Uvod

1 delec ... navadna DE (ODE)

n delcev $\xrightarrow[\text{limita}]{\text{kontinuumska}}$ polje ... parcialna DE (PDE)

$n \to \infty$ zvezno sredstvo

princip najmanjše akcije $\longrightarrow$ integral <u>lokalne</u> gostote $\xrightarrow{\text{Euler-Lagr. en.}}$ PDE

kavzalnost

Robni (začetni) pogoj (ZP) nam šele izloči enolično rešitev

Tip: PDE

- najenostavnejše le z prvimi odvodi (red r je najvišji odvod)

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \vec{v}) = 0$$ kontinuitetna enačba (enačba za ohranitev mešanja)

ρ in $\vec{v}$ neznani funkciji – potrebujemo še zvezo med ρ in $\vec{v}$ (1. red)

enačba 2. reda

Najpomembnejši primeri

1. Valovna enačba

$$\nabla^2 u = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$ (hiperbolična)

Klein-Gordon $\nabla^2 u = \frac{1}{c^2} u_{tt} + m^2 u$

2. Difuzijska enačba (parabolična)

$$\nabla^2 u = \frac{1}{D}\frac{\partial u}{\partial t}$$

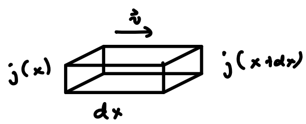
3. Laplacova enačba (eliptična)

$$\nabla^2 u = 0$$

PDE 1. reda

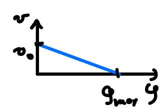
Primer: dinamika prometa

$\rho(x,t)$ (gostota prometa), $v(x,t)$



$$\frac{\partial \rho}{\partial t} + \frac{\partial j}{\partial x} = 0 \qquad j = \rho v = \underbrace{\rho v(\rho)}_{j(\rho)}$$

$\hookrightarrow$ npr. $v(\rho) = v_0\left(1 - \frac{\rho}{\rho_{max}}\right)$



$$\rho_t + j_\rho \rho_x = 0$$

$$c(\rho) = \frac{dj}{d\rho} = v_0\left(1 - \frac{2\rho}{\rho_{max}}\right)$$

$$\rho_t + \rho_x\, c(\rho) = 0$$

Kakšne so rešitve?

- Linearna verzija $c = \text{konst}$ ZP $\rho(x, t=0) = \rho_0(x)$

$$\rho_t + c\,\rho_x = 0$$

Enosmerna valovna enačba
Advekcijska enačba

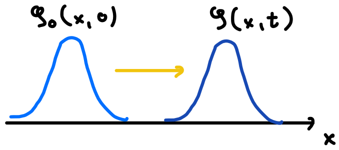
enačba v premikajočem sistemu $x(t)$: $\rho(x(t), t)$

$$\frac{d\varrho}{dt} = \varrho_x \frac{dx}{dt} + \varrho_t = \varrho_x \frac{dx}{dt} - c\,\varrho_x$$

V sistemu kjer je $\dot{x}(t) = c$ je enačba ker $\frac{d\varrho}{dt} = 0$

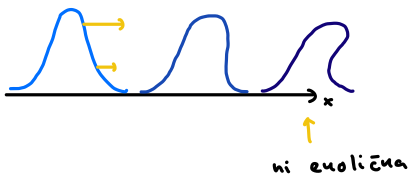
$\downarrow$ $x = x_0 + ct$ ← karakteristika

$\downarrow$ $\varrho(x,t) = \varrho(t=0) = \varrho_0 = \varrho_0(x_0) = \varrho_0(x - ct)$



- Nelinearna verzija
  $c(\varrho) = \varrho$

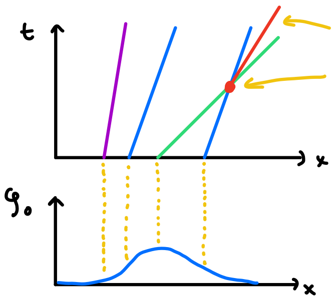
$\varrho_t + \varrho\varrho_x = 0$ (Burger's equation)



$\frac{dx}{dt} = \varrho$ $\quad x(t) = x_0 + \varrho_0(x_0)t$

Hitrost odvisna od amplitude

$\downarrow$ obrnemo



udarni val (w hitrost ud. vala)

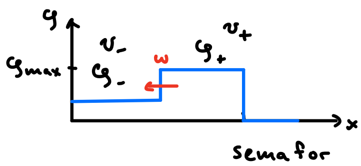
singularnost

$x_0(x,t)$

$\downarrow$

$\varrho(x,t) = \varrho_0(x_0(x,t))$ ← rešitev

Hitrost udarnega vala



V sistemu w:

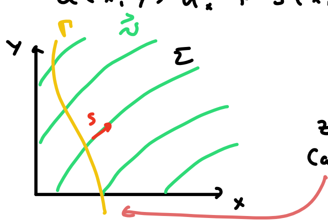
$$j_- = j_+$$
$$\varrho_-(v_- - w) = \varrho_+(v_+ - w)$$
$$\Downarrow$$
$$w = \frac{\varrho_+ v_+ - \varrho_- v_-}{\varrho_+ - \varrho_-}$$

Semafor: $v_+ = 0$ $\quad \varrho_+ = \varrho_{max} \Rightarrow$ $\quad w = -\frac{\varrho_- v_-}{\varrho_{max} - \varrho_-}$

w je lahko bistveno večja od v_-, zato pride do naleta

Metoda karakteristik za PDE 1. reda

$a(x,y)\,u_x + b(x,y)\,u_y = \phi(x,y,u)$ $\quad (\vec{v} = (a,b)\,;\ (\vec{v}\cdot\nabla)u = \phi)$



Krivulja $x(s)$, $y(s)$

ZP $u(\Gamma)$

Cauchyjev problem

$$\frac{du}{ds} = u_x \frac{dx}{ds} + u_y \frac{dy}{ds}$$

Če izberemo $\underbrace{\frac{dx}{ds} = a(x,y) \quad \text{in} \quad \frac{dy}{ds} = b(x,y)}_{\text{karakteristike } \Sigma}$, potem je DE za u na krivulji $(x(s), y(s))$ enaka

$$\frac{du}{ds} = \phi(x,y,u) = \phi(x(s), y(s), u) = \phi(s,u) \qquad \text{ODE}$$

- na Σ preide PDE 1. reda v ODE (integriramo)
- rešitev se propagira vzdolž Σ
- za $\phi = 0 \Rightarrow \frac{du}{ds} = 0$ oz. Σ so izohipse $u(x,y)$
- da je Cauchy na Γ korekten ne sme biti tangenten na Σ

Karakteristike in PDE 2. reda

Cauchy: na Γ imamo dan $\quad \vec{x} = (x, y, \dots) \quad \dim \vec{x} = m \quad \dim \Gamma = m-1$

$u(\Gamma)$, $\frac{\partial u}{\partial n}(\Gamma)$

Kako propagiramo rešitev?

- ODE

$$a(x)\, u'' = \phi(x, u, u')$$

Cauchy: $u(0)$, $u'(0)$ $\leftarrow \Gamma \quad \dim\ 1-1=0$

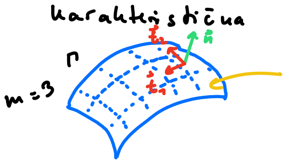
$$u(dx) = u(0) + u'(0)\,dx + \dots$$

$$u'(dx) = u'(0) + \underbrace{u''(0)}_{\text{iz enačbe}} dx + \dots$$

OK, razen v točkah $a(x) = 0$ (karakteristika)

- PDE (2. red)

karakteristična hiperploskev Σ: drugih odvodov se ne da izračunati iz nižjih



$u(\Gamma)$
$\frac{\partial u}{\partial \vec{n}}(\Gamma)$ Cauchy

$$A_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + (\text{nižji odvodi}) = 0$$

Potrebujemo vse druge odvode: $\underbrace{\frac{\partial^2 u}{\partial n^2}}_{\text{iz enačbe}}$, $\underbrace{\frac{\partial^2 u}{\partial n\, \partial t_i}, \quad \frac{\partial^2 u}{\partial t_i \partial t_j}}_{\text{iz Cauchyja}}$

Če ta izraz v neki točki zapišemo v „lastni bazi":

$$\underbrace{(\dots)}\frac{\partial^2 u}{\partial n^2} + (\dots) \frac{\partial^2 u}{\partial t_i \partial t_j} \qquad \dots \text{ normalni odvod}$$

Iz koordinat x_i bi šli radi v koordinate t_i, t_j, n:

$$\frac{\partial}{\partial x_j} = \frac{\partial}{\partial n}\frac{\partial n}{\partial x_i} + \frac{\partial}{\partial t_j}\frac{\partial t_j}{\partial x_i} \qquad \text{(v resnici le zamenjave spremenljivk)}$$

$\sum A_{ij} n_i n_j = 0$ oz. $\vec{n} A \vec{n}$ (to je neodvisno od koord. sistema)

↳ lahko je v splošnem odvisna od lege v prostoru.

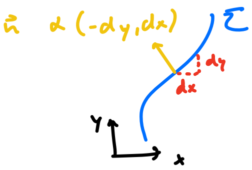
oz. $f(\vec{x}) = 0$; $n_i = \frac{\partial f}{\partial x_i}$

$$\Rightarrow A_{ij} \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_j} = 0$$

Na Σ (dim $n-1$) $\frac{\partial^2 u}{\partial n^2}$ ne nastopa $\Rightarrow$ Cauchy se v $\perp$ smeri ne da propagirati (rešitev se propagira le vzdolž Σ). Da je Cauchy korekten, Γ ne sme biti tangentna na Σ. To pomeni, da se nam na Σ enačba poenostavi, ker imamo 1 odvod manj. To pa še ne pomeni, da jo bomo znali rešiti.

Primer: $n=2$

$$\vec{x} = (x,y), \quad a u_{xx} + 2b u_{xy} + c u_{yy} + (\text{nižji}) = 0 \quad \Rightarrow \quad A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$



$\rightarrow$ Da zapišemo Σ je dovolj, da povemo, kako se smerni koeficient spreminja s prostorom.

