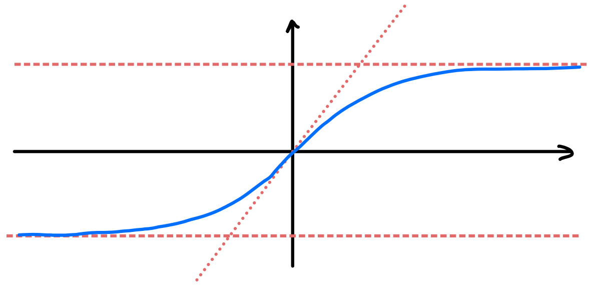
① Kodre 20/26

Skiciraj $\quad \mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt$



Lastnosti funkcije:
- monotonost
- limiti $\quad \mathrm{erf}(\infty) = 1 \quad , \mathrm{erf}(-\infty) = -1$
- $\mathrm{erf}(0) = 0$
- liha
- odvod $\quad \mathrm{erf}'(x) = \frac{2}{\sqrt{\pi}} e^{-x^2} \qquad \mathrm{erf}'(0) \doteq 1{,}13$

• $\int_{-\infty}^{\infty} e^{-t^2} dt = 2 \int_0^{\infty} \frac{1}{2} \frac{1}{\sqrt{u}} e^{-u} du = \int_0^{\infty} u^{-\frac{1}{2}} e^{-u} du = \Gamma(\tfrac{1}{2}) = \sqrt{\pi}$

$u = t^2$

$du = 2t\,dt$

$dt = \frac{du}{2\sqrt{u}}$

• Alternativa

$$\left(\int_{-\infty}^{\infty} e^{-t^2} dt \right)^2 = \int_{-\infty}^{\infty} e^{-t^2} dt \int_{-\infty}^{\infty} e^{-u^2} du = \iint_{-\infty}^{\infty} e^{-(t^2+u^2)} du\, dt = \int_0^{2\pi}\int_0^{\infty} e^{-r^2} r\, dr\, d\varphi = 2\pi \int_0^{\infty} e^{-r^2} r\, dr$$

$$= 2\pi \frac{1}{2} \int_0^{\infty} e^{-u} du = \pi$$

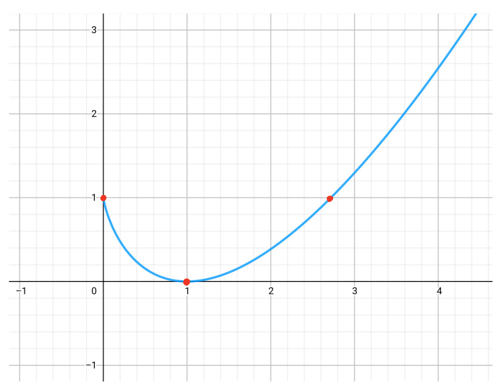
Box - Mullerjev algoritem za generiranje naključnih števil po Gaussu.

68% verjetnosti leži znotraj σ intervala

$$0{,}68 = \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\sigma}^{\sigma} e^{-\frac{x^2}{2\sigma^2}} dx \underset{\frac{x^2}{2\sigma^2} = t^2}{=} \frac{1}{\sqrt{2\pi}\,\sigma} \sigma\sqrt{2} \int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} e^{-t^2} dt = \frac{2}{\sqrt{\pi}} \int_0^{\frac{1}{\sqrt{2}}} e^{-t} dt = \mathrm{erf}\left(\frac{1}{\sqrt{2}}\right)$$

② Kodre 19/8

Za $x \geq 0$ skiciraj $\quad y = x \ln x - x + 1$



Lastnosti funkcije

$y(1) = 0$

$y(e) = 1$

$y(2) \doteq 0{,}39$

$y(0) = \mathrm{L'Hop} = 1$

$y'(x) = \ln x$

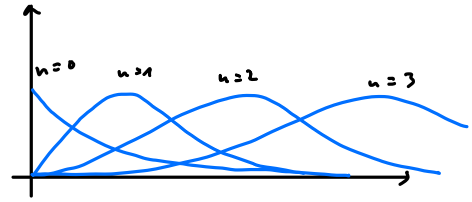
$$y = \int_1^x \ln t\, dt \approx \sum_{t=1}^{x} \ln t = \ln(x!) \approx x \ln x - x + 1$$

$\Rightarrow \quad x! = x^x e^{-x} \cdot e \qquad$ Stirling $\quad x! \approx \sqrt{2\pi x}\, x^x e^{-x}$

Slab približek

Izpeljava Stirlingove formule

$$n! = \int_0^\infty \underbrace{x^n e^{-x}}_{f(x)}\, dx$$



Aproks. z Gaussovko

$$f'(x) = n x^{n-1} e^{-x} - x^n e^{-x} = 0$$
$$(n-x)\, x^{n-1} e^{-x} = 0$$
$$x_{max} = n$$

Alternativno povrečje

$$\langle x\rangle = \frac{\int_0^\infty x \cdot x^n e^{-x}\, dx}{n!} = \frac{(n+1)!}{n!} = n+1$$

V lim $n\to\infty$ sta enaki

Gauss s centrom x_{max} in višino $f(max) = n^n e^{-n}$

$$\sigma^2 = \langle x^2\rangle - \langle x\rangle^2$$
$$\langle x^2\rangle = \frac{\int x^2 x^n e^{-x}\, dx}{n!} = \frac{(n+2)!}{n!} = (n+2)(n+1)$$
$$\sigma^2 = n+1$$

Ujemanje 2. odvodov

$$A e^{-\frac{x^2}{2\sigma^2}} \approx A\left(1 - \frac{x^2}{2\sigma^2}\right)$$
$$G''(0) = -\frac{A}{\sigma^2}$$

$$f''(x) = n(n-1)x^{n-2}e^{-x} - 2n\, x^{n-1}e^{-x} + x^n e^{-x}$$
$$f''(x_{max}=n) = n(n-1)\, n^{n-2} e^{-n} - 2n^n e^{-n} + n^n e^{-n} = -n^{n-1}e^{-n}$$

$$A = n^n e^{-n} = f(x_{max})$$
$$\frac{A}{\sigma^2} = n^{n-1} e^{-n}$$

$$n! = \int_0^\infty x^n e^{-x}\, dx \overset{\text{Aprox. z Gauss.}}{\approx} \int_0^\infty A\, e^{-\frac{(x-n)^2}{\sigma^2}}\, dx = A\sqrt{2\pi}\,\sigma = n^n e^{-n}\sqrt{2\pi \frac{n^n e^{-n}}{n^{n-1}e^{-n}}} = n^n e^{-n}\sqrt{2\pi n}$$

delamo se, da je $-\infty$

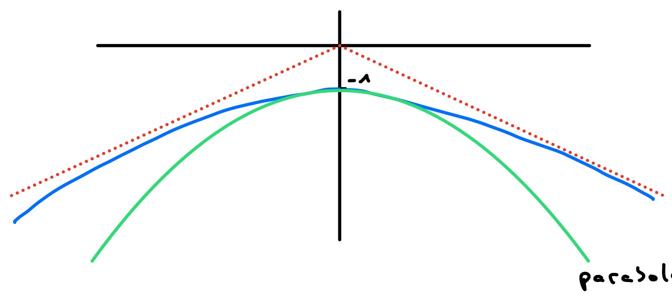
3 Skiciraj funkcijo $f(x) = \frac{x}{e^{-x}-1} + \frac{x}{2}$

$$f(0) = \text{L'Hop} = -1$$

$$f(x) = \frac{x}{2}\left(\frac{2}{e^{-x}-1} + 1\right) = \frac{x}{2}\,\frac{2+e^{-x}-1}{e^{-x}-1} = \frac{x}{2}\,\frac{e^{-x}+1}{e^{-x}-1} = -\frac{x}{2}\,\frac{1}{\tanh(\frac{x}{2})} = \frac{\text{liha}}{\text{liha}} = \text{soda}$$

$$f(x\to 0) \doteq -\frac{x}{2}\,\frac{1}{\frac{x}{2}} = -1$$

$$f(x\to\pm\infty) = -\frac{x}{2}\,\frac{1}{\pm 1} = -\frac{|x|}{2}$$



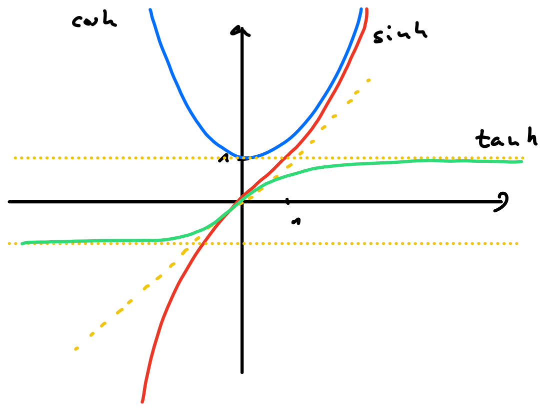
$$f(x) = \frac{x}{2}\,\frac{2}{e^{-x}-1} + \frac{x}{2} \approx$$

$$\approx x\,\frac{1}{1-x+\frac{x^2}{2!}-\frac{x^3}{3!}+\ldots-1} + \frac{x}{2}$$

$$= -\frac{1}{1-\underbrace{\left(\frac{x}{2}-\frac{x^2}{3!}+\frac{x^3}{4!}-\ldots\right)}_{\varepsilon}} + \frac{x}{2} \overset{\text{Geo. vrsta}}{=} -(1+\varepsilon+\varepsilon^2+\ldots) \doteq -1-\frac{x}{2}+\frac{x^2}{3!}-\left(\frac{x}{2}\right)^2+\frac{x}{2} = -1-\frac{x^2}{12}$$

Ⓣ

Hiperbolične funkcije



$$\sinh(x) = \frac{e^{x}-e^{-x}}{2} = x + \frac{x^3}{3!} + \frac{x^5}{5!}$$

$$\cosh(x) = \frac{e^{x}+e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!}$$

$$\tanh(x) = \frac{e^{x}-e^{-x}}{e^{x}+e^{-x}} = \frac{e^{2x}-1}{e^{2x}+1}$$

Vrsta $\tanh x$

$$\tanh' x = \left(\frac{\sinh x}{\cosh x}\right)' = \frac{\sinh' x \cosh x - \cosh' x \sinh x}{\cosh^2 x} = \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} = 1 - \tanh^2 x$$

$$y(x) = \tanh x$$

$$y' = 1 - y^2 \qquad y = \int_0^x (1-y^2)\,dx$$

Picardova iteracija

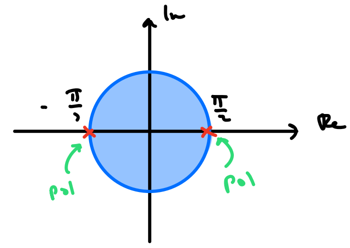
$$y^{(0)} = 0 \qquad \text{Prvi približek}$$
$$y^{(1)} = \int_0^x (1-0^2)\,dx = x$$
$$y^{(2)} = \int_0^x (1-x^2)\,dx = x - \frac{x^3}{3}$$
$$y^{(3)} = \int_0^x \left(1-\left(x-\frac{x^3}{3}\right)^2\right)dx = x - \frac{x^3}{3} + \frac{2}{15}x^5 - \frac{x^7}{63}$$ ni še prav
$$y^{(4)} = \ldots = x - \frac{x^3}{3} + \frac{2}{15}x^5 - \frac{17}{315}x^7 + \mathcal{O}(x^9)$$

Konvergenčni radij

$$\sinh x = \frac{\sin ix}{i} \qquad \cosh x = \cos ix$$

$$\tanh x = \frac{\tan ix}{i} \qquad \tan x \text{ ima pol pri } x = \pm\frac{\pi}{2}$$

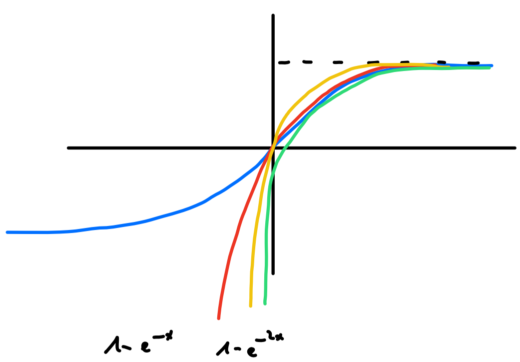
konv. radij $\quad |x| < \frac{\pi}{2}$



Asimptotska vrsta za $\tanh x$

$$\tanh x = \frac{e^{x}-e^{-x}}{e^{x}+e^{-x}} = \frac{1-\overbrace{e^{-2x}}^{\varepsilon}}{1+\underbrace{e^{-2x}}_{\varepsilon}} = \frac{1-\varepsilon}{1+\varepsilon} = (1-\varepsilon)(1-\varepsilon+\varepsilon^2-\varepsilon^3+\varepsilon^4-\ldots) =$$

$$= 1 - 2\varepsilon + 2\varepsilon^2 - 2\varepsilon^3 + 2\varepsilon^4 - \ldots = 1 - 2e^{-2x} + 2e^{-4x} - 2e^{-6x} + \ldots$$

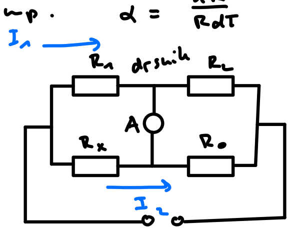


Konvergira za $|\varepsilon| < 1$

$$e^{-2x} < 1$$
$$x > 0$$

④ Kodre 23 / 11+12

Določi natančnost meritve upora z Wheatstonovim mostičkom in natančnost merjenja temp. $\alpha = \frac{dR}{R\,dT}$



$R_1 = R x$ $\quad R_2 = R(1-x)$

R_0 znan

$R_x = ?$ $\quad \delta x = 0{,}3 \frac{mm}{1m}$

Želimo $R_1 I_1 = R_x I_2$

$R_2 I_1 = R_0 I_2$

$$\frac{R_1}{R_2} = \frac{R_x}{R_0}$$

$$R_x = R_0 \frac{x}{1-x}$$ Iščemo zvezo med δR_x in δx

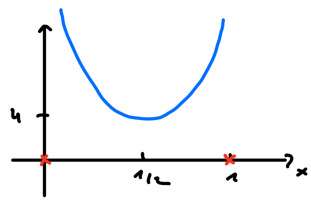
$d\Big\{$ $\ln R_x = \ln R_0 + \ln x - \ln(1-x)$ → rel. popravka napake

$$\frac{dR_x}{R_x} = \frac{dx}{x} + \frac{dx}{1-x}$$

$dx = dx$

$d(\ln x) = \frac{dx}{x}$

$$\frac{\delta R_x}{R_x} = \delta x \frac{1}{x(1-x)}$$



Min. napaka pri $x = \frac{1}{2}$

kjer je $\frac{\delta R_x}{R_x} = 4\delta x = 1{,}2 \cdot 10^{-3}$

Napaka pri meritvi temperature (

$$\frac{\delta R_x}{R_x} = \frac{\delta R_0}{R_0} = \alpha\,\delta T \quad ; \quad \alpha = 0.004\,K^{-1}$$

$$= \frac{\delta x}{x(1-x)} \approx 4\delta x$$ Dolžina stopinje ?

$$\delta x = \frac{\delta l}{l}$$

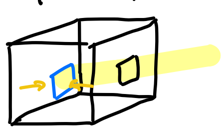
$$\delta T = \frac{4}{\alpha}\delta x$$

$$\frac{\delta l}{\delta T} = \frac{\alpha l}{4} = \dots = 1 \frac{mm}{K}$$

Min. temp. razlika $\Delta T = 0{,}3K$

⑤ Kodre 23 / 13

Počrnjena ploščica v vakuumu s temp. 293K, svetimo $j_0 = 0{,}1 \frac{W}{m^2}$ (pravokotno). $\Delta T = ?$



Prejeta moč: $\qquad$ Oddana moč

$$P = \underbrace{2S\sigma T_0^4}_{\text{stene segreva obe strani}} + \underbrace{S j_0}_{\text{sonce}} \quad = \quad P = 2S\sigma(T_0 + \delta T)^4$$

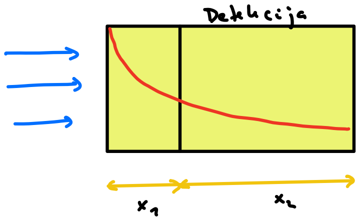
$$2S\sigma T_0^4 + S j_0 = 2S\sigma(T_0 + \delta T)^4$$

$$\frac{\cancel{S} j_0}{\delta T} = 2\cancel{S}\sigma \underbrace{\frac{\left((T_0 - \delta T)^4 - T_0^4\right)}{\delta T}}_{\text{odvod}} \qquad /:\delta T$$

$$\frac{j_0}{\delta T} = 2\sigma \frac{d}{dT} T^4 \Big|_{T=T_0} = 2\sigma 4 T_0^3 = 8\sigma T_0^3$$

$$\delta T = \frac{j_0}{8\sigma T_0^3} = 8{,}8\,mK$$

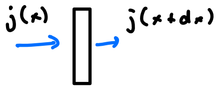
6 Kodre 24/9



Detektor z absorbanco μ je iz 2 delov, v x_1 ne zaznavamo, v x_2 šteje. Pri kakem μ zaznamo največ?

Obstoja max?
Najprej $j(x)$

$$j(x+dx) - j(x) = -\mu\, dx\, j(x)$$



$$\frac{dj}{dx} = -\mu j$$

$$\ln j/j_0 = -\mu x$$

$$j = j_0 e^{-\mu x}$$

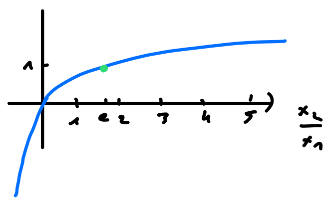
Koliko se absorbira med x_1 in $x_2 + x_1$

$$\Delta j = j(x_1+x_2) - j(x_1) = j_0\left(e^{-\mu(x_1+x_2)} - e^{-\mu x_1}\right)$$

$$\frac{d\Delta j}{d\mu} = j_0\left(-(x_1+x_2)\, e^{-\mu(x_1+x_2)} + x_1 e^{-\mu x_1}\right) = 0$$

$$j_0 e^{-\mu x_1}\left(x_1 - (x_1+x_2)e^{-\mu x_2}\right) = 0$$

$$e^{-\mu x_2} = \frac{x_1}{x_1+x_2} = \frac{1}{1+\frac{x_2}{x_1}}$$



$$\mu = \frac{1}{x_2}\ln\left(1+\frac{x_2}{x_1}\right)$$

7 Samozatemnitvena stekla

Kako izgleda absorbcija svetlobe, če je koeficient absorbcije odvisen od gostote j.
Model $\mu = \mu_0 + \xi j = \mu_0(1+kj)$

$$\frac{dj}{dx} = -\mu j = -\mu_0 j - \xi j^2$$

$$\int_{j_0}^{j}\frac{dj}{j+kj^2} = \int_0^x -\mu_0\, dx = -\mu_0 x$$

Parcialni ulomki

$$\int_{j_0}^{j}\frac{1}{j} - \frac{k}{1+kj}\, dj = \ln\frac{j}{j_0} - \ln\left(\frac{1+kj}{1+kj_0}\right) = -\mu_0 x$$

$$\frac{j}{1+kj} = \frac{j_0}{1+kj_0}e^{-\mu x_0} \Rightarrow j = \frac{\frac{j_0}{1+kj_0}e^{-\mu_0 x}}{1+\frac{kj_0}{1+kj_0}e^{-\mu_0 x}} = \frac{j_0 e^{-\mu_0 x}}{1+kj_0\left(1-e^{-\mu_0 x}\right)}$$

$$\approx j_0 e^{-\mu_0 x}\left(1 - kj_0\left(1-e^{-\mu_0 x}\right) + (kj_0)^2\left(1-e^{-\mu_0 x}\right)^2 \ldots\right)$$

Alternativa: iteracija

$$\frac{dj}{j} = -\mu\, dx$$ Prvi žarek naleti na sveža očala

$$j^{(0)} = j_0\, e^{-\mu_0 x}$$

Drugi žarek že vidi spremenjen μ

$$\int_{j_0}^{j} \frac{dj^{(1)}}{j} = -\int_0^x \left(\mu_0 + k j_0\, e^{-\mu_0 x}\right) dx$$

$$\ln \frac{j}{j_0} = -\mu_0 x + k j_0 \left(-\frac{1}{\mu_0} e^{-\mu_0 x} + \frac{1}{\mu_0}\right)$$

$$j = j_0\, e^{-\mu_0 x}\, e^{-kj_0(1-e^{-\mu_0 x})} \approx j_0 e^{-\mu_0 x}\left(1 - kj_0\left(1-e^{-\mu_0 x}\right) + \frac{1}{2}(kj_0)^2\left(1-e^{-\mu_0 x}\right)^2 \dots\right)$$

narobe

Alternativa:

$$\frac{dj}{dx} = -\mu j = -\mu_0 j - k\mu_0 j^2$$

$$\underbrace{\frac{dj}{dx} + \mu_0 j}_{\substack{\text{linearno}\\ \text{z očitno rešitvijo}}} = -\underbrace{k\mu_0 j^2}_{\substack{\text{nehomogeni del}\\ \sim \text{partikularna rešitev}}}$$

$$j^{(0)} = j_0\, e^{-\mu_0 x}$$ homogena rešitev

Iteracija

$$\frac{dj}{dx} + \mu_0 j = -k\mu_0\, j^{(0)2} = -k\mu_0 j_0^2 e^{-2\mu_0 x}$$

$$j^{(1)} = \text{hom} + \text{part} = C e^{-\mu_0 x} + A e^{-2\mu_0 x}$$

$$A(-2\mu_0)\, e^{-2\mu_0 x} + \mu_0 A e^{-2\mu_0 x} = -k\mu_0 j_0^2 e^{-2\mu_0 x}$$

$$A = kj_0^2$$

$$j^{(1)} = (j_0 - kj_0^2)\, e^{-\mu_0 x} + kj_0^2\, e^{-2\mu_0 x} = j_0\, e^{-\mu_0 x}\left(1 - kj_0\left(1 - e^{-\mu_0 x}\right)\right)$$

8 Kodre 27/1

Razvij $\tan x$ na 3 načine:

- kar odvajamo
- $\frac{\sin x}{\cos x}$
- inverz $\arctan y = x$

$$f(x) = \tan x$$
$$f'(x) = \frac{1}{\cos^2 x}$$
$$f''(x) = \frac{2\sin x}{\cos^3 x}$$
$$f'''(x) = \dots$$

$$f(x) = 0 + 1x + 0 + \frac{2}{3!}x^3 + \dots$$

Kvocient

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \sigma(x^7)$$
$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \sigma(x^6)$$
$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \sigma(x^4)$$

$+\sigma(x^6)$

$$\tan x = \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} + \sigma(x^7)}{1 + (-\frac{x^2}{2!} + \frac{x^4}{4!} + \sigma(x^6))} = \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \sigma(x^7)\right)\left(1 - \left(-\frac{x^2}{2!} + \frac{x^4}{4!} + \sigma(x^6)\right) + \left(-\frac{x^2}{2!} + \frac{x^4}{4!} + \sigma(x^6)\right)^2 \dots\right)$$

$$= \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \sigma(x^7)\right)\left(1 + \frac{x^2}{2!} - \frac{x^4}{4!} + \sigma(x^6) + \left(\frac{x^2}{2}\right)^2 + \underset{\sigma(x^6)}{\text{ostali so v}}\right)$$

$$= \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \sigma(x^7)\right)\left(1 + \frac{x^2}{2!} + \frac{5x^4}{24} + \sigma(x^6)\right)$$

$$= x + \frac{x^3}{3} + \frac{2}{15}x^5 + \sigma(x^7)$$

Inverz

$$x = \operatorname{arctg} y = \int_0^y \frac{du}{1+u^2} = \int_0^y 1 - u^2 + u^4 - u^6 + u^8 - \dots \, du =$$

$$x = y - \frac{1}{3}y^3 + \frac{1}{5}y^5 - \frac{1}{7}y^7 + \sigma(y^9)$$

$$y = x + \frac{1}{3}y^3 - \frac{1}{5}y^5 + \frac{1}{7}y^7 + \sigma(y^9)$$

$$y = x + \frac{1}{3}\left(x + \frac{1}{3}y^3 - \frac{1}{5}y^5 + \sigma(y^7)\right)^3 - \frac{1}{5}\left(x + \frac{1}{3}y^3 + \sigma(y^5)\right)^5 + \frac{1}{7}\left(x + \sigma(y^3)\right)^7 + \sigma(y^9)$$

$$y = \dots$$

9 Razvoj funkcije $E_1(x)$

$$E_1(x) = \int_x^\infty \frac{e^{-t}}{t}\,dt \qquad \left(E_i = \int_{-\infty}^x \frac{e^t}{t}\,dt \right)$$

Asimptotične vrste (razvoj okoli ∞) $\quad x \sim \infty \quad \frac{1}{x} \sim 0$

$$u = t - x \qquad du = dt$$

$$E_1(x) = \int_0^\infty \frac{e^{-(u+x)}}{u+x}\,du = e^{-x}\int_0^\infty \frac{e^{-u}}{u+x}\,du = \frac{e^{-x}}{x}\int_0^\infty \frac{e^{-u}}{\underbrace{\tfrac{u}{x}}_{\text{majhno}}+1}$$

$$= \frac{e^{-x}}{x}\int_0^\infty e^{-u}\left(1 - \frac{u}{x} + \left(\frac{u}{x}\right)^2 - \left(\frac{u}{x}\right)^3 + \sigma(x^{-4})\right)du$$

$$= \frac{e^{-x}}{x}\left(0! - \frac{1!}{x} + \frac{2!}{x^2} - \frac{3!}{x^3} + \dots\right)$$

Divergentna vrsta ampak uporabna do nekega člena.

