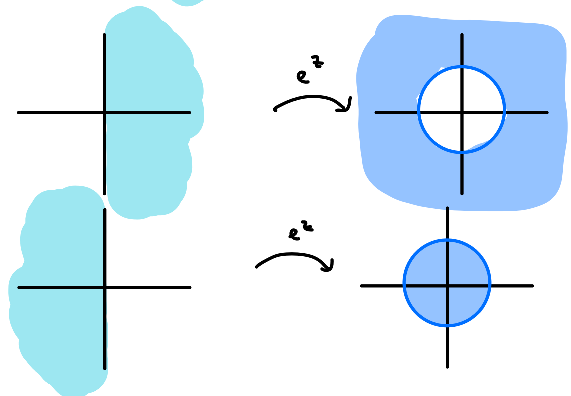
$a \in [\pi + 2k\pi, 2\pi + 2k\pi]$

2) f cela, $\operatorname{Im}(f(z)) > 0 \quad \forall z \in \mathbb{C}$

Dokaži: f konstantna.



$e^{g(f(z))}$ je konst. po L. izr.



$$e^{g(f(z))} = e^{if(z)} = c \qquad c \in \mathbb{C}$$

$$i f(z) = \ln c + 2k\pi i$$

$$f(z) = \frac{1}{i}\ln c + 2\pi k(z)$$

$k \in \mathbb{Z}$, če k = konst. saj je f zvezna

obstoj in enoličnost

3) $\Omega = \mathbb{C}$ ali $\Omega = \mathbb{C} \setminus \{0\}$

Ali $\exists!$ $f: \Omega \to \mathbb{C}$ hol., da $\forall n \in \mathbb{N}$ velja

a) $f(\frac{1}{n}) = \exp\left(\frac{n^2 - 3n + 2}{n^3}\right)$

b) $f(\frac{1}{n}) = \exp\left(\frac{n^3 - 3n + 2}{n^2}\right)$ $\qquad A = \{\frac{1}{n}, n \in \mathbb{N}\}$

a) $\Omega = \mathbb{C} \setminus \{0\}$

$f(\frac{1}{n}) = e^{\frac{1}{n} - \frac{3}{n^2} + \frac{2}{n^3}}$ $\qquad f(z) = e^{z - 3z^2 + 2z^3}$ hol na Ω $\Rightarrow$ Da

Ni ena sama

$$f(z) = e^{z - 3z^2 + 2z^3} + \underbrace{\sin\left(\frac{\pi}{z}\right)}_{\text{ima ničle v } z = \frac{1}{n}}$$ hol. na Ω ni enolična

$\Omega = \mathbb{C}$

$f(\frac{1}{n}) = e^{\frac{1}{n} - \frac{3}{n^2} + \frac{2}{n^3}}$ $\qquad f(z) = e^{z - 3z^2 + 2z^3}$

Princip identičnosti: če je $g(z)$ še ena funk. $g: \Omega \to \mathbb{C}$, $g(\frac{1}{n}) = e^{\frac{1}{n} - \frac{3}{n^2} + \frac{2}{n^3}}$

$\Rightarrow f, g$ se ujemata na množici $A = \{\frac{1}{n}, n \in \mathbb{N}\}$, A ima stekališče v Ω $\Rightarrow g = f$ na Ω

(b) $f(u) = e^{u - \frac{3}{u} + \frac{2}{u^2}}$

$\Omega = \mathbb{C} \setminus \{0\}$

$f(z) = e^{\frac{1}{z} - 3z + 2z^2}$ holom. na Ω

$g(z) = e^{1/z - 3z + 2z^2} + \sin\frac{\pi}{z}$ ni enolična

$\Omega = \mathbb{C}$

$f(z) = e^{1/z - 3z + 2z^2}$ ni holom na Ω

$\lim\limits_{z \to 0} f(z) = \lim\limits_{u \to \infty} f(\frac{1}{u}) = \infty$ ne obstaja $\Rightarrow$ ni zvezna v točki $z=0$

Residuum

- Če je α izolirana singularnost funkcije f in $f(z) = \sum_{-\infty}^{\infty} c_n(z-\alpha)^n$ njen Laurentov razvoj okrog α, potem je $\text{Res}(f,\alpha) = c_{-1}$ *residuum* funkcije f v točki α.
- Če ima f pol stopnje največ m v točki α, je $\boxed{\text{Res}(f,\alpha) = \frac{((z-\alpha)^m f(z))^{(m-1)}(\alpha)}{(m-1)!}}$.

nima luknj

- Če je D enostavno povezano območje, f holomorfna na $D \setminus \{\alpha_1, \ldots, \alpha_k\}$ in γ sklenjena krivulja v D, potem je $\oint_\gamma f(z)dz = 2\pi i \sum_{j=1}^{k} \text{Res}(f,\alpha_j) I_\gamma(\alpha_j)$. indeks
- Naj bo R holomorfna funkcija s končno singularnostmi v zgornji polravnini in brez singularnosti na realni osi.
  (1) Če ima R ničlo stopnje vsaj 2 v neskončnosti (tj. obstaja limita $\lim\limits_{z\to\infty} z^2 f(z)$), potem je
  $$\int_{-\infty}^{\infty} R(x)dx = 2\pi i \sum_{\text{Im}\,\alpha>0} \text{Res}(R,\alpha).$$
  (2) Če ima R ničlo stopnje vsaj 1 v neskončnosti (tj. obstaja limita $\lim\limits_{z\to\infty} zf(z)$), potem je
  $$\int_{-\infty}^{\infty} R(x)e^{ix}dx = 2\pi i \sum_{\text{Im}\,\alpha>0} \text{Res}(R(z)e^{iz},\alpha).$$

(1) Določi izolirane singularnosti in njih residue:

(a) $f(z) = \frac{z^2+4z+5}{z^2+z}$

$z_1 = 0$ pol st. 1 $\quad \text{Res}(f,0) = \frac{((z-0)^1 f(z))^{(0)}(0)}{(1-1)!} = \left(\cancel{z}\,\frac{z^2+4z+5}{\cancel{z^2}+1}\right)(0) = 5$

$z_2 = -1$ pol st. 1 $\quad \text{Res}(f,-1) = \frac{((z+1)f(z))(-1)}{0!} = \frac{z^2+4z+5}{z}\Big|_{z=-1} = -2$

(b) $f(z) = \frac{e^z}{\text{sh}\, z}$

Singularnosti: $\text{sh}\, z = 0$

$z_k = k\pi i$, $k \in \mathbb{Z}$ poli? katere stopnje?

Trik: množimo z $(z-\alpha)^m$ in sledimo ali $\exists$ limita $z \to \alpha$

$k=0$ $\quad \lim\limits_{z\to 0} z f(z) = \lim\limits_{z\to 0} \frac{ze^z}{\text{sh}\, z} \overset{\text{L'Hop}}{=} \lim\limits_{z\to 0} \frac{e^z + ze^z}{\text{ch}\, z} = 1$

k poljuben $\quad \lim\limits_{z\to z_k} (z-z_k)^1 f(z) = \lim\limits_{z\to z_k} \frac{(z-k\pi i)e^z}{\text{sh}\, z} \overset{\text{L'Hop}}{=} \lim\limits_{z\to z_k} \frac{e^z(1+z-k\pi i)}{\text{ch}\, z} = \frac{e^{k\pi i}}{\cos k\pi} = \frac{(-1)^k}{(-1)^k} = 1$

Gre za pol največ st. 1

$\text{Res}(f, z_k) = \frac{\lim_{z\to z_k}((z-z_k)^1 f(z))}{0!} = 1$

(c) $f(z) = \dfrac{ze^z}{(z+1)^{34}}$

$z=-1$ pol st. 34 $\Rightarrow \operatorname{Res}(f,-1) = \dfrac{\left((z+1)^{34} f(z)\right)^{(33)}\Big|_{z=-1}}{33!} = \,?$

Razvoj u Laurentovu vrstu

$$f(z) = \frac{e^z z}{(z+1)^{34}} = \frac{1}{(z+1)^{34}} e^{(z+1)-1}\left((z+1)-1\right)$$

$$= \frac{1}{(z+1)^{34}} \frac{1}{e}\left(1 + (z+1) + \frac{1}{2!}(z+1)^2 + \dots\right)\left((z+1)-1\right) =$$

$$= \frac{1}{e(z+1)^{34}}\left((z+1) + (z+1)^2 + \frac{(z+1)^3}{2!} + \dots - 1 - (z+1) - \frac{(z+1)^2}{2!} - \dots\right)$$

$$\operatorname{Res}(f,-1) = \frac{1}{e}\left(\frac{1}{32!} - \frac{1}{33!}\right)$$

(2) Izračunaj

(a) $\displaystyle\int_0^{2\pi} \frac{dx}{1-2a\cos x + a^2}$ $a \in \mathbb{C}$, $|a| \neq 1$

$$\oint_{|z|=1} f(z)\,dz = \int_0^{2\pi} f(e^{it})\, i e^{it}\, dt$$

$x \in [0, 2\pi]$

$z = e^{ix}$

$dz = ie^{ix}dx$ $dx = \dfrac{dz}{iz}$

$$\cos x = \operatorname{ch} ix = \frac{e^{ix}+e^{-ix}}{2} = \frac{z+\frac{1}{z}}{2}$$

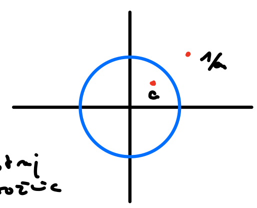
$$\Rightarrow \int_0^{2\pi}\frac{dx}{1-2a\cos x+a^2} = \oint_{|z|=1}\frac{\frac{dz}{iz}}{1-2a\frac{z+1/z}{2}+a^2} = \oint\frac{dz}{i(z-az^2-a+a^2z)} = \oint\frac{i\,dz}{az^2-(a^2+1)z+a} = \dots$$

Singularnosti
$$z_{1,2} = \frac{(a^2+1)\pm\sqrt{(a^2+1)^2-4a^2}}{2a} = \frac{a^2+1\pm\sqrt{a^4-2a^2+1}}{2a} = \frac{a^2+1\pm(a^2-1)}{2a}$$

$z_1 = a$ $z_2 = \dfrac{1}{a}$

$$\dots = \oint\frac{i\,dz}{a(z-a)(z-1/a)}$$

po izrazu uzimamo u sing. unutar kružnice



ako je $|a|<1$

ako je $|a|<1$ $= 2\pi i \operatorname{Res}(f, a)$

$$= 2\pi i\frac{(z-a)\frac{i}{a(z-a)(z-1/a)}\Big|_{z=a}}{0!} = -2\pi\frac{1}{a^2-1} = \frac{2\pi}{1-a^2}$$

ako je $|a|>1$ $= 2\pi i \operatorname{Res}\left(f, \frac{1}{a}\right)$

$$= 2\pi i\frac{\frac{1}{-ia(z-a)}\Big|_{z=\frac{1}{a}}}{0!} = \frac{2\pi}{a^2-1}$$

(b) $\int_0^{2\pi} (\cos x)^n \, dx = \dots \qquad n \in \mathbb{N}$

$x \in [0, 2\pi)$

$z = e^{ix}$ sing. $z = 0$ pol st. $(n+1)$

$$\dots = \oint \underbrace{\left(\frac{z + 1/z}{2}\right)^n \frac{dz}{iz}}_{f(z)} = 2\pi i \, \mathrm{Res}\,(f, 0) = \dots$$

$\mathrm{Res}\,(f,0) = ?$

$$f(z) = \frac{1}{i2^n} \frac{1}{z}\left(z + \frac{1}{z}\right)^n = \frac{1}{2^n i} \frac{1}{z}\left(z^n + \binom{n}{1} z^{n-1}\frac{1}{z} + \dots + \binom{n}{n-1} z \frac{1}{z^{n-1}} + \frac{1}{z^n}\right)$$

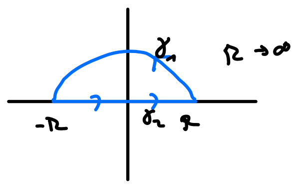
$$= \frac{1}{i2^n} \frac{1}{z}\left(z^n + \binom{n}{1} z^{n-2} + \dots + \binom{n}{n-1} z^{-n+2} + z^{-n}\right)$$

Iščemo člen c_{-1}, torej v oklepaju z^0

$$\mathrm{Res}\,(f,0) = \begin{cases} 0 & ; \ n \text{ liho} \\ \frac{1}{i2^n}\binom{n}{n/2} & ; \ n \text{ sodo} \end{cases}$$

$$\dots = 2\pi i \begin{cases} 0 \\ \frac{1}{i2^n}\binom{n}{n/2} \end{cases} = \begin{cases} 0 \\ \frac{\pi}{2^{n-1}}\binom{n}{n/2} \end{cases}$$

