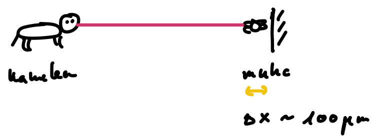
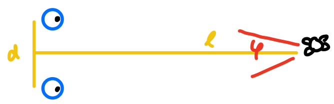
Ocenjevanje razdalje

Kameleon

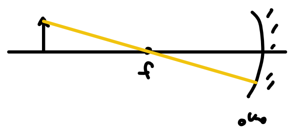


Kako
- Stereoskopski vid



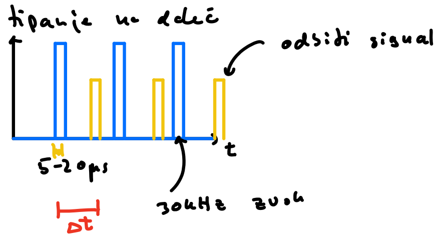
$\ell = \frac{d}{\varphi}$

- Akomodacija leče

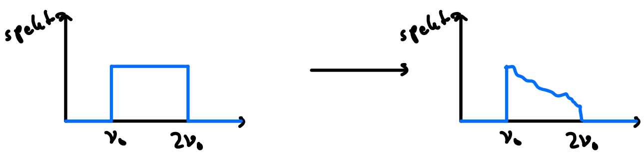


- Stereoskopsko poslušanje lege

Netopir
- aktivni sonar
- določi lahko $\vec{r}$, $\vec{v}$ in sestavo snovi
- tipanje na daleč



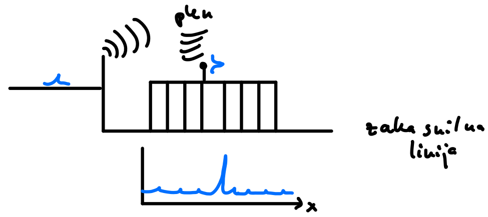
- FM sunki (sestava snovi)



- Dopplerjev zamik

$\nu = \nu_0 (1 \pm 2 \frac{v}{c})$

Meritev razdalje preko zakasnilne linije



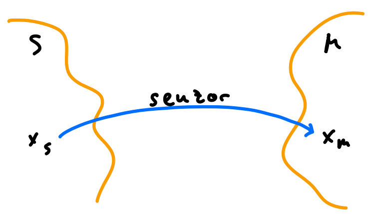
$1 = A + R$

$A = A(\nu)$

Absorbcija je zelo odvisna od ν.

Optimalno filtriranje

- Iščemo optimalen predpis za optimizacijo realnega sistema S in model skega sistema M.



Za opis dinamike uporabimo linearne DE.

Zahteve:
- šibka sklopitev S in M (meritev ne sme vplivati na S).
- x_m berljiva količina
- ocena stopnje usklajenosti

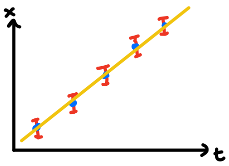
$\lim_{t \to \infty} \langle (x_m - x_s)^2 \rangle = ?$

$\langle \dots \rangle$ ensembelsko povprečje

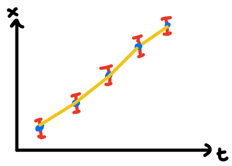
- dinamika za x_s in x_m naj bo kar se da podobna

Premo enakomerno gibanje

1D $v = \text{konst}$

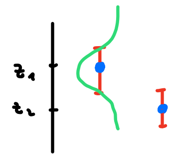


ne poznamo dinamike



Optimalno združevanje

2 ločeni opazovanji. $x \to \bar{z}_1, \sigma_1$; $\bar{z}_2, \sigma_2$

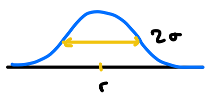


Meritev $z = x + r$

merilni šum

naključna spremenljivka → porazdeljena po Gaussovi porazdelitvi

$$\frac{dp}{dr} = \frac{1}{\sqrt{2\pi}\sigma} e^{-r^2/2\sigma^2}$$

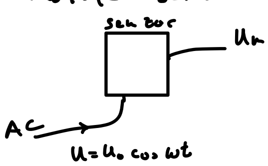


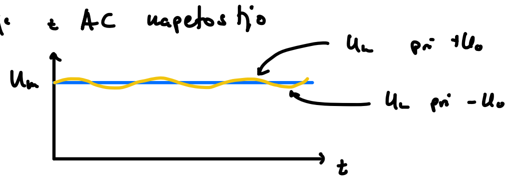
$$\langle r \rangle = \int_{-\infty}^{\infty} \frac{dp}{dr}\, r \, dr$$

$$\langle r \rangle = 0 \qquad \langle r^2 \rangle \neq 0 = \sigma^2 \quad \Rightarrow \quad \delta r = \sigma$$

Brumov šum (niso vsi šumi porazdeljeni Gaussovsko)

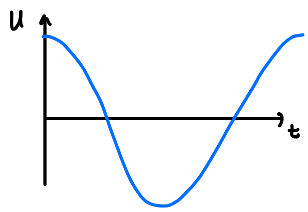
- motnje zaradi napajanja z AC napetostjo

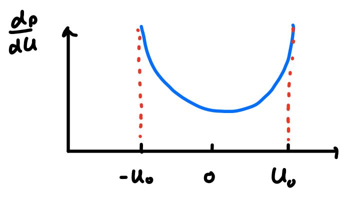




$$dU = -U_0 \omega \sin \omega t \, dt$$

$$\frac{dt}{dU} = -\frac{1}{U_0 \omega \sin \omega t}$$





$$\frac{dp}{dt} = \frac{1}{T/2}$$

$$\frac{dp}{du} \neq \text{konst}$$

$$\frac{dp}{du} = \frac{dp}{dt}\left|\frac{dt}{du}\right| = \frac{1}{T/2}\,\frac{1}{U_0 \omega \sqrt{1-\cos^2 \omega t}}$$

$$= \frac{1}{T/2\ \omega}\,\frac{1}{\sqrt{U_0^2 - U^2}} = \frac{1}{\pi \sqrt{U_0^2 - U^2}}$$

Če imamo več prispevkov je združitev (konvolucija) Gaussovska (Centralni limitni teorem)

Kako združimo meritvi z_1, z_2

Povprečevanje

$$\bar{z} = \frac{1}{N}\sum z_i \qquad \frac{dp}{dz_i} = N(x, \sigma) \qquad \overline{(z_i - x)} = \bar{r} = 0 \qquad \overline{(\bar{z} - x)} = 0$$

$$\overline{(z_i - x)^2} = \frac{1}{N^2}\overline{(\sum \underbrace{z_i - x}_{r_i})^2} = \frac{1}{N^2}\left(\overline{\sum (z_i - x)^2} + \underbrace{\overline{\sum_i \sum_j (z_i - x)(z_j - x)}}_{=0}\right) = \frac{1}{N^2} N \sigma^2 = \frac{\sigma^2}{N}$$

nekorelirani vrednosti

Naj bo $\bar{z}_1 = \frac{1}{N}\sum z_i$ $\quad \sigma_1 = \frac{\sigma}{\sqrt{N}}$ $\quad N = \frac{\sigma^2}{\sigma_1^2}$

$\bar{z}_2 = \frac{1}{M}\sum z_i$ $\quad \sigma_2 = \frac{\sigma}{\sqrt{M}}$ $\quad M = \frac{\sigma^2}{\sigma_2^2}$

N+M

$$\bar{z}_3 = \frac{1}{N+M}\sum z_i = \frac{1}{N+M}\left(\sum_1^N z_i + \sum_{N+1}^{N+M} z_i\right) =$$

$$\sigma_3^2 = \frac{\sigma^2}{M+N} = \frac{\sigma^2}{\sigma^2/\sigma_1^2 + \sigma^2/\sigma_2^2} \Rightarrow \boxed{\frac{1}{\sigma_3^2} = \frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}}$$

$$= \frac{N}{N+M}\bar{z}_1 + \frac{M}{N+M}\bar{z}_2$$

$$= \frac{\sigma_3^2}{\sigma_1^2}\bar{z}_1 + \frac{\sigma_3^2}{\sigma_2^2}\bar{z}_2$$

$$\boxed{\bar{z}_3 = \frac{\sigma_2^2}{\sigma_1^2+\sigma_2^2}\bar{z}_1 + \frac{\sigma_1^2}{\sigma_1^2+\sigma_2^2}\bar{z}_2} = \underbrace{\overset{\text{stari izmerek}}{\bar{z}_1}}_{\text{rekurzivno, online združevanje podatkov}} + \underbrace{\frac{\sigma_1^2}{\sigma_1^2+\sigma_2^2}\overbrace{(\bar{z}_2 - \bar{z}_1)}^{\text{inovacija}}}_{\text{popravek starega izmerka}}$$

Kvadratna forma

$$2J(x) = \frac{(\bar{z}_1 - x)^2}{\sigma_1^2} + \frac{(\bar{z}_2 - x)^2}{\sigma_2^2}$$

sami nekorelirani prispevki

$\bar{z}_1 - x$ .... porazdeljeno po $N(0, \sigma_1)$

$\bar{z}_2 - x$ .... porazdeljeno po $N(0, \sigma_2)$

$\frac{\bar{z}_1 - x}{\sigma_1}$ ... porazdeljeno po $N(0, 1)$

Zahtevam $\frac{d}{dx} 2J(x) = 0$

$$\frac{-2(z_1 - x)}{\sigma_1^2} - \frac{2(z_2 - x)}{\sigma_2^2} = 0$$

$$x\left(\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}\right) = \frac{z_1}{\sigma_1^2} + \frac{z_2}{\sigma_2^2}$$

Optimalen x : $\underset{z_3}{\overset{\shortparallel}{x}} = \frac{\frac{z_1}{\sigma_1^2} + \frac{z_2}{\sigma_2^2}}{\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}} = \sigma_3^2\left(\frac{z_1}{\sigma_1^2} + \frac{z_2}{\sigma_2^2}\right)$ ✓✓

Ali ima tako združevanje min σ_3^2, ali je to res optimalno združevanje

(z_1, σ_1), (z_2, σ_2)

Naj bo $z_3 = \alpha z_1 + \beta z_2 = x + r$

$$\alpha(x + r_1) + \beta(x + r_2) = x + r$$

$$\alpha + \beta = 1 \qquad \alpha = 1 - \beta$$

$$\alpha r_1 + \beta r_2 = r$$

$$\alpha r_1 + (1-\alpha) r_2 = r$$

$$\langle r \rangle = \alpha\langle r_1\rangle + (1-\alpha)\langle r_2\rangle \qquad 0 = 0 \ ✓$$

$$\langle r^2 \rangle = \alpha^2\langle r_1^2\rangle + (1-\alpha)^2\langle r_2^2\rangle + 2\alpha(1-\alpha)\langle r_1, r_2\rangle$$

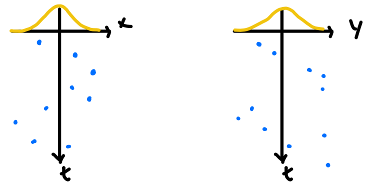
$\langle r_1, r_2\rangle = 0$ če nista korelirani

$$\sigma^2 = \alpha^2\sigma_1^2 + (1-\alpha)^2\sigma_2^2$$

Optimalno $\frac{d\sigma^2}{d\alpha} = 0 = 2\alpha\sigma_1^2 - 2(1-\alpha)\sigma_2^2 \Rightarrow \frac{\alpha}{1-\alpha} = \frac{\sigma_2^2}{\sigma_1^2} \Rightarrow \alpha = \frac{\sigma_2^2}{\sigma_1^2+\sigma_2^2}$ $\quad \beta = \frac{\sigma_1^2}{\sigma_1^2+\sigma_2^2}$ ✓

Korelacija med izmerki

Imejmo dve seti meritev x, y



$\overline{r_x} = \overline{r_y} = 0$

$\overline{r_x^2} = \sigma_x^2$

$\overline{r_y^2} = \sigma_y^2$

Def: kovarianca $\sigma_{xy} = \overline{(x-\bar{x})(y-\bar{y})}$

$= \rho\, \sigma_x \sigma_y$ $\quad \rho$ ... korelacijski koeficient

$-1 \le \rho \le 1$

$\overline{r_x r_y} \neq 0$

obstaja korelacija/povezava med šumom x in y meritve

$$\sigma_{xy} = \frac{1}{N}\sum_i (x_i - \bar{x})(y_i - \bar{y}) = \frac{1}{N}\sum_i \left(x_i y_i - \bar{x} y_i - x_i \bar{y} + \bar{x}\bar{y}\right)$$

$$= \frac{1}{N}\sum_i x_i y_i - \frac{\bar{x}}{N}\sum_i y_i - \frac{\bar{y}}{N}\sum_i x_i + \bar{x}\bar{y} =$$

$$\boxed{\sigma_{xy} = \overline{xy} - \bar{x}\,\bar{y}} \qquad \rho = \frac{\sigma_{xy}}{\sigma_x \sigma_y}$$

Združevanje koreliranih ocen

šum (prej imenovan r)

$\left.\begin{matrix}(z_1, \sigma_1)\\ (z_2, \sigma_2)\end{matrix}\right\} \rho$ $\qquad z_1 = x + w_1 \quad \langle w_1^2\rangle = \sigma_1^2$

$\qquad z_2 = x + w_2 \quad \langle w_2^2 \rangle = \sigma_2^2 \qquad \langle w_1 w_2 \rangle = \rho\, \sigma_1 \sigma_2$

Šum ene meritve zapišemo kot kombinacijo šuma druge meritve in neodvisnega dela.

$$w_1 = \alpha w_2 + \underset{\text{neodvisen del}}{w} \qquad \langle w^2 \rangle = \sigma_w^2 \qquad \langle w \rangle = 0 \qquad \langle w w_2 \rangle = 0$$

$$\rho\, \sigma_1 \sigma_2 = \langle w_1 w_2 \rangle = \langle (\alpha w_2 + w) w_2 \rangle = \alpha \langle w_2^2 \rangle + \langle w w_2 \rangle = \alpha\, \sigma_2^2$$

$$\boxed{\alpha = \rho \frac{\sigma_1}{\sigma_2}}$$

$$\sigma_1^2 = \langle w_1^2 \rangle = \langle (\alpha w_2 + w)^2 \rangle = \alpha^2 \langle w_2^2 \rangle + 2\alpha \langle w w_2 \rangle + \langle w^2 \rangle = \alpha^2 \sigma_2^2 + \sigma_w^2$$

$$\boxed{\sigma_1^2 = \rho^2 \sigma_1^2 + \sigma_w^2} \qquad \sigma_1^2 (1 - \rho^2) = \sigma_w^2$$

Kvadratna forma

$$2J(x) = \left(\frac{w_2}{\sigma_2}\right)^2 + \left(\frac{w}{\sigma_w}\right)^2 = \frac{w_2^2}{\sigma_2^2} + \frac{(w_1 - \alpha w_2)^2}{\sigma_1^2 (1-\rho^2)} = \frac{w_2^2}{\sigma_2^2} + \frac{w_1^2 - 2\alpha w_1 w_2 + \alpha^2 w_2^2}{\sigma_1^2 (1 - \rho^2)} =$$

$$= \frac{w_2^2}{\sigma_2^2} + \frac{\rho^2 \frac{\sigma_1^2}{\sigma_2^2} w_2^2 - 2\rho \frac{\sigma_1}{\sigma_2} w_1 w_2 + w_1^2}{\sigma_1^2 (1-\rho^2)}$$

$$= \frac{w_2^2}{\sigma_2^2}\left(1 + \frac{\rho^2}{1-\rho^2}\right) + \frac{w_1^2}{\sigma_1^2} \frac{1}{1-\rho^2} - \frac{2\rho\, w_1 w_2}{(1-\rho^2)\,\sigma_1 \sigma_2}$$

$$\boxed{2J(x) = \left(\frac{w_2^2}{\sigma_2^2} + \frac{w_1^2}{\sigma_1^2} - \frac{2\rho\, w_1 w_2}{\sigma_1 \sigma_2}\right) \frac{1}{1-\rho^2}}$$

$w_1 = z_1 - x$

$w_2 = z_2 - x$

Za optimalno združevanje $\frac{dJ}{dx} = 0$

$$\frac{dJ}{dx} = \frac{-2(z_2-x)}{\sigma_2^2} + \frac{-2(z_1-x)}{\sigma_1^2} - \frac{2\varrho}{\sigma_1\sigma_2}\left(2x-(z_1+z_2)\right) = 0$$

$$\frac{z_2}{\sigma_2^2}+\frac{z_1}{\sigma_1^2} - \frac{\varrho}{\sigma_1\sigma_2}(z_1+z_2) = x\left(\frac{1}{\sigma_2^2}+\frac{1}{\sigma_1^2}-\frac{2\varrho}{\sigma_1\sigma_2}\right)$$

Optimalna skupna ocena $x = \boxed{z_3 = \underbrace{\left(\frac{1}{\sigma_2^2}+\frac{1}{\sigma_1^2}-\frac{2\varrho}{\sigma_1\sigma_2}\right)^{-1}}\left(\frac{z_2}{\sigma_2^2}+\frac{z_1}{\sigma_1^2}-\frac{\varrho}{\sigma_1\sigma_2}(z_1+z_2)\right)}$

$$\hookrightarrow \quad \sigma_3^2 = \left(\frac{1}{\sigma_1^2}+\frac{1}{\sigma_2^2}-\frac{2\varrho}{\sigma_1\sigma_2}\right)^{-1}(1-\varrho^2)$$

Skupna disperzija

Mejni primeri:

1. $\varrho = 0 \qquad z_3 = \left(\frac{1}{\sigma_1^2}+\frac{1}{\sigma_2^2}\right)^{-1}\left(\frac{z_1}{\sigma_1^2}+\frac{z_2}{\sigma_2^2}\right)$ ✓

2. $\varrho = 1 \qquad z_3 = \left(\frac{\sigma_2^2+\sigma_1^2-2\sigma_1\sigma_2}{\sigma_1^2\sigma_2^2}\right)^{-1}\left(\frac{z_1\sigma_2^2+z_2\sigma_1^2-(z_1+z_2)\sigma_1\sigma_2}{\sigma_1^2\sigma_2^2}\right)$

Popolna korelacija $z_1 = z_2$

$$= \frac{z_1\sigma_2(\sigma_2-\sigma_1)-z_2\sigma_1(\sigma_2-\sigma_1)}{(\sigma_2-\sigma_1)^2} = \frac{z_1\sigma_2-z_2\sigma_1}{\sigma_2-\sigma_1} = z_1 = z_2$$

Če dve isti meritvi združimo, dobimo enako oceno.

3. $\sigma_1=\sigma_2=\sigma \qquad z_3 = (2-2\varrho)^{-1}\left(z_2+z_1-\varrho(z_1+z_2)\right) = \frac{z_1+z_2}{2}$

Merjenje konstantne skalarne količine

Dotok meritev $z_i = x + r_i$, r_i – merilni šum. Merilni šum je nekoreliran, torej $\langle r_i r_j\rangle = \delta_{ij}\sigma^2$, i,j ... šumi so različnih časih. Šum v vsakem
↑ idealizacija
trenutku je nepovezan s šumom v prejšnjih korakih.

