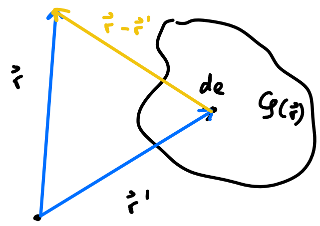
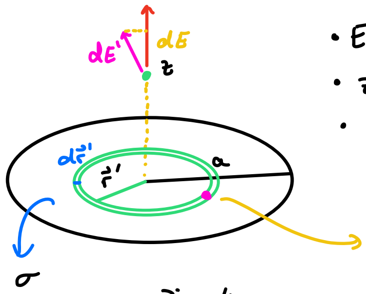
(T) El. polje porazdelitve nabojev $\rho(\vec r\,')$ prostorninske gostote naboja

$$\vec E(\vec r) = \frac{1}{4\pi\varepsilon_0}\int \underbrace{\frac{\overbrace{\rho(\vec r\,')\, d^3\vec r\,'}^{de}}{|\vec r-\vec r\,'|^2}}_{\text{prispevek točkastega naboja}} \; \underbrace{\frac{\vec r-\vec r\,'}{|\vec r-\vec r\,'|}}_{\text{smer}}$$



(1) El. polje nabite okrogle plošče



- $E(z) = ?$ $\vec r = (0, 0, z)$
- $z \gg a$ $\frac{a}{z} \ll 1$
- $z \ll a$ $\frac{a}{z} \gg 1$

$de = \sigma\, 2\pi r'\, dr'$

površinska gostota

$$dE' = \frac{1}{4\pi\varepsilon_0}\frac{2\pi r'\sigma\, dr'}{r'^2+z^2}$$

$$dE = \frac{1}{\cancel{4\