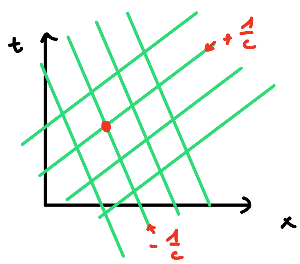
na Σ: $\quad a(dy)^2 + c(dx)^2 - 2b\,dx\,dy = 0 \qquad \Big/ : (dx)^2$

$$a\left(\frac{dy}{dx}\right)^2 - 2b\left(\frac{dy}{dx}\right) + c = 0$$

$$\frac{dy}{dx} = \frac{b \pm \sqrt{b^2 - ac}}{a} \quad \rightarrow \text{ naklon je lahko imaginaren ali realen}$$

① $b^2 - ac > 0 \Rightarrow$ 2 realni karakteristiki $\Rightarrow$ hiperbolična enačba

npr. $\quad u_{xx} = \frac{1}{c^2} u_{tt} \quad \Rightarrow \quad \vec{x} = (x,t) \qquad A = \begin{pmatrix} 1 & 0 \\ 0 & -1/c^2 \end{pmatrix} \qquad \frac{dt}{dx} = \pm \frac{1}{c}$



② $b^2 - ac = 0 \Rightarrow$ ena realna karakteristika $\Rightarrow$ parabolična enačba

npr. $\quad u_{xx} = u_t$

③ $b^2 - ac < 0 \Rightarrow$ ni realne karakteristike $\Rightarrow$ eliptična enačba $\qquad u_{xx} + u_{yy} = 0$

Če je mat A diagonalizabilna, lahko zapišemo:

$$a_1 \frac{\partial^2 u}{\partial x_1} + a_2 \frac{\partial^2 u}{\partial x_2} + \ldots = 0$$

Tu enostavno vidimo, kakšnega tipa je enačba:

① a_1 in a_2 različnega predznaka $\rightarrow$ Hip.
② $a_1 = 0$ $\rightarrow$ Par.
③ a_1 in a_2 enakega predznaka $\rightarrow$ Elip.

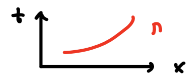
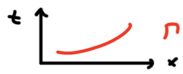
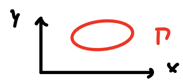
Robni pogoji

Najpogostejši tipi:

- Cauchy: u, $\frac{\partial u}{\partial n}$ na Γ → valovne enačbe
- Dirichletov: u na Γ → difuzijske enačbe
- Neumannov: $\frac{\partial u}{\partial n}$ na Γ

Korektno definiran problem imamo, ko je rešitev enolična in stabilna. Stabilnost pomeni, da majhne spremembe začetnega pogoja na Γ prinese majhno spremembo rešitve.

Omejene domene D: $\exists\ \delta$ in ε da velja: če je $|u_1(\Gamma) - u_2(\Gamma)| < \varepsilon \Rightarrow$
$\Rightarrow |u_1(\vec{x}) - u_2(\vec{x})| < \delta$ za $\forall\ \vec{x} \in D$

PDE	Dovoljeni z.p.	oblika Γ
Hiperbolične	Cauchy	
Parabolične	Dirichlet ali Neumann	
Eliptične	Dirichlet ali Neumann	

Primer: nekorektno definiran problem

$$u_{xx} + u_{yy} = 0 \qquad \text{Cauchy na } \Gamma(y=0)$$

$$\left.\begin{aligned} u_1(x,y) &= 0 \\ u_2(x,y) &= 1/k \sin kx \operatorname{ch} ky \end{aligned}\right\} \text{ oboje zadošča naši enačbi}$$

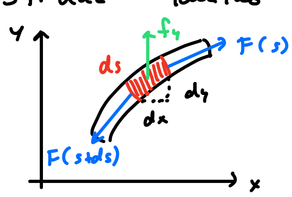
$$u_1(x,0) = 0, \quad u_{1y}(x,0) = 0$$
$$u_2(x,0) = \sin(kx)/k, \quad u_{2y}(x,0) = 0$$

$$|u_1(x,0) - u_2(x,0)| < 1/k \xrightarrow{k\to\infty} 0$$
$$|u_1(x,y) - u_2(x,y)| < \frac{\operatorname{ch}(ky)}{k} \xrightarrow{k\to\infty} \infty \quad \text{to rešitev ni stabilna}$$

Valovna enačba

Izpeljava valovne enačbe

ⓐ Struna = idealno (ni sil za zvijanje) gibka tanka vrv ⇒ vse sile so v tang. smeri.



$$f_y = \frac{dF_y}{ds} \qquad \mu = \frac{dm}{ds}$$

↳ (gostota zun. sil, npr. teža)

$$\mu\, ds\, x_{tt} = f_x\, ds + \underbrace{\frac{\partial}{\partial s}\left(F \frac{\partial x}{\partial s}\right) ds}_{F_x(s+ds) - F_x(s)}$$

$|:ds$

$F_x(s) = F \frac{dx}{ds}$ ... x-komp. sile

$$\boxed{\begin{aligned} \mu\, x_{tt} &= \frac{\partial}{\partial s}\left(F(s) \frac{\partial x}{\partial s}\right) + f_x \\ \mu\, y_{tt} &= \frac{\partial}{\partial s}\left(F(s) \frac{\partial y}{\partial s}\right) + f_y \end{aligned}}$$ Gibalni enačbi

$$ds^2 = dx^2 + dy^2 \iff \left|\frac{d\vec{r}}{ds}\right| = 1$$

Če je s enota dolžine, je to pogoj za neraztegljivost. To je tudi implicitni pogoj za $F(s)$ (sile moramo izbrati ravno pravšnje, da se koščki vrvi med sabo ne raztegujejo)

Pomemben primer: majhni nakloni $\frac{dy}{ds} \ll 1$ (v majhni odmiki) $\frac{dx}{ds} \approx 1$

$f_x = 0$ in $x_{tt} = 0$ (ni zun. sil v x-smeri in vrv ne pospešuje v x-smeri)

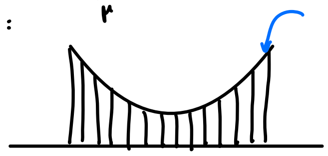
$\Downarrow$

$F(s) = \text{konst.} = F_0$ sila s katero je struna napeta

$$\mu\, y_{tt} = F_0 \frac{\partial^2 y}{\partial s^2} + f_y \approx F_0\, y_{xx} + f_y \quad /:\mu$$

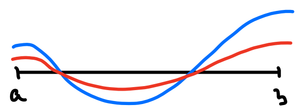
$$y_{tt} = \underset{\frac{F}{\mu}}{\underset{\parallel}{c^2}}\, y_{xx} + \frac{f_y}{\mu}$$ val. en. za nihanje strune

Viseč most:



parabola (ker je obremenitev konst. na dx. Če bi bila na ds bi dobili verižnico.)

Energija nihanja strune ($f_y = 0$)



$\beta \in [0,1)$

$$\boxed{E = \frac{1}{2}\mu\int_a^b y_t^2\, dx + \underbrace{\frac{1}{2}F\int_a^b y_x^2\, dx}_{V}}$$

Energija potrebna za deformacijo strune

$$V = \text{delo za deformacijo} = \int_0^1 F_y\, du = \int d\beta\, y \int_a^b dx\, \frac{dF_y}{dx} = \dots$$

$u = \beta y$

$du = d\beta\, y$

$$(\mu\, y_{tt})\, dx = \underbrace{(F\, y_{xx} + f_y)\, dx}_{dF_y}$$

$$\dots = \int_0^1\int_a^b d\beta\, dx\, y\,(-F\beta\, y_{xx} - f_y) = -\int_a^b dx\, \frac{1}{2} y F y_{xx} - \int_a^b \underbrace{y f_y\, dx}_{\text{delo proti zunanji sili}} =$$

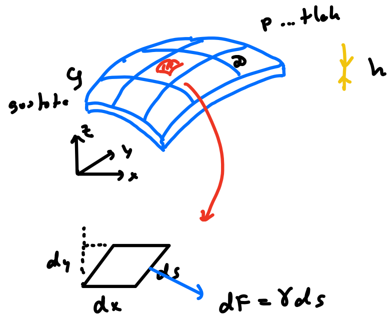
$$\overset{f_y=0}{=} -\frac{1}{2}F\int_a^b y\, y_{xx}\, dx = -\frac{1}{2}F\int_a^b (y\, y_x)_x - y_x^2\, dx =$$

$$= \underbrace{-\frac{1}{2}F\, y\, y_x\Big|_a^b}_{0} + \frac{1}{2}F\int_a^b y_x^2\, dx = \frac{1}{2}F\int_a^b y_x^2\, dx$$

če $y(a) = y(b)$ vpeta struna

ali: $y_x(a) = y_x(b)$ prosta struna

b) Opna



- Napetost γ $\quad (\gamma ds = F)$
- majhni nakloni (v x,y smeri se ne premika)

$$\int_{\mathcal{D}} \left(z_t(x,y,t_2) - z_t(x,y,t_1) \right) \varrho \cdot h \, dx\, dy =$$

$$= \iint \underbrace{p \, dx\, dy}_{F} \, dt + \int_{t_1}^{t_2} \oint_{\partial\mathcal{D}} dF \underbrace{\frac{\partial z}{\partial n}}_{\text{projekcija v } z \text{ smer}} dt$$

$$dF_z = dF \frac{\partial z}{\partial n} = \gamma ds \frac{\partial z}{\partial n} = \gamma ds (\nabla z \cdot \vec{n}) = \gamma (z_x dy - z_y dx) = \gamma \vec{A} \, d\vec{s}$$

$$\vec{n}\, ds = (dy, -dx)$$

$d\vec{s} = (dx, dy)$ $\qquad \vec{A} = (-z_y, z_x)$

Stokes $\qquad \oint_{\partial\mathcal{D}} \gamma \vec{A} \, d\vec{s} = \int_{\mathcal{D}} \gamma (\nabla \times \vec{A}) \, dx\, dy$

$$\iint_{\mathcal{D}\ t_1}^{t_2} z_{tt} \varrho h \, dx\, dy\, dt = \int_{\mathcal{D}} \int_{t_1}^{t_2} p \, dx\, dy\, dt + \int_{\mathcal{D}} \int_{t_1}^{t_2} \gamma (\nabla \times \vec{A})_z \, dx\, dy\, dt$$

$$\Rightarrow \varrho h z_{tt} = p + \gamma (z_{xx} + z_{yy})$$

$$z_{tt} = \frac{p}{\varrho h} + \underbrace{\frac{\gamma}{\varrho h}}_{c^2} \nabla^2 z$$

c) Hidrodinamika - zvok

Navier - Stokes $\qquad \varrho \underbrace{\left(\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} \right)}_{\text{pospešek}} = -\nabla p + \eta \nabla^2 \vec{v} + \vec{f}$

↳ gostota zun. sil

Kontinuitetna enačba $\qquad \frac{\partial \varrho}{\partial t} + \nabla \cdot (\varrho \vec{v}) = 0$

Enačba stanja $\qquad p = p(\varrho)$

Predpostavke
$\delta p = p - p_0 \qquad \delta p \ll p_0$
$\delta \varrho = \varrho - \varrho_0 \qquad \delta \varrho \ll \varrho_0$
$f = 0 \qquad \eta = 0$
lineariziramo, vzamemo le lin. člen razvoja

$$\varrho_0 \vec{v}_t + 0 = -\nabla \delta p$$
$$\delta \varrho_t + \varrho_0 \nabla \cdot \vec{v} = 0$$
$$\frac{\delta \varrho}{\varrho_0} = \chi \delta p \qquad \text{en. stanja}$$