Shema za sledenje

os časa $n\cdot T$: $\hat x_n = \frac{1}{n}\sum z_i$

$\hat\sigma_n^2 = \frac{\sigma^2}{n}$ (če so $\sigma_i = \sigma$ in $\varrho = 0$)

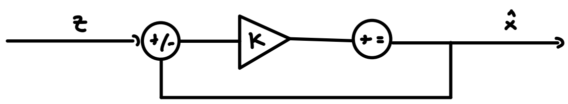
os časa $(n+1)T$: $\hat x_{n+1} = \frac{1}{n+1}\sum^{n+1} z_i = \frac{1}{n+1}\left(\sum^n z_i + z_{n+1}\right) =$

$$= \frac{n}{n+1}\hat x_n + \frac{1}{n+1}z_{n+1} = \hat x_n + \frac{1}{n+1}(z_{n+1}-\hat x_n)$$

$$\Rightarrow \hat x_{n+1} = \hat x_n + \frac{1}{n+1}(z_{n+1}-\hat x_n)$$

$$\hat\sigma_{n+1}^{-2} = \hat\sigma_n^{-2} + \sigma^{-2}$$

$$\hat x_{n+1} = \hat x_n + \frac{\hat\sigma_{n+1}^2}{\sigma^2}(z_{n+1}-\hat x_n)$$



Shema sledenja konstanti

$K(t_i) = \frac{\hat\sigma_{n+1}^2}{\sigma^2}$

Ocena konvergence $\hat x \to x$

čas vzorčenja $T \to 0$

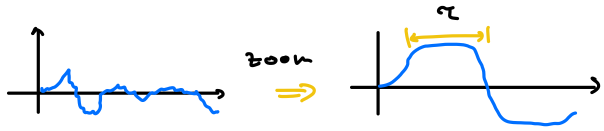
$\lim T\to 0$	
$\hat x_n$	$\hat x(t)$
$\hat\sigma_n^2$	$\hat\sigma_x^2(t)$
z_n	$z(t)$

$$\lim_{T\to 0}\frac{\hat x_{n+1}-\hat x_n}{T} = \dot{\hat x}(t) = \frac{\hat\sigma_{n+1}^2}{\sigma^2 T}(z_n-\hat x_n)$$

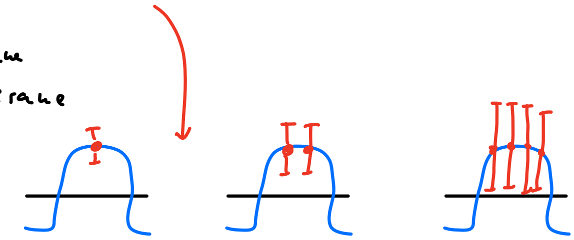
$$\dot{\hat x}(t) = \frac{\hat\sigma_x^2}{\sigma^2 T}\left(z(t)-\hat x(t)\right) = \frac{\hat\sigma_x^2}{R(t)}\left(z(t)-\hat x(t)\right) + \dots$$

naj bo $\lim \sigma^2 T = R(t) < \infty$

Šum $r(t)$



τ ... čas minimalne fluktuacije

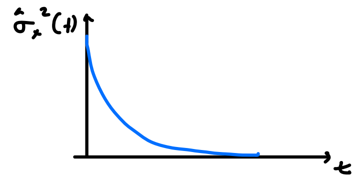


$T \gg \tau$ meritve nekorelirane

$T \ll \tau$ meritve so korelirane

Disperzija združenih meritev v kontinuumski sliki

$$\frac{\hat\sigma_{n+1}^2-\hat\sigma_n^2}{T} = \frac{1}{T}\left(\frac{\hat\sigma_n^2\sigma^2}{\sigma^2+\hat\sigma_n^2} - \frac{\hat\sigma_n^2(\sigma^2+\sigma_i^2)}{\sigma^2+\hat\sigma_n^2}\right) = -\frac{\hat\sigma_n^4}{(\hat\sigma_n^2 T - \sigma^2 T)} \quad \xrightarrow{\lim_{T\to 0}} \quad \dot{\hat\sigma}_x^2 = -\frac{(\hat\sigma_x^2)^2}{R}$$

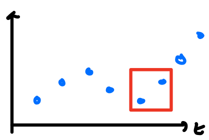


Merjenje skalarne spremenljivke

① Ne poznamo dinamike za $x(t)$;

$\dot x(t) = z(t)$ $\quad \dot{\hat\sigma}_x^2 = r(t)$ $\quad$ pozabljamo prejšnje meritve

② Nekaj vemo o dinamiki $x(t)$



③ Kako opišemo dinamiko?

z LDE prvega reda $\dot x(t) = A x(t) + c(t)$ linearna diferencialna enačba

diskretno $\frac{x_{n+1}-x_n}{T} = A x_n + c_n \Rightarrow x_{n+1} = \underbrace{(1+AT)}_{\phi_n} x_n + \underbrace{c_n T}_{c_n}$

Postopek optimalne sinhronizacije

ob nT: $\hat x_n$, $\hat\sigma_n^2$

napovedna ocena $\quad \boxed{\bar x_{n+1} = \phi_n \hat x_n + c_n}$

napoved disperzije $\quad \boxed{\bar\sigma_{n+1}^2 = \phi_n^2 \hat\sigma_n^2}$

$$\langle(\bar x_{n+1} - x_{n+1})^2\rangle = \langle(\phi_n\hat x_n + c_n - \phi_n x_n - c_n)^2\rangle = \phi_n^2\langle(\hat x_n - x_n)^2\rangle = \phi_n^2\hat\sigma_n^2$$

ob $(n+1)T$ nov izmerek z_{n+1}, σ

izostrena ocena (združimo meritev in napoved):

$$\hat x_{n+1} = \bar x_{n+1} + \frac{\hat\sigma_{n+1}^2}{\sigma^2}(z_{n+1} - \bar x_{n+1})$$

$$\hat\sigma_{n+1}^{-2} = \bar\sigma_{n+1}^{-2} + \sigma^{-2}$$

Nove oznake

$\hat\sigma_{n+1}^2 \longrightarrow P_{n+1}$ Kovarianca izostrene ocene (kovariančna matrika ocene)

$\bar\sigma_{n+1}^2 \longrightarrow M_{n+1}$ Kovarianca napovedi (kov. mat. napovedi)

$K_{n+1} = \frac{\hat\sigma_{n+1}^2}{\sigma^2} = \frac{P_{n+1}}{\sigma^2}$ Ojačevalni faktor

$$M_{n+1} = \phi_n^2 P_n$$

$$P_{n+1} = \frac{M_{n+1}\sigma^2}{M_{n+1}+\sigma^2} = M_{n+1} - \frac{M_{n+1}^2}{M_{n+1}+\sigma^2}$$

Dinamični šum

Φ_n in c_n v diferenčni sliki ne popisuje popolno
+ (...)

$$x_{n+1} = \Phi_n x_n + c_n + \Gamma_n w_n$$

w_n obravnavajmo kot šum $\langle w_n w_{n'} \rangle = \delta_{nn'} Q_n$
nekoreliran (bel) šum

V vezilnem sistemu
$\hat{x}_{n+1} = \Phi_n \hat{x}_n + c_n$ (ni vpliva šuma)
vpliva pa na kovarianco

$$M_{n+1} = \langle (\bar{x}_{n+1} - x_{n+1})^2 \rangle = \langle (\Phi_n \hat{x}_n + c_n - \Phi_n x_n - c_n - \Gamma_n w_n)^2 \rangle =$$
$$= \langle (\Phi_n (\hat{x}_n - x_n) - \Gamma_n w_n)^2 \rangle = \Phi_n^2 P_n + \Gamma_n^2 \langle w_n^2 \rangle - 2\Gamma_n \Phi_n \underbrace{\langle (\hat{x}_n - x_n) w_n \rangle}_{=0 \text{ nekorelirani}}$$
$$= \Phi_n^2 P_n + \Gamma_n^2 Q_n$$

kovarianca naraša

Prehod v kontinuirano sliko

v sistemu M: $\hat{x}_{n+1} = \bar{x}_{n+1} + K_{n+1}(z_{n+1} - \bar{x}_{n+1})$

$\uparrow$

$\Phi_n \hat{x}_n + c_n$

$$\lim \frac{\hat{x}_{n+1} - \hat{x}_n}{T} = \frac{(\Phi_n - 1)}{T} \hat{x}_n + \frac{c_n}{T} + \frac{P_{n+1}}{T\sigma^2}(z_{n+1} - \bar{x}_{n+1})$$

$\Phi_n = 1 + A(nT)T$

$c_n = C(nT)T$

$$\dot{\hat{x}}(t) = \underbrace{A(t)\hat{x}_n + C(t)}_{\text{dinamika}} + \frac{P(t)}{R(t)} \underbrace{(z(t) - \hat{x}(t))}_{\text{inovacija}}$$

$$\underset{\text{izostrena}}{P_{n+1}} = \underset{\text{napovedna}}{M_{n+1}} - \frac{M_{n+1}^2}{M_{n+1} + \sigma^2}$$

$$= \Phi_n^2 P_n + \Gamma_n^2 Q_n - \frac{(\Phi_n^2 P_n + \Gamma_n^2 Q_n)^2}{M_{n+1} + \sigma^2}$$

$\lim T \to 0$

$$\frac{P_{n+1} - P_n}{T} = \frac{(\Phi_n + 1)(\Phi_n - 1)}{T} P_n + \frac{\Gamma_n^2 Q_n}{T} - \frac{\Phi_n^4 P_n^2 + \overbrace{\Gamma_n^4 Q_n^2}^{\to 0} + \overbrace{2\Gamma_n^2 Q_n \Phi_n^2 P_n}^{\to 0}}{\underbrace{M_{n+1} T}_{\to 0} + \sigma^2 T}$$

$\to A$

$$\lim \frac{\Gamma_n^2 Q_n T}{T^2} \qquad \lim Q_n T \to Q(t) \qquad \lim \frac{\Gamma_n^2}{T^2} \to \Gamma^2(t)$$

$$\dot{P}(t) = 2A(t)P(t) + \Gamma^2(t) Q(t) - \frac{P^2(t)}{R(t)}$$

$$\dot{P} = \underbrace{2AP}_{\text{dinamika}} + \underbrace{\Gamma^2 Q}_{\text{povečevanje}} - \underbrace{\frac{P^2}{R}}_{\text{zmanjševanje / ostrenje } P}$$

povečevanje P zaradi din. šuma

zmanjševanje / ostrenje P zaradi dotoka novih meritev

$$\lim \frac{\Gamma_n}{T} \to \Gamma(t)$$
$$\lim Q_n T \to Q(t)$$

Vektorske spremenljivke

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$\hat{\vec{x}}$ ocena k vektorju $\vec{x}$

Disperzije $\langle (\hat{x}_i - x_i)^2 \rangle = \sigma_i^2$

$\langle (\hat{x}_i - x_i)(\hat{x}_j - x_j) \rangle = \sigma_{ij}$

Zgled: $\bar{x} = x + m_x$ šum $\qquad x(t+T) = x(t) + vT$

$\bar{v} = v + m_v$

Ostrenje vektorske spremenljivke

$z = x + r \qquad$ r... merilni šum $\qquad \langle r^2 \rangle = \sigma^2$

(merimo samo lego) $\qquad \langle m_x r \rangle = 0$

$\langle m_x m_v \rangle \neq 0$

$$\begin{aligned}
\hat{x} = x + \hat{p}_x &= a_{xx}\bar{x} + a_{xv}\bar{v} + b_x z \\
&= a_{xx}(x + m_x) + a_{xv}(v + m_v) + b_x(x + r) \\
&= x(a_{xx} + \underbrace{a_{xv}}_{=0 \text{ px ni odv. od } v} + b_x) + a_{xx}m_x + a_{xv}m_v + b_x r
\end{aligned}$$

$\Rightarrow a_{xx} + b_x = 1 \qquad \hat{p}_x = a_{xx}m_x + b_x r = a_{xx}m_x + (1 - a_{xx})r$

$$\langle p_x^2 \rangle = a_{xx}^2 \underbrace{\langle m_x^2 \rangle}_{\sigma_x^2} + (1-a_{xx})^2 \underbrace{\langle r^2 \rangle}_{\sigma_r^2} + 2a_{xx}(1-a_{xx}) \underbrace{\langle m_x r \rangle}_{=0}$$

$$\frac{d\langle p_x^2 \rangle}{da_{xx}} = 2a_{xx}\sigma_x^2 - 2(1-a_{xx})\sigma_r^2 = 0 \quad \Rightarrow a_{xx} = \frac{\sigma_r^2}{\sigma_x^2 + \sigma_r^2}$$

$$\Rightarrow b_x = \frac{\sigma_x^2}{\sigma_x^2 + \sigma_r^2}$$

$$\begin{aligned}
\hat{v} = v + \hat{p}_v &= a_{vx}\bar{x} + a_{vv}\bar{v} + b_v z = \\
&= a_{vx}(x + m_x) + a_{vv}(v + m_v) + b_v(x + r) \\
&= \underbrace{a_{vv}}_{=1} v + x\underbrace{(a_{vx} + b_v)}_{=0} \underbrace{+ a_{vx}m_x + a_{vv}m_v + b_v r}_{\hat{p}_v}
\end{aligned}$$

$$\begin{aligned}
\langle \hat{p}_v^2 \rangle &= a_{vx}a_{vv}\langle m_x m_v \rangle + a_{vx}^2\langle m_x^2 \rangle + \langle m_v^2 \rangle + b_v^2\langle r^2 \rangle \\
&= 2a_{vx}\langle m_x m_v \rangle + a_{vx}^2\sigma_x^2 + \sigma_v^2 + a_{vx}^2\sigma_r^2
\end{aligned}$$

$$\frac{d\langle \hat{p}^2 \rangle}{da_{vx}} = 2\langle m_x m_v \rangle + 2a_{vx}(\sigma_x^2 + \sigma_r^2) = 0 \qquad a_{vx} = \frac{-\langle m_x m_v \rangle}{(\sigma_x^2 + \sigma_r^2)}$$

$$\hat{x} = \bar{x} + \frac{\sigma_x^2}{\sigma_x^2 + \sigma_r^2}(z - \bar{x})$$

optimalno

$$\hat{v} = \bar{v} + \frac{\langle m_x m_v \rangle}{\sigma_x^2 + \sigma_r^2}(z - \bar{x})$$

Kovariančna matrika

$M = \langle (\bar{x} - x)(\bar{x} - x)^T \rangle \qquad \bar{x} = \vec{\bar{x}} \qquad x = \vec{x}$

$$M = \begin{bmatrix} \sigma_1^2 & \sigma_{12} & \cdots \\ \sigma_{21} & \sigma_2^2 & \\ \vdots & & \sigma_3^2 \\ & & & \ddots \end{bmatrix}$$

kovariančna matrika napovedi

P... kovariančna matrika izostrene ocene

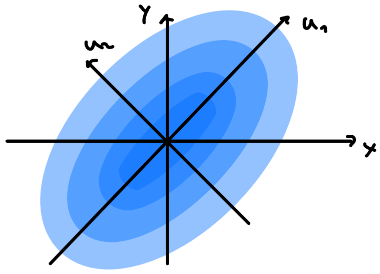
Za verjetnostno gostoto vzamemo večrazsežno Gaussovo porazdelitev

$$p(\vec{x}) = \frac{1}{\sqrt{(2\pi)^n}} \frac{1}{\det M} \exp\left(-\frac{1}{2}(\bar{x}-x)^T M^{-1}(\bar{x}-x)\right)$$

vektor

Primer

$$p(\bar{x},\bar{y}) = \frac{1}{2\pi}\frac{1}{\sigma_x\sigma_y\sqrt{1-\varphi^2}} \exp\left(\frac{(\bar{x}-x)^2}{\sigma_x^2} + \frac{(\bar{y}-y)^2}{\sigma_y^2} - \frac{2\varphi(\bar{x}-x)(\bar{y}-y)}{\sigma_x\sigma_y}\right)$$



$\bar{x},\bar{y} \to u_1, u_2$ $\qquad \vec{u} = \Omega\,\vec{x}$

ortogonalna transformacija

M_u ... diagonalna

$$M_u = \Omega M \Omega^T$$

$$p(\bar{u}) = \prod_i \frac{1}{\sqrt{2\pi}}\frac{1}{\sigma_i}\exp\left(-\frac{1}{2}\frac{(\bar{u}_i-u_i)^2}{\sigma_i^2}\right)$$

Merjenje več spremenljivk

$$\begin{bmatrix} z_1 \\ \vdots \\ z_s \end{bmatrix} = \vec{z} \xrightarrow[\text{okno}]{\text{matrika senzorjev } H} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \vec{x}$$

$$\bar{z} = H\vec{x} + \vec{r}$$

meritni šum $\qquad R = \langle \vec{r}\vec{r}^T \rangle$ kovariančna matrika senzorskega šuma

Primer H: $\quad \vec{x} = \begin{bmatrix} x \\ v \end{bmatrix}$

$$x = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ v \end{bmatrix}$$

$$v = \begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ v \end{bmatrix}$$

$$\begin{bmatrix} x \\ v \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ v \end{bmatrix}$$

$\begin{bmatrix} \alpha & \beta \\ \nu & 1 \end{bmatrix}$ $\qquad$ s(e) senzor, ker meri x in v hkrati

Ostenje vektor skih meritev

$$\left.\begin{array}{l} s \\ \vec{z} \\ \vec{z} = H\vec{x} + \vec{r} \end{array}\right\} \longrightarrow \left\{\begin{array}{l} M \\ \bar{x} \\ \hat{x} \end{array}\right.$$

$R = \langle \vec{r}\vec{r}^T \rangle$

$M = \langle (\bar{x}-x)(\bar{x}-x)^T \rangle = \langle m\, m^T \rangle \qquad \bar{x} = x + m$

$P = \langle (\hat{x}-x)(\hat{x}-x)^T \rangle = \langle \hat{p}\hat{p}^T \rangle \qquad \hat{x} = x + \hat{p}$