4 (a) $\int_0^\infty \frac{dx}{a^4 + x^4}$, $a > 0$



$f(z) = \frac{1}{a^4 + z^4}$

Singularnosti $f(z)$

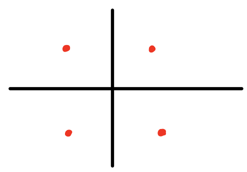
$a^4 + z^4 = 0$

$z^4 = -a^4$

$z^2 = \pm i a^2$

$z = \pm\sqrt{\pm i}\, a = \pm\sqrt{e^{\pm\frac{\pi i}{2}}}\, a = \pm e^{\pm\frac{\pi i}{4}} a$

$z = a e^{\frac{\pi i}{4} + \frac{2k\pi i}{4}} \qquad k = 0, \dots, 3 \qquad 1$ st.



$$\mathrm{Res}\,(f, a e^{\frac{\pi i}{4}}) = \frac{1}{0!}\left(f(z)(z - z_0)\right)\Big|_{z_0} = \lim_{z \to z_0} \frac{z - a e^{\frac{\pi i}{4}}}{a^4 + z^4} \overset{\text{L'H}}{=} \lim_{z \to z_0} \frac{1}{4z^3} = \frac{1}{4a^3} e^{-\frac{3}{4}\pi i}$$

$$\mathrm{Res}\,(f, a e^{\frac{3\pi i}{4}}) = \lim_{z \to z_1} \frac{z - a e^{\frac{3}{4}\pi i}}{a^4 + z^4} \overset{\text{L'H}}{=} \frac{1}{4z^3} = \frac{1}{4a^3} e^{-\frac{9}{4}\pi i}$$

$$\int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz = \oint f(z)\,dz = 2\pi i \left(\mathrm{Res}\,(f, z_0) + \mathrm{Res}\,(f, z_1)\right) = \frac{\pi i}{2a^3}\left(e^{-\frac{3}{4}\pi i} + e^{-\frac{9}{4}\pi i}\right)$$

$$= \frac{\pi i}{2a^3}\left(-\cancel{\frac{\sqrt{2}}{2}} - i\frac{\sqrt{2}}{2} + \cancel{\frac{\sqrt{2}}{2}} - i\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}\pi}{2a^3}$$

$$\int_{\gamma_1} f(z)\,dz \xrightarrow[R \to \infty]{?} 0$$

$$\left|\int_{\gamma_1} f(z)\,dz\right| \le \ell(\gamma_1) \max_{z \in \gamma_1} |f(z)| = \pi R \max_{z \in \gamma_1} \frac{1}{|a^4 + z^4|} \le \pi R \max \frac{1}{|z^4| - |a^4|} = \frac{\pi R}{R^4 - |a^4|} \xrightarrow{R \to \infty} 0$$

$$\Rightarrow \int_0^\infty \frac{1}{a^4 + x^4}\,dx = \frac{1}{2} \cdot \frac{\sqrt{2}\pi}{2a^3} = \frac{\sqrt{2}\pi}{4a^3}$$

Ocena deluje, če $\exists \lim z^2 f(z)$

ⓑ $$\int_{-\infty}^{\infty} \frac{\cos x}{x^2+4x+5}\,dx = \mathrm{Re}\int_{-\infty}^{\infty} \frac{e^{ix}}{x^2+4x+5}\,dx$$

$$f(z) = \frac{e^{iz}}{z^2+4z+5} = \frac{e^{iz}}{(z+2)^2+1} = \frac{e^{iz}}{(z+2+i)(z+2-i)}$$

$$\mathrm{Res}(f, -2+i) = \frac{e^{iz}}{z+2+i}\Big|_{z=-2+i} = -i\,\frac{\cos 2}{2e} - \frac{\sin 2}{2e}$$

$$\int_{-R}^{R} f(x)\,dx + \int_{\gamma_1} f(z)\,dz = 2\pi i\,\mathrm{Res}(f,-2+i) = \underbrace{\frac{\pi\cos 2}{e}}_{\text{rezultat}} - \frac{\pi \sin 2}{e}\,i$$

$z = a+bi \qquad b>0$

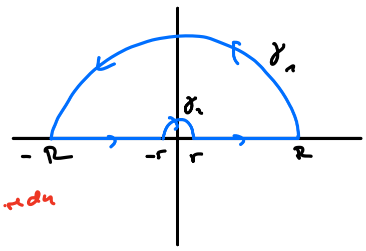
$$\left|\int_{\gamma_1} f(z)\,dz\right| \le \ell(\gamma_1)\max_{z\in\gamma_1}|f(z)| = \pi R \max \frac{|e^{iz}|}{|z^2+4z+5|} = \pi R \max \frac{|e^{ia}e^{-b}|}{|z+2+i||z+2-i|} \le$$

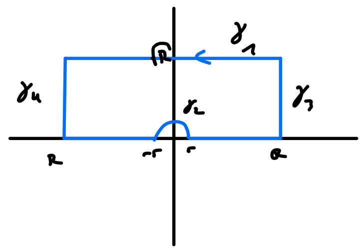
$$\le \pi R \max \frac{e^{-b}}{(|z|-|2+i|)(|z|-|2-i|)} \le \pi R \frac{1}{(R-3)(R-3)} \to 0$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\cos x}{x^2+4x+5}\,dx = \frac{\pi}{e}\cos 2$$

ⓒ $$\int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx = \mathrm{Im}\int_{-\infty}^{\infty} \frac{e^{ix}}{x}\,dx$$

1. $\sin x \to \mathrm{Im}\, e^{ix}$
2. pol na Re osi
3. imenovalec st. le 1 — ni v redu





$$\int_{\gamma_1} f(z)\,dz + \int_{-R}^{-r} f(x)\,dx + \int_{\gamma_2} f(z)\,dz + \int_{r}^{R} f(x)\,dx = 2\pi i \cdot 0 = 0$$

$$\int_{\gamma_1} = \int_{\gamma_3} + \int_{\gamma_4}$$

$$\left|\int_{\gamma_1} f(z)\,dz\right| \le 2R \max_{|z|=R}\left|\frac{e^{iz}}{z}\right| = 2R\,\frac{e^{-b}}{R} = 2e^{-\sqrt{R}} \to 0$$

$$\left|\int_{\gamma_3} f(z)\,dz\right| \le \sqrt{R}\max_{z\in\gamma_3}\frac{e^{-b}}{|z|} \le \sqrt{R}\,\frac{1}{R} \to 0 \qquad \text{Podobno za } \gamma_4$$

$$-\int_{\gamma_2} f(z)\,dz = \int_0^{\pi} \frac{e^{ire^{i\varphi}}}{re^{i\varphi}}\, rie^{i\varphi}\,d\varphi = i\int_0^{\pi} e^{ire^{i\varphi}}\,d\varphi \to \pi i$$

$z = re^{i\varphi} \qquad \varphi \in (0,\pi)$

$$\int_0^{\pi} \lim_{r\to 0} e^{ire^{i\varphi}}\,d\varphi = \int_0^{\pi} 1\,d\varphi = \pi$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx = \pi$$

ⓓ $$\int_{-\infty}^{\infty} \frac{\cos x}{x^2+2i}\,dx = \int_{-\infty}^{\infty} \frac{e^{ix}+e^{-ix}}{2x^2+4i} = \int_{-\infty}^{\infty} \frac{e^{ix}dx}{2(x^2+2i)} + \int_{-\infty}^{\infty} \frac{e^{-ix}dx}{2(x^2+2i)} = \int_{-\infty}^{\infty} \frac{e^{ix}dx}{2(x^2+2i)} + \int_{-\infty}^{\infty} \frac{e^{it}dt}{2(t^2+2i)} = \int_{-\infty}^{\infty} \frac{e^{ix}dx}{x^2+2i} =$$

ali $t=-x$

$=$

DN

ⓔ $$\int_{-\infty}^{\infty} \frac{1-\cos x}{x^2}\,dx = \int_{-\infty}^{\infty} \frac{2\sin^2 x/2}{x^2}\,dx = \int_{-\infty}^{\infty} \left(\frac{\sin t}{t}\right)^2 dt = \mathrm{Re}\lim_{\substack{R\to\infty\\ r\to 0}}\left(\int_{-R}^{-r} \frac{1-e^{ix}}{x^2}\,dx + \int_{r}^{R} \frac{1-e^{ix}}{x^2}\,dx\right) =$$

$$\int_{r}^{R} f(x)\,dx + \int_{\gamma} f(z)\,dz + \int_{-R}^{-r} f(x)\,dx + \int_{\delta} f(z)\,dz = 2\pi i \cdot 0$$

$z = a+bi$

$$\left|\int_{\gamma} f(z)\,dz\right| \le \pi R \max_{z\in\gamma}\left|\frac{1-e^{iz}}{z^2}\right| = \pi R\,\frac{|1-e^{ai-b}|}{R^2} \le \frac{\pi}{R} \to 0$$

$$\int_\sigma f(z)\,dz = \int_0^\pi \frac{1-e^{ire^{i\varphi}}}{r^2e^{2i\varphi}}\, r i e^{i\varphi}\,d\varphi = \int_0^\pi i\,\frac{1-e^{ire^{i\varphi}}}{re^{i\varphi}}\,d\varphi \xrightarrow[r\to 0]{\text{L'Hop}} i\int_0^\pi \frac{-ie^{i\varphi}e^{ire^{i\varphi}}}{e^{i\varphi}}\,d\varphi =$$

$z = re^{i\varphi}$, $\varphi \in [0,\pi]$, $dz = rie^{i\varphi}d\varphi$

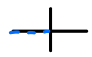
$$\xrightarrow{r\to 0} \int_0^\pi e^{ire^{i\varphi}}\,d\varphi = \int_0^\pi 1\,d\varphi = \pi$$

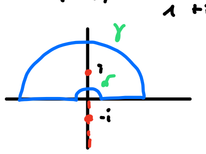
$$\int_{-\infty}^{\infty} \frac{1-\cos x}{x^2}\,dx = \pi$$

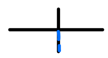
f) $\displaystyle\int_0^\infty \frac{\ln x}{1+x^2}\,dx$

Nasvet: integriraj po robu kolobarja v zg. polravnini, pri čemer logaritem def. v kompleksni ravnini, zarezani po negativni imaginarni osi.

$$f(z) = \frac{\log z}{1+z^2}$$

 $\ln|z| + i\arg z \qquad \arg z \in (-\pi,\pi)$



 $\arg \in \left(-\frac{\pi}{2}, \frac{7\pi}{2}\right)$

$$\int_\gamma f(z)\,dz + \int_{-R}^{-r} f(z)\,dz - \int_\sigma f(z)\,dz + \int_r^R f(x)\,dx = 2\pi i \operatorname{Res}(f, i)$$

$$\int_{-R}^{-r} f(z)\,dz \underset{-t=z}{=} \int_r^R \frac{\log(-t)}{1+t^2} = \int_r^R \frac{\ln t + i\pi}{1+t^2}\,dt = \int_r^R \frac{\ln t}{1+t^2} + \pi i \int_0^\infty \frac{1}{1+t^2}\,dt$$

$$\left|\int_\gamma f(z)\,dz\right| \le \pi R \max \left|\frac{\log z}{1+z^2}\right| = \pi R \max \left|\frac{\ln|z| + i\arg z}{1+z^2}\right| \le \pi R \max \frac{|\ln|z|| + |i\arg z|}{||z|^2-1|} \le$$

$$\pi R\, \frac{\ln R + \pi}{R^2+1} \xrightarrow{R\to\infty} 0$$

$$\left|\int_\sigma f(z)\,dz\right| \le \pi r \max_{z\in\sigma} \left|\frac{\log|z| + i\arg z}{1+z^2}\right| \le \pi r\,\frac{|\ln r| + \pi}{1-r^2} \overset{\text{L'Hop}}{=} \pi^2 r + \frac{\frac{1}{r}}{\frac{1}{r^2}} = r(\pi^2+1) \to 0$$