Alternativa: per partes $E_1(x)$

$$\int_x^\infty \frac{e^{-t}}{t}\,dt = -\frac{1}{t}e^{-t}\Big|_x^\infty - \int_x^\infty e^{-t}\frac{1}{t^2}\,dt = \frac{e^{-x}}{x} - \int_x^\infty \frac{e^{-t}}{t^2}\,dt \overset{\text{per partes}}{=} \dots$$

(10) Razvoj Debyeve c_v $\qquad c_v = \frac{3R}{M} D\left(\frac{T}{\theta}\right)$

$$D(x) = 12x^3 \int_0^{1/x} \frac{u^3\,du}{e^u - 1} - \frac{3}{x(e^{1/x}-1)}$$

(a) $x \to \infty$ $\quad \frac{1}{x}$ majhen $\quad u$ ne preseže $\frac{1}{x}$

$\Rightarrow$ smemo razviti po u

$$\frac{1}{e^u-1} = \frac{1}{1+u+\frac{u^2}{2!}+\frac{u^3}{3!}+\sigma(u^4)-1} = \frac{1}{u}\left(\frac{1}{1+\underbrace{\frac{u}{2!}+\frac{u^2}{3!}+\sigma(u^3)}_{\varepsilon}}\right) =$$

$$= \frac{1}{u}\left(1-\left(\frac{u}{2!}+\frac{u^2}{3!}+\sigma(u^3)\right)+\left(\frac{u}{2!}+\frac{u^2}{3!}+\sigma(u^3)\right)^2+\sigma(u^3)\right) =$$

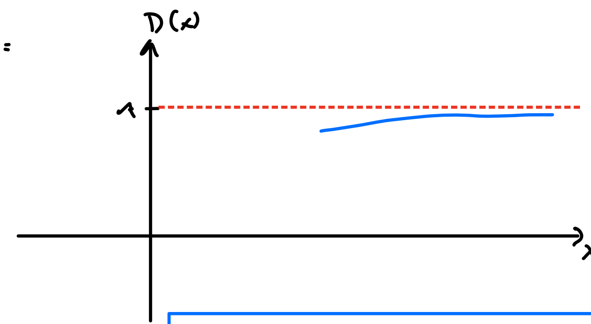
$$= \frac{1}{u}\left(1-\frac{u}{2}-\frac{u^2}{6}+\frac{u^2}{4}+\sigma(u^3)\right) = \frac{1}{u}\left(1-\frac{u}{2}-\frac{u^2}{12}+\sigma(u^3)\right)$$

$$D(x) = 12x^3 \int_0^{1/x} u^2-\frac{u^3}{2}+\frac{u^4}{12}+\sigma(u^5)\,du - \frac{3}{x}\left(\frac{1}{1/x}-\frac{1}{2}+\frac{1}{x\,12}+\sigma(x^{-3})\right) =$$

$$= 12x^3\left(\frac{1}{3}\frac{1}{x^3}-\frac{1}{8}\frac{1}{x^4}+\frac{1}{12\cdot 5}\frac{1}{x^5}+\sigma(x^{-6})\right)-3+\frac{3}{2x}-\frac{1}{4x^2}+\sigma(x^{-3}) =$$

$$= 4-\frac{3}{2}\frac{1}{x}+\frac{1}{5}\frac{1}{x^2}-3+\frac{3}{2x}-\frac{1}{4x^2}+\sigma(x^{-3}) =$$

$$= 1-\frac{1}{20x^2}+\sigma(x^{-3})$$



(b) $x \to 0$

$$\int_0^\infty \frac{u^3\,du}{e^u-1} = \zeta(4)\,\Gamma(4) = 3!\,\frac{\pi^4}{90} = \frac{\pi^4}{15}$$

Alternativno

$$= \int_0^\infty \frac{e^{-u}u^3}{1-\underbrace{e^{-u}}_{\text{majhen}}} = \int_0^\infty e^{-u}u^3\left(1+e^{-u}+e^{-2u}+e^{-3u}+\dots\right)du =$$

$$= \int_0^\infty e^{-u}u^3\,du + \int_0^\infty e^{-2u}u^3\,du + \int_0^\infty e^{-3u}u^3\,du + \dots$$

$$= 3! + \frac{1}{2^4}3! + \frac{1}{3^4}3! + \frac{1}{4^4}3! + \dots$$

$$= 3!\left(1+\frac{1}{2^4}+\frac{1}{3^4}+\dots\right) = 3!\,\zeta(4)$$

Riemannova - Zeta funkcija

$$\zeta(s) = \sum_{n=1}^\infty \frac{1}{n^s}$$

$$\zeta(2) = 1+\frac{1}{2^2}+\frac{1}{3^2}+\dots = \frac{\pi^2}{6}$$

$$\zeta(s) = \frac{1}{\Gamma(s)}\int_0^\infty \frac{u^{s-1}}{e^u-1}\,du$$

$$D(x) \sim 12x^3\,\frac{\pi^4}{15}$$

Ostanek:

$$D(x) = 12x^3\left(\int_0^\infty - \underbrace{\int_{1/x}^\infty\right)\frac{u^3\,du}{e^u-1} - \frac{3}{x(e^{1/x}-1)}}_{\text{ostanek ne splet ni učinkic:}}$$

11 Razvoj sferične besslove funkcije okrog 0
$\downarrow$ rešijo to dif. en.

$$x^2 y'' + 2x y' + (x^2 - n(n+1))y = 0$$

Nastavek: $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$

S tem ignoriramo singularno rešitev

$$x^2 (\underline{2a_2} + 3\cdot 2 a_3 x + 4\cdot 3 a_4 x^2 + 5\cdot 4 a_5 x^3 + \dots) + 2x(\underline{a_1} + \underline{2a_2 x} + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4)$$
$$+ (x^2 - n(n+1))(\underline{a_0} + \underline{a_1 x} + \underline{a_2 x^2} + a_3 x^3 + \dots) = 0$$

$$-n(n+1) a_0 = 0$$
$$(2 - n(n+1))a_1 = 0$$
$$(2 + 2\cdot 2 - n(n+1))a_2 = -a_0$$
$$(3\cdot 2 + 2\cdot 3 - n(n+1))a_3 = -a_1$$
$$(4\cdot 3 + 2\cdot 4 - n(n+1))a_4 = -a_2$$
$$\vdots$$
$$(\ell(\ell-1) + 2\ell - n(n+1))a_\ell = -a_{\ell-2}$$
$$(\ell(\ell+1) - n(n+1))a_\ell = -a_{\ell-2}$$

rekurzija

Skače po dve: lihe in sode
Vodilni člen je x^n ($\ell = n$ prvi neničelni)

$$j_n(x) = x^n + a_{n+2} x^{n+2} + \dots$$

$$a_\ell = -\frac{a_{\ell-2}}{\ell(\ell+1) - n(n+1)}$$

Primer $n=0$

$a_0 = 1$

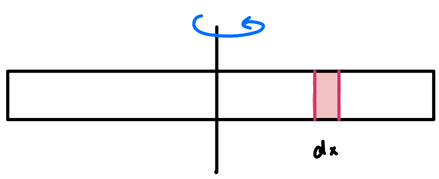
$a_2 = -\frac{1}{2\cdot 3}$

$a_4 = +\frac{1}{2\cdot 3\cdot 4\cdot 5}$

$\vdots$

$j_0 = \frac{\sin x}{x}$

12 Zataljeno epruvetko polno pline centrifugiramo
Iščemo $\rho(x)$.



2NZ $\quad dm\, \omega^2 x = S(p(x+dx) - p(x))$

$$\rho S \omega^2 x\, dx = S\, dp$$

$$\rho \omega^2 x = \frac{dp}{dx} \qquad p = \rho \frac{R}{M} T$$
$$dp = d\rho \frac{R}{M} T$$

$$\int_0^x \omega^2 x\, dx = \frac{R}{M} T \int_{\rho_0}^{\rho} \frac{d\rho}{\rho}$$

$$\omega^2 \frac{x^2}{2} = \frac{R}{M} T \ln \frac{\rho}{\rho_0}$$

$$\rho = \rho_0 e^{\frac{M}{RT}\omega^2 \frac{x^2}{2}} = \rho_0 e^{u^2} \qquad u^2 = \frac{M\omega^2 x^2}{2RT}$$

Problem: ϱ_0 ni znan, določen je z globalnim pogojem (ohranitev mase)

zato ni erf je pa $A\,\mathrm{erf}(ix)$

Pogoj

$$m = S\int_{-R}^{R} \varrho(x)\,dx = S\varrho_0 \frac{R}{u_{max}} \underbrace{\int_{-u_{max}}^{u_{max}} e^{u^2}\,du}$$

$$u_{max} = \sqrt{\frac{m}{2RT}}\,\omega R$$

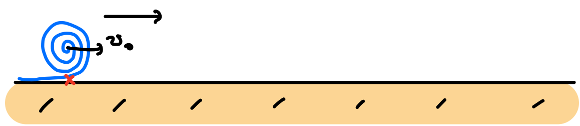
Dawsonov integral ... $D(x) = e^{-x^2}\int_0^x e^{u^2}\,du$

Če R ni prevelik

$$\int_0^x e^{u^2}\,du \sim \int_0^x 1+u^2+\frac{u^4}{2!}+\frac{u^6}{3!}+\dots\,du$$

$$= x+\frac{x^3}{3}+\frac{x^5}{5\cdot 2!}+\frac{x^7}{7\cdot 3!}+\dots$$

13 Kako se odvije tepih $(x(t))$



$$J = \frac{3}{2}mr^2 = \frac{3}{2}\sigma\pi r^4 \qquad \sigma = \frac{dm}{dS}$$

- $\Gamma = J\omega \sim r^3 v \Rightarrow v \sim r^{-3}$

  $W_k \sim J\omega^2 \sim r^4 r^{-8} \sim r^{-4} \qquad \omega \sim r^{-4}$

  Ohranitev $\Gamma \to W_k$ divergira ???

  Problem: kdaj se ohrani Γ? Ko $M=0$ ker pa ni res

  Pospeševanje kotalja. Sila podlage ne gre skozi središče role

- Ohranitev energije

$$W = \frac{1}{2}J\omega^2 + mgr$$

$$W = \frac{3}{4}\sigma\pi r^4\omega^2 + gr\sigma\pi r^2 \qquad v = r\omega$$

$$W = \frac{3}{4}\sigma\pi\, r^2 v^2 + \sigma\pi g r^3$$

$$v = \sqrt{\frac{W-\sigma\pi g r^3}{\frac{3}{4}\sigma\pi r^2}}$$

Rešimo $r(t)$ oz. $r(x)$

debelina tepiha

$$\frac{d}{dt}\quad S = \pi r^2 = h(\underbrace{\ell - x}_{\text{prosti del}})$$

$$\pi 2r\dot r = -h\dot x = -hv$$

$$-\frac{2\pi r\dot r}{h} = \sqrt{\frac{W-\sigma\pi g r^3}{\frac{3}{4}\sigma\pi r^2}}$$

$$\dot r = 0 \Rightarrow W = \sigma\pi g\, r_x^3$$

r_x ... referenčni r oz. začetni r

$$r_x = \sqrt[3]{\frac{W}{\sigma\pi g}}$$

$$u = \frac{r}{r_x} \le 1$$

$$\dot u\, r_x = -\frac{h}{2\pi u^2 r_x}\sqrt{\frac{W(1-u^3)}{\frac{3}{4}\sigma\pi r_x^2}}$$

$$u^2\dot u = -\underbrace{\frac{h}{\pi r_x^3}\sqrt{\frac{W}{3\sigma\pi}}}_{\lambda}\sqrt{1-u^3}$$

$$\int_{\frac{r_0}{r_x}}^{\frac{r}{r_x}}\frac{u^2\,du}{\sqrt{1-u^3}} = -\int_0^t \lambda\, dt$$

$$1-u^3 = w$$
$$-3u^2\,du = dw$$

$$\frac{1}{3}\int_{w_0}^{w} w^{-\frac{1}{2}}\,dw = \lambda\int_0^t dt$$

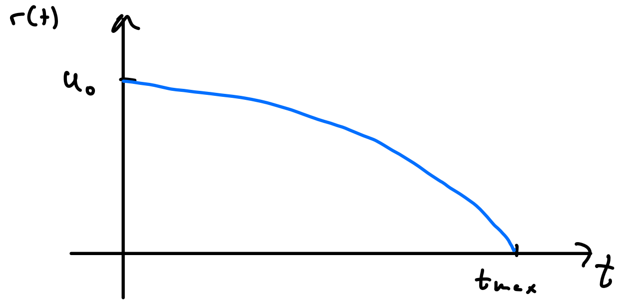
$$\frac{2}{3}\left(\sqrt{w}-\sqrt{w_0}\right) = \lambda t$$

$$\frac{2}{3}\left(\sqrt{1-u^3}-\sqrt{1-u_0^3}\right) = \lambda t$$

$$\frac{r}{r_x} = u = \sqrt[3]{1-\left(\sqrt{1-u_0^3}+\frac{3}{2}\lambda t\right)^2}$$

čas odvijanja

$$t = \frac{2}{3\lambda}\left(1-\sqrt{1-u_0^3}\right)$$



Kaj če $g \to 0$

$$W = \frac{3}{4}\sigma\pi\, r^2 v^2$$

$$v = \sqrt{\frac{W}{\frac{3}{4}\sigma\pi r^2}}$$

$$\pi 2r\dot r = -h\dot x = -hv$$

$$v = -\frac{2\pi r\dot r}{h} = \sqrt{\frac{4W}{3\sigma\pi}}\,\frac{1}{r}$$

$$\int_{r_0}^{r} r^2\,dr = -\underbrace{\frac{h}{2\pi}\sqrt{\frac{4W}{3\sigma\pi}}}_{\Lambda}\int_0^t dt \qquad \left[\frac{m^3}{s}\right]$$

$$\frac{r^3}{3} = \frac{r_0^3}{3} - \Lambda t$$

$$r = \sqrt[3]{r_0^3 - 3\Lambda t}$$

$$t_{max} = \frac{r_0^3}{3\Lambda}$$

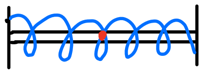
$$x = l - \frac{\pi r^2}{h}$$

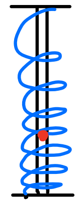
14 2× vpeta vzmet

$m = 1\,kg$

$k = 100\,\frac{N}{m}$

Premik središča ?





Hookov zakon

$$\frac{F}{S} = E\frac{dy}{dx}$$

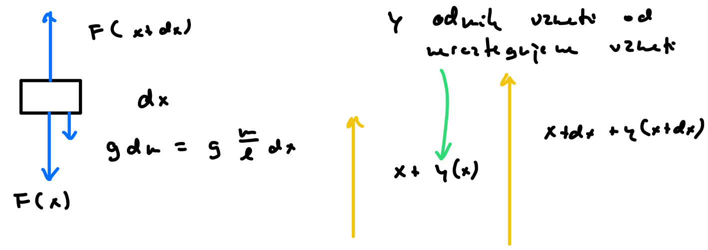
y... kumulativni raztezek

x... štje navojev

$$F = ky$$

$$F = \left(k\frac{l}{dx}\right)dy$$

$$F = kl\frac{dy}{dx}$$



$$F(x+dx) = F(x) + gm\frac{dx}{l}$$

$$\frac{dF}{dx} = \frac{gm}{l}$$

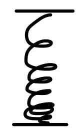
$$\Downarrow$$

$$F = F(0) + \frac{gm}{l}x$$

Robni pogoji:

$y(0) = 0$

$y(L) = 0$



$F(0)$, $F(L)$ sta neznanki

$$F = kl\frac{dy}{dx} = F(0) + \frac{mgx}{l}$$

$$\int_0^y dy = \int_0^x \frac{F(0)}{kl} + \frac{mgx}{kl^2}\,dx \qquad x \in [0, L]$$

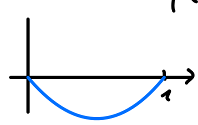
$$y = \frac{F(0)}{kl}x + \frac{mg\,x^2}{kl^2\,2} \qquad \text{Robni pogoji}$$

$$0 = \frac{F(0)}{k}\frac{l}{l} + \frac{mg}{2k}\frac{l^2}{l^2}$$

$$F(0) = -\frac{mg}{2} \quad \Rightarrow \quad F(L) = \frac{mg}{2}$$

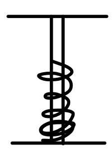
$$\Rightarrow y(x) = -\frac{mg}{2kl}x + \frac{mg}{2kl^2}x^2 \qquad y\left(\frac{x}{2}\right) = ?$$

$$= \frac{mg}{2k}\frac{x}{l}\left(\frac{x}{l} - 1\right)$$



$$y\left(\frac{l}{2}\right) = -\frac{mg}{8k} = \ldots = -\frac{1}{80}\,m = -1,25\,cm$$

- Če na vrhu vzmet ni vpeta

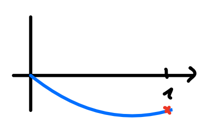


$$F(l) = 0$$

$$F(l) = F(0) + \frac{mg}{l}l = 0$$

$$\Rightarrow F(0) = -mg$$

$$y(x) = -\frac{mg}{kl}x + \frac{mg\,x^2}{2kl^2} = \frac{mg}{2k}\frac{x}{l}\left(\frac{x}{l} - 2\right)$$

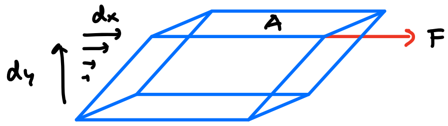


$$y\left(\frac{l}{2}\right) = -\frac{3}{8}\frac{mg}{k}$$

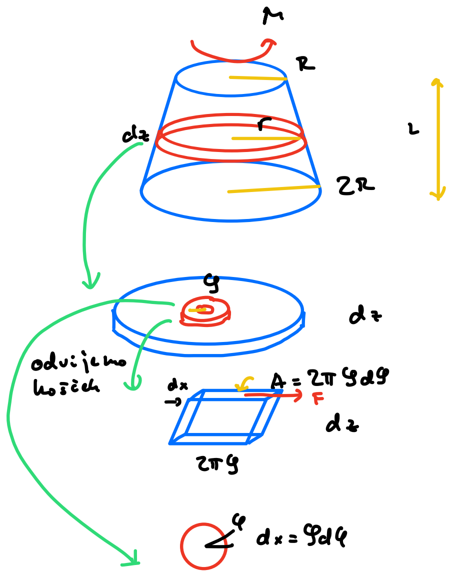
15 Vrtež stožca

Prisekan stožec iz gume, polmer $2R$ in R, višina L.

Zasuk =? če delujemo z M, strižni modul G



$$\frac{F}{S} = G\,\frac{dx}{dy}$$



φ ... absolutni zasuk
dφ ... razlika zasukov plošč dz

ρ dφ ... strig kolobarja plošč

$$r = 2R - R\,\frac{z}{l}$$

$$\frac{dF}{dS} = G\,\frac{dx}{dz}$$

$$\frac{dF}{2\pi\rho\, d\rho} = G\,\frac{\rho\, d\varphi}{dz}$$

$$dM = \rho\, dF = G 2\pi\, \rho^3\, d\rho\, \frac{d\varphi}{dz}$$

$$M = 2\pi G\, \frac{d\varphi}{dz} \int_0^r \rho^3 d\rho$$

$$M = \underbrace{\frac{1}{2}\pi G r^4}_{\text{Torzijski koeficient cilindra}}\, \frac{d\varphi}{dz}$$

konst. (3.N.Z., navor se prenaša med plastmi)

$$d\varphi = \frac{2M}{\pi G\, r^4}\, dz \qquad r = 2R - R\frac{z}{l}$$

$$dr = -\frac{R}{l}\, dz$$

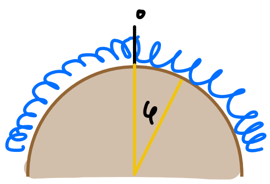
$$d\varphi = -\frac{2M}{\pi G}\,\frac{l}{R}\,\frac{dr}{r^4}$$

$$\int d\varphi = \Delta\varphi = -\frac{2Ml}{\pi R G}\int_{2R}^{R}\frac{dr}{r^4} = \frac{2Ml}{3\pi R G}\left(\frac{1}{R^3} - \frac{1}{8R^3}\right)$$

$$\Delta\varphi = \frac{7\, M\, l}{12\pi\, G R^4}$$

16 Slinky na klodu

Težka vzmet ničelne dolžine 0 povežemo na polkrožni valj : k, m, r



$F(0) = ?$

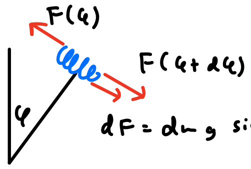
pogoj, da seže do tal

Zaradi: simetrije obravnavamo le eno stran

$$F = k\,\frac{dy}{dx} \qquad x \in (0,1)$$

Parametrizacija po kotu

kumulativen raztezek ↪ $y = r\varphi$ = položaj, kjer je tanek lok vzmeti



$dF = dm\, g \sin\varphi$ ← Sila odvisna od raztezka

$$F(\varphi + d\varphi) - F(\varphi) = -dm\, g \sin\varphi = -m\, dx\, g \sin\varphi$$

$$dF = -m g \sin\varphi \, dx$$

$$F = k \frac{dy}{dx} = k r \frac{d\varphi}{dx}$$

$$k r \frac{d^2\varphi}{dx^2} = -m g \sin\varphi$$

$$E = \frac{1}{2}\left(\frac{kr}{m}\right)^2 \dot\varphi^2 - g \frac{kr}{m} \cos\varphi = \text{konst.}$$

$$E(0) = E(\varphi_{max})$$

$x = 0$ sila $F_0 = kr \left.\frac{d\varphi}{dx}\right|_0$ $\quad$ $x = 1$ $\quad$ $F = 0 = kr \left.\frac{d\varphi}{dx}\right|_1$

položaj $\varphi = 0$ $\quad$ $\varphi_{max} = ?$

$$E(0) = E(1)$$

$$\frac{1}{2}\left(\frac{kr}{m}\right)^2 \dot\varphi^2\Big|_0 - g\frac{kr}{m} = -g\frac{kr}{m}\cos\varphi_{max}$$

Splošen φ

$$\frac{1}{2}\left(\frac{kr}{m}\right)^2 \dot\varphi^2 - g\frac{kr}{m}\cos\varphi = -g\frac{kr}{m}\cos\varphi_{max} \Rightarrow \text{eliptični integral}$$

Da seže do tal:

$\varphi_{max} = \frac{\pi}{2}$

$x = 1$

$$\frac{1}{2}\left(\frac{kr}{m}\right)^2 \dot\varphi^2\Big|_0 - g\frac{kr}{m} = 0$$

$$\dot\varphi\Big|_0 = \sqrt{\frac{2mg}{kr}}$$

$$F_0 = k r \dot\varphi\Big|_0 = \sqrt{2mgkr}$$

$$\dot\varphi = \sqrt{\frac{2gm}{kg}}\sqrt{\cos\varphi - \cos\varphi_0}$$

$$\int_0^{\varphi} \frac{d\varphi}{\sqrt{\cos\varphi - \cos\varphi_0}} = \sqrt{\frac{2gm}{kr}}\; x$$

za $x = 1$

$$\underbrace{\int_0^{\varphi_0} \frac{d\varphi}{\sqrt{\cos\varphi - \cos\varphi_0}}}_{\dots} = \sqrt{\frac{2gm}{kr}}$$

$$\dots = F\left(\frac{\pi}{2}, \sin\frac{\varphi_0}{2}\right) = \underbrace{K\left(\sin\frac{\varphi_0}{2}\right) = \sqrt{\frac{gm}{kr}}}$$

Eliptični integral prve vrste

To poda φ_0 (raztezek) v odvisnosti od g, m, k, r

17 Poišči: maksimum $1+xy$ na enotskem krogu

- Lagrang. mult.
- Eksplicitna oblika (izrazimo vez)
- Parametrizacija

• λ način

$$g(x,y,\lambda) = 1+xy - \lambda(x^2+y^2-1)$$

$y=0 \Rightarrow x=0$ ker smo na krožnici

$$\begin{aligned} &\text{Vez} && x^2+y^2=1 \\ &\frac{\partial g}{\partial x} && y - 2\lambda x = 0 \\ &\frac{\partial g}{\partial y} && x - 2\lambda y = 0 \end{aligned}$$

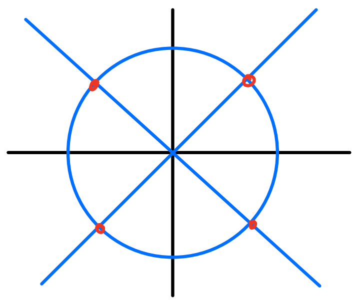
$$y = 2\lambda x, \quad x = 2\lambda y \quad y = 4\lambda^2 y \iff \lambda = \pm\frac{1}{2} \text{ ali } \cancel{y=0}$$

$$\lambda = \pm\frac{1}{2}$$

$$x = \pm y$$

$+ \quad x = y = \frac{1}{\sqrt{2}}$

$- \quad -x = y = \frac{1}{\sqrt{2}}$



• Eksplicitno

$$x^2+y^2=1$$
$$y = \pm\sqrt{1-x^2}$$

$$g(x) = f(x, \sqrt{1-x^2}) = 1 \pm x\sqrt{1-x^2}$$

Pazi: več vej rešitev

kaj če je rešitev na robu

$$\frac{dg}{dx} = \pm\sqrt{1-x^2} \pm \frac{x(-2x)}{2\sqrt{1-x^2}} = 0$$

$$\sqrt{1-x^2} = \frac{x^2}{\sqrt{1-x^2}}$$

$$1-x^2 = x^2$$

$$x = \pm\frac{\sqrt{2}}{2} \qquad y = \pm\sqrt{1-x^2} = \pm\frac{\sqrt{2}}{2}$$

• Parametrizacija vezi

$$x^2+y^2=1 \iff y = \sin\varphi \quad x = \cos\varphi$$

$$g(\varphi) = 1 + \cos\varphi \sin\varphi$$

$$g_\varphi = \cos 2\varphi = 0 \qquad 2\varphi = \frac{\pi}{2} + k\pi$$

$$\varphi = \frac{\pi}{4} + \frac{\pi}{2}k = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$$

(18) Magnetno polje polneskončne žice

Biot-Savartov zakon $\vec H = \frac{I}{4\pi}\int \frac{d\vec r'\times(\vec r-\vec r')}{|\vec r-\vec r'|^3}$

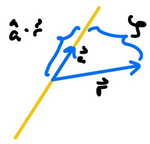
$\vec r'$ po žici $\quad \vec r$ točka polja

$\vec r' = t\hat a \;;\; t\in(0,\infty)$

$d\vec r' = \hat a\,dt$

$$\vec H = \frac{I}{4\pi}\int_0^\infty \frac{dt\,\hat a\times(\vec r - t\hat a)}{|\vec r-t\hat a|^3} =$$

$$= \frac{I}{4\pi}\,\hat a\times\vec r\int_0^\infty \frac{dt}{(r^2+t^2-2t\,\vec r\cdot\hat a)^{3/2}} = \frac{I}{4\pi}\,\hat a\times\vec r\int_0^\infty\frac{dt}{\big((\underbrace{t-\vec r\cdot\hat a}_{u})^2+\underbrace{r^2-(\vec r\cdot\hat a)^2}_{\rho^2}\big)^{3/2}} =$$



