T

Ortogonalne krivočrtne koordinate in ∇

Koordinate q_i , bazni vektorji $\hat{e}_i$

$$\nabla = \sum_i \hat{e}_i \frac{1}{h_i} \frac{\partial}{\partial q_i} \qquad h_i = \left| \frac{\partial \vec{r}}{\partial q_i} \right| \qquad \hat{e}_i = \frac{1}{h_i} \frac{\partial \vec{r}}{\partial q_i}$$

h_i skalarni faktor

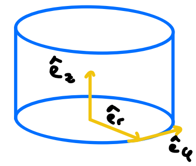
Gradient $\nabla f = \sum \hat{e}_i \frac{1}{h_i} \frac{\partial f}{\partial q_i}$ $\quad f$... skalarna funkcija

Divergenca $\nabla \cdot \vec{v} = \sum_i \hat{e}_i \frac{1}{h_i} \frac{\partial}{\partial q_i} \sum_j \hat{e}_j v_j$ $\quad \vec{v} = \sum \hat{e}_i v_i$ vektorska funkcija

Rotor $\nabla \times \vec{v} = \sum_i \hat{e}_i \frac{1}{h_i} \frac{\partial}{\partial q_i} \times \sum_j \hat{e}_j v_j$

Laplace $\nabla^2 f = \nabla \cdot (\nabla f) = \sum_i \hat{e}_i \frac{1}{h_i} \frac{\partial}{\partial q_i} \cdot \left(\sum_j \hat{e}_j \frac{1}{h_j} \frac{\partial f}{\partial q_j} \right)$

$\nabla^2 \vec{v} = \nabla \cdot (\nabla \vec{v}) = \sum_i \hat{e}_i \frac{1}{h_i} \frac{\partial}{\partial q_i} \left(\sum_j \hat{e}_j \frac{1}{h_j} \frac{\partial}{\partial q_j} \otimes \left(\sum_k \hat{e}_k v_k \right)\right)$ nastane vektor



Cilindrične koordinate

$$\vec{v} = v_r \hat{e}_r + v_\varphi \hat{e}_\varphi + v_z \hat{e}_z$$

$$\nabla \cdot \vec{v} = \sum_i \hat{e}_i \frac{1}{h_i} \frac{\partial}{\partial q_i} \cdot (v_r \hat{e}_r + v_\varphi \hat{e}_\varphi + v_z \hat{e}_z)$$

$$= \hat{e}_r \frac{\partial}{\partial r} (v_r \hat{e}_r + v_\varphi \hat{e}_\varphi + v_z \hat{e}_z)$$
$$+ \hat{e}_\varphi \frac{1}{r} \frac{\partial}{\partial \varphi} (v_r \hat{e}_r + v_\varphi \hat{e}_\varphi + v_z \hat{e}_z)$$
$$+ \hat{e}_z \frac{\partial}{\partial z} (v_r \hat{e}_r + v_\varphi \hat{e}_\varphi + v_z \hat{e}_z)$$
$$= \frac{\partial v_r}{\partial r} + 0 + 0 + 0 + 0 + 0 +$$
$$\frac{1}{r} v_r + 0 + \frac{1}{r} \frac{\partial v_\varphi}{\partial \varphi} + 0 + 0 +$$
$$0 + 0 + 0 + 0 + \frac{\partial v_z}{\partial z}$$
$$= \frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{1}{r} \frac{\partial v_\varphi}{\partial \varphi} + \frac{\partial v_z}{\partial z}$$
$$= \frac{1}{r} \frac{\partial (r v_r)}{\partial r} + \frac{1}{r} \frac{\partial v_\varphi}{\partial \varphi} + \frac{\partial v_z}{\partial z}$$

$$\vec{r} = (r \cos\varphi, r \sin\varphi, z)$$
$$h_r = \left|\frac{\partial \vec{r}}{\partial r}\right| = (\cos\varphi, \sin\varphi, 0) = 1$$
$$h_\varphi = \left|\frac{\partial \vec{r}}{\partial \varphi}\right| = (-r \sin\varphi, r\cos\varphi, 0) = r$$
$$h_z = \left|\frac{\partial \vec{r}}{\partial z}\right| = (0, 0, 1) = 1$$

$$\frac{\partial \hat{e}_r}{\partial \varphi} = \frac{\partial}{\partial \varphi} (\cos\varphi, \sin\varphi, 0) = (-\sin\varphi, \cos\varphi, 0) = \hat{e}_\varphi$$
$$\frac{\partial \hat{e}_\varphi}{\partial \varphi} = \frac{\partial}{\partial \varphi} (-\sin\varphi, \cos\varphi) = (-\cos\varphi, -\sin\varphi) = -\hat{e}_r$$
$$\frac{\partial \hat{e}_r}{\partial r} = 0 \qquad \frac{\partial \hat{e}_\varphi}{\partial r} = 0$$

$$\nabla \times \vec{v} = \left(\hat{e}_r \frac{\partial}{\partial r} + \frac{1}{r} \hat{e}_\varphi \frac{\partial}{\partial \varphi} + \hat{e}_z \frac{\partial}{\partial z} \right) \times (v_r \hat{e}_r + v_\varphi \hat{e}_\varphi + v_z \hat{e}_z)$$

$$\left.\begin{aligned} &= 0 + \frac{\partial v_\varphi}{\partial r} \hat{e}_z - \frac{\partial v_z}{\partial r} \hat{e}_\varphi + \\ &- \frac{1}{r} \frac{\partial v_r}{\partial \varphi} \hat{e}_z + 0 + \frac{1}{r} \frac{\partial v_\varphi}{\partial \varphi} \hat{e}_z + \frac{\partial v_r}{\partial z} \hat{e}_\varphi - \frac{\partial v_\varphi}{\partial z} \hat{e}_r \end{aligned}\right\} \text{Verjetno narobe}$$

$$= \left(\frac{1}{r} \frac{\partial v_z}{\partial \varphi} - \frac{\partial v_\varphi}{\partial z} \right) \hat{e}_r + \left(\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r} \right) \hat{e}_\varphi + \left(\frac{\partial v_\varphi}{\partial r} - \frac{1}{r} \frac{\partial v_r}{\partial \varphi} + \frac{v_\varphi}{r} \right) \hat{e}_z$$

Sferične koordinate

$$x = r \sin\theta \cos\varphi \qquad h_r \hat{e}_r = \frac{\partial \vec{r}}{\partial r} = \sin\theta \cos\varphi \,\hat{e}_x + \sin\theta \sin\varphi \,\hat{e}_y + \cos\theta \,\hat{e}_z$$
$$y = r \sin\theta \sin\varphi \qquad h_\varphi \hat{e}_\varphi = \frac{\partial \vec{r}}{\partial \varphi} = -r \sin\theta \sin\varphi \,\hat{e}_x + r \sin\theta \cos\varphi \,\hat{e}_y + 0$$
$$z = r \cos\theta \qquad h_\theta \hat{e}_\theta = \frac{\partial \vec{r}}{\partial \theta} = r \cos\theta \cos\varphi \,\hat{e}_x + r \cos\theta \sin\varphi \,\hat{e}_y - r \sin\theta \,\hat{e}_z$$

$$h_r = 1 \qquad \hat{e}_r = (\sin\theta \cos\varphi, \sin\theta \sin\varphi, \cos\theta)$$
$$h_\varphi = r \sin\theta \qquad \hat{e}_\varphi = (-\sin\varphi, \cos\varphi, 0)$$
$$h_\theta = r \qquad \hat{e}_\theta = (\cos\theta \cos\varphi, \cos\theta \sin\varphi, -\sin\theta)$$

$$\frac{\partial \hat{e}_r}{\partial \theta} = (\cos\theta \cos\varphi, \cos\theta \sin\varphi, -\sin\theta) = \hat{e}_\theta$$
$$\frac{\partial \hat{e}_\theta}{\partial \theta} = (-\sin\theta \cos\varphi, -\sin\theta \sin\varphi, -\cos\theta) = -\hat{e}_r$$
$$\frac{\partial \hat{e}_r}{\partial \varphi} = (-\sin\theta \sin\varphi, \sin\theta \cos\varphi, 0) = \sin\theta\, \hat{e}_\varphi$$
$$\frac{\partial \hat{e}_\theta}{\partial \varphi} = (-\cos\theta \sin\varphi, \cos\theta \cos\varphi, 0) = \cos\theta\, \hat{e}_\varphi$$
$$\frac{\partial \hat{e}_\varphi}{\partial \varphi} = (-\cos\varphi, -\sin\varphi, 0) = -\sin\theta\, \hat{e}_r - \cos\theta\, \hat{e}_\theta$$

1 Zapiši simetrični del deformacijskega tenzorja v krivočrtnih koordinatah

$$u_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}\right)$$

$$\nabla = \sum_i \hat{e}_i \frac{1}{h_i}\frac{\partial}{\partial q_i} \qquad u = \sum_i u_i \hat{e}_i$$

$$\nabla u = \left(\sum_l \hat{e}_l \frac{1}{h_l}\frac{\partial}{\partial q_l}\right) \otimes \left(\sum_k u_k \hat{e}_k\right)$$
$$= \sum_{lk} \hat{e}_l \otimes \hat{e}_k \frac{1}{h_l}\frac{\partial u_k}{\partial q_l} + \sum_{lk} \hat{e}_l \otimes \frac{\partial \hat{e}_k}{\partial q_l}\frac{1}{h_l} u_k$$
$$= \sum_{lk} \hat{e}_l \otimes \hat{e}_k \frac{\partial u_k}{h_l \partial q_l} + \sum_{lk} u_k \sum_m \Gamma^m_{kl}\, \hat{e}_l \otimes \hat{e}_m$$

Kristoflov simbol

$$\frac{\partial \hat{e}_j}{h_i \partial q_i} = \sum_k \Gamma^k_{ij}\, \hat{e}_k$$

$$(\nabla u)_{ij} = \hat{e}_i \cdot (\nabla u) \cdot \hat{e}_j$$
$$= \frac{\partial u_j}{h_i \partial q_i} + \sum_k u_k \Gamma^j_{ki}$$

$$u_{ij} = \frac{1}{2}\left((\nabla u)_{ij} + (\nabla u)_{ji}\right)$$

$$u_{ij} = \frac{1}{2}\left(\frac{\partial u_j}{h_i \partial q_i} + \sum_k u_k \Gamma^j_{ki} + \frac{\partial u_i}{h_j \partial q_j} + \sum_k u_k \Gamma^i_{kj}\right)$$

$$u_{ij} = \frac{1}{2}\left(\frac{\partial u_j}{h_i \partial q_i} + \frac{\partial u_i}{h_j \partial q_j} + \sum_k u_k \left(\Gamma^j_{ik} + \Gamma^i_{jk}\right)\right)$$

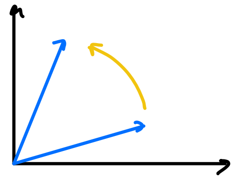
2 Zapiši deformacijski tenzor v sfernih koordinatah

$$\nabla u = \left(\hat{e}_r \frac{1}{1}\frac{\partial}{\partial r} + \hat{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \hat{e}_\varphi \frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi}\right) \otimes \left(u_r \hat{e}_r + u_\theta \hat{e}_\theta + u_\varphi \hat{e}_\varphi\right) =$$
$$= \left(\hat{e}_r \otimes \hat{e}_r \frac{\partial u_r}{\partial r} + 0 + \hat{e}_r \otimes \hat{e}_\theta \frac{\partial u_\theta}{\partial r} + 0 + \hat{e}_r \otimes \hat{e}_\varphi \frac{\partial u_\varphi}{\partial r} + 0\right) +$$
$$+ \left(\frac{1}{r}\hat{e}_\theta \otimes \hat{e}_r \frac{\partial u_r}{\partial \theta} + \frac{1}{r}\hat{e}_\theta \otimes \underbrace{\frac{\partial \hat{e}_r}{\partial \theta}}_{\hat{e}_\theta} u_r + \frac{1}{r}\hat{e}_\theta \otimes \hat{e}_\theta \frac{\partial u_\theta}{\partial \theta} + \frac{1}{r}\hat{e}_\theta \otimes \underbrace{\frac{\partial \hat{e}_\theta}{\partial \theta}}_{-\hat{e}_r} u_\theta\right.$$
$$\left. + \frac{1}{r}\hat{e}_\theta \otimes \hat{e}_\varphi \frac{\partial u_\varphi}{\partial \theta} + \frac{1}{r}\hat{e}_\theta \otimes \underbrace{\frac{\partial \hat{e}_\varphi}{\partial \theta}}_{0} u_\varphi\right) +$$
$$+ \frac{1}{r\sin\theta}\left(\hat{e}_\varphi \otimes \hat{e}_r \frac{\partial u_r}{\partial \varphi} + \hat{e}_\varphi \otimes \underbrace{\frac{\partial \hat{e}_r}{\partial \varphi}}_{\sin\theta\, \hat{e}_\varphi} u_r + \hat{e}_\varphi \otimes \hat{e}_\theta \frac{\partial u_\theta}{\partial \varphi} + \hat{e}_\varphi \otimes \underbrace{\frac{\partial \hat{e}_\theta}{\partial \varphi}}_{\cos\theta\, \hat{e}_\varphi} u_\theta +\right.$$
$$\left. \hat{e}_\varphi \otimes \hat{e}_\varphi \frac{\partial u_\varphi}{\partial \varphi} + \hat{e}_\varphi \otimes \underbrace{\frac{\partial \hat{e}_\varphi}{\partial \varphi}}_{-\sin\theta\, \hat{e}_r - \cos\theta\, \hat{e}_\theta} u_\varphi\right)$$