↳ stisljivost

$$\delta \varrho = \varrho_0 \chi \delta p \Rightarrow \delta \varrho_t = \varrho_0 \chi \delta p_t$$

$$\delta p_t \varrho_0 \chi + \varrho_0 \nabla \cdot \vec{v} = 0 \qquad / \frac{d}{dt}$$
$$\chi \delta p_{tt} + \nabla \cdot \vec{v}_t = 0$$

$$\Rightarrow \chi \delta p_{tt} - \frac{1}{\varrho_0} \nabla (\nabla \cdot \delta p) = 0 \Rightarrow \underbrace{\varrho_0 \chi}_{1/c^2} \delta p_{tt} = \nabla^2 \delta p$$

Naj bo $\vec{v} = -\nabla \varphi$ Prepišimo enačbo iz δp na $\vec{v}$

$$\Rightarrow \quad \rho_0 (-\nabla \varphi_t) = -\nabla \delta p \quad \Rightarrow \quad \delta p = \rho_0 \varphi_t$$

$$\delta \rho_t + \rho_0 (-\nabla^2 \varphi) = 0 \quad \Rightarrow \quad \varphi_{tt} = c^2 \nabla^2 \varphi$$

Valovna enačba v 1D

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

$f(x,t) = $ konst., $\quad A_{ij} f_i f_j = 0 \quad$ Metoda karakteristik

$$\left(\frac{\partial f}{\partial t}\right)^2 - c^2 \left(\frac{\partial f}{\partial x}\right)^2 = 0$$

$$\Big(\underbrace{\frac{\partial f}{\partial t} - c \frac{\partial f}{\partial x}}_{p = x + ct}\Big)\Big(\underbrace{\frac{\partial f}{\partial t} + c \frac{\partial f}{\partial x}}_{r = x - ct}\Big) = 0$$

→ Pri metodi karakteristik odpade drugi odvod po normali

$$x, t \to r, p$$

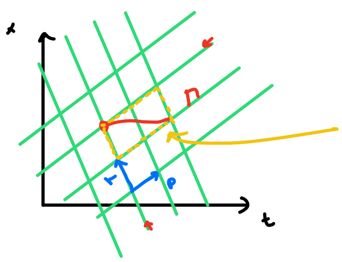
$$c^2 \frac{\partial^2 u}{\partial r \partial p} = 0$$

$$\Downarrow$$

$u(x,t) = g(x - ct) + h(x + ct)$ Splošna rešitev valovne enačbe

g in h sta določeni z začetnimi pogoji

"dva potujoča vala"



tak Π določa rešitev u v

ZP. za Π pri $t = 0$

$$u(x,0) = g(x) + h(x)$$

$$\underbrace{u_t(x,0)}_{\text{poznamo}} = -c g'(x) + c h'(x)$$

integriramo

$$\frac{1}{2c} \int_{x-ct}^{x+ct} u_t(y,0)\, dy = \frac{1}{2} \left(h(x+ct) - h(x-ct) + g(x-ct) - g(x+ct) \right)$$

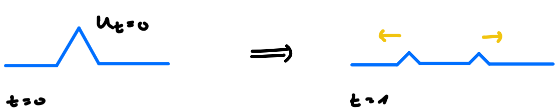
Rešimo sistem enačb

$$\boxed{\begin{aligned} u(x,t) &= \frac{1}{2}\left[u(x-ct, 0) + u(x+ct, 0) \right] + \bar{v} t \\ \bar{v} t &= \frac{1}{2c} \int_{x-ct}^{x+ct} u_t(y,0)\, dy \end{aligned}}$$

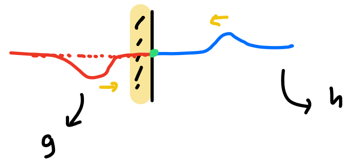
d'Alembertova rešitev

Ali val. en. dovoljuje negladke rešitve? Da

začetni pogoj



Polneskončna struna



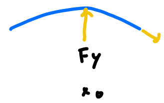
Struno liho preslikamo, s čimer dobimo ponovno neskončno struno

Sodo preslikavo vzamemo če je prijemališče gibajoče.

Nehomogeno sredstvo

ⓐ Diskretna sila

$$u_{tt} = c^2 u_{xx} + \frac{F_y}{\mu}\,\delta(x-x_0) \Big/ \int_{x_0-\epsilon}^{x_0+\epsilon} dx \Rightarrow$$



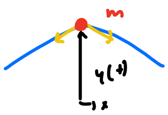
$$\sigma(u_{tt}\, 2\epsilon) = c^2\, u_x\Big|_{x_0-\epsilon}^{x_0-\epsilon} + \frac{F_y}{\mu}$$

$\epsilon \to 0$

$$0 = c^2\, u_x\Big|_{x_0^-}^{x_0^+} + \frac{F_y}{\mu}$$

levi in desni odvod nista enaka

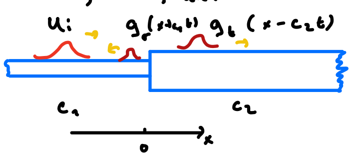
ⓑ Diskretna masa



$$m\ddot{y} = F u_x\Big|_{x_0^-}^{x_0^+}$$ Gibalna enačba kroglice

Ortogonalnost lastnih funkcij $\langle X_m, X_n \rangle = \delta_{m,n} = \int \mu X_m^* X_n dx + m X_m'(x_0) X_n(x_0)$

ⓒ Nehomogena struna



Pogoji:
- u zvezna
- $\frac{\partial u}{\partial x}$ zvezen

$u_i(x,t) = g(x-c_1 t)$

Pogoji

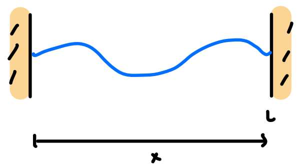
- $g_i(0-c_1 t) + g_r(0+c_1 t) = g_t(0-c_2 t)$
- $g_i'(-c_1 t) + g_r'(c_1 t) = g_t'(-c_2 t)$ $\quad \Big| \int_{-\infty}^{t} dt$

na spodnji meji so člani 0

$$-\frac{1}{c_1} g_i(-c_1 t) + \frac{1}{c_1} g_r(c_1 t) = -\frac{1}{c_2} g_t(-c_2 t)$$

$$g_r(-s) = \frac{c_2 - c_1}{c_2 + c_1}\, g_i(s) \qquad g_t\left(\frac{c_2}{c_1} t\right) = \frac{2 c_2}{c_1 + c_2}\, g_i(s)$$

ⓓ Končna struna – Separacija spremenljivk



$$c^2 u_{xx} = u_{tt}$$

$$u(x,t) = X(x)\, T(t)$$

$$c^2\, T(t)\, X''(x) = X(x)\, T''(t)$$

$$c^2 \frac{X''(x)}{X(x)} = \frac{T''(t)}{T(t)} = -\omega_n^2 = -c^2 k^2 \qquad \omega_n^2 = c^2 k^2$$

$$X'' + k^2 X = 0 \qquad\qquad T'' + \omega_n^2 T = 0$$

$$X = \gamma_n \cos(kx) + \delta_n \sin(kx) \qquad\qquad T = \alpha_n \cos(\omega_n t) + \beta_n \sin(\omega_n t)$$

Robni pogoj $\quad X(0) = X(L) = 0 \quad \Rightarrow \gamma_n = 0$

$$k_n = \frac{n\pi}{L} \qquad n = 1, 2, \dots$$

$$X_n = \sqrt{\frac{2}{L}} \sin(k_n x) \qquad k_n = \frac{n\pi}{L} \qquad \text{Lastne funkcije}$$

↑ normalizacija

$$\langle X_n | X_m \rangle = \int_0^L X_n^* X_m \, dx = \delta_{mn} \qquad \text{Ortogonalne funkcije}$$

$$u(x,t) = \sum_n X_n(x)\, T_n(t) = \sum_n X_n(x) \left(\alpha_n \cos(\omega_n t) + \beta_n \sin(\omega_n t) \right)$$

$$\omega_n = c k_n = c \frac{n\pi}{L}$$

$$f(x) = \sum_n a_n X_n(x) \qquad a_n = \langle X_n | f \rangle$$

ⓔ Nehomogena enačba

$$u_{tt} = c^2 u_{xx} + f(x,t)$$

r.p. homogeni | fiksni konci strune

z.p. $\quad u(x,0) = \varphi(x) = \sum_n \varphi_n X_n(x)$

$\quad u_t(x,0) = \psi(x) = \sum_n \psi_n X_n(x)$

$$f(x,t) = \sum_n f_n(t)\, X_n(x)$$

$$u(x,t) = \sum_n u_n(t)\, X_n(x)$$

Vstavimo v enačbo

$$\sum_n \ddot{u}_n(t) X_n(x) = \sum_n -\omega_n^2 u_n(t) X_n(x) + f_n X_n(x) \qquad \Big| \int_0^L X_m^* dx \qquad \ddot{X}_n(x) = -k^2 X_n(x)$$

$$\ddot{u}_m = -\omega_m^2 u_m + f_m$$

$$\ddot{u}_m + \omega^2 u_m = f_m$$

$$u_m = u_{Hom} + u_{Part} \qquad u_m(0) = \varphi_m \qquad \dot{u}_m(0) = \psi_m$$

$$u_{Part} = \frac{1}{\omega_n} \int_0^t \sin(\omega_m (t-\tau))\, f_n(\tau)\, d\tau$$

$$u_m(t) = u_m^{Hom} + \left(\varphi_m \cos \omega t + \frac{\psi}{\omega_n} \sin \omega_n t \right)$$

4 Nehomogeni robni pogoj

$$c^2 u_{xx} = u_{tt}$$

r.p. $\quad u(0,t) = \mu_1(t)$

$\quad\quad u(L,t) = \mu_2(t)$

Trik: $\quad u(x,t) = \tilde{u}(x,t) + \underbrace{\mu_1 + \frac{x}{L}(\mu_2 - \mu_1)}_{\text{zado šče robnim pogojem}}$

$$c^2 \tilde{u}_{xx} = \tilde{u}_{tt} + \Big[\underbrace{\ddot{\mu}_1 + \frac{x}{L}(\ddot{\mu}_2 - \ddot{\mu}_1)}_{f(x,t)}\Big]$$

$$c^2 \tilde{u}_{xx} = \tilde{u}_{tt} + f(x,t)$$

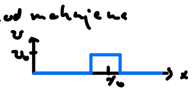
Nehomogena en. a s homogenimi robnimi pogoji.

$$\tilde{u}(0,t) = 0 = \tilde{u}(L,t)$$

Primer: klavir

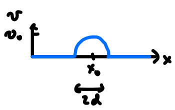
Strune nedomahnjene

ⓐ hitrost



kvadratasto profil kladiva

ⓑ hitrost



$v_0 \cos\left(\frac{x - x_0}{d}\frac{\pi}{2}\right)$

$E_n = ?$

$X_n = \sqrt{\frac{2}{L}} \sin k_n x \qquad k_n = \frac{n\pi}{L} \qquad T_n = A \sin \omega_n t + \underset{\substack{\parallel \\ 0}}{B} \cos \omega_n t$ ker more $T_n(0) = 0$

$$u(x,t) = \sum_n a_n X_n T_n$$

ⓐ $\quad a_n = \frac{v_0}{\omega_n} \int_{x_0 - d}^{x_0 + d} X_n \, dx \qquad$ podobno za b del

$$E_n = \frac{1}{2}\mu \int u_t^2 \, dx + \frac{1}{2}\mu c^2 \int u_x^2 \, dx = \frac{1}{4} m \, \omega_n^2 \left(a_n \sqrt{\frac{2}{L}}\right)^2$$