$du = dt$

$$= \frac{I}{4\pi}\int_{-\vec r\cdot\hat a}^\infty \frac{du}{(u^2+\rho^2)^{3/2}} \overset{\text{Bronštein}}{=} \frac{I}{4\pi}\,\hat a\times\vec r\, \frac{u}{\rho^2\sqrt{\rho^2+u^2}}\Bigg|_{-\vec r\cdot\hat a}^{\infty} =$$

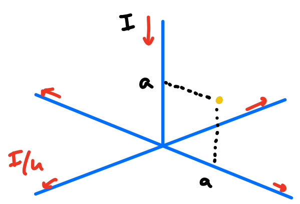
$$= \frac{I}{4\pi}\frac{\hat a\times\vec r}{\rho^2}\Big(1+\frac{\vec r\cdot\hat a}{\underbrace{\sqrt{\rho^2+(\vec r\cdot\hat a)^2}}_{r}}\Big)$$

$$\rho^2 = r^2\hat a^2 - (\vec r\cdot\hat a)^2 = r^2(1-\cos^2\varphi) = r^2\sin^2\varphi = \|\hat a\times\vec r\|^2$$

$$= \frac{I}{4\pi}\frac{\hat a\times\vec r}{\|\hat a\times\vec r\|^2}\Big(1+\frac{\vec r\cdot\hat a}{r}\Big)$$

(19) 2. kol 2011/12

Dolgi ⊥ žici na vodoravni ravnini, dodana polžica ⊥ na obe v presečišču. I po navpični, ki se razdeli na 4 krake. $\vec H(a,0,a)=?$



$$\vec H(\vec r) = -\frac{I}{4\pi}\frac{\hat e_z\times\vec r}{\|\hat e_z\times\vec r\|^2}\Big(1+\frac{\hat e_z\cdot\vec r}{r}\Big) + \frac14\frac{I}{4\pi}\Big[\frac{\hat e_x\times\vec r}{|\hat e_x\times\vec r|^2}\Big(\cancel{1}+\frac{\hat e_x\cdot\vec r}{r}\Big) + \frac{-\hat e_x\times\vec r}{|\hat e_x\times\vec r|^2}\Big(\cancel{1}-\frac{\hat e_x\cdot\vec r}{r}\Big)$$
$$+ \frac{\hat e_y\times\vec r}{|\hat e_y\times\vec r|^2}\Big(1+\frac{\hat e_y\cdot\vec r}{r}\Big) + \frac{-\hat e_y\times\vec r}{|\hat e_y\times\vec r|^2}\Big(1+\frac{-\hat e_y\cdot\vec r}{r}\Big) =$$

$$= -\frac{I}{4\pi}\frac{\hat e_z\times\vec r}{\|\hat e_z\times\vec r\|^2}\Big(1+\frac{\hat e_z\cdot\vec r}{r}\Big) + \frac14\frac{I}{4\pi}\Big(2\frac{\hat e_x\times\vec r}{|\hat e_x\times\vec r|^2}\frac{\hat e_x\cdot\vec r}{r} + 2\frac{\hat e_y\times\vec r}{|\hat e_y\times\vec r|^2}\frac{\hat e_y\cdot\vec r}{r}\Big)$$

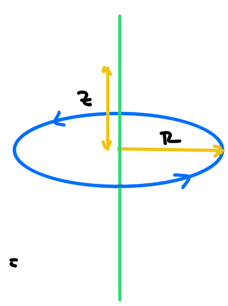
$\vec r = (a,0,a) = a\hat e_x + a\hat e_z \qquad \hat e_x\times\hat e_y = \hat e_z \qquad \hat e_y\times\hat e_z=\hat e_x \qquad \hat e_z\times\hat e_x=\hat e_y$

$$\vec H(\vec r) = -\frac{I}{4\pi}\frac{a\hat e_y}{|a\hat e_y|^2}\Big(1+\frac{a}{\sqrt{a^2+a^2}}\Big) + \frac12\frac{I}{4\pi}\Big(\frac{-a\hat e_y}{|a\hat e_y|^2}\frac{a}{\sqrt{a^2+a^2}} + \cdots 0\Big) =$$

$$= -\frac{I}{4\pi}\frac{\hat e_y}{a}\Big(1+\frac{1}{\sqrt2}\Big) + \frac12\frac{I}{4\pi}\Big(-\frac{e_y}{a}\frac{1}{\sqrt2}\Big) = \frac{I}{4\pi}\frac{\hat e_y}{a}\Big(1+\frac{3}{2\sqrt2}\Big)$$

(20) Kodre 54/2

Magnetno polje krožne zanke v osi, dokaži, da je integral H enak I



$\vec r' = R(\sin\varphi,\cos\varphi,0)$

$d\vec r' = R(\cos\varphi,-\sin\varphi,0)\,d\varphi$

$\vec r = (0,0,z)$

$$\vec H(r) = \frac{I}{4\pi}\oint\frac{d\vec r'\times(\vec r-\vec r')}{\|\vec r-\vec r'\|^3} = \frac{I}{4\pi}\oint_0^{2\pi}\frac{d\varphi\, R(\cos\varphi,-\sin\varphi,0)\times R(-\sin\varphi,-\cos\varphi,z)}{(R^2\sin^2\varphi+R^2\cos^2\varphi+z^2)^{3/2}} =$$

$$\begin{vmatrix} i & j & k\\ \cos\varphi & -\sin\varphi & 0\\ -\sin\varphi & -\cos\varphi & z\end{vmatrix} = (-z\sin\varphi,\ z\cos\varphi,\ -1)$$

$$= \frac{I}{4\pi} \int_0^{2\pi} \frac{Rz(\cos\varphi, \sin\varphi, 0) - R^2(0,0,-1)}{\sqrt{R^2+z^2}^3} \overset{\substack{x \text{ komp.} \to 0 \\ y \text{ komp.} \to 0}}{=} \frac{I}{4\pi} \int_0^{2\pi} d\varphi \frac{(0,0,R^2)}{\sqrt{R^2+z^2}^3} = \frac{I}{2} \hat{e}_z \frac{R^2}{(R^2+z^2)^{3/2}}$$

Polarni k. s.

$$\hat{e}_r \times \hat{e}_\varphi = \hat{e}_z$$

$$\frac{\partial \hat{e}_r}{d\varphi} = \hat{e}_\varphi \qquad \frac{\partial \hat{e}_\varphi}{d\varphi} = -\hat{e}_r$$

$\hat{e}_r, \hat{e}_\varphi$ sta funkcija φ

$$\frac{4\pi}{I} \vec{H} = \int_0^{2\pi} \frac{R\hat{e}_\varphi \times (z\hat{e}_z - R\hat{e}_r)\, d\varphi}{\sqrt{R^2+z^2}^3}$$

$$= \int_0^{2\pi} \frac{Rz\,\hat{e}_r + R^2 \hat{e}_z}{\sqrt{R^2+z^2}^3} d\varphi$$

$$= \int_0^{2\pi} \frac{Rz(\cos\varphi, \sin\varphi, 0)}{\sqrt{R^2+z^2}^3} d\varphi = 0$$

$$+ \frac{R^2}{\sqrt{R^2+z^2}^3} \int_0^{2\pi} \hat{e}_z d\varphi = \frac{2\pi R^2}{\sqrt{R^2+z^2}^3} \hat{e}_z$$

Amperov zakon

$$\int_{-\infty}^{\infty} H\,dz \overset{?}{=} I$$

$$\frac{I}{2}\int_{-\infty}^{\infty} \frac{R^2}{(R^2+z^2)^{3/2}} dz = \frac{I}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cancel{R^2}}{(\cancel{R^2}(1+\tan^2 t))^{3/2}} \frac{\cancel{R}}{\cos^2 t} dt = \frac{I}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{\left(\frac{1}{\cos^2 t}\right)^{3/2}} \frac{1}{\cos^2 t} dt = \frac{I}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos t\, dt = I$$

$z = R\tan t$

$dz = \frac{R}{\cos^2 t} dt$

21 Kodre 54/7

Izračunaj H v središču in gorišču eliptične tokovne zanke

Sredina elipse

$\vec{r}(0,0,z)$

$\vec{r}' = (a\cos\varphi, b\sin\varphi, 0)$

$d\vec{r}' = (-a\sin\varphi, b\cos\varphi, 0)\, d\varphi$

$$\vec{H} = \frac{I}{4\pi} \int_0^{2\pi} \frac{(-a\sin\varphi, b\cos\varphi, 0) \times (-a\cos\varphi, -b\sin\varphi, z)}{(a^2\cos^2\varphi + b^2\sin^2\varphi + z^2)^{3/2}} d\varphi$$

$$\begin{vmatrix} i & j & k \\ -a\sin\varphi, & b\cos\varphi, & 0 \\ -a\cos\varphi, & -b\sin\varphi, & z \end{vmatrix}$$

$$(\dots, \dots, ab)$$

$$H_z = \frac{Iab}{4\pi} \int_0^{2\pi} \frac{1}{(a^2\cos^2\varphi + b^2\sin^2\varphi + z^2)^{3/2}} d\varphi$$

$$= \frac{Iab}{\pi} \int_0^{\pi/2} \frac{d\varphi}{(a^2+z^2 + (b^2-a^2)\sin^2\varphi)^{3/2}} =$$

$$= \frac{Iab}{\pi} (a^2+z^2)^{-3/2} \int_0^{\pi/2} \frac{d\varphi}{(1-k^2\sin^2\varphi)^{3/2}} \qquad k^2 = \frac{b^2-a^2}{a^2+z^2}$$

$$\dots \frac{Iab}{\pi(a^2+z^2)^{3/2}} \frac{E(k)}{1-k^2}$$

Eliptični integral

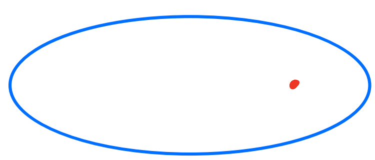
$$K(k) = \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}}$$

$$E(k) = \int_0^{\pi/2} \sqrt{1-k^2\sin^2\varphi}\, d\varphi$$

Gorišče lege

goriščni parameter $p = \frac{b^2}{a}$ ekscentričnost $e = \sqrt{1-\left(\frac{b}{a}\right)^2}$

$$\underset{\text{krog}}{0} < e < \underset{\text{parabola}}{1}$$



$$r = \frac{p}{1-e\cos\varphi}$$ (deluje tudi za parabolo in hiperbolo)

$$\vec{r} = (0,0,0) \qquad \vec{r}' = r\hat{e}_r \qquad d\vec{r}' = dr\,\hat{e}_r + r\,d\hat{e}_r = \left(\frac{\partial r}{\partial\varphi}\right) d\varphi\,\hat{e}_r + r\hat{e}_\varphi\, d\varphi \qquad b < a$$

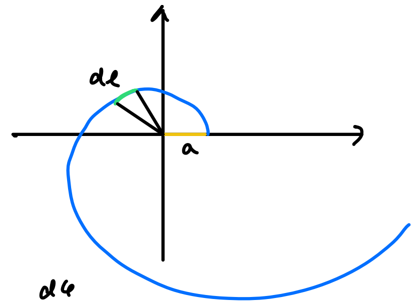
$$\vec{H} = \frac{I}{4\pi} \int_{-\pi}^{\pi} \frac{-d\vec{r}' \times \vec{r}'}{|\vec{r}'|^3} = \frac{I}{4\pi} \int_{-\pi}^{\pi} \frac{-r\hat{e}_\varphi \times r\hat{e}_r}{r^3} d\varphi = \frac{I}{4\pi} \int_{-\pi}^{\pi} \frac{e_z}{r} d\varphi = \frac{I}{4\pi} \hat{e}_z \int_{-\pi}^{\pi} \frac{1-e\cos\varphi}{p} d\varphi = \frac{I}{2p} \hat{e}_z = \frac{Ia}{2b^2} \hat{e}_z$$

(22) 2. kol 2014/15

Določi potencial in el. polje v središču logaritemske spirale

$r = a e^{k\varphi}$, dolga je l, $0<\varphi, \varphi_{max}$, dolžinska gostota naboja je podana $\mu = \frac{de}{dl}$

Naj bo $4\pi\varepsilon_0 = 1$



Potencial točkastega naboja $V = \frac{e}{r}$

$$V = \int \frac{de}{r} = \int \frac{dl\,\mu}{r}$$

$$dl = \|\vec{r}\,'\| \qquad \vec{r}\,' = a e^{k\varphi}\,\hat{e}_r \qquad d\vec{r}\,' = a e^{k\varphi} k\,\hat{e}_r + a e^{k\varphi}\,\hat{e}_\varphi \quad d\varphi$$

$$\|d\vec{r}\,'\| = a e^{k\varphi}\sqrt{k^2+1}\;d\varphi$$

$$V = \int_0^{\varphi_0} \frac{\mu\, dl}{r} = \mu \int_0^{\varphi_0} \frac{a e^{k\varphi}\sqrt{k^2+1}}{a e^{k\varphi}}\,d\varphi = \mu\sqrt{k^2+1}\,\varphi_0$$

$$l = \int dl = \int_0^{\varphi_0} a e^{k\varphi}\sqrt{k^2+1}\,d\varphi = a\sqrt{k^2+1}\,\frac{1}{k}\left(e^{k\varphi_0}-1\right)$$

$$\varphi_0 = \frac{1}{k}\ln\left(\frac{lk}{a\sqrt{k^2+1}}+1\right)$$

$$dE = \frac{-de}{r^2}\,\frac{\vec{r}}{r}$$

$$\vec{E} = -\int_0^{\varphi_0}\frac{\mu\,dl}{r^2}\,\hat{e}_r = -\mu\int_0^{\varphi_0}\frac{\sqrt{k^2+1}\,\cancel{a e^{k\varphi}}\,d\varphi}{a^{\cancel{2}}\, e^{2k\varphi}}\,\hat{e}_r = -\frac{\mu\sqrt{k^2+1}}{a}\int_0^{\varphi_0} e^{-k\varphi}(\cos\varphi, \sin\varphi)\,d\varphi$$

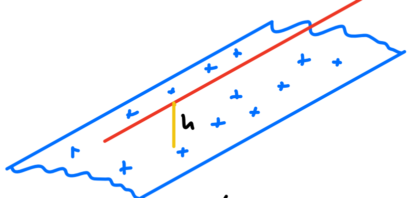
pretvorimo vektorje 2D v kompleksna števila $\vec{E} = \vec{E}_x + i\vec{E}_y$

$$\vec{E} = -\frac{\mu\sqrt{k^2+1}}{a}\int_0^{\varphi_0} e^{-k\varphi}(\cos\varphi + i\sin\varphi)\,d\varphi = -\frac{\mu\sqrt{k^2+1}}{a}\int_0^{\varphi_0} e^{(-k+i)\varphi}\,d\varphi = \underbrace{-\frac{\mu\sqrt{k^2+1}}{a}}_{Q}\,\frac{1}{i-k}\left(e^{(-k+i)\varphi_0}-1\right)$$

$$= Q\,\frac{\left(k+i-(k+i)\left(\cos\varphi\, e^{-k\varphi_0} + i\sin\varphi\, e^{-k\varphi_0}\right)\right)}{k^2+1} = \frac{-Q}{k^2+1}\left[\left(k - k\cos\varphi_0\, e^{-k\varphi_0} + \sin\varphi_0\, e^{-k\varphi_0}\right) + i\left(1-\cos\varphi_0\, e^{-k\varphi_0} - k\sin\varphi_0\, e^{-k\varphi_0}\right)\right]$$

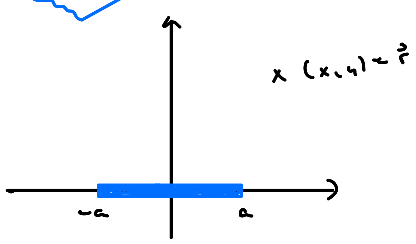
(23) 2. kol 2010/11

Trak iz dielektrika z gostoto naboja σ in širino $2a$ in žička nad sredino traka.
Izračunaj silo na dolžinsko enoto.

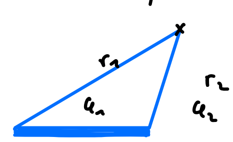


$$\vec{E}_1 = \frac{\mu}{2\pi\varepsilon_0 r^2}\,\vec{r}$$

Parametrizacija
$\vec{r}\,' = (t, 0)$; $t \in (-a, a)$



$$d\vec{E} = \frac{\sigma\,dx}{2\pi\varepsilon_0}\,\frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^2}$$
Q



$$\vec{E} = Q\int_{-a}^{a}\frac{(x-t, y)}{(x-t)^2+y^2}\,dt$$

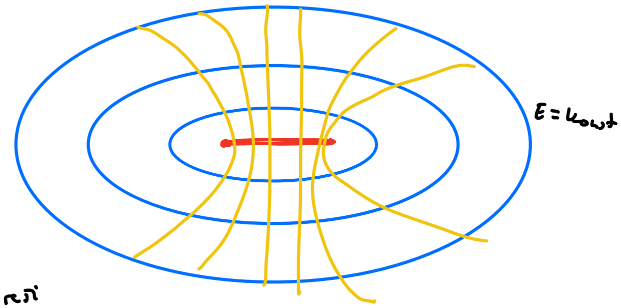
$$\vec{E}_x = Q\int_{-a}^{a}\frac{x-t}{\underbrace{(x-t)^2+y^2}_{u}}\,dt = -Q\,\frac{1}{2}\int_{(x+a)^2+y^2}^{(x-a)^2+y^2}\frac{du}{u} = Q\ln\sqrt{\frac{(x+a)^2+y^2}{(x-a)^2+y^2}} = Q\ln\frac{r_1}{r_2}$$

$du = -(x-t)\,2\,dt$

$$\vec E_y = Q\int_{-a}^{a}\frac{y\,dt}{(a-t)^2+y^2} \overset{u=t-x}{=} Q\int_{-a-x}^{a-x}\frac{y\,du}{u^2+y^2} = Q\,\frac{1}{y}\,y\arctan\frac{u}{y}\Big|_{-a-x}^{a-x} = Q\left(\arctan\frac{a+x}{y} - \arctan\frac{x-a}{y}\right)$$

$$\vec E_y = Q(\varphi_1 - \varphi_2) = Q\,\delta\varphi$$

zorni kot



Eliptični koordinatni sistem

$\vec E = -\nabla V$

$\nabla^2 V = 0$

V = harmonična funkcija

Re in Im del komp. funkcij reši

ta sistem enačb

$V(r) = Q\ln z$

Potencial žice

$V(z) = \mathrm{Re}(Q\ln z) \qquad z = re^{i\varphi}$

EL polje žice

$E = \nabla V$

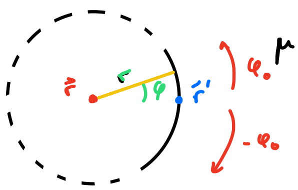
$E_x = \frac{\partial\,\mathrm{Re}V}{\partial x} \qquad E_y = \frac{\partial\,\mathrm{Re}V}{\partial y} = -\frac{\partial\,\mathrm{Im}V}{\partial x}$

Cauchy

$$\vec E = \frac{\partial}{\partial x}(\mathrm{Re}V, -\mathrm{Im}V) = \frac{\partial \bar V}{\partial x}$$

$$\vec E^* = Q\int_{-a}^{a}\frac{dz'}{z-z'} = Q\ln z-z'\Big|_{-a}^{a} = Q\ln\frac{z+a}{z-a}$$

24 Polje krožnega loka



μ ... gostota naboja

$$\vec E = \frac{de}{4\pi\varepsilon_0}\frac{(\vec r-\vec r\,')}{|\vec r-\vec r\,'|^3}$$

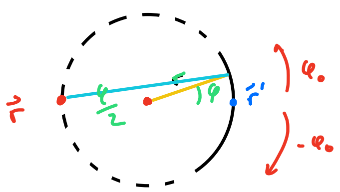
$4\pi\varepsilon_0 \to 1$

$\vec r\,' = (r\cos\varphi, r\sin\varphi)$

$\vec r = (0,0)$

$de = \mu|d\vec r\,'| = \mu r\,d\varphi$

$$E_x = \int\frac{\mu r\,d\varphi\,(0 - r\cos\varphi)}{r^3} = -\frac{\mu}{r}\int_{-\varphi_0}^{\varphi_0}\cos\varphi\,d\varphi = -2\frac{\mu}{r}\sin\varphi_0$$



$\vec r = (-r, 0)$

$\vec r\,' = (r\cos\varphi, r\sin\varphi)$

$\vec r - \vec r\,' = -r(1+\cos\varphi, \sin\varphi)$

$\quad = -r\left(2\cos^2\frac{\varphi}{2}, 2\sin\frac{\varphi}{2}\cos\frac{\varphi}{2}\right)$

$$E_x = \int_{-\varphi_0}^{\varphi_0}\frac{-\mu r\,d\varphi\; r\,2\cos^2\frac{\varphi}{2}}{\left|2r\cos\frac{\varphi}{2}\sqrt{\cos^2\frac{\varphi}{2}+\sin^2\frac{\varphi}{2}}\right|^3} = -\frac{\mu r^2}{(2r)^3}\,2\int_{-\varphi_0}^{\varphi_0}\frac{d\varphi}{\cos\varphi/2} = -\frac{\mu}{4r}\int_{-\varphi_0}^{\varphi_0}\frac{d\varphi}{\cos\varphi/2} = -\frac{\mu}{4r}\,2\ln\left|\tan\left(\frac{\varphi}{4}+\frac{\pi}{4}\right)\right|\Big|_{-\varphi_0}^{\varphi_0} =$$

$$= -\frac{\mu}{r}\ln\frac{1+\tan\frac{\varphi_0}{4}}{1-\tan\frac{\varphi_0}{4}}$$

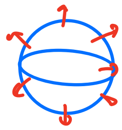
(25) Kodre 65/6

ρ = konst. gostota emersijskih izvirov v zemlji $\qquad j_r = \frac{1}{3} r \rho$

Pokaži, da velja kontinuitetna enačba

Integralska oblika

$$\oint_{\partial D} \vec{j}\cdot d\vec{S} = j_r\, 4\pi r^2 = \frac{4}{3}\pi r^3 \rho = \int \rho\, dV = \rho V$$



Diferencialna oblika

$$\mathrm{div}\, \vec{j} = q \quad \text{gostota izvirov}$$

Sferične koordinate

$$\mathrm{div} \quad \vec{\nabla}\cdot\vec{j} = \frac{\partial j_x}{\partial x} + \frac{\partial j_y}{\partial y} + \frac{\partial j_z}{\partial z}$$

$$\mathrm{grad} \quad \vec{\nabla} u = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}\right)$$

https://en.wikipedia.org/wiki/Del_in_cylindrical_and_spherical_coordinates

Kodre: A.2 poslovenj $\qquad \nabla$ = del (ang.)

A.10 enačba

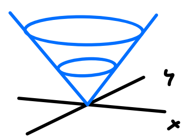
$$\vec{\nabla}\cdot\vec{v} = \frac{1}{r^2}\frac{\partial(r^2 v_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(v_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\varphi}{\partial\varphi}$$

$$\vec{\nabla}\cdot\vec{j} = \frac{1}{r^2}\frac{\partial(r^2 j_r)}{\partial r} = \frac{1}{r^2}\frac{\partial(r^2 \frac{1}{3} r q)}{\partial r} = \frac{1}{r^2} q r^2 = q$$

(26) Brez komponent dokaži $\quad \vec{\nabla}\cdot\left(\frac{\vec{r}}{r}\right) = \frac{2}{r}$

$$\mathrm{grad}\,(r^a) = \vec{\nabla} r^a$$

$$\mathrm{div}\,(r^a \vec{r}) = \vec{\nabla}\cdot(r^a \vec{r})$$



$$\nabla r^a = a r^{a-1} \underbrace{\nabla r}_{\text{enotski vektor v smeri iz izhodišča}} = a r^{a-1}\frac{\vec{r}}{r} \qquad r = |\vec{r}|$$

$$\boxed{\nabla r^a = a r^{a-2}\vec{r}}$$

$$\vec{\nabla}\cdot(r^a \vec{r}) = (\nabla r^a)\cdot\vec{r} + r^a \nabla\cdot\vec{r}$$

$$= a r^{a-2}\vec{r}\cdot\vec{r} + r^a d \qquad d = \text{št. dimenzij}$$

$$\boxed{\vec{\nabla}\cdot(r^a\vec{r}) = r^a(a+d)}$$

v resnici $\neq 0$ ampak je na središču delta funkcija

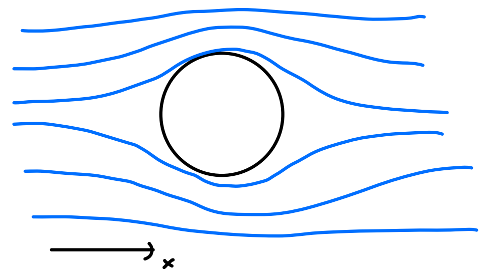
$$a = -3 \quad \Rightarrow \quad \nabla\cdot\Big(\underbrace{\frac{\vec{r}}{r^3}}_{\vec{g},\,\vec{E}}\Big) = 0 \quad = \delta(\vec{r})$$

$$\nabla\cdot\left(\frac{\vec{r}}{r}\right) = r^{-1}(-1+3) = \frac{2}{r}$$

(27) Kodre 62/13

Tok tekočine v_0 v x smeri. V tok postavimo valj z radijem r_0. Hitrostni potencial

$$u = v_0 \cdot x \left(1 + \frac{1}{2}\left(\frac{r_0}{r}\right)^3\right)$$



- Pokaži, da radialna komponenta pri $r=r_0$ izgine
- $p(\varphi)$ na površini

$$\vec{v} = \nabla u \quad \text{def. potenciala}$$

Cilindrične k.

$$\nabla u = \underbrace{\frac{\partial u}{\partial r}}_{v_r} \hat{e}_r + \underbrace{\frac{1}{r}\frac{\partial u}{\partial \varphi}}_{v_\varphi} \hat{e}_\varphi = v$$

$$u = v_0\, r \cos\varphi \left(1 + \frac{1}{2}\left(\frac{r_0}{r}\right)^3\right)$$

$$v_r = \frac{du}{dr} = v_0 \cos\varphi \left(1 - \left(\frac{r_0}{r}\right)^3\right)$$

$$\text{Pri } r_0 = r \Rightarrow v_r = 0$$

Tlak iz Bernoullijeve en.

$$\text{Rabimo } v^2 = v_r^2 + v_\varphi^2$$

$$v_\varphi = \frac{1}{r}\frac{\partial u}{\partial \varphi} = -v_0 \sin\varphi \left(1 + \frac{1}{2}\left(\frac{r_0}{r}\right)^3\right)$$

$$v_\varphi\Big|_{r=r_0} = -v_0 \sin\varphi\, \frac{3}{2}$$

$$p_\infty + \frac{\rho v_0^2}{2} = p(\varphi) + \frac{\rho v_\varphi^2}{2}$$

$$p(\varphi) = p_\infty + \frac{\rho}{2} v_0^2 \left(1 - \frac{9}{4}\sin^2\varphi\right)$$

(28) Kodre 91/1

$$\vec{v}(r) = \frac{1}{r^n}\left(\vec{a}\, r^2 - \alpha (\vec{a}\cdot\vec{r})\vec{r}\right)$$

Za katera n, α je polje brezvrtinčno in brezizvirno?