Zapišimo komponente tenzorja in ga simetrizirajmo

$$u_{rr} = \frac{\partial u_r}{\partial r} \qquad u_{r\theta} = \frac{1}{2}\left(\frac{\partial u_\theta}{\partial r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r}\right)$$
$$u_{\theta\theta} = \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} \qquad u_{r\varphi} = \frac{1}{2}\left(\frac{\partial u_\varphi}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial u_r}{\partial \varphi} - \frac{u_\varphi}{r}\right)$$
$$u_{\varphi\varphi} = \frac{u_r}{r} + \frac{\cot\theta}{r} u_\theta + \frac{\partial u_\varphi}{\partial \varphi} \qquad u_{\varphi\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_\varphi}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial \varphi} - \frac{\cot\theta}{r} u_\varphi\right)$$

③ Izračunaj def. tenzor če telo zavrtimo okoli z osi



$$R = \begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\vec{x}\,' = R\vec{x}$$

$$\vec{u} = \vec{x}\,' - \vec{x} = R\vec{x} - \vec{x} = (R - I)\vec{x} \qquad \vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\vec{u} = \begin{bmatrix} (\cos\varphi - 1)\,x - \sin\varphi\, y \\ \sin\varphi\, x + (\cos\varphi - 1)\, y \\ 0 \end{bmatrix}$$

$$u_{xx}^{lin} = \frac{\partial u_x}{\partial x} = \cos\varphi - 1$$
$$u_{yy}^{lin} = \frac{\partial u_y}{\partial y} = \cos\varphi - 1$$
$$u_{xy}^{lin} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) = \frac{1}{2}\left(-\sin\varphi + \sin\varphi\right) = 0$$

$$u^{lin} = \begin{bmatrix} \cos\varphi - 1 & 0 & 0 \\ 0 & \cos\varphi - 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$u_{xx}^{kvad} = \frac{1}{2}\sum_k \left(\frac{\partial u_k}{\partial x}\right)^2 = \frac{1}{2}\left((\cos\varphi - 1)^2 + (\sin\varphi)^2\right) = 1 - \cos\varphi$$
$$u_{yy}^{kvad} = \frac{1}{2}\sum_k \left(\frac{\partial u_k}{\partial y}\right)^2 = \frac{1}{2}\left((-\sin\varphi)^2 + (\cos\varphi - 1)^2\right) = 1 - \cos\varphi$$
$$u_{xy}^{kvad} = \frac{1}{2}\sum_k \frac{\partial u_k}{\partial x}\frac{\partial u_k}{\partial y} = \frac{1}{2}\left((\cos\varphi - 1)\sin\varphi - \sin\varphi(\cos\varphi - 1)\right) = 0$$

$$u = u^{lin} + u^{kvad} = 0$$

④ Primeri σ_{ij}

Izotropni tlak

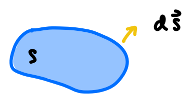
$$\sigma_{ij} = -p\,\delta_{ij}$$

sila normala

Volumska gostota sile na delček

$$f_i = \frac{\partial}{\partial x_j}\sigma_{ij} = -\frac{\partial}{\partial x_j} p\,\delta_{ij} = -\frac{\partial p}{\partial x_i} \qquad \vec{f} = -\nabla p$$

Sila na območje



$$F_i = \oint_S \sigma_{ij}\, dS_j \qquad \text{po definiciji}$$
$$= -\oint p\,\delta_{ij}\, dS_j$$
$$= -\oint p\, dS_i$$

$$F_i = \oint_S \sigma_{ij}\, dS_j = \int \frac{\partial}{\partial x_j}\sigma_{ij}\, dV$$
$$= -\int \frac{\partial}{\partial x_j} p\,\delta_{ij}\, dV = -\int \frac{\partial p}{\partial x_i}\, dV$$

p je hidrostatičen tlak $\quad p(r) = \rho\,\vec{g}\cdot\vec{r} = \rho\, g_i r_i$

$$F_i = -\oint \rho\, g_j r_j\, dS_i \qquad F_i = -\int \rho\, g_j \underbrace{\frac{\partial r_j}{\partial x_i}}_{\delta_{ij}}\, dV = -\int \rho\, g_i\, dV = -\rho\, g_i V$$

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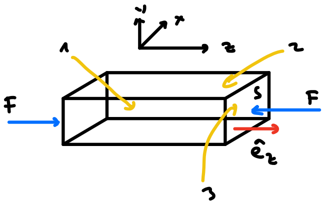


Cauchyjev pogoj

$$f_i = \frac{\partial}{\partial x_j}\sigma_{ij} = 0$$

$$\sigma_{ij} = konst.$$

Robni pogoji

$$F_i = \oint \sigma_{ij}\, dS_j$$

3 $\quad \sigma_{zz} = \frac{F_z}{S_z} = \frac{F}{S} \qquad \sigma_{xz} = \frac{F_x}{S_z} = 0 \qquad \sigma_{yz} = \frac{F_y}{S_z} = 0$

1 $\quad \sigma_{ix} = 0$

2 $\quad \sigma_{iy} = 0$

Hookov zakon

$$u_{ij} = \frac{1}{2\mu}\left(\sigma_{ij} - \frac{\lambda}{2\mu+3\lambda}\sigma_{kk}\delta_{ij}\right)$$

$$u_{xx} = \frac{1}{2\mu}\left(0 - \frac{\lambda}{2\mu+3\lambda}\sigma_{zz}\right) \qquad u_{xy} = 0 \qquad u_{xz} = 0 \qquad u_{yz} = 0$$

$$= -\frac{\lambda}{2\mu(2\mu+3\lambda)}\frac{F}{S} = u_{yy}$$

$$u_{zz} = \frac{1}{2\mu}\left(\sigma_{zz} - \frac{\lambda}{2\mu+3\lambda}\sigma_{zz}\right) = \frac{(\mu+\lambda)}{\mu(2\mu+3\lambda)}\frac{F}{S}$$

Iščemo ΔV

$$\frac{\Delta V}{V} = u_{kk}$$

$$\frac{\Delta V}{V} = -\frac{2\cdot\lambda}{2\mu(2\mu+3\lambda)}\frac{F}{S} + \frac{(\mu+\lambda)}{\mu(2\mu+3\lambda)}\frac{F}{S}$$

$$= \frac{1}{2\mu+3\lambda}\frac{F}{S}$$

Iščemo E

$$\frac{F}{S} = E\frac{\Delta l}{l} = E\,u_{zz} = E\frac{(\mu+\lambda)}{\mu(2\mu+3\lambda)}\frac{F}{S}$$

$$E = \frac{\mu(2\mu+3\lambda)}{\mu+\lambda}$$

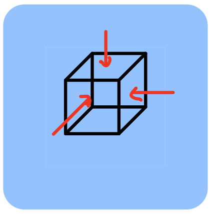
Iščemo Poissonovo razmerje σ

$$\sigma = -\frac{\Delta x/x}{\Delta z/z} = -\frac{u_{xx}}{u_{zz}} = -\frac{-\frac{\lambda}{2\mu(2\mu+3\lambda)}\frac{F}{S}}{\frac{(\mu+\lambda)}{\mu(2\mu+3\lambda)}\frac{F}{S}} = \frac{\lambda}{2(\lambda+\mu)}$$

Obratno $\qquad \mu = \frac{E}{2(1+\sigma)} \qquad \lambda = \frac{E\sigma}{(1-2\sigma)(1+\sigma)}$

$$\frac{\Delta V}{V} = \frac{F}{S}\frac{1}{2\mu+3\lambda} = \frac{F}{S}\frac{1}{\frac{E}{(1+\sigma)} + \frac{3E\sigma}{(1-2\sigma)(1+\sigma)}} = \frac{F}{S}\frac{(1-2\sigma)}{E}$$

⑥ Kocko potopimo v tekočino s tlakom p



$$\sigma_{ij} = -p\,\delta_{ij}$$

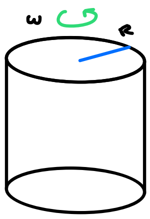
$$\frac{\Delta V}{V} = u_{kk} = \qquad\qquad u_{ij} = \frac{1}{2\mu}\left(\sigma_{ij} - \frac{\lambda}{2\mu+3\lambda}\,\sigma_{kk}\,\delta_{ij}\right)$$

$$= 3\,\frac{1}{2\mu}\left(-p - \frac{\lambda}{2\mu+3\lambda}\,(-3p)\right)$$

$$= -\frac{3p}{2\mu}\left(1 - \frac{3\lambda}{2\mu+3\lambda}\right) =$$

$$= -\frac{\lambda}{2\mu+3\lambda}\,p = -\frac{3(1-2\sigma)}{E}\,p = -\chi_T p$$

⑦ Homogen valj se vrti z ω. Koliko se razširi valj? Privzemimo, da se valj ne raztegne v smeri osi vrtenja.



Navierova enačba z E in σ

$$\rho\ddot{\vec u} = \vec f^{z} + \frac{E}{2(1+\sigma)}\left(\nabla^2\vec u + \frac{1}{1-2\sigma}\nabla(\nabla\cdot\vec u)\right)$$

$$\vec u = u_r(r)\,\hat e_r = u(r)\,\hat e_r$$

$$\vec f^{zun} = \frac{F_c}{V} = \frac{m\omega^2 r}{V}\,\hat e_r = \rho\omega^2 r\,\hat e_r$$

$$\nabla\,\nabla\cdot\vec u = \nabla\times(\nabla\times\vec u) + \nabla\cdot\nabla\cdot\vec u = \nabla\times\nabla\times\vec u + \nabla^2\vec u$$

$$0 = \rho\omega^2 r\,\hat e_r + \frac{E}{2(1+\sigma)}\Big(-\nabla\times\underbrace{\nabla\times\vec u}_{=0\ \text{ker}\ \vec u = u(r)} + \underbrace{\left(1+\frac{1}{1-2\sigma}\right)}_{\frac{2-2\sigma}{1-2\sigma}}\nabla(\nabla\cdot\vec u)\Big)$$

$$0 = \rho\omega^2 r\,\hat e_r + \frac{E\,(1-\sigma)}{(1+\sigma)(1-2\sigma)}\underbrace{\nabla(\nabla\cdot\vec u)}_{\text{vemo da kaže v}\ \hat e_r}$$

$$\rho\omega^2 r = -\frac{E(1-\sigma)}{(1+\sigma)(1-2\sigma)}\,\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\,r\,u(r)\right) \qquad \alpha = \frac{(1+\sigma)(1-2\sigma)}{(1-\sigma)}\,\frac{\rho\omega^2}{E}$$

$$-\alpha r = \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\,r\,u(r)\right)$$

$$-\alpha\frac{r^2}{2} + A = \frac{1}{r}\frac{\partial}{\partial r}(r\,u(r)) \quad \left(= \nabla u = u_{rr}\right)$$

$$-\alpha\frac{r^4}{8} + A\frac{r^2}{2} + B = r\,u(r)$$

$$u(r) = -\alpha\frac{r^3}{8} + A\frac{r}{2} + B\frac{1}{r}$$

RP
- $B = 0$
- Na plašču ni ploskovnih sil, $\sigma_{rr} = 0$

površinsko porazdeljena sila z normalo v smeri r.