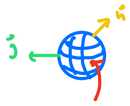
ⓐ $\quad E_n = m v_0^2 \, 4 \sin^2\left(\frac{n\pi x_0}{L}\right)\left(\frac{\sin \delta_n}{\delta_n}\right)^2 \left(\frac{d}{L}\right)^2 \qquad \delta_n = \frac{n\pi d}{L}$

ⓑ $\quad E_n = m v_0^2 \frac{16}{\pi^2} \sin^2\left(\frac{n\pi x_0}{L}\right)\left(\frac{\cos \delta_n}{1 - \left(\frac{2\delta_n}{\pi}\right)^2}\right)^2 \left(\frac{d}{L}\right)^2$

Difuzijska enačba

Izpeljava

Ohranitev energije



q

grelec $\left[\frac{W}{m^3}\right]$

$$\oint \vec{j}\, d\vec{S} = -\frac{dQ}{dt} = \underbrace{\int q\, dV}_{\text{izvor}} - \int \rho c_p \frac{\partial T}{\partial t}\, dV \qquad \text{kontinuitetna enačba}$$

pretok toka

$$\| $$

$$\int \nabla \vec{j}\, dV$$

$$\Rightarrow \quad \nabla \vec{j} = q - \rho c_p \frac{\partial T}{\partial t}$$

Fourierov zakon $\quad \vec{j} = -\lambda \nabla T \qquad \lambda = \text{konst.}$

$$\Rightarrow \quad -\lambda \nabla^2 T = q - \rho c_p \frac{\partial T}{\partial t}$$

$$\boxed{\frac{\partial T}{\partial t} = D \nabla^2 T + \frac{q}{\rho c_p} \; ; D = \frac{\lambda}{\rho c_p}}$$

za idealni plin $D \sim \bar{v} \bar{\ell}$

V premikajočem sredstvu $\quad \frac{\partial T}{\partial t} \longrightarrow \frac{\partial T}{\partial t} + (\vec{v} \cdot \nabla) T$

Pogosti RP :

$$T = 0$$

$$\frac{\partial T}{\partial n} \sim \vec{j} = 0$$

$$\lambda \frac{\partial T}{\partial n} = \Lambda (T - T_z) \quad \Rightarrow \quad T' + \alpha T = 0$$

Neskončni prostor

1D, brez izvorov

$$\frac{\partial T}{\partial t} = D \nabla^2 T$$

Invariantna na $\begin{array}{l} x \mapsto kx \\ t \mapsto k^2 t \end{array}$

Če je tudi zač. pogoj skaliramo invarianten, ima končna rešitev tudi takšno lastnost

$$\underset{\text{naj bo T rešitev}}{T(x,t)} = T(kx, k^2 t) \overset{\text{naj bo } k = \frac{1}{\sqrt{t}}}{=} T\left(\frac{x}{\sqrt{t}}, 1\right) = f(s)$$

$$s = \frac{x}{\sqrt{t}}$$

$$f' \frac{\partial s}{\partial t} = D f'' \left(\frac{\partial s}{\partial x}\right)^2$$

$$-f' \frac{s}{2t} = D f'' \frac{1}{t}$$

$$\int \frac{f''}{f'} = -\int \frac{s}{2D}$$

$$\ln f' = -\frac{s^2}{4D} + C_1$$

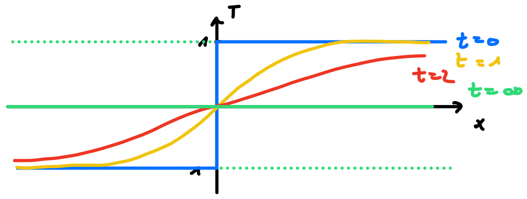
$$f' = e^{-\frac{s^2}{4D} + c_1}$$

$$f(s) = \tilde{C}_1 \int_0^s e^{-\frac{s^2}{4D}} ds + C_2$$

$\tilde{C}_1, C_2$ iz z.p.

Naj bo $C_2 = 0$ $\quad \tilde{C}_1 = \frac{1}{\sqrt{\pi D}}$

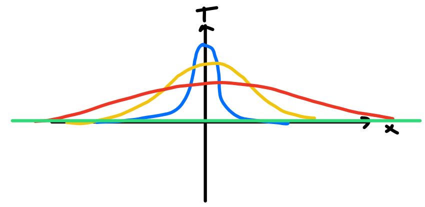
$$T_1(x,t) = \frac{1}{\sqrt{\pi D}} \int_0^{x/\sqrt{t}} e^{-s^2/4D} ds = f(s) = \underline{\mathrm{erf}\left(\frac{x}{\sqrt{4Dt}}\right)}$$



Ker je en. linearna na njej lahko izvajamo lin. operacije in spet dobimo rešitev

$$F(x,t) = \frac{1}{2}\frac{\partial}{\partial x} T_1(x,t) = \underline{\frac{1}{\sqrt{4\pi D t}} e^{-x^2/4Dt}}$$

tudi rešitev



Nova rešitev (superpozicija Gaussovih funkcij)

$$T(x,t) = \int_{-\infty}^{\infty} g(x_0) F(x-x_0,t)\, dx_0 = g * F$$

$$\lim_{t\to 0} F(x-x_0,t) = \delta(x-x_0)$$

Reši z.p. $T(x,0) = g(x)$ $\quad$ Za poljuben z.p. $g(x)$ lahko izračunamo $T(x,t)$.

Greenova funkcija

$$G(x,x_0,t) = \frac{1}{\sqrt{4\pi D t}} e^{-(x-x_0)^2/4Dt}\, \Theta(t)$$

Heavyside

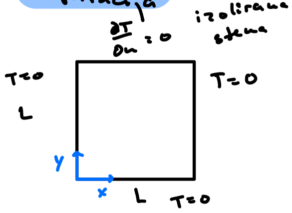
$$\left(\frac{\partial}{\partial t} - D\nabla^2\right) G = \delta(x-x_0)\,\delta(t)$$

Rešitev v nehomogenem sredstvu

$$\frac{\partial T}{\partial t} - D\nabla^2 T = f(x,t)$$

$$T(x,t) = g * G + \int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} dx_0\, f(x_0,\tau)\, G(x,x_0,t-\tau)$$

Separacija



$$T(x,y,t) = T(x,y)\, u(t)$$

$$\frac{\partial T}{\partial t} = D\nabla^2 T$$

$T(x,0) = f(x)$

$$D\frac{\nabla^2 T}{T} = \frac{\dot{u}}{u} = -\lambda \quad \text{cel. konstanta}$$

naj bo $D=1$

① $\lambda = 0$

$\Rightarrow \quad u(t) = \text{konst.}$

$\nabla^2 T = 0 \quad \Rightarrow \quad T(\vec{r}) = 0$ ✗ neuporabno

② $\lambda < 0$

$\Rightarrow \quad u(t) = A\, e^{|\lambda| t}$ ne pričakujemo eksponentnega naraščanja temperature

$T(x,y) = X(x)\, Y(y)$

skalarje $\underbrace{\frac{X''}{X}}_{k^2} + \underbrace{\frac{Y''}{Y}} = -\lambda$

$\frac{X''}{X} = \pm k^2 \quad \Rightarrow \quad X_+(x) = A\,\mathrm{sh}(kx) + B\,\mathrm{ch}(kx)$

$X_-(x) = A\sin(kx) + B\cos(kx)$

Ena izmed $\frac{X''}{X}$ in $\frac{Y''}{Y}$ more biti pozitivna, $X_+(x)$

✗ naših robnih pogojev se ne da zadostiti

③ $\lambda > 0$

$u(t) = A\, e^{-\lambda t}$

Krajevni del: oba kvocienta negativna

$X'' + k_x^2 X = 0 \qquad X = \alpha \sin k_x x + \beta \cos k_x x$

Naš primer: x • $\beta = 0$

• $\sin(kL) = 0 \qquad k = \frac{n\pi}{L} \qquad n = 1, 2, \ldots$

$X_n(x) = \sqrt{\frac{2}{L}} \sin \frac{n\pi}{L} x$

y • $\beta = 0$

$\frac{d}{dy} \sin(kL) = 0$

$\cos kL = 0 \qquad k = \frac{(n + 1/2)\pi}{L} \qquad n = 0, 1, 2, \ldots$

$Y_n(y) = \sqrt{\frac{2}{L}} \sin \frac{(n+1/2)\pi}{L} y$

$-\lambda = -k_x^2 - k_y^2$

$\lambda_{n,m} = \left(\frac{\pi}{L}\right)^2 \left(n^2 + (m + 1/2)^2 \right)$

Splošna rešitev

$T(\vec{r}, t) = \sum_{m,n} c_{mn} X_n(x) Y_m(y) \underbrace{e^{-D\lambda_{mn} t}}$

↳ iz z.p. $f(\vec{r})$ ⟶ eksp. pojemanje = ekvilibracija

Laplaceova enačba

$$\nabla^2 \varphi = 0$$

- stacionarna difuzijska
- EMP
- hidrodinamika

  neviskozna in nestisljiva $\qquad \vec{\omega} = \nabla \times \vec{v}$ se ohranja (vrtinčnost)

  $$\vec{\omega} = 0 = \nabla \times \vec{v} \longrightarrow \vec{v} = -\nabla \varphi \text{ in } \nabla \vec{v} = 0 \rightarrow \nabla^2 \varphi = 0$$

- minimalne ploskve

- $\nabla^2 \varphi$ je ukrivljenost (v eni smeri pozitivna, v drugi pa nasprotna ukrivljenost)
- ekstrem $\int dx\,dy\, \nabla^2 \varphi$
- harmonične funkcije, vrednost v točki je enaka povprečni vrednosti v okolici,
  ni lokalnih ekstremov (le na robovih)
- predpisan odvod na robu (Neumannov r.p.) ne more biti poljuben
  veljati mora $\int_{\partial D} \frac{\partial \varphi}{\partial n} dS = 0 = \int_{\partial D} \nabla \varphi \, d\vec{S} = \int_D \nabla^2 \varphi \, dV = 0$

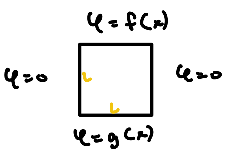
Nehomogena je Poissonova enačba $\quad \nabla^2 \varphi = \rho$

Simetrične rešitve: $\quad$ v 2D $\quad \varphi \sim \ln r$

$\qquad$ v 3D $\quad \varphi \sim \frac{1}{r}$

analitične funkcije in konformne preslikave rešijo enačbo.