Sestavimo vektorske signale $\qquad \hat{x} = A\bar{x} + Bz \qquad$ A, B matriki kateri komp. predstavljajo uteži

$$\hat{x} = x + \hat{p}$$

$$x + \hat{p} = A(x+m) + B(Hx + r)$$

$$= \underbrace{(A + BH)}_{I} x + \underbrace{Am + Br}_{\hat{p}}$$

$A + BH = I$ vez

$$\langle \hat{p}\hat{p}^T \rangle = \langle (Au + Br)(u^T A^T + r^T B^T) \rangle$$
$$= A \underbrace{\langle u u^T \rangle}_{M} A^T + B \underbrace{\langle r r^T \rangle}_{R} B^T + A \underbrace{\langle u r^T \rangle}_{=0} B^T + B \underbrace{\langle r u^T \rangle}_{=0} A^T$$

$$\boxed{P = AMA^T + BRB^T}$$

P minimiziramo glede na A, B ob pogoju $A + BH - I = 0$

$$\tilde{P} = AMA^T + BRB^T - (A + BH - I)\lambda$$

Lagrangeov multiplikator

minimiziramo $\operatorname{tr} \tilde{P}$

$$\frac{d \operatorname{tr} \tilde{P}}{dA} = 2AM - \lambda^T = 0 \qquad A = \frac{\lambda^T}{2} M^{-1} \qquad A^T = M^{-1} \frac{\lambda}{2}$$

$$\frac{d \operatorname{tr} \tilde{P}}{dB} = 2BR - (H\lambda)^T = 0 \qquad B = \frac{\lambda^T}{2} H^T R^{-1} \qquad B^T = R^{-1} \frac{H\lambda}{2}$$

$$\frac{d \operatorname{tr} \tilde{P}}{d\lambda} = (A + BH - I)^T = 0$$

$$A^T + H^T B^T - I = 0$$
$$M^{-1} \frac{\lambda}{2} + H^T R^{-1} H \frac{\lambda}{2} - I = 0$$
$$(M^{-1} + H^T R^{-1} H) \frac{\lambda}{2} = I$$
$$\frac{\lambda}{2} = (M^{-1} + H^T R^{-1} H)^{-1}$$

$$\Rightarrow P = AMA^T + BRB^T$$
$$= \frac{\lambda^T}{2} M^{-1} M M^{-1} \frac{\lambda}{2} + \frac{\lambda^T}{2} H^T R^{-1} R R^{-1} H \frac{\lambda}{2}$$
$$= \frac{\lambda^T}{2} \underbrace{[M^{-1} + H^T R^{-1} H]}_{(\frac{\lambda}{2})^{-1}} \frac{\lambda}{2} = \frac{\lambda^T}{2} = \frac{\lambda}{2}$$

$$\Rightarrow \boxed{P = (M^{-1} + H^T R^{-1} H)^{-1}}$$ ostrenje

$$P P^{-1} = I = P M^{-1} + P H^T R^{-1} H \qquad / P\cdot \qquad / \cdot M$$
$$M = P + P H^T R^{-1} H M$$
$$M = P (I + H^T R^{-1} H M) \qquad / \cdot H^T$$
$$MH^T = P (H^T + H^T R^{-1} H M H^T)$$
$$MH^T = PH^T (\underbrace{I}_{R^{-1}R} + R^{-1} H M H^T)$$
$$MH^T = PH^T R^{-1} (R + HMH^T)$$
$$PH^T R^{-1} = MH^T (R + HMH^T)^{-1}$$
$$P = M - PH^T R^{-1} HM$$
$$P = M - \underbrace{MH^T (R + HMH^T)^{-1} HM}_{\text{ostrenje}}$$

$$\hat{x} = A\tilde{x} + Bz$$
$$= \frac{\lambda^T}{2} M^{-1} \tilde{x} + \frac{\lambda^T}{2} H^T R^{-1} z$$
$$= \frac{\lambda^T}{2} (M^{-1} \tilde{x} + H^T R^{-1} z + H^T R^{-1} H \tilde{x} - H^T R^{-1} H \tilde{x})$$
$$= \frac{\lambda^T}{2} (\underbrace{(M^{-1} + H^T R^{-1} H)}_{(\frac{\lambda}{2})^{-1}} \tilde{x} + H^T R^{-1} (z - H\tilde{x}))$$

$$\boxed{\hat{x} = \tilde{x} + \underbrace{PH^T R^{-1}}_{\text{utež}} \underbrace{(z - H\tilde{x})}_{\text{inovacija}}}$$

Vključimo dinamiko in dinamični šum

Diskretno $nT \longrightarrow (n+1)T$

M ; $\bar{x}_{n+1} = \Phi_n \hat{x}_n + c_n$

S ; $x_{n+1} = \Phi_n x_n + c_n + \Gamma_n w_n$

$$M_{n+1} = \langle (\bar{x}_{n+1} - x_{n+1})(\bar{x}_{n+1} - x_{n+1})^T \rangle$$
$$= \langle (\Phi_n \hat{x}_n + c_n - \Phi_n x_n - c_n - \Gamma_n w_n)(\Phi_n(\hat{x}_n - x_n) - \Gamma_n w_n)^T \rangle =$$
$$= \langle (\Phi_n(\hat{x}_n - x_n) - \Gamma_n w_n)((\hat{x}_n - x_n)^T \Phi_n^T - w_n^T \Gamma_n^T) \rangle =$$
$$= \Phi_n \langle (\hat{x}_n - x_n)(\hat{x}_n - x_n) \rangle \Phi_n^T + \Gamma_n \langle w_n w_n^T \rangle \Gamma_n^T$$
$$+ \Phi_n \underbrace{\langle (\hat{x}_n - x_n) v_n^T \rangle}_{=0} \Gamma_n^T + 0$$

$$\boxed{M_{n+1} = \Phi_n P_n \Phi_n^T + \Gamma_n Q_n \Gamma_n^T}$$ kovariančna matrika napovedi

$$P_{n+1}^{-1} = M_{n+1}^{-1} + H^T R^{-1} H$$

Zvezno

- dinamični šum w
- merilni šum r

nekorelirana

	$\lim T \to 0$
$\hat{x}_{n+1}$	$\hat{x}(t)$
$\bar{x}_{n+1}$	$\hat{x}(t)$
$T R_n$	$R(t)$
z_n	$z(t)$
P_n	$P(t)$
M_n	$P(t)$

$$\dot{\hat{x}} = \frac{\hat{x}_{n+1} - \hat{x}_n}{T} = \frac{\Phi \hat{x}_n + c_n - \hat{x}_n}{T} + \frac{K_{n+1}(z_{n+1} - H\bar{x}_{n+1})}{T}$$
$$= \frac{(\Phi_n - I)\hat{x}_n}{T} + \frac{c_n}{T} + \frac{P_{n+1} H^T R_{n+1}^{-1}}{T}(z_{n+1} - H \bar{x}_{n+1})$$

$$\Phi_n = I + A T$$

$$\dot{\hat{x}} = A(t)\hat{x}(t) + c(t) + P H^T R^{-1}(t)\left(z(t) - H\hat{x}(t)\right)$$

$$P_{n+1} - P_n = \Phi_n P_n \Phi_n^T + \Gamma_n Q_n \Gamma_n^T - M_{n+1} H^T (H M_{n+1} H^T + R_{n+1})^{-1} H M_{n+1} - P_n$$

$$= (I + AT) P_n (I + AT)^T + \Gamma_n Q_n \Gamma_n^T - M_{n+1} H^T (H M_{n+1} H^T + R_{n+1})^{-1} H M_{n+1} - P_n$$

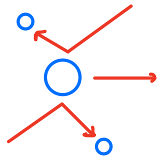
$$= P_n + AT P_n + P_n T A^T + AT P_n T A^T + \Gamma_n Q_n \Gamma_n^T - P_n - M_{n+1} H^T (H M_{n+1} H^T + R_{n+1})^{-1} H M_{n+1}$$

$$\frac{P_{n+1} - P_n}{T} \overset{T \to 0}{=} A P_n + P_n A^T + \underbrace{A P_n A^T T}_{\to 0} + \underbrace{\left(\frac{\Gamma_n}{T}\right)(Q_n T)\left(\frac{\Gamma_n^T}{T}\right)}_{\Gamma(t)\, Q(t)\, \Gamma^T(t)} - P H^T R(t)^{-1} H P$$

$$\boxed{\dot{P} = AP + PA^T + \Gamma Q \Gamma^T - P H^T R^{-1} H P}$$

dinamični razvoj kovariančne matrike
v zveznem primeru
Riccatijeva enačba

Primer: Brownovo gibanje koloidnega delca



Dinamika

$$m\ddot{x} = -6\pi r \eta \dot{x} + F_x(t)$$

naključne sile zaradi diskretnih trkov

$$\frac{1}{m^2}\langle F_x(t) F_x(t')\rangle = Q\,\delta(t-t')$$

Kalmanov filter za sledenje delca $\quad \vec{x} = \begin{bmatrix} x \\ v \end{bmatrix}$

$t = 0$; $\quad P(0) = P_0$

$\bar{x}(0) = 0$

$t > 0$; $\quad$ dinamika

$$\dot{x} = v$$

$$\dot{v} = -\frac{1}{\tau} v + \frac{F_x(t)}{m} \quad \Rightarrow \quad \frac{d}{dt}\begin{bmatrix} x \\ v \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & -1/\tau \end{bmatrix}}_{A}\begin{bmatrix} x \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\frac{F_x}{m}$$

$$\dot{\vec{x}} = A\vec{x} + \Gamma \vec{w}$$

$$P = \begin{bmatrix} \langle x^2\rangle & \langle xv\rangle \\ \langle xv\rangle & \langle v^2\rangle \end{bmatrix}$$

$$\dot{P} = AP + PA^T + \Gamma Q \Gamma^T$$

$$\dot{P} = \begin{bmatrix} 0 & 1 \\ 0 & -1/\tau \end{bmatrix}\begin{bmatrix} \langle x^2\rangle & \langle xv\rangle \\ \langle xv\rangle & \langle v^2\rangle \end{bmatrix} + PA^T + \begin{bmatrix} 0 \\ 1 \end{bmatrix} Q \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$\dot{P} = \begin{bmatrix} \langle xv\rangle & \langle v^2\rangle \\ -1/\tau\,\langle xv\rangle & -\frac{1}{\tau}\langle v^2\rangle \end{bmatrix} + \begin{bmatrix} \langle xv\rangle & -1/\tau\,\langle xv\rangle \\ \langle v^2\rangle & -1/\tau\,\langle v^2\rangle \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & Q \end{bmatrix}$$

$$\dot{P} = \begin{bmatrix} 2\langle xv\rangle & \langle v^2\rangle - \frac{1}{\tau}\langle xv\rangle \\ \langle v^2\rangle - \frac{1}{\tau}\langle xv\rangle & -\frac{2}{\tau}\langle v^2\rangle + Q \end{bmatrix} = \frac{d}{dt}\begin{bmatrix} \langle x^2\rangle & \langle xv\rangle \\ \langle xv\rangle & \langle v^2\rangle \end{bmatrix}$$

Rešujemo sistem enačb

$$\frac{d}{dt}\langle v^2\rangle = -\frac{2}{\tau}\langle v^2\rangle + Q$$

? Stacionarna rešitev $\quad \frac{d}{dt}\langle v^2\rangle = 0, \quad t\to\infty \quad \Rightarrow \langle v^2\rangle_\infty = Q\frac{\tau}{2}$

↳ termodinamsko ravnovesje pri temp. T

$$\frac{1}{2}m\langle v^2\rangle = \langle W_k\rangle = \frac{1}{2}k_BT$$

$$\Rightarrow \langle v^2\rangle = \frac{k_BT}{m}$$

$$\Rightarrow \frac{k_BT}{m} = \frac{Q\tau}{2} \qquad Q = \frac{2}{\tau}\frac{k_BT}{m}$$

Korelacijski člen

$$\underbrace{\frac{d}{dt}\langle xv\rangle}_{=0} = \langle v^2\rangle - \frac{1}{\tau}\langle xv\rangle$$

stac. rešitev

$$\Rightarrow \langle xv\rangle_\infty = \tau\langle v^2\rangle = \frac{\tau}{m}k_BT$$

$$\frac{d}{dt}\langle x^2\rangle = 2\langle xv\rangle$$

Stacionarna rešitev ne obstaja

$$\langle x^2(t)\rangle - \langle x^2(0)\rangle = 2\int\langle xv\rangle\,dt = 2\boxed{\frac{\tau k_BT}{m}}\,t \quad \Rightarrow \quad x^2 - x_0^2 = 2Dt$$

↳ D

Meritev lege

$$\dot{P} = \begin{bmatrix} 2\langle xv\rangle & \langle v^2\rangle - \frac{1}{\tau}\langle xv\rangle \\ \langle v^2\rangle - \frac{1}{\tau}\langle xv\rangle & -\frac{2}{\tau}\langle v^2\rangle + Q \end{bmatrix} - PH^T R^{-1} H P$$

$$z = H\hat{x} + r$$

$$z = [1\ 0]\begin{bmatrix} x \\ v\end{bmatrix} + r$$

$$\langle r r^T\rangle = R = \text{skalar } \sigma^2$$

$$PH^T R^{-1} H P = \begin{bmatrix} \langle x^2\rangle & \langle xv\rangle \\ \langle xv\rangle & \langle v^2\rangle \end{bmatrix} \underbrace{\begin{bmatrix} 1 \\ 0\end{bmatrix} \frac{1}{R} [1\ 0]}_{\begin{bmatrix} 1/R & 0 \\ 0 & 0\end{bmatrix}} \begin{bmatrix} \langle x^2\rangle & \langle xv\rangle \\ \langle xv\rangle & \langle v^2\rangle \end{bmatrix}$$

$$= \frac{1}{R}\begin{bmatrix} \langle x^2\rangle^2 & \langle x^2\rangle\langle xv\rangle \\ \langle xv\rangle\langle x^2\rangle & \langle xv\rangle^2 \end{bmatrix}$$

$$\dot{P} = \begin{bmatrix} 2\langle xv\rangle & \langle v^2\rangle - \frac{1}{\tau}\langle xv\rangle \\ \langle v^2\rangle - \frac{1}{\tau}\langle xv\rangle & -\frac{2}{\tau}\langle v^2\rangle + Q \end{bmatrix} - \frac{1}{R}\begin{bmatrix} \langle x^2\rangle^2 & \langle x^2\rangle\langle xv\rangle \\ \langle xv\rangle\langle x^2\rangle & \langle xv\rangle^2 \end{bmatrix}$$

Stacionarna rešitev $\dot{P} = 0$

$$\langle xv\rangle = \frac{1}{2R}\langle x^2\rangle^2$$

$$\langle v^2\rangle - \frac{1}{\tau}\langle xv\rangle = \frac{1}{R}\langle x^2\rangle\langle xv\rangle$$

$$-\frac{2}{\tau}\langle v^2\rangle + Q = \frac{1}{R}\langle xv\rangle^2 \quad \Rightarrow \quad \langle v^2\rangle = \frac{\tau}{2}Q - \frac{\tau}{2R}\langle xv\rangle^2$$

$$\frac{\tau}{2}Q - \frac{\tau}{2R}\langle xv\rangle^2 - \frac{1}{\tau}\langle xv\rangle = \frac{1}{R}\langle x^2\rangle\langle xv\rangle$$

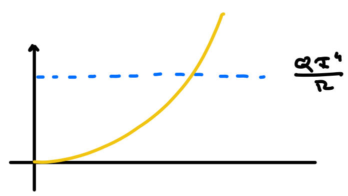
$$\frac{\tau}{2}Q - \frac{\tau}{2R}\frac{1}{4R^2}\langle x^2\rangle^4 - \frac{1}{2R\tau}\langle x^2\rangle^2 - \frac{1}{2R^2}\langle x^2\rangle^3 = 0$$

Naj bo $y = \langle x^2\rangle \frac{\tau}{R}$

$$\frac{\tau}{2}Q - \frac{R}{8\tau^3}y^4 - \frac{R}{2\tau^3}y^2 - \frac{R}{2\tau^3}y^3 = 0$$

$$\frac{1}{4}y^4 + y^3 + y^2 - \frac{\tau^4}{R}Q = 0$$

$$\frac{Q\tau^4}{R} = \frac{1}{4}y^4 + y^3 + y^2$$

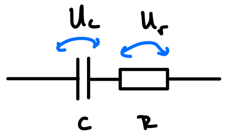


Primer "grobe meritve", R velik, y majhen

$$\frac{Q\tau^4}{R} = y^2 \quad \Rightarrow \quad y = \sqrt{\frac{Q}{R}}\,\tau^2 = \langle x^2\rangle\frac{\tau}{R}$$

$$\langle x^2\rangle_\infty = \sqrt{\frac{Q}{R}}\,\tau R = \tau\sqrt{QR}$$

Primer: merjenje napetosti na RC členu



$\Sigma u_i = 0 \qquad U_R + U_c = 0$

$-IR - \frac{e}{C} = 0$

$\Rightarrow \dot{u}_c + \frac{1}{\tau} u_c = 0 \qquad \tau = RC$

din. šum

$\dot{u} = -\frac{1}{\tau} u + w(t) \qquad \langle w(t) w(t') \rangle = Q\, \delta(t-t')$

$\Rightarrow A = -\frac{1}{\tau} \qquad \Gamma = 1$

Pomerimo U_c

$z = u + r \qquad \langle r(t)\, r(t') \rangle = R\, \delta(t-t')$

$\hat{u}$ ocena k U v sistemu s

$\langle (\hat{u} - u)^2 \rangle = P$ kov. matrika

Kalman $\qquad \dot{P} = 2AP + \Gamma^2 Q - \frac{P^2}{R}$

① Stacionarna rešitev $\qquad P(t \to \infty) = P_\infty$

$$-\frac{P^2}{R} + 2AP + \Gamma^2 Q = 0$$

$$-\frac{P^2}{R} - \frac{2}{\tau} P + Q = 0$$

$$P_{1,2} = \frac{\frac{2}{\tau} \pm \sqrt{\frac{4}{\tau^2} + \frac{4}{R} Q}}{-2/R} = -\frac{R}{\tau} \pm \underbrace{\sqrt{\frac{R^2}{\tau^2} + QR}}_{\alpha} = \frac{R}{\tau}\left(-1 \pm \sqrt{1 + \frac{Q\tau^2}{R}}\right)$$

② Splošna rešitev

$$\frac{dp}{p^2/R + \frac{2p}{\tau} - Q} = -dt = \frac{dp\, R}{(p + \frac{R}{\tau} - \alpha)(p + \frac{R}{\tau} + \alpha)} =$$

$$= R\left(\frac{B}{p + \frac{R}{\tau} + \alpha} + \frac{D}{p + \frac{R}{\tau} - \alpha}\right) dp$$

$\Rightarrow B = -D$

$B\left(\frac{R}{\tau} - \alpha\right) - B\left(\frac{R}{\tau} + \alpha\right) = 1$

$-2B\alpha = 1$

$B = -\frac{1}{2\alpha}$

$D = \frac{1}{2\alpha}$

$$= \frac{R}{2\alpha}\left(\frac{1}{p + \frac{R}{\tau} - \alpha} - \frac{1}{p + \frac{R}{\tau} + \alpha}\right) dp$$

$$\Rightarrow -\frac{2\alpha t}{R} = \ln \frac{p + \frac{R}{\tau} - \alpha}{p + \frac{R}{\tau} + \alpha}\Bigg|_{P_0}^{P(t)}$$

$$\frac{P + R/\tau - \alpha}{P + R/\tau + \alpha} = \underbrace{\frac{P_0 + R/\tau - \alpha}{P_0 + R/\tau + \alpha}}_{\to 1} e^{-2\alpha t/R}$$

Za $P_0(t \to 0) = \infty$

$$\Rightarrow P = -\frac{R}{\tau} + \alpha\left(\frac{1 + e^{-2\alpha t/R}}{1 - e^{-2\alpha t/R}}\right)$$

P_∞ za majhen Q

$$P_\infty = -\frac{R}{\tau} + \alpha = -\frac{R}{\tau} + \frac{R}{\tau}\sqrt{1+\frac{Q\tau^2}{R}} = \frac{R}{\tau}\left(-1+1+\frac{1}{2}\frac{Q\tau^2}{R}+\dots\right) = \frac{Q\tau}{2}$$

V sistemu M

$$\dot{\hat u} = -\frac{1}{\tau}\hat u + \underbrace{k(t)}(z-\hat u)$$

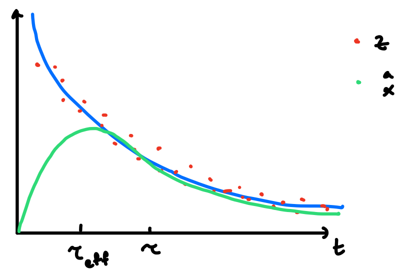
$$\frac{P_\infty}{R} = \frac{Q\tau}{2R}$$

$$\dot{\hat u} = -\frac{1}{\tau}\hat u + \frac{Q\tau}{2R}(z-\hat u) = \underbrace{\left(-\frac{1}{\tau} - \frac{Q\tau}{2R}\right)}_{-\frac{1}{\tau_{eff}}}\hat u + \frac{Q\tau}{2R} z$$

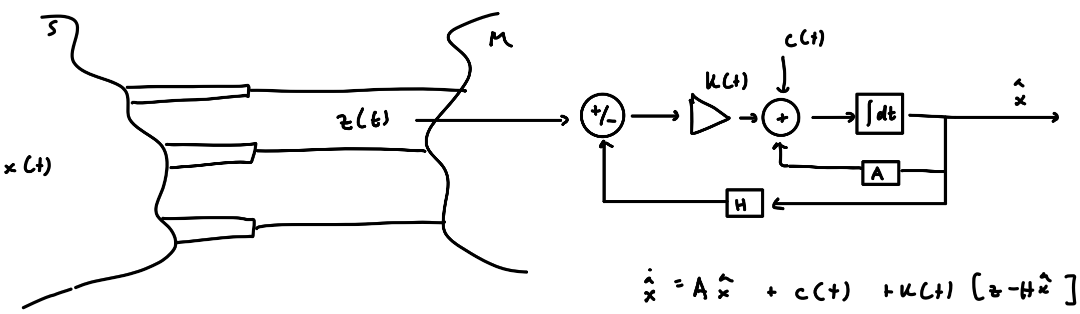
$Q = 0 \qquad \tau_{eff} = \tau$

$Q \to \infty \qquad \tau_{eff} \to 0$

Eksponentno padajoč šum $\qquad z = e^{-t/\tau} + r(t)$



Poenostavitve Kalmanove sheme in povratna zanka



$$\dot{\hat x} = A\hat x + c(t) + k(t)\left[z - H\hat x\right]$$

Inovacija $k(t)(z - H\hat x)$ stopnja sinhronizacije med S in M

Kadar sta S in M usklajena $\langle \underbrace{(z - H\hat x)}_{\text{bel šum}} \rangle = 0$

$\Rightarrow$ k je lahko karkoli, saj $z - H\hat x = 0$, torej je $k = k_\infty$ ok.