Hooke $$\sigma_{ij} = \frac{E}{1+\sigma}\left(u_{ij} + \frac{\sigma}{1-2\sigma}\,\underbrace{u_{kk}}_{\nabla u\,!}\,\delta_{ij}\right)$$

$$0 = \sigma_{rr} = \frac{E}{1+\sigma}\left(u_{rr} + \frac{\sigma}{1-2\sigma}\,\nabla u\right)$$

$$u_{zz} = 0 \qquad u_{\varphi\varphi} = \frac{\partial u_\varphi}{r\,\partial\varphi} + \frac{\overset{=u}{u_r}}{r} = \frac{u}{r} \qquad\qquad u_{rr} = \frac{\partial u}{\partial r} = -\alpha\frac{r^2 3}{8} + \frac{A}{2}$$

$$u_{kk} = 0 + \frac{u}{r} - \alpha\frac{r^2 3}{8} + \frac{A}{2} = -\alpha\frac{r^2}{8} + \frac{A}{2} - \alpha\frac{r^2 3}{8} + \frac{A}{2}$$

$$= -\alpha\frac{r^2}{2} + A$$ Kar je enako kot ∇u kot smo že izračunali.

$$\sigma_{rr} = \frac{E}{1+\sigma}\left(-\alpha\frac{3r^2}{8} + \frac{A}{2} + \frac{\sigma}{1-2\sigma}\left(-\alpha\frac{r^2}{2} + A\right)\right)\Big|_{r=R} = 0$$

$$A\left(\frac{1}{2} + \frac{\sigma}{1-2\sigma}\right) = \alpha\frac{R^2}{2}\left(\frac{3}{4} + \frac{\sigma}{1-2\sigma}\right)$$

$$A = \alpha\frac{R^2}{4}\,\frac{3 + \frac{4\sigma}{1-2\sigma}}{1 + \frac{2\sigma}{1-2\sigma}} = \frac{\alpha R^2}{4}(3-2\sigma)$$

$$u(r) = -\alpha\frac{r^3}{8} + \frac{\alpha R^2}{8}(3-2\sigma)\,r$$

$$u(r) = \frac{\alpha}{8}\left((3-2\sigma)\,rR^2 - r^3\right)$$

Ali je smiselno privzeti, da v z smeri ni deformacije.

$$\sigma_{zz} = \frac{E}{1+\sigma}\left(0 + \frac{\sigma}{1-2\sigma}\left(-\alpha\frac{r^2}{2} + A\right)\right)$$

$$= -\frac{E\sigma}{(1+\sigma)(1-2\sigma)}\underbrace{\left(A - \alpha\frac{r^2}{2}\right)}_{\text{pri } r=R:\ R^2(1-2\sigma)>0\ \checkmark} \neq 0$$

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$R_1 = 2{,}5\,\mathrm{cm}$

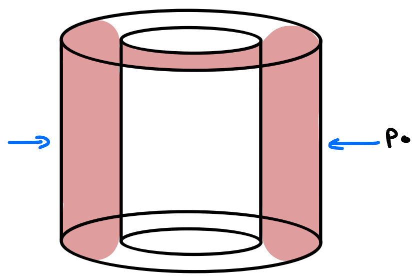
$R_2 = 7{,}5\,\mathrm{cm}$ plašč izolacije

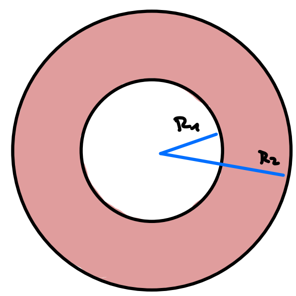
$p_0 = 10^6\,\mathrm{Pa}$

$E = 10^7\,\mathrm{Pa}$ za plašč

$\sigma = 0{,}4$

$k = 0{,}8$ koeficient trenja





Za koliko zasukamo plašč, da le-ta ravno zdrsne?

gostota volumske sile — pri nas $=0$

$$\frac{E}{2(1+\sigma)}\left(\nabla^2\vec u + \frac{1}{1-2\sigma}\nabla(\nabla\cdot\vec u)\right) + \vec f^{\,z} = 0$$

$$\nabla^2\vec u + \frac{1}{1-2\sigma}\nabla(\nabla\cdot\vec u) = 0$$

$\vec u \neq \vec u(z,\varphi)$

$\vec u = \vec u(r)$

$\vec u = u_r(r)\,\hat e_r + u_\varphi(r)\,\hat e_\varphi = \vec u_1 + \vec u_2$

$$\nabla\nabla = \nabla\times\nabla\times + \nabla^2$$

$$\nabla(\nabla\cdot\vec u) - \nabla\times\nabla\times\vec u + \frac{1}{1-2\sigma}\nabla(\nabla\cdot\vec u) = 0$$

$$\frac{2-2\sigma}{1-2\sigma}\nabla(\nabla\cdot\vec u) = \nabla\times\nabla\times\vec u$$

$$\nabla\cdot\vec v = \frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{1}{r}\frac{\partial v_\varphi}{\partial\varphi} + \frac{\partial v_z}{\partial z} \Rightarrow \nabla\cdot\vec u_2 = 0$$

$\nabla\times\nabla\times\vec u_1 = 0$ od prejšnič vemo

$$\frac{2-2\sigma}{1-2\sigma}\underbrace{\nabla(\nabla\cdot\vec{u}_1)}_{\substack{\text{kaže v}\\ \hat{e}_r \text{ smeri}}} = \underbrace{\nabla\times\nabla\times\vec{u}_2}_{\substack{\text{kaže v}\\ \hat{e}_\varphi \text{ smer}}} \quad \Rightarrow \quad 2 \text{ enačbi}$$

$$\nabla\times\vec{v} = \left(\frac{1}{r}\frac{\partial v_z}{\partial\varphi} - \frac{\partial v_\varphi}{\partial z}\right)\hat{e}_r + \left(\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}\right)\hat{e}_\varphi + \frac{1}{r}\left(\frac{\partial(r v_\varphi)}{\partial r} - \frac{\partial v_r}{\partial\varphi}\right)e_z$$

$$\nabla\times\vec{u}_2 = \frac{1}{r}\frac{\partial(r u_\varphi)}{\partial r}\hat{e}_z$$

- $\nabla\times\nabla\times\vec{u}_2 = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial(r u_\varphi)}{\partial r}\right)\hat{e}_\varphi = 0$
- $\nabla(\nabla\cdot\vec{u}_1) = \frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}(r u_r)\hat{e}_r = 0$

Rešujemo te enačbi

Rešujemo

- $\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}(r u_r) = 0$

$$\nabla u_r = \frac{1}{r}\frac{\partial}{\partial r}(r u_r) = A$$
$$\frac{\partial}{\partial r}(r u_r) = Ar$$
$$r u_r = \frac{A}{2}r^2 + B$$
$$u_r = \frac{A}{2}r + \frac{B}{r}$$

$$-\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial(r u_\varphi)}{\partial r}\right) = 0$$
$$\frac{1}{r}\frac{\partial(r u_\varphi)}{\partial r} = C$$
$$\frac{\partial}{\partial r} r u_\varphi = Cr$$
$$r u_\varphi = C\frac{r^2}{2} + D$$
$$u_\varphi = \frac{C}{2}r + \frac{D}{r}$$

Robni pogoji

1. $u_\varphi\big|_{r=R_1} = 0$
2. $u_r\big|_{r=R_1} = 0$
3. p_0 sila na zunanjosti p, $\quad \sigma_{rr} = -p_0$ pri $r = R_2$ — sila kaže notri, normala pa ven
4. k; $\quad \sigma_{\varphi r} = k\sigma_{rr} \quad \Leftrightarrow \quad \sigma_{\varphi r}\,dS = f_{tan} = k f_r = \sigma_{rr}\,dS$

1. $0 = \frac{A}{2}R_1 + \frac{B}{R_1} \quad \Rightarrow \quad B = -\frac{A}{2}R_1^2$
2. $0 = \frac{C}{2}R_1 + \frac{D}{R_1}$

3. Hooke: $\quad \sigma_{ij} = \frac{E}{1+\sigma}\left(u_{ij} + \frac{\sigma}{1-2\sigma}u_{kk}\delta_{ij}\right)$

$$\sigma_{rr} = \frac{E}{1+\sigma}\left(\underbrace{u_{rr}}_{\frac{\partial u_r}{\partial r}} + \frac{\sigma}{1-2\sigma}\underbrace{u_{kk}}_{\nabla u_r = A}\right)$$

$$u_{rr} = \frac{E}{1+\sigma}\left(\frac{A}{2} - \frac{B}{R_2^2} + \frac{\sigma}{1-2\sigma}A\right) = -p_0$$

$$\frac{E}{1+\sigma}\left(\frac{A}{2} + \frac{A}{2}\frac{R_1^2}{R_2^2} + \frac{\sigma}{1-2\sigma}A\right) = -p_0$$

$$A = \frac{-\frac{p_0}{2}\frac{1+\sigma}{E}}{\left(1 + \frac{R_1^2}{R_2^2} + \frac{2\sigma}{1-2\sigma}\right)} = -\frac{p_0}{2E}\frac{1+\sigma}{\left(1 + \frac{R_1^2}{R_2^2} + \frac{2\sigma}{1-2\sigma}\right)} \quad \Rightarrow \ldots B$$

4. $\sigma_{\varphi r} = k\sigma_{rr}\Big|_{r=R_1}$

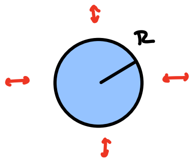
$$\frac{E}{1+\sigma}u_{\varphi r} = k\frac{E}{1+\sigma}\left(\frac{1}{2}A - \frac{B}{R_1^2} + \frac{A\sigma}{1-2\sigma}\right)$$

$$u_{\varphi r} = \frac{1}{2}\left(\frac{\partial u_\varphi}{\partial r} - \frac{u_\varphi}{r} - \frac{\partial u_r}{r\partial\varphi}\right) = \frac{1}{2}\left(\frac{C}{2} - \frac{D}{r^2} - \frac{1}{r}\left(\frac{C}{2}r + \frac{D}{r}\right)\right) = -\frac{D}{r^2}$$

$$-\frac{D}{R_1^2} \quad \ldots$$

$$\varphi = \frac{u_\varphi(R_2)}{R_2}$$

9 Lastna nihanja žogice, radialno nihanje $\circ \leftrightarrow \bigcirc$



$$\vec{u} = u(r)\,\hat{e}_r$$

Navierova enačba (dinamična)

$$\rho \ddot{\vec{u}} = \frac{E}{2(1+\sigma)} \left(\nabla^2 \vec{u} + \frac{1}{1-2\sigma} \nabla \nabla \cdot \vec{u} \right)$$

...

$$\nabla \times \vec{v} = \frac{1}{r\sin\theta} \left(\frac{\partial \sin\theta\, v_\varphi}{\partial\theta} - \frac{\partial v_\theta}{\partial\varphi} \right) \hat{e}_r \qquad \Rightarrow \nabla \times u(r)\hat{e}_r = 0$$

$$+ \frac{1}{r} \left(\frac{1}{\sin\theta} \frac{\partial v_r}{\partial\varphi} - \frac{\partial r v_\varphi}{\partial r} \right) \hat{e}_\theta$$

$$+ \frac{1}{r} \left(\frac{\partial r v_\theta}{\partial r} - \frac{\partial v_r}{\partial\theta} \right) \hat{e}_\varphi$$

$$\ldots \qquad \nabla\nabla\cdot = \nabla^2 + \nabla\times\nabla\times$$

$$\rho \ddot{\vec{u}} = \frac{E}{2(1+\sigma)} \Big(-\nabla\times\underbrace{\nabla\times\vec{u}}_{=0} + \left(\frac{1}{1-2\sigma} + 1\right) \nabla\nabla\cdot\vec{u} \Big)$$

$$\rho \ddot{\vec{u}} = \frac{E(1-\sigma)}{(1+\sigma)(1-2\sigma)} \nabla\nabla\cdot\vec{u}$$

Nastavek $\quad \vec{u}(\vec{r},t) = \vec{u}(\vec{r})\, e^{i\omega t}$

$$-\omega^2 \rho\, \vec{u}(\vec{r}) = \frac{E(1-\sigma)}{(1+\sigma)(1-2\sigma)} \nabla\nabla\cdot\vec{u}(\vec{r})$$

$$\nabla\nabla\cdot\vec{u} + \underbrace{\frac{\rho\,\omega^2}{E} \frac{(1+\sigma)(1-2\sigma)}{(1-\sigma)}}_{k^2} \vec{u} = 0$$

$$\nabla\nabla\cdot\vec{u} + k^2 \vec{u} = 0 \qquad \vec{u} = u(r)\,\hat{e}_r$$

$$\frac{d}{dr}\left(\frac{1}{r^2} \frac{d}{dr}(r^2 u) \right) + k^2 u = 0$$

$$\frac{d}{dr}\left(2\frac{u}{r} + \frac{du}{dr} \right) + k^2 u = 0$$

$$-2\frac{u}{r^2} + \frac{2}{r}\frac{du}{dr} + \frac{d^2u}{dr^2} + k^2 u = 0$$

$$\frac{d^2u}{dr^2} + \frac{2}{r}\frac{du}{dr} + \left(k^2 - \frac{2}{r^2}\right) u = 0 \qquad \Big/ : k^2$$

Besslova DE
$$\frac{d^2}{dr^2} z_\ell + \frac{2}{r}\frac{d}{dr} z_\ell + \left(1 - \frac{\ell(\ell+1)}{r^2}\right) z_\ell = 0$$

$$\frac{d^2u}{dk^2r^2} + \frac{2}{kr}\frac{du}{dkr} + \left(1 - \frac{2}{k^2r^2}\right) u = 0$$

$$\Rightarrow 2 = \ell(\ell+1) \Rightarrow \ell = 1$$

$$u(r) = z_1(kr) = A\, j_1(kr) + B\, n_1(kr)$$

Besslova sferična — Neumannova

↳ divergira v izhodišču $\Rightarrow B = 0$

$$u(r) = A\, j_1(kr)$$

k določimo iz R.P.