Separacija v kartezičnih koordinatah



$$\varphi = X(x)\, Y(y)$$

$$\underbrace{\frac{X''}{X}}_{+} + \underbrace{\frac{Y''}{Y}}_{-} = 0 \quad \text{konst}$$

$\Rightarrow$ v "homogeni" smeri oscilirajoče funkcije

$$X_n = \sqrt{\frac{2}{L}} \sin \frac{n\pi}{L} x$$

v smeri y:

$$Y(y) = \alpha \,\mathrm{sh}\left(\frac{n\pi}{L} y\right) + \beta \,\mathrm{ch}\left(\frac{n\pi}{L} y\right)$$

$$\varphi = \sum_{n=1}^{\infty} X_n(x) \left(a_n \,\mathrm{sh}\left(\frac{n\pi}{L} y\right) + b_n \,\mathrm{ch}\left(\frac{n\pi}{L} y\right) \right) = \ldots$$

a_n, b_n določimo iz $f(x)$, $g(x)$

boljše

$$u_n = \mathrm{sh}\left(\frac{n\pi}{L} y\right) \mathrm{ch}\left(\frac{n\pi}{L} 0\right) - \mathrm{ch}\left(\frac{n\pi}{L} y\right) \mathrm{sh}\left(\frac{n\pi}{L} 0\right)$$

$$v_n = \mathrm{sh}\left(\frac{n\pi}{L} y\right) \mathrm{ch}\left(\frac{n\pi}{L} L\right) - \mathrm{ch}\left(\frac{n\pi}{L} y\right) \mathrm{sh}\left(\frac{n\pi}{L} L\right)$$

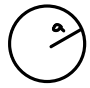
$$u_n(0) = 0 \qquad v_n(L) = 0$$

$$\ldots = \sum_n X_n(x) \left(c_n u_n + d_n v_n \right)$$

c_n ← f, $\quad d_n$ ← g

Separacija v polarnih koordinatah, 2D

$\nabla^2 \psi = 0$ $\quad\quad$ R.P. $\quad \psi(a, \varphi) = f(\varphi)$



$$\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \varphi^2} = 0$$

$$\psi = R(r)\, \Phi(\varphi)$$

$$\frac{\Phi''}{\Phi} = r^2 \frac{R''}{R} + r\frac{R'}{R} = -m^2$$

↳ $\Phi(\varphi) = e^{im\varphi}$ ← sinusi in kosinusi $\quad m = \pm 1, \pm 2, \dots$

$\quad\Phi(\varphi) = a_0 + b_0 \varphi \quad\quad m = 0$

↳ $R(r) = r^{\pm m} \quad\quad m \neq 0$

$\quad R(r) = 1, \ln r \quad\quad m = 0$

Splošna rešitev

$$\psi(r, \varphi) = \underbrace{(A_0 + B_0 \ln r)(a_0 + b_0 \varphi)}_{m=0} + \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} \left(A_m r^{|m|} + B_m r^{-|m|}\right) e^{im\varphi}$$

Sturm-Liouvillov problem

Kdaj nam lastne rešitve DE dajo ortonormiran sistem?

- Lin. operator $\mathcal{L}$ $\quad$ (npr $\mathcal{L} = -\frac{d^2}{dx^2}$)
- Skalarni produkt $\quad \langle u, v\rangle = \int u^* v \, dx$

Def. adjungiranega operatorja $\quad \langle u, \mathcal{L} v\rangle = \langle \mathcal{L}^+ u, v\rangle \quad \Rightarrow \mathcal{L}$ = simetričen

Sebiadjungirani $\quad \mathcal{L} = \mathcal{L}^+ \Leftrightarrow D(\mathcal{L}) = D(\mathcal{L}^+) \quad\quad D(\mathcal{L}^+) \supset D(\mathcal{L})$

Na končno dim. prostoru ni razlike med simetričnimi in sebiadjungiranimi operatorji.

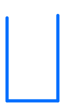
⇓

realne lastne vrednosti

⇓

lastni vekt. so ortogonalni in kompletni

Primer: simetričen a ne sebiadjungiran



$\hat{p} = -i\hbar \frac{\partial}{\partial x}$ $\quad\quad$ ni sebiadjungiran na intervalu

$H = \frac{p^2}{2m}$ $\quad\quad$ spekter $\hat{p} \to \lambda_n \to E_n = \frac{\lambda_n^2}{2m}$ ne deluje

Sturm-Liouvillov problem

$\mathcal{L} u(x) = (p(x) u')' - q(x) u$ $\quad p, q \in \mathbb{R}$ $\quad x \in [a,b]$ končni interval

Lastni problem $\quad \mathcal{L} u = -\lambda w(x) u$

utež, $w \geq 0$

Iščemo λ in lastne funkcije

Npr: dif. en. $\quad (\lambda(x) T(x))' = \varrho(x) c_p(x) \frac{\partial T}{\partial t}$ $\quad$ Nastavek $T = u e^{-\lambda t}$

$(\lambda(x) u'(x))' = -\varrho c_p u \lambda$

$p(x) \sim \lambda(x)$

$w(x) \sim \varrho(x) c_p(x)$

$q(x) = 0$

Kdaj je $\mathcal{L} = \mathcal{L}^+$?

$$u^* \mathcal{L} v - (\mathcal{L} u)^* v = u^* ((pv')' - qv) - ((pu^{*\prime})' - qu^*) v =$$

$$= (u^* p v')' - u^{*\prime}(pv') - (pu^{*\prime} v)' + (pu^{*\prime}) v'$$

$$= (p(u^* v' - v u^{*\prime}))'$$

Integriramo

$$\langle u, \mathcal{L} v\rangle - \langle \mathcal{L} u, v\rangle = p(u^* v' - v u^{*\prime})\Big|_a^b \qquad \text{Greenova formula}$$

$\mathcal{L}$ je sebi adjungiran če:

- $\alpha_a u(a) + \beta_a u'(a) = 0$

  in $\quad \alpha_b u(b) + \beta_b u'(b) = 0$

  Posebna primera:

  - Dirichletov pogoj $u(a) = u(b) = 0$
  - Neumannov pogoj $u'(a) = u'(b) = 0$

- Periodični robni pogoji

  $u(a) = u(b) \quad \neq 0$

  Regularni S-L problem $\Rightarrow \lambda_n \in \mathbb{R}$, štetno neskončno, u_n so ortogonalne in kompletne. u_n ima ne več ničel kot u_{n+1}.

Posledice:

- $\langle u, \mathcal{L} v\rangle - \langle \mathcal{L} u, v\rangle = 0$

  za dve lastni

  $\langle u, -\lambda_v w v\rangle - \langle -\lambda_u w u, v\rangle$ $\quad$ w poznamo, ga mi določimo

  $(\lambda_u - \lambda_v) \int_a^b u^* v w \, dx = 0$

  za $\lambda_u \neq \lambda_v \Rightarrow$ ortogonalnost z utežjo

  $$\langle u, v\rangle_w = \int_a^b u^* v w \, dx$$

• Kompaktnost

$$f(x) = \sum c_n u_n(x)$$

$$c_n = \langle u_n, f \rangle_w$$

za končno vsoto $\sum_n^N c_n u_n(x)$ je to najboljši približek f.

• Variacijska formulacija

$$S = \int_a^b (p u'^2 + q u^2)\, dx \quad \text{ob vezi} \quad \int_a^b (p u'^2 + q u^2)\, dx = 1$$

ekstremi S dajo lastne vrednosti

• Asimptotika u_n in λ_n, $n \gg 1$

$$u_n(x) = A(x)\, e^{ik(x)\, x}$$

$$= \frac{1}{\sqrt[4]{w p}}\, e^{\pm i \int_0^x \sqrt{\frac{\lambda w}{p}}\, dy}$$

$$\mathcal{L} u = (p u')' - q u = p' u' + p u'' - q u$$

Če $\mathcal{L} u = f u'' + g u' + k u$

če $g \neq f'$ enačbo pomnožimo z $\frac{1}{f} \exp\left(\int \frac{g}{f} dx\right)$

lahko dobimo SL problem.

Cilindrični koordinatni sistem

Helmholtzova enačba

$$\nabla^2 u + \lambda u = 0 \qquad \text{Nastavek} \quad u(r, \varphi, z) = R(r)\, \Phi(\varphi)\, Z(z)$$

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2} + \frac{\partial^2 u}{\partial z^2} + \lambda u = 0$$

$$Z \Phi R'' + \frac{1}{r} Z \Phi R' + \frac{1}{r^2} Z R \Phi'' + R \Phi Z'' + \lambda R \Phi Z = 0 \qquad / R\Phi Z$$

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\underbrace{\frac{\Phi''}{\Phi}}_{= -m^2} + \underbrace{\frac{Z''}{Z}}_{= \pm \beta^2} + \lambda = 0$$

v φ smeri $\quad \frac{\Phi''}{\Phi} = -m^2 \quad$ za $m = 0 \quad \Phi = \begin{Bmatrix} 1 \\ \varphi \end{Bmatrix}$ linearna kombinacija odvisna od ZP

$m \neq 0 \quad \Phi = \begin{Bmatrix} \sin m\varphi \\ \cos m\varphi \end{Bmatrix}$

v z smeri: $\quad \frac{Z''}{Z} = \begin{Bmatrix} +\beta^2 \\ -\beta^2 \end{Bmatrix} \quad \begin{Bmatrix} \operatorname{sh} \beta z \\ \operatorname{ch} \beta z \end{Bmatrix} \quad \begin{Bmatrix} \sin \beta z \\ \cos \beta z \end{Bmatrix} \qquad$ za $\beta = 0 \quad \begin{Bmatrix} 1 \\ z \end{Bmatrix}$

v r smeri $\quad r^2 R'' + r R' + ((\lambda \pm \beta^2) r^2 - m^2) R = 0 \qquad$ Besselova DE

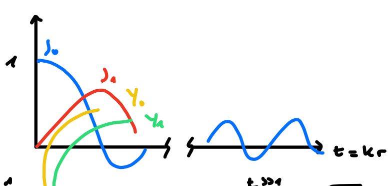
• $\lambda \pm \beta^2 = 0$: enako kot rešitve Laplaceove v polarnih

$\Rightarrow R = \begin{Bmatrix} r^m \\ r^{-m} \end{Bmatrix}$ oz. za $m = 0 \quad \begin{Bmatrix} 1 \\ \ln r \end{Bmatrix}$

• $\lambda \pm \beta^2 > 0$: $\quad \lambda \pm \beta^2 \equiv k^2$

$R = \begin{Bmatrix} J_m(kr) \\ Y_m(kr) \end{Bmatrix}$ Besslove funkcije / Neumannove funkcije

oscilirajoči



Če J_ν je $\nu \in \mathbb{R}$, če J_m je $m \in \mathbb{Z}$

$t \ll 1$

$J_\nu \sim t^\nu$

$Y_\nu \sim \frac{1}{t^\nu} \quad (Y_0 \sim \ln t)$

$t \gg 1$

$J_\nu \simeq \sqrt{2/\pi t}\, \cos\left(t - \frac{1}{2}\nu\pi - \frac{\pi}{4}\right)$

$Y_\nu \simeq \sqrt{2/\pi t}\, \sin\left(t - \frac{1}{2}\nu\pi - \frac{\pi}{4}\right)$

"stoječi valovi"