Senzor ⊐— kot univerzalni merilni sistem

Za senzor želimo

1. na izhodu napetost $\hat{x} = u(t)$
2. odvisen samo od ene količine (x)
3. senzor naj odpravi čim več merilnega šuma
4. senzor naj čim manj vpliva nazaj na opazovani sistem
5. $\hat{x}(t) \to u(t)$ berljive količine

preko diferencialne enačbe

Red senzorja

Red senzorja je enak reda diferencialne enačbe ki povezuje $z(t)$ in $\hat{x}(t)$. n-ti red senzorja $(n>0)$ obravnavamo kot idealen (optimalen) sledilni sistem za spremenljivko $x(t)$ v sistemu S, katere dinamika se spreminja kot:

$$\frac{d^n}{dt^n} x(t) = 0 + w(t)$$

Senzor 1. reda

$$\dot{x} = w(t) \qquad x \approx \text{konst.} \qquad \langle w^2 \rangle = Q$$

$$z = x + r(t) \qquad \langle r^2 \rangle = R$$

Kalman S: $\dot{x} = 0 + w$, $A = 0$, $C = 0$, $\Gamma = 1$

M: $\dot{\hat{x}} = k(z - \hat{x})$, $\dot{p} = -P^2/R + Q$, $k = P/R$

$\hat{x}$ ... ocena na izhodu senzorja

konst. ojačevalni faktor $k(t) \to k_\infty = P_\infty / R$

$$\dot{p} = 0 \qquad P_\infty = \sqrt{QR} \qquad k_\infty = \sqrt{Q/R}$$

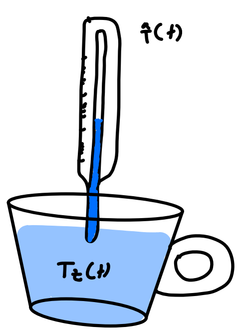
$$\underbrace{\frac{1}{k_\infty}}_{\tau} \dot{\hat{x}} + \hat{x} = z(t)$$

$$\tau \dot{\hat{x}} + \hat{x} = z(t)$$

D.E. 1 reda za senzor 1. reda

Je optimalen indikator

Primer: termometer



$$P = -\frac{\lambda S (T_z - T)}{d} = \frac{dQ}{dt} = -mc_p \frac{dT}{dt}$$

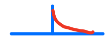
$$-mc_p \frac{dT}{dt} = -\frac{\lambda S}{d}(T_z - T)$$

$$\underbrace{\frac{mc_p d}{\lambda S}}_{\kappa} \dot{T} = T_z - T$$

$$\kappa \dot{T} + T = T_z$$ enačba senzorja 1. reda

$z(t) = ?$, (sistematične napake, prehodno obdobje)

tipični vhodi $z(t)$:

1. $z(t) = \delta(t)$ 
2. $z(t) = H(t)$ 
3. $z(t) = \alpha t$
4. $z(t) = \cos \omega t$

• odziv senzorja

1. red $\quad \tau \dot{\hat{x}} + \hat{x} = z$

• $z(t) = \delta(t)$

homogeni del

$$\tau \dot{\hat{x}} + \hat{x} = 0 \qquad \hat{x} = C e^{-t/\tau}$$

$$\int_{-\varepsilon}^{\varepsilon} \tau \frac{d\hat{x}}{dt}\, dt + \int_{-\varepsilon}^{\varepsilon} \hat{x}\, dt = \int_{-\varepsilon}^{\varepsilon} \delta(t)\, dt$$

$$\tau\left(\hat{x}(\varepsilon) - \hat{x}(-\varepsilon)\right) + 0 = 1$$

zahtevamo $\quad x(t<0), \dots, x^{(n)}(t<0) = 0$

$$\Rightarrow \hat{x}(0) = \frac{1}{\tau}$$

$$\Rightarrow \hat{x}(t) = \frac{1}{\tau} e^{-t/\tau}$$

• $z(t) = \alpha t$

$$\tau \dot{x} + x = \alpha t$$

$$x_p = At + B$$

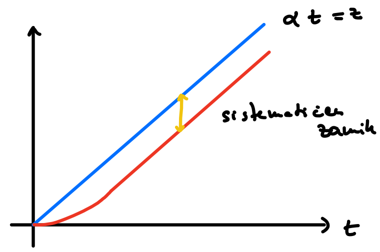
$$\tau A + At + B = \alpha t \quad \Rightarrow \alpha = A \qquad B = -\alpha\tau$$

$$x_p = \alpha(t - \tau)$$

$$x = C e^{-t/\tau} + \alpha(t - \tau)$$

$$x(0) = C - \alpha\tau = 0 \qquad C = \alpha\tau$$

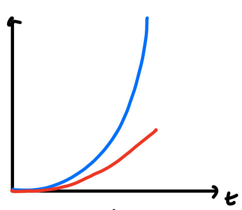
$$x(t) = \alpha\tau e^{-t/\tau} + \alpha(t - \tau)$$



Senzor sledi z zamikom

• $z(t) = \beta t^2$

izkaže se



Ne sledi

Senzor 1. reda najbolje sledi konstanti, višjim redom polinom slabo sledi ali ne sledi $(n \geq 2)$

2. red

$$z(t) \longrightarrow \boxed{II} \longrightarrow \hat{x}(t) = x(t)$$

$$\frac{d^2x}{dt^2} = 0 + w$$

$$\dot{x} = v \qquad \dot{v} = 0 + w \qquad \frac{d}{dt}\begin{bmatrix} x \\ v \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w \qquad \langle w^2 \rangle = Q$$

$$\dot{x} = A \quad x + \Gamma \, w$$

V sistemu M $\qquad z = x + r \Rightarrow H = [1, 0]$ merimo le x

$$\frac{d}{dt}\begin{bmatrix} \hat{x} \\ \hat{v} \end{bmatrix} = A \begin{bmatrix} \hat{x} \\ \hat{v} \end{bmatrix} + PH^T R^{-1} (z - H\hat{x}) \qquad P = \begin{bmatrix} P_{11} & P_{21} \\ P_{21} & P_{22} \end{bmatrix}$$

$$\dot{P} = \underbrace{AP + PA^T} + \underbrace{\Gamma Q \Gamma^T} - PH^T R^{-1} HP$$

$$\dot{P} = \begin{bmatrix} 2p_{12} & p_{22} \\ p_{22} & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & Q \end{bmatrix} - \frac{1}{R}\begin{bmatrix} p_{11}^2 & p_{11}p_{12} \\ p_{11}p_{12} & p_{12}^2 \end{bmatrix}$$

Stac. rešitve:

$$2p_{12} = \frac{1}{R} p_{11}^2 \qquad p_{11} = \sqrt{2R\sqrt{QR}}$$

$$p_{22} = \frac{1}{R} p_{11} p_{12} \qquad p_{22} = \frac{1}{R}\sqrt{2R\sqrt{QR}}\sqrt{QR} = \sqrt{2Q}\sqrt{\sqrt{QR}}$$

$$Q = \frac{1}{R} p_{12}^2 \qquad p_{12} = \sqrt{QR}$$

$$K_\infty = P_\infty H^T R^{-1} = \frac{1}{R}\begin{bmatrix} p_{11} \\ p_{12} \end{bmatrix} =$$

$$\dot{\hat{x}} = \hat{v} + \frac{1}{R} p_{11} (z - \hat{x}) \quad \xrightarrow{\text{odvod}} \quad \ddot{\hat{x}} = \dot{\hat{v}} + \frac{1}{R} p_{11} (\dot{z} - \dot{\hat{x}})$$

$$\dot{\hat{v}} = \frac{1}{R} p_{12} (z - \hat{x}) \qquad \ddot{\hat{x}} = \frac{1}{R} p_{12} (z - \hat{x}) + \frac{1}{R} p_{11} (\dot{z} - \dot{\hat{x}})$$

$$\boxed{\ddot{\hat{x}} + \frac{p_{11}}{R}\dot{\hat{x}} + \frac{p_{12}}{R}\hat{x} = \frac{p_{11}}{R}\dot{z} + \frac{p_{12}}{R} z}$$ enačba oscilatorja 2. reda

Std. oblika $\qquad \ddot{\hat{x}} + 2\zeta\omega\dot{\hat{x}} + \omega^2\hat{x} = \underbrace{2\zeta\omega}\dot{z} + \omega^2 z$ $\qquad$ dodatni člen to člen zamenjuje

$$\omega^2 = \frac{p_{12}}{R} = \sqrt{\frac{Q}{R}} \qquad \zeta = \frac{1}{\sqrt{2}}$$

koeficient dušenja

Specifični vhodi $\quad z(t)$

- $z(t) = \delta(t) \qquad \ddot{x} + 2\zeta\omega\dot{x} + \omega^2 x = \omega^2\delta(t)$

$\Rightarrow x(t) = G(t)$ $\quad$ greenova funkcija

z. p. $\quad x(0) = ? \qquad \dot{x}(0) = ?$

$$\int_{-\varepsilon}^{\varepsilon} \ddot{x}\, dt + 2\zeta\omega\int_{-\varepsilon}^{\varepsilon}\dot{x}\, dt + \omega^2\int_{-\varepsilon}^{\varepsilon} x\, dt = \omega^2 \underbrace{\int_{-\varepsilon}^{\varepsilon}\delta(t)\, dt}_{1}$$

$$\dot{x}(\varepsilon) - \dot{x}(-\varepsilon) + \underbrace{2\zeta\omega\left(x(\varepsilon) - x(-\varepsilon)\right)}_{\text{pri } t<0 \;\Rightarrow\; 0} + 0 = \omega^2$$

$$\dot{x}(0) + 2\zeta\omega x(0) = \omega^2 \qquad x(0) = 0 \qquad \dot{x}(0) = \omega^2$$

Homogeni del $\qquad x = e^{\lambda t}$

$$\Rightarrow \lambda^2 + 2\zeta\omega\lambda + \omega^2 = 0$$

$$\lambda_{1,2} = \frac{-2\zeta\omega \pm \sqrt{4\zeta^2\omega^2 - 4\omega^2}}{2} = -\zeta\omega \pm \omega\sqrt{\zeta^2 - 1} = -\omega\left(\zeta \mp \sqrt{\zeta^2 - 1}\right)$$

$$x(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$

Z.P. $x(0)=0 \Rightarrow c_1 = -c_2$

$$\dot{x}(0)=\omega^2 = c_1\lambda_1 + c_2\lambda_2 = \omega^2$$

$$c_1\lambda_1 - c_1\lambda_2 = \omega^2$$

$$c_1 = \frac{\omega^2}{\lambda_1-\lambda_2} =$$

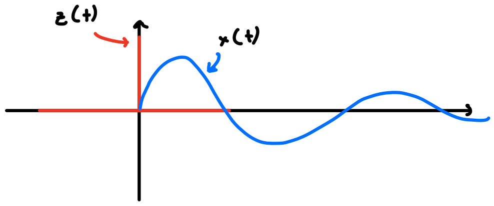
$$x(t) = \frac{\omega^2}{\lambda_1-\lambda_2}\left(e^{\lambda_1 t} - e^{\lambda_2 t}\right)$$

Naj bo $\zeta = \frac{1}{\sqrt{2}}$ optimalen

$$\frac{\omega^2}{\lambda_1-\lambda_2} = \frac{\omega^2}{\omega\, 2\sqrt{\zeta^2-1}} = \frac{\omega}{i\sqrt{2}}$$

$$x(t) = \frac{\omega}{\sqrt{2}i}\left(e^{i\omega/\sqrt{2}\,t} - e^{-i\omega/\sqrt{2}\,t}\right)e^{-\omega/\sqrt{2}\,t}$$

$$x(t) = \omega\sqrt{2}\,\sin\frac{\omega}{\sqrt{2}}t\; e^{-\frac{\omega}{\sqrt{2}}t}$$



Kaj če ζ ni optimalen ($\zeta \neq \frac{1}{\sqrt{2}}$)?

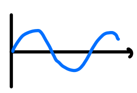
$$\lambda_{1,2} = -\omega\left(\zeta \mp i\sqrt{1-\zeta^2}\right)$$

$$x(t) = \frac{\omega^2}{\lambda_1-\lambda_2}\left(e^{\lambda_1 t} - e^{\lambda_2 t}\right) = \frac{\omega}{2\sqrt{\zeta^2-1}}\left(e^{\omega\sqrt{\zeta^2-1}\,t} - e^{-\omega\sqrt{\zeta^2-1}\,t}\right)e^{-\zeta\omega t}$$

$$= \frac{\omega}{\sqrt{1-\zeta^2}}\sin\omega\sqrt{1-\zeta^2}\,t\; e^{-\zeta\omega t}$$

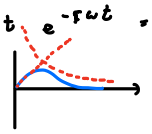
za $\zeta = 0$ $\quad x(t) = \omega\sin\omega t$

brez dušenja



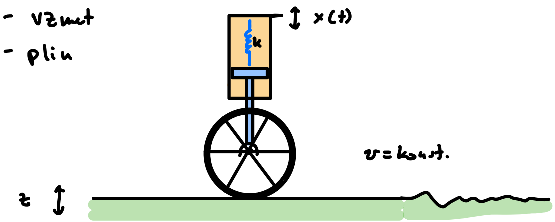
$\zeta \gg 1$ tedaj $\quad x(t) = \frac{\omega}{2\sqrt{\zeta^2-1}}\, 2\omega\sqrt{\zeta^2-1}\; t\; e^{-\zeta\omega t} = \omega^2 t\, e^{-\zeta\omega t}$

močno dušenje



Primer senzorja 2. reda: amortizer

- vzmet
- plin



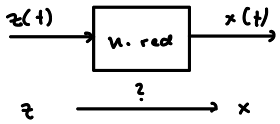
$$\sum F = m\ddot{x} = -k(x-z) - D\eta(\dot{x}-\dot{z})$$

$$\ddot{x} + \frac{D\eta}{m}\dot{x} + \frac{k}{m}x = \frac{D\eta}{m}\dot{z} + \frac{k}{m}z$$

$$2\zeta\omega = 2\zeta\sqrt{\frac{k}{m}} = \frac{D\eta}{m}$$

za optimalen $\zeta = 1/\sqrt{2} \Rightarrow D\eta = \sqrt{2km}$

Prenosna funkcija



integralske transformacije

Laplaceova transformacija

$$\mathcal{L}(f(t)) = F(s) = \int_0^\infty e^{-st} f(t)\, dt \qquad s \in \mathbb{C}$$

$f(t)$	$\mathcal{L} f(t) = F(s)$
1	$\frac{1}{s}$
e^{at}	$\frac{1}{s-a}$
$f(t)\, e^{at}$	$F(s-a)$
$\frac{d}{dt} f(t)$	$s F(s)$
t	$1/s^2$
$\delta(t)$	1
$e^{i\omega t}$	$\frac{1}{s-i\omega}$
$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$

Lastnosti

$$\mathcal{L}(1) = \int_0^\infty e^{-st}\, dt = -\frac{1}{s}(0-1) = \frac{1}{s}$$

$$\mathcal{L}(e^{at}) = \int_0^\infty e^{-st} e^{at}\, dt = \int_0^\infty e^{-(s-a)t}\, dt = \frac{1}{s-a}$$

$$\mathcal{L}(f(t)e^{at}) = \int_0^\infty e^{-(s-a)t} f(t)\, dt = F(s-a)$$

$$\mathcal{L}(f'(t)) = \int_0^\infty \frac{d}{dt} f(t)\, e^{-st}\, dt \overset{\text{per partes}}{=} f(t)\, e^{-st}\Big|_0^\infty + \int_0^\infty s\, e^{-st} f(t)\, dt$$

$$= -f(0) + s F(s) = s F(s) \qquad \text{naj bo } f(0) = \dot{f}(0) = \dots = 0$$

$$\mathcal{L}\, e^{i\omega t} = \mathcal{L} \cos \omega t + i\, \mathcal{L} \sin \omega t = \frac{1}{s-i\omega} = \frac{1}{s^2+\omega^2}(s+i\omega)$$

Prenosne funkcije senzorjev

n, m reda

$$\frac{d^{(n)}}{dt^{(n)}} x + a_{n-1} \frac{d^{(n-1)}}{dt^{(n-1)}} x + \dots + a_1 \frac{d}{dt} x + a_0 x = \frac{d^{(m)}}{dt^{(m)}} z + \dots + b_1 \frac{d}{dt} z + b_0 z$$

$\mathcal{L}$

$$s^n x(s) + a_{n-1} s^{n-1} x(s) + \dots + s\, a_1\, x(s) + a_0 x(s) = s^m z(s) + \dots + b_1 s\, z(s) + b_0 z(s)$$

$$\Rightarrow \boxed{\frac{x(s)}{z(s)} = H(s) = \frac{s^m + b_{m-1} s^{m-1} + \dots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0}} \qquad \text{Prenosna funkcija}$$

Primer senzorja 1. reda

$$\tau \dot{x} + x = z(t)$$

$\mathcal{L}$

$$\tau s\, x(s) + x(s) = z(s)$$

$$\frac{x(s)}{z(s)} = \frac{1}{1+\tau s}$$

2. red

$$(s^2 + 2\zeta\omega s + \omega^2)\, x(s) = \omega^2 z(s)$$

$$\frac{x(s)}{z(s)} = H(s) = \frac{\omega^2}{s^2 + 2\zeta\omega s + \omega^2}$$

Primer uporabe

a) $z(t) = \delta(t)$

$z(s) = 1$

$$H(s) = \frac{1}{1+\tau s} \qquad x(s) = H(s)\, z(s) = \frac{1}{1+\tau s}$$

$$x(t) = \mathcal{L}^{-1} \frac{1}{\tau} \frac{1}{\frac{1}{\tau}+s} = \frac{1}{\tau} e^{-t/\tau}$$