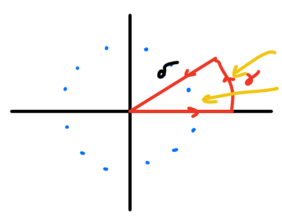
$$\operatorname{Res}(f, i) \overset{\text{pol 1. st.}}{=} \lim \left(f(z)(z-i)\right) = \lim_{z\to i} \frac{\log z}{z+i} = \frac{\log i}{2i} = \frac{\ln 1 + i\frac{\pi}{2}}{2i} = \frac{\pi}{4}$$

$$2I + \pi i \int_0^\infty \frac{dt}{1+t^2} = 2\pi i\,\frac{\pi}{4} \quad\Rightarrow\quad I = 0$$

g) $\displaystyle\int_0^\infty \frac{1}{1+x^n}\,dx \qquad n \ge 2$

Nasvet: integriraj funkcijo po robu krožnega izseka, ki vsebuje 1 pol te funkcije

$$f(z) = \frac{1}{1+z^n}$$



$$z^n = -1 \qquad z = \sqrt[n]{-1} = e^{\frac{\pi i}{n} + \frac{2k\pi i}{n}} \qquad k = 0, \dots, n-1$$

$$\int_\gamma f(z)\,dz \;(\to 0) + \int_0^\infty f(x)\,dx + \int_\sigma f(z)\,dz$$

$$\int_0^\infty \frac{1}{1+x^n}\,dx = \frac{\pi}{n\sin \pi/n}$$

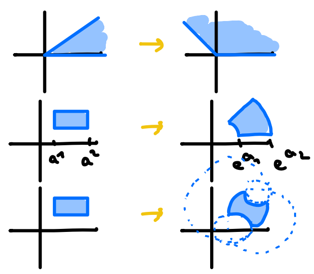
DN

Princip maksimuma in biholomorfne preslikave

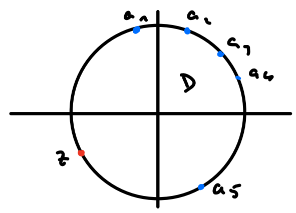
odprt in povezan

- Če je f nekonstantna in holomorfna na območju D, potem $|f|$ ne doseže maksimuma na D. Če je še f zvezna na $\bar{D}$ in je območje D omejeno, potem f na $\bar{D}$ doseže maksimum na ∂D.
- Funkcija $f(z) = z^t$, $t > 0$, biholomorfno preslika območje $\{z \in \mathbb{C} \mid r_1 < |z| < r_2,\ \varphi_1 < \arg z < \varphi_2\}$ na $\{z \in \mathbb{C} \mid r_1^t < |z| < r_2^t,\ t\varphi_1 < \arg z < t\varphi_2\}$.
- Funkcija $f(z) = e^z$ biholomorfno preslika pravokotnik $\{z \in \mathbb{C} \mid \operatorname{Re} z \in (a_1, a_2),\ \operatorname{Im} z \in (\varphi_1, \varphi_2)\}$ na $\{z \in \mathbb{C} \mid e^{a_1} < |z| < e^{a_2},\ \varphi_1 < \arg z < \varphi_2\}$.
- *Möbiusove transformacije* $f(z) = \frac{az+b}{cz+d}$, $ad \neq bc$, biholomorfno preslikajo $\mathbb{C} \cup \{\infty\}$ nase, so določene s slikami treh točk in ohranjajo množico $\{\text{premice v } \mathbb{C}\} \cup \{\text{krožnice v } \mathbb{C}\}$.
- Vse biholomorfne preslikave iz kroga $D(0,1)$ nase so Möbiusove transformacije oblike $f(z) = \omega \cdot \frac{z-\alpha}{1-\bar{\alpha}z}$, $|\alpha| < 1$, $|\omega| = 1$.

Ohranitev kotov pri transformaciji



① $a_1, \dots, a_n \in S^1$ (enotska krožnica) Dokaži: $\exists\, z \in S^1$; produkt razdalj $|z-a_1|\cdot|z-a_2|\cdot \dots \cdot|z-a_n| \geq 1$



Def. funkcijo $f(z) = (z-a_1)(z-a_2)\dots$

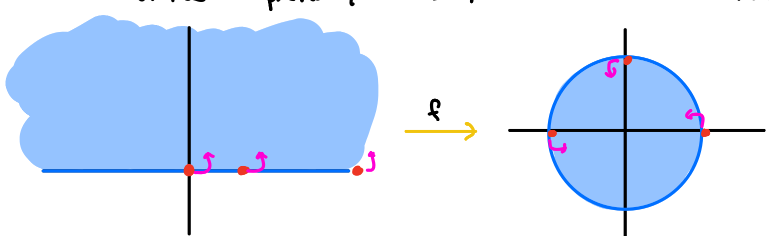
Princip maksima: $|f|$ doseže maksimum na $S^1 = \partial D$

$|f(z_0)| \geq 1$?

$$|f(0)| = |a_1|\cdot|a_2|\cdot \dots \cdot|a_n| = 1$$

$$|f(z_0)| \geq |f(0)| = 1$$

② Biholomorfno preslikaj zg. polravnino na enotski disk.



premica → krožnica

$$f(z) = \frac{az+b}{cz+d}$$

Izberemo si 3 točke in povemo kam se preslikajo

$0 \to 1 \Rightarrow \frac{b}{d} = 1$ $d = b$

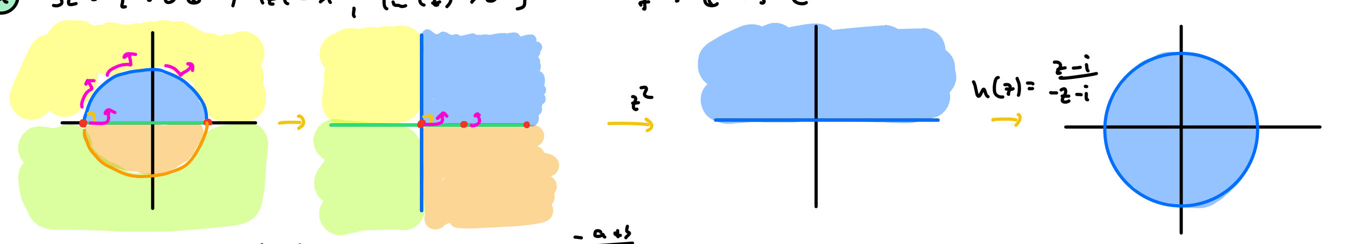
$1 \to i \Rightarrow \frac{a+b}{c+d} = i$ $a+b = i(-a+b)$, $b = a\frac{-1-i}{1-i} = -ia$

$\infty \to -1 \Rightarrow \frac{a}{c} = -1$ $c = -a$

$$f(z) = \frac{az - ia}{-az - ia} = \frac{z-i}{-z-i}$$

③ Poišči bihol. presl. iz območja Ω na enotski disk

ⓐ $\Omega = \{z \in \mathbb{C};\ |z| < 1,\ \operatorname{Im}(z) > 0\}$ $f: \hat{\mathbb{C}} \to \hat{\mathbb{C}}$



$-1 \to 0$

$0 \to 1$

$1 \to \infty$

$\frac{-a+b}{-c+d} = 0 \Rightarrow a = b$

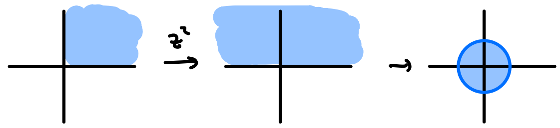
$\frac{b}{d} = 1 \Rightarrow b = d$

$c + d = 0$ $c = -d = -b$

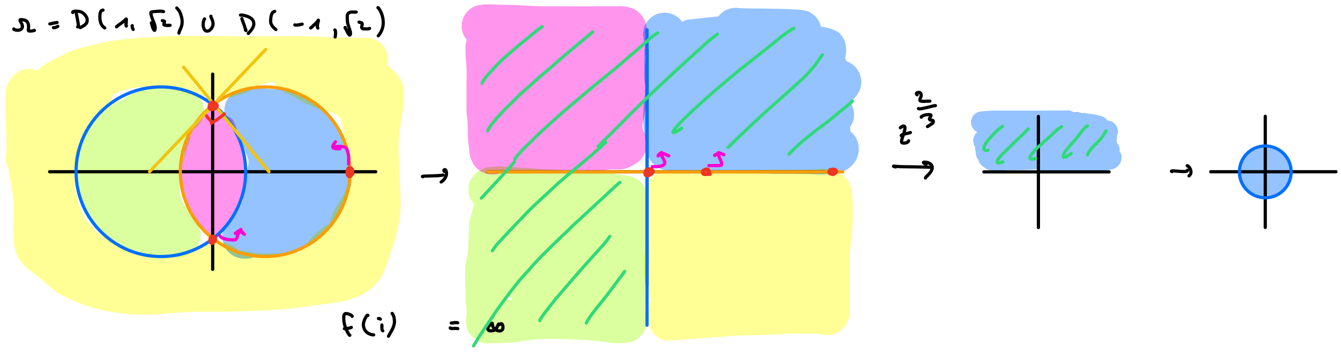
$$\frac{bz+b}{-bz+b} = \frac{z+1}{-z+1}$$

$$\Rightarrow F(z) = h(g(f(z))) = \frac{\left(\frac{z+1}{-z+1}\right)^2 - i}{-\left(\frac{z+1}{-z+1}\right)^2 - i}$$

ⓑ $\Omega = \{z \in \mathbb{C} \, ; \, \arg z \in (0, \frac{\pi}{2})\}$



ⓒ $\Omega = D(1, \sqrt{2}) \cup D(-1, \sqrt{2})$



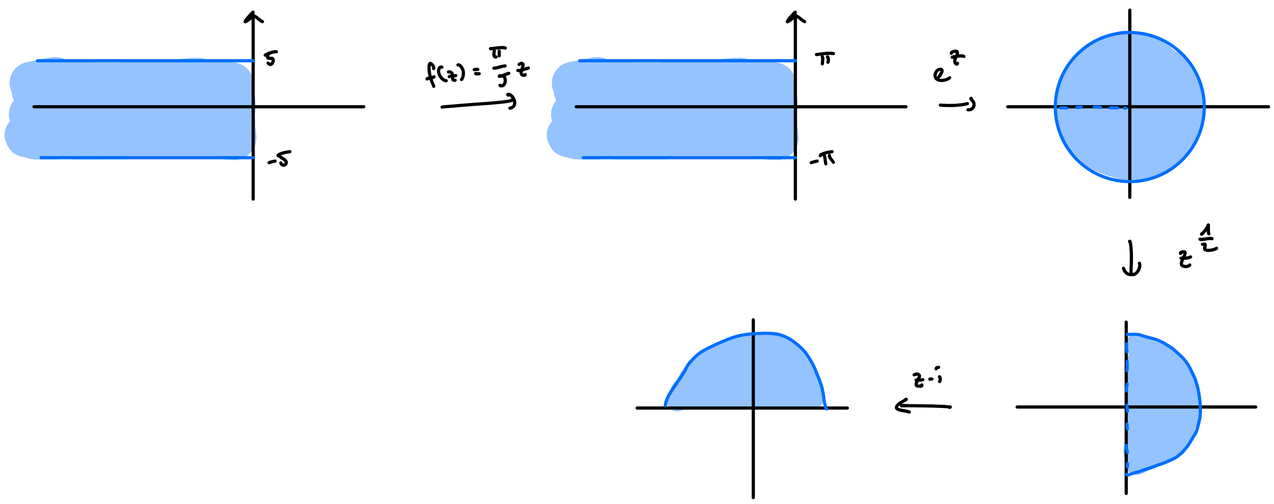
$f(i) = \infty$

$f(-i) = 0$

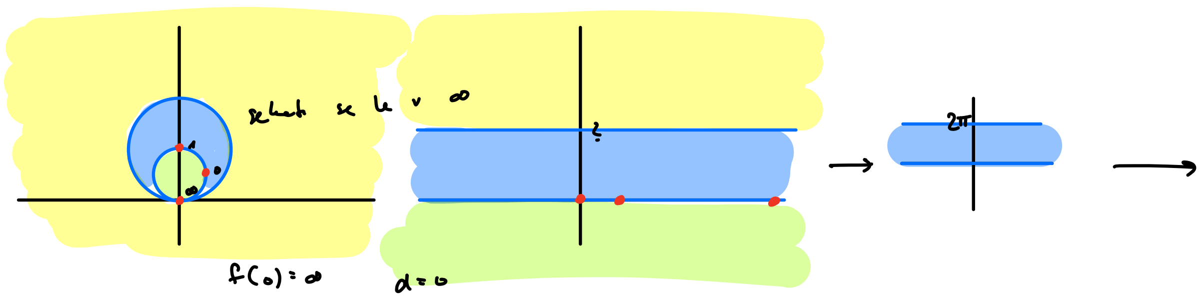
$f(1+\sqrt{2}) = 1$

$\Rightarrow f(z)$

ⓓ $\Omega = \{z \in \mathbb{C} \, ; \, \operatorname{Re}(z) < 0, \, \operatorname{Im}(z) \in (-5, 5)\}$



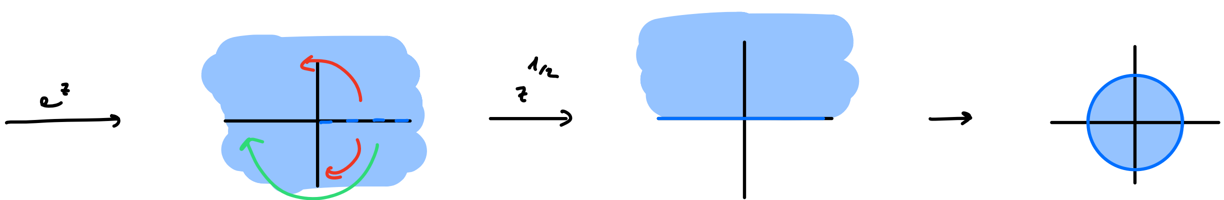
ⓔ $\Omega = \{z \in \mathbb{C} \, , \, |z - i| > 1 \, , \, |z - 2i| < 2\}$



$f(0) = \infty$ $\quad d = 0$

$f(1+i) = 0$ $\quad a + ia + b = 0 \Rightarrow f \Rightarrow f(4i)$ vrednost vzporednice

$f(2i) = 1$ $\quad \frac{2ia + b}{2ic} = 1$



s korenom lahko zapiramo kakor želim iz obeh strani ali pa samo iz ene.