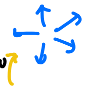
$$\text{Brezizvirno}: \quad \operatorname{div}\vec{v} = \nabla\cdot\vec{v} = 0$$

$$\begin{aligned}
\nabla\cdot\vec{v} &= (\nabla r^{-n})\cdot\left(\vec{a}\, r^2 - \alpha(\vec{a}\cdot\vec{r})\vec{r}\right) + r^{-n}\, \nabla\cdot\left(\vec{a}\, r^2 - \alpha(\vec{a}\cdot\vec{r})\vec{r}\right) = \\
&= -n r^{-n-2}\, \vec{r}\cdot\left(\vec{a}\, r^2 - \alpha(\vec{a}\cdot\vec{r})\vec{r}\right) + r^{-n}\left(\vec{a}\cdot\nabla r^2 - \alpha\left(\nabla(\vec{a}\cdot\vec{r})\cdot\vec{r} + (\vec{a}\cdot\vec{r})\,\nabla\cdot\vec{r}\right)\right) = \\
&= -n r^{-n-2}\, \vec{r}\cdot\left(\vec{a}\, r^2 - \alpha(\vec{a}\cdot\vec{r})\vec{r}\right) + r^{-n}\left(\vec{a}\, 2r - \alpha\left(\vec{a}\cdot\vec{r} \quad (\vec{a}\cdot\vec{r})\cdot 3\right)\right) = \\
&= (\vec{a}\cdot\vec{r})\left(-n r^{-n} + n\alpha r^{-n} + 2r^{-n} - \alpha\cdot 4 r^{-n}\right) = \\
&= (\vec{a}\cdot\vec{r})\, r^{-n}\left(n(\alpha-1) + 2 - 4\alpha\right)
\end{aligned}$$

$$\Rightarrow \quad n(\alpha - 1) = 4\alpha - 2$$

Brezvrtinčnost : $\mathrm{rot}\,\vec{v} = 0$

nima vrtincev



$$\nabla \times \vec{v} = (\nabla r^{-n}) \times \left(\vec{a}\, r^2 - \alpha\, (\vec{a}\cdot\vec{r})\vec{r}\right) + r^{-n}\, \nabla \times \left(\vec{a}\, r^2 - \alpha\, (\vec{a}\cdot\vec{r})\vec{r}\right) =$$

$$= -n r^{-n-2}\, \vec{r} \times \left(\vec{a}\, r^2 - \alpha\, (\vec{a}\cdot\vec{r})\vec{r}\right) + r^{-n} \left(\nabla r^2 \times \vec{a} - \alpha \left(\nabla(\vec{a}\cdot\vec{r}) \times \vec{r} + (\vec{a}\cdot\vec{r})\, \nabla \times \vec{r} \right)\right) =$$

$$= -n r^{-n-2} \left((\vec{r}\times\vec{a}) r^2 - \alpha (\vec{a}\cdot\vec{r})\, \cancel{\vec{r}\times\vec{r}}^{\,0} \right) + r^{-n} \left(2\vec{r}\times\vec{a} - \alpha \left(\vec{a}\times\vec{r} + \cancel{(\vec{a}\cdot\vec{r})}^{\,0} \cdot \emptyset \right)\right) =$$

$$= (\vec{a}\times\vec{r})\, r^{-n} (n - 2 - \alpha)$$

$$\Rightarrow \qquad n = 2 + \alpha$$

Brezvrtinčno in brezizvirno

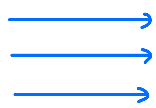
$$n(\alpha - 1) = 4\alpha - 2$$
$$n = 2 + \alpha$$

$$(2+\alpha)(\alpha - 1) = 4\alpha - 2$$
$$2\alpha + \alpha^2 - 2 - \alpha = 4\alpha - 2$$
$$\alpha^2 - 3\alpha = 0$$
$$\alpha = 0 \qquad \alpha = 3$$
$$n = 2 \qquad n = 5$$

$$\vec{v}(r) = \frac{1}{r^n}\left(\vec{a}\, r^2 - \alpha\, (\vec{a}\cdot\vec{r})\vec{r}\right)$$

$\alpha = 0 \quad n = 2$

$$\vec{v}(r) = \vec{a}$$

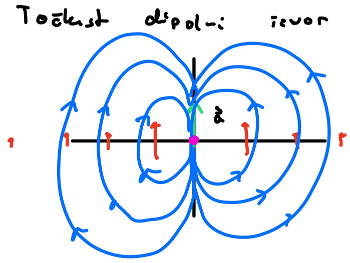


$\alpha = 3 \quad n = 5$

$$\vec{v}(r) = \frac{\vec{a}}{r} - \frac{5}{r^3}(\vec{a}\cdot\vec{r})\vec{r}$$

Polje dipola

Točkast dipol – izvor pri $\vec{r} = 0$



(29) Voda 7815

Gostota sile $\vec{f} \overset{?}{=} (\mu - 1)\mu_0 \nabla\left(\frac{H^2}{2}\right)$

Izotropna snov, nehomogeno polje

$$\vec{M} = \vec{p}_m \times \vec{B}$$ ni funkcija x, y, z

$$W_m = -\vec{p}_m \cdot \vec{B}$$

$$\vec{F} = -\nabla W_m = \nabla(\vec{p}_m \cdot \vec{B}) =$$

$$\Updownarrow$$

$$F_i = \nabla_i \sum_j p_j B_j = \sum_j p_j \underbrace{\nabla_i B_j}_{?} = \dots$$

zun. polje $\vec{B} = \mu_0 \vec{H}$

Velja je:

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{H} = 0 \quad \text{(ni tokov, ker delamo z dipolom)}$$

$$\begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ B_x & B_y & B_z \end{vmatrix} = (\partial_y B_z - \partial_z B_y,\ \partial_z B_x - \partial_x B_z,\ \partial_x B_y - \partial_y B_x) = 0$$

$$\vec\nabla\times\vec H=0 \quad\Leftrightarrow\quad \nabla_i B_j = \nabla_j B_i$$

$$\nabla_i B_j - \nabla_j B_i = \begin{bmatrix} 0 & \partial_x B_y-\partial_y B_x & -(\partial_z B_x - \partial_x B_z) \\ & 0 & \partial_y B_z - \partial_z B_y \\ \text{Antisimetrična} & & 0\end{bmatrix} = \begin{bmatrix} 0 & (\nabla\times B)_z & -(\nabla\times B)_y \\ \text{Antisim.} & 0 & (\nabla\times B)_x \\ & & 0\end{bmatrix}$$

Torej

$$F_i = \dots = p_j \nabla_j B_i$$

$\Rightarrow$ $\boxed{\vec F = (\vec p\cdot\vec\nabla)\vec B}$ Usmerjeni odvod, odvod v smeri $\vec p$. Koliko se B spremeni, če greš v smeri $\vec p$.

$$(\hat e_x\cdot\vec\nabla)\vec u = \frac{\partial u}{\partial x}$$

Imamo paramagnetno snov

$$\vec M = (\mu-1)\vec H = \frac{\vec p_m}{V} \quad \text{magnetizacija}$$

$$\Rightarrow \quad \vec F = (V(\mu-1)\vec H\cdot\nabla)\mu_0\vec H$$

$$\frac{\vec F}{V} = \mu_0(\mu-1)(\vec H\cdot\vec\nabla)\vec H$$

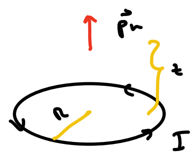
Kodre $\nabla\left(\frac{\vec v^2}{2}\right) = (\vec v\cdot\nabla)\vec v + \vec v\times\underbrace{(\nabla\times\vec v)}$

$\nabla\times H=0$ če ni tokov

$$\vec f = \mu_0(\mu-1)\nabla\left(\frac{\vec H^2}{2}\right)$$

30 Kodre 11

Sila na točkast dipol usmerjen v smeri osi krožne tokovne zanke

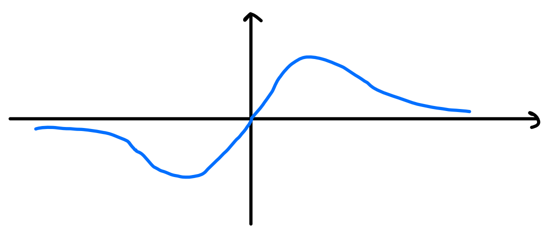


$$\vec F = (\vec p_m\cdot\vec\nabla)\vec B$$
$$\vec F_z = p(\hat e_z\cdot\nabla)\vec B = p\frac{\partial}{\partial z}\vec B_z = p\frac{\partial B_z}{\partial z}$$

$\vec B$ zanke

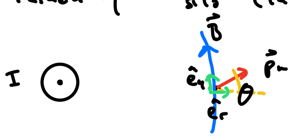
$$B_z = \frac{\mu_0 I}{2}\frac{R^2}{(R^2+z^2)^{3/2}}$$

$$\vec F_z = p\frac{\mu_0 I}{\cancel{2}}R^2\frac{(-\frac32)\cancel{2}z}{(R^2+z^2)^{5/2}} = -\frac32 p\mu_0 I\frac{R^2 z}{(R^2+z^2)^{3/2}}$$



31 Kodre 74/13

Izračunaj silo (in navor) tokovnega vodnika na točkast magnetni dipol.



S polarnimi baznimi vektorji:

$$\vec B = \frac{\mu_0 I}{2\pi r}\hat e_\varphi \qquad \vec p = p(\hat e_r\cos\theta + \hat e_\varphi\sin\theta) = \hat e_r(\vec p\cdot\hat e_r) + \hat e_\varphi(\vec p\cdot\hat e_\varphi)$$

$$(\vec p\cdot\vec\nabla) = (\vec p\cdot\hat e_r)[\underbrace{\hat e_r\cdot\nabla}_{\frac{\partial}{\partial r}}] + (\vec p\cdot\hat e_\varphi)[\underbrace{\hat e_\varphi\cdot\nabla}_{\frac1r\frac{\partial}{\partial\varphi}}]$$

enota

$$\vec F = (\vec p\cdot\nabla)\vec B = \frac{\mu_0 I}{2\pi}\left((\vec p\cdot\hat e_r)\frac{\partial}{\partial r}\left(\frac{\hat e_\varphi}{r}\right) + (\vec p\cdot\hat e_\varphi)\frac{1}{r}\frac{\partial}{\partial\varphi}\left(\frac{\hat e_\varphi}{r}\right)\right) =$$

$$\frac{\partial \hat e_\varphi}{\partial\varphi} = -\hat e_r$$

$$= \frac{\mu_0 I}{2\pi}\left((\vec p\cdot\hat e_r)\left(-\frac{1}{r^2}\right)\hat e_\varphi + (\vec p\cdot\hat e_\varphi)\frac{1}{r^2}(-\hat e_r)\right)$$

$$= -\frac{\mu_0 I}{2\pi r^2}\left((\vec p\cdot\hat e_r)\hat e_\varphi + (\vec p\cdot\hat e_\varphi)\hat e_r\right)$$

$$\vec M = \vec p\times\vec B = \frac{\mu_0 I}{2\pi r}\left((\vec p\cdot\hat e_r)\hat e_r + (p\cdot\hat e_\varphi)\hat e_\varphi\right)\times\hat e_\varphi =$$

$$= \frac{\mu_0 I}{2\pi r}(\vec p\cdot\hat e_r)\hat e_z$$

Kartezično

$$\vec B = \frac{\mu_0 I}{2\pi}\left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0\right)$$

$$(\vec p\cdot\nabla)\vec B = \frac{\mu_0 I}{2\pi}\left(p\cos\theta\frac{\partial}{\partial x}\left(\frac{-y}{x^2+y^2}\right) + p\sin\theta\frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right), p\cos\theta\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) + p\sin\theta\frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2}\right)\right) =$$

$$= \frac{\mu_0 p I}{2\pi}\left(\cos\theta\frac{2xy}{(x^2+y^2)^2} + \sin\theta\left(-\frac{1}{x^2+y^2} + \frac{2y^2}{(x^2+y^2)^2}\right), \cos\theta\left(\frac{1}{x^2+y^2} - \frac{2x^2}{(x^2+y^2)^2}\right) + \sin\theta\left(-\frac{2xy}{(x^2+y^2)^2}\right)\right) =$$

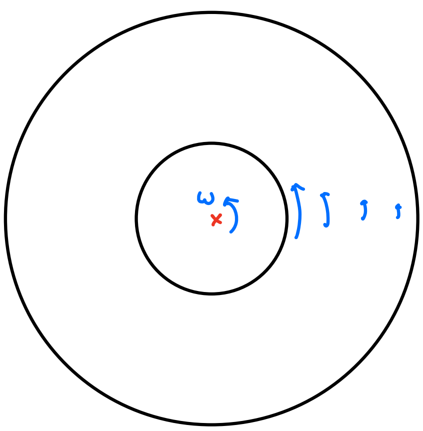
$$= \frac{\mu_0 p I}{2\pi}\left(\cos\theta\left[\frac{2xy}{(x^2+y^2)^2}, \frac{y^2-x^2}{(x^2+y^2)^2}\right] + \sin\theta\left[\frac{y^2-x^2}{(x^2+y^2)^2}, -\frac{2xy}{(x^2+y^2)^2}\right]\right) =$$

Velja za vse x, y, θ je kot na x os.

Izberimo $y=0$ $x=r$

$$(\vec p\cdot\nabla)\vec B = \frac{\mu_0 p I}{2\pi}\left(\cos\theta\left(0, -\frac{1}{r^2}\right) + \sin\theta\left(-\frac{1}{r^2}, 0\right)\right) = -\frac{\mu_0 p I}{2\pi r^2}(\sin\theta, \cos\theta)$$

32 Določi hitrostno polje v plasti med vrtečim se in stacionarnim valjem



Navier - Stokesova enačba

$$\boxed{\rho\frac{d\vec v}{dt} = -\nabla p + \eta\nabla^2\vec v}$$

∇^2 na vektorju ni enak ∇^2 na skalarju razen v kartezičnih.

Substancialni odvod

$$\frac{d\vec v}{dt} = \frac{\partial\vec v}{\partial t} + \frac{\partial\vec v}{\partial x}\frac{dx}{dt} + \frac{\partial\vec v}{\partial y}\frac{dy}{dt} + \frac{\partial\vec v}{\partial z}\frac{dz}{dt} = \underbrace{\frac{\partial\vec v}{\partial t}}_{\emptyset \text{ v stacionarnih pogojih}} + (\vec v\cdot\nabla)\vec v$$

$$\rho(\vec v\cdot\nabla)\vec v = -\nabla p + \eta\nabla^2\vec v$$

($\nabla\cdot\vec v = 0$, nestisljivost)

$$\frac{\partial}{\partial x}\frac{dx}{dt} + \frac{\partial}{\partial y}\frac{dy}{dt} + \frac{\partial}{\partial z}\frac{dz}{dt}$$

$$\int\rho\,\nabla\cdot\vec v\,dV = \oint\rho\vec v\cdot d\vec S = \oint d\Phi_m$$

Simetrija:

$$\vec v = v(r)\,\hat e_\varphi$$

$$\rho\Big(v(r)\underbrace{\hat e_\varphi\cdot\nabla}_{\frac{1}{r}\frac{\partial}{\partial\varphi}}\Big)v(r)\,\hat e_\varphi = -\nabla p + \eta\nabla^2\vec v$$

$$\rho \frac{v^2}{r} \frac{\partial e_\varphi}{\partial \varphi} = -\rho \underbrace{\frac{v^2}{r}}\hat{e}_r$$

radialni pospešek

$$-\rho \frac{v^2}{r}\hat{e}_r = -\nabla p + \eta\left(\nabla^2 v(r) - \frac{v(r)}{r^2}\right)\hat{e}_\varphi$$

le v r-smeri

Naš primer

Vodn A.104
$$\nabla^2 \vec{v} = \left(\nabla^2 \cancel{v_r} - \cancel{\frac{v_r}{r^2}} - \frac{2}{r^2}\frac{\partial v_\varphi}{\partial \varphi}\right)\hat{e}_r + \left(\nabla^2 v_\varphi - \frac{v_\varphi}{r^2} + \frac{2}{r^2}\cancel{\frac{\partial v_r}{\partial \varphi}}\right)\hat{e}_\varphi$$

Vodn A.103
$$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\cancel{\frac{\partial^2 f}{\partial \varphi^2}}$$

$$r: \quad -\rho\frac{v^2}{r} = -\frac{\partial p}{\partial r} \qquad \varphi: \quad 0 = \nabla^2 v - \frac{v}{r^2}$$

$$0 = \frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2}$$

Eulerjeva dif. en.

$t = \ln r$ ali

nastavek $v = A r^l$

$$0 = A l(l-1) r^{l-2} + l A r^{l-2} - A r^{l-2}$$

$$l^2 - l + l - 1 = 0$$

$$l = \pm 1$$

$$v = ar + \frac{b}{r} \qquad a, b \text{ iz robnih pogojev}$$

$$v(r_1) = v_0 \qquad v(r_2) = 0$$

$$v_0 = ar_1 + \frac{b}{r_1} \qquad 0 = ar_2 + \frac{b}{r_2} \qquad b = -ar_2^2$$

$$v_0 = ar_1 - \frac{ar_2^2}{r_1}$$

$$a = \frac{v_0}{r_1 - r_2^2/r_1} \Rightarrow v = v_0 \frac{\frac{r}{r_1} - \frac{r_2}{r_1}\frac{r_2}{r}}{\left(\frac{r_2}{r_1}\right)^2 - 1}$$

Tlak
$$\frac{\partial p}{\partial r} = \rho\frac{v^2}{r} = \rho\frac{1}{r}\left(v_0\frac{\frac{r}{r_1} - \frac{r_2}{r_1}\frac{r_2}{r}}{\left(\frac{r_2}{r_1}\right)^2 - 1}\right)^2$$

$$\Delta p = \frac{\rho v^2}{\left(\left(\frac{r_2}{r_1}\right)^2 - 1\right)^2}\int_{r_1}^{r} \frac{r}{r_1^2} + \left(\frac{r_2}{r_1}\right)^2\frac{r_2^2}{r^3} - 2\frac{r_2^2}{r_1^2}\frac{1}{r}\, dr$$

$$\Delta p = \frac{\rho v^2}{\left(\left(\frac{r_2}{r_1}\right)^2 - 1\right)^2}\left(\frac{r^2 - r_1^2}{2r_1^2} - \frac{1}{2}\left(\frac{r_2}{r_1}\right)^2\left(\frac{1}{r^2} - \frac{1}{r_1^2}\right) - 2\left(\frac{r_2}{r_1}\right)^2 \ln\frac{r}{r_1}\right)$$

33 $$\vec{A} = \frac{\mu_0}{2\pi}\left(\frac{\vec{m}\times\vec{n}}{r^2} - \frac{2(\vec{m}\times\vec{r})(\vec{r}\cdot\vec{n})}{r^4}\right) \qquad \nabla\times\vec{A} = ? \qquad \nabla\cdot\vec{A} = ? \qquad \vec{m}, \vec{n} \text{ konst.}$$

$$\nabla\times(f\vec{v}) = \nabla f\times\vec{v} + f\nabla\times\vec{v} \qquad \vec{a}\times(\vec{b}\times\vec{c}) = \vec{b}(\vec{a}\cdot\vec{c}) - \vec{c}(\vec{a}\cdot\vec{b})$$
$$\nabla\cdot(f\vec{v}) = \nabla f\cdot\vec{v} + f\nabla\cdot\vec{v}$$

$$\nabla\times\vec{A} = \frac{\mu_0}{2\pi}\left(\nabla(r^{-2})\times(\vec{m}\times\vec{n}) - 2\nabla(r^{-4}(\vec{r}\cdot\vec{n}))\times(\vec{m}\times\vec{r}) - 2r^{-4}(\vec{r}\cdot\vec{n})\nabla\times(\vec{m}\times\vec{r})\right)$$

$$= \frac{\mu_0}{2\pi}\left(-2\frac{\vec{r}}{r^4}\times(\vec{m}\times\vec{n}) - 2\left[\nabla(r^{-4})(\vec{r}\cdot\vec{n}) + r^{-4}\nabla(\vec{r}\cdot\vec{n})\right]\times(\vec{m}\times\vec{r}) - 2r^{-4}(\vec{r}\cdot\vec{n})2\vec{m}\right)$$

$$= -\frac{\mu_0}{\pi}\left(r^{-4}\vec{r}\times(\vec{m}\times\vec{n}) + \left[(-4)r^{-6}\vec{r}(\vec{r}\cdot\vec{n}) + r^{-4}\vec{n}\right]\times(\vec{m}\times\vec{r}) + r^{-4}(\vec{r}\cdot\vec{n})2\vec{m}\right)$$

$$= -\frac{\mu_0}{\pi r^4}\left(\vec{m}(\vec{r}\cdot\vec{n}) - \vec{n}(\vec{r}\cdot\vec{m}) - \frac{4}{r^2}(\vec{r}\cdot\vec{n})\vec{r}\times(\vec{m}\times\vec{r}) + \vec{n}\times(\vec{m}\times\vec{r}) + 2(\vec{r}\cdot\vec{n})\vec{m}\right)$$

$$= -\frac{\mu_0}{2r^4}\left(\vec{m}(\cancel{\vec{r}}\cdot\vec{n}) - \vec{n}(\vec{r}\cdot\vec{m}) - 4(\vec{r}\cdot\cancel{\vec{n}})\vec{m} - \frac{4}{r^2}(\vec{r}\cdot\vec{n})(\vec{m}\cdot\vec{r})\vec{r} + \vec{m}(\cancel{\vec{n}\cdot\vec{r}}) - \vec{r}(\vec{m}\cdot\vec{n}) + 2(\vec{r}\cancel{\vec{n}})\vec{m}\right)$$

$$= -\frac{\mu_0}{2r^4}\left(\frac{4}{r^2}(\vec{r}\cdot\vec{n})(\vec{r}\cdot\vec{m})\vec{r} - \vec{n}(\vec{r}\cdot\vec{m}) - \vec{r}(\vec{m}\cdot\vec{n})\right)$$

$$\nabla\cdot\vec A = \frac{\mu_0}{2\pi}\Big(\nabla(r^{-2})\cdot(\vec m\times\vec n) - 2\nabla\big(r^{-4}(\vec r\cdot\vec n)\big)\cdot(\vec m\times\vec r) - 2r^{-4}(\vec r\cdot\vec n)\,\nabla\cdot(\vec m\times\vec r)\Big)$$

$\nabla\cdot(\vec m\times\vec r)=0$

$$= \frac{\mu_0}{2\pi}\Big(-2r^{-4}\,\vec r\cdot(\vec m\times\vec n) - 2\big((-4)r^{-6}\vec r\,(\vec r\cdot\vec n)+r^{-4}\vec n\big)\cdot(\vec m\times\vec r)\Big)$$

$$= -\frac{\mu_0}{\pi r^4}\Big(\vec r\cdot(\vec m\times\vec n) \quad -4r^{-2}\underbrace{\vec r\cdot(\vec m\times\vec r)}_{0}(\vec r\cdot\vec n) + \vec n\cdot(\vec m\times\vec r)\Big)$$

$$-\vec r\cdot(\vec m\times\vec n)$$

$$=0$$

(34) Določi vir potenciala

$$U = U_0 \ln\frac{x-f+\sqrt{(x-f)^2+y^2+z^2}}{x+f+\sqrt{(x+f)^2+y^2+z^2}} \qquad \vec E = -\nabla U \qquad \nabla\cdot\vec E = \rho = -\nabla^2 U$$

$$U = U_0\ln\frac{\hat e_x\cdot(\vec r-\vec f)+\|\vec r-\vec f\|}{\hat e_x\cdot(\vec r+\vec f)+\|\vec r+\vec f\|} \qquad \vec f = (f,0,0)$$

$$\nabla U = U_0\Big(\nabla\ln\big(\hat e_x\cdot(\vec r-\vec f)+\|\vec r-\vec f\|\big) - \nabla\ln\big(\hat e_x\cdot(\vec r+\vec f)+\|\vec r+\vec f\|\big)\Big) =$$

$$= U_0\left(\frac{\nabla(\hat e_x\cdot(\vec r-\vec f)) + \nabla\|\vec r-\vec f\|}{\hat e_x\cdot(\vec r-\vec f)+\|\vec r-\vec f\|} - \text{—||—}\right) = \qquad f\to -f$$

$$= U_0\left(\frac{\hat e_x + \frac{\vec r-\vec f}{\|\vec r-\vec f\|}}{\hat e_x\cdot(\vec r-\vec f)+\|\vec r-\vec f\|} - \text{—||—}\right)$$

$$\nabla\cdot(\nabla U) = \nabla^2 U = U_0\left(\nabla\cdot\left(\frac{1}{\hat e_x\cdot(\vec r-\vec f)+\|\vec r-\vec f\|}\Big(\hat e_x+\frac{\vec r-\vec f}{\|\vec r-\vec f\|}\Big)\right) - \text{—||—}\right)$$

$$= U_0\left(-\frac{\Big(\hat e_x+\frac{\vec r-\vec f}{\|\vec r-\vec f\|}\Big)^2}{\big(\hat e_x\cdot(\vec r-\vec f)+\|\vec r-\vec f\|\big)^2} + \frac{\nabla\cdot\Big(\frac{\vec r-\vec f}{\|\vec r-\vec f\|}\Big)}{\hat e_x\cdot(\vec r-\vec f)+\|\vec r-\vec f\|} - \text{—||—}\right) \qquad \nabla\cdot r^n\vec r = (n+3)r^n, \quad \nabla\cdot\frac{\vec r}{r} = 2\frac{1}{r}$$

$$= U_0\left(-\frac{1+2\frac{\hat e_x\cdot(\vec r-\vec f)}{\|\vec r-\vec f\|}+1}{(\dots)^2} + \frac{\frac{2}{\|\vec r-\vec f\|}\big(\hat e_x\cdot(\vec r-\vec f)+\|\vec r-\vec f\|\big)}{(\dots)^2} - \text{—||—}\right)$$