2. $z(t) = \begin{cases} z_0 & ; t > 0 \\ 0 & ; t < 0 \end{cases}$

$$z(s) = \frac{z_0}{s} \qquad x(s) = H(s)\, z(s) = z_0 \frac{1}{s} \frac{1}{1+\tau s} \overset{\text{parcialni ulomki}}{=} z_0 \left(\frac{1}{s} - \frac{\tau}{1+\tau s} \right)$$

$$x(t) = z_0 - z_0 e^{-t/\tau} = z_0 \left(1 - e^{-t/\tau} \right)$$

2. red

$$H(s) = \frac{\omega^2}{s^2 + 2\xi\omega s + \omega^2} = \frac{\omega^2}{(s+\xi\omega)^2 + \omega^2(1-\xi^2)} = \frac{\omega\sqrt{1-\xi^2}}{(s+\xi\omega)^2 + \omega^2(1-\xi^2)} \quad \frac{\omega}{\sqrt{1-\xi^2}}$$

$$x(t) = \mathcal{L}^{-1}(H(s) \ast \delta(t)) = \frac{\omega}{\sqrt{1-\xi^2}} \sin \sqrt{1-\xi^2}\, \omega t \quad e^{-\xi\omega t}$$

Odziv senzorja na periodične signale / Bodejevi diagrami

$$\xrightarrow{z(t)} \boxed{H(s)} \xrightarrow{x(t)}$$

$z = z_0 e^{i\omega t} \qquad x = x_0 e^{i\omega t} e^{i\delta}$

$$\frac{d}{dt} e^{i\omega t} = i\omega e^{i\omega t} \quad / \mathcal{L}$$

$$s \mathcal{L} e^{i\omega t} = i\omega \mathcal{L} e^{i\omega t} \Rightarrow s = i\omega$$

$s \in \mathbb{C}$, za periodične signale, $s = i\omega$, $H(s) = H(i\omega)$

$$H(i\omega)\, s(i\omega) = x(i\omega)$$

$$H(i\omega)\, z_0\, \mathcal{L}(e^{i\omega t}) = x_0\, \mathcal{L}(e^{i\omega t})\, e^{i\delta}$$

$$H(i\omega) = \frac{x_0}{z_0} e^{i\delta} \qquad H(i\omega) \in \mathbb{C}$$

$$\tan \delta = \frac{\operatorname{Im} H(i\omega)}{\operatorname{Re} H(i\omega)}$$

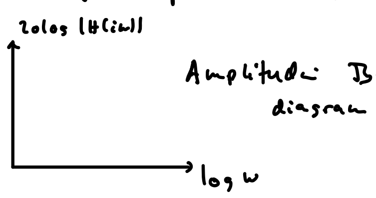
V splošnem

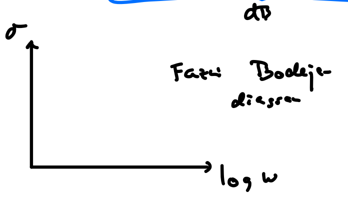
$$H(i\omega) = \frac{\prod_i |1 + i\omega\tau_i| e^{i\delta_i} \prod_j \left| \left(\frac{i\omega}{\omega_j}\right)^2 + \frac{2\xi_j i\omega}{\omega_j} + 1 \right| e^{i\Delta_j}}{\prod_k |1 + i\omega\tau_k| e^{i\delta_k} \prod_l \left| \left(\frac{i\omega}{\omega_l}\right)^2 + \frac{2\xi_l i\omega}{\omega_l} + 1 \right| e^{i\Delta_l}}$$

Imamo

- razmerje amplitud → množenje in deljenje delnih amplitud
- fazni premik → seštevanje in odštevanje faznih zamikov

Def. prikazuje razmerje amplitud kot $\underbrace{20 \log_{10} (|H(i\omega)|)}_{dB}$



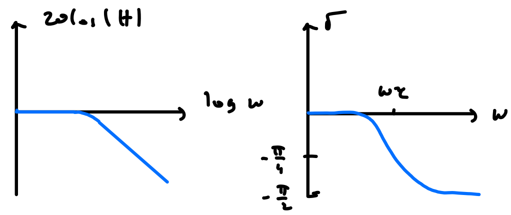


Sistem 1. reda

$$H(s) = \frac{1}{1+\tau s} = \frac{1}{1+i\omega\tau} = H(i\omega) = \frac{1-i\omega\tau}{1+\omega^2\tau^2}$$

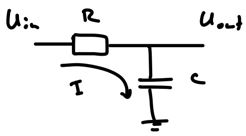
$$|H(i\omega)| = \frac{1}{\sqrt{1+\omega^2\tau^2}}$$

$$\tan\delta = -\omega\tau \qquad \delta = -\arctan\omega\tau$$



Primer:

1 Low pass filter



$$Z_C = \frac{1}{i\omega C} \qquad Z_R = R \qquad Z_L = i\omega L$$

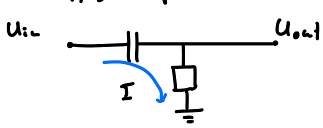
$$Z = R + \frac{1}{i\omega C} = \frac{1+i\omega RC}{i\omega C}$$

$$U_{in} = I Z$$

$$U_{out} = I Z_C = I\frac{1}{i\omega C} = U_{in}\frac{Z_C}{Z} = \frac{1}{1+i\omega RC}U_{in}$$

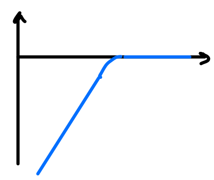
$$\frac{U_{out}}{U_{in}} = \frac{1}{1+i\omega\tau} \qquad RC = \tau$$

2 High pass filter



$$U_{out} = IR = \frac{U_{in}}{Z}R = U_{in}\frac{i\omega RC}{1+i\omega RC}$$

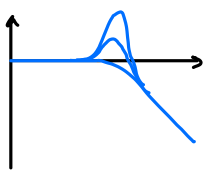
$$\Rightarrow \frac{U_{out}}{U_{in}} = H = \frac{i\omega\tau}{1+i\omega\tau}$$

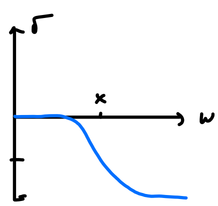


Sistem 2. reda

$$H(\overset{i\omega}{s}) = \frac{1}{s^2/\omega_0^2 + 2\frac{\xi}{\omega_0}s + 1}$$

$$\tan\delta = \frac{-2\xi x}{1-x^2} \qquad x = \frac{\omega}{\omega_0}$$

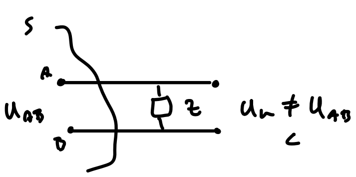




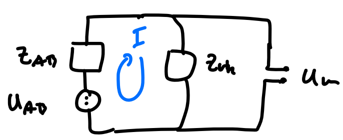
Vpliv senzorja na opazovani sistem

Eno od meril je moč oz. energija (za enkratne meritve) ki jo črpamo iz S za delovanje senzorja.

Theveninov izrek: Vsako el. vezje, ki ga sestavljajo lin. elementi (R, C, L) in napetostni vir in ga pomerimo v poljubnih točkah vezja A in B lahko nadomestimo z generatorjem napetosti U_{AB} in notranjo upornostjo Z_{AB}.



→



Z_{vh} ... vhodna impedanca

$Z = Z_{AB} + Z_{vh}$, $I = \frac{U_{AB}}{Z} = \frac{U_{AB}}{Z_{AB} + Z_{vh}}$

$$U_{vh} = I Z_{vh} = U_{AB}\left(\frac{Z_{vh}}{Z_{AB}+Z_{vh}}\right) = U_{AB}\frac{1}{1+\frac{Z_{AB}}{Z_{vh}}} \Rightarrow U_{vh} < U_{AB}$$

Idealno $Z_{AB} \ll Z_{vh}$

$$P_{vh} = \frac{U_{vh}^2}{Z_{vh}} = \frac{U_{AB}^2 Z_{vh}^2}{Z_{vh}(Z_{vh}+Z_{AB})^2} = U_{AB}^2 \frac{Z_{vh}}{(Z_{vh}+Z_{AB})^2}$$

P_{max}:

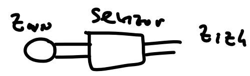
$$\frac{dP_{vh}}{dZ_{vh}} = U_{AB}^2\left(\frac{1}{(Z_{vh}+Z_{AB})^2} - \frac{2Z_{vh}}{(Z_{vh}+Z_{AB})^3}\right) = 0 \Rightarrow Z_{AB} = Z_{vh}$$

$$P_{max} = \frac{U_{AB}^2}{4Z_{AB}}$$

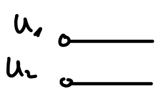
$$P_{vh} = U_{AB}^2 \frac{Z_{vh}}{(Z_{vh}+Z_{AB})^2} = P_{max}\frac{4Z_{vh}Z_{AB}}{(Z_{vh}+Z_{AB})^2} = P_{max}\frac{4\frac{Z_{AB}}{Z_{vh}}}{\left(1+\frac{Z_{AB}}{Z_{vh}}\right)^2}$$

$$\Rightarrow \frac{P}{P_{max}} \approx 4\frac{Z_{AB}}{Z_{vh}} \qquad P \ll P_{max} \Leftrightarrow Z_{vh} \gg Z_{AB}$$

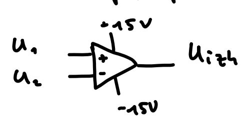
Posplošitev:
- vhodna impedanca $Z_{vh} \to \infty$
- izhodna impedanca $Z_{izh} \to 0$



Primer: instrumentacijski ojačevalnik (diferencialni preamp)



3× opamp

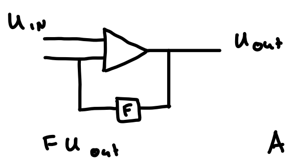


$A_{DC} \sim 10^8$ ojačevalni faktor

$U_{izh} = A_{DC}(u_1 - u_2)$

$H(s) = A_{DC} H_{LPF}$

Negativna povratna zanka



$$A(U_{in} - F U_{out}) = U_{out}$$

$$A U_{in} = (AF + 1) U_{out}$$

$$\frac{U_{out}}{U_{in}} = \frac{1}{F + 1/A} = \frac{1}{F}$$

A velik

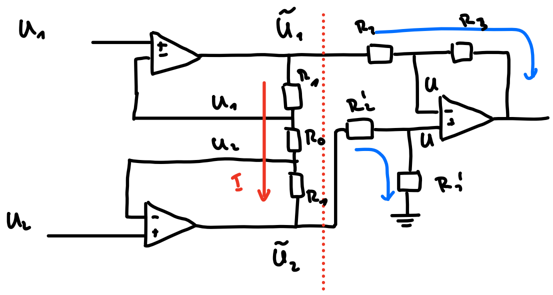
Buffer Pri $F = 1$ $H = 1$

$U_{in} = U_{out}$

$I_{in} = I_{out} = 0$

$R_{in} = \infty$

Instrumentacijski ojačevalec



$$I = \frac{\hat{U}_1 - U_1}{R_1} = \frac{\hat{U}_2 - U_2}{R_1} = \frac{U_1 - U_2}{R_0}$$

$\Rightarrow \hat{U}_1 + \hat{U}_2 = U_1 + U_2$ vsota se ohranja

$$\frac{(\tilde{U}_1 - \tilde{U}_2) + (U_2 - U_1)}{R_1} = \frac{2(U_1 - U_2)}{R_0}$$

$$\hat{U}_1 - \tilde{U}_2 = \left(2\frac{R_1}{R_0} + 1\right)(U_1 - U_2)$$

$$\Delta\tilde{U} = \left(2\frac{R_1}{R_0} + 1\right)\Delta U$$

Desni del

- $\dfrac{\hat{U}_1 - U}{R_2} = \dfrac{U - U_{out}}{R_3}$
- $\dfrac{\tilde{U}_2 - U}{R_2'} = \dfrac{U}{R_3'}$

$$\frac{\hat{U}_2}{R_2'} = U\left(\frac{1}{R_2'} + \frac{1}{R_3'}\right) = U\frac{R_2' + R_3'}{R_2' R_3'}$$

$$\frac{R_3'}{R_2' + R_3'}\hat{U}_2 = U$$

$$\frac{R_3}{R_2}(\hat{U}_1 - U_0) = U - U_{out}$$

$$-\frac{R_3}{R_2}\hat{U}_1 + \left(\frac{R_3}{R_2} + 1\right)U = U_{out}$$

$$-\frac{R_3}{R_2}\left(\tilde{U}_1 - \underbrace{\left(1 + \frac{R_2}{R_3}\right)\frac{R_3'}{R_2' + R_3'}}_{c}\hat{U}_2\right) = U_{out}$$

($\frac{R_3'}{R_2' + R_3'}$ is also marked k')

če je $c = 1$

$$-\frac{R_3}{R_2}(\hat{U}_1 - \hat{U}_2) = U_{out}$$

$$-\frac{R_3}{R_2}\Delta\hat{U} = U_{out}$$

$$-\frac{R_3}{R_2}\left(2\frac{R_1}{R_0} + 1\right)\Delta U = U_{out}$$

Kdaj je $c = 1$

$$\left(1 + \frac{R_2}{R_3}\right)\frac{R_3'}{R_2' + R_3'} = 1$$

$$\frac{1 + \frac{R_2}{R_3}}{1 + \frac{R_2'}{R_3'}} = 1$$

$$1 + \frac{R_2}{R_3} = 1 + \frac{R_2'}{R_3'}$$

$$\frac{R_2}{R_3} = \frac{R_2'}{R_3'}$$

Kaj če to ne drži povsem

$$k'\left(1 + \frac{R_2}{R_3}\right) = \frac{1+\varepsilon}{1-\varepsilon} \qquad \varepsilon \to 0$$

$$U_{out} = -\frac{R_1}{R_2}\left(\tilde{u}_1 - \frac{1+\varepsilon}{1-\varepsilon}\,\tilde{u}_2\right)$$

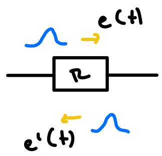
$$= -\frac{R_1}{R_2}\,\frac{1}{1-\varepsilon}\left(\tilde{u}_1 - \tilde{u}_2 - \varepsilon\,(\underbrace{\tilde{u}_1 + \tilde{u}_2}_{\text{common mode}})\right)$$

CMRR ... common mode rejection ratio

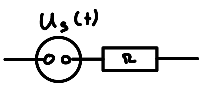
$$= \frac{A\,(u_1 - u_2)}{A'\,(u_1 + u_2)} \sim 10^6$$

ojačanje

Termični šumi upornikov



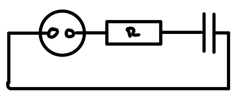
$\Downarrow$



$$\frac{de}{dt} = I(t)$$

$U_s(t) = I(t)\,R$ šumeča napetost

$$\langle U_s(t)\rangle = 0$$
$$\langle U_s^2\rangle \neq 0$$



$$U_s - IR - \frac{e}{C} = 0$$

$e = C\,U_c$

$\frac{de}{dt} = C\,\dot{U}_c$

$\tau = RC$

$$U_s - U_c = RC\,\dot{U}_c$$

$$\dot{U}_c = -\frac{1}{\tau}U_c + \frac{U_s}{\tau}$$

Kalmanova dinamika za U_c,
$U_s(t)$ je dinamični šum

$$\dot{x} = Ax + \Gamma w$$

$\langle \hat{u}_c^2\rangle = P$

$\langle U_s^2\rangle = Q$

$$A = -\frac{1}{\tau} \qquad \Gamma = \frac{1}{\tau}$$

$$\dot{P} = 2AP + \Gamma Q \Gamma^T$$
$$\dot{P} = -\frac{2}{\tau}P + \frac{1}{\tau^2}Q$$

Stac. rešitev $\quad -\frac{2}{\tau}P + \frac{1}{\tau^2}Q = 0$

$$Q = 2\tau P_\infty$$

Termodinamsko ravnovesje

$$\langle W_c\rangle = \frac{1}{2}C\langle u_c^2\rangle = \frac{1}{2}k_B T$$

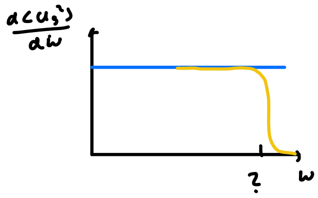
1 en. prost. stopnje $\Rightarrow$ $Q = 2R\,k_B T$

beli šum

Napetosti tako merimo do mV

Spektralna gostota termičnega šuma?

Ali imamo res beli šum? $\langle U_s(t)\, U_s(t+\tau)\rangle = \text{const.}\ \delta(\tau)$



$$\int_0^{\infty} \frac{d\,U_s^2}{d\omega}\, d\omega \to \infty$$

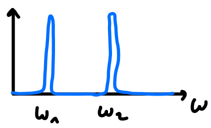
$$P_{\text{moč}} \to \infty$$

Želim $P < \infty$, zato

mora obstajati cut off

$$P^{\text{moč}} = \frac{\langle U_s^2\rangle}{R} \qquad \frac{dP}{d\omega} = \frac{d\langle U_s^2\rangle}{R\, d\omega}$$

(1) Posamezne frekvence obravnavamo neodvisno



$$U = U_0(\omega_1)\cos\omega_1 t + U_0(\omega_2)\cos(\omega_2 t + \delta)$$

$$\langle U^2\rangle = U_0^2(\omega_1)\,\tfrac{1}{2} + \tfrac{1}{2}\,U_0^2(\omega_2) + \overbrace{\langle U_0(\omega_1)\,U_0(\omega_2)\cos\omega_1 t\,\cos\omega_2 t+\delta\rangle}^{0}$$

$$= \sum_i \langle u_i^2\rangle \longrightarrow \int \underbrace{\frac{d}{d\omega}\langle u^2\rangle}_{\text{spektralna gostota šuma napetosti}}\, d\omega$$

Wiener - Hinčinov izrek

Spektralna gostota šuma je Fourierova transformacija avtokorelacijske funkcije

$$c(\tau) = \int_{-\infty}^{\infty} u^*(t)\, u(t+\tau)\, dt$$

F.T. $$u(t) = \int_{-\infty}^{\infty} u_\nu\, e^{-2\pi\nu i t}\, d\nu$$

$$u^*(t) = \int_{-\infty}^{\infty} u_{\nu'}^*\, e^{2\pi\nu' i t}\, d\nu'$$

$$c(\tau) = \iiint_{-\infty}^{\infty} u_{\nu'}^*\, e^{2\pi\nu' i t}\, u_\nu\, e^{-2\pi\nu i t}\, e^{-2\pi i \nu \tau}\, d\nu' d\nu dt$$

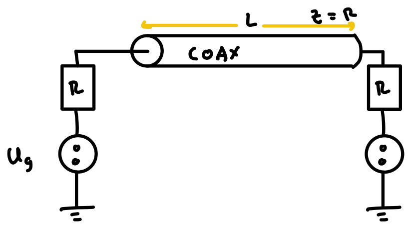
$$= \iiint u_{\nu'}^*\, u_\nu\, \underbrace{e^{2\pi i(\nu'-\nu)t}\, dt}_{\delta(\nu-\nu')}\, e^{-2\pi i\nu\tau}\, d\nu' d\nu$$

$$c(\tau) = \int |u_\nu|^2\, e^{-2\pi i\nu\tau}\, d\nu$$

$$|u_\nu|^2 = \frac{1}{\pi}\int e^{-i\omega\tau}\, c(\tau)\, d\tau$$

$$\frac{d\langle u^2\rangle}{d\omega} = \frac{1}{\pi}\int \underbrace{\langle u(t)\, u(t+\tau)\rangle}_{2kRT\ \delta(\tau)}\, e^{-i\omega\tau}\, d\tau = \frac{2kTR}{\pi} = \text{konst}$$

Termodinamsko ravnovesje (šum meritev) Nyquistova izpeljava



Št. nihanj v coax kablu

$c = \lambda \nu \qquad \lambda = \frac{c}{\nu} = \frac{2\pi}{\omega} c$

$L = n \frac{\lambda}{2} = n \frac{\pi}{\omega} c$

$n = \frac{L\omega}{\pi c}$

$\frac{dn}{d\omega} = \frac{L}{\pi c}$ št. valovanj na frekvenčni interval

Energija načina valovanja

$E(\omega) = \hbar\omega \frac{1}{e^{\hbar\omega/kT} - 1}$

Bose-Einstein
za zasedenost pri T

$dP = R\, d\langle I^2\rangle = \frac{1}{2} \underbrace{\frac{L}{\pi c} d\omega}_{dn} \frac{c}{L} E(\omega) = \dots$

zaradi potujočega val.
en. pot. val $= \frac{1}{2}$ en. stoječega val.