Harmonične funkcije

$\Delta u = u_{xx} + u_{yy} + u_{zz} = 0$

$f: D \to \mathbb{C}$ holomorfna, $f = u + iv \to u, v$ harmonični

$u: D^{\subseteq \mathbb{R}^2} \to \mathbb{C}$ harmonična $\Rightarrow \exists f: D \to \mathbb{C}$ hol, Re $f = u$

D enostavno povezano (brez lukenj)

(4) $U, V \subseteq \mathbb{C}$, $u: V \to \mathbb{R}$ harmonična, $f: U \to \mathbb{C}$ holom. Dokaži da je $u \circ f$ harmonična

$u(x,y)$

$f(x+iy) = v(x,y) + i\, w(x,y)$

$$u(f(x,y)) = u(v(x,y), w(x,y))$$

$$u(f(x,y))_x = u_x(v(x,y), w(x,y)) \cdot v_x(x,y) + u_y(v,w)\, w_x(x,y)$$

$$u(f(x,y))_{xx} = (u_{xx}(v,w)\, v_x + u_{xy}(v,w)\, w_x)\, v_x + u_x(v,w)\, v_{xx} + (u_{yx}(v,w)\, v_x + u_{yy}(v,w)\, v_x) w_x + u_y(v,w)\, w_{xx}$$

$$\Delta u(f(v,w)) = u(f)_{xx} + u(f)_{yy} = u_{xx} v_x^2 + 2u_{xy} w_x v_x + u_{yy} w_x^2 + u_x v_{xx} + u_y w_{xx}$$
$$+ u_{xx} v_y^2 + 2u_{xy} w_y v_y + u_{yy} w_y^2 + u_x v_{yy} + u_y w_{yy} =$$

$v_x = w_y$
$v_y = -w_x$ $\quad v \ldots$ hol.
$u_{xx} + u_{yy} = 0 \quad u \ldots$ harmonična

$$= \cancel{u_{xx}}\, w_x^2 + 2\cancel{u_{xy} w_x w_y} + u_{yy} v_x^2 + \cancel{u_x v_{xx}} + \cancel{u_y v_{xx}} + u_{xx} w_x^2 - 2\cancel{u_{xy} w_y w_x} + u_{yy} w_y^2 +$$
$$\cancel{u_x v_{yy}} + \cancel{u_y w_{yy}} = 0$$

(1) $\Omega \subseteq \mathbb{R}^2$, $u: \Omega \to \mathbb{R}$ harmonična

$D(a, R) \subseteq \Omega$

Dokaži: $\quad u(a) = \dfrac{1}{\pi r^2} \displaystyle\iint_D u(x,y)\, dx\, dy \qquad$ Povprečna vrednost za disk

Velja za krožnico $\quad u(a) = \dfrac{1}{2\pi r} \displaystyle\oint_{\partial D} u(x,y)\, ds$

$$\iint_{D(a,R)} u(x,y)\, dx\, dy = \int_0^R \int_0^{2\pi} u(x_0 + r\cos\varphi, y_0 + r\sin\varphi)\, r\, d\varphi\, dr = \int_0^R dr \oint_{\partial D} u(x,y)\, ds = \int_0^R 2\pi r\, dr\; u(a) =$$

$x = x_0 + r\cos\varphi \qquad \varphi \in [0, 2\pi]$

$y = y_0 + r\sin\varphi \qquad r \in [0, R]$

$a = (x_0, y_0)$

$$= 2\pi\, u(a) \frac{R^2}{2} = \pi R^2\, u(a)$$

Dirichletov problem

D (omejeno) območje v $\mathbb{R}^2$

$f: \partial D \to \mathbb{R}$ zvezna

Iščemo $u: \bar{D} \to \mathbb{R}$

$u: D \to \mathbb{R}$ harmonična

$u|_{\partial D} = f$

Rešitev:

$$u(\vec{r}_0) = \int_{\partial D} \underbrace{\partial_{\vec{n}}\, G(\vec{r}, \vec{r}_0)}_{\text{Poissonovo jedro}} f(\vec{r})\, ds$$

smerni / normalni odvod

$\iint \quad dS$ ali

Greenova funkcija za $D \subseteq \mathbb{R}^2$

$G(\vec{r}, \vec{r}_0): \bar{D} \times D \to \mathbb{R}$ z naslednjima lastnostima:

① $G(\vec{r}, \vec{r}_0) = \dfrac{1}{2\pi} \log|\vec{r} - \vec{r}_0| + u(\vec{r}, \vec{r}_0)$,

u harmonična na D

(kot funkcija $\vec{r}$)

$[\Delta G = \delta_{\vec{r}, \vec{r}_0}$

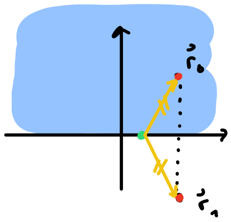
② $G(\vec{r}, \vec{r}_0)|_{\vec{r} \in \partial D} = 0$

Za $D \subseteq \mathbb{R}^3$ velja isto le:

$$G(\vec{r}, \vec{r}_0) = -\frac{1}{4\pi|\vec{r} - \vec{r}_0|} + u(\vec{r}, \vec{r}_0)$$

5 Poišči Greenovo funkcijo

(a) $D = \mathbb{R} \times (0,\infty)$



$$G(\vec r, \vec r_0) = \frac{1}{2\pi} \log|\vec r - \vec r_0| + u(\vec r, \vec r_0)$$

- u harmonične
- na robu se izniči z logaritmom

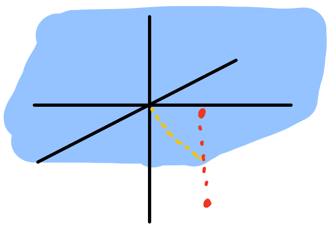
zrcaljena točka $\vec r_0$

$\frac{1}{2\pi}\log|\vec r - \vec r_1|$ harmonična na D (nima pola v D)

$$\vec r \in \partial D \quad \frac{1}{2\pi}\log|\vec r - \vec r_0| - \frac{1}{2\pi}\log|\vec r - \vec r_1| = 0$$

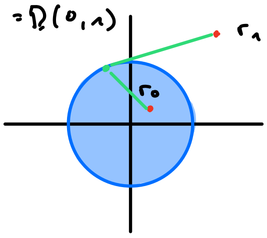
$$G(\vec r, \vec r_0) = \frac{1}{2\pi}\log|\vec r - \vec r_0| - \frac{1}{2\pi}\log|\vec r - \vec r_1|$$

$D = \mathbb{R}^2 \times (0,\infty)$



$$G(\vec r, \vec r_0) = -\frac{1}{4\pi|\vec r - \vec r_0|} + \frac{1}{4\pi|\vec r - \vec r_1|}$$

(b) $D = D(0,1)$



$$|\vec r_0||\vec r_1| = 1$$

$$\vec r_1 = \frac{\vec r_0}{|\vec r_0|^2}$$

$$G(\vec r, \vec r_0) = \frac{1}{2\pi}\log|\vec r - \vec r_0| - \frac{1}{2\pi}\log(|\vec r - \vec r_1||\vec r_0|)$$

Na robu

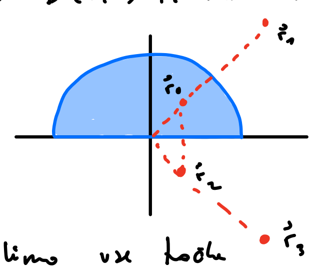
$$|\vec r - \vec r_0| = |\vec r_0|\left|\vec r - \frac{\vec r_0}{|\vec r_0|^2}\right|$$

$$|\vec r - \vec r_0|^2 = |\vec r_0|^2\left|\vec r - \frac{\vec r_0}{|\vec r_0|^2}\right|^2$$

$$\overset{1}{\cancel{\vec r\vec r}} - 2\vec r\vec r_0 + \vec r_0\vec r_0 = \frac{1}{\vec r_0\vec r_0}\left(\overset{1}{\cancel{\vec r\vec r}}\,|\vec r_0|^4 - 2|\vec r_0|^2\vec r\vec r_0 + \vec r_0\vec r_0\right)$$

$$0 = 0$$

(c) $D = D(0,1) \cap (\mathbb{R} \times (0,\infty))$



Zrcalimo vse točke skozi vse robove

$$\vec r_1 = \frac{\vec r_0}{|\vec r_0|^2}$$

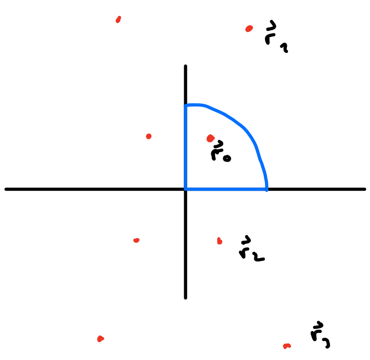
$$\vec r_3 = \frac{\vec r_2}{|\vec r_2|^2}$$

krajšata na $\cap$

$$G(\vec r, \vec r_0) = \frac{1}{2\pi}\log|\vec r - \vec r_0| - \frac{1}{2\pi}\log(|\vec r_0||\vec r - \vec r_1|) - \frac{1}{2\pi}\log|\vec r - \vec r_2| + \frac{1}{2\pi}\log(|\vec r_0||\vec r - \vec r_3|)$$

krajšata na $-$

(d)