Hankelove funkcije

$H_\nu^{(1)} = J_\nu + i\,Y_\nu$ "potujoči valovi za $t>>1$" valovi navzven $e^{-i\omega t}$

$H_\nu^{(2)} = J_\nu - i\,Y_\nu$ valovi noter $e^{+i\omega t}$

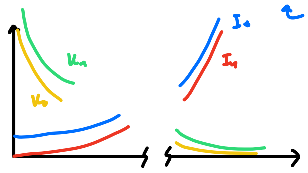
$$e^{ix\sin\theta} = \sum_{m=-\infty}^{\infty} e^{im\theta} J_m(x)$$

ravni val cilindrični valovi

• $\lambda = \beta^2 < 0$: $\lambda = \beta^2 = -k^2$

$$R = \left\{ \begin{matrix} K_m(kr) \\ I_m(kr) \end{matrix} \right\}$$ modificirane Besselove funkcije

I_0 eksponentno narašča



$t<<1$

$I_\nu \sim t^\nu$

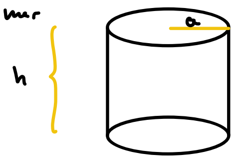
$K_\nu \sim \frac{1}{t^\nu}$ $K_0 \sim \ln t$

$t>>1$

$I_\nu \simeq \sqrt{\frac{\pi}{2t}}\, e^{+t}$

$K_\nu \simeq \sqrt{\frac{\pi}{2t}}\, e^{-t}$

Primer



RP $u(r, z=0,h, \varphi) = 0$

$u(r=a, z, \varphi) = 0$

$$\nabla^2 u + \lambda u = 0$$

v φ: $\Phi = \left\{ \begin{matrix} \sin m\varphi \\ \cos m\varphi \end{matrix} \right\}$ za $m=0$ pa 1

v z: $Z = \sin \frac{n\pi}{h} z$ ker je 0 na zgornji in spodnji ploskvi

$n = 1, 2, \dots$

v r: $R = J_m(kr)$

RP. na plašču

$J_m(ka) = 0$ ξ p-ta ničla J_m.

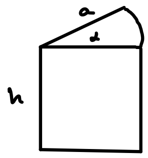
$ka = \xi_{mp}$

Lastne funkcije

$$\boxed{\begin{aligned} u_{m,n,p} &= J_m\left(\frac{\xi_{mp}}{a} r\right) \sin\frac{n\pi}{h} z \; e^{im\varphi} \\ \lambda_{m,n,p} &= \left(\frac{\xi_{mp}}{a}\right)^2 + \left(\frac{n\pi}{h}\right)^2 \end{aligned}}$$

$$\int_0^a r\, J_m\left(\frac{\xi_{mp}}{a} r\right) J_m\left(\frac{\xi_{mp'}}{a} r\right) dr \propto \delta_{pp'}$$ $\perp$ pri enakem m, različnih p.

$$\int_0^a r\, J_r^2\left(\frac{\xi_{mp}}{a} r\right) dr = \frac{a^2}{2} \tan^2(\xi_{mp})$$



enaki robni pogoji

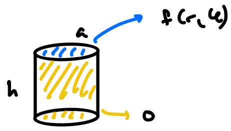
v φ: $\sin n\frac{\pi\varphi}{\alpha}$ $\qquad \nu = n\frac{\pi}{\alpha} \in \mathbb{R}$ $\qquad$ neceli indeksi

v r $\qquad J_\nu\left(\frac{\xi_{\nu p}}{a} r\right)$

Laplacova enačba

$\nabla^2 u = 0$

Primer



v φ: $\begin{cases} \sin m\varphi \\ \cos m\varphi \end{cases}$ $\qquad m = 0, 1, \dots$

v r: $\left\{ J_m\left(\frac{\xi_{n,m}}{a} r\right)\right.$

v z: $\begin{cases} \operatorname{ch} \frac{\xi_{n,m}}{a} z \\ \operatorname{sh} \frac{\xi_{n,m}}{a} z \end{cases}$

$$u(\vec{r}) = \sum J_m\left(\frac{\xi_{mn}}{a} r\right) \operatorname{sh}\left(\frac{\xi_{mn}}{a} z\right)\left(a_{mn} \sin m\varphi + b_{mn} \cos m\varphi\right)$$

R.P. $\quad f(r,\varphi) = \sum J_m\left(\frac{\xi_{mn}}{a} r\right) \operatorname{sh}\left(\frac{\xi_{mn}}{a} h\right)\left(a_{mn} \sin m\varphi + b_{mn} \cos m\varphi\right)$

$\langle \quad \rangle_w = \int \quad r\,dr\,d\varphi$ $\qquad a_{m,n} = \dfrac{\langle J_n \sin n\varphi, f\rangle_w}{\operatorname{sh}(\dots)\langle J_n \sin n\varphi, J_n \sin n\varphi\rangle_w}$ $\qquad \int J_n \sin n\varphi\, r\,dr\,d\varphi$

Sferični koordinatni sistem

$\nabla^2 u + \lambda u = 0$ $\qquad \nabla^2 u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) - \frac{L^2}{r^2} u$

$$L^2 = -\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial}{\partial\theta}\right) - \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}$$

kot vrtilne količine

$\vec{L} = \vec{r} \times \vec{p}$ $\qquad \vec{p} = -i\nabla$ $\qquad L_z = -\frac{\partial}{\partial\varphi}$

Separacijski nastavek $\qquad u = R(r)\,\Theta(\theta)\,\Phi(\varphi)$

v φ: $\qquad \Phi = \begin{Bmatrix} \sin m\varphi \\ \cos m\varphi \end{Bmatrix}$

v θ: $\qquad L^2 Y = l(l+1) Y$ $\qquad t = \cos\theta$

$$\frac{d}{dt}\left((1-t^2)\frac{d\Theta}{dt}\right) + \left((l+1)l - \frac{m^2}{1-t^2}\right)\Theta = 0$$

$\Theta = \begin{cases} P_l^m(\cos\theta) \\ Q_l^m(\cos\theta) \end{cases}$ $\qquad$ Legendrovi polinomi $\qquad Q_l^m$ za cele indekse divergira na polih

skupaj $\qquad Y_{lm}(\theta,\varphi) \sim P_l^m e^{im\varphi}$ $\qquad$ Sferični harmoniki

$$\int Y_{lm}^* Y_{l'm'}\, d\Omega = \delta_{ll'}\delta_{mm'}$$

$d\Omega = d\varphi \sin\theta\, d\theta$

$$Q_0^0 = \frac{1}{2}\ln\frac{1+t}{1-t}, \qquad Q_1^0 = \frac{t}{2}\ln\frac{1+t}{1-t} - 1$$

v r: $\frac{1}{r^2}\frac{\partial}{\partial r} r^2 R' - \frac{l(l+1)}{r^2} R + \lambda R = 0$

$Z = \sqrt{r} R$

$\Rightarrow r^2 Z'' + r Z' + (\lambda r^2 - (l+\frac{1}{2})^2) Z = 0$ Besselova DE

$\lambda = 0$ $R = \begin{cases} r^l \\ r^{-(l+1)} \end{cases}$ $u = \sum_{lm} c_{lm} \begin{Bmatrix} r^l \\ r^{-(l+1)} \end{Bmatrix} Y_{lm}(\theta, \varphi)$

$\lambda < 0$ $-\lambda \in k^2$

$R = \begin{cases} i_l(kr) \\ k_l(kr) \end{cases}$ modificirane sferne Besselove funkcije

divergira v ∞

divergira v 0

$i_0(x) = \frac{\sinh(x)}{x}$ $k_0(x) = \frac{e^{-x}}{x}$

$\lambda > 0$ $k^2 = \lambda$ $R = \begin{cases} j_l(kr) \\ Y_l(kr) \end{cases} = \begin{cases} \sqrt{\frac{\pi}{2kr}} J_{l+1/2}(kr) \\ \sqrt{\frac{\pi}{2kr}} Y_{l+1/2}(kr) \end{cases}$ za $r \to 0$ $Y_l \sim \frac{1}{x^{l+1}}$

oscilirajoči

$y_0 = -\frac{\cos x}{x}$ $j_0 = \frac{\sin x}{x}$

Primer: 3D ∞ pot. jama

$-\frac{\hbar^2}{2m}\nabla^2 \psi = E\psi$ RP $\psi(r=a) = 0$ $\psi(r\to 0) < \infty$

$\nabla^2 \psi + \lambda \psi = 0$ $\lambda = \frac{2mE}{\hbar^2}$

v θ, φ: periodičnost, končnost na Ω $\Rightarrow$ Y_{lm} s celimi l, m

v R: potrebujemo ničle $\Rightarrow$ $\lambda = 0$, $\lambda < 0$ $\times$ $\Rightarrow$ $\lambda > 0$ in $j_l(kr) = j_l(\xi_{l,n} \frac{r}{a})$

$\psi_{n,l,m} \propto j_l(\xi_{ln}\frac{r}{a}) Y_{lm}(\theta, \varphi)$ $E_{n,l,m} = \frac{\hbar^2}{2m}\left(\frac{\xi_{ln}}{a}\right)^2$

Greenove funkcije

$\mathcal{L}_{\vec{r}} u(\vec{r}) = f(\vec{r})$

↳ linearen operator

formalno $u = \mathcal{L}^{-1} f$

$\mathcal{L}_{\vec{r}} G(\vec{r}, \vec{r}_0) = \delta(\vec{r} - \vec{r}_0)$

↳ Greenova funkcija ali fundamentalna rešitev

$\delta(\vec{r}) = \frac{\delta(r)}{4\pi r^2} = \delta(x)\delta(y)\delta(z)$

$\int \delta(\vec{r} - \vec{r}_0) g(\vec{r}) d\vec{r} = g(\vec{r}_0)$

$u(\vec{r}) = \int G(\vec{r}, \vec{r}_0) f(\vec{r}_0) d\vec{r}_0$

Že vidimo G:

- difuzijska enačba $G(x, x_0, t) = \frac{1}{\sqrt{4\pi D t}} e^{-(x-x_0)^2/4Dt} H(t)$

$\left(\frac{\partial}{\partial t} - D\nabla_x^2\right) G(x, x_0, t) = \delta(x - x_0)\delta(t)$

V 3D je $G(\vec{r}, \vec{r}_0, t) = G(x, x_0, t) G(y, y_0, t) G(z, z_0, t)$

Reši z.p. $u(x, t=0) = h(x)$

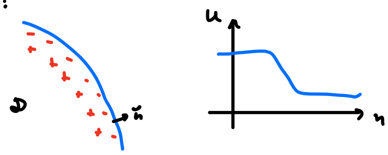
$$u(x,t) = \int G(x, x_0, t)\, h(x_0)\, dx_0$$

- Poissonova enačba $\nabla^2 u = f(\vec r)$ nehomogenost

$$G(\vec r, \vec r_0) = -\frac{1}{4\pi|\vec r - \vec r_0|} \qquad \nabla^2 G(\vec r, \vec r_0) = \delta(\vec r - \vec r_0)$$

rešitev $u(\vec r) = \int G(\vec r, \vec r_0)\, f(\vec r_0)\, d\vec r_0$

Primer:



Helmholtzova enačba v neskončnem prostoru

$$(\nabla^2 \pm k^2)\, G = \delta(\vec r - \vec r_0)$$

ⓐ 2D $\qquad u \propto \{r^m, r^{-m}\}\{\sin m\varphi, \cos m\varphi\}$

sferna simetrija

$k = 0$ iščemo krožno invariantno $\Rightarrow m = 0 \Rightarrow A + B\ln r \Rightarrow G = B \ln r$

$$\nabla^2 G = \delta(r)$$

$$\int \nabla^2 G\, dV = \int \delta\, dV = 1$$

$$\int \frac{\partial G}{\partial n}\, dS = 1$$

$$\frac{\partial G}{\partial n} = \frac{\partial G}{\partial r} = \frac{B}{r} \qquad \int \frac{B}{r}\, dS = \frac{B}{r}\, 2\pi r = 1 \Rightarrow B = \frac{1}{2\pi}$$

$$\Rightarrow \quad G(\vec r, \vec r_0) = \frac{1}{2\pi} \ln|\vec r - \vec r_0|$$

$+k^2 > 0$

$\cancel{J_0}$, Y_0, $H_0^{(1)}$, $H_0^{(2)}$

singularni v izhodišču

stoječe valovanje $\qquad G = \frac{1}{4} Y_0(k|\vec r - \vec r_0|)$

potujoče val. navzven $\qquad G = -\frac{i}{4} H_0^{(1)}(k|\vec r - \vec r_0|)$

potujoče val. noter $\qquad G = \frac{i}{4} H_0^{(2)}(k|\vec r - \vec r_0|)$

$-k^2 < 0 \qquad G = -\frac{1}{2\pi} K_0(k|\vec r - \vec r_0|)$

ⓑ 3D

$k = 0 \qquad G = -\frac{1}{4\pi|\vec r - \vec r_0|}$

$+k^2 > 0 \qquad j_0, y_0, h_0^{(1)}, h_0^{(2)}$ Sferične Besselove in Hankelove

$G = -\frac{1}{4\pi|\vec r - \vec r_0|} e^{\pm ik|\vec r - \vec r_0|}$ potujoče valovanje $\qquad G(\vec r, \vec r_0) = -\frac{1}{2\pi|\vec r - \vec r_0|} \cos k|\vec r - \vec r_0|$ stoječe

$-k^2 < 0 \qquad G(\vec r, \vec r_0) = -\frac{e^{-k|\vec r - \vec r_0|}}{4\pi|\vec r - \vec r_0|}$

Končna domena

D in r. p. na ∂D

$$G = G_\infty + g$$

na ∞ domeni

$$\mathcal{L} G = \mathcal{L} G_\infty + \underbrace{\mathcal{L} g}_{=0} = \delta(\vec{r}-\vec{r}_0)$$

$\mathcal{L} g = 0$ iščemo g, homogena enačba in komplicirani robni pogoj na ∂D

Npr. za Dirichlet $G(\vec{r}_{\partial D}, \vec{r}_0) = 0$

$$\Rightarrow \quad g(\vec{r}_{\partial D}, \vec{r}_0) = -G_\infty(\vec{r}_{\partial D}, \vec{r}_0)$$

$$\vec{r} = \vec{r}_{\partial D} = \vec{r}_s \in \partial D$$

Za rešitev uporabimo Greenovo formulo

a) $\mathcal{L} = \nabla^2 + q(\vec{r})$

Greenova formula
$$\boxed{\int_D (u \mathcal{L} v - v \mathcal{L} u)\, dV = \int_{\partial D} \left(u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n}\right) dS}$$

- Dirichlet

$\mathcal{L} u = f(\vec{r})$ in $u(\vec{r}_s) = h(\vec{r}_s)$ nehomogena enačba in robni pogoj

$\mathcal{L} G = \delta(\vec{r}-\vec{r}_0)$ in $G(\vec{r}_{\partial D}, \vec{r}_0) = 0$

$v = G$ $\quad u$ - poljuben

$$\int_D (u \underbrace{\mathcal{L} G}_{\delta(\vec{r}-\vec{r}_0)} - G \underbrace{\mathcal{L} u}_{f(\vec{r})})\, dV = \int_{\partial D} \left(u \frac{\partial G}{\partial n} - \underbrace{G}_{=0} \frac{\partial u}{\partial n}\right) dS$$

$$u(\vec{r}_0) - \int_D G(\vec{r}, \vec{r}_0) f(\vec{r})\, d\vec{r} = \int_{\partial D} u(\vec{r}_s) \frac{\partial G(\vec{r}_s, \vec{r}_0)}{\partial \vec{r}_s}\, d\vec{S}_s$$

simetrična $G(\vec{r}, \vec{r}_0) = G(\vec{r}_0, \vec{r})$ $\qquad \vec{r} \leftrightarrow \vec{r}_0$

$$\boxed{u(\vec{r}) = \int_D G(\vec{r}, \vec{r}_0) f(\vec{r}_0)\, d\vec{r}_0 + \int_{\partial D} \underbrace{h(\vec{r}_s)}_{\text{rob}} \underbrace{\frac{\partial G(\vec{r}, \vec{r}_s)}{\partial \vec{r}_s}}_{\text{plast dipolov}}\, d\vec{S}_s}$$

- Neumann

$\nabla^2 u = f \quad (q(\vec{r}) = 0)$

$\left.\frac{\partial u}{\partial n}\right|_{\partial D} = h(\vec{r})$ $\qquad$ Vez $\int_D f(\vec{r}_0)\, d\vec{r}_0 = \int_{\partial D} h(\vec{r}_s)\, dS_s$

Neumannova Greenova funkcija G_N

$$\nabla^2 G_N = \delta(\vec{r}-\vec{r}_0) - \frac{1}{V} \quad \text{in} \quad \left.\frac{\partial G_N}{\partial n}\right|_{\partial D} = 0$$

prostornina prostora D

iz Greenove formule dobimo:

$$\boxed{u(\vec{r}) = \int_D G_N(\vec{r}, \vec{r}_0) f(\vec{r}_0)\, d\vec{r}_0 - \int_{\partial D} \underbrace{h(\vec{r}_s)\, G_N(\vec{r}, \vec{r}_s)}_{\text{plast nabojev}}\, dS_s + (\text{konst.})}$$

(b) Difuzijska enačba

$$\mathcal{L} = \frac{\partial}{\partial t} - D\nabla^2$$

$$\mathcal{L}u = f(\vec r, t) \qquad\qquad \mathcal{L}G(\vec r, \vec r_0, t) = \delta(\vec r - \vec r_0)\,\delta(t)$$

RP $\quad u(\vec r_b, t) = h(\vec r_b, t)$ Dirichlet $\qquad G(\vec r_b, \vec r_0, t) = 0$

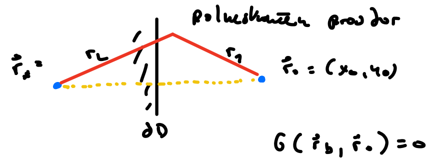
ZP $\quad u(\vec r, 0) = g(\vec r)$

$$\boxed{u(\vec r, t) = \int_D\int_0^\infty f(\vec r_0, \tau)\, G(\vec r, \vec r_0, t-\tau)\, d\vec r_0\, d\tau - D\int_{\partial D}\int_0^\infty h(\vec r_s, \tau)\,\frac{\partial G(\vec r, \vec r_0, t-\tau)}{\partial \vec r_s}\, d\vec S_r\, d\tau + \int_D g(\vec r_0)\, G(\vec r, \vec r_0, t)\, d\vec r_0}$$

Primer:

$$G = G_\infty + g$$
$$\mathcal{L} = \nabla^2$$

$$G_\infty = \frac{1}{2\pi}\ln|\vec r - \vec r_0|$$



$$G(\vec r_b, \vec r_0) = 0$$

Ideja: zrcaljenje

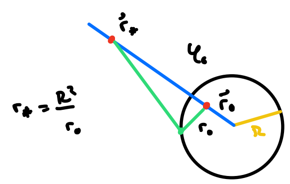
$$g = -\frac{1}{2\pi}\ln|\vec r - \vec r_*|$$

$$\Rightarrow \quad G = \frac{1}{2\pi}\ln\left|\frac{\vec r - \vec r_0}{\vec r - \vec r_*}\right| = \frac{1}{2\pi}\ln\frac{r_1}{r_2}$$

na $r_1 = r_2 \Rightarrow G = 0$

na robu

Primer $\quad \mathcal{L} = \nabla^2$, notranjost kroga



$$G(\partial D, \vec r_0) = 0$$
$$G = G_\infty + g \qquad\qquad G_\infty = \frac{1}{2\pi}\ln|\vec r - \vec r_0|$$
$$\mathcal{L}g = \nabla^2 g = 0$$

$$g(r = R, \varphi, r_0, \varphi_0) = -\frac{1}{4\pi}\ln\left(R^2 + r_0^2 - 2Rr_0\cos(\varphi - \varphi_0)\right) =$$

$$= -\frac{1}{4\pi}\ln\left(R^2\left(1 + \frac{r_0^2}{R^2} - 2\frac{r_0}{R}\cos(\varphi - \varphi_0)\right)\right) = \ldots$$

Splošna rešitev $\quad g = c_0 + \sum_{m=1}^\infty r^m\left(c_m\cos m\varphi + d_m\sin m\varphi\right)$

$x < 1 \qquad \ln(1 + x^2 - 2x\cos\alpha) = -\sum_{n=1}^\infty \frac{2}{n}x^n\cos n\alpha \qquad *$

$$\ldots = -\frac{1}{4\pi}\left(\ln R^2 - \sum_n \frac{2}{n}\left(\frac{r_0}{R}\right)^2 \underbrace{\cos m(\varphi - \varphi_0)}_{\cos m\varphi\cos m\varphi_0 + \sin m\varphi\sin m\varphi_0}\right) =$$

$\Rightarrow$ iz primerjave dobimo d_m, c_m $\quad\ldots$ DN

vrsta za g je iste oblike kot $*$ in jo lahko nazaj seštejemo

$$g = -\frac{1}{4\pi}\ln\left(R^2 + \left(\frac{r_0 r}{R}\right)^2 - 2rr_0\cos(\varphi - \varphi_0)\right) = -\frac{1}{4\pi}\ln\left(\frac{r_0^2}{R^2}|\vec r - \vec r_*|^2\right)$$

Razvoj po lastnih funkcijah

$\mathcal{L} G = \delta(\vec r - \vec r_0)$ $\qquad$ $\mathcal{L} u_n = \lambda_n u_n$

lastne funkcije
za iste r.p.