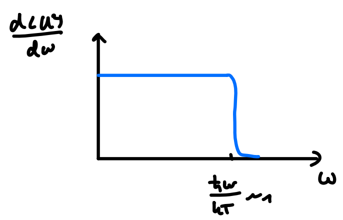
$I = \frac{u}{2R} \qquad \langle I^2\rangle = \frac{\langle u^2\rangle}{4R^2} \qquad d\langle I^2\rangle = \frac{1}{4R^2} \langle u^2\rangle$

$\dots = \frac{R\, d\langle u^2\rangle}{4R^2} = \frac{\hbar\omega}{2\pi} \frac{1}{e^{\hbar\omega/kT} - 1} d\omega$

$\Rightarrow \frac{d\langle u^2\rangle}{d\omega} = \frac{2R\hbar\omega}{\pi} \frac{1}{e^{\hbar\omega/kT} - 1}$

Pri $\hbar\omega \ll kT \qquad \frac{d\langle u^2\rangle}{d\omega} = \frac{2R\hbar\omega}{\pi} \frac{1}{1 + \frac{\hbar\omega}{kT} - 1} = \frac{2RkT}{\pi}$

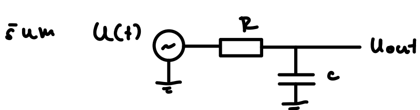
$\hbar\omega \gg kT \qquad \frac{d\langle u^2\rangle}{d\omega} = 0$



$\hbar\omega \sim kT \overset{T=300K}{\Rightarrow} 10^{13} - 10^{14}$ Hz

Naš približek da je šum nekoreliran (bel šum) je dober

Širjenje termičnega šuma skozi vezje



$u_{out}(s) = H(s)\, u(s) \qquad s = i\omega$

$u_{out}^2(i\omega) = |H(i\omega)|^2 u^2(i\omega)$

$\frac{d}{d\omega} \langle u_{out}^2\rangle = |H(i\omega)|^2 \frac{d}{d\omega} \langle u^2(i\omega)\rangle$

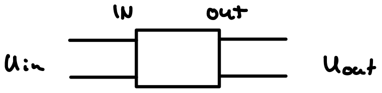
$\boxed{\frac{d}{d\omega} \langle u_{out}^2(i\omega)\rangle = |H(i\omega)|^2 \frac{2k_B T R}{\pi}}$ Nyquistov izrek

$R\,|H(i\omega)|^2 = \mathrm{Re}(Z_0)$

izhodna impedanca vezja

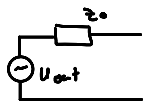
$$\frac{d}{d\omega}\langle U_{out}^2\rangle = \frac{2k_BT}{\pi}\,\mathrm{Re}\,Z_0$$

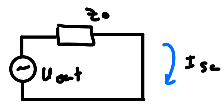
Vhodna / izhodna impedanca



Thevenin $\longrightarrow$

OUT

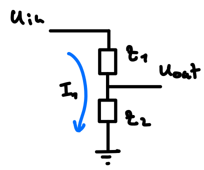


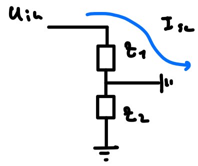


OC - open circuit

SC - short circuit

Napetostni delilnik





$U_{in} = Z_{in} I_1$

$Z_{in} = Z_1 + Z_2$

$U_{out} = I_1 Z_2 = \frac{U_{in} Z_2}{Z_1 + Z_2}$

$Z_0 I_{sc} = U_{out}$

$I_{sc} = \frac{U_{in}}{Z_1}$

Združimo enačbe

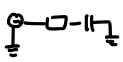
$$U_{out} = \frac{Z_2}{Z_1+Z_2} Z_1 I_{sc}$$

$$= \underbrace{\left(\frac{1}{Z_1} + \frac{1}{Z_2}\right)^{-1}}_{Z_0} I_{sc}$$

$Z_0^{-1} = \sum Z_i^{-1}$

$Z_{in} = \sum Z_i$

RC - člen



$$Z_{out} = \left(\frac{1}{R} + i\omega C\right)^{-1} = \frac{R(1-i\omega RC)}{1+\omega^2R^2C^2} \qquad \mathrm{Re}(Z_{out}) = \frac{R}{1+\omega^2R^2C^2}$$ ✓

Meritev konstantnih količin | statistika

Šum $\quad r = z - Hx \quad$ meritni šum $\quad r \sim N(0,\sigma)$

Izmerki $\{z_i\}_n$ privzamemo da so porazdeljeni Gaussovsko po $N(a,\sigma)$

Iščemo a, σ.

Ko $n \to \infty$ naredimo histogram in fitamo

n omejen

- sestavimo iz vzorca $\{z_i\}_n$ vzorčni statistiki

$\bar{z} = \frac{1}{n}\sum z_i$ povprečje

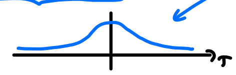
$s^2 = \underbrace{\frac{1}{n-1}}\sum (z_i - \bar{z})^2$ varianca

št. prost. stopenj (eno smo že uporabili z povprečjem)

T statistika (ocenjujemo a)

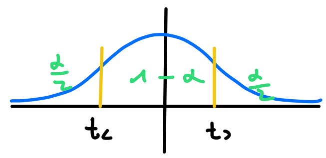
$$\boxed{T = \frac{\bar{z}-a}{\sqrt{s^2}}\sqrt{n}}$$ odvisna od parametra a

Studentova por. $S(n) = \frac{dP}{dT} = \frac{1}{\sqrt{n-1}\, B(\frac{n-1}{2}, 1/2)} \underbrace{\left(1 + \frac{T^2}{n-1}\right)^{-n/2}}$ normirana



Normiran, univerzalna porazdelitev

Postopek:
- $\{z_i\}$, n
- Izberemo α
- $\bar{z}$, s^2
- T
- tabele (n, α) $\quad P(t > t_{mejni}) = \alpha$



$\alpha = 1\%$, 5%, 10%

če $T > t_{mejni} = t_D$

zavržemo a, ker je izven stopnje tveganja α

$T < t_{mejni}$

ne zavržemo a, ker je znotraj stopnje tveganja

Interval zaupanja

$$\int_{t_L}^{t_D} \frac{dP}{dT}\, dT = 1-\alpha$$

$T \in [t_L, t_D]$ na stopnji zaupanja $1-\alpha$

izven tega intervala pričakujemo T z verjetnostjo α

$$t_L = \frac{\bar{z} - a_L}{s}\sqrt{n} \qquad t_D = \frac{\bar{z} - a_D}{s}\sqrt{n}$$

$a \in [a_L, a_D]$

če je a znotraj intervala ne moremo zavreči,

če pa je zunaj ga intervala, lahko a zavrnemo

na stopnji α.

Porazdelitev χ^2 (ocenjujemo σ)

$\{z_i\}$, n $\quad$ privzamemo por. po $N(a, \sigma)$

$\bar{z}$, s^2

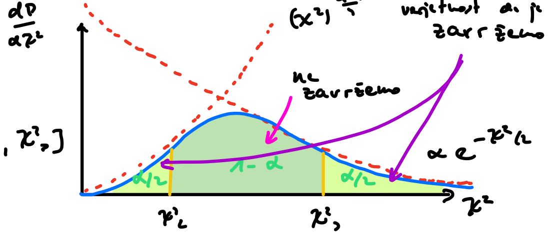
Radi bi ocenili σ

$$\boxed{\chi^2 = (n-1)\frac{s^2}{\sigma^2}}$$ je $f(\sigma)$

$\frac{dP}{d\chi^2}$ pa ni odvisna od σ^2, je univerzalna in normirana

$$\frac{dP}{d\chi^2} = \frac{1}{2^{\frac{n-1}{2}}\Gamma(\frac{n-1}{2})}(\chi^2)^{\frac{n-3}{2}} e^{-\chi^2/2}$$

Z verjetnostjo $1-\alpha$ pričakujemo, da bo $\chi^2 \in [\chi^2_L, \chi^2_D]$



Najverjetneje, da smo izbrali napačno σ, še vedno pa obstaja verjetnost da je pravilna. zavržemo σ

Def. Vsota kvadratov n neodvisnih, naključno porazdeljenih, standardiziranih normalno porazdeljenih spremenljivk je porazdeljena po zakonu χ^2 z n prostostnimi stopnjami

$$\chi^2 = \sum x^2$$

$$s^2 = \frac{1}{n-1}\sum (z_i - \bar z)^2$$

$\{z_i\}$

$$z_i \sim N(\bar z, \sigma)$$

$$(z_i - \bar z) \sim N(0, \sigma)$$

$$\frac{z_i - \bar z}{\sigma} \sim N(0,1)$$

$$\sum_n \left(\frac{z_i - \bar z}{\sigma}\right)^2 = (n-1)\frac{s^2}{\sigma^2} = \chi^2(n-1)$$

Def. x por. po $N(0,1)$, Y por. po $\chi^2(n)$

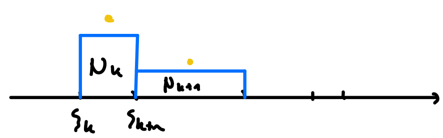
$\Rightarrow$ $T = \frac{x}{\sqrt{Y/n}}$ por. po $S(n)$

$$T = \frac{\bar z - a}{s}\sqrt{n} = \frac{\bar z - a}{\sigma\sqrt{\frac{s^2}{\sigma^2}}}\sqrt{n} = \underbrace{\frac{\bar z - a}{\sigma/\sqrt{n}}}_{N(0,1)} \frac{1}{\sqrt{s^2/\sigma^2\,(n-1)}}\sqrt{n-1} \;\Big\}\; \chi^2(n-1) \Rightarrow \text{por. po } S(n-1)$$

Oslikovni testi (ali je res por. po N)

$\{z_i\}$, n $\qquad \frac{dP}{dz}$ privzeti porazdelitveni zakon

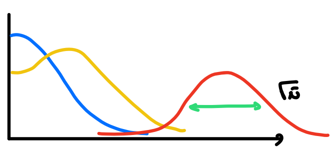
- kam lahko padejo izmerki
- sestavimo razrede
- brez prekrivanj in lukenj sestavimo g razredov
- N_k izmerkov pade v k-ti razred



$$N p_k = N \int_{z_k}^{z_{k+1}} \frac{dP}{dz}\,dz$$

Če smo si dobro izbrali $\frac{dP}{dz}$ pričakujemo, da je $N_k - N p_k$ majhen (meritni šum)

Štejemo dogodke, $\bar N$, $\frac{dp}{dN} = \frac{\bar N^N e^{-\bar N}}{N!}$ (Poisson)



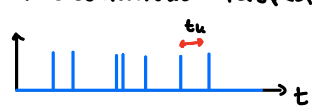
št. izmerkov v enem razredu

$(\bar N > 20)$

za dovolj velik $\bar N$ skoraj Gaussova porazdelitev

$$\chi^2 = \sum_{k=1}^{g} \frac{(N_k - N p_k)^2}{N p_k} \sim \chi^2(g-1) \qquad \text{Pearsonov } \chi^2 \text{ test}$$

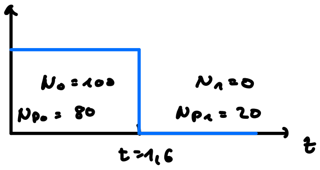
Primer: radioaktivni razpad



$\{t_k\}$ meritve

Po $N = 100$ izmerkih, izmerimo, da je $t_k < 1{,}6\,s$.
Ali lahko na stopnji tveganja α zavrnemo $\tau = 1s$

$$\frac{dP}{dt} = \frac{1}{\tau} e^{-t/\tau} \qquad H \text{ (hipoteza)}: \tau = 1$$

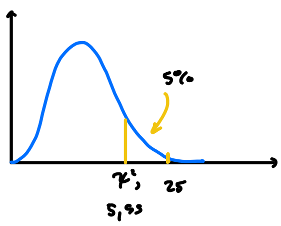


$$P_0 = \int_0^{1,6} \frac{1}{\tau} e^{-t/\tau} dt = 1 - e^{-1,6/1} = 0{,}8$$

$$P_1 = \int_{1,6}^{\infty} \frac{1}{\tau} e^{-t/\tau} dt = 1 - 0{,}8 = 0{,}2$$

$$\chi^2 = \frac{(100-80)^2}{80} + \frac{(0-20)^2}{20} = 25$$

Tabele $\quad \chi^2_\alpha (2-1) = 5{,}99$
$\alpha = 5\%$



$\Rightarrow \quad 25 > 5{,}99 \quad$ hipotezo zavržemo

Fisherjev test (iščemo optimalne $\tau, a, \sigma, \ldots$)

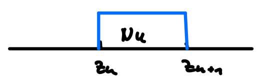
Želimo testirati privzeti porazdelitveni zakon, pri čemer parametre zakona določimo optimalno iz izmerkov.

$$\frac{dP}{dz}(\underbrace{a_1, a_2, \ldots, a_m}_{\text{neznani parametri}})$$

$$p_k = \int_{z_{k-1}}^{z_k} \frac{dP}{dz}(a_1, \ldots, a_m)\, dz = p_k(a_1, \ldots, a_m) \qquad \text{verjetnost za } k\text{-ti razred}$$

Sestavimo funkcijo zanesljivosti (likelyhood)

$$L^* = \prod_{k=1}^{g} (p_k(a_i))^{N_k}$$



$$\frac{\partial L^*}{\partial a_i} = 0 \Rightarrow a_i \text{ optimalni}$$

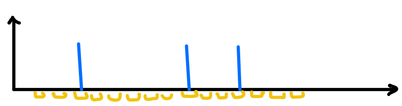
m enačb

$$p_k^*(a_i^*) \to \text{Pearson } \chi^2$$

$$\chi^2 = \sum \frac{(N_k - Np_k^*)^2}{Np_k^*} \qquad \text{por po } \chi^2(g-1-m)$$

Posplošitev L

$$L = \prod \frac{dP}{dz}(z_k, a_1, \ldots, a_m)$$



Primer: razpad

$$\{t_k\}_n \qquad \frac{dP}{dt} = \frac{1}{\tau} e^{-t/\tau} \qquad L = \prod \frac{1}{\tau} e^{-t_k/\tau} = \frac{1}{\tau^n} e^{-\sum t_k/\tau}$$

$$\ln L = -n \ln \tau - \frac{1}{\tau} \sum_k t_k$$

$$\frac{\partial \ln L}{\partial \tau} = -n\frac{1}{\tau} - \frac{1}{\tau}\sum t_n = 0$$

$$\tau = \frac{1}{n}\sum t_n \qquad \text{optimalen } \tau$$

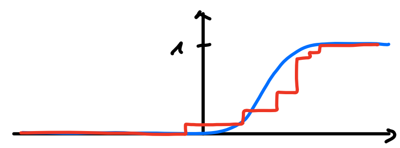
Test Kolmogorova

Kumulativni test

$\{z_i\}_n$, $\frac{dp}{dz} = ?$ testiramo

Kumulativna funkcija $F(z) = \int_{-\infty}^{z} \frac{dp}{dz}\, dz$

Eksperimentalna kum. funkcija $f(z) = \frac{\text{št. izmerkov} < z}{n}$

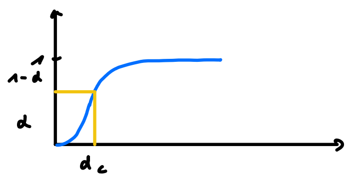


Testiramo max odmik

$$D = \sup |F(z) - f(z)|$$

Kolmogorov $P(D\sqrt{n} < d) = \sum_{k=-\infty}^{\infty} (-1)^k e^{-2k^2 d^2}$

standardizirana (univerzalna



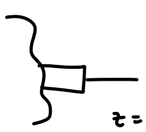
Tabela

$d_c(5\%) = 1{,}36$

$d_c(1\%) = 1{,}65$

$d_c(0{,}1\%) = 1{,}96$

Metoda najmanjših kvadratov



$z = Hx + r$

$H(t)$, $H(f(t))$

Meritve

$$\begin{bmatrix} z(t_1) \\ z(t_2) \\ \vdots \\ z(t_n) \end{bmatrix} = \begin{bmatrix} x_0 f_0(t_1) + x_1 f_1(t_1) + \dots x_m f_m(t_1) + r_1 \\ x_0 f_0(t_2) + \dots \\ \dots \end{bmatrix}$$

$$= \begin{bmatrix} f_0(t_1) & f_1(t_1) & \dots & f_m(t_1) \\ & \ddots & & \vdots \\ f_0(t_n) & \dots & & f_m(t_n) \end{bmatrix} \underbrace{\begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_m \end{bmatrix}}_{\text{iščemo}} + \begin{bmatrix} r_1 \\ \vdots \\ r_n \end{bmatrix}$$

$$\vec{z} = H\vec{x} + \vec{r}$$

Kalman $(\vec{z} - H\vec{x})^T R (\vec{z} - H\vec{x}) = 2J(\vec{x})$ minimalno

↳ kov. mat. ↳ por. po $\chi^2(n-m-1)$

Predpostavimo $R = \sigma^2 I$

Kalman

$$\hat{x} = \bar{x} + P H^T R^{-1} (\vec{z} - H\bar{x})$$

$$P^{-1} = M^{-1} + H^T R^{-1} H$$

$$R = \sigma^2 I$$

$\bar{x}$ - nepoznan $= 0$

$M^{-1} = 0$

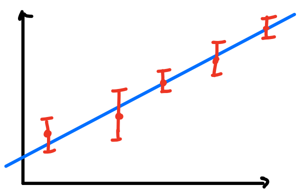
$$P^{-1} = H^T H \frac{1}{\sigma^2}$$

$$\boxed{P = \sigma^2 (H^T H)^{-1}}$$

$$\hat{x} = \sigma^2 (H^T H)^{-1} H^T \frac{1}{\sigma^2} \vec{z}$$

$$\Rightarrow \boxed{\hat{x} = (H^T H)^{-1} H^T \vec{z}}$$

1 Linearen primer



$$z = Hx + r = x_0 t + x_1 + r$$

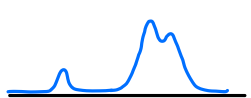
$$\vec{x} = \begin{bmatrix} x_0 \\ x_1 \end{bmatrix} \qquad H = \begin{bmatrix} t_1 & 1 \\ \vdots & \\ t_n & 1 \end{bmatrix} \qquad H^T H = \begin{bmatrix} \sum t_i^2 & \sum t_i \\ \sum t_i & n \end{bmatrix}$$

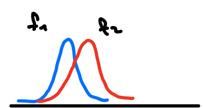
$$(H^T H)^{-1} = \frac{1}{n \sum t_i^2 - (\sum t_i)^2} \begin{bmatrix} n & -\sum t_i \\ -\sum t_i & \sum t_i^2 \end{bmatrix}$$

$$H^T z = \begin{bmatrix} \sum t_i z_i \\ \sum z_i \end{bmatrix}$$

$$\hat{x} = \begin{bmatrix} x_0 \\ x_1 \end{bmatrix} = \frac{1}{n \sum t_i^2 - (\sum t_i)^2} \begin{bmatrix} n \sum t_i z_i - \sum t_i \sum z_i \\ -\sum t_i z_i \sum t_i + \sum z_i \sum t_i^2 \end{bmatrix}$$