Ni površinske sile na robu $\quad \sigma_{rr} = 0$

$$\sigma_{rr} = \frac{E}{1+\sigma}\Big(u_{rr} + \frac{\sigma}{1-2\sigma} \underbrace{u_{kk}}_{\nabla u} \Big) = 0 \qquad \nabla u = \frac{1}{r^2}\frac{d}{dr} r^2 u$$

$$0 = \frac{d}{dr} j_1(kr) + \frac{\sigma}{1-2\sigma} \frac{1}{r^2} \frac{d}{dr} r^2 j_1(kr)$$

$$k \frac{d}{dkr} j_1(kr) + \frac{\sigma}{1-2\sigma} \frac{k}{(kr)^2} \frac{d}{dkr} \frac{\partial (kr)^2 j_1(kr)}{\partial (kr)} \Big|_{r=R} = 0$$

$x = kr$ $$\frac{d j_1(x)}{dx} + \frac{\sigma}{1-2\sigma}\frac{1}{x^2}\frac{\partial x^2 j_1(x)}{\partial x}\Big|_{r=R} = 0$$

Zveze za Bessel. f.

$$-x^{-l} z_{l+1} = (x^{-l} z_l)'$$
$$x^{l+1} z_{l-1} = (x^{l+1} z_l)'$$

Vemo $j_0 = \frac{\sin x}{x}$

$$j_1 = -x^0 (x^{-0} j_0)' = -j_0' = -\frac{x\cos x - \sin x}{x^2} = -\frac{\cos x}{x} + \frac{\sin x}{x^2}$$

$$\frac{d}{dx}(x^2 j_1) = x^2 j_0 = x\sin x$$

$$\frac{d}{dx}\left(-\frac{\cos x}{x}+\frac{\sin x}{x^2}\right) + \frac{\sigma}{1-2\sigma}\frac{1}{x^2}\, x\sin x = 0$$

$$\frac{\sin x}{x} + \frac{\cos x}{x^2} + \frac{\cos x}{x^2} - 2\frac{\sin x}{x^3} + \frac{\sigma}{1-2\sigma}\frac{\sin x}{x} = 0$$

$$\frac{\cos x}{x} - \frac{\sin x}{x^2} + \frac{1-\sigma}{2(1-2\sigma)}\sin x = 0$$

$r = R$
$x = kr = kR$

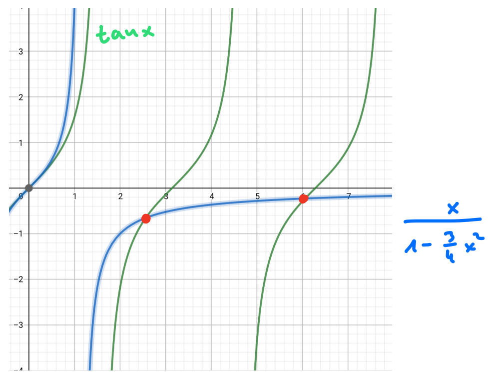
$$\frac{\cot x}{x} - \frac{1}{x^2} + \frac{1-\sigma}{2(1-2\sigma)} = 0 \qquad \Big/ \frac{x^2}{\cot x}$$

$$x - \tan x + \frac{1-\sigma}{2(1-2\sigma)}x^2 \tan x = 0$$

$$\tan x = \frac{x}{1-\underbrace{\frac{1-\sigma}{2(1-2\sigma)}}\,x^2}$$

Ocenimo $\sigma \approx \frac{1}{4}$

$$= \frac{3/4}{2(1-1/2)} = \frac{3}{4}$$



Za $x \gg 1$ so presečišča $k\pi$

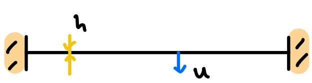
$x_i = 2.563,\ 6.059,\ 9.280,\ 12.459$

$$\omega_i = \frac{x_i}{R}\sqrt{\frac{E(1-\sigma)}{\rho(1+\sigma)(1-2\sigma)}}$$

$R = 5\,\mathrm{cm}$

$\omega_i = 90\,\mathrm{kHz}, \ldots$

10 Kako se poveši okrogla plošča na robu togo vpeta



$$D\nabla^2\nabla^2 u - P = 0$$

$P \to$ površinsko porazdeljena sila

$$D = \frac{Eh^3}{12(1-\sigma)^2}$$

$P = \mu g$
$= \rho g h$

μ... površinska gostota

R.P. $u(r=R) = 0$ $\quad \frac{du}{dr}\Big|_R = 0$

$u(\vec{r}) = u(r)$ cilindrična simetrija

$$\nabla^2\nabla^2 u = \nabla\cdot\nabla(\nabla\cdot\nabla\cdot u) =$$ $\nabla\cdot u = \frac{1}{r}\frac{\partial}{\partial r} r u_r + \dots$

$$= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)\right)\right) = \frac{\varrho g h}{D} = \alpha$$

$$r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)\right) = \alpha\frac{r^2}{2} + C_1$$ $C_1 = 0$ ker u divergira

$$\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)\right) = \alpha\frac{r}{2}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) = \alpha\frac{r^2}{4} + C_2$$

$$\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) = \alpha\frac{r^3}{4} + C_2 r$$

$$r\frac{\partial u}{\partial r} = \alpha\frac{r^4}{16} + \frac{C_2 r^2}{2} + \overset{=0}{C_3}$$

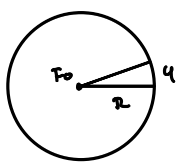
$$u = \alpha\frac{r^4}{64} + C_2\frac{r^2}{4} + C_4$$

R.P. $u|_{r=R} = 0 \Rightarrow C_4 = -\frac{\alpha R^4}{64} + \frac{\alpha R^4}{32} = \frac{\alpha R^4}{64}$

$\frac{\partial u}{\partial r}\Big|_{r=R} = 0 \Rightarrow C_2 = -\frac{\alpha R^2}{8}$

$$u(r) = \frac{\alpha}{64}\left(r^4 - 2R^2r^2 + R^4\right) = \frac{\alpha}{64}\left(r^2 - R^2\right)^2$$

11 Točkasta sila na sredino plošče



$$\frac{d\psi}{dl} = \frac{1}{R} \quad (\Leftarrow l = \psi R)$$

$$D\nabla^2\nabla^2 u = p$$
$$= F_0\,\delta(r)\frac{1}{2\pi r}$$ ✓

$$\int p\,dS = 2\pi\int F_0\,\delta(r)\frac{1}{2\pi r}\,r\,dr = F_0\int\delta(r)\,dr = F_0$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)\right)\right) = \frac{F_0}{2\pi D}\frac{\delta(r)}{r} = \alpha\frac{\delta(r)}{r}$$

$$r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)\right) = \alpha + \overset{0}{\underset{\parallel}{}}C_1$$

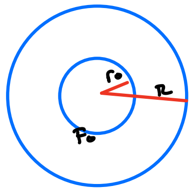
$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) = \alpha\ln\frac{r}{R} + C_1$$

$$r\frac{\partial u}{\partial r} = \alpha\int\ln\left(\frac{r}{R}\right) + \frac{C_1 r^2}{2} + C_2$$

$$u = \alpha\left(\frac{r^2}{4}\ln\left(\frac{r}{R}\right) - \frac{1}{4}r^2\right) + C_1\frac{r^2}{4} + C_3$$

RP $u|_R = 0$ $$\frac{\partial^2 u}{\partial r^2} + \underbrace{\frac{\partial\psi}{\partial l}}_{\frac{1}{R}}\frac{\partial u}{\partial r}\sigma = 0$$

(12)



$$P = \frac{F_0}{2\pi r}\,\delta(r-r_0)$$

$$D\nabla^2\nabla^2 u = \frac{F_0}{2\pi}\,\frac{\delta(r-r_0)}{r}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)\right)\right) = \frac{F_0}{2\pi D}\,\frac{\delta(r-r_0)}{r}$$

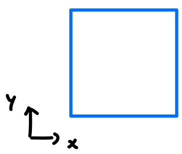
$$r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)\right) = \alpha H(r-r_0)$$

$$\int f(x)\,H(x-x_0)\,dx = F(x)-F(x_0)\,H(x-x_0)$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) = \alpha\ln\left(\frac{r}{r_0}\right)H(r-r_0) + A$$

⋮

(13) Prislonjena kvadratna plošča



$$u(x,y) = \sum_{m,n} d_{m,n}\sin\left(m\pi\frac{x}{a}\right)\sin\left(n\pi\frac{x}{a}\right)$$

Energija $$F = \frac{1}{2}D\int dS\left((\nabla^2 u)^2 + 2(1-\sigma)\Big(\underbrace{\left(\frac{\partial^2 u}{\partial x\partial y}\right)^2 - \frac{\partial^2 u}{\partial x^2}\frac{\partial^2 u}{\partial y^2}}_{\text{prevede se na rob}}\Big)\right) - \int dS\,uP$$

$\int$ po površini da 0 z variacijo

RP. $u|_{rob} = 0$

$$\frac{\partial^2 u}{\partial n^2} + \sigma\frac{\partial u}{\partial l}\frac{\partial u}{\partial n} = 0$$

$u(x,y)$, vzamemo le prvi člen $n,m=1,1$

$$u(x,y) = a_{11}\sin\frac{\pi x}{a}\sin\frac{\pi y}{a}$$

$P = \rho g h$

$$F = \frac{1}{2}D\iint_0^a dx\,dy\left(\left(-2\left(\frac{\pi}{a}\right)^2\sin\frac{\pi x}{a}\sin\frac{\pi y}{a}\,a_{11}\right)^2 - \iint_0^a a_{11}\sin\frac{\pi x}{a}\sin\frac{\pi y}{a}\,P\,dx\,dy\right.$$

$$= \frac{1}{2}D\,a_{11}^2\left(\frac{\pi}{a}\right)^4 - a_{11}\rho g h\left(\frac{a}{\pi}\right)^2\cdot 4$$

$$= \frac{D a_{11}^2}{2}\frac{\pi^4}{a^2} - a_{11}\rho g h\frac{a^2}{\pi^2}4 = a_{11}\left(\frac{D\pi^4 a_{11}}{2a^2} - \rho g h\frac{4a^2}{\pi^2}\right)$$

Zahtevamo δF je minimalna

$$\frac{dF}{da_{11}} = 0 \Rightarrow 0 = D a_{11}\frac{\pi^4}{a^2} - \rho g h\frac{a^2}{\pi^2}4 \Rightarrow a_{11} = \frac{4\rho g h a^4}{D\pi^6}$$

Pomik na sredini

$$u\left(\frac{a}{2},\frac{a}{2}\right) = \frac{4a^4\rho g h}{D\pi^6}\sin\frac{\pi}{2}\sin\frac{\pi}{2}$$

$$= \frac{a^4\rho g h}{D\pi^6}\begin{cases}4 & a_{11}\\ 3{,}8977 & a_{11}\ a_{31}\ a_{13}\\ 3{,}89892 & +a_{33}\end{cases}$$

⋮

14 Deform. prosta energija f za kristal tetragonalne in kubične simetrije

$$f = \frac{1}{2} K_{ijkl} u_{ij} u_{kl}$$

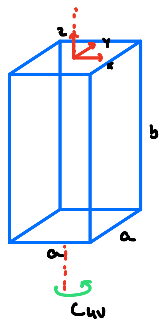
Simetrije

$K_{ijkl} = K_{klij}$

$K_{ijkl} = K_{jikl}$

$K_{klij} = K_{lkij}$

$\Bigg\}$ ker je u simetričen



tetragonalne

a) Simetrijska grupa

C_{4v}

4 števna os — zrcalni ravnini $x=0$, $y=0$

Zrcaljenje:

- $x \to -x$, $y \to y$, $z \to z$ npr $K_{xzzz} \to -K_{xzzz}$ $\Rightarrow K_{xzzz} = 0$
- $x \to x$, $y \to -y$, $z \to z$

Vsi ki imajo liho število indeksov x ali y so nič

Rotacije

- $90°$

$x \to y$, $y \to -x$, $z \to z$

$K_{xxxx} \overset{=}{\to} K_{yyyy}$

$K_{xxzz} \overset{=}{\to} K_{yyzz}$

$K_{xzxz} \overset{=}{\to} K_{yzyz}$

$$f = \frac{1}{2} K_{xxxx}(u_{xx}^2 + u_{yy}^2) + \frac{1}{2} K_{zzzz} u_{zz}^2$$

$$+ \frac{1}{2} K_{xxzz}(u_{xx} + u_{yy}) u_{zz} \cdot 2 \qquad + \frac{1}{2} K_{xxyy}(u_{xx} u_{yy}) \cdot 2$$