8 členov

6 Reši Dirichletov problem $\Delta u = 0$ $\quad u|_{\partial D} = f$

(a) $D = D(0,1)$ $\qquad f(x,y) = x^2$

$$G(\vec r, \vec r_0) = \frac{1}{2\pi} \log |\vec r - \vec r_0| - \frac{1}{2\pi} \log |\vec r_0| \, |\vec r - \frac{\vec r_0}{|\vec r_0|^2}|$$

$$u(\vec r_0) = \oint_{\partial D} \partial_{\vec n} G(\vec r, \vec r_0) \, x^2 \, ds$$

$$G(x, y, x_0, y_0) = \frac{1}{2\pi} \log \sqrt{(x-x_0)^2 + (y-y_0)^2} - \frac{1}{2\pi} \log\left(\sqrt{x_0^2 + y_0^2} \sqrt{\dots}\right)$$

$$\partial_{\vec n} G = \nabla G \cdot \vec n$$

Obstaja hitrejši način - ugibanje rešitve

$u(x,y) = x^2 - y^2$ $\qquad$ rob : $x^2 + y^2 = 1$

$u(x,y) = x^2 - (1 - x^2) = 2x^2 - 1$ //

$u(x,y) = \frac{1}{2}(x^2 - y^2 + 1)$ $\qquad$ harmonična ✓

rob: $\frac{1}{2}(x^2 - (1 - x^2) + 1) = x^2$

(b) $D = \mathbb{R}^2 \times (0, \infty)$, $\quad f = x^2 + y^2$

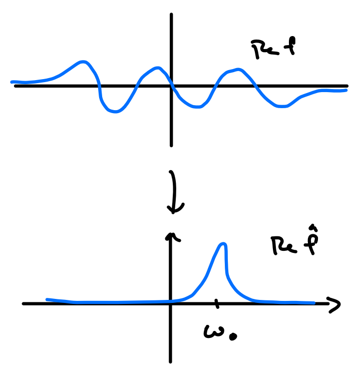
rob : $z = 0$

$u(x, y, z) = x^2 + y^2 - 2z^2$ $\qquad$ harmonična ✓

rob ✓

Fourierova transformacija

- $f \in L^1(\mathbb{R}) \Rightarrow \hat f(\omega) = \widehat{f(x)}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} \, dx$
- $f \in L^1(\mathbb{R}) \Rightarrow \hat f$ zvezna , $\hat f(\omega) \xrightarrow{|\omega| \to \infty} 0$
- $\widehat{f(x) e^{iax}} = \hat f(\omega - a)$ $\qquad a \in \mathbb{R}$
- $\widehat{f(ax)} = \frac{1}{a} \hat f(\frac{\omega}{a})$ $\qquad a > 0$
- $\widehat{f(x-a)} = e^{-ia\omega} \hat f(\omega)$
- f odvedljiva in $f' \in L^1(\mathbb{R}) \Rightarrow \hat{f'} = i\omega \hat f$
- $\widehat{f * g} = \sqrt{2\pi} \, \hat f \cdot \hat g$
- $f \in \mathcal{S}(\mathbb{R}) \Rightarrow \hat f \in \mathcal{S}(\mathbb{R})$
- $\hat{\hat f}(x) = f(-x)$ $\quad$ skoraj povsod
- $\langle f, g \rangle = \langle \hat f, \hat g \rangle = \int f \cdot \bar g \, dx$ $\qquad$ (celo na $L^2(\mathbb{R})$)



f	$\hat f$
$e^{-\frac{x^2}{2}}$	$e^{-\frac{\omega^2}{2}}$
$e^{-\lvert x \rvert}$	$\sqrt{\frac{2}{\pi}} \frac{1}{1+\omega^2}$
$\chi_{[-a,a]}$	$\sqrt{\frac{2}{\pi}} \frac{\sin(a\omega)}{\omega}$

1 $f(x) = \dfrac{e^{ix}}{x^2 + 2x + 5}$ $\qquad \hat f(\omega) = ?$

$$\hat f(\omega) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \frac{e^{ix}}{x^2 + 2x + 5} e^{-i\omega x} \, dx = \dots$$

$$\hat f(\omega) = \widehat{\frac{e^{ix}}{x^2+2x+5}} = \left(\widehat{\frac{1}{x^2+2x+5}}\right)(\omega-1) = \widehat{\frac{1}{(x+1)^2+4}} = e^{-i(-1)(\omega-1)}\,\widehat{\frac{1}{x^2+4}}(\omega-1)$$

$$= e^{i(\omega-1)}\,\frac14\,\widehat{\frac{1}{(\frac{x}{2})^2+1}}(\omega-1) = \frac14 e^{i(\omega-1)}\,2\,\widehat{\frac{1}{1+x^2}}\left(\frac{\omega-1}{1/2}\right) = \frac12 e^{i(\omega-1)}\sqrt{\frac{\pi}{2}}\,e^{-|2(\omega-1)|} =$$

$$= \sqrt{2\pi}\,e^{i(\omega-1)}\,e^{-|2(\omega-1)|}$$

② ⓐ $\widehat{e^{-ax^2}} = ?$ $\qquad a>0$

ⓑ $\widehat{e^{-x^2}\cos(2x)} = ?$

ⓐ $$\widehat{e^{-ax^2}} = \widehat{e^{-\frac{(\sqrt{2a}\,x)^2}{2}}} = \frac{1}{\sqrt{2a}}\,\widehat{e^{-\frac{x^2}{2}}}\left(\frac{\omega}{\sqrt{2a}}\right) = \frac{1}{\sqrt{2a}}\,e^{-\frac{\omega^2}{4a}}$$

ⓑ $$\cos\alpha = \frac{e^{i\alpha}+e^{-i\alpha}}{2}$$

$$\widehat{e^{-x^2}\frac{e^{2ix}}{2}} + \widehat{e^{-x^2}\frac{e^{-2ix}}{2}} = \frac12\,\widehat{e^{-x^2}}(\omega-2) + \frac12\,\widehat{e^{-x^2}}(\omega+2) = \frac12\frac{1}{\sqrt2}e^{-\frac{(\omega-2)^2}{4}} + \frac12\frac{1}{\sqrt2}e^{-\frac{(\omega+2)^2}{4}} =$$

$$= \frac{1}{2\sqrt2}\left(e^{-\frac{\omega^4}{4}+\omega-1} + e^{-\frac{\omega^4}{4}-\omega-1}\right) = \frac{1}{\sqrt2}\,e^{-\frac{\omega^4}{4}-1}\,\operatorname{ch}\omega$$

③ $a>0$ $\qquad f_a(x) = \dfrac{a}{\pi(a^2+x^2)}$

Dokaži $\qquad a,b>0 \Rightarrow f_a * f_b = f_{a+b}$

$$(f_a * f_b)(x) = \int_{-\infty}^{\infty} f_a(t)\,f_b(x-t)\,dt = \dots$$

$$f_a * f_b \overset{?}{=} f_{a+b} \qquad /\ \widehat{\ }$$

$$\sqrt{2\pi}\,\hat f_a\cdot\hat f_b \overset{?}{=} \hat f_{a+b}$$

$$\hat f_a = \widehat{\frac{1/a}{\pi(1+(\frac{x}{a})^2)}} = \frac{1}{\pi}\,\widehat{\frac{1}{1+x^2}}\left(\frac{\omega}{1/a}\right) = \frac{1}{\sqrt{2\pi}}\,e^{-|a\omega|}$$

$$\cancel{\sqrt{2\pi}}\,\frac{1}{\cancel{\sqrt{2\pi}}}\,e^{-a|\omega|}\,\frac{1}{\sqrt{2\pi}}\,e^{-b|\omega|} \overset{\checkmark}{=} \frac{1}{\sqrt{2\pi}}\,e^{-(a+b)|\omega|}$$

④ $f,g\in\mathcal{S}(\mathbb{R}) \Rightarrow \widehat{fg} = \dfrac{1}{\sqrt{2\pi}}\,\hat f * \hat g$

$$\widehat{fg} = \frac{1}{\sqrt{2\pi}}\,\hat f * \hat g \qquad /\ \widehat{\ }$$

$$\widehat{\widehat{fg}} = \frac{1}{\sqrt{2\pi}}\,\widehat{\hat f * \hat g}$$

$$fg(-x) = \frac{1}{\sqrt{2\pi}}\,\sqrt{2\pi}\,\hat{\hat f}\cdot\hat{\hat g}$$

$$f(-x)\cdot g(-x) = f(-x)\cdot g(-x)$$

⑤ Poišči funkcijo $f:\mathbb{R}\to\mathbb{R}$ da $\displaystyle\int_{-\infty}^{\infty} f(t)\,f(x-t)\,dt = \frac{1}{1+x^2}$ $\quad\forall x\in\mathbb{R}$

$$f*f(x) = \frac{1}{1+x^2} \qquad /\ \widehat{\ }$$

$$\widehat{f*f} = \widehat{\frac{1}{1+x^2}}$$

$$\sqrt{2\pi}\,\hat f^2(\omega) = \sqrt{\frac{\pi}{2}}\,e^{-|\omega|}$$

$$\hat f = \pm\frac{1}{\sqrt2}\,e^{-|\frac{\omega}{2}|}$$

$$\hat f(\omega) = \pm\frac{1}{\sqrt2}\,2\,e^{-|\omega|}\,(2\omega)$$

$$f(-x) = \pm\sqrt2\,\frac{1}{1+4x^2}\sqrt{\frac{2}{\pi}}$$

$$f(x) = \pm\frac{2}{\sqrt\pi}\,\frac{1}{1+4x^2}$$

(6) $f \in L^1(\mathbb{R})$ zv. odv., $x^2 f(x)$ in $x^2 f'(x)$ omejeni

(a) Dokaži, da predpis $g(x)=\sum_{k\in\mathbb{Z}} f(x+k)$ definira zvezno odv. funkcijo g s periodo 1

Konvergenca vrste za $\forall x \in \mathbb{R}$
$\sum_k f(x+k)$

Omejimo $x \in [0,1]$

Weierstrassov M test $\quad \sum |f(x+k)| = \sum |f_k| \le \sum \frac{M}{(x+k)^2} \le \sum_{k\ge 1} \frac{M}{k^2} + \sum_{k\le -2} \frac{M}{(1+k)^2} + \frac{M}{x^2} + \frac{M}{(x-1)^2}$

enakomerna konv. na $(0,1)$

$x=0 \quad \sum_{k\in\mathbb{Z}} |f(k)| \le |f(0)| + \sum_{k\neq 0} \frac{M}{k^2} < \infty$

enako je $x=1 \Rightarrow$ konv. na $[0,1]$

Periodičnost
$g(x+1)=g(x)$ ✓ $\Rightarrow$ konvergira na $\mathbb{R}$

Zvezna odvedljivost g
- $f(x)=\sum f_n(x)$ enak. konv. $\Rightarrow$ f_n zvezne $\Rightarrow$ f zvezna $\Rightarrow$ g zv. na $(0,1)$
  g zv. na $(1,2)$ ...
  g zv. na $(-\frac{1}{2},\frac{1}{2})$ (isto dalje)
- $f(x)=\sum f_n(x)$ konv., $\sum f_n'(x)$ enak. konv. $\Rightarrow$ $f(x)$ odv. in $f'(x)=\sum f_n'(x)$

(b) Določi Fourierove koeficiente c_n razvoja razvoja g v Fourierovo vrsto $\sum_{n\in\mathbb{Z}} c_n e^{2n\pi i x}$

ortogonalna baza

(T) Fourierova vrsta
$f \in L^1(-L,L)$
$$f(x) = \frac{a_0}{2} + \sum_{n\ge 1} a_n \cos\frac{n\pi x}{L} + \sum_{n\ge 1} b_n \sin\frac{n\pi x}{L} = \sum_{n\in\mathbb{Z}} c_n e^{\frac{n\pi i}{L}x} \qquad c_n \in \mathbb{C}$$
$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx \qquad c_n = \frac{\langle u, u_n\rangle}{\langle u_n, u_n\rangle} \qquad \langle f, g\rangle = \int f\cdot\bar{g}$$
$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx$$

Če je f zv. odv. je Four. razvoj povsod enak funkciji f.

$g \in L^1(-\frac{1}{2},\frac{1}{2}) \qquad \{e^{n\pi i x/L},\ n\in\mathbb{Z}\}$ ortogonalna baza

razvijemo po $e^{2n\pi i x}$
$$c_n = \frac{\langle g, e^{2n\pi i x}\rangle}{\langle e^{2n\pi i x}, e^{2n\pi i x}\rangle} = \frac{\int_{-L}^{L} g(x)\overline{e^{2n\pi i x}}\,dx}{\int_{-L}^{L} e^{2n\pi i x}\overline{e^{2n\pi i x}}\,dx} = \frac{\int_{-L}^{L} g(x) e^{-2n\pi i x}\,dx}{\int_{-L}^{L} e^{2n\pi i x} e^{-2n\pi i x}\,dx} =$$
$$= \frac{1}{2L}\int_{-L}^{L} g(x) e^{-2n\pi x i}\,dx = \int_{-1/2}^{1/2}\sum_{k\in\mathbb{Z}} f(x+k)\, e^{-2n\pi x i}\,dx = \sum_{k\in\mathbb{Z}}\int_{-1/2}^{1/2} f(x+k)\, e^{-2n\pi x i}\,dx =$$
$$\overset{t=x+k}{=} \sum_{k\in\mathbb{Z}} \int_{-1/2+k}^{1/2+k} f(t)\, e^{-2n\pi i(t-k)}\,dt \overset{e^{2n\pi k i}=1,\ \in\mathbb{Z}}{=} \int_{-\infty}^{\infty} f(t)\, e^{-2n\pi t i}\,dt = \sqrt{2\pi}\,\hat{f}(2\pi n)$$

(c) Dokaži, da velja

$$\sum_{k\in\mathbb{Z}} f(k) = \sqrt{2\pi} \sum_{k\in\mathbb{Z}} \hat{f}(2\pi k)$$

enakost velja povsod, ker je g zv. odv.

$$g(x) = \sum_{n\in\mathbb{Z}} c_n e^{2n\pi i x}$$

$$g(x) = \sum_{k\in\mathbb{Z}} f(x+k) = \sum_{k\in\mathbb{Z}} \sqrt{2\pi}\, \hat{f}(2\pi k)\, e^{2k\pi x i}$$

$$g(0) = \sum_{k\in\mathbb{Z}} f(k) = \sqrt{2\pi} \sum_{k\in\mathbb{Z}} \hat{f}(2\pi k)$$