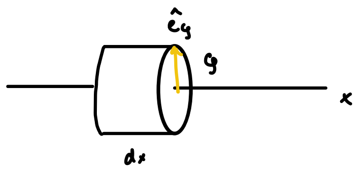
$$= 2U_0\left(\frac{-1-\hat e_x\cdot\frac{\vec r-\vec f}{\|\vec r-\vec f\|} + \hat e_x\cdot\frac{\vec r-\vec f}{\|\vec r-\vec f\|}+1}{(\dots)^2} - \text{—||—}\right) = 0$$

$\to = 0$, če smo na x osi

→ Ni: prostorskih izvorov

→ Dif. oblika ne pomaga

→ Gaussov izrek $\quad \varepsilon = \varepsilon_0 = 1$



$$de = \oint \vec E\cdot d\vec S = \iint \vec E\cdot\hat e_\rho\,\rho\,d\varphi\,dx$$

$$\frac{de}{dx} = -\int_0^{2\pi}\nabla U\cdot\hat e_\rho\,\rho\,d\varphi = -\nabla U\cdot\hat e_\rho\,\rho\,2\pi$$

$$\frac{de}{dx} = -2\pi\rho\,U_0\left(\frac{\Big(\cancel{\hat e_x}+\frac{\vec r-\vec f}{\|\vec r-\vec f\|}\Big)\cdot\hat e_\rho}{\hat e_x\cdot(\vec r-\vec f)+\|\vec r-\vec f\|} - \frac{\Big(\hat e_x+\frac{\vec r+\vec f}{\|\vec r+\vec f\|}\Big)\cdot\hat e_\rho}{\hat e_x\cdot(\vec r+\vec f)+\|\vec r+\vec f\|}\right)\cdot$$

$$= -2\pi\rho\left(\frac{(\vec r-\vec f)\cdot\hat e_\rho}{\hat e_x\cdot(\vec r-\vec f)|\vec r-\vec f| + |\vec r-\vec f|^2} - \frac{(\vec r+\vec f)\cdot\hat e_\rho}{\hat e_x\cdot(\vec r+\vec f)|\vec r+\vec f|+|\vec r+\vec f|^2}\right)$$

$$= -2\pi\rho^2\left(\frac{1}{(x-f)\|\vec r-\vec f\| + (x-f)^2+\rho^2} - \frac{1}{(x+f)\|\vec r+\vec f\| + (x+f)^2+\rho^2}\right)$$

$$= -2\pi\rho^2\left(\frac{1}{(x-f)\sqrt{(x-f)^2+\rho^2}+(x-f)^2+\rho^2} - \frac{1}{(x+f)\sqrt{(x+f)^2+\rho^2}+(x+f)^2+\rho^2}\right)$$

$$\overset{\rho\to 0}{\approx} \rho^2\left(\frac{1}{(x-f)|x-f|\big(1+\frac{1}{2}\frac{\rho^2}{(x-f)^2}\big)+(x-f)^2+\rho^2} - \frac{1}{(x+f)|x+f|\big(1+\frac{1}{2}(\frac{\rho}{x+f})^2\big)+(x+f)^2+\rho^2}\right)$$

Če je imenovalec $\neq 0$ $\to \frac{de}{dx}\to 0$

• $x > f$ $\quad |x-f| = x-f$

$|x+f| = x+f$

$$\frac{de}{dx} = -2\pi g^2 \left(\frac{1}{(x-f)^2 + \frac{1}{2}g^2 + (x-f)^2 + g^2} - \dots \right)$$

$$= -2\pi \left(\frac{1}{2(x-f)^2 + \frac{3}{2}g^2} - \frac{1}{2(x+f)^2 + \frac{3}{2}g^2} \right) \to 0$$

• $x < -f$ $\quad |x-f| = -(x-f)$

$|x+f| = -(x+f)$

$$\frac{de}{dx} = -2\pi g^2 \left(\frac{1}{-(\cancel{x-f})^2 - \frac{1}{2}g^2 + (\cancel{x-f})^2 + g^2} - \frac{1}{-(\cancel{x+g})^2 - \frac{1}{2}g^2 + (\cancel{x+g})^2 + g^2} \right) \to 0$$

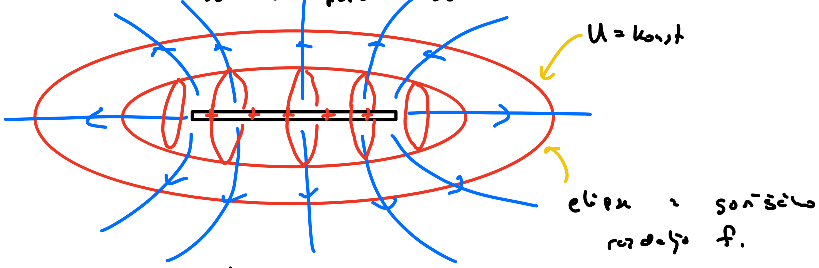
• $-f < x < f$ $\quad |x-f| = -(x-f)$

$|x+f| = x+f$

$$\frac{de}{dx} = -2\pi g^2 \left(\frac{1}{(x-f)^2 + \frac{1}{2}g^2 + (x-f)^2 + g^2} - \frac{1}{-(x+g)^2 - \frac{1}{2}g^2 + (x+g)^2 + g^2} \right)$$

$$= -2\pi g^2 \left(\frac{2}{g^2} - \frac{1}{2(x+f)^2 + \frac{3}{2}g^2} \right) \to 2(-2\pi)$$

Izvori so na palici dolžine $2f$



35 $\quad \vec{A} = c\left(\vec{a} \ln \left| \big(\overbrace{\vec{r} - D(\vec{a}\times\vec{b})}^{\text{premik}}\big)\times\vec{a} \right| - \vec{b} \ln |\vec{r}\times\vec{b}| \right)$

$\|\vec{a}\| = \|\vec{b}\| = 1 \quad$ kont $\vec{a}, \vec{b}$ $\qquad \vec{B} = \nabla\times\vec{A}$

$$\nabla\times\vec{A} = c\left(\nabla \ln\left|\vec{r} - D(\vec{a}\times\vec{b})\big)\times\vec{a}\right| \times\vec{a} - \nabla \ln|\vec{r}\times\vec{b}| \times\vec{b} \right) =$$

$$= c\left(\frac{\nabla\left|(\vec{r} - D(\vec{a}\times\vec{b}))\times\vec{a}\right|}{\left|(\vec{r} - D(\vec{a}\times\vec{b}))\times\vec{a}\right|}\times\vec{a} - \frac{\nabla|\vec{r}\times\vec{b}|}{|\vec{r}\times\vec{b}|}\times\vec{b} \right) =$$

Stranski račun

$$\nabla|\vec{v}| = \nabla\sqrt{\vec{v}^2} = \frac{1}{2}\frac{\nabla v^2}{|v|}$$

$$(\vec{a}\times\vec{b})^2 = a^2b^2(1-\cos^2) = a^2b^2 - (\vec{a}\cdot\vec{b})^2$$

$$= \frac{c}{2}\left(\frac{\nabla\big(((\vec{r} - D(\vec{a}\times\vec{b}))\times\vec{a})^2\big)}{|(\vec{r} - D(\vec{a}\times\vec{b}))\times\vec{a}|^2}\times\vec{a} - \frac{\nabla\big(r^2 \overset{=1}{b^2} - (\vec{r}\cdot\vec{b})^2\big)}{|\vec{r}\times\vec{b}|^2}\times\vec{b} \right)$$

$$= \frac{c}{2}\left(\frac{\nabla\big(((\vec{r} - D(\vec{a}\times\vec{b}))\times\vec{a})^2\big)}{|(\vec{r} - D(\vec{a}\times\vec{b}))\times\vec{a}|^2}\times\vec{a} - \frac{\big(2\vec{r} \overbrace{- 2(\vec{r}\cdot\vec{b})\cdot\vec{b}}^{=0}\big)\times\vec{b}}{|\vec{r}\times\vec{b}|^2} \right)$$

Po vzoru

$$= \frac{c}{2}\left(\frac{2(\vec{r} - D(\vec{a}\times\vec{b}))\times\vec{a}}{|(\vec{r} - D(\vec{a}\times\vec{b}))\times\vec{a}|^2} - \frac{2(\vec{r}\times\vec{b})}{|\vec{r}\times\vec{b}|^2} \right)$$

$$= c\left(\frac{\vec{r}\times\vec{a} - D(\vec{a}\times\vec{b})\times\vec{a}}{|-\text{u}-|^2} - \frac{\vec{r}\times\vec{b}}{|\vec{r}\times\vec{b}|^2} \right)$$

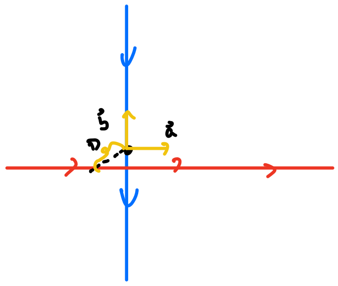
$$= c\left(\frac{\vec{r}\times\vec{a} - D\vec{b}}{|\vec{r}\times\vec{a} - D\vec{b}|^2} - \frac{\vec{r}\times\vec{b}}{|\vec{r}\times\vec{b}|^2} \right.$$

Ker sta $\vec{a}$ in $\vec{b}$ enotska in $\perp$, lahko izberemo koordinatni sistem

$\vec{a} = (1,0,0) \qquad \vec{b} = (0,1,0)$

$$\nabla \times \vec{A} = c\left(\frac{(0,\, z-D,\, -y)}{(z-D)^2+y^2} - \frac{(-z,\, 0,\, x)}{x^2+z^2}\right)$$

Žica v smeri $-\vec{b}$ + žica v smeri $\vec{a}$ premaknjena v $r_0 = D(\vec{a}\times\vec{b})$



36 $\vec{u} = ax\cdot\hat{e}_x - (y - gxz)\hat{e}_y - (z + gxy)\,\hat{e}_z = (ax,\, -y+gxz,\, -z-gxy)$

Določi a, da bo v razvoju po g veljalo

$$\vec{u}\cdot(\nabla\times\vec{u}) = \lambda\|\vec{u}\|^2 + o(g^2)$$

$$\nabla\times\vec{u} = \begin{vmatrix} i & j & k \\ \partial x & \partial y & \partial z \\ ax & -y+gxz & -z-gxy \end{vmatrix} = (-gx-gx,\ 0+gy,\ gz-0) = g(-2x,\, y,\, z)$$

$$\vec{u}\cdot(\nabla\times\vec{u}) = g(ax,\, -y+gxz,\, -z-gxy)\cdot(-2x,\, y,\, z) =$$

$$= g\left(-2ax^2 - y^2 + \cancel{gxyz} - z^2 - \cancel{gxyz}\right) = -g\,(2ax^2+y^2+z^2)$$

$$\|\vec{u}\|^2 = (ax)^2 + y^2 - 2\cancel{gxyz} + (gxz)^2 + z^2 + 2\cancel{gxyz} + (gxy)^2$$

$$= a^2x^2 + y^2 + z^2 + g^2x^2(y^2+z^2)$$

$\lambda = -g \qquad a = 2$ ali $a = 0$

37 Kodre 79/2

Izpelji: kontinuitetno enačbo za pointingsov vektor $\vec{P} = \vec{E}\times\vec{H}$

Povezava med div. in časovnih odvodom.

$$\nabla\cdot\vec{P} = \nabla\cdot(\vec{E}\times\vec{H}) = (\nabla\times\vec{E})\cdot\vec{H} - \vec{E}\cdot(\nabla\times\vec{H}) = -\frac{\partial\vec{B}}{\partial t}\cdot\vec{H} - \vec{E}\cdot\left(\vec{j} + \frac{\partial\vec{D}}{\partial t}\right)$$

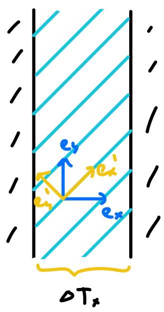
$$= -\mu_0\vec{H}\frac{\partial\vec{H}}{\partial t} - \varepsilon_0\vec{E}\cdot\frac{\partial\vec{E}}{\partial t} - \vec{E}\cdot\vec{j} = -\frac{\partial}{\partial t}\frac{\mu_0H^2}{2}\cdot\frac{\partial}{\partial t}\frac{\varepsilon_0E^2}{2} - \vec{E}\cdot\vec{j}$$

$$\nabla\cdot\vec{P} = -\frac{\partial}{\partial t}\underbrace{W_{EM}}_{\text{energijska gostota}} - \underbrace{\vec{E}\cdot\vec{j}}_{\text{Joulova toplota}} \qquad \frac{U\cdot I}{dS} = \frac{U\cdot I}{V}$$

Izvor toka energije = sprem. en. polja + nastanek en. iz toka

38 Safirna ploščica z anizotropno toplotno prevodnostjo: $\lambda_{\parallel} = 36 \frac{W}{mK}$, $\lambda_{\perp} = 32 \frac{W}{mK}$
lastne smer pod kotom $45°$. $a = 5\,mm$, $l = 100\,mm$, $\Delta T_x = 10\,K$

1. $\Delta T_y = ?$ če je toplotno izolirano v y smeri
2. $j_y = ?$ če je toplotni kratek stik



$$\vec{j} = -\underline{\lambda} \cdot \nabla T$$

Tenzor lahko razvijemo po bazi, dobimo matriko

- Aktivna rotacija:

$$R\hat{e}_x = \hat{e}_x' = \cos\varphi\, \hat{e}_x + \sin\varphi\, \hat{e}_y$$
$$R\hat{e}_y = \hat{e}_y' = -\sin\varphi\, \hat{e}_x + \cos\varphi\, \hat{e}_y$$

Izberemo bazo A $\quad \hat{e}_x = (1,0)_A \quad \hat{e}_y = (0,1)_A \quad R = \begin{bmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{bmatrix}_{A\to A}$

- Pasivna rotacija

$$\vec{v} = v_x \hat{e}_x + v_y \hat{e}_y = v_x R^T \hat{e}_x' + v_y R^T \hat{e}_y'$$
$$= v_x (\cos\varphi\, \hat{e}_x' - \sin\varphi\, \hat{e}_y') + v_y (\sin\varphi\, \hat{e}_x' + \cos\varphi\, \hat{e}_y')$$
$$= (v_x \cos\varphi + v_y \sin\varphi)\hat{e}_x' + (-v_y \sin\varphi + v_x \cos\varphi)\hat{e}_y'$$

$$\begin{bmatrix} v_x' \\ v_y' \end{bmatrix}_B = \begin{bmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{bmatrix}_A^B \begin{bmatrix} v_x \\ v_y \end{bmatrix}_A$$

$$\begin{bmatrix} j_x \\ j_y \end{bmatrix}_A = -\underline{\lambda}_A \begin{bmatrix} \partial_x T \\ \partial_y T \end{bmatrix}_A$$

$$\begin{bmatrix} j_x \\ j_y \end{bmatrix}_A = -\big[R\big]_{B\to A} \begin{bmatrix} \lambda_{\parallel} & \\ & \lambda_{\perp} \end{bmatrix}_B \big[R^T\big]_{A\to B} \begin{bmatrix} \partial_x T \\ \partial_y T \end{bmatrix}_A$$

$$= -\begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix} \begin{bmatrix} \lambda_{\parallel} & 0 \\ 0 & \lambda_{\perp} \end{bmatrix} \begin{bmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{bmatrix} \nabla T$$

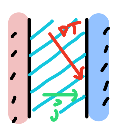
$$= -\begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix} \begin{bmatrix} \lambda_{\parallel}\cos\varphi & \lambda_{\parallel}\sin\varphi \\ -\lambda_{\perp}\sin\varphi & \lambda_{\perp}\cos\varphi \end{bmatrix} \nabla T$$

$$\begin{bmatrix} j_x \\ j_y \end{bmatrix} = -\begin{bmatrix} \lambda_{\parallel}\cos^2\varphi + \lambda_{\perp}\sin^2\varphi & (\lambda_{\parallel} - \lambda_{\perp})\sin\varphi\cos\varphi \\ (\lambda_{\parallel} - \lambda_{\perp})\sin\varphi\cos\varphi & \lambda_{\parallel}\sin^2\varphi + \lambda_{\perp}\cos^2\varphi \end{bmatrix} \begin{bmatrix} \partial_x T \\ \partial_y T \end{bmatrix}$$

Sistem dveh enačb.

1. Če je $\perp$ izoliran $\quad j_y = 0 \qquad \partial_x T = \frac{\Delta T_x}{a}$
2. Če ni $\perp$ izoliran: $\quad j_x, j_y \quad , \quad \partial_y T = 0$

1.

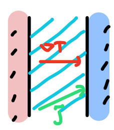


$$j_y = 0 = -\lambda_{xy}\, \partial_x T - \lambda_{yy}\, \partial_y T$$

$$\partial_y T = -\frac{\lambda_{xy}}{\lambda_{yy}}\, \partial_x T \qquad \text{Inducirana temp. razlika}$$

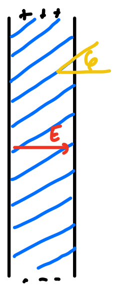
$$j_x = -\lambda_{xx}\, \partial_x T - \lambda_{xy}\, \partial_y T = -\frac{\lambda_{xx}\lambda_{yy} - \lambda_{xy}^2}{\lambda_{yy}}\, \partial_x T = -\frac{\lambda_{\parallel}\lambda_{\perp}}{\lambda_{\parallel}\sin^2\varphi + \lambda_{\perp}\cos^2\varphi}\, \partial_x T$$

2.



$$\partial_y T = 0 \qquad \begin{bmatrix} j_x \\ j_y \end{bmatrix} = \begin{bmatrix} -\lambda_{xx}\, \partial_x T \\ -\lambda_{xy}\, \partial_x T \end{bmatrix}$$

39 Koda 32/5 – kondenzator ki pušča



$\rho_\parallel = 15\ \Omega m$

$\rho_\perp = 10\ \Omega m$

$\varphi = 45°$

$\vec{j} = ?$

$U = 100V$

$a = 1cm$

$U = E_x a$

$E_y = ?$

$\vec{j} = \underline{\underline{\sigma}}\,\vec{E}$ $\qquad \underline{\underline{\sigma}} = \underline{\underline{\rho}}^{-1}$

$\vec{E} = \underline{\underline{\rho}}\,\vec{j}$ $\qquad \vec{j} = j_x \hat{e}_x$ (ne premi sor-dol)

$j_x \hat{e}_x = \underline{\underline{\sigma}}\,\vec{E} \quad / \cdot \hat{e}_x$

$j_x = \hat{e}_x \cdot (\underline{\underline{\sigma}}\,\vec{E})$

če bi imeli podatek $E_y = 0$

$\vec{E} = j_x\, \underline{\underline{\rho}} \cdot \hat{e}_x$

$\hat{e}_x \cdot \vec{E} = j_x\, \hat{e}_x \cdot \underline{\underline{\rho}}\,\hat{e}_x$

če imamo samo E_x

$E_x = j_x\, \hat{e}_x \underline{\underline{\rho}}\,\hat{e}_x$

$\hat{e}_x \underline{\underline{\rho}}\,\hat{e}_x$ je kar x-x komponenta tenzorja

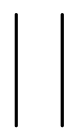
lahko izračunamo v poljubnem sistemu ker dobimo skalar

$E_x = j_x \rho_{xx}$

$E_x = j_x (\cos^2\varphi\, \rho_\parallel + \sin^2\varphi\, \rho_\perp)$

$$\begin{bmatrix} \cos\varphi \\ -\sin\varphi \end{bmatrix} \begin{bmatrix} \rho_\parallel & 0 \\ 0 & \rho_\perp \end{bmatrix} \begin{bmatrix} \cos\varphi \\ -\sin\varphi \end{bmatrix}$$

Če imamo kratek stik



$\vec{E} = E_x \hat{e}_x = \frac{U_0}{a} \hat{e}_x$

$\vec{j}\,\hat{e}_x = j_x = (\hat{e}_x \underline{\underline{\sigma}}\,\hat{e}_x) E_x$

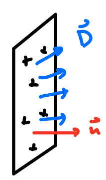
$\hat{e}_x \underline{\underline{\sigma}}\,\hat{e}_x = \sigma_x = \rho_\parallel^{-1} \cos^2\varphi + \rho_\perp^{-1} \sin^2\varphi$

$j_x = (\rho_\parallel^{-1} \cos^2\varphi - \rho_\perp^{-1} \sin^2\varphi)\, E_x = 83,3$

40 Kapaciteta kondenzatorja z dielektrikom

Določi C, če je v kondenzatorju dielektrik s tenzorjem $\underline{\underline{\varepsilon}}$.

$C = \frac{e}{U} = ?$ $\qquad e = \oint \vec{D} \cdot d\vec{S} =$



$= \vec{D} \cdot (\vec{n} S) = S \varepsilon_0 (\underline{\underline{\varepsilon}}\,\vec{E})\,\vec{n} =$

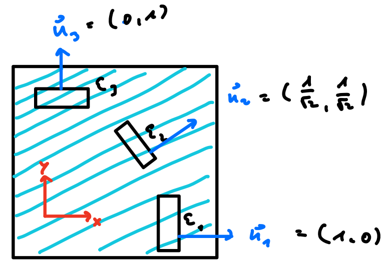
$= S \varepsilon_0 (\vec{n}\, \underline{\underline{\varepsilon}}\,\vec{E}) =$ $\qquad$ Privzemimo $\vec{E} = \frac{U_0}{d} \vec{n}$

$e = S \varepsilon_0 \frac{U_0}{d} \underbrace{(\vec{n}\, \underline{\underline{\varepsilon}}\,\vec{n})}_{\varepsilon_{eff}}$

$\varepsilon_{eff} = \varepsilon_\parallel \cos^2\varphi + \varepsilon_\perp \sin^2\varphi$ (2D)

$C = \frac{\varepsilon_{eff}\,\varepsilon_0 S}{d}$

(41) 3 ploščice iz dielektrika izrezan 1 m 3. last. os



Smer
$0°$ $\varepsilon_1 = 7,8$
$45°$ $\varepsilon_2 = 7,2$
$90°$ $\varepsilon_3 = 7,6$

Določi lastn ε in smeri lastnih osi

$$\underline{\varepsilon} = ? = \begin{bmatrix} a & d & 0 \\ d & b & 0 \\ 0 & 0 & \varepsilon_\perp \end{bmatrix}$$

$$\varepsilon_1 = \vec{u}_1 \underline{\varepsilon} \vec{u}_1 = a = 7,8$$

$$\varepsilon_2 = \vec{u}_2 \underline{\varepsilon} \vec{u}_2 = \frac{1}{2} \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} a & d \\ d & b \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{2}(a+b+2d) \qquad d = \varepsilon_2 - \frac{a+b}{2} = -0,5$$

$$\varepsilon_3 = \vec{u}_3 \underline{\varepsilon} \vec{u}_3 = b = 7,6$$

Iščemo lastne vrednosti $\begin{bmatrix} 7,8 & -0,5 \\ -0,5 & 7,6 \end{bmatrix}$

V 2x2 velja $tr\, \underline{\varepsilon} = \underbrace{\varepsilon_A + \varepsilon_B}_{\text{lastni vrednosti}} = 7,4$ $\quad det\, \underline{\varepsilon} = \varepsilon_A \varepsilon_B = 17,43$

$\Rightarrow \quad \varepsilon_A = 7,19 \qquad \varepsilon_B = 4,2$

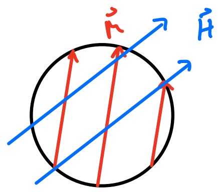
Lastni vektorji: $(\underline{\varepsilon} - \varepsilon_A I)\, \vec{v}_A = 0$

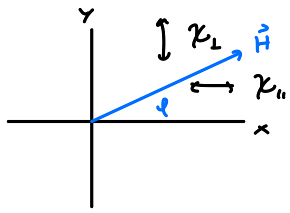
$$\vec{v}_A = \begin{bmatrix} 1 \\ 1,22 \end{bmatrix} \qquad \vec{v}_B = \begin{bmatrix} -1,22 \\ 1 \end{bmatrix} \qquad \vec{v}_B \perp \vec{v}_A$$

$\varphi = 50,7°$ $\qquad \varphi = 50,7° + 90°$

(42) Kroglica bizmut v $\vec{B}$. Bizmut ima susceptibilnost $\mu_{\parallel}$, $\mu_\perp$. Določi navor na kroglico v odvisnosti od kota. $\vec{H}$ = homogen

(torque) $\vec{T} = \vec{p}_m \times \vec{B} \leftarrow$ zun. polje $\qquad \vec{p}_m = V \cdot \vec{M} = V(\underline{\mu} - I)\vec{H} = V\, \underline{\underline{\chi}}\, \vec{H}$





$$\chi = \begin{bmatrix} \chi_\parallel & & \\ & \chi_\perp & \\ & & \chi_\perp \end{bmatrix}$$

$$\vec{H} = (\cos\varphi, \sin\varphi, 0) H$$

$$\vec{M} = \chi \vec{H} = (\chi_\parallel \cos\varphi, \chi_\perp \sin\varphi, 0) H$$

$$\vec{T} = \mu_0 V H^2 (\underbrace{\chi_\parallel \cos\varphi, \chi_\perp \sin\varphi, 0}_{\chi H}) \times (\underbrace{\cos\varphi, \sin\varphi, 0}_{H})$$

$$\vec{T} = \mu_0 V H^2\, \hat{e}_z \cos\varphi \sin\varphi\, (\chi_\parallel - \chi_\perp)$$