(ker K_{zzxx})

$$+ \frac{1}{2} K_{xyxy}(u_{xy}^2 + u_{yx}^2) \cdot 2 + \frac{1}{2} K_{xzxz}(u_{xz}^2 + u_{zx}^2 + u_{yz}^2 + u_{zy}^2) \cdot 2$$

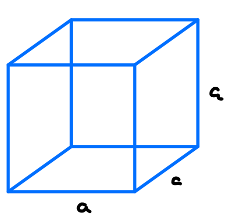
Hookov zakon

$$\sigma_{ij} = \frac{\partial f}{\partial u_{ij}}$$

Newtonov zakon (Navierove enačbe)

$$\partial_j \sigma_{ij} + f_i^z = \rho \ddot{u}_i$$

b) kubične



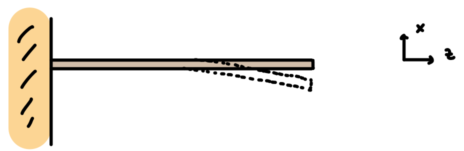
- rotacije $90°$

$x \to x$	$x \to -z$	$K_{xxxx} \to K_{zzzz}$
$y \to z$	$y \to y$	$K_{xxzz} \to K_{xxyy}$
$z \to -y$	$z \to x$	$K_{xzxz} \to K_{xyxy}$

$$f = \frac{1}{2} K_{xxxx}(u_{xx}^2 + u_{yy}^2 + u_{zz}^2) + \frac{1}{2} K_{xxyy}(u_{xx} u_{yy} + u_{xx} u_{zz} + u_{yy} u_{zz}) \cdot 2$$

$$+ \frac{1}{2} K_{xyxy}(u_{xy}^2 + u_{yx}^2 + u_{xz}^2 + u_{zx}^2 + u_{yz}^2 + u_{zy}^2) \cdot 2$$

15 Za koliko se pod lastno težo upogne polica, dolžine ℓ, mase m. Kakšna je sila in navor na steno.



$$EI_z\, x^{(4)} - F_z \ddot{x} - \dot{F}_z \dot{x} - k_x = 0 \qquad F_z = 0$$

$$k_x = -\frac{mg}{\ell}$$

$$x^{(4)} = -\frac{mg}{\ell E I_z} = \alpha$$

$$x = \alpha \frac{z^4}{24} + A z^3 + B z^2 + C z + D$$

RP

$z=0$
- $x(0) = 0$
- $\dot{x}(0) = 0$ ker je uzidana polica

$z=a$
- $\vec{F} = 0$ $\quad EI_z x^{(3)} - \dot{x} \overset{=0}{F_z} + F_x = 0$, $\quad F_x = -EI_z x^{(3)} = 0$
- $\vec{M} = 0$ $\quad M_y = E I_z \ddot{x} = 0$

- $D = 0$
- $C = 0$
- $\alpha \frac{4 \cdot 3}{24} L^2 + A\, 3 \cdot 2 L + B 2 = 0 \qquad B = -\frac{1}{2}\left(\alpha \frac{1}{12} L^2 + A 6 L\right) = \frac{1}{4} \alpha L^2$
- $\alpha \frac{2}{2} L + A\, 3 \cdot 2 = 0 \qquad A = -\alpha \frac{1}{6} L = -\frac{\alpha L}{6}$

$$x = \alpha\left(\frac{z^4}{24} - \frac{L}{6} z^3 + \frac{L^2}{4} z^2\right)$$

Navor in sila na steno

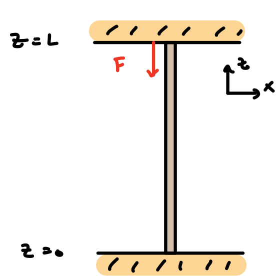
1. $F_x = -mg \qquad M_y = -mg \frac{L}{2}$

2. $M_y = EI_z \ddot{x}\Big|_{z=0} = EI_z\, \alpha \frac{L^2}{24}\, 2 = -\frac{mg}{\ell E I_z} \ell^2 EI_z \frac{2}{4} = -\frac{mg\ell}{2}$

$F_x = -EI_z x^{(3)} = -EI_z\, 6\left(-\frac{\alpha \ell}{6}\right) = EI_z \alpha \ell = -\frac{mg}{\ell E I_z} \ell E I_z = -mg$

16 Eulerjeva nestabilnost

Dolgo lahko polico, z dolžino L, vertikalno upeto, obtežimo s silo F



$$EI\, x^{(4)} - F_z \ddot{x} - \dot{F}_z \dot{x} - k_x = 0$$

$$k_x = 0\,, \quad \dot{F}_z = 0$$

$$x^{(4)} + \underbrace{\frac{|F_z|}{EI}}_{k^2} \ddot{x} = 0 \qquad u = \ddot{x}$$

$$\ddot{u} + k^2 u = 0$$

$$\ddot{x} = u = A' \sin kz + B' \cos kz$$

$$x = D \sin kz + C \cos kz + A + Bz$$

RP

- $x(z=0)=0 \quad \Rightarrow \quad A + C = 0$
- $x(z=L)=0 \quad \Rightarrow \quad D\sin kL + BL = 0 \quad \Rightarrow \quad \frac{B}{D} = -\frac{\sin kL}{L} = 0 \quad \Rightarrow B=0$

Vrtljivo vpetje $M=0$

$M_y = EI\ddot{x}$

- $\ddot{x}(0) = 0 \quad \Rightarrow \quad C=0 \,,\; A=0$
- $\ddot{x}(L)=0 \quad \Rightarrow \quad \sin kL = 0 \quad \Rightarrow \quad kL = \pi n \qquad k = \frac{n\pi}{L}$

$$x = D \sin \frac{n\pi}{L} z$$

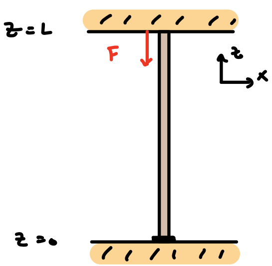
Izračun sile

$$k^2 = \frac{|F_z|}{EI} \qquad \text{najmanjša sila pri } n=1$$

$$\frac{\pi^2}{L^2} = \frac{|F_z|}{EI}$$

$$|F_z| = EI\left(\frac{\pi}{L}\right)^2$$

(17) Enako kot prej, spodaj fiksno vpeto, zgoraj prosto



$$x = A + Bz + C\cos kz + D\sin kz$$

Pojavi se tudi prečna sila v x smeri, ki je ne poznamo.

R.P.
1.) $x(z=0)=0$
2.) $x(z=L)=0$
3.) $\ddot{x}(z=L)=0$
4.) $\dot{x}(z=0)=0$ togo vpetje

1.) $A = -C$

4.) $B + Dk = 0 \qquad B = -Dk$

3.) $Ck^2\cos kL + Dk^2 \sin kL = 0$

$C = -D\tan kL$

2.) $A + BL + C\cos kL + D\sin kL = 0$

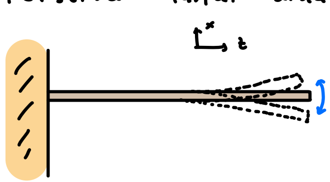
$D\tan kL - DkL - D\tan kL \cos kL + D\sin kL = 0$

$\tan kL - kL - \sin kL + \sin kL = 0$

$\tan kL = kL \quad \Rightarrow \quad$ Prva rešitev $\quad |F_z| = \frac{(4{,}49)^2 EI}{L^2}$

$$\Rightarrow \quad x = D\left(\tan kL - kz - \tan kL \cos kz + \sin kz\right)$$

(18) Poiščimo lastno nihanje palice (dolžina L)



Rešimo dinamično enačbo

$$EI\,u^{(4)} - F_z\ddot{u} - \dot{F}_z\dot{u} - k_x$$

$$EI\,u^{(4)} - F_z\ddot{u} - \dot{F}_z\dot{u} - k_x = -\rho\frac{\partial^2 u}{\partial t^2}$$

$$EI\,u^{(4)} = -\rho\frac{\partial^2 u}{\partial t^2} \qquad k_x = 0 \,,\; F_z = 0$$

Nastavek $u = v(z)\, e^{-i\omega t}$

$$EI\, v^{(4)} e^{-i\omega t} = + \rho\, \omega^2\, v\, e^{-i\omega t}$$

$$v^{(4)} - \underbrace{\frac{\rho \omega^2}{EI}}_{\alpha^4} v = 0 \qquad \left(\text{nastavek } v = e^{\lambda z} \Rightarrow \lambda^4 - \alpha^4 = 0 \Rightarrow \lambda_{1,2} = \pm \alpha \quad \lambda_{3,4} = \pm i \alpha \right)$$

$$v(z) = A \operatorname{sh} \alpha z + B \operatorname{ch} \alpha z + C \sin \alpha z + D \cos \alpha z$$

RP
- $v(0) = 0 \Rightarrow B + D = 0 \qquad B = -D$
- $\dot{v}(0) = 0 \Rightarrow A + C = 0 \qquad A = -C$

Pri $z = L$ ni sil in navorov
- $\ddot{v}(L) = 0 \Rightarrow A \operatorname{sh} \alpha L + B \operatorname{ch} \alpha L + A \sin \alpha L + B \cos \alpha L = 0$

sile $EI_2 \dddot{v} + F_x - \dot{v} F_z = 0$

$F_z = 0 \Rightarrow F_x = -EI \dddot{v} = 0$

- $\dddot{v}(L) = 0 \Rightarrow A \operatorname{ch} \alpha L + B \operatorname{sh} \alpha L + A \cos \alpha L - B \sin \alpha L = 0$

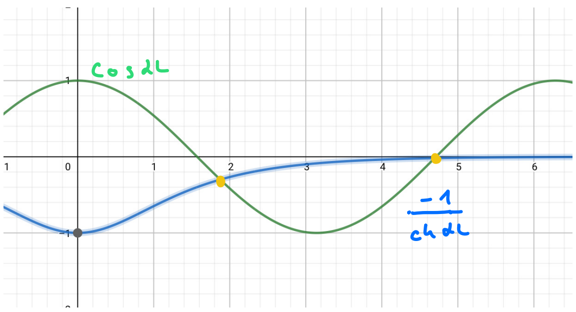
$$\begin{pmatrix} \operatorname{sh} \alpha L + \sin \alpha L & \operatorname{ch} \alpha L + \cos \alpha L \\ \operatorname{ch} \alpha L + \cos \alpha L & \operatorname{sh} \alpha L - \sin \alpha L \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = 0$$

det = 0

$$\operatorname{sh}^2 \alpha L - \sin^2 \alpha L - \operatorname{ch}^2 \alpha L - 2 \operatorname{ch} \alpha L \cos \alpha L - \cos^2 \alpha L = 0$$

$$-1 - 1 - 2 \operatorname{ch} \alpha L \cos \alpha L = 0$$

$$\operatorname{ch} \alpha L \cos \alpha L = -1$$



$$\cos \alpha L = \frac{-1}{\operatorname{ch} \alpha L}$$

$\alpha L = 1{,}875 \,,\quad 4{,}694 \,,\quad 7{,}855$

$\alpha L = (n + \tfrac{1}{2}) \pi$ za $n \gg 1$

$$\alpha^4 = \frac{\rho \omega^2}{EI} \qquad \omega_n = \alpha_n^2 \sqrt{\frac{EI}{\rho}}$$

Hidrodinamika

(13) $\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0$ kontinuitetna enačba

$$\rho \frac{d\vec{v}}{dt} = \rho \left(\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} \right) = -\nabla p \qquad \text{Eulerjeva enačba}$$

$$\frac{\partial}{\partial t} \vec{g} = \frac{\partial}{\partial t} \rho \vec{v}$$

$\vec{g}$ ... gostota gib. kol.