2 Luščenje vrhov v spektrih





i-ti kanal

$$z_i = (A_1 f_1 + A_2 f_2)_i \qquad f_1, f_2 \text{ znano}$$

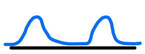
$$H = \begin{bmatrix} f_1(1) & f_2(1) \\ \vdots & \vdots \\ f_1(n) & f_2(n) \end{bmatrix} \qquad \vec{x} = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$$

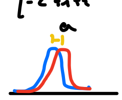
MNK:

$$\hat{x} = (H^T H)^{-1} H^T \vec{z}$$

$$P = (H^T H)^{-1} \sigma^2$$

$$P = \sigma^2 \begin{bmatrix} \sum f_1^2 & \sum f_1 f_2 \\ \sum f_1 f_2 & \sum f_2^2 \end{bmatrix}^{-1} = \frac{\sigma^2}{\sum f_1^2 \sum f_2^2 - (\sum f_1 f_2)^2} \begin{bmatrix} \sum f_2^2 & -\sum f_1 f_2 \\ -\sum f_1 f_2 & \sum f_1^2 \end{bmatrix}$$

1.) dobro separirane vrhova 

2.) močno prekrita vrhova 

$f_1 \sim f_2$

$$\sum f_1 f_2 \approx 0 \qquad P = \sigma^2 \begin{bmatrix} \frac{1}{\sum f_1^2} & 0 \\ 0 & \frac{1}{\sum f_2^2} \end{bmatrix}$$

P postane velik

$\rho_{12} = -1$ ker $P_{12} < 0$

Praktično: kdaj lahko ločimo vrhova? ko $a > \text{FWHM} \frac{1}{3}$

Lock-in detekcija

$z = Hx + r$

harmonična motnja $\sin(\omega t + \delta)$

$z = x_0 \sin(\omega t + \delta)$ iščemo x_0

MNK $z(t)$ gosta meritev $\{z(0), z(t_1), \dots\}$

$$H = \begin{bmatrix} \sin\delta \\ \sin \omega t_1 + \delta \\ \vdots \end{bmatrix}$$

$$\hat{x}_0 = (H^T H)^{-1} H^T z = \frac{\sum \sin(\omega t_i + \delta)\, z(t_i) \cdot \Delta t}{\sum \sin^2(\omega t_i + \delta) \cdot \Delta t} \overset{\Delta t \to 0}{=} \frac{\int_0^\tau \sin(\omega t + \delta)\, z(t)\, dt}{\int_0^\tau \sin^2(\omega t + \delta)\, dt}$$

$$= \frac{2}{T} \int_0^\tau \sin(\omega t + \delta)\, z(t)\, dt$$

Napaka amplitude

$\lim \sigma^2 \Delta t \to R$

$$P = \frac{\sigma^2 \Delta t}{\int_0^\tau \sin^2(\omega t + \delta)\, dt} = \frac{2R}{T}$$

Oblika frekvenčnega okna pri lock-in detekciji

Ref. $A(t) = A_\omega \cos \omega_0 t$ $\omega_0 =$ delovna frekvenca

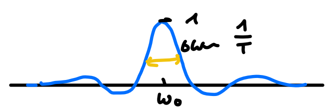
prepuščeni frekvenčni del ? $d(\omega)$

$$d(\omega) = \frac{2}{T} A_\omega \int \cos \omega_0 t \, \underbrace{\cos \omega t}_{z(t)} \, dt$$

$$= \frac{A_\omega}{T} \int_0^T \cos(\omega - \omega_0) t + \cos(\omega + \omega_0) t \, dt$$

$$= \frac{A_\omega}{T} \left(\frac{\sin(\omega - \omega_0) T}{\omega - \omega_0} + \underbrace{\frac{\sin(\omega + \omega_0) T}{\omega + \omega_0}}_{\approx 0} \right)$$

$$= A_\omega \frac{\sin(\omega - \omega_0) T}{(\omega - \omega_0) T}$$

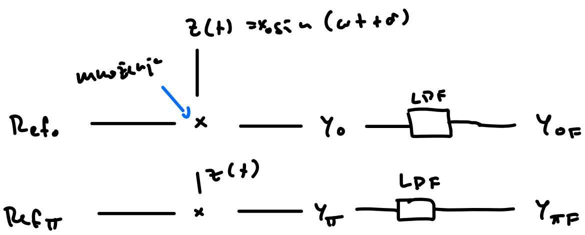


Lock-in faze

Lock-in amplifier

Ref$_0$ $= R \sin \omega t$

Ref$_{\pi/2}$ $= R \cos \omega t$



$$Y_0 = x_0 R \sin(\omega t + \delta) \sin \omega t = \frac{1}{2} x_0 R \left(\cos \delta - \cos(2\omega t + \delta) \right) \qquad Y_{0F} = \frac{1}{2} x_0 R \cos \delta$$

$$Y_\pi = \frac{1}{2} x_0 R \left(\sin \delta + \sin(2\omega t + \delta) \right) \qquad Y_{\pi F} = \frac{1}{2} x_0 R \sin \delta$$

$$\frac{1}{2} R x_0 = \sqrt{Y_{0F}^2 + Y_{0\pi}^2}$$

Lock-in, kaj moduliramo na senzorju

$s(t) \rightarrow V_{out}(s)$

spre. s od katere je odvisen vhod senzorja

s - stimulus

$A \ll \bar{s}$

$s(t) = \bar{s} + A \sin \Omega t$

$V(s(t)) = V(\bar{s}) + \left.\frac{dV}{ds}\right|_{\bar{s}} A \sin \Omega t$

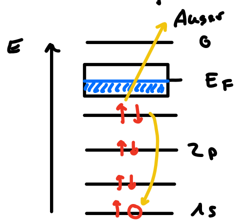
Izhod moduliran kot odvod

$V_{out} \propto \left.\frac{dV}{ds}\right|_{\bar{s}}$

Lock-in: izlušči Fourierovo komponento ref. frekvence

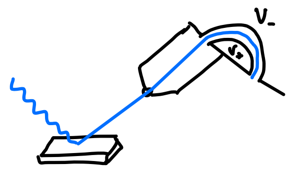
$$V(s) = V(\bar{s}) + \frac{dV}{ds} A \sin \Omega t + \frac{dV^2}{d^2 s} A^2 \sin 2\Omega t$$

Primer Odkrivanje Augerjevih elektronov



valenčni pas

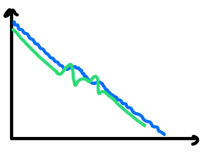
Želimo določiti W_{kin} Auger. e^-



retardni potencial

če pride e^- čez pri V im. pravo energijo

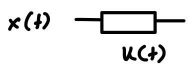
s stimulus je V



- navadno merjenje
- lock in

Stabilnost povratne zanke

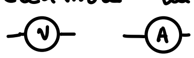
- optimalen merilni sistem → partikularna zgradba



$K(z - Hx)$

- realni merilni sistem → univerzalen (glede na $x(t)$)
  → neoptimalen (tranzienti, offset / sistematična napaka)
  → ohranjamo princip povratne zanke
  → primerno dušenje

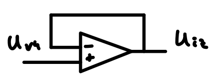
Zgled: električni merilni sistem



$$H(s) = \frac{\alpha\left(1 + 2\zeta \frac{s}{\omega_0}\right)}{1 + 2\zeta\frac{s}{\omega_0} + \frac{s^2}{\omega_0^2}}$$

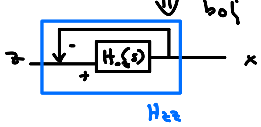
- stasiken
- dovolj hiter

⇒ napetostni sledilnik



unity gain opamp

⇓ bolj splošno



H_{oz} ... H odprte zanke

H_{zz} ... H zaključene zanke

$$H(s)\,(z-x) = x$$
$$H z = (H+1)\,x$$

$$H_{zz} = \frac{x}{z} = \frac{H_{oz}}{1+H_{oz}}$$ Nestabilno za $H \to -1$

$$H(i\omega) = \underbrace{|H(i\omega)|}_{1}\,\underbrace{e^{i\varphi}}_{-1}$$

$$\tan\varphi = \frac{\operatorname{Im} H(i\omega)}{\operatorname{Re} H(i\omega)} \Rightarrow \varphi \to \pm\pi$$

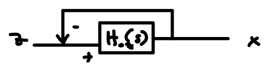
Naš zgled

unity gain $\left(\frac{x}{z}\right)_{oz} \xrightarrow{\omega\to 0} A_{DC} \sim 10^6 \qquad \xrightarrow{\omega\to\infty} 0$

$\left(\frac{x}{z}\right)_{zz} \xrightarrow{\omega\to 0} 1 \qquad \xrightarrow{\omega\to\infty} 0$

Stabilnost? Delni valovi, ki se širijo skozi tako vezje ob neg. pov. zanki

Gledam pri $\varphi = \pm\pi$ iščem $f = |H(i\omega)| = ?$



1. prehod $x = f z$; $z - (-f z) = z + f z = (1+f) z$
2. prehod $x = (1+f) f z$, $z - (-(1+f) f z) = (1+f+f^2)\,z$

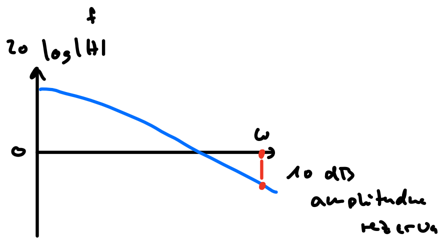
$\vdots$

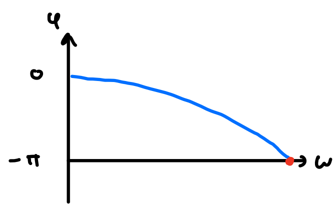
n-ti prehod $(1 + f + \dots + f^n)\,z = \frac{1-f^n}{1-f}\,z$

konvergira za $f<1$ (stabilnostni kriterij)

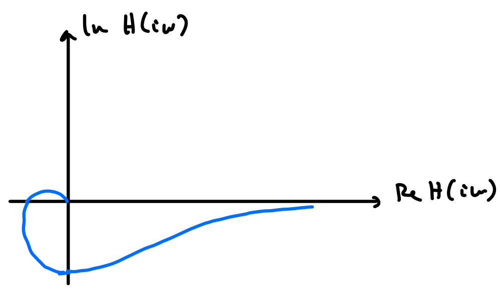
$f<1$ če smo pri $\varphi = \pm\pi$

Bode





Nyquist

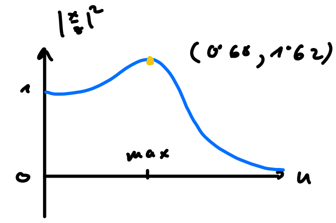


Pogoj za primerno dušenje
opt. sistem 2. reda

$$\left(\frac{x}{z}\right) = \frac{1 + 2\zeta \frac{s}{\omega_0}}{1 + 2\zeta\frac{s}{\omega_0} + \frac{s^2}{\omega_0^2}}$$

$$\left|\frac{x}{z}\right|^2 (i\omega) = \frac{1 + 4\zeta^2 \left(\frac{\omega}{\omega_0}\right)^2}{\left(1 - \frac{\omega^2}{\omega_0^2}\right)^2 + 4\zeta^2 \left(\frac{\omega}{\omega_0}\right)^2}$$

$$u = \frac{\omega}{\omega_0}$$



pri $\zeta = \frac{1}{\sqrt{2}}$

$$\left|\frac{x}{z}\right|^2 = \frac{1 + 2u^2}{(1-u^2)^2 + 2u^2} = \frac{1+2u^2}{1+u^4}$$

max =?

$$\frac{4u(1+u^4) - (1+2u^2)4u^3}{(1+u^4)^2} = \frac{4u^5 + 4u - 4u^3 - 8u^5}{(1+u^4)^2} = \frac{4u(-u^4 - u^2 + 1)}{(1+u^4)^2} = 0$$

$$u^4 + u^2 - 1 = 0 \quad \Rightarrow \quad u_{1,2}^2 = -\frac{1}{2} \pm \frac{\sqrt{5}}{2} \overset{+}{=} 0,681$$

$$M^2 = \left|\frac{x}{z}\right|^2 (u_1) = 1,62$$

Pogoj $\left|\frac{x}{z}\right|_{oz} < 1,3$

Zaključena zanka $\left|\frac{x}{z}\right|_{zz} = M^2 = \left|\frac{H}{1+H}\right|^2 = \left|\frac{\xi + i\eta}{1 + \xi + i\eta}\right|^2 = \frac{\xi^2 + \eta^2}{(1+\xi)^2 + \eta^2}$

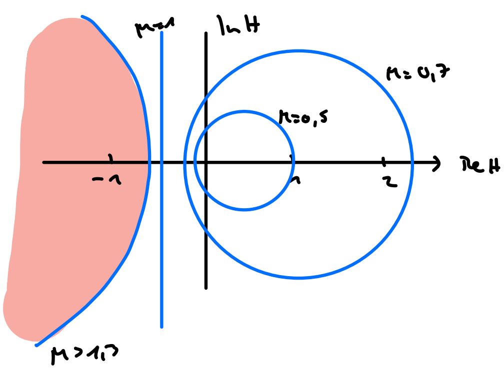
$$H = H_{oz} \in \mathbb{C} = \xi + i\eta$$

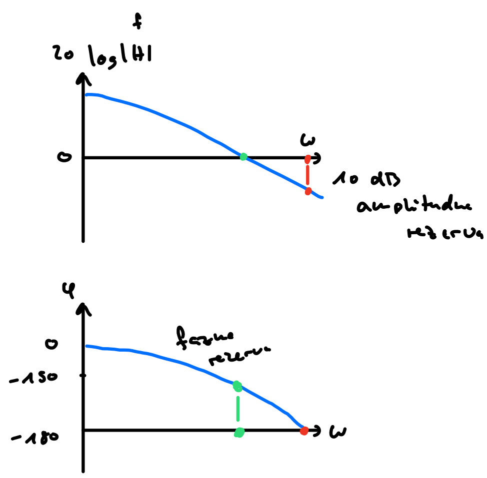
$$M^2(1+\xi)^2 + M^2\eta^2 = \xi^2 + \eta^2$$

$$M^2 = \xi^2(1-M^2) + \eta^2(1-M^2) - M^2 2\xi$$

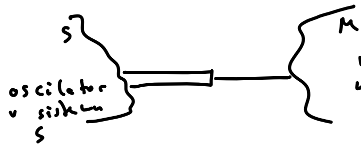
$$\frac{M^2}{(1-M)^2} = \xi^2 + \eta^2 - \frac{M^2}{(1-M)^2} 2\xi$$

$$\frac{M^2}{(1-M)^2}\left(1 + \frac{M^2}{(1-M)^2}\right) = \left(\frac{M}{1-M^2}\right)^2 = \left(\xi - \frac{M^2}{(1-M)^2}\right)^2 + \eta^2 \quad \text{Krožnica}$$



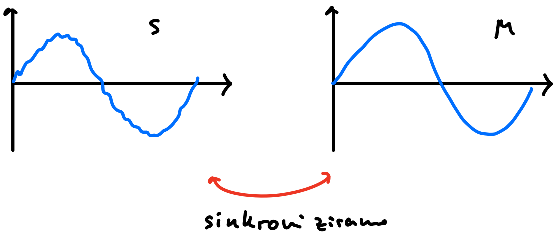


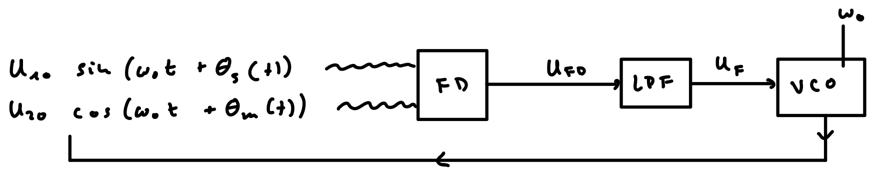
Merjenje frekvence, PLL fazno vpeta zanka



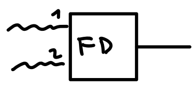
napetostno kontroliran modelski oscilator

Osnovna shema
1.) Fazni detektor FD
2.) Filter LPF
3.) VCO - voltage controlled oscillator





1. Fazni detektor



$$U_{FD} = k\langle U_{10}U_{20}\sin(\omega_0 t + \theta_1)\cos(\omega_0 t + \theta_2)\rangle$$

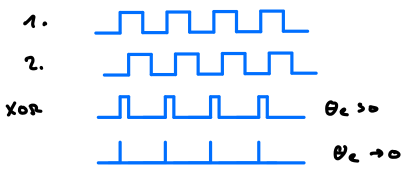
časovno povprečje

$$= k\, U_{10}U_{20}\left(\langle \sin(\theta_1 - \theta_2)\rangle + \underbrace{\langle \sin(2\omega_0 t + \theta_1 + \theta_2)\rangle}_{=0}\right)$$

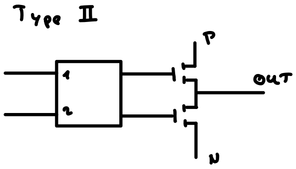
$$= k_{FD}\ \sin(\theta_1 - \theta_2) \qquad \theta_e = \theta_1 - \theta_2$$

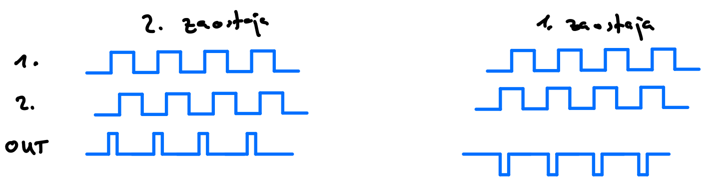
$$= k_{FD}\ \sin\theta_e \doteq k_{FD}\ \theta_e$$

1.) Type I, XOR gate

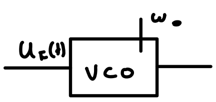


2.) Type II





2. VCO



$U_{20}\cos(\omega_0 t + \theta_2(t))$

zahtevamo monotonost izhoda

$$\omega_2(t) = \omega_0 + k_0\, U_F(t)$$

$$\omega_2 t = \omega_0 t + \theta_2(t) \qquad \left(\tfrac{d}{dt}\right.$$

$$\omega_2 = \omega_0 + \dot\theta_2(t) = \omega_0 + k_0\, U_F(t)$$

$$\Rightarrow\ s\,\Theta_2(s) = k_0\, U_F(s) \qquad \Rightarrow\ \frac{\Theta_2(s)}{U_F(s)} = \frac{k_0}{s}$$

3. Regulacijski filter

$$F(s) = \frac{U_F(s)}{U_{FD}(s)}$$

$\Rightarrow$ PLL $\qquad \theta_2(s) = \frac{K_o}{s} u_{FD}(s)\, F(s)$

$$\theta_2(s) = \frac{K_o K_{FD}}{s} F(s) \left(\theta_1(s) - \theta_2(s)\right)$$

$$\theta_1(s) = \theta_e(s) \left(1 + \frac{K_o K_{FD}}{s} F(s)\right)$$

$$\frac{\theta_e}{\theta_1} = \frac{1}{1 + \frac{K_o K_{FD}}{s} F(s)}$$

Določimo optimalen filter (Kalman)