→ Posebni primeri
- $\chi_\parallel = \chi_\perp \Rightarrow \vec{T} = 0$
- $\varphi = 0°$ ali $90° \Rightarrow \vec{T} = 0$
- $\chi_\perp > \chi_\parallel$; $\varphi \ll 1$ $\quad T_z < 0$ $\quad$ Navor vleče večjo μ proti $\vec{H}$

Tenzorji so sodi za rotacije. Rotacija za π da isti tenzor

43 Določi pogoje za a–f, da potencial u ne bo imel prostorskih izvorov

$$u = r^{-p}\left(ax^2 + by^2 + cz^2 + dxy + eyz + fxz\right)$$

$$Q_{ij} = \begin{bmatrix} a & d/2 & f/2 \\ d/2 & b & e/2 \\ f/2 & e/2 & c \end{bmatrix} \qquad u = r^{-p}\, r_j\, Q_{jk}\, r_k \qquad \boxed{\nabla_i r_j = \delta_{ij}}$$

$\not\exists$ izvorov $\Leftrightarrow$ $\nabla\cdot(\nabla u) = 0$

$$E_i : \nabla_i u = \nabla_i r^{-p} r_j Q_{jk} r_k = Q_{jk}\nabla_i r^{-p} r_j r_k = Q_{jk}\left((-p\, r^{-p-2} r_i)\, r_j r_k + r^{-p} r_k \delta_{ij} + r^{-p} r_j \delta_{ik}\right) =$$

$$= -p\, r^{-p-2}\, r_i\,(r_j Q_{jk} r_k) + r^{-p}\left(Q_{ik} r_k + Q_{ji} r_j\right) =$$

ker je sim. matrika

$$\boxed{\delta_{ij} Q_{jk} = Q_{ik}} \qquad \boxed{\delta_{ij}\, r_i = r_j}$$

$$= -p\, r^{-p-2}\, r_i\,(r_j Q_{jk} r_k) + 2 r^{-p} Q_{ik} r_k$$

Divergenca: skalarni produkt = podvojeni indeks

$$\nabla_i E_i = \nabla_i\left(-p\, r^{-p-2}\, r_i\,(r_j Q_{jk} r_k) + 2 r^{-p} Q_{ik} r_k\right) =$$

$$= -p\, Q_{jk}\left((\nabla_i r^{-p-2})\, r_i r_j r_k + r^{-p-2}\left(\delta_{ii} r_j r_k + r_i \delta_{ij} r_k + r_i r_j \delta_{ik}\right)\right) +$$

$$+ 2 Q_{ik}\left((\nabla_i r^{-p}) r_k + r^{-p}\delta_{ik}\right) = \qquad \boxed{\delta_{ii} = 3}$$

$$= -p\, Q_{jk}\left((-p-2)\, r^{-p-4}\, r_i r_i r_j r_k + r^{-p-2}\left(3 r_j r_k + r_j r_k + r_k r_j\right)\right) +$$

$$+ 2 Q_{ik}\left(-p\, r^{-p-2} r_i r_k + r^{-p}\delta_{ik}\right) = \qquad \boxed{Q_{ik}\delta_{ik} = \underbrace{Q_{ii}}_{\mathrm{tr}\,Q}}$$

$$= p(p+2)\, r^{-p-2}(r_j Q_{jk} r_k) - p\, r^{-p-2}\, 5\,(r_j Q_{jk} r_k) + 2\left(-p\, r^{-p-2}\, r_i Q_{ik} r_k + r^{-p} Q_{ii}\right)$$

$$= \vec r\, Q\, \vec r\; r^{-p-2}\left(p^2 + 2p - 5p - 2p\right) + 2 r^{-p}\,\mathrm{tr}\,Q$$

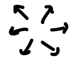
Če hočemo $\nabla\cdot\vec E = 0 \Rightarrow p^2 - 5p = 0$ in $\mathrm{tr}\,Q = 0$

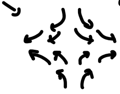
$p(p-5) = 0$ $\qquad a+b+c = 0$

- $p = 0$

  $u = \vec r\, Q\, \vec r$

  in $\mathrm{tr}\,Q = 0$

  Proti primer: $Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $u = x^2 + y^2$  div $\neq 0$

  Primer $Q = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ $u = x^2 - y^2$  div $= 0$

- $p = 5$ $\qquad \mathrm{tr}\,Q = 0$

  $$u = \frac{\vec r\, Q\, \vec r}{r^5} \qquad Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

  $$u = \frac{x^2 + y^2 - 2z^2}{r^5} = \frac{x^2 + y^2 + z^2 - 3z^2}{r^5} = \frac{r^2 - 3\,\vec r\cdot e_z}{r^5} \qquad \text{Kvadropol}$$

$\Rightarrow$ $\mathrm{tr}\,Q \neq 0$ $\qquad \nabla^2 = r^{-p-2}\left(\vec r\, Q\, \vec r\,(p^2 - 5p) + 2\,\mathrm{tr}\,Q\, r^2\right) = 0$

$Q = Q_{ij} = g\,\delta_{ij}$ izotropen (diagonalen)

$$r^{-p-2}\left(r_i \delta_{ij} r_j\, g\,(p^2-5p) + 2\cdot 3\, g\, r^2\right)$$

$$= r^{-p-2} r^2 g \quad (p^2-5p+6)=0$$

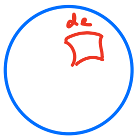
- $p=2 \Rightarrow U = r^{-2}(\vec r Q \vec r) = r^{-2} r^2 g = g$

konst. potencial, ni polja

- $p=3 \Rightarrow U = r^{-3} g\, r^2 = \frac{g}{r}$

polje monopola

Vnaj pomeni Q?



Gaussov izrek

$$de = \vec E \cdot d\vec S = \nabla_i U \;\frac{r_i}{r}\cdot r^2\, d\Omega$$

$$\frac{de}{d\Omega} = r\, r_i\, \nabla_i U = r\, r_i\left(-p\, r^{-p-2} r_i\,(r_j Q_{jk} r_k) + 2 r^{-p} Q_{ik} r_k\right) =$$

$$= r\left(p\, r^{-p}\, \vec r Q \vec r + 2 r^{-p} r_i Q_{jk} r_j\right) =$$

$$= r\; \vec r Q \vec r\; (p+2)\, r^{-p}$$

$p=5$

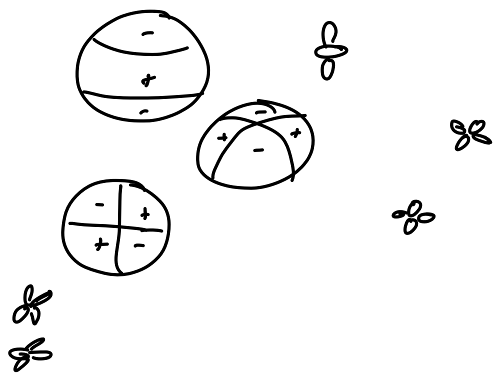
$\begin{bmatrix} 1 & & \\ & 1 & \\ & & -2 \end{bmatrix}$ $\quad x^2+y^2-2z^2$

$\begin{bmatrix} 1 & & \\ & -1 & \\ & & 0 \end{bmatrix}$ $\quad x^2-y^2$

$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad 2xy$

$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$



Q ima 5 neodvisnih komponent

$Y_{2m} \Leftrightarrow$ sferični harmoniki $l=2 \Leftrightarrow d$ orbitale

44

$$\vec P = \vec E \times \vec H$$

$$\nabla\cdot\vec P = -\frac12 \frac{\partial}{\partial t}\left(\mu_0 H^2 + \varepsilon E^2\right) - \underbrace{\vec E\cdot\vec j}_{\frac{\partial P}{\partial V}}$$

Izpelji Cauchyjevo enačbo za EM napetostni tenzor

$$\vec g = \vec D \times \vec B$$

Max. en.

$$\nabla\times\vec H = \vec j + \frac{\partial \vec D}{\partial t}$$ ← premikalni tok

$$\frac{\partial \vec g}{\partial t} = \frac{\partial \vec D}{\partial t}\times\vec B + \vec D\times\frac{\partial \vec B}{\partial t} =$$

$$= \underbrace{(\nabla\times\vec H - \vec j)_j \times \vec B_k}_{\varepsilon_{ijk}} - \vec D\times(\nabla\times\vec E)$$

$$= -\vec j\times\vec B + \varepsilon_{ijk}(\nabla\times H)_j B_k - \varepsilon_{ijk} D_j\,(\nabla\times E)_k =$$

$$= -\vec j\times\vec B + \underbrace{\varepsilon_{ijk}\,\varepsilon_{jlm}}_{\substack{\varepsilon_{jki}\ \varepsilon_{jlm} \\ \delta_{kl}\delta_{im} - \delta_{km}\delta_{il}}}(\nabla_l H_m)\, B_k - \underbrace{\varepsilon_{ijk}}_{\varepsilon_{kij}} D_j\, \varepsilon_{klm}\nabla_l E_m =$$

$$\boxed{\varepsilon_{ijk}\,\varepsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}}$$

$$= -(\vec j\times\vec B)_i + (\delta_{hk}\delta_{il} - \delta_{kl}\delta_{ih}) B_k \nabla_l H_h - (\delta_{il}\delta_{jh} - \delta_{ih}\delta_{jl}) D_j \nabla_l E_h$$

$k=l \quad i=h$

$$= -(\vec j\times\vec B)_i + B_k\nabla_k H_i - B_k \nabla_i H_k - D_j\nabla_i E_j + D_j \nabla_j E_i =$$

$\vec B=\mu\mu_0\vec H \qquad \vec D = \varepsilon\varepsilon_0\vec E$

$$= -(\vec j\times\vec B)_i + \mu\mu_0(H_k\nabla_k H_i - H_k\nabla_i H_k) + \varepsilon\varepsilon_0(E_j\nabla_j E_i - E_j\nabla_i E_j)$$

$$H_k\nabla_i H_k = \tfrac12\nabla_i(H_kH_k) \qquad \vec\nabla\cdot\vec H = 0$$

$$H_k\nabla_k H_i = \nabla_k(H_kH_i) - (\nabla_k H_k)H_i$$

$$E_j\nabla_j E_i = \nabla_j(E_jE_i) - (\underbrace{\nabla_j E_j}_{\frac{\rho}{\varepsilon\varepsilon_0}})E_i$$

$\nabla_i = \nabla_k\delta_{ik}$

$$= -(\vec j\times\vec B)_i + \mu\mu_0\left(\nabla_k(H_kH_i) - \nabla_k\left(\tfrac{H^2}{2}\right)\delta_{ik}\right) + \varepsilon\varepsilon_0\left(\nabla_k(E_kE_i) - \nabla_k\left(\tfrac{E^2}{2}\right)\delta_{ik} - \tfrac{\rho}{\varepsilon\varepsilon_0}E_i\right)$$

$$= -\rho\vec E - \vec j\times\vec B + \nabla_k\Big(\underbrace{\mu\mu_0\left(H_kH_i - \tfrac{H^2}{2}\delta_{ik}\right) + \varepsilon\varepsilon_0\left(E_kE_i - \tfrac{E^2}{2}\delta_{ki}\right)}_{\sigma_{ki}\ =\ \text{napetostni tenzor}}\Big)$$

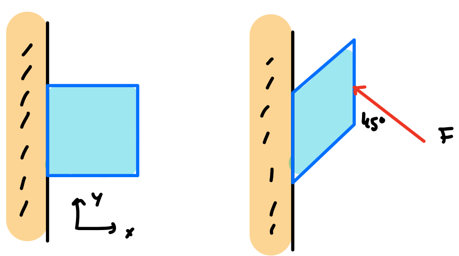
gostota gibalne količine — gostota sil — divergenca

$$\frac{\partial \vec g}{\partial t} = -\vec f + \nabla\sigma$$

$$\frac{\partial g_{em}}{\partial t} + \frac{\partial g_{meh}}{\partial t} = \nabla\sigma \qquad \sigma = \mu\mu_0\left(\vec H\otimes\vec H - \tfrac{H^2}{2}I\right) + \varepsilon\varepsilon_0\left(\vec E\otimes\vec E - \tfrac{E^2}{2}I\right)$$

45 Gumijasta kocka

Določi tenzor deformacije b×b kock



Tenzor tlaka

$$-\underline{p} = \begin{pmatrix} \frac{F_x}{S_x} & \frac12\left(\frac{F_y}{S_x}+\frac{F_x}{S_y}\right) \\ \frac12\left(\frac{F_y}{S_x}+\frac{F_x}{S_y}\right) & \frac{F_y}{S_y}\end{pmatrix}$$

p = simetričen ? ⇕ Ohranitev vrtilne kol.

$$-\underline p = \begin{pmatrix} \frac{F\sqrt2}{2b^2} & -\frac{F\sqrt2}{4b^2}\\ -\frac{F\sqrt2}{4b^2} & 0\end{pmatrix} = \frac{F\sqrt2}{2b^2}\begin{pmatrix}1 & -1/2\\ -1/2 & 0\end{pmatrix}$$

ni sile na y ploskev

sled — Deformacijski tenzor

Hookov zakon $\quad p_{ij} = -\lambda\,\varepsilon_{kk}\delta_{ij} - 2G\varepsilon_{ij}$

Lamejev parameter — Strižni modul

$$p_{ii} = -\lambda\varepsilon_{kk}\,3 - 2G\varepsilon_{ii} = -\varepsilon_{ii}(3\lambda+2G)$$

$$\varepsilon_{ii} = \frac{-p_{ii}}{3\lambda+2G}$$

Izrazimo ε_{ij} $\quad 2G\varepsilon_{ij} = -\lambda\varepsilon_{kk}\delta_{ij} - p_{ij} = -p_{ij} + \frac{\lambda}{3\lambda+2G}\delta_{ij}p_{ii}$

$$\varepsilon_{ij} = \underbrace{\frac{\lambda}{2G(3\lambda+2G)}}\delta_{ij}\,p_{ii} - \frac{1}{2G}p_{ij}$$

Poissonovo št → $\frac{\nu}{E}$ ← Youngov modul

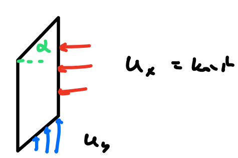
$p_{ii} \to \frac{F\sqrt2}{2b^2} \qquad p_{ij} \to \frac{F\sqrt2}{2b^2}\begin{bmatrix}1 & -1/2\\ -1/2 & 0\end{bmatrix}$

$$\varepsilon_{ij} = \begin{bmatrix}\frac{\nu}{E} - \frac{1}{2G} & +\frac12\frac{1}{2G}\\ +\frac12\frac1{2G} & \frac{\nu}{E}\end{bmatrix}\frac{F\sqrt2}{2b^2}$$

$\varepsilon_{xx} = \frac{\partial u_x}{\partial x}$ = shrček v x smeri

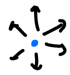
$\varepsilon_{yy} = \frac{\partial u_y}{\partial y}$ = shrček v y smeri

$$\varepsilon_{xy} = \frac{F\sqrt2}{2b^2}\frac12\frac1{2G} = \frac12\Big(\underbrace{\frac{\partial u_x}{\partial y}}_{0} + \frac{\partial u_y}{\partial x}\Big) = \frac12\tan\alpha$$



(46) El. potencial naboja v anizotropnem dielektriku

$$\nabla\cdot\vec{D} = \rho_e = e\,\delta(\vec{r})$$



$$\nabla\cdot(\underline{\underline{\varepsilon}}\varepsilon_0\vec{E}) = e\,\delta(\vec{r})$$

$$\vec{\nabla}\times\vec{E} = 0 \quad \text{(Faradayev zakon)}$$

Lahko zapišem $\vec{E} = -\nabla U$

Določi $U(\vec{r})$

$$-\varepsilon_0\nabla_i\,\overset{\text{konst.}}{\varepsilon_{ij}}\,\nabla_j U = e\,\delta(\vec{r})$$

$$-\varepsilon_0\varepsilon_{ij}\,\nabla_i\nabla_j U = e\,\delta(\vec{r})$$

Gremo v lastni sistem. Da ne bo mešanih odvodov.

$$\varepsilon_x\frac{\partial^2 U}{\partial x^2} + \varepsilon_y\frac{\partial^2 U}{\partial y^2} + \varepsilon_z\frac{\partial^2 U}{\partial z^2} = -\frac{e}{\varepsilon_0}\delta(x)\delta(y)\delta(z)$$

$$x' = \frac{1}{\sqrt{\varepsilon_x}}x \qquad \frac{\partial}{\partial x} = \frac{\partial x'}{\partial x}\frac{\partial}{\partial x'}$$

$$y' = \frac{1}{\sqrt{\varepsilon_y}}y \qquad \text{Nove spremenljivke}$$

$$z' = \frac{1}{\sqrt{\varepsilon_z}}z$$

$$\frac{\partial^2 U}{\partial x'^2} + \frac{\partial^2 U}{\partial y'^2} + \frac{\partial^2 U}{\partial z'^2} = -\frac{e}{\varepsilon_0}\delta(x'\sqrt{\varepsilon_x})\delta(y'\sqrt{\varepsilon_y})\delta(z'\sqrt{\varepsilon_z}) \qquad \delta(ax) = \frac{1}{a}\delta(x)$$

$$\nabla'^2 U = -\frac{e}{\varepsilon_0\sqrt{\varepsilon_x\varepsilon_y\varepsilon_z}}\delta(x')\delta(y')\delta(z')$$

To je enačba za naboj v vakuumu.

$$U = \frac{e}{4\pi\varepsilon_0\sqrt{\varepsilon_x\varepsilon_y\varepsilon_z}}\frac{1}{r'} = \frac{e}{4\pi\varepsilon_0\underbrace{\sqrt{\varepsilon_x\varepsilon_y\varepsilon_z}}_{\det\varepsilon}}\frac{1}{\sqrt{\underbrace{x'^2+y'^2+z'^2}_{x\frac{1}{\varepsilon_x}x + y\frac{1}{\varepsilon_y}y + z\frac{1}{\varepsilon_z}z}}}$$

$$U = \frac{e}{4\pi\varepsilon_0\sqrt{\det\underline{\underline{\varepsilon}}}}\frac{1}{\sqrt{\vec{r}\,\underline{\underline{\varepsilon}}^{-1}\vec{r}}}$$

$$\vec{E} = -\nabla U \qquad \nabla_i r_j = \delta_{ij}$$

$$= -\frac{e}{4\pi\varepsilon_0\sqrt{\det\underline{\underline{\varepsilon}}}}\nabla_i\sqrt{r_k\varepsilon^{-1}_{kj}r_j}^{\,-1} = -\frac{e}{4\pi\varepsilon_0\sqrt{\det\underline{\underline{\varepsilon}}}}\left(-\frac{1}{2}\right)\sqrt{r_k\varepsilon^{-1}_{kj}r_j}^{\,-\frac{3}{2}}\left(\delta_{ik}\varepsilon^{-1}_{kj}r_j + r_k\varepsilon^{-1}_{kj}\delta_{ij}\right)$$

simetričen ε

$$= -\frac{e}{4\pi\varepsilon_0\sqrt{\det\underline{\underline{\varepsilon}}}}\left(-\frac{1}{2}\right)\sqrt{r_k\varepsilon^{-1}_{kj}r_j}^{\,-\frac{3}{2}}\left(\varepsilon^{-1}_{ij}r_j + r_k\varepsilon^{-1}_{ki}\right) =$$

$$\varepsilon^{-1}_{ij}r_j + \varepsilon^{-1}_{ki}r_k = 2\varepsilon^{-1}_{ij}r_j$$

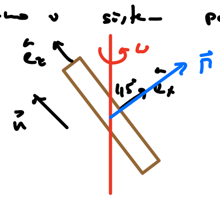
$$= \frac{e}{4\pi\varepsilon_0\sqrt{\det\underline{\underline{\varepsilon}}}\,|\vec{r}\,\varepsilon^{-1}\vec{r}|^{3/2}}\underline{\underline{\varepsilon}}^{-1}\vec{r}$$

(47) Palica pod kotom 45°. Določi smer $\vec{\Gamma}$ in gibanje če to spustimo ($g=0$, v vesolju)

$$\vec{\Gamma} = \underline{\underline{J}}\,\vec{\omega}$$

Gremo v sistem palice (palica v $\hat{e}_z$);

$$J = \begin{bmatrix} J_0 & 0 & 0 \\ 0 & J_0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$



$$\vec{\omega} = \frac{1}{\sqrt{2}}(1,0,1)\,\omega_0$$

$$\vec{\Gamma} = \frac{J_0 \omega_0}{\sqrt{2}} \begin{bmatrix} 1 & & \\ & 1 & \\ & & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \frac{J_0 \omega_0}{\sqrt{2}} (1, 0, 0)$$

Zapišimo raje vse z vektorjem $\vec{n}$ v smeri police

$$J = \begin{bmatrix} J_0 & & \\ & J_0 & \\ & & J_0 \end{bmatrix} - J_0 \underbrace{\begin{bmatrix} 0 & & \\ & 0 & \\ & & 1 \end{bmatrix}}_{e_z \otimes e_z}$$

$$J_{ij} = J_0 (\delta_{ij} - a\, n_i n_j) \qquad J = J_0 (I - a\, \vec{n} \otimes \vec{n})$$

$a = 1$ za tanko polico

Ohranitev vrtilne kol.

$$\dot{\Gamma}_i = 0 = \dot{J}_{ij} \omega_j + J_{ij} \dot{\omega}_j = - a J_0 (\dot{n}_i n_j + n_i \dot{n}_j) \omega_j + J_{ij} \dot{\omega}_j = 0$$

Pri vrtenju je časovni odvod kar rotacija
$\dot{\vec{n}} = \vec{\omega} \times \vec{n}$

Nazaj na vektorje $\quad 0 = -a J_0 \left((\vec{\omega} \times \vec{n})(\vec{n} \cdot \vec{\omega}) + \vec{n} (\cancel{\vec{\omega} \times \vec{n}}) \cdot \vec{\omega} \right) + J \dot{\vec{\omega}}$

Glejmo komponento v smeri $\vec{n}$ $\qquad / \cdot \vec{n}$

$\vec{n} \cdot \underline{\underline{J}}\, \dot{\vec{\omega}} = 0$

$J_0 \vec{n} (I - a \vec{n} \otimes \vec{n}) \dot{\vec{\omega}} = 0$

$J_0 (\vec{n} \cdot \dot{\vec{\omega}} - a\, \vec{n} \cdot \dot{\vec{\omega}}) = 0 \quad \Rightarrow \quad (1 - a)\, \vec{n} \cdot \dot{\vec{\omega}} = 0$

$\dot{\vec{\omega}}$ v smeri $\vec{n}$ je konst.

$$-a J_0 (\vec{\omega} \times \vec{n})(\vec{\omega} \cdot \vec{n}) = - J_0 (\dot{\vec{\omega}} - a\, \vec{n} (\vec{n} \cdot \dot{\vec{\omega}}))$$

$$\dot{\vec{\omega}} = a (\vec{\omega} \times \vec{n})(\vec{\omega} \cdot \vec{n})$$
$$\dot{\vec{n}} = \vec{\omega} \times \vec{n}$$

48 Določi elastične konstante izotropne ter enoosne elastične snovi s pomočjo simetrijskih argumentov.

Imamo deformacijski tenzor $\quad \varepsilon_{kl} = \frac{1}{2} (\nabla_k u_l + \nabla_l u_k)$

tlačni tenzor $\quad p_{ij}$

Splošna zveza $\quad p_{ij} = - C_{ijkl}\, \varepsilon_{kl} \qquad 3^4 = 81$ parametrov

Ohranitev vrtilne količine $\Leftrightarrow p_{ij} = p_{ji}$

ε je že simetričen $\quad \varepsilon_{kl} = \varepsilon_{lk}$

p ima 6 parametrov

ε ima 6 parametrov

$\Rightarrow$ 36 parametrov

$$C_{ijkl} = C_{jikl} = C_{ijlk} = C_{jilk}$$

kotna hitrost

Taylorjev razvoj $\qquad u_l (\vec{r}) = u_l (0) + (r_k \nabla_k) u_l = \vec{u}(0) + (\varepsilon_{kl} + \omega_{kl}) r_k$

antisimetrični del / rotacija

Delo pri premiku

$$A = \int \oint \dot{u}_i \, \underbrace{dF_i}_{pS = F} \, dt = \int \oint (\dot{\varepsilon}_{ki} + \dot{\omega}_{ki}) r_k\, p_{ij}\, dS_j\, dt$$

$$= -\int \oiint (\dot{\varepsilon}_{ki} + \dot{\omega}_{ki})\, r_k\, C_{ijmn}\, \varepsilon_{mn}\, dS_j\, dt$$

Gaussov izrek $\oiint = \iiint \mathrm{div}$

$$= -\int C_{ijmn}\, \varepsilon_{mn} (\dot{\varepsilon}_{ki} + \dot{\omega}_{ki}) \iiint \underbrace{\partial_j r_k\, dV}_{\delta_{jk} V}\, dt$$

$$= -V\int C_{ijmn}\, \varepsilon_{mn} (\dot{\varepsilon}_{ji} + \cancel{\dot{\omega}_{ji}}^{\,0})\, dt$$

ker antisim. · sim = 0

$$= -V\int \dot{\varepsilon}_{ji}\, C_{ijmn}\, \varepsilon_{mn}\, dt$$

$$\varepsilon_{ij} = \varepsilon_{ij}^{(0)} + \frac{t}{t_{max}} (\varepsilon_{ij}^{(1)} - \varepsilon_{ij}^{(0)})$$

$$A = -\frac{1}{2} V (\varepsilon_{ij}^{(1)} - \varepsilon_{ij}^{(0)})\, C_{ijmn} (\varepsilon_{mn}^{(1)} + \varepsilon_{mn}^{(0)})$$

Zdaj pa:

↳ $0 \to \varepsilon^A \to \varepsilon^A + \varepsilon^D$ "funkcija stanja"

↳ $0 \to \varepsilon^D \to \varepsilon^A + \varepsilon^D$ delo neodvisno od vrstnega reda deformacij

$$A_{AD} = -\frac{1}{2} V \underline{\underline{\varepsilon}}^A \underline{\underline{\underline{\underline{C}}}}\, \varepsilon^A - \frac{1}{2} V (\varepsilon^A + \varepsilon^D - \varepsilon^A)\, C\, (\varepsilon^A + \varepsilon^D + \varepsilon^A)$$

$$= -\frac{1}{2} V (\varepsilon^A C \varepsilon^A + 2\varepsilon^D C \varepsilon^A + \varepsilon^D C \varepsilon^D)$$

$$A_{DA} = -\frac{1}{2} V (\varepsilon^A C \varepsilon^A + 2\varepsilon^A C \varepsilon^D + \varepsilon^D C \varepsilon^D)$$

$$\Rightarrow \quad \varepsilon_{ij}^A\, C_{ijkl}\, \varepsilon_{kl}^D = \varepsilon_{ij}^D\, C_{ijkl}\, \varepsilon_{kl}^A$$

$$C_{ijkl} = C_{klij}$$

$$p = C\varepsilon$$

$$\begin{array}{ccc} & 6\times 6 \text{ tudi sim} & \\ \begin{bmatrix} p_{xx} \\ p_{yy} \\ p_{zz} \\ p_{xy} \\ p_{xz} \\ p_{yz} \end{bmatrix} = & \begin{bmatrix} \ddots & & \\ & 21 \text{ komponent} & \\ & & \ddots \end{bmatrix} & \begin{bmatrix} \varepsilon_{xx} \\ \vdots \\ \varepsilon_{yz} \end{bmatrix} \end{array}$$

Splošno asimetrično sredstvo ima max 21 elastičnih konstant.