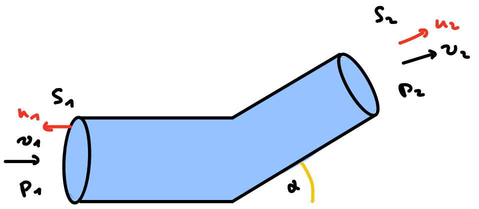
$$\frac{\partial}{\partial t} \rho v_i = \frac{\partial \rho}{\partial t} v_i + \rho \frac{\partial v_i}{\partial t}$$

$$= -v_i \frac{\partial}{\partial x_j} \rho v_j - \frac{\partial}{\partial x_i} p - \rho v_j \frac{\partial}{\partial x_j} v_i = -\partial_i p - \partial_j (\rho v_i v_j)$$

$$= -\partial_j p\, \delta_{ij} - \partial_j (\rho v_i v_j) = -\partial_j (p \delta_{ij} + \rho v_i v_j) = -\nabla \cdot \Pi$$

$$\Rightarrow \frac{\partial g_i}{\partial t} + \partial_j \underbrace{(p \delta_{ij} + \rho v_i v_j)}_{\Pi_{ij}} = 0 \quad \Leftrightarrow \quad \frac{\partial \vec{g}}{\partial t} + \nabla \cdot \Pi = 0$$

(20) Nestisljiva idealna tekočina, sila na cev



$$\frac{\partial \vec G}{\partial t} = 0 \quad \text{gib. kol.} \qquad \text{stacionarni tok}$$

$$\vec G = \int \vec g \, dV$$

$$\frac{\partial \vec G}{\partial t} = \frac{\partial}{\partial t}\int \vec g \, dV = \int dV \frac{\partial \vec g}{\partial t} = -\int dV \, \nabla\cdot\Pi = -\oint \Pi \, d\vec S$$

$$\frac{\partial G_i}{\partial t} = -\oint dS_j \, \pi_{ij} = 0$$

$$\vec v \perp \vec S$$

$$0 = -\int_1 dS_j \left(p_1 \delta_{ij} + \rho v_{1i} v_{1j}\right) - \int_2 dS_j \left(p_2 \delta_{ij} + \rho v_{2i} v_{2j}\right) - \underbrace{\int_? dS_j \, p \, \delta_{ij}}_{\text{sila kolena na tekočino}} + 0$$

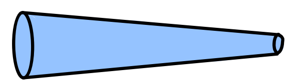
$= -F$

$$x: \quad 0 = +S_1 \rho v_1^2 + S_1 p_1 - S_2 p_2 \cos\alpha - S_2 v_2^2 \rho \cos\alpha - F_x$$

$$y: \quad 0 = -S_2 p_2 \sin\alpha - S_2 \rho v_2^2 \sin\alpha - F_y$$

$$\begin{bmatrix} F_x \\ F_y \end{bmatrix} = \begin{bmatrix} -S_2 \cos\alpha \left(p_2 + v_2^2 \rho\right) + S_1 \left(p_1 + v_1^2 \rho\right) \\ -S_2 \sin\alpha \left(p_2 + v_2^2 \rho\right) \end{bmatrix}$$

Poseben primer $\alpha = 0$, $S_2 < S_1$ (šoba)



$p_2 = 0$

$$F_y = 0 \qquad F_x = S_1 \left(p_1 + v_1^2 \rho\right) - S_2 \left(p_2 + v_2^2 \rho\right)$$

Ohranitev pretoka $\quad S_1 v_1 = S_2 v_2$

Bernoullijeva en. $\quad p_1 + \frac{1}{2}\rho v_1^2 = p_2 + \frac{1}{2}\rho v_2^2$

$$\rho v_1^2 = 2(p_2 - p_1) + \rho v_2^2 = 2(p_2 - p_1) + \rho \left(\frac{S_1}{S_2}\right)^2 v_1^2$$

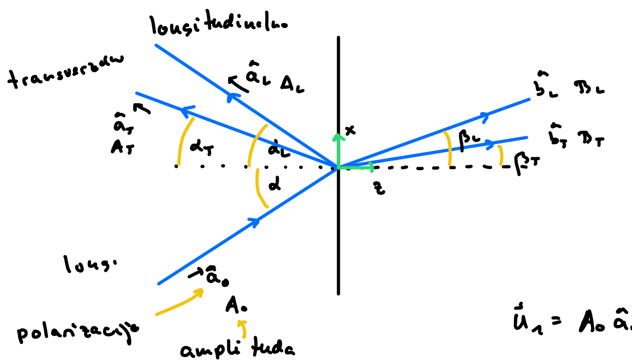
$$\rho v_1^2 = \frac{2(p_2 - p_1)}{1 - \left(\frac{S_1}{S_2}\right)^2} = \frac{2 p_1}{\left(\frac{S_1}{S_2}\right)^2 - 1}$$

$$F_x = S_1 \left(p_1 + \frac{2p_1}{\left(\frac{S_1}{S_2}\right)^2 - 1}\right) - S_2 \left(0 + \left(\frac{S_1}{S_2}\right)^2 \frac{2p_1}{\left(\frac{S_1}{S_2}\right)^2 - 1}\right)$$

$$= p_1 S_1 \left(1 - \frac{2}{\left(\frac{S_1}{S_2}\right)^2 - 1} \left(\frac{S_1}{S_2} - 1\right)\right)$$

$$= p_1 S_1 \frac{\frac{S_1}{S_2} - 1}{\frac{S_1}{S_2} + 1} = p_1 S_1 \frac{S_1 - S_2}{S_1 + S_2}$$

21



$$\vec{u}_1 = A_0 \hat{a}_0 e^{i(\vec{k}_0\cdot\vec{r} - \omega t)} + A_L \hat{a}_L e^{i(\vec{k}_L\cdot\vec{r} - \omega t)} + A_T \hat{a}_T e^{i(\vec{k}_T\cdot\vec{r} - \omega t)}$$

$$\vec{u}_2 = B_L \hat{b}_L e^{i(\vec{q}_L\cdot\vec{r} - \omega t)} + B_T \hat{b}_T e^{i(\vec{q}_T\cdot\vec{r} - \omega t)}$$

RP $\quad u_1(z=0,t) = u_2(z=0,t)$

$$A_0 \hat{a}_0 e^{ik_{0x}x} + A_L \hat{a}_L e^{ik_{Lx}x} + A_T \hat{a}_T e^{ik_{Tx}x} = B_L \hat{b}_L e^{iq_{Lx}x} + B_T \hat{b}_T e^{iq_{Tx}x}$$

Eksponenti morajo biti enaki : $k_{0x} = k_{Lx} = k_{Tx} = q_{Lx} = q_{Tx}$

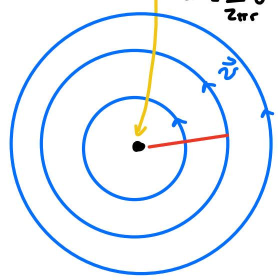
$k = \frac{\omega}{c}$ $\qquad k_0 \sin\alpha_0 = k_L \sin\alpha_L = k_T \sin\alpha_T = q_L \sin\beta_L = q_T \sin\beta_T$

Lomni zakon $\qquad \frac{\sin\alpha_0}{c_{1L}} = \frac{\sin\alpha_L}{c_{1L}} = \frac{\sin\alpha_T}{c_{1T}} = \frac{\sin\beta_L}{c_{2L}} = \frac{\sin\beta_T}{c_{2T}}$

22

Vrtinčna nit s cirkulacijo Γ (idealni vrtinec)

$\omega = \frac{\Gamma}{2\pi r}\delta(r)$



$$\Gamma = \oint \vec{v}\cdot d\vec{l} = v\, 2\pi r \qquad v = \frac{\Gamma}{2\pi r}$$

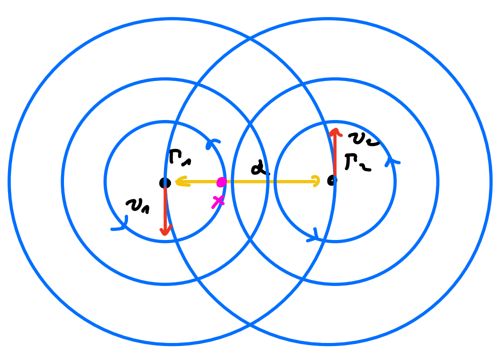
Analogija: $\vec{v} \leftrightarrow \vec{H}$

$\nabla\times\vec{v} = \vec{\omega} \quad \leftrightarrow \quad \nabla\times\vec{H} = \vec{j}_e$

$$\vec{v}(\vec{r}) = \frac{1}{4\pi}\int d^3r' \frac{\omega(\vec{r}')\times(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}$$ Biot-Savart za poljuben ω (porazdelitev)

$$= \frac{\Gamma}{4\pi}\int \frac{d\vec{l}\times(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}$$ za nit (žica)

Dinamika dveh niti (Γ_1, Γ_2) na razdalji d



Začne se vrteti okoli x

$$v_1 = \omega x \qquad v_2 = \omega(d-x)$$

$$\omega = \frac{v_1}{x} = \frac{v_2}{d-x}$$

$$v_2 x = v_1 d - v_1 x$$

$$\omega = \frac{v_1+v_2}{d} \qquad x = d\,\frac{v_1}{v_1+v_2} \qquad v = \frac{\Gamma}{2\pi r}$$

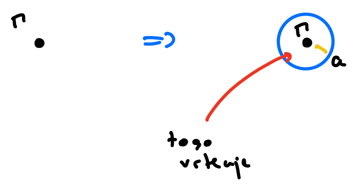
$$\omega = \frac{\Gamma_2+\Gamma_1}{2\pi d^2} \qquad x = d\,\frac{\Gamma_2}{\Gamma_1+\Gamma_2}$$

Primer $\Gamma_1 = \Gamma_2$ $\quad x = \frac{d}{2}$

$\Gamma_1 = -\Gamma_2 \quad \omega = 0 \;,\; x \to \infty$

$$v = x\omega = \frac{\Gamma_2}{2\pi d}$$

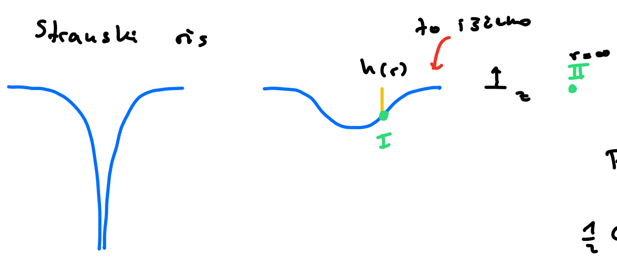
27 Višinski profil vrtinca



$r > a \qquad \vec{v} = \frac{\Gamma}{2\pi r}\hat{e}_\varphi$

$r < a \qquad \vec{v} = \omega r\,\hat{e}_\varphi \qquad$ toga rotacija

Stranski ris



Določimo ω, zveznost

$$\frac{\Gamma}{2\pi a} = \omega a \qquad \omega = \frac{\Gamma}{2\pi a^2}$$

Bernoullijeva enačba

$$\frac{1}{2}\rho v^2 + \rho g z = \text{konst}$$

$$\frac{1}{2}v^2 + h = \frac{1}{2}v^2 + \frac{p}{\rho} + gz = \text{konst}$$

$r > a \qquad \overset{\mathrm{I}}{\frac{1}{2}\rho v^2 + \rho g h} = \overset{\mathrm{II}}{0}$

$$h = -\frac{1}{2}\frac{v^2}{g} = -\frac{\Gamma^2}{8\pi^2 r^2 g} \propto \frac{1}{r^2}$$

$r < a \qquad \nabla \times \vec{v} \neq 0 = \text{konst.}$

uporabimo Eulerjevo enačbo

$$\rho\frac{d\vec{v}}{dt} = \rho\Big(\underbrace{\frac{\partial \vec{v}}{\partial t}}_{=0} + (\vec{v}\cdot\nabla)\vec{v}\Big) = -\nabla p + \rho\vec{g}$$

$\Leftrightarrow$ stacionarni vrtinec

$$(\vec{v}\cdot\nabla)\vec{v} = \omega r\frac{\partial}{r\,\partial\varphi}(\omega r\hat{e}_\varphi) = \omega^2 r\frac{\partial \hat{e}_\varphi}{\partial\varphi} = \omega^2 r(-\hat{e}_r) = -\omega^2 r\,\hat{e}_r$$

centripetalna sila

$$-\rho\omega^2 r\,\hat{e}_r = -\nabla p - \rho g\,\hat{e}_z$$

$z: \quad 0 = -\frac{\partial p}{\partial z} - \rho g \qquad\qquad r: \quad -\rho\omega^2 r = -\frac{\partial p}{\partial r}$

$p(r,z) = -\rho g z + C(r) \qquad\qquad \rho\omega^2 r = \rho g\frac{\partial h}{\partial r}$

RP na gladini je $p = 0 \qquad\qquad h(r) = \frac{\omega^2 r^2}{2g} + D$

$C(r) = \rho g h(r) \qquad\qquad$ zveznost

$p(r,z) = \rho g\,(h(r) - z)$

$$\frac{\omega^2 a^2}{2g} + D = -\frac{\Gamma^2}{8\pi^2 a^2 g}$$

$$D = -\left(\frac{\Gamma^2}{8\pi^2 a^2 g} + \frac{\omega^2 a^2}{2g}\right) = -\frac{\Gamma^2}{4\pi^2 g a^2}$$