(9) $f(\vec{x}) = \begin{cases} 1 & |\vec{x}| \le a \\ 0 & \text{sicer} \end{cases}$ $\quad f: \mathbb{R}^3 \to \mathbb{C}$ $\quad$ določi $\hat{f}$

$$\boxed{\hat{f}(\vec{\omega}) = \frac{1}{\sqrt{2\pi}^n} \int_{\mathbb{R}^n} f(\vec{x})\, e^{-i\langle \vec{\omega}, \vec{x}\rangle}\, dV(\vec{x})}$$

$\boxed{f(\vec{x}) \text{ odvisna le od } |\vec{x}| \Rightarrow \hat{f}(\vec{\omega}) \text{ odvisna le od } |\vec{\omega}|}$

$$\langle \vec{\omega}, \vec{x}\rangle = \left\langle \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}, \begin{pmatrix} r\sin\theta\cos\varphi \\ r\sin\theta\sin\varphi \\ r\cos\theta \end{pmatrix} \right\rangle$$

$$\hat{f}(\vec{\omega}) = \frac{1}{\sqrt{2\pi}^3} \iiint_{\mathbb{R}^3} f(\vec{x})\, e^{-i\langle\vec{\omega},\vec{x}\rangle}\, dV(\vec{x}) \overset{\text{Sferične}}{=} \frac{1}{\sqrt{2\pi}^3} \int_0^a dr \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \cdot 1\, e^{-\langle\vec{\omega},\vec{x}\rangle i}\, r^2\sin\theta$$

$$\hat{f}(0,0,\omega) = \frac{1}{\sqrt{2\pi}^3} \int_0^a\int_0^{2\pi}\int_0^{\pi} r^2\sin\theta\, e^{-r\omega\cos\theta i}\, d\theta\, d\varphi\, dr \overset{t=-\cos\theta}{=} \frac{1}{\sqrt{2\pi}} \int_0^a r^2 dr \int_{-1}^{1} e^{r\omega i t}\, dt =$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^a r^2 dr\, \frac{1}{i\omega r} e^{r\omega i t}\Big|_{-1}^{1} = \frac{1}{i\omega\sqrt{2\pi}} \int_0^a r\left(e^{i\omega r} - e^{-i\omega r}\right) dr = \frac{2}{\omega\sqrt{2\pi}} \int_0^a r\sin(\omega r)\, dr =$$

$u = r \quad \sin\omega r\, dr = dv$

$du = dr \quad -\frac{1}{\omega}\cos\omega r = v$

$$= \frac{2}{\omega\sqrt{2\pi}} \left(-\frac{r}{\omega}\cos\omega r\Big|_0^a + \frac{1}{\omega}\int_0^a \cos\omega r\, dr\right) = \frac{2}{\omega^2\sqrt{2\pi}}\left(-a\cos\omega a + \frac{1}{\omega}\sin\omega a\right)$$

zarotiramo vektor

$$\hat{f}(\vec{\omega}) = \hat{f}(0,0,|\vec{\omega}|) = \frac{2}{\omega^2\sqrt{2\pi}}\left(\frac{1}{\omega}\sin\omega a - a\cos\omega a\right)$$

Kompleksne diferencialne enačbe

$$y'' + a(z)\, y' + b(z)\, y = 0 \qquad a, b \text{ holomorfni okrog } d$$

$$y = \sum_{n\ge 0} c_n (z-d)^n$$

(1) Poišči vse rešitve v okolici točke $z_0 = 0$

(a) $(1+z^2)\, y'' - 4z\, y' + 6y = 0 \quad \Rightarrow \quad a = -\frac{4z}{1+z^2} \qquad b = \frac{6}{1+z^2}$

$$y = \sum_{n\ge 0}^{\infty} c_n z^n \qquad y' = \sum_{n\ge 1}^{\infty} c_n n z^{n-1} \qquad y'' = \sum_{n\ge 2}^{\infty} n(n-1) c_n z^{n-2}$$

$$(1+z^2)\sum_{n\ge 2}^{\infty} n(n-1) c_n z^{n-2} - 4z\sum_{n\ge 1}^{\infty} c_n n z^{n-1} + 6\sum_{n\ge 0}^{\infty} c_n z^n = 0$$

$$\sum_{n\ge 2}^{\infty} n(n-1) c_n z^{n-2} + \sum_{n\ge 2}^{\infty} n(n-1) c_n z^n - 4\sum_{n\ge 1}^{\infty} c_n n z^n + 6\sum_{n\ge 0}^{\infty} c_n z^n = 0$$

z^0: $\quad c_2 \cdot 2 \cdot 1 + 6c_0 = 0 \qquad c_2 = -3c_0$

z^1: $\quad c_3\, 3\cdot 2 - 4c_1 \cdot 1 + 6c_1 = 0 \qquad c_3 = -\frac{1}{3} c_1$

$z^{k\ge 2}$: $\quad (k+2)(k+1)\, c_{k+2} + k(k-1)\, c_k - 4k\, c_k + 6c_k = 0 \qquad c_{k+2} = c_k \frac{4k - 6 - k(k-1)}{(k+2)(k+1)}$

$$c_{k+2} = -c_k \frac{k^2 - 5k + 6}{(k+2)(k+1)} = -\frac{(k-2)(k-3)}{(k+2)(k+1)}\, c_k$$

$n=2 \quad c_4 = 0$

$n=3 \quad c_5 = 0$

$n=4 \quad c_6 = c_4 = 0$

$\vdots$

$$\Rightarrow y = c_0 + c_1 - 3c_0 z^2 - \frac{1}{3} c_1 z^3 = c_0(1-3z^2) + c_1\left(1 - \frac{z^3}{3}\right)$$

(b) $2y'' - zy' - 2y = 0 \qquad y(0) = 0$

$$y = \sum_{n=0}^{\infty} c_n z^n \qquad \Downarrow \qquad c_0 = 0$$

$$2\sum n(n-1) c_n z^{n-2} - \sum n c_n z^n - 2\sum c_n z^n = 0$$

$$2(n+2)(n+1) c_{n+2} - n c_n - 2c_n = 0$$

$$c_{n+2} = c_n \frac{(n+2)}{2(n+2)(n+1)} = c_n \frac{1}{2(n+1)}$$

Vsi sodi $c_{2n} = 0 \Leftarrow c_2 = c_0 \frac{1}{2\cdot 1} = 0 \qquad c_3 = c_1 \frac{1}{2\cdot 2} = \frac{c_1}{4}$

$$c_5 = c_1 \frac{1}{2^2}\frac{1}{2\cdot 4}$$

$$c_7 = c_1 \frac{1}{2^3}\frac{1}{2\cdot 4\cdot 6} = c_1 \frac{1}{2^3}\frac{1}{6!!} = c_1 \frac{1}{2^3}\frac{1}{2^3}\frac{1}{3!}$$

$$c_{2n+1} = c_1 \frac{1}{2^n}\frac{1}{(2n)!!} = c_1 \frac{1}{2^{2n}}\frac{1}{n!}$$

$$y = c_1 z + \frac{c_1}{2^2\, 1!} z^3 + \frac{c_1}{2^4\, 2!} z^5 + \frac{c_1}{2^6\, 3!} z^7 + \frac{c_1}{2^8\, 4!} z^9 + \dots =$$

$$= c_1 z\left(1 + \left(\frac{z^2}{4}\right) + \frac{\left(\frac{z^2}{4}\right)^2}{2!} + \frac{\left(\frac{z^2}{4}\right)^3}{3!} + \dots\right) = c_1 z e^{\frac{z^2}{4}}$$

(2) Poišči $y(z)$ na zarezani okolici točke 0, ki zadošča DE

(a) $2zy'' + (1-2z^2)y' - 4zy = 0$

$y'' + p(z)y' + q(z)y = 0$

Če sta p in q holom. je isto kot prejšnje naloge

Če nista holom. (imata pol)

$p(z)$ ima pol st. 1

$q(z)$ nima pola

$p(z)$ pol st ≤ 1 v $z = a$

in $q(z)$ pol st ≤ 2 v $z = a$

je nastavek $y = (z-a)^r \sum_{n=0}^{\infty} c_n (z-a)^n$

$c_0 \neq 0 \qquad r \in \mathbb{C}$

$$y = z^r \sum_{n=0}^{\infty} c_n z^n = \sum c_n z^{n+r}$$

$$2z\sum c_n (n+r)(n+r-1) z^{n+r-2} + (1-2z^2)\sum c_n (n+r) z^{n+r-1} - 4z\sum c_n z^{n+r} = 0$$

$$\sum 2(n+r)(n+r-1) c_n z^{n+r-1} + c_n (n+r) z^{n+r-1} - 2c_n (n+r) z^{n+r+1} - 4c_n z^{n+r+1} = 0$$

$$\sum c_n z^{n+r-1}(n+r)(2(n+r-1)+1) - \sum c_n z^{n+r+1}(2(n+r)+4) = 0 \qquad / : z^r$$

$n=0 \quad z^{-1} \qquad c_0\, r(2r-1) = 0 \qquad$ ker $c_0 \neq 0 \quad \Rightarrow \quad r_1 = 0 \quad r_2 = \frac{1}{2}$

Ker $r_1 - r_2 \notin \mathbb{Z}$

$n=1 \quad z^0 \qquad c_1 (r+1)(2r+1) = 0$

dobimo za r_1 in r_2 dve lin. neodv. rešitvi.

z^{k+1}: $c_{k+1}(k+r+1)(2k+2r+1) - c_{k-1}(2k+2r+2) = 0$

$n = k+1$ $\qquad$ $n = k-1$

(1) $r=0$: $c_1 = 0$

$$c_{k+1}(k+1)(2k+1) - c_{k-1}(2k+2) = 0$$

$$c_{k+1} = c_{k-1}\frac{(2k+2)}{(k+1)(2k+1)} = \frac{2}{2k+1}c_{k-1}$$

Lihi $c_{2k+1} = 0$ $\qquad$ $c_{2k} = \dots$ $\Rightarrow$ $y = c_0 + \frac{2c_0}{3}z^2 + \frac{2^2 c_0}{3\cdot 7}z^4 + \dots$

(2) $r = \frac{1}{2}$ $\qquad$ $c_1 = 0$ $\qquad$ $c_{k+1} = \dots = c_{k-1}\frac{1}{k+1}$ $\Rightarrow$ $c_2 = c_0\frac{1}{2}$ $\quad$ $c_4 = c_0\frac{1}{2\cdot 4}$

$$c_{2k} = c_0\frac{1}{(2k)!!} = c_0\frac{1}{2^k k!}$$

$$y = z^{\frac{1}{2}}c_0\left(1 + \frac{1}{2}z^2 + \frac{1}{2^2 2!}z^4 + \frac{1}{2^3}\frac{1}{3!}z^6 + \dots\right)$$

$$y = z^{\frac{1}{2}}c_0 e^{\frac{z^2}{2}}$$

Končna rešitev je $y = Ay_1 + By_2$

(b) $z^2y'' - z^2y' + (z-2)y = 0$

$$y = z^r\sum_{n=0}^{\infty}c_n z^n$$

$\dots$ $\qquad$ $r_1 = -1$ $\qquad$ $r_2 = 2$ $\qquad$ $r_1 - r_2 \in \mathbb{Z}$

(1) Vzamemo manjšega $r_1 = -1$

z^0 : $c_0(r+1)(r-2) = 0$ $\qquad$ $r=-1$, $r=2$

z^{k+1} : $c_k((k+r)(k+r-1)-2) - c_{k-1}(k+r-2) = 0$

$r = -1$

(upamo, da dobimo dve rešitvi, če dobimo le eno, računamo še za drugi r, upamo da sta lin. neodv.
c_n visita lin. neodv. rešitvi lahko dobimo z determinanto Wronskega
$y_2 = y_1 u(z)$ $\Rightarrow$ dam v DE in rešim enačbo za u

$$c_k((k-1)(k-2)-2) - c_{k-1}(k-3) = 0$$

$$c_k = c_{k-1}\frac{(k-3)}{k(k-3)} = \frac{c_{k-1}}{k}$$ kritičen k v primeru $k=3$

$c_0 \neq 0$ $\qquad$ $c_1 = \frac{c_0}{1}$ $\qquad$ $c_2 = \frac{c_0}{1\cdot 2}$ $\qquad$ $c_4 = \frac{c_3}{4}$ $\qquad$ $c_5 = \frac{c_3}{5\cdot 4}$ $\qquad \dots \qquad$ $c_n = \frac{6c_3}{n!}$