Izotropno sredstvo

→ invariantno na vse rotacije

C_{ijkl}
- 0: vsi skalarji
- 1: noben vektor
- 2: samo δ_{ij}
- 3: samo ε_{ijk} (vendar antisim.)

$$\parallel \quad C_{ijkl} = \lambda\, \delta_{ij}\delta_{kl} + \mu_1 \delta_{ik}\delta_{jl} + \mu_2 \delta_{il}\delta_{jk}$$

$$\parallel \quad C_{klij} = \lambda\, \delta_{ij}\delta_{kl} + \mu_1 \delta_{ik}\delta_{jl} + \mu_2 \delta_{il}\delta_{jk} \quad ✓$$

$$C_{jikl} = \lambda\, \delta_{ij}\delta_{kl} + \mu_1 \delta_{jk}\delta_{il} + \mu_2 \delta_{jl}\delta_{ik}$$

samo če $\lambda_1 = \lambda_2$

$$C_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk})$$

$$-P_{ij} = C_{ijkl} \; \varepsilon_{kl} = \lambda \; \delta_{ij} \; \varepsilon_{kk} + \mu \; (\varepsilon_{ij} + \varepsilon_{ji})$$

Enoosno sredstvo

- invariatno mora biti na rotacijo okrog $\vec{a}$

rang

0: vsi skalarji

1: a_i

2: δ_{ij} , $a_i a_j$

4: $a_i \varepsilon_{jkl}$ antisim $\times$

$$\begin{aligned} C_{ijkl} = \; & \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) + \alpha (a_i a_j \delta_{kl} + \delta_{ij} a_k a_l) \\ & + \beta (a_i a_k \delta_{jl} + a_i a_l \delta_{jk} + a_j a_k \delta_{il} + a_j a_l \delta_{ik}) \\ & + \gamma \; a_i a_j a_k a_l \end{aligned}$$

$$\begin{aligned} \underline{P} = - \underline{\underline{C}} \, \underline{\varepsilon} = \; & - \lambda I \operatorname{Tr} \underline{\varepsilon} - 2\mu \underline{\varepsilon} - \alpha (\vec{a} \otimes \vec{a} \operatorname{tr} \underline{\varepsilon} + I (\vec{a} \underline{\varepsilon} \vec{a})) \\ & - 2\beta (\vec{a} \otimes \underline{\varepsilon} \vec{a} + \underline{\varepsilon} \vec{a} \otimes \vec{a}) - \gamma \; \vec{a} \otimes \vec{a} \; (\vec{a} \underline{\varepsilon} \vec{a}) \end{aligned}$$

49 Torzijsko nihalo

$$J \ddot{\varphi} + D \varphi = M_0 \sin(\omega t) \qquad \omega(t=0) = \dot{\varphi}_0 = 0$$

$$\ddot{\varphi} + \frac{D}{J} \varphi = \frac{M_0}{J} \operatorname{Im}(e^{i\omega t}) \qquad \varphi(t=0) = \varphi_0 = 0$$

$$\frac{D}{J} = \Omega^2$$

$$\varphi = \varphi_{Hom} + \varphi_{part}$$

$$\varphi_{Hom} = A e^{i\Omega t} \qquad \varphi_{PART} = B e^{i\omega t}$$

$$-B\omega^2 e^{i\omega t} + \Omega^2 B e^{i\omega t} = \frac{M_0}{J} e^{i\omega t}$$

$$B = \frac{M_0}{J} \frac{1}{\Omega^2 - \omega^2}$$

Z.P. pogoji:

$$\varphi = \operatorname{Im}(\varphi_{Hom} + \varphi_{part}) = A \sin \Omega t + B \sin \omega t$$

$$\varphi(0) = 0 \Rightarrow \operatorname{Im} A = 0$$

$$\dot{\varphi} = \dot{\varphi}_H + \dot{\varphi}_P = A i \Omega e^{i\Omega t} + B i \omega e^{i\omega t}$$

$$0 = \operatorname{Im}(A i \Omega \cdot 1 + B i \omega \cdot 1) = 0 \Rightarrow A = -B \frac{\omega}{\Omega}$$

$$\Rightarrow \varphi = \operatorname{Im}\left(\frac{M_0}{J(\Omega^2-\omega^2)} \left(e^{i\omega t} - \frac{\omega}{\Omega} e^{i\Omega t} \right)\right) = \frac{M_0}{J(\Omega^2-\omega^2)} \left(\sin \omega t - \frac{\omega}{\Omega} \sin \Omega t \right)$$

(50) Gibanje kaplice ob širjenju zvoka

$v_z = v_0 \cos(\omega t)$ hitrost zvoka   zač. pog.   zač. hitrost $v(0) = u_0$

$$m\dot v = F_v = 6\pi r\eta\,(v_z - v)$$

$$\dot v + v\underbrace{\frac{6\pi r\eta}{m}}_{\frac{1}{\tau}} = \frac{6\pi r\eta}{m} v_z$$

$$\dot v + \frac{v}{\tau} = \frac{v_z}{\tau} = \frac{v_0}{\tau}\cos(\omega t) = \frac{v_0}{\tau} e^{i\omega t}$$

Homogena rešitev $v_H = A e^{-\frac{t}{\tau}}$ $v_p = B e^{i\omega t}$

Vstavimo part. $B i\omega \cancel{e^{i\omega t}} + \frac{B}{\tau}\cancel{e^{i\omega t}} = \frac{1}{\tau} v_0 \cancel{e^{i\omega t}}$

$$B = \frac{v_0}{\tau}\left(i\omega + \frac{1}{\tau}\right)^{-1} = \frac{v_0}{i\omega\tau + 1}$$

Splošna rešitev $v = \mathrm{Re}\left(A e^{-\frac{t}{\tau}} + B e^{i\omega t}\right)$

$$u_0 = A + \mathrm{Re}(B) \qquad A = u_0 - \mathrm{Re}(B)$$

$$\mathrm{Re}(B) = \mathrm{Re}\frac{v_0(1 - i\omega\tau)}{1 + (\omega\tau)^2} = \frac{v_0}{1+(\omega\tau)^2}$$

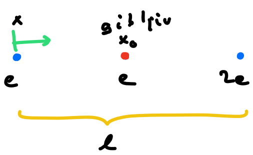
$$\Rightarrow \quad v = \left(u_0 - \frac{v_0}{1+(\omega\tau)^2}\right) e^{-\frac{t}{\tau}} + \mathrm{Re}\left(B e^{i\omega t}\right)$$

$$\mathrm{Re}\left(B e^{i\omega t}\right) = \mathrm{Re}\frac{v_0(1 - i\omega\tau)(\cos\omega t + i\sin\omega t)}{1 + (\omega\tau)^2}$$

$$= \frac{v_0(\cos\omega t + \omega\tau\sin\omega t)}{1 + (\omega\tau)^2}$$

$$v = \left(u_0 - \frac{v_0}{1+(\omega\tau)^2}\right) e^{-\frac{t}{\tau}} + \frac{v_0(\cos\omega t + \omega\tau\sin\omega t)}{1 + (\omega\tau)^2}$$

(51) Iščemo frekvenco nihanja okrog ravnovesja za sistem



ravnovesje $\Leftrightarrow \oint \vec E\, d\vec s = 0$

Reševanje z energijo

$$U = \frac{e^2}{4\pi\varepsilon_0}\left(\frac{1}{r_1} + \frac{2}{r_2}\right)$$ Ne poznamo ravnovesja in frekvence

$\vec r = (x_0 + x,\ y)$ x, y odmiki x_0 ravnovesje

$$U = \underbrace{\frac{e^2}{4\pi\varepsilon_0}}_{\text{naj bo } 1}\left(\frac{1}{\sqrt{(x_0+x)^2 + y^2}} + \frac{2}{\sqrt{(l - x - x_0)^2 + y^2}}\right)$$

če smo v ravnovesju $U = U_0 + 0x + 0 \cdot y + O(x^2, y^2)$

če nismo v ravnovesju $U = U_0 + ax + by + \dots$

Razvijemo do kvadratnega člena

$$U = \frac{1}{x_0\sqrt{1+2\frac{x}{x_0}+\frac{x^2}{x_0^2}+\frac{y^2}{x_0^2}}} + \frac{2}{|l-x_0|\sqrt{1-\frac{2x}{|l-x_0|}+\frac{x^2+y^2}{|l-x_0|^2}}} \qquad (1+\varepsilon)^{-\frac12} = 1-\frac12\varepsilon+\frac38\varepsilon^2$$

$$U = \frac{1}{x_0}\left(1-\frac{x}{x_0}-\frac12\,\frac{x^2+y^2}{x_0^2}+\frac38\left(2\frac{x}{x_0}\right)^2+\dots\right)+\frac{2}{|l-x_0|}\left(1+\frac{x}{|l-x_0|}-\frac12\,\frac{x^2+y^2}{(l-x_0)^2}+\frac38\left(2\frac{x}{l-x_0}\right)^2\right)$$

$$U = \frac{1}{x_0}+\frac{2}{|l-x_0|} + x\underbrace{\left(\frac{2}{|l-x_0|^2}-\frac{1}{x_0^2}\right)}_{0 \text{ za ravnovesje}} + x^2\left(\frac32\left(\frac{1}{x_0^3}+\frac{2}{(l-x_0)^3}\right)-\frac12\left(\frac{1}{x_0^3}+\frac{2}{(l-x_0)^3}\right)\right)$$
$$+\, y^2\left(-\frac12\left(\frac{1}{x_0^3}+\frac{2}{(l-x_0)^3}\right)\right)$$

Pogoj ravnovesja $\quad \frac{2}{|l-x_0|^2}-\frac{1}{x_0^2}=0 \quad\Rightarrow\quad x_0\sqrt2 = l-x_0 \quad\Rightarrow\quad x_0=\frac{l}{1+\sqrt2}$

Nihanje

$$U = U_0 + \frac{x^2}{2}A + \frac{y^2}{2}B$$

$$A = 2\left(\frac{1}{x_0^3}+\frac{2}{(l-x_0)^3}\right)\frac{e^2}{4\pi\varepsilon_0} \qquad B=-\left(\frac{1}{x_0^3}+\frac{2}{(l-x_0)^3}\right)\frac{e^2}{4\pi\varepsilon_0}$$

Polna energija $\quad W = \frac{m\dot x^2}{2}+A\frac{x^2}{2}+\frac{m\dot y^2}{2}+B\frac{y^2}{2}$

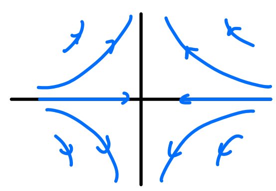
x smer $\quad \omega_x^2 = \frac{A}{m}$

$$x = x_0\cos\omega_x t + \frac{v_{x0}}{\omega_x}\sin\omega_x t$$

y smer $\quad \omega_y^2=\frac{B}{m}<0 \qquad \omega_y = i\sqrt{\frac{|B|}{m}} \qquad \sin i\omega t \leftrightarrow i\,\mathrm{sh}\,\omega t$

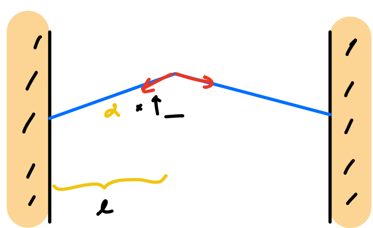
$\cos i\omega t \leftrightarrow \mathrm{ch}\,\omega t$

$$y = y_0\,\mathrm{ch}\,\omega_y t + \frac{v_{y0}}{\omega_y}\,\mathrm{sh}\,\omega_y t$$



Enako tudi v z smeri.

52 Izpelji enačbo za transverzalno nihanje uteži na ohlapni elastiki



ni napeta pri $x=0$

$$F = k\Delta l = k\left(\sqrt{l^2+x^2}-l_0\right)$$

$$m\ddot x = -2F\frac{x}{\sqrt{l^2+x^2}} = -2kx\left(1-\frac{l_0}{\sqrt{l^2+x^2}}\right)$$

$$\ddot x = -2\frac{k}{m}x\left(1-\frac{l_0}{l\sqrt{1+(\frac{x}{l})^2}}\right) \overset{\text{Taylor}}{=} -2\frac km x\left(1-\frac{l_0}{l}\left(1-\frac12\left(\frac xl\right)^2\right)\right)$$

Če je $l>l_0$, potem je $\ddot x = -2\frac km x\left(1-\frac{l_0}{l}\right)$

Če je $l=l_0$, potem je $\ddot x = -\frac km\frac{x^3}{l_0^2}$ nelinearna enačba

→ odvisno od amplitude

$$v\frac{dv}{dx} = -\frac km\frac{x^3}{l_0^2}$$

$$v\,dv = -\frac km\frac{x^3}{l_0^2}\,dx$$

$$\frac{v^2}{2}+\frac14\frac km\frac{x^4}{l_0^2} = \text{konst.} = \frac{v_0^2}{2}+\frac14\frac km\frac{x_0^4}{l_0^2}$$ energija

Če začnem v skrajni legi $v_0 = 0$

$$v = \sqrt{\frac{k}{2ml_0^2}\left(x_0^4 - x^4\right)}$$

$$\int_0^{x_0} \frac{dx}{\sqrt{\frac{k}{2ml_0^2}\left(x_0^4 - x^4\right)}} = \int_0^{t_0} dt$$

$$t_0 = 4\,\frac{1}{\sqrt{\frac{k x_0^4}{2ml_0^2}}} \int_0^{x_0} \frac{dx}{\sqrt{1-\left(\frac{x}{x_0}\right)^4}} \qquad \frac{x}{x_0} = u$$

$$t_0 = 4\sqrt{\frac{2m}{k}}\,\frac{l_0}{x_0} \underbrace{\int_0^1 \frac{du}{\sqrt{1-u^4}}}_{K(-1) \,=\, 1{,}311} \qquad \omega_0 = \sqrt{\frac{k}{m}}$$

$$t_0 = 4\sqrt{2}\,\frac{l_0}{x_0\,\omega_0}\,K(-1)$$

$$t_0 \to \infty \qquad x_0 \to 0$$

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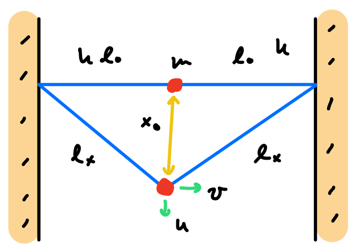
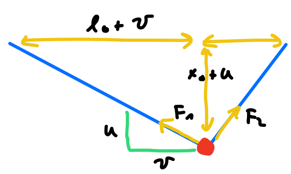


Poišči lastne frekvence nihanja povezanih elastik z utežjo



Zapis sil

$$F_1 = k\left(l_1 - l_0\right) \qquad l_1 = \sqrt{(l_0+v)^2 + (x_0+u)^2} \qquad l_2 = \sqrt{(l_0-v)^2 + (x_0+u)^2}$$

$$F_2 = k\left(l_2 - l_0\right)$$

$$\vec{F}_1 = -F_1\,\frac{(x_0+u,\; l_0+v)}{l_1} \Big\} \; \frac{\vec{r}}{r}$$

$$\vec{F}_2 = -F_2\,\frac{(x_0+u,\; -(l_0-v))}{l_2}$$

$$m\begin{bmatrix}\ddot{u}\\ \ddot{v}\end{bmatrix} = -k\left(\frac{l_1-l_0}{l_1}\right)\begin{bmatrix}x_0+u\\ l_0+v\end{bmatrix} - k\left(\frac{l_2-l_0}{l_2}\right)\begin{bmatrix}x_0+u\\ -l_0+v\end{bmatrix} + \begin{bmatrix}g\\ 0\end{bmatrix}m \qquad l_1 = l_1(u,v) \qquad l_2 = l_2(u,v)$$

Razvoj do linearnega reda v u in v

$$\boxed{(1+\varepsilon)^n = 1 + n\varepsilon}$$

$$\frac{1}{l_1} \overset{\text{razvoj}}{=} \frac{1}{\sqrt{(l_0+v)^2 + (x_0+u)^2}} = \frac{1}{\sqrt{l_0^2 + x_0^2 + 2l_0 v + 2x_0 u + \mathcal{O}(u^2, v^2)}} \approx$$

$$= \frac{1}{\sqrt{l_0^2 + x_0^2}}\,\frac{1}{\sqrt{1 + \frac{2(l_0 v + x_0 u)}{l_0^2 + x_0^2}}} = \frac{1}{l_x}\left(1 - \frac{1}{2}\,2\,\frac{(l_0 v + x_0 u)}{l_x^2}\right)$$

dolžina elastike v ravnovesju l_x

$$\frac{1}{l_2} = \frac{1}{l_x}\left(1 - \frac{(-l_0 v + x_0 u)}{l_x^2}\right)$$

$$m\begin{bmatrix}\ddot{u}\\ \ddot{v}\end{bmatrix} = -k\left(\frac{l_1-l_0}{l_1}\right)\begin{bmatrix}x_0+u\\ l_0+v\end{bmatrix} - k\left(\frac{l_2-l_0}{l_2}\right)\begin{bmatrix}x_0+u\\ -l_0+v\end{bmatrix} + \begin{bmatrix}g\\ 0\end{bmatrix}m$$

$$m\begin{bmatrix}\ddot{u}\\ \ddot{v}\end{bmatrix} = -k\left(1 - \frac{l_0}{l_x} + \frac{l_0}{l_x^3}\left(l_0 v + x_0 u\right)\right)\begin{bmatrix}x_0+u\\ l_0+v\end{bmatrix} - k\left(1 - \frac{l_0}{l_x} + \frac{l_0}{l_x^3}\left(-l_0 v + x_0 u\right)\right)\begin{bmatrix}x_0+u\\ -l_0+v\end{bmatrix} + \begin{bmatrix}g\\ 0\end{bmatrix}m$$

Obdržimo le linearne in konstantne člene

Ravnovesni posoj: konstantni člen je 0.

$$0 = -k\left(1-\frac{l_0}{l_x}\right)\begin{bmatrix} x_0 \\ l_0 \end{bmatrix} - k\left(1-\frac{l_0}{l_x}\right)\begin{bmatrix} x_0 \\ -l_0 \end{bmatrix} + \begin{bmatrix} g \\ 0 \end{bmatrix} m$$

u smer: $-k\left(1-\frac{l_0}{l_x}\right)2x_0 + mg = 0 \Rightarrow x_0$ nas ne zanima

Za nihanje pogledamo linearni člen

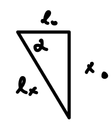
$$m\begin{bmatrix} \ddot{u} \\ \ddot{v} \end{bmatrix} = \overset{=0}{\begin{bmatrix} konst \end{bmatrix}} - k\left(2\left(1-\frac{l_0}{l_x}\right)\begin{bmatrix} u \\ v \end{bmatrix} + \frac{l_0}{l_x^3}\begin{bmatrix} (\cancel{l_0 v} + x_0 u)x_0 + (\cancel{-l_0 v} + x_0 u)x_0 \\ (l_0 v + \cancel{x_0 u})\, l_0 - (-l_0 v + \cancel{x_0 u})\, l_0 \end{bmatrix}\right)$$

$$\begin{bmatrix} \ddot{u} \\ \ddot{v} \end{bmatrix} = -2\underbrace{\frac{k}{m}}_{\omega_0^2}\left(\left(1-\frac{l_0}{l_x}\right)\begin{bmatrix} u \\ v \end{bmatrix} + \frac{l_0}{l_x^3}\begin{bmatrix} x_0^2\, u \\ l_0^2\, v \end{bmatrix}\right)$$

$$\ddot{u} = \underbrace{-2\omega_0^2\left(1-\frac{l_0}{l_x}+\frac{l_0 x_0^2}{l_x^3}\right)}_{\omega_u^2} u$$

$$\ddot{v} = \underbrace{-2\omega_0^2\left(1-\frac{l_0}{l_x}+\frac{l_0^3}{l_x^3}\right)}_{\omega_v^2} v$$

Pokličimo $\cos\alpha = \frac{l_0}{l_x}$ $\quad \sin\alpha = \frac{x_0}{l_x}$



$$\omega_u^2 = 2\omega_0^2\left(1-\cos\alpha+\cos\alpha\,\sin^2\alpha\right) = 2\omega_0^2\left(1-\cos^3\alpha\right)$$
$$\omega_v^2 = 2\omega_0^2\left(1-\cos\alpha+\cos^3\alpha\right)$$

Limite

$\alpha = 0$ $\quad \omega_u^2 \to 0$

$\quad\quad\quad \omega_v^2 \to 2\omega_0^2(1-1+1) = 2\omega_0^2$ ✓

$\alpha = 90°$ $\quad \omega_u^2 = 2\omega_0^2$

$\quad\quad\quad \omega_v^2 = 2\omega_0^2$

(54) Razmnoževanje zajcev

$$\dot{N} = \lambda N^\alpha$$

rešitev obnašanje za različne α.

$$\int_{N_0}^{N}\frac{dN}{N^\alpha} = \int_0^t \lambda\, dt$$

$$\frac{1}{(1-\alpha)} N^{1-\alpha}\Big| = \lambda t$$

$$N = N_0\left(1 + \frac{1-\alpha}{N_0^{1-\alpha}}\lambda t\right)^{\frac{1}{1-\alpha}} \qquad \alpha \neq 1$$

Za $\alpha = 1$ poseben primer

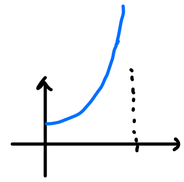
$$N = N_0\left(1+\frac{t}{t_0}\right)^{\frac{1}{1-\alpha}} \qquad t_0 = \frac{N_0^{1-\alpha}}{\lambda(1-\alpha)}$$

$\alpha \to 1 \qquad t_0 \to \infty$

$$N = N_0\left(\left(1+\frac{t}{t_0}\right)^{t_0}\right)^{\frac{\lambda}{N_0^{1-\alpha}}} \xrightarrow{\lim} N_0\, e^{\lambda}$$

$\alpha = 1$ ... mitoza zajcev 

$\alpha = 2$ ... mahljačno srečevanje zajcev $\quad N = \dfrac{N_0}{1 - N_0 \lambda t}$



$\alpha < 1 \quad$ Naj bo $\alpha = \frac{1}{\beta}$

$$N = N_0 \left(1 + \frac{1 - \frac{1}{\beta}}{N^{1-1/\beta}} t \lambda\right)^{\frac{1}{1-1/\beta}}$$

Polinom ali reda pot. potenca

$\alpha = \frac{1}{2} \quad N = N_0 \left(1 + \dfrac{\lambda t}{2\sqrt{N_0}}\right)^2$

Kapaciteta okolja

$$\dot{N} = \lambda N (N_{max} - N)$$

Logistični problem

→ Rodnost $\lambda(N_{max} - N)$ pada

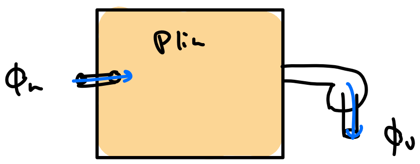
→ Parjenje zajcev z luknjami

- Srečanje N zajcev z $N_{max} - N$ praznimi prostori/luknjami v prostoru

55 Naloga 115/5 Pampa

Črpalka vleče Φ_V, iščemo $p(t)$, če ima posoda luknjo in dotok Φ_m

$T =$ konst.