Rezultat zgoraj bi lahko dobili z Bernoullijevo enačbo za vrtinčenje

$$\frac{1}{2}\rho v^2 + \rho g\, h(r) = \frac{1}{2}\rho v^2 + \rho g z + p$$

$$p = \rho g\,(h(r) - z)$$

$$\Rightarrow h(r) = \left(\frac{\Gamma}{2\pi a^2}\right)^2 \frac{r^2}{2g} - \left(\frac{\Gamma^2}{8\pi^2 a^2 g} + \frac{a^2}{2g}\left(\frac{\Gamma}{2\pi a^2}\right)^2\right)$$

$$= \frac{\Gamma^2}{4\pi^2 g a^2}\left(\frac{r^2}{2a^2} - 1\right)$$

24 2D potencialni tok v $\mathbb{C}$

- $\nabla\cdot\vec{v} = 0$, $\vec{v} = \nabla\phi$, $\nabla^2\phi = 0$
- Tokovna funkcija

konst. na tokovnici

$$v_x = \frac{\partial\psi}{\partial y} \qquad v_y = -\frac{\partial\psi}{\partial x}$$

$$(\vec{v}\cdot\nabla)\psi = v_x\frac{\partial\psi}{\partial x} + v_y\frac{\partial\psi}{\partial y} = -v_x v_y + v_x v_y = 0$$

$$\left.\begin{aligned} v_x &= \frac{\partial\phi}{\partial x} \\ v_y &= \frac{\partial\phi}{\partial y}\end{aligned}\right\} \quad \left.\begin{aligned}\frac{\partial\psi}{\partial y} &= \frac{\partial\phi}{\partial x} \\ \frac{\partial\psi}{\partial x} &= -\frac{\partial\phi}{\partial y}\end{aligned}\right\} \quad \begin{aligned}\nabla^2\phi &= 0 \\ \nabla^2\psi &= 0\end{aligned}$$

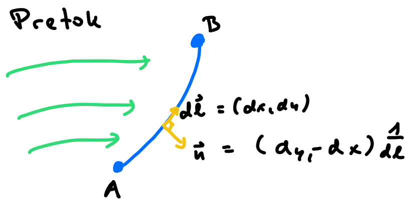
$\hookrightarrow$ $w(z) = \phi + i\psi$ $\qquad z = x + iy$

$$\frac{dw}{dz} = \frac{\partial\phi}{\partial x} + i\frac{\partial\psi}{\partial x} = v_x - i v_y$$

$$\parallel$$

$$\frac{dw}{dz} = \frac{\partial\phi}{\partial iy} + i\frac{\partial\psi}{\partial iy} = -i v_y + v_x$$

Pretok



$$Q = \rho\int_A^B \vec{v}\cdot\vec{n}\, dl$$

$$= \rho\int_A^B v_x\, dy - v_y\, dx$$

$$= \rho\int_A^B \frac{\partial\psi}{\partial y}dy + \frac{\partial\psi}{\partial x}dx$$

$$= \rho\int_A^B d\psi = \rho(\psi(B) - \psi(A))$$

ψ je na tokovnici konstanta

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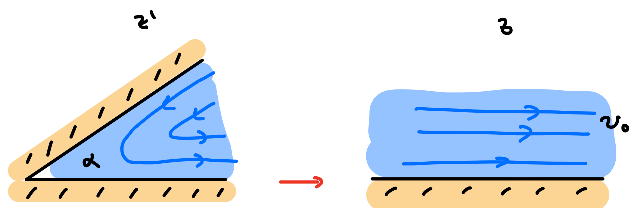

$w(z) = v_0 z$

$w'(z') = w(z(z'))$ $\qquad f(z) = z^{\pi/\alpha}$

$= v_0 z'^{\frac{\pi}{\alpha}}$ $\qquad$ preslika prvo mejo v drugo

Iščemo $\vec{v}$

$$z = re^{i\varphi} \qquad w = v_0 r^{\frac{\pi}{\alpha}} e^{i\varphi\frac{\pi}{\alpha}} = \phi + i\psi = v_0 r^{\frac{\pi}{\alpha}}\left(\cos\varphi\frac{\pi}{\alpha} + i\sin\varphi\frac{\pi}{\alpha}\right)$$

$$\vec{v} = \nabla\phi = \left(\frac{\partial\phi}{\partial r}, \frac{1}{r}\frac{\partial\phi}{\partial\varphi}\right) = \nabla v_0 r^{\frac{\pi}{\alpha}}\cos\varphi\frac{\pi}{\alpha} = \left(v_0\frac{\pi}{\alpha}r^{\frac{\pi}{\alpha}-1}, -v_0 r^{\frac{\pi}{\alpha}-1}\sin\varphi\frac{\pi}{\alpha}\;\frac{\pi}{\alpha}\right)$$

$$= v_0\frac{\pi}{\alpha}r^{\frac{\pi}{\alpha}-1}\left(\underset{v_r}{1}, \underset{v_\varphi}{-\sin\varphi\frac{\pi}{\alpha}}\right)$$

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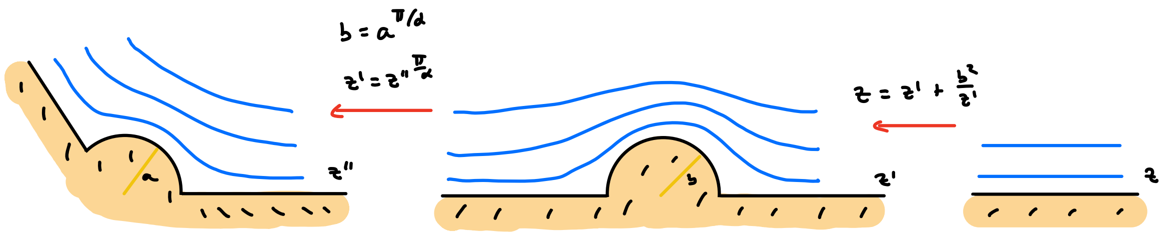

$$w(z'') = v_0\, z(z'(z'')) = v_0 \left(z' + \frac{b^2}{z'} \right) = v_0 \left(z''^{\pi/\alpha} + \frac{b^2}{z''^{\pi/\alpha}} \right) = v_0 \left(z''^{\pi/\alpha} + \left(\frac{a^2}{z''}\right)^{\pi/\alpha} \right)$$

$$= v_0 \left(r^{\pi/\alpha} e^{i\frac{\pi}{\alpha}\varphi} + a^{2\frac{\pi}{\alpha}} r^{-\pi/\alpha} e^{-i\frac{\pi}{\alpha}\varphi} \right) \qquad \phi = v_0 \left(r^{\pi/\alpha} + a^{2\frac{\pi}{\alpha}} r^{-\pi/\alpha} \right) \cos \frac{\pi}{\alpha}\varphi$$

$$v_r = \frac{\partial \phi}{\partial r} \qquad v_\varphi = \frac{1}{r} \frac{\partial \phi}{\partial \varphi}$$

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$\nabla^2 \phi = 0$ Splošna rešitev

$$\phi = \begin{cases} \{ r^m, r^{-m} \} \{ \cos m\varphi, \sin m\varphi \} & m \neq 0 \\ \{ 1, \ln r, \varphi \} & m = 0 \end{cases}$$

$$\nabla^2 \phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial \phi}{\partial r} \right) + \frac{\partial^2 \phi}{r^2 \partial \varphi^2}$$

Izvir: $\phi = \frac{Q}{2\pi} \ln r$ Ali je to res izvir? Računamo pretok

$$\oint \vec{v} \cdot \vec{n}\, dl = \qquad v_r = \frac{Q}{2\pi r} \qquad v_\varphi = 0$$

$$= \frac{Q}{2\pi r}\, 2\pi r = Q$$

Vrtinec: $\phi = \frac{\Gamma \varphi}{2\pi}$ $\qquad \vec{v} = \nabla \phi = \frac{\Gamma}{2\pi r} \hat{e}_\varphi$

Cirkulacija $\oint \vec{v}\, d\vec{s} = \int_0^{2\pi} \frac{\Gamma}{2\pi r} r\, d\varphi\, (\hat{e}_\varphi \cdot \hat{e}_\varphi) = \Gamma$

Izvir v $\mathbb{C}$:

$$w(z) = \frac{Q}{2\pi} \ln z = \frac{Q}{2\pi} \ln r e^{i\varphi} = \frac{Q}{2\pi} (\ln r + i\varphi)$$

Pretok $\psi(b) - \psi(a) = \psi(2\pi) - \psi(0) = \frac{Q}{2\pi}(2\pi - 0) = Q$

Vrtinec v $\mathbb{C}$:

$$w(z) = -i \frac{\Gamma}{2\pi} \ln z = \frac{\Gamma}{2\pi} (\varphi - i \ln r)$$

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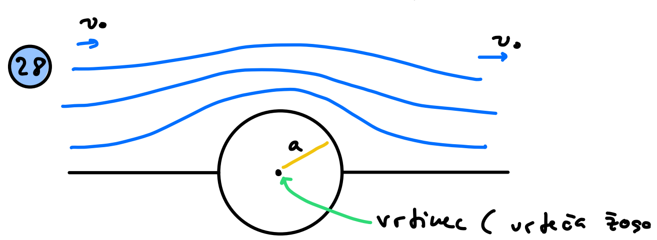


$$w(z) = v_0 \left(z + \frac{a^2}{z} \right) - i \frac{\Gamma}{2\pi} \ln z$$

$$= v_0 \left(r e^{i\varphi} + a^2 \frac{1}{r} e^{-i\varphi} \right) - i \frac{\Gamma}{2\pi} (\ln r + i\varphi)$$

$$\phi = v_0 r \cos\varphi + \frac{a^2}{r} v_0 \cos\varphi + \frac{\Gamma}{2\pi}\varphi$$

$$v_r = v_0 \cos\varphi - \frac{a^2}{r^2} v_0 \cos\varphi \qquad v_\varphi = \frac{1}{r}\left(-v_0 r \sin\varphi - \frac{a^2}{r} v_0 \sin\varphi + \frac{\Gamma}{2\pi} \right)$$

$$v_r = v_0 \cos\varphi \left(1 - \frac{a^2}{r^2} \right) \qquad v_\varphi = -v_0 \sin\varphi \left(1 + \frac{a^2}{r^2} \right) + \frac{\Gamma}{2\pi r}$$

Ker je $\nabla \times \vec{v} = 0$ lahko uporabimo Bernulijevo enačbo

$$\frac{1}{2}\rho v^2 + p = \frac{1}{2}\rho v_0^2 + \overset{p(x\to\infty)}{0}$$

$$p = \frac{1}{2}\rho\left(v_0^2 - v^2\right)$$

$$\frac{F_x}{l}\Big|_{r=a} = -\int_0^{2\pi} a\,d\varphi \cos\varphi\, p =$$

$$= -\int_0^{2\pi} a\cos\varphi\,\frac{1}{2}\rho\left(v_0^2 - \left(-v_0\sin\varphi\left(1+\frac{a^2}{a^2}\right) + \frac{\Gamma}{2\pi a}\right)^2\right)d\varphi$$

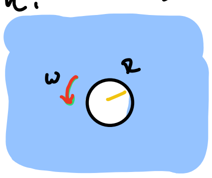
$$= -\frac{a\rho}{2}\int_0^{2\pi}\cos\varphi\,d\varphi\left(v_0^2 - \frac{\Gamma^2}{4\pi^2a^2} - 4v_0^2\sin^2\varphi + 4v_0\sin\varphi\,\frac{\Gamma}{2\pi a}\right) = 0$$

$$\frac{F_y}{l}\Big|_{r=a} = -\int_0^{2\pi} a\,d\varphi\sin\varphi\, p =$$

$$= -\frac{a\rho}{2}\int_0^{2\pi}\sin\varphi\,d\varphi\left(v_0^2 - \frac{\Gamma^2}{4\pi^2a^2} - 4v_0^2\sin^2\varphi + 4v_0\sin\varphi\,\frac{\Gamma}{2\pi a}\right)$$

$$= -\frac{a\rho}{2}\,\frac{4v_0\Gamma}{2\pi a}\int_0^{2\pi}\sin^2\varphi\,d\varphi = -\rho\Gamma v_0$$

29 η, ρ stacionarno polje



$$\rho(\vec{v}\cdot\nabla)\vec{v} = -\nabla p + \eta\nabla^2\vec{v} + \vec{f}^z \qquad \text{Navier-Stokes}$$

$$\vec{v} = v(r)\,\hat{e}_\varphi \qquad \nabla\nabla\cdot = \nabla^2 + \nabla\times\nabla\times$$

$$\nabla\times\vec{v} = \left(\frac{1}{r}\frac{\partial v_z}{\partial\varphi} - \frac{\partial v_\varphi}{\partial z}\right)\hat{e}_r + \left(\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}\right)\hat{e}_\varphi + \left(\frac{1}{r}\frac{\partial(r v_\varphi)}{\partial r} - \frac{1}{r}\frac{\partial v_r}{\partial\varphi}\right)\hat{e}_z$$

$$\nabla\times\vec{v} = \frac{1}{r}\frac{\partial(r v(r))}{\partial r}\hat{e}_z$$

$$\nabla\times\nabla\times\vec{v} = -\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}(r v(r))\,\hat{e}_\varphi$$

$$(\vec{v}\cdot\nabla)\vec{v} = v(r)\frac{1}{r}\frac{\partial}{\partial\varphi}\left(v(r)\,\hat{e}_\varphi\right) = \frac{v}{r}\left(\overset{0}{\overset{\shortparallel}{\frac{\partial v}{\partial\varphi}}}\hat{e}_\varphi + v(-\hat{e}_r)\right) = -\frac{v^2}{r}\hat{e}_r$$