Dobimo dve rešitvi

$$y = \frac{1}{z}\left(c_0 + c_0 z + \frac{c_0}{2}z^2 + \frac{6}{3!}c_3 z^3 + \frac{6c_3}{4!}z^4 + \frac{6c_3}{5!}z^5 + \dots\right) =$$

$$= \frac{1}{z}\left(c_0\left(1+z+\frac{z^2}{2}\right) + 6c_3\left(e^z - 1 - z - \frac{z^2}{2}\right)\right) =$$

$$= \frac{1}{z}\left((c_0 - 6c_3)\left(1+z+\frac{z^2}{2}\right) + 6c_3e^z\right) = \frac{1}{z}\left(\tilde{c}_0\left(1+z+\frac{z^2}{2}\right) + \tilde{c}_1 e^z\right)$$

Besselove funkcije

$$J_\nu(z) = \sum_{n=0}^{\infty} (-1)^n \frac{\left(\frac{z}{2}\right)^{2n+\nu}}{n!\,\Gamma(n+\nu+1)} \qquad \nu \in \mathbb{C}$$

- J_ν je ena od rešitev Besselove DE $\quad z^2 y'' + z y' + (z^2-\nu^2) y = 0$
- $\nu \notin \mathbb{Z} \Rightarrow J_{-\nu}$ je druga (lin. neodv.) rešitev (če je $\nu \in \mathbb{Z}$ pa vzamemo Neumannovo funkcijo Y_ν)
- $2 J_\nu' = J_{\nu-1} - J_{\nu+1}$
- $\frac{2\nu}{z} J_\nu = J_{\nu-1} + J_{\nu+1}$
- $n \in \mathbb{Z} \Rightarrow J_{-n}(z) = (-1)^n J_n(z)$
- $\sum_{n\in\mathbb{Z}} J_n(z)\, t^n = e^{\frac{z}{2}\left(t-\frac{1}{t}\right)}$

(2) S pomočjo rodovne fn. izpelji: $\quad 2J_n'(z) = J_{n-1}(z) - J_{n+1}(z) \quad n \in \mathbb{Z}$

$$\sum_{n\in\mathbb{Z}} J_n(z)\, t^n = e^{\frac{z}{2}\left(t-\frac{1}{t}\right)} \qquad \Big/ \frac{d}{dz}$$

$$\sum_{n\in\mathbb{Z}} J_n'(z)\, t^n = e^{\frac{z}{2}\left(t-\frac{1}{t}\right)} \frac{1}{2}\left(t-\frac{1}{t}\right) \qquad \Big/ \cdot 2$$

$$2\sum J_n'\, t^n = \sum J_n\, t^n \left(t-\frac{1}{t}\right) = \sum J_n(z)\left(t^{n+1} - t^{n-1}\right)$$

pri t^n: $\quad 2J_n'(z) = J_{n-1}(z) - J_{n+1}(z)$

(3) $a, b \geq 0$ različni ničli J_ν. Dokaži $\int_0^1 J_\nu(ax)\, J_\nu(bx)\, x\, dx = 0$ $\quad (\langle J_\nu(ax), J_\nu(bx)\rangle = 0)$

Namig:
- $u(x) = J_\nu(ax) \quad v(x) = J_\nu(bx)$
- poišči DE ki jim zadoščata u in v.
- izračunaj $(x(u'v - uv'))'$

$$x^2 J_\nu''(x) + J_\nu'(x) + (x^2-\nu^2) J_\nu(x) = 0 \qquad \Big/ x \mapsto ax$$
$$a^2x^2 J_\nu''(ax) + J_\nu'(ax) + (a^2x^2-\nu^2)\underbrace{J_\nu(ax)}_{u} = 0$$

$$u'(x) = J_\nu'(ax)\, a \qquad u''(x) = J_\nu''(ax)\, a^2$$

$$x^2 u''(x) + x u'(x) + (a^2x^2-\nu^2) u(x) = 0 \Rightarrow u''x + u' = -\left(a^2 x - \frac{\nu^2}{x}\right) u$$
$$x^2 v''(x) + x v'(x) + (b^2x^2-\nu^2) v(x) = 0 \Rightarrow v''x + v' = -\left(b^2 x - \frac{\nu^2}{x}\right) v$$

$$(x(u'v-uv'))' = u'v - uv' + x(u''v + v'u' - u'v' - uv'') =$$
$$= v(u' + xu'') - u(v' + xv'') =$$
$$= v\left(-a^2x + \frac{\nu^2}{x}\right)u - u\left(-b^2x + \frac{\nu^2}{x}\right)v = -a^2x\,vu + b^2x\,uv = xuv(b^2-a^2)$$

$$\int_0^1 J_\nu(ax)\, J_\nu(bx)\, x\,dx = \int_0^1 uvx\,dx = \frac{1}{b^2-a^2}\int_0^1 (x(u'v-uv'))'\,dx = \frac{1}{b^2-a^2}\, x(u'v-uv')\Big|_0^1 =$$

$$= \frac{1}{b^2-a^2}\left(u'(1)\underset{J_\nu(b)=0}{v(1)} - \underset{J_\nu(a)=0}{u(1)}v'(1)\right) = 0$$

(4) $J_{n+\frac{1}{2}}(x) = (-1)^n x^n \sqrt{\frac{2x}{\pi}} \left(\frac{1}{x}\frac{d}{dx}\right)^n \left(\frac{\sin x}{x}\right)$

Dokaz z indukcijo $n=0$ $J_{1/2}(x) = \sqrt{\frac{2x}{\pi}}\,\frac{\sin x}{x}$

$n \to n+1$ $J_{n+\frac{3}{2}}(x) = (-1)^{n+1} x^{n+1} \sqrt{\frac{2x}{\pi}} \left(\frac{1}{x}\frac{d}{dx}\right)^{n+1} \left(\frac{\sin x}{x}\right)$

$$(-1)^{n+1} x^{n+1} \sqrt{\frac{2x}{\pi}}\, \frac{1}{x}\frac{d}{dx}\Big(\underbrace{\left(\frac{1}{x}\frac{d}{dx}\right)^n \left(\frac{\sin x}{x}\right)}_{\shortparallel} \Big)$$

$$J_{n+\frac{1}{2}}(x)\ (-1)^n\ x^{-n} \sqrt{\frac{\pi}{2x}}$$

$$= -\cancel{(-1)^{n+1}}\, x^{n+1} \sqrt{\cancel{\frac{2x}{\pi}}}\ \frac{1}{x} \left(\cancel{(-1)^n} \sqrt{\cancel{\frac{\pi}{2}}}\ x^{-n-1/2}\, J_{n+\frac{1}{2}}(x)\right)' = \ \overset{\text{odvajam}}{\dots}$$

Ortogonalni polinomi (Legendrovi polinomi)

- $P_n(x) = \frac{1}{2^n n!}\left(\frac{d}{dx}\right)^n \left((x^2-1)^n\right)$ polinom stopnje n
  P_{lih} ... liha funk.
  P_{sod} ... soda funk.
- P_n je ena od rešitev Legendrove DE
  $(z^2-1)y'' + 2zy' - n(n+1)y = 0$
- Rodovna funkcija
  $\sum_{n=0}^{\infty} P_n(z)\, t^n = \frac{1}{\sqrt{1-2zt+t^2}}$ (konv. za dovolj majhne t)
- $(n+1)P_{n+1}(x) = (2n+1)\,x\,P_n(x) - n P_{n-1}(x)$
- $\int_{-1}^{1} P_m(x) P_n(x)\,dx = \delta_{mn}\frac{2}{2n+1}$
- $f \in L^2(-1,1) \Rightarrow f(x) = \sum_{n=0}^{\infty} a_n P_n(x)$ $a_n = \frac{\langle f, P_n\rangle}{\langle P_n, P_n\rangle} = \frac{2n+1}{2}\int_{-1}^{1} f\,P_n\,dx$

(1) Izračunaj $P_0,\ P_1,\ P_2$

(a) $P_0(x) = \frac{1}{2^0 0!}\cdot 1 = 1$
$P_1(x) = \frac{1}{2^1\cdot 1!}(x^2-1)' = x$
$P_2(x) = \frac{1}{2}(3x\,x - 1\cdot 1) = \frac{3}{2}x^2 - \frac{1}{2}$

(b) $f \in L^2(-1,1)$, $f(x) = x^2$ razvij po Legendrovih polinomih

$f(x) = a_0 P_0(x) + a_1 P_1(x) + a_2 P_2(x)$
$x^2 = a_0 + a_1 x + a_2 \frac{3}{2}x^2 - \frac{a_2}{2}$ $a_1 = 0$
$a_0 - \frac{a_2}{2} = 0$ $a_0 = \frac{1}{3}$
$x^2 = \frac{1}{3}P_0 + \frac{2}{3}P_2$ $1 = a_2\frac{3}{2}$ $a_2 = \frac{2}{3}$

(c) Za $n \geq 0$ izračunaj $\int_{-1}^{1} P_n(x)\,dx$ in $\int_{-1}^{1} x^2 P_n(x)\,dx$

$$\int_{-1}^{1} x^2 P_n(x)\,dx = \int_{-1}^{1}\left(\frac{1}{3}P_0 + \frac{2}{3}P_2\right) P_n(x)\,dx = \begin{cases} n=0 & \frac{1}{3}\,\frac{2}{2\cdot 0+1} = \frac{2}{3} \\ n=2 & \frac{2}{3}\,\frac{2}{2\cdot 2+1} = \frac{4}{15} \\ \text{sicer} & 0 = 0 \end{cases}$$

$$\int_{-1}^{1} P_n(x)\,dx = \int_{-1}^{1} P_0(x) P_n(x)\,dx = \begin{cases} n=0; & 2 \\ n\neq 0; & 0 \end{cases}$$

(d) Izračunaj $P_n(1)$ in $P_n(-1)$

$$\sum_{n=0}^{\infty} P_n(z)\, t^n = \frac{1}{\sqrt{1-2zt+t^2}} \qquad z=1$$

$$\sum_{n=0}^{\infty} P_n(1)\, t^n = \frac{1}{\sqrt{t^2-2t+1}} = \frac{1}{\underbrace{1-t}_{>0}} = 1+t+t^2+\dots \Rightarrow P_n(1) = 1$$

$$P_n(-1) = \begin{cases} -1 & n \text{ lih} \\ 1 & n \text{ sod} \end{cases} = (-1)^n$$

(e) $a \in (-1,1)$ Izračunaj $\int_{-1}^{1} \frac{P_{20}(x)}{\sqrt{1-2ax+a^2}}\,dx$

$$\int_{-1}^{1} \frac{P_{20}(x)}{\sqrt{1-2ax+a^2}}\,dx = \int_{-1}^{1} P_{20}(x) \sum_{n=0}^{\infty} P_n(x)\, a^n\, dx \overset{|a|<1}{=} \sum_{n=0}^{\infty} a^n \int_{-1}^{1} P_{20}(x) P_n(x)\,dx = a^{20} \frac{2}{2\cdot 20+1}\, \delta_{n\,20}$$

$$= a^{20} \frac{2}{41}$$

(2) Razvij funkcijo $f(x) \in L^2(-1,1)$ $f(x) = \begin{cases} 0 & x<0 \\ 1 & x>0 \end{cases}$ po Legendrovih polinomih

$$f(x) = \sum_{n=0}^{\infty} a_n P_n(x) \qquad a_n = \frac{2n+1}{2} \int_{-1}^{1} f(x) P_n(x) = \frac{2n+1}{2} \int_0^1 P_n(x)\,dx$$

$$\int_0^1 \sum_{n=0}^{\infty} P_n(x)\, t^n dx = \int_0^1 \frac{1}{\sqrt{t^2-2xt+1}}\,dx$$

$$\sum t^n \int_0^1 P_n(x)\,dx \overset{\substack{u=1+t^2-2xt \\ du=-2t\,dx}}{=} -\frac{1}{2t}\int_{S_1}^{S_2} \frac{1}{\sqrt{u}}\,du = -\frac{1}{t}\sqrt{u}\Big|_{S_1}^{S_2} = -\frac{1}{t}\left(\sqrt{1+t^2-2t} - \sqrt{1+t^2}\right) =$$

$$\overset{|t|<1}{=} \frac{1}{t}\left(\sqrt{1+t^2} - 1 + t\right) = \frac{1}{t}\left(\cancel{1} + \binom{1/2}{1}t^2 + \binom{1/2}{2}t^4 + \binom{1/2}{3}t^6 + \dots - \cancel{1} + t\right) =$$

$$= 1 + \binom{1/2}{1}t + \binom{1/2}{2}t^3 + \binom{1/2}{3}t^5 + \dots + \dots \binom{1/2}{\frac{n+1}{2}}t^n$$

$$\int_0^1 P_n(x) = \begin{cases} n=0 & 1 \\ n \text{ sod} & 0 \\ n \text{ lih} & \binom{1/2}{\frac{n+1}{2}} \end{cases} \qquad a_n = \frac{2n+1}{2}\begin{cases} n=0 & 1 \\ n \text{ sod} & 0 \\ n \text{ lih} & \binom{1/2}{\frac{n+1}{2}} \end{cases}$$

Parcialne dif. en. (2. reda, linearne)

(1) Poišči rešitev $u(x,t): [0,\pi] \times (0,\infty) \to \mathbb{R}$ enačbe $u_t = d\, u_{xx}$ $(d>0)$ pri robnih pogojih $u(0,t) = u(\pi,t) = 0$ in zač. pogoju $u(x,0) = f(x)$, kjer je

(a) $f(x) = \sin x$

(b) $f(x) = \begin{cases} 0 & x<\pi/2 \\ 1 & x \geq \pi/2 \end{cases}$

(c) $f(x) = \cos x$

NI: homogen pogoj

Homogena pogoja

u_1, u_2 zadoščata $\Rightarrow$ tudi $\lambda_1 u_1 + \lambda_2 u_2$ zadošča tem pogojem

$$u(x,t) = X(x)\, T(t)$$

$$u_t = d\, u_{xx}$$
$$X T' = d\, T X''$$

Homogeni pogoji

$u(0,t) = X(0)\, T(t) = 0 \Rightarrow X(0) = 0$

$u(\pi,t) = X(\pi)\, T(t) = 0 \Rightarrow X(\pi) = 0$

$$\frac{1}{d}\frac{T'}{T} = \frac{X''}{X} = \text{konst.} = \lambda$$

X: $\qquad X'' = \lambda X$

(1) $\lambda > 0$ $\qquad X = A e^{\sqrt{\lambda}x} + B e^{-\sqrt{\lambda}x} = A \operatorname{ch}\sqrt{\lambda}x + B \operatorname{sh}\sqrt{\lambda}x$

Pogoji: $X(0) = 0 \Rightarrow A = 0 \qquad 0 = X(\pi) \Rightarrow B = 0 \qquad \Rightarrow X = 0$

② $\lambda = 0$ $X'' = 0$ $X = Ax + B$

Posoji: $X(0) = 0$ $B = 0$ $X(\pi) = 0 \Rightarrow A = 0 \Rightarrow X = 0$

③ $\lambda < 0$ $X = A\cos\sqrt{-\lambda}\,x + B\sin\sqrt{-\lambda}\,x$

Posoji: $X(0) = 0 \Rightarrow A = 0$ $X(\pi) = 0$ $B\sin\sqrt{-\lambda}\,\pi = 0$ $B \neq 0$

$\sqrt{-\lambda}\,\pi = k\pi$

$\lambda = -k^2$

$X = B\sin\sqrt{-\lambda}\,x$

$X_k = B\sin kx$

T: $T' = \lambda\alpha T$

$T = Ae^{\lambda\alpha t}$

$T_k = A_k e^{-\alpha k^2 t}$

$$u(x,t) = \sum_{k=1}^{\infty} X_k T_k = \sum_{k=1}^{\infty} A_k e^{-\alpha k^2 t}\sin kx$$

Začetni posoj $u(x,0) = f(x)$

$u(x,0) = \sum_{k=1}^{\infty} A_k \sin kx$ razvijemo $f(x)$ na $[0,\pi]$ po sinusih

ⓐ $f(x) = \sin x$ $\sin x = \sum_{k=1}^{\infty} A_k \sin kx$ $A_1 = 1$ $A_{k\geq 2} = 0$

$u(x,t) = e^{-\alpha t}\sin x$

ⓑ $f(x) = \begin{cases} 0 & x < \pi/2 \\ 1 & x \geq \pi/2 \end{cases}$

$$b_k = \frac{2}{\pi}\int_0^{\pi} f(x)\sin\frac{k\pi x}{\pi}\,dx$$

$$= \frac{2}{\pi}\int_{\pi/2}^{\pi}\sin kx\,dx =$$

$$= \frac{2}{k\pi}\left(-\cos kx\right)\Big|_{\pi/2}^{\pi} =$$

$$= \frac{2}{k\pi}\left(\cos\frac{k\pi}{2} - \cos k\pi\right)$$

$f: [-L, L] \to \mathbb{R}$

$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\frac{k\pi x}{L} + \sum_{k=1}^{\infty} b_k \sin\frac{k\pi x}{L}$

$a_k = \frac{1}{L}\int_{-L}^{L} f(x)\cos\ldots$

$b_k = \frac{1}{L}\int_{-L}^{L} f(x)\sin\ldots$

$f: [0, L] \to \mathbb{R}$ želimo razviti po sinusih $\Rightarrow$ razširimo do lihe funk.

$f(x) = \sum b_k \sin\ldots$ $b_k = \frac{2}{L}\int_0^L f(x)\sin\frac{k\pi x}{L}\,dx$

če razvijemo po cos razširimo do sode funk.

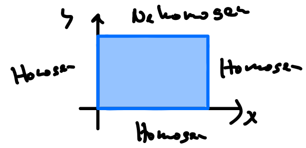
$f(x) = \sum b_k \sin kx$

$$u(x,t) = \sum_{k=1}^{\infty} b_k e^{-\alpha k^2 t}\sin kx = \sum_{k=1}^{\infty}\frac{2}{\pi k}\left(\cos\frac{k\pi}{2} - \cos k\pi\right)e^{-\alpha k^2 t}\sin kx$$

ⓒ $f(x) = \cos x = \sum b_k \sin kx$

$$b_k = \frac{2}{\pi}\int_0^{\pi}\cos x\,\sin kx\,dx = \ldots$$

② Poišči $u: [0,1]\times[0,1] \to \mathbb{R}$ harmonično



ⓐ $u(x,0)=0 \qquad u(x,1) = \cos \pi x \qquad u_x(0,y) = u_x(1,y) = 0$

ⓑ $u(x,0)=0 \qquad u(x,1) = \cos \pi x \qquad u_x(0,y) = u_x(1,y) = 1$

(namig: nove sprem. $v = u - x$)

ⓐ $u_{xx} + u_{yy} = 0$

$u = X(x)Y(y)$

$X''Y + XY'' = 0$

Homogeni robni pogoji

$X'(0)\,Y(y) = 0 \qquad X'(0) = 0 = X'(1)$

$X'(1)\,Y(y) = 0$

$$\frac{X''}{X} = -\frac{Y''}{Y} = \lambda$$

Ker imamo hom. pogoje vzamemo X

① $\lambda > 0 \Rightarrow X = A \operatorname{ch}\sqrt{\lambda}x + B\operatorname{sh}\sqrt{\lambda}x \qquad X' = A\sqrt{\lambda}\operatorname{sh}\sqrt{\lambda}x + B\sqrt{\lambda}\operatorname{ch}\sqrt{\lambda}x \Rightarrow X = 0$

② $\lambda = 0 \Rightarrow X = Ax + B \qquad X' = A \Rightarrow A = 0 \qquad X = B$

③ $\lambda < 0 \Rightarrow X = A\cos\sqrt{-\lambda}x + B\sin\sqrt{-\lambda}x \qquad X' = -A\sqrt{-\lambda}\sin\sqrt{-\lambda}x + B\sqrt{-\lambda}\cos\sqrt{-\lambda}x$

$X'(0) = 0 \Rightarrow B = 0$

$X'(1) = 0 \Rightarrow 0 = -A\sqrt{-\lambda}\sin\sqrt{-\lambda}\,1$

$\sqrt{-\lambda} = k\pi$

$\lambda = -k^2\pi^2 \qquad k \in \mathbb{N}$

$$X_k = A\cos k\pi x$$

Rešitve 2 in 3 lahko združimo $X_k = A\cos k\pi x \qquad k \in \mathbb{N}_0$

Y: $-\dfrac{Y''}{Y} = \lambda$

$Y'' = -\lambda Y$

$Y'' = (k\pi)^2 Y \qquad k \geq 0$

$k = 0 \qquad Y = A_0 y + B_0$

$k \geq 1 \qquad Y = A_k e^{k\pi y} + B_k e^{-k\pi y} = A_k \operatorname{sh} k\pi y + B_k \operatorname{ch} k\pi y$

$$u(x,y) = \sum_{k=0}^{\infty} X_k Y_k = (A_0 y + B_0) + \sum_{k=1}^{\infty} (A_k \operatorname{sh} k\pi y + B_k \operatorname{ch} k\pi y)\cos k\pi x$$

$u(x,0) = 0 = B_0 + \sum_{k=1}^{\infty} A_k \cos k\pi x \Rightarrow B_0 = 0 \quad A_k = 0 \ \forall k$

$u(x,1) = \cos\pi x = A_0 + \sum_{k=1}^{\infty} \cos k\pi x\,(B_k \operatorname{sh} k\pi y) \Rightarrow A_0 = 0 \quad B_1 \operatorname{sh}\pi = 1 \qquad B_{k\geq 2} = 0$

$$u(x,t) = \frac{1}{\operatorname{sh}\pi}\cos\pi x \operatorname{sh}\pi y$$

ⓑ $v = u - x \Rightarrow$ prevedemo na problem za funk v

$v_x = u_x - 1$

$v_{xx} = u_{xx}$

$v_{yy} = u_{yy}$

$v_{yy} + v_{xx} = 0$

$u(x,0) = 0 \Rightarrow v(x,0) = u(x,0) - x = -x$

$u(x,1) = \cos\pi x \Rightarrow v(x,1) = \cos\pi x - x$

$u_x(0,y) = 1 \Rightarrow v_x(0,y) = 1 - 1 = 0$

$u_x(1,y) = 1 \Rightarrow v_x(1,y) = 0$

hom. rešitvi

...

3

$u(x,t): [0,\pi] \times [0,\infty) \to \mathbb{R}$

$u_{xx} = u_{tt} + 2u_t$

$u(0,t) = u(\pi,t) = 0$

$u(x,0) = f(x)$

$u_t(x,0) = g(x)$

$$u = XT$$

$$X''T = XT'' + 2XT'$$

$$\frac{X''}{X} = \frac{T''}{T} + 2\frac{T'}{T} = \lambda$$

$$u(0,t) = X(0)\,T(t) = 0 \qquad X(0) = 0 = X(\pi)$$

$$X'' = \lambda X \quad \Rightarrow \quad \text{enako kot prej}$$

$$\lambda < 0 \qquad X_k = \sin kx$$

$$T: \quad T'' + 2T' + k^2 T = 0$$

$$\lambda_{1,2} = \frac{-2 \pm \sqrt{4 - 4k^2}}{2} = -1 \pm \sqrt{1-k^2} \qquad k \geq 1$$

$$k = 1 \qquad \lambda_{1,2} = -1 \qquad T_1 = A_1 e^{-t} + B_1 t e^{-t}$$

$$k \geq 2 \qquad \lambda_{1,2} = -1 \pm i\sqrt{k^2-1} \quad \Rightarrow \quad T_k = A e^{-t} \cos\sqrt{k^2-1}\,t + B e^{-t} \sin\sqrt{k^2-1}\,t$$

$$u = \sum X_k T_k = (A_1 e^{-t} + B_1 t e^{-t}) \sin x + \sum_{k=2}^{\infty} (A_k e^{-t} \cos\sqrt{k^2-1}\,t + B_k e^{-t} \sin\sqrt{k^2-1}\,t) \sin kx$$

Začetni pogoji:

$$u(x,0) = f(x) = A_1 \sin x + \sum_{k=2}^{\infty} A_k \sin kx = \sum_{k=1}^{\infty} A_k \sin kx$$

$$A_k = \frac{2}{\pi} \int_0^{\pi} f(x) \sin kx \, dx$$

$$u_t(x,0) = g(x) = (-A_1 + B_1 + 0) \sin x + \sum_{k=2}^{\infty} (-A_k + B_k \sqrt{k^2-1}) \sin k\pi$$

$$-A_1 + B_1 = \frac{2}{\pi} \int_0^{\pi} g(x) \sin x \, dx$$

$$-A_k + B_k \sqrt{k^2-1} = \frac{2}{\pi} \int_0^{\pi} g(x) \sin kx \, dx \qquad k \geq 2$$