Začnemo s časovnim odvodom mase

$$\frac{dm}{dt} = \Phi_m - \rho \Phi_V = \Phi_m - \frac{m}{V} \Phi_V$$

$$\frac{dm}{dt} + \underbrace{\frac{\Phi_V}{V}}_{\frac{1}{\tau}} m = \Phi_m$$

Partikularna rešitev

- ko so vsi odvodi $= 0$

$\frac{1}{\tau} m_\infty = \Phi_m \quad m_\infty = \Phi_m \tau$

$$m = \tau \Phi_m + A e^{-\frac{t}{\tau}}$$

Plinska enačba $\Rightarrow p = \dfrac{RT}{VM}\left(\tau \Phi_m + A e^{-\frac{t}{\tau}}\right)$

$\dot{y} + ky = C + Dt + Et^2 + \dots$

→ nastavek $\quad y_{hom} = e^{\lambda t}$

$\quad y_{part} = \alpha + \beta t + \gamma t^2 + \dots$

→ separacija spremenljivk

$\displaystyle\int \frac{dy}{C - ky} = \int dt$

→ $\dot{y} = 0 \Rightarrow$ part. rešitev

(samo če ostaja stac. stanje)

če se nekje ustali

→ Menjava spremenljivk / premik izhodišča

$\dot{y} + k(\underbrace{y - \tfrac{C}{k}}_{u}) = 0$

Ob $t = 0 \quad p = p_0$

$$p = \frac{RT\tau\Phi_m}{VM} + \left(p_0 - \frac{\tau R T \Phi_m}{VM}\right) e^{-t/\tau}$$

56 Naloga 116/8 V posodo s posoljeno vodo priteka čista voda. Mešanica teče čez rob. Koncentracija barvila (t)

↳ Φ_V

m ... masa barvila v posodi $\qquad c = \frac{m}{V}$

$$dm = +c_1\,dV - c\,dV \qquad /: dt\,V$$

trenutna koncentracija (c); koncentracija barvila v dotoku (c_1)

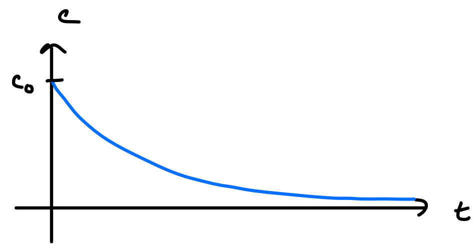
$$\frac{dc}{dt} = (c_1 - c)\frac{1}{V}\frac{dV}{dt} \qquad V = \text{konst}$$

$$\frac{dc}{dt} = -\frac{\Phi_V}{V}(c - c_1) \qquad u = c - c_1$$

$$\dot{u} = -\underbrace{\frac{\Phi_V}{V}}_{\hat{=}\,\frac{1}{\tau}} u$$

$$u = A e^{-\frac{t}{\tau}}$$

$$c = c_1 + (c_0 - c_1)e^{-\frac{t}{\tau}}$$



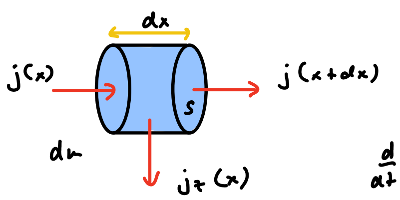
57 Naloga 123/5

Cev zelo pelje v sneg, po njej teče topla voda ($v = 1{,}5\,\frac{m}{s}$, $r = 1$ cm, debelina $b = 1$ mm, $\lambda = 0{,}8\,\frac{W}{mK}$, $T_0 = 60°C$). Izračunaj temp. profil vzdolž cevi v stac. stanju. $T(x) = ?$

T odvisen le od x $\qquad T(y) = \text{konst}$

Toplota se prevaja tudi po vodi

$T_z = 0°C$



Energijski zakon

$$\frac{d}{dt}(dW_n) = -S\,j(x+dx) + S\,j(x) - 2\pi r\,dx\,j_z(x)$$

$$\frac{d}{dt}(dW_n) = dm\,c\,\frac{dT}{dt} = -S\frac{dj}{dx}dx - \lambda\frac{T-T_z}{b}2\pi r\,dx$$

$-\lambda'\frac{dT}{dx}$ toplotna prev. vode

Delimo z $dV = S\,dx$ $\qquad \rho c\frac{dT}{dt} = \lambda'\frac{d^2T}{dx^2} - \frac{2\pi r}{bS}\lambda(T - T_z)$

↑ gibanje s hitrostjo

$$\rho c\Big(\underbrace{\frac{\partial T}{\partial t}}_{=0} + \frac{\partial T}{\partial x}\frac{dx}{dt}\Big) = \lambda\frac{\partial^2 T}{\partial x^2} - \frac{r\lambda}{br^2}\underbrace{(T - \overset{0}{T_z})}_{T-T_z=T}$$

stacionarno stanje

$$\rho c v\frac{\partial T}{\partial x} = \lambda'\frac{\partial^2 T}{\partial x^2} - \frac{2\lambda}{br}T$$

$$D\frac{\partial^2 T}{\partial x^2} - \frac{\partial T}{\partial x} - \frac{1}{x_0}T = 0 \qquad D = \frac{\lambda'}{\rho c v} = \text{„Difuzijska konstanta"}$$

$$x_0 = \frac{br\rho cv}{2\lambda}$$

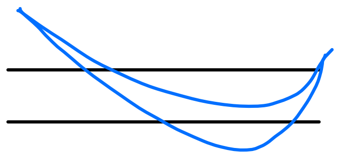
Nastavek $T = e^{kx}$

$$Dk^2 - k - \frac{1}{x_0} = 0 \quad \Leftarrow \text{karakteristični polinom}$$

$$k = \frac{1 \pm \sqrt{1 + \frac{4D}{x_0}}}{2D}$$

$k_- < 0$ ← ta je v redu rešitev

$k_+ > 0$



Če imamo levi in desni robni pogoj sodelujeta obe rešitvi

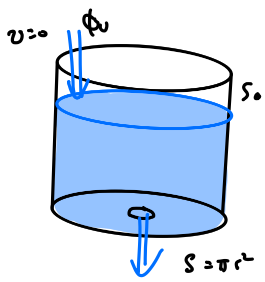
Z polneskončno cev $T(x \to \infty) < \infty$

uporabimo le k_- rešitev. $= T_z$

$$T = T_0 \, e^{\frac{1 - \sqrt{1 + 4D/x_0}}{2D} x}$$

- $D \to 0$ $\quad k_- = \dfrac{1 - (1 + \frac{1}{2}\frac{4D}{x_0})}{2D} = -\dfrac{1}{x_0}$ $\qquad T = T_0 e^{-\frac{x}{x_0}}$
- $v \to 0$ Stoječa voda

58 V valjasto posodo priteka 1 l/s $(S_0 = 1\,m^2)$



$$\underline{S_0 \dot h = \Phi_V - vS}$$ Ohranitev pretoka

Bernoulli: $\frac{1}{2}\rho \dot h^2 + \rho g h + p = 0$

$$\frac{1}{2}\rho \dot h^2 + \rho g h = \frac{1}{2}\rho v^2 \quad \Big| : \frac{1}{2}\rho$$

$$\underline{\dot h^2 = v^2 - 2gh}$$

$$\dot h^2 = \left(\frac{S_0}{S}\dot h - \frac{\Phi_V}{S}\right)^2 - 2gh$$

$$\dot h^2\left(1 - \left(\frac{S_0}{S}\right)^2\right) + 2\dot h \frac{S_0}{S^2}\Phi_V + \left(2gh - \frac{\Phi_V^2}{S^2}\right) = 0$$

Rešimo kvadratno

$$\dot h = \frac{\frac{S_0 \Phi_V}{S^2} - \sqrt{\frac{\Phi_V^2}{S^2} + 2gh\left(\frac{S_0^2}{S^2} - 1\right)}}{\left(\frac{S_0}{S}\right)^2 - 1} \Rightarrow \text{rešimo integral}$$

Poenostavitev $\quad S \ll S_0$

$$\dot h = \frac{\Phi_V}{S_0} - \frac{S}{S_0}\sqrt{2gh}$$

h_∞ ko je $\dot h = 0$ $\qquad h_\infty = \dfrac{\Phi_V^2}{2gS^2}$

$$x = \frac{h}{h_\infty} \Big\{ \; 1 \text{ ravnovesje}$$

$$h_\infty \dot x = \frac{\Phi_V}{S_0}(1 - \sqrt{x})$$

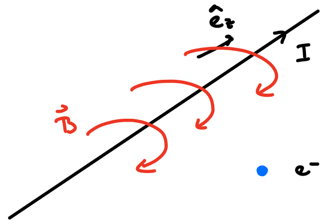
$$\underbrace{\frac{S_0 h_\infty}{\Phi_V}}_{\tau} \dot x = 1 - \sqrt{x} \qquad u = \frac{t}{\tau}$$

$$\Rightarrow \frac{dx}{du} = 1 - \sqrt{x}$$

$$u = \int_{x_0}^{x} \frac{dx}{1-\sqrt{x}} \overset{y=\sqrt{x}}{=} \int_{\sqrt{x_0}}^{\sqrt{x}} \frac{2y\,dy}{1-y} =$$

$$= 2\int \left(\frac{1}{1-y} - 1 \right) dy = 2\left(\ln \frac{\sqrt{x_0}-1}{\sqrt{x}-1} - \sqrt{x} + \sqrt{x_0} \right) = \frac{t}{\tau}$$

59 Elektron in žica



Dobi zakonitosti gibanja elektrona in min/max radij glede na žico.

Ohranja se:
- energija (W_k)
- vrtilna količina okoli z
- translacijska simetrija vzdolž z $\Leftrightarrow$ ohranitev?

$$\vec{B} = -\frac{\mu_0 I}{2\pi r}\hat{e}_\varphi \qquad \hat{e}_r \times \hat{e}_\varphi = \hat{e}_z$$

$$\begin{matrix} z & r & \varphi \\ \varphi & z & r \end{matrix}$$

$$m\dot{\vec{v}} = e\vec{v} \times \vec{B} \qquad \text{2. N.Z.}$$

$$\vec{v} = v_r\hat{e}_r + v_\varphi \hat{e}_\varphi + v_z \hat{e}_z$$

$$\dot{v}_r\hat{e}_r + v_r\dot{\hat{e}}_r + \dot{v}_\varphi\hat{e}_\varphi + v_\varphi\dot{\hat{e}}_\varphi + \dot{v}_z\hat{e}_z = -\underbrace{\frac{e\mu_0 I}{2\pi m}}_{Q}\frac{1}{r}\left(v_r\hat{e}_z - v_z\hat{e}_r\right)$$

$$\dot{\hat{e}}_r = \frac{\partial \hat{e}_r}{\partial \varphi}\frac{\partial \varphi}{\partial t}, \quad \frac{\partial \hat{e}_r}{\partial \varphi} = \hat{e}_\varphi \qquad \dot{\hat{e}}_\varphi = \frac{\partial \hat{e}_\varphi}{\partial \varphi}\frac{d\varphi}{dt}, \quad \frac{\partial \hat{e}_\varphi}{\partial \varphi} = -\hat{e}_r$$

z: $\dot{v}_z + \frac{Q}{r}v_r = 0$

r: $\dot{v}_r - v_\varphi\dot{\varphi} = \frac{Q}{r}v_z \qquad v_\varphi = r\dot{\varphi}$

φ: $v_r\dot{\varphi} + \dot{v}_\varphi = 0 \quad \Big) \cdot r$

$$v_r r\dot{\varphi} + r\dot{v}_\varphi = 0$$

$$v_r v_\varphi + r\dot{v}_\varphi = 0 \quad \Big) \text{ odvod produkta} \qquad v_r = \dot{r}$$

$$\frac{d}{dt}(r v_\varphi) = 0$$

$$r v_\varphi = \gamma = \frac{\text{vrtilna količina}}{m}$$

$$\dot{v}_r - \frac{v_\varphi^2}{r} = \frac{Q}{r}v_z$$

$$v_z = Q^{-1}\left(r\dot{v}_r - \frac{\gamma^2}{r^2}\right)$$

$$\dot{v}_z = -Q\frac{\dot{r}}{r} \qquad \Big| \int$$

$$v_z = -Q\ln r + \text{konst} \qquad r_0 \text{ naj bo položaj kjer } v_z = 0$$

$$v_z = -Q\ln\frac{r}{r_0}$$

$$-Q\ln\frac{r}{r_0} = \frac{1}{Q}\left(r\dot{v}_r - \frac{\gamma^2}{r^2}\right)$$

$$\dot{v}_r = \frac{1}{r}\left(\frac{\gamma^2}{r^2} - Q^2\ln\frac{r}{r_0}\right) = \frac{dv_r}{dr}\frac{dr}{dt} = v_r\frac{dv_r}{dr}$$

$$\int v_r\,dv_r = \int \frac{\gamma^2}{r^3} - Q^2\frac{1}{r}\ln\frac{r}{r_0}\,dr$$

$$\frac{v_r^2}{2} + \text{konst} = -Q^2\int \ln\frac{r}{r_0}\, d\ln\frac{r}{r_0} + \int \gamma^2 r^{-3}\, dr = -\frac{1}{2}\underbrace{Q^2 \ln^2\frac{r}{r_0}}_{v_z^2} - \frac{1}{2}\underbrace{\frac{\gamma^2}{r^2}}_{v_\varphi^2}$$

$$\Rightarrow \frac{v_r^2}{2} + \frac{v_z^2}{2} + \frac{v_\varphi^2}{2} = \text{konst.}$$ Energija = konst.

Kako dobim r_{min}, r_{max}

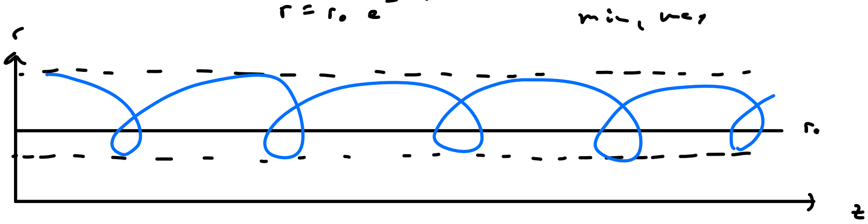
$v_r = \dot r = 0$

$$\frac{1}{2} Q^2 \ln^2\frac{r}{r_0} + \frac{\gamma^2}{2r^2} = \frac{v_0^2}{2}$$

$\gamma = 0$ ... ravninsko gibanje

$$\frac{1}{2} Q^2 \ln^2\frac{r}{r_0} = \frac{v_0^2}{2}$$

$$r = r_0\, e^{\pm Q v_0}$$ min, max



(60) Določi trajektorijo gibanja linearno dušenega elektrona v magnetnem polju.

$$m\dot{\vec v} = e\vec v \times \vec B - \eta\cdot\vec v$$
dušenje

Gre $\perp$ na $\vec B$ (2D

$$m\dot v_x = -\eta v_x + eB v_y$$
$$m\dot v_y = -\eta v_y - eB v_x$$

→ kompleksni spremenljivki
→ 2x2 matrika + lastne vrednosti
→ prehod na 1 enačbo 2. reda → karakteristični polinom

$$m\begin{bmatrix}\dot v_x\\ \dot v_y\end{bmatrix} = \begin{bmatrix}-\eta & eB\\ -eB & -\eta\end{bmatrix}\begin{bmatrix}v_x\\ v_y\end{bmatrix}$$

Nastavek $\vec v = A_1 \hat v_1 e^{\lambda_1 t} + A_2 \hat v_2 e^{\lambda_2 t}$

$$\det(\underline{A} - \lambda I) = 0$$

$$(-\eta-\lambda)^2 + (eB)^2 = 0$$
$$-\eta - \lambda = \pm eBi$$
$$\lambda = -\eta \pm ieB$$

λ_+ $$\underline{A} - \lambda I = \begin{bmatrix}-ieB & eB\\ -eB & -ieB\end{bmatrix}\vec v = 0$$

$v_+ = (1, i)$ $\qquad$ $v_- = (1, -1)$

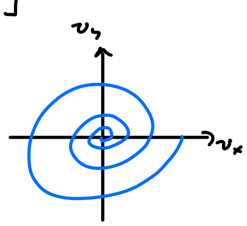
$$\vec v = \begin{bmatrix} A_+\cdot 1\, e^{-\frac{\eta t}{m}} e^{i\frac{eB}{m}t} + A_-\cdot 1\cdot e^{-\frac{\eta}{m}t} e^{-i\frac{eB}{m}t} \\ A_+\, i\, e^{-\frac{\eta t}{m}} e^{i\frac{eB}{m}t} + A_-(-i)\, e^{-\frac{\eta}{m}t} e^{-i\frac{eB}{m}t}\end{bmatrix}$$

$$A_- = A_+^* e^{i\delta} = \bar A_+ e^{i\delta}$$

$$\vec v = 2A_+ e^{-\beta t}\begin{bmatrix}\cos(\omega_c t + \delta)\\ -\sin(\omega_c t + \delta)\end{bmatrix}$$ krožnica

manjša se radij

$$\beta = \frac{\eta}{m} \qquad \omega_c = \frac{eB}{m}$$



Kompleksne spr.

$$z = v_x + i\, v_y \quad \Leftrightarrow \quad z = (1, i)\,(v_x, v_y)$$

1. en. + i·2. en. :

$$m(\dot v_x + i\,\dot v_y) = -\eta\,(v_x + i v_y) + eB\,(v_y - i v_x)$$

$$m\dot z = -\eta z - ieB z \quad \Rightarrow \quad z = z_0\, e^{(-\beta - i\omega_c)t}$$

(6.1) Določi trajektorijo / gibanje elektrona v magnetnem polju v harmonskem potencialu

$$m\ddot{\vec r} = e\dot{\vec r} \times \vec B - k\vec r$$

← vzamet

$$\ddot{\vec r} = -\underbrace{\frac{k}{m}}_{\omega_0^2}\vec r + \underbrace{\omega_c}_{\frac{eB}{m}}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\dot{\vec r}$$

Možnosti
- nastavek $\vec r = \vec r_{1,2}\, e^{\lambda_{1,2} t} \Rightarrow 2\times 2$ matrik
- 1 enač. 4 red.
- 4×4 matrika 1. red.

$$\lambda^2 \vec r_{1,2} + \omega_0^2 \vec r_{1,2} - \omega_c \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\lambda\,\vec r_{1,2} = 0$$

$$\begin{bmatrix} \lambda^2 + \omega_0^2 & -\omega_c\lambda \\ +\omega_c\lambda & \lambda^2 + \omega_0^2 \end{bmatrix}\underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\text{lastni vektorji}} = 0$$

$$\det = 0 \Rightarrow (\lambda^2 + \omega_0^2)^2 + \omega_c^2\lambda^2 = 0 \Rightarrow 4 \text{ lastne vrednosti}$$

$$\lambda = \pm\sqrt{\frac{-(2\omega_0^2 + \omega_c^2) \pm \sqrt{4\omega_0^2\omega_c^2 + \omega_c^4}}{2}}$$

Če se nam zdi da so rotacije nastavek $\lambda \to i\lambda$

Poskusimo s kompleksnimi:

$$\begin{aligned} (\lambda^2 + \omega_0^2)\,x - \omega_c\lambda y &= 0 \quad \cdot 1 \\ (\lambda^2 + \omega_0^2)\,y + \omega_c\lambda x &= 0 \quad \cdot i \end{aligned} \Big] +$$

$$(\lambda^2 + \omega_0^2)\,z + i\omega_c\lambda z = 0$$

$$(\lambda^2 + i\omega_c\lambda + \omega_0^2)\,z = 0$$

$$\lambda_{1,2} = \frac{-i\omega_c \pm i\sqrt{\omega_c^2 + 4\omega_0^2}}{2} \quad \to \text{imaginarno} \to \text{periodično gibanje}$$

→ to je le en last. vekt., kaj pa $(1, i)$

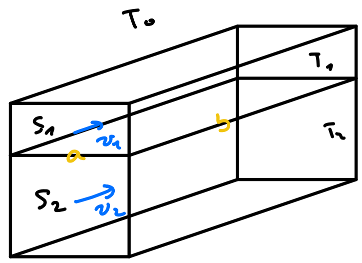
$$\begin{aligned} (\lambda^2 + \omega_0^2)\,x - \omega_c\lambda y &= 0 \quad \cdot 1 \\ (\lambda^2 + \omega_0^2)\,y + \omega_c\lambda x &= 0 \quad \cdot (-i) \end{aligned} \Big] +$$

$$(\lambda^2 + \omega_0^2)(x - iy) - i\omega_c\lambda(x - iy) = 0$$

$$(\lambda^2 + \omega_0^2 - i\omega_c\lambda)\,z^* = 0$$

$$\lambda_{3,4} = \frac{i\omega_c \pm i\sqrt{\omega_c^2 + 4\omega_0^2}}{2}$$

62 2 cevi s pravokotnih presekov, hitrosti v_1 in v_2, preseka S_1 in S_2, širina stika a dolžina stika b, ena cev prevaja še v zunanjost



Podobno kot prej, le da sta dve cevi
$T_1(x)$, $T_2(x)$
Zanemarim prevajanje po vodi

Izmenjava toplote

$$dm_1 c \left(\cancel{\frac{\partial T_1}{\partial t}} + (v_1 \nabla) T_1 \right) = - \frac{dx\, a \lambda (T_1 - T_2)}{b} - \frac{dx\, a \lambda' (T_1 - T_0)}{b}$$

0 stac. stanje

novo izhodišče

$$dm_2 c \left(\cancel{\frac{\partial T_2}{\partial t}} + (v_2 \nabla) T \right) = - \frac{dx\, a \lambda}{b} (T_2 - T_1)$$

$$\rho c S_1 v_1 \frac{\partial T_1}{\partial x} = - \frac{a\lambda}{b} (T_1 - T_2) - \frac{a \lambda'}{b} T_1$$

$$\rho c S_2 v_2 \frac{\partial T_2}{\partial x} = - \frac{a\lambda}{b} (T_2 - T_1)$$

$$\underbrace{\underbrace{\frac{\rho c b}{a}}_{Q} \begin{bmatrix} S_1 v_1 & 0 \\ 0 & S_2 v_2 \end{bmatrix}}_{\underline{M}} \begin{bmatrix} \frac{\partial T_1}{\partial x} \\ \frac{\partial T_2}{\partial x} \end{bmatrix} = \underbrace{\begin{bmatrix} -\lambda - \lambda' & \lambda \\ \lambda & -\lambda \end{bmatrix}}_{\underline{\Lambda}} \begin{bmatrix} T_1 \\ T_2 \end{bmatrix}$$

Nastavek $\begin{bmatrix} T_1 \\ T_2 \end{bmatrix} = A_1 \vec{v}_1 e^{k_1 x} + A_2 \vec{v}_2 e^{k_2 x}$

$$\underline{M} \vec{T}' = \underline{\Lambda} \vec{T}$$

$$(\underline{\Lambda} - k \underline{M}) \vec{T} = 0$$

$$\left(- \begin{bmatrix} Q S_1 v_1 k & 0 \\ 0 & Q S_2 v_2 k \end{bmatrix} + \begin{bmatrix} -\lambda - \lambda' & \lambda \\ \lambda & -\lambda \end{bmatrix} \right) \vec{T} = 0$$

$$\det \begin{bmatrix} Q S_1 v_1 k + \lambda + \lambda' & -\lambda \\ -\lambda & Q S_2 v_2 k + \lambda \end{bmatrix} = 0$$

$$(\underbrace{Q S_1 v_1}_{\Phi_1} k + \lambda + \lambda')(\underbrace{Q S_2 v_2}_{\Phi_2} k + \lambda) - \lambda^2$$

Rešimo primer $v_1 S_1 = v_2 S_2$

$$\Phi_1 \Phi_2 k^2 + (\Phi_1 \lambda + \Phi_2 (\lambda + \lambda')) k + \lambda^2 + (\lambda + \lambda') \lambda = 0$$

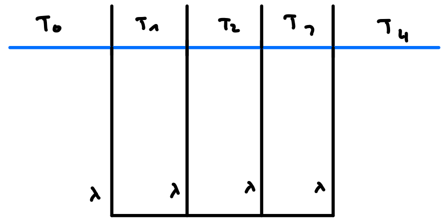
$$\Rightarrow k = \frac{1}{2 \Phi_1 \Phi_2} \left(-(\Phi_1 \lambda + \Phi_2 (\lambda + \lambda')) \pm \sqrt{(\ )^2 - 4 \Phi_1 \Phi_2 \lambda \lambda'} \right)$$

$\Rightarrow$ lastni vektorji

$\underline{M} \dot{x} = \underline{\Lambda} x$ lahko rešimo na 3 načine

1. $\det(\underline{M} k - \underline{\Lambda}) = 0$
2. $\dot{\vec{x}} = \underline{M}^{-1} \underline{\Lambda} \vec{x}$ $\quad \det(\underline{M}^{-1} \underline{\Lambda} - k I) = 0$ $\quad$ // slabost: inverz
3. $\sqrt{\underline{M}} \sqrt{\underline{M}} \dot{\vec{x}} = \underline{\Lambda} \vec{x}$ $\quad \dot{\vec{x}} = \underbrace{\sqrt{\underline{M}}^{-1} \underline{\Lambda} \sqrt{\underline{M}}^{-1}}_{\text{je simetrična}} \vec{x}$ $\quad$ // slabost $\sqrt{\underline{M}}$

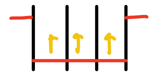
63 1. izpit 2013/7



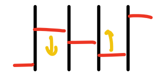
- Mešanje v predelkih je dobro
- Določi potek temperature

(a) $T_0 = 30°C = T_4$

$T_1 = T_2 = T_3 = 10°C$



(b) $T_0 = 20°$ $T_2 = 40°C$ $T_3 = 70°C$ $T_4 = 70°C$ $T_5 = 50°C$



Iščemo lastne načine

$$\rho c a \frac{\partial T_1}{\partial t} = \frac{\lambda}{\delta}\left((-T_1+T_2) + (-T_1+T_L)\right)$$
$$\rho c a \frac{\partial T_2}{\partial t} = \frac{\lambda}{\delta}\left((-T_2+T_3) + (-T_2+T_1)\right)$$
$$\rho c a \frac{\partial T_3}{\partial t} = \frac{\lambda}{\delta}\left((-T_3+T_R) + (-T_3+T_2)\right)$$

$$\tau \begin{bmatrix} \dot T_1 \\ \dot T_2 \\ \dot T_3 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} + \begin{bmatrix} T_L \\ 0 \\ T_R \end{bmatrix}$$

(a) $$\begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} = \overset{\text{part}}{\begin{bmatrix} 30°C \\ 30°C \\ 30°C \end{bmatrix}} + \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}(t)$$

(b) $$\begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} = \overset{\text{part}}{\begin{bmatrix} 20°C \\ 30°C \\ 40°C \end{bmatrix}} + \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}(t)$$

$$\tau \dot{\vec u} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix} \vec u \qquad \vec u = \vec u_0 e^{\lambda t}$$

⇒ Ugonemo lihi lastni način

$$\vec u = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \qquad \tau \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ 2 \end{pmatrix}$$ ⇒ lastna vrednost $\lambda = \frac{-2}{\tau}$

Rešitev je $A \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} e^{-\frac{2t}{\tau}}$

Sodi način $u_1 = u_3$

$$\tau \begin{pmatrix} \dot u_1 \\ \dot u_2 \\ \dot u_1 \end{pmatrix} = \begin{pmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_1 \end{pmatrix}$$

$$\tau \dot u_1 = -2u_1 + u_2$$
$$\tau \dot u_2 = 2u_1 - 2u_2$$

$$2\tau \dot u_1 = -4u_1 + 2u_2$$
$$\tau \dot u_2 = 2u_1 - 2u_2$$

$$\det\left(\begin{bmatrix} -4 & 2 \\ 2 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 2\tau & 0 \\ 0 & \tau \end{bmatrix} \right) = 0$$

$$(-4 - 2\lambda\tau)(-2 - \lambda\tau) - 4 = 0$$
$$\lambda = -\frac{1}{\tau}(2 \pm \sqrt 2)$$

$$\overset{\lambda_+}{\hat u_1 = (1, -\sqrt 2)} \qquad \overset{\lambda_-}{\hat u_2 = (1, +\sqrt 2)}$$