$\nabla\,\nabla\cdot\vec{v} = 0$ nestisljiva tekočina

$$-\frac{\rho v^2}{r}\hat{e}_r = -\frac{\partial p}{\partial r}\hat{e}_r + \eta\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}(r v)\,\hat{e}_\varphi$$

$$\hat{e}_\varphi: \quad \frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}(r v) = 0$$

$$\frac{\partial}{\partial r}(r v) = A r$$

$$r v = \frac{A}{2}r^2 + B$$

$$v = \frac{A}{2}r + \frac{B}{r}$$

RP 1 $\quad v(r\to\infty) = 0 \quad \Rightarrow \quad A = 0$

RP 2 $\quad v\big|_{r=R} = \omega R \quad \omega R = \frac{B}{R} \quad \Rightarrow \quad B = \omega R^2$

$$v = \frac{\omega R^2}{r}$$

$\hat{e}_r$ $\qquad \frac{\rho v^2}{r} = \frac{\partial p}{\partial r}$

$$p = \int \frac{\rho}{r} \omega^2 R^4 \frac{1}{r^2}\, dr$$

$$p = p_0 - \frac{\rho \omega^2}{2} \frac{R^4}{r^2}$$

S kolikšno močjo moramo vrteti gred

1. način $\quad$ Iščemo viskozno silo ($\sigma_{\varphi r}$)

$$\sigma_{ij}^v = 2\eta v_{ij} \qquad v_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)$$

$$v_{rr} = \frac{\partial v_r}{\partial r} \qquad v_{\varphi r} = \frac{1}{2}\left(\frac{\partial v_\varphi}{\partial r} - \frac{v_\varphi}{r} + \frac{\partial v_r}{r\partial \varphi}\right) \qquad v_{\varphi\varphi} = \frac{\partial v_\varphi}{r\partial\varphi} + \frac{v_r}{r}$$

$$v_{\varphi r} = \frac{1}{2}\left(\frac{\partial}{\partial r}\frac{\omega R^2}{r} - \frac{\omega R^2}{r^2} + 0\right) = -\frac{\omega R^2}{r^2} \qquad v_{rr} = 0 \qquad v_{\varphi\varphi} = 0$$

$$\sigma_{\varphi r}^v = -\frac{2\eta\omega R^2}{r^2}$$

$$\frac{dF_\varphi}{l} = \sigma_{\varphi r}^v R\, d\varphi$$

Navor $\quad \frac{dM}{l} = \sigma_{\varphi r}^v R^2\, d\varphi \qquad \frac{M}{l} = 2\pi R^2 \sigma_{\varphi r}^v \Big|_R$

$$\frac{M}{l} = -4\pi\eta\omega R^2$$

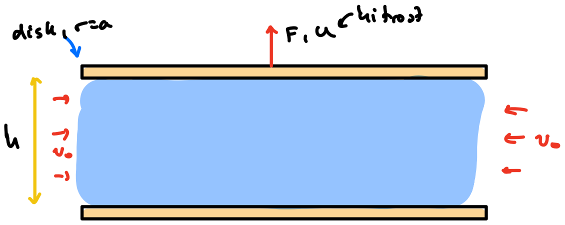
$$\frac{P}{l} = \frac{M\omega}{l} = -4\pi\eta\omega^2 R^2$$

2. način $\quad$ Disipacija energije

$$P = \int dV\, \sigma_{ij}^v v_{ij} = l \int dS\, \sigma_{ij}^v v_{ij} = 2\eta l \int dS\, v_{ij} v_{ij} = 2\eta l \int dS \left(v_{rr}^2 + v_{\varphi\varphi}^2 + v_{\varphi r}^2 + v_{r\varphi}^2\right)$$

$$\frac{P}{l} = 4\eta \int_R^\infty 2\pi r\, dr \left(\frac{\omega R^2}{r^2}\right)^2 = 8\pi\eta\, \omega^2 R^4 \int_R^\infty \frac{1}{r^3}\, dr = +4\pi\eta\omega^2 R^4 \left(\frac{1}{R^2} - 0\right) = 4\pi\eta\omega^2 R^2$$

30 $\quad$ Nestisljiva viskozna tekočina



$\dot{h} = u$

$$\rho\left(\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v}\right) = -\nabla p + \eta \nabla^2 \vec{v} \qquad \text{Navier-Stokes}$$

$$\nabla \cdot \vec{v} = 0 \qquad \text{Kontinuitetna en.}$$

Pretok $\quad \pi a^2 = v_0 h 2\pi a$

$$v_0 = \frac{a}{2}\frac{u}{h} \gg u$$

Ocena za Reynoldsovo število

$$Re \doteq \frac{\rho |(\vec{v}\cdot\nabla)\vec{v}|}{\eta |\nabla^2 \vec{v}|} \doteq \frac{\rho \frac{v_0^2}{a}}{\eta \frac{v_0}{h^2}} = \frac{\rho v_0}{\eta}\frac{h^2}{a} = \frac{\rho}{\eta}\frac{u}{2}\frac{a}{h}\frac{h^2}{a} = \frac{\rho u h}{2\eta}$$

Želimo $Re \ll 1 \iff u \ll \frac{\eta}{\rho h}$ $\qquad$ Za $Re \ll 1$ je $(\vec{v}\cdot\nabla)\vec{v} \doteq 0$

$\tau \doteq \frac{h}{u}$

Čas preleta $\quad \tau' = \frac{a}{v_0}$

$$v_0 \doteq \frac{ua}{h}$$

$$St = \frac{\left|\frac{d\vec{v}}{dt}\right|}{|(\vec{v}\cdot\nabla)\vec{v}|} \sim \frac{\tau'}{\tau} = \left(\frac{h}{u}\frac{v_0}{a}\right)^{-1} \sim \left(\frac{u a h}{a^2}\right)^{-1} \doteq 1$$

$\Rightarrow \quad \frac{\partial \vec{v}}{\partial t} \doteq 0 \quad \Rightarrow$ Stokesova enačba $\quad -\nabla p + \eta \nabla^2 \vec{v} = 0$

$$\vec{v} = \vec{v}(r, z, t) = v_r(r,z,t)\,\hat{e}_r + v_z(r,z,t)\,\hat{e}_z$$

$h \ll a \quad \Rightarrow \quad \frac{\partial}{\partial r} \ll \frac{\partial}{\partial z} \quad \Rightarrow \quad \nabla^2 \doteq \frac{\partial^2}{\partial z^2}$

$\Rightarrow \quad v_z \ll v_r \quad \Rightarrow \quad \nabla^2 v = \frac{\partial^2}{\partial z^2}(v_r\,\hat{e}_r) + 0$

$\nabla p = \eta \frac{\partial^2}{\partial z^2}(v_r \hat{e}_r) \Rightarrow \quad r: \quad \frac{\partial p}{\partial r} = \eta \frac{\partial^2 v_r}{\partial z}$

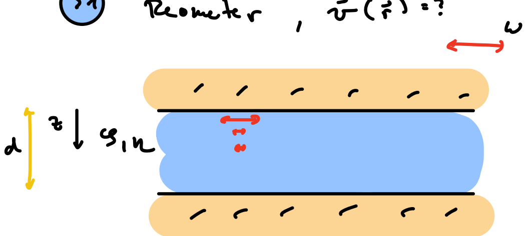
$z: \quad \frac{\partial p}{\partial z} = \eta\, 0$

Kontinuitetna enačba $\quad \nabla\cdot\vec{v} = 0$

$$0 = \frac{1}{r}\frac{\partial}{\partial r} r v_r + \frac{\partial v_z}{\partial z}$$

Rešimo sistem enačb.

(1) Reometer, $\vec{v}(\vec{r}) = ?$ $\quad$ vdorna globina $\ll d$



$$\vec{v} = v(z,t)\,\hat{e}_x$$

$$\rho\frac{\partial \vec{v}}{\partial t} + \underbrace{\rho(\vec{v}\cdot\nabla)\vec{v}}_{=0} = \underbrace{-\nabla p}_{=0} + \eta\nabla^2\vec{v}$$

$$\hat{e}_x\,\hat{e}_x \frac{\partial}{\partial x} v(z) = 0$$

$x: \quad \rho\frac{\partial v}{\partial t} = \eta\frac{\partial^2 v}{\partial z^2} \quad$ Difuzijska en.

Nastavek $\quad v = v_0\, e^{i(kz - \omega t)}$

$$-\rho i \omega = -\eta k^2$$

$$k = \pm\sqrt{i\frac{\rho\omega}{\eta}} = \pm\sqrt{\frac{\rho\omega}{2\eta}}(1+i)$$

$$v = A\, e^{i\left(-\omega t + \sqrt{\frac{\rho\omega}{2\eta}}(1+i) z\right)} + B\, e^{i\left(-\omega t - \sqrt{\frac{\rho\omega}{2\eta}}(1+i) z\right)}$$

$$= e^{-i\omega t}\left(A\, e^{i\sqrt{\frac{\rho\omega}{2\eta}} z}\, e^{-\sqrt{\frac{\rho\omega}{2\eta}} z} + \underset{0}{\underset{\shortparallel}{B}}\, e^{-i\sqrt{\frac{\rho\omega}{2\eta}} z}\, \underline{e^{\sqrt{\frac{\rho\omega}{2\eta}} z}}\right)$$

RP omejenost

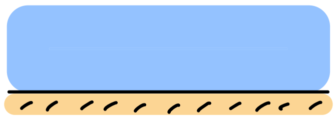
$$v(z=0) = v_0$$

$$v = v_0\, e^{i\left(\sqrt{\frac{\rho\omega}{2\eta}} z - \omega t\right)}\, e^{-\sqrt{\frac{\rho\omega}{2\eta}} z}$$

Vdorna globina $\quad \sqrt{\frac{2\eta}{\rho\omega}}$

Fazni zamik $\quad \Delta\varphi(z) = \sqrt{\frac{\rho\omega}{2\eta}}\, z$

32 Kako s časom raste vrtinčna plast



RP $\vec{v}(z=0) = \vec{v}_0$

Helmholtzova en. za vrtinčnost

$$\frac{d\vec{\omega}}{dt} = (\vec{\omega}\cdot\nabla)\vec{v} + \frac{\eta}{\rho}\nabla^2\vec{\omega} \qquad \vec{\omega} = \nabla\times\vec{v}$$

$$\vec{v} = v(z,t)\,\hat{e}_x \qquad \vec{\omega} = \nabla\times\vec{v} = \frac{\partial v}{\partial z}\,\hat{e}_y = \omega\,\hat{e}_y$$

$$\frac{\partial\omega}{\partial t}\hat{e}_y + \underbrace{(\vec{v}\cdot\nabla)\vec{\omega}}_{=0} = \underbrace{(\vec{\omega}\cdot\nabla)\vec{v}}_{=0} + \underbrace{\frac{\eta}{\rho}}_{D}\frac{\partial^2}{\partial z^2}\omega\,\hat{e}_y$$

$$\frac{\partial\omega}{\partial t} = D\frac{\partial^2\omega}{\partial z^2}$$

RP $\omega(z, t=0) = \frac{\partial v}{\partial z}(z, t=0) = -v_0\,\delta(z)$

$\uparrow \int\limits_0 \omega\, dz = v$

$$\frac{\partial\omega}{\partial t} - D\frac{\partial^2\omega}{\partial z^2} = -v_0\,\delta(z)\,\delta(t)$$

$$\omega(z,t) = \frac{v_0}{\sqrt{4\pi D t}}\,e^{-\frac{z^2}{4Dt}}$$

Rešitev je Greenova funkcija.

Velja za $-\infty < z < \infty$

Prezrcalimo na $z<0$, $v(z=0, t=0) = 2v_0$

$$\omega(z,t) = \frac{2v_0}{\sqrt{4\pi D t}}\,e^{-\frac{z^2}{4Dt}}$$

Širina mejne plasti $z^2 \propto 4Dt$

$$z \propto \sqrt{4Dt} = \sqrt{4\frac{\eta}{\rho}t} = \sqrt{4\frac{\eta}{\rho}\frac{\ell}{v}}$$

$$\frac{z}{\ell} = \sqrt{4\frac{\eta}{\rho\ell v}} = 2\sqrt{\frac{1}{Re}}$$