θ_s .. faza v sistemu s

Dinamika:

$$\dot{\theta}_s = \omega_s$$

$$\dot{\omega}_s = 0 + w \qquad \langle w(t)\, w(t') \rangle = \delta(t-t')\, Q \qquad \text{bel šum}$$

Izmerek faze

$$\theta_1(t) = z + r(t) \qquad \langle r(t)\, r(t') \rangle = \delta(t'-t)\, R$$

V sistemu μ:

$$\dot{\hat{\theta}}_s = \hat{\omega}_s + K_\theta (z - \hat{\theta}_s)$$

$$\dot{\hat{\omega}}_s = 0 + K_\omega (z - \hat{\theta}_s)$$

$$\ddot{\hat{\theta}}_s = \dot{\hat{\omega}}_s + K_\theta (\dot{z} - \dot{\hat{\theta}}_s) = K_\omega (z - \hat{\theta}_s) + K_\theta (\dot{z} - \dot{\hat{\theta}}_s)$$

$$\ddot{\hat{\theta}}_s + K_\theta \dot{\hat{\theta}}_s + K_\omega \hat{\theta}_s = K_\theta \dot{z} + K_\omega z$$

$$(s^2 + K_\theta s + K_\omega)\, \theta_s(s) = (K_\omega + K_\theta s)\, z$$

$$\theta_s = \frac{K_\omega + K_\theta s}{s^2 + K_\theta s + K_\omega}\, z$$

$$s^2 \theta_s = K_\theta s (z - \theta_s) + K_\omega (z - \theta_s)$$

$$\theta_s = \frac{1}{s} \left(\frac{K_\omega}{s} + K_\theta\right)(z - \theta_s)$$

Očitno $\quad \hat{\theta}_s \rightarrow \theta_2$

$\qquad\quad z \rightarrow \theta_1$

$$\theta_2 = \frac{1}{s} \underbrace{K_o K_{FD}\, F(s)}\, (\theta_1 - \theta_2)$$

$$\theta_2 = \frac{1}{s} \overbrace{\left(\frac{K_\omega}{s} + K_\theta\right)}\, (z - \theta_s)$$

$\Rightarrow \quad F(s) = \frac{1}{K_o K_{FD}} \left(\frac{K_\omega}{s} + K_\theta\right) \qquad$ optimalna oblika filtra

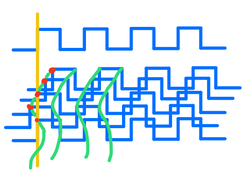
Primer: $\quad F(s) = \frac{1}{1 + \tau s} \qquad$ RC člen

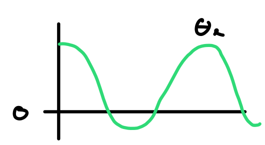
$$\frac{\theta_e}{\theta_1} = \frac{1}{1 + \frac{K_o K_{FD}}{s} \frac{1}{1+\tau s}} = \frac{s + \tau s^2}{s + \tau s^2 + K_o K_{FD}} = \frac{s^2 + s/\tau}{s^2 + s/\tau + K_o K_{FD}/\tau}$$

Želim $\tau \to \infty$, da dobro poreže visoke frekvence, problem ker dušilnega člena ni več

θ_1

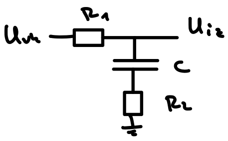
θ_2





ker je neoptimalno dušeno (ni stabilen)

Modificiran RC filter



$$\frac{U_{iz}}{R_2 + \frac{1}{i\omega C}} = \frac{U_{vh}}{R_1+R_2+\frac{1}{i\omega C}} \Rightarrow H = \frac{U_{iz}}{U_{vh}} = \frac{i\omega C}{i\omega C(R_1+R_2)+1} \frac{1}{i\omega C}(R_2 i\omega C+1)$$

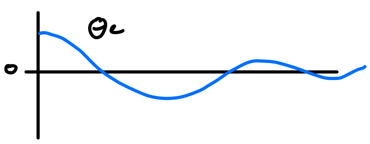
$$H = \frac{1+i\omega\tau_2}{1+i\omega(\tau_2+\tau_1)} = \frac{1+s\tau_2}{1+s(\tau_1+\tau_2)}$$

$$\frac{\theta_e}{\theta_1} = \frac{1}{1+\frac{K_oK_{FD}}{s}\,\frac{1+s\tau_2}{1+s(\tau_1+\tau_2)}}$$

$$= \frac{s+s^2(\tau_1+\tau_2)}{s+s^2(\tau_1+\tau_2)+K_oK_{FD}+s\tau_2K_oK_{FD}}$$

$$= \frac{s^2 + s/(\tau_1+\tau_2)}{s^2 + s\underbrace{(1+\tau_2K_oK_{FD})/(\tau_1+\tau_2)}_{\text{dušilni člen}} + \underbrace{\frac{K_oK_{FD}}{\tau_1+\tau_2}}_{\omega_N^2}}$$

optimalno dušenje $\quad 2\zeta\omega_N = \frac{1+\tau_2K_oK_{FD}}{\tau_1+\tau_2} \qquad \tau_2K_oK_{FD} \gg 1$



$$\sqrt{2}\,\omega_N = \omega_N^2\tau_2 \Rightarrow \tau_2 = \frac{\sqrt{2}}{\omega_N}$$

Omejitve delovanja PLL in stabilnost

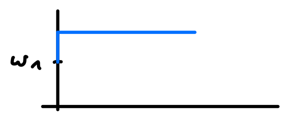
$\theta_e = \theta_1 - \theta_2 \qquad U_o = K_{FD}\sin\theta_e \doteq K_{FD}\theta_e$

lahko krava

$$K_{FD}\theta_e \rightarrow K_{FD}\sin\theta_e$$
$$K_{FD}\dot\theta_e \rightarrow K_{FD}\cos\theta_e\,\dot\theta_e$$

$$\ddot\theta_e + \frac{1+K_oK_{FD}\tau_2\cos\theta_e}{\tau_1+\tau_2}\dot\theta_e + \frac{K_oK_{FD}\sin\theta_e}{\tau_1+\tau_2} = \ddot\theta_1 + \frac{\dot\theta_1}{\tau_1+\tau_2}$$ Nelin. prenosna funkcija

1. Frekvenčni skok $\quad \dot\theta_1 = \Delta\omega \quad \frac{d}{dt}\Delta\omega = 0 = \ddot\theta_1$



Odgovor zanke. Iščemo stacionarno rešitve $\ddot\theta_e = 0,\ \dot\theta_e = 0\ \big|_{t\to\infty}$

$$\frac{K_oK_{FD}}{\tau_1+\tau_2}\sin\theta_e = \frac{\Delta\omega}{\tau_1+\tau_2}$$

$$\sin\theta_e = \frac{\Delta\omega}{K_oK_{FD}}$$

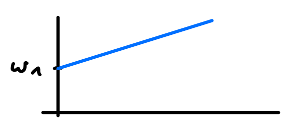
Če rešitev obstaja, imamo stac. rešitev.

$$1 > \frac{\Delta\omega}{K_oK_{FD}} \Rightarrow K_oK_{FD} > \Delta\omega$$
$$K_oK_{FD} > \Delta\omega_{max}$$

2. Kako hitro pa se lahko vhodna frekvenca odmika?

$\frac{d}{dt}\Delta\omega \neq 0 \qquad \Delta\omega = 0$

$\ddot\theta_1 \neq 0 \qquad \dot\theta_1 = 0$



Stabilnost, $\dot{\theta}_e = 0$, $\ddot{\theta}_e = 0$

$$\frac{K_o K_{FD}}{\tau_1+\tau_2} \sin\theta_e = \frac{d}{dt}\Delta\omega$$

$$\frac{K_o K_{FD}}{\tau_1+\tau_2} > \frac{d}{dt}\Delta\omega \qquad \frac{d}{dt}\Delta\omega_{max} = \frac{K_o K_{FD}}{\tau_1+\tau_2} = \omega_N^2$$

Dinamika uklapljanja PLL

$\sin\theta_e \neq \theta_e$; naj bo $\theta_e(t) = \Delta\omega t$

FD $\quad U_{FD} = U_{FD} \sin(\Delta\omega t)$ — LPF — $U_F = |F(i\Delta\omega)|\, K_o K_{FD} \sin(\Delta\omega t)$ —

VCO — $\omega_2(t) = \omega_0 + \underbrace{K_o K_{FD} |F(i\Delta\omega)|}_{(\Delta\omega)_0} \sin(\Delta\omega t)$

ω_0 — amplituda

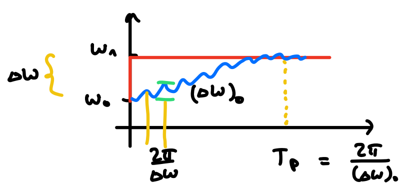
Kriterij za stabilnost

$$\Delta\omega \le \Delta\omega_L = K_o K_{FD} |F(i\Delta\omega_L)|$$

$$|F(i\omega_L)| = \frac{1+i\omega\tau_2}{1+i\omega(\tau_1+\tau_2)} \doteq \frac{\tau_2}{\tau_1+\tau_2}$$

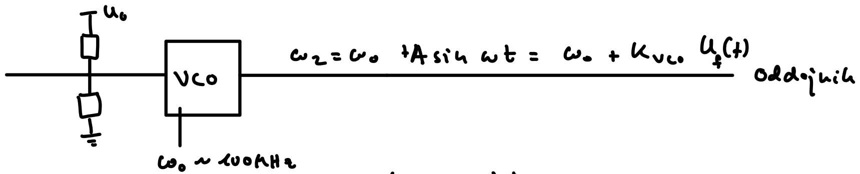
pri $b \to \infty$

$$\Rightarrow \Delta\omega_L = \frac{K_o K_{FD} \tau_2}{\tau_1+\tau_2}$$

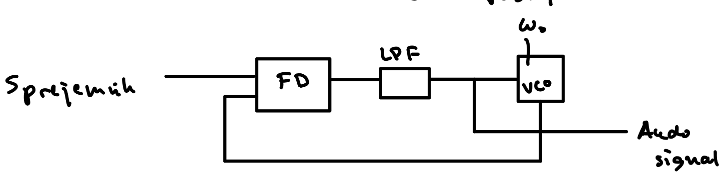


Frekvenčna modulacija / demodulacija

Audio signal (20 – 20 kHz)



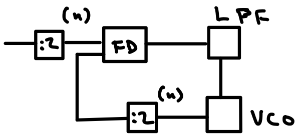
izbira postaje



Če sta oba VCO sinhronizirana → ju poganja enak signal $U_f(t)$

Frekvenčni sintesizer (generator)

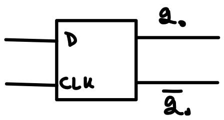
quartz $\rightarrow \omega_0$

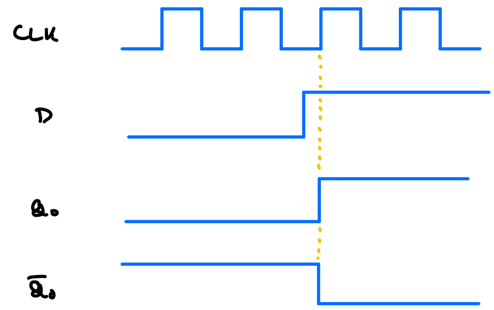


delitev frek. z 2

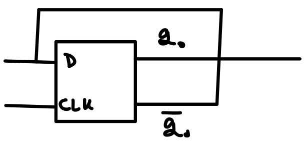
Frekvenčni delilnik $\boxed{:2}$

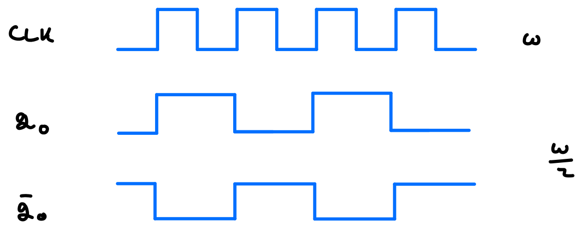
D-flip flop





Na rising edge CLK nastavi $Q_0 = D$ in $\bar{Q}_0 = -D$





Merjenje majhnih premikov

1. uporovni senzorji premika (uporovni lističi)
2. Kondenzatorski senzor
3. Piezo senzor
4. Induktivni senzor

1. Uporovni

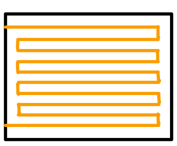
$$R = \rho \frac{l}{S} \qquad dR = \frac{dl\,\rho}{S} + \frac{d\rho\, l}{S} - \frac{\rho l}{S^2}\, dS$$

$$\frac{dR}{R} = \frac{dl}{l} + \frac{d\rho}{\rho} - \frac{dS}{S}$$

$$\frac{dS}{S} = -2\mu \frac{dl}{l}$$ Poissonovo število $\mu \sim \frac{1}{2}$

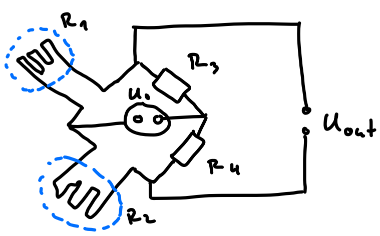
$$\frac{dR}{R} = \frac{dl}{l}\left(1 + 2\mu + \underbrace{\frac{d\rho/\rho}{dl/l}}_{\text{piezo rezistivnost}}\right) \sim 1$$

$$= 3\,\frac{dl}{l}$$



Uporovni lističi

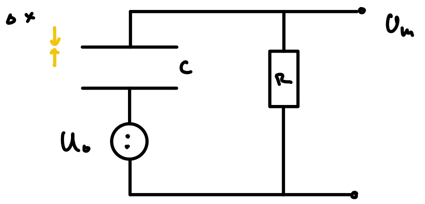
Vežemo na Wheatstonov mostiček



$$U_{out} = U_0 \frac{R_2 R_3 - R_1 R_4}{(R_1 + R_2)(R_3 + R_4)}$$

$$= U_0 \frac{\Delta R}{2R} \frac{R'}{2R'} = U_0 \frac{\Delta R}{4R}$$

② Kondenzatorski senzor



$$U_c + U_R + U_0 = 0$$
$$\dot U_c = -\dot U_R$$
$$U_c \sim -U_0$$

$$C = \frac{\varepsilon \varepsilon_0 S}{x}$$

$$dC = -\frac{\varepsilon \varepsilon_0 S}{x^2}\, dx$$

$$dC = -\left(\frac{C_0}{x_0}\right) dx$$

$$\frac{dC}{dt} = -\left(\frac{C_0}{x_0}\right)\frac{dx}{dt}$$

$$I = \frac{de}{dt} = \frac{d}{dt}(C U_c)$$

$$U_R = IR = R\left(U_c \frac{dC}{dt} + C\frac{dU_c}{dt}\right)$$

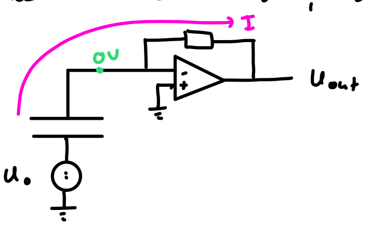
$$U_R = R\left(\left(\frac{C_0}{x_0}\right)\dot x U_0 - C\dot U_R\right) \quad \Big\} \mathcal{L}$$

$$U_R = \frac{RCU_0}{x_0} sx - RC\, sU_R$$

$$U_R(1+RCs) = \left(\frac{U_0}{x_0}\right) RCs\, X$$

$$H(s) = \frac{U_R}{x} = \frac{U_0}{x_0}\,\frac{\tau s}{1+\tau s} \qquad \tau = RC$$

Problem $U_c \sim -U_0$, dodamo op amp

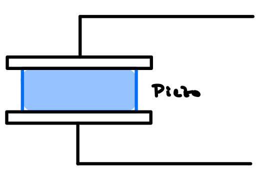


$$U_0 + U_c = 0$$
$$U_0 - \frac{e}{C} = 0$$
$$CU_0 = e \quad \Big/ \frac{d}{dt}$$
$$U_0 \dot C = I = -\frac{U_{out}}{R}$$
$$U_0 \frac{C_0}{x_0}\dot x = -\frac{U_{out}}{R} \quad \Big\} \mathcal{L}$$

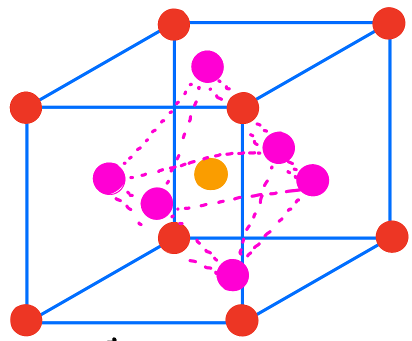
$$\frac{U_0 RC}{x_0} sx = U_{out}$$

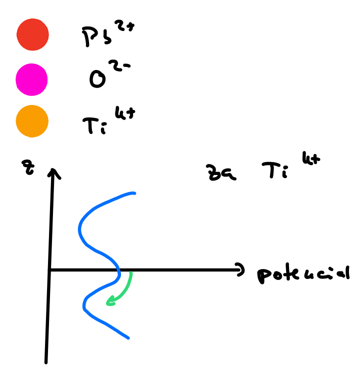
$$H = \frac{U_{out}}{x} = \frac{U_0}{x_0}\tau s$$

③ Piezo senzor



$PbTiO_3$, $SrTiO_3$





ob deformaciji pride do nastanka p_e
$\Rightarrow$ polarizacije $\vec P$

$$\int \vec P\, d\vec S = \int \vec D\, d\vec S = e$$

Linearni približek

$\vec{P}_e = \underline{\underline{d}}\,\underline{T}$

piezo napetostni tenzor — napetostni tenzor (mehanski)

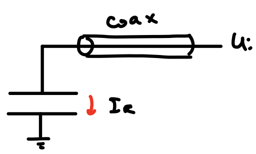
$P_i = \sum_{jk} d_{ijk} T_{jk}$

Zaradi simetrij $T_{jk} \to T_m$, $m = 1, \dots, 6$ trije raztezi in tri rotacije

$P_i = \sum d_{im} T_m$

Naboj v i smeri $\quad e_i = P_i S = \sum d_{im} T_m S = \sum d_{im} F_m$

Merimo napetost na kristalu



$e_3 = d_{33} F_3$

$\frac{de}{dt} = I = I_R + I_{cp} + I_{ck}$

tok skozi piezo zaradi upornosti; kapacitivnost piezo.; kabel

$d_{33} \dot{F}_3 = \frac{u_i}{R} + C_p \dot{u}_i + C_k \dot{u}_i$

$R d_{33} \dot{F}_3 = u_i + (C_p + C_k) R \dot{u}_i \quad \big\} \mathcal{L}$

$R d_{33} s F_3 = u_i \left(1 + (C_p + C_k) R s\right)$

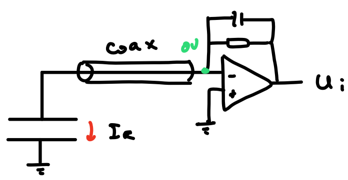
Hooke $\quad \frac{F}{S} = E \frac{\Delta x}{x}$

$R d_{33} E S \frac{1}{x_0} s \Delta x = u_i \left(1 + (C_p + C_k) R s\right)$

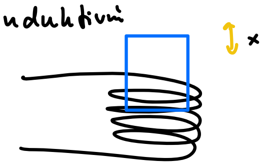
$H = \frac{u_i}{\Delta x} = \frac{d_3 E R S}{x_0} \cdot \frac{s}{1 + (C_p + C_k) R s}$

Problem: odvisno od kabla

Problem rešimo z op amp.



(4) Induktivni



$\dot{\phi}_m = (LI)^{\cdot} = I\dot{L}$