Razvoj po u_n

$$\delta(\vec r - \vec r_0) = \sum_n c_n u_n(\vec r) = \sum_n u_n^*(\vec r_0)\, u_n(\vec r) \qquad c_n = \langle u_n | \delta \rangle_w \overset{w=1}{=} u_n^*(\vec r_0)$$

$$G(\vec r, \vec r_0) = \sum_n \frac{u_n^*(\vec r_0)}{\lambda_n} u_n(\vec r)$$ za normirane u_n

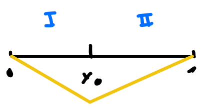
Primer: krog $\qquad G \sim \sum_{mn} \frac{J_m(\frac{\xi_{mn}}{R} r)\, J_m(\frac{\xi_{mn}}{R} r_0) \cos(m(\varphi - \varphi_0))}{\xi_{mn}^2 J_m'^2(\xi_{mn})}$

Kako poiščemo $G = G_\infty + g$:
1. homogena enačba za g (zrcaljenje)
2. g razvijemo po lastnih funkcijah
3. lepljenje

Primer: lepljenje

$\mathcal{L} = \frac{d^2}{dx^2}$ na $x \in [0,1]$

$\mathcal{L} G = \delta(x - x_0)$ $\qquad$ Dirichlet



v I. $\quad G'' = 0 \Rightarrow$ premica

v II. $\quad G'' = 0 \Rightarrow$ premica

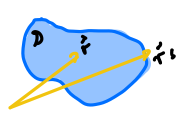
$$\Rightarrow \quad G(x, x_0) = \frac{1}{2}\begin{cases} -x(1 - x_0) & x < x_0 \\ -x_0(1 - x) & x > x_0 \end{cases}$$

Sipanje in G_∞

$\mathcal{L}$ je ∇^2 ali $\nabla^2 + k^2$. Rešujemo $\mathcal{L} u = 0$ in poznamo G_∞, $\mathcal{L} G_\infty(\vec r, \vec r_0) = \delta(\vec r - \vec r_0)$.

$$\int_V (u \mathcal{L} v - v \mathcal{L} u)\, dV = \int_{\partial V} \left(u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n}\right) dS$$

a) Notranji problem

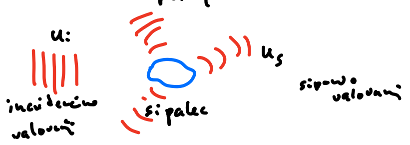


$$u(\vec r) = \int_{\partial D} dS_s \left(u(\vec r_s) \frac{\partial G_\infty}{\partial \vec n_s} - \frac{\partial u(\vec r_s)}{\partial \vec n_s} G_\infty(\vec r, \vec r_s)\right)$$

b) Zunanji problem

$$u(\vec r) = \int dS_s \left(u(\vec r_s) \frac{\partial G_\infty}{\partial \vec n_s} - \frac{\partial u}{\partial \vec n_s} G_\infty\right) - \int dS_\infty \left(u(\vec r_\infty) \frac{\partial G}{\partial \vec n_\infty} - \frac{\partial u}{\partial \vec n_\infty} G(\vec r_\infty)\right)$$

Za sipanje



zaradi ohranitve energije $u_s \sim \frac{1}{r}$
$\Rightarrow$ v enačbi bomo imeli je vpadni člen

$$u = u_i + u_s$$
$$\hookrightarrow \int d\vec{s}_s \left(u(\vec{r}_s) \frac{\partial G_0}{\partial \vec{r}_s} - \frac{\partial u}{\partial \vec{r}_s} G_0(\vec{r}_s) \right)$$

Primer: sipanje na akustično trdi sferi

$$\varphi(\vec{r}, t) \qquad \frac{1}{c^2} \frac{\partial^2 \varphi}{\partial t^2} = \nabla^2 \varphi$$

$$\varphi = a(\vec{r}) e^{-i\omega t}$$
$$\Downarrow$$
$$\nabla^2 u + k^2 u = 0$$
$$u = u_s + u_i$$

Trdnost RP $\quad \frac{\partial \varphi}{\partial n}\Big|_{r_0} = 0$

$$u_i = e^{ikz}$$

Splošna rešitev $\quad u_s = \sum_{l=0}^{\infty} c_l \, h_l^{(1)}(kr) \, P_l(\cos\theta)$

Ravno valovanje zapišemo v sferičnih valovih:

$$e^{ikz} = \sum_{l=0}^{\infty} i^l (2l+1) \, j_l(kr) \, P_l(\cos\theta)$$

R.P. $\quad 0 = \frac{\partial u}{\partial n}\Big|_{r_0} \quad \Rightarrow \quad c_l = -(2l+1) \, i^l \frac{j_l'(ka)}{h_l^{(1)'}(ka)}$

$$\Rightarrow \quad u = e^{ikz} + \sum (-2l+1) \, i^l \frac{j_l'(ka)}{h_l^{(1)'}(ka)} \, h_l^{(1)}(kr) \, P_l(\cos\theta)$$

Integralska formulacija

$\mathcal{L} u - \varepsilon V(\vec{r}) \, u(\vec{r}) = 0 \quad$ z nekimi RP

če poznamo G za $\mathcal{L}$ z danimi RP

$$u(\vec{r}) = h(\vec{r}) + \varepsilon \int G(\vec{r}, \vec{r}_0) \, V(\vec{r}_0) \, u(\vec{r}_0) \, d^3\vec{r}_0$$

rešitev homogene en. $\mathcal{L} h = 0$, ki ustreza RP

Fredholmove integralske en.

Če je $\varepsilon \ll 1$, lahko rešimo perturbativno:

Neumannova oz. Bornova vrsta

1. red $\quad u_1(\vec{r}) = h(\vec{r}) + \varepsilon \int G(\vec{r}, \vec{r}_0) \, V(\vec{r}_0) \, \underline{h(\vec{r}_0)} \, d^3\vec{r}_0$

2. red $\quad u_2(\vec{r}) = u_1(\vec{r}) + \varepsilon^2 \iint G(\vec{r}, \vec{r}_0) \, G(\vec{r}_0, \vec{r}_1) \, V(\vec{r}_0) \, V(\vec{r}_1) \, h(\vec{r}_1) \, d^3\vec{r}_0 \, d^3\vec{r}_1$

$\vdots$

Naj bo $\hat{K} u = \int G(\vec r, \vec r_0)\, V(\vec r_0)\, u(\vec r_0)\, d^3\vec r_0$

$\Rightarrow \quad u = h + \varepsilon K u \quad$ oz $\quad (I - \varepsilon K)\, u = h$

formalen zapis $\quad u = (I - \varepsilon K)^{-1} h$

razvoj do Neumannove vrste, ki ima končni konvergenčni radij

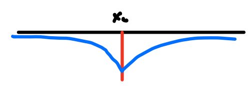
Obstaja tudi Fredholmova vrsta, ki vedno konvergira.

Primer: vezana stanja $(E<0)$ v $V(x)$:

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)\right) u(x) = E\, u(x) \qquad -\frac{2mE}{\hbar^2} = \kappa^2$$

$$\underbrace{\left(\frac{d^2}{dx^2} - \kappa^2\right)}_{\mathcal{L}} u(x) = \frac{2m V(x)}{\hbar^2}\, u(x)$$

G za $\mathcal{L}$ $\qquad G = -\frac{e^{-\kappa|x-x_0|}}{2\kappa}$



$h(x)$ je rešitev homogene enačbe $\mathcal{L} h = 0 \quad \Rightarrow \quad h = c_1 e^{-\kappa x} + c_2 e^{\kappa x}$

za naš primer $c_1 = c_2 = 0$

$$u(x) = -\frac{m}{\hbar^2 \kappa}\int_{-\infty}^{\infty} e^{-\kappa|x-x_0|}\, V(x_0)\, u(x_0)\, dx_0$$

za $V(x_0) \sim \delta(x_0) \qquad u(x) = \int \dots = u(0)\, e^{-\kappa|x|}$

Primer: sipalna stanja v 3D

Vpadni ravni val $e^{i\vec k \vec r}$ se sipa na $V(\vec r)$

$k^2 = \frac{2mE}{\hbar^2} \qquad (\nabla^2 + k^2)\, u(\vec r) = \frac{2m V(\vec r)}{\hbar^2}\, u(\vec r)$

ga poznamo

$$\Rightarrow G(\vec r, \vec r_0) = -\frac{e^{i\vec k(\vec r - \vec r_0)}}{4\pi|\vec r - \vec r_0|}$$

$$u(\vec r) = \underbrace{e^{i\vec k \vec r}}_{\text{vpadni val}} - \underbrace{\frac{m}{2\pi\hbar^2}\int \frac{e^{i\vec k(\vec r - \vec r_0)}}{|\vec r - \vec r_0|}\, V(\vec r_0)\, u(\vec r_0)\, d^3\vec r_0}_{\text{sipano valovanje}}$$

Variacijska formulacija

$\mathcal{L} u = f$

$$S = \frac{1}{2}\langle \mathcal{L}u | u \rangle - \langle f | u \rangle$$ ekstremi so rešitve

$$\delta S = \frac{1}{2}\langle \mathcal{L}\delta u | u\rangle + \frac{1}{2}\langle \mathcal{L}u | \delta u\rangle - \langle f | \delta u\rangle =$$ variacija

$$= \frac{1}{2}\langle \delta u | \mathcal{L}u\rangle + \frac{1}{2}\langle \mathcal{L}u | \delta u\rangle - \langle f | \delta u\rangle =$$

$$= \langle \mathcal{L}u - f | \delta u\rangle = 0$$ kar potrjuje ansatz za S.

ⓐ Laplace $\mathcal{L} = -\nabla^2$, $f=0$

$$S = -\frac{1}{2}\int \nabla^2 u \, u \, dV = -\frac{1}{2}\int \underbrace{\nabla(\nabla u u)}_{0 \text{ za ustrezne r.p.}} dV + \frac{1}{2}\int (\nabla u)^2 dV = \frac{1}{2}\int (\nabla u)^2 dV$$

ⓑ Poisson $-\nabla^2 u = f$

$$S = \frac{1}{2}\int (\nabla u)^2 dV - \int f u \, dV$$

ⓒ Helmholtz $\nabla^2 u + k^2 u = 0$

$$S = \frac{1}{2}\int (\nabla u)^2 dV - \frac{1}{2}\int k^2 u^2 dV$$

vezani ekstrem za $S_0 = \int (\nabla u)^2 dV$ in vez $\int u^2 dV = 1$

k^2 je Lagrangev multiplikator

Vrednost S_0 v ekstremu je enaka k^2

Za minimalno vrednost k^2 velja

$$k^2_{min} = \min\left(\frac{\int (\nabla u)^2 dV}{\int u^2 dV}\right)$$

Rayleigh-Ritz

Osnova za numerične metode

Primer: osnovna frekvenca opne

$$(\nabla^2 + k^2) u = 0 \qquad u \sim J_0(\xi_{01} r) \qquad k^2 = \xi_{01}^2 = 5{,}783\ldots$$

Nastavek $u = A(1-r^2)$

$$k^2_{min} = \min \frac{\int \nabla^2 u \, dS}{\int u^2 \, dS} = \min \frac{-\int \nabla^2 r^2 \, dS}{\int (1-r^2) \, dS} = 6$$

Nastavek $u = A(1-r^2) + B(1-r^2)^2$

1 prost parameter

$$k^2_{min} = \ldots = 5{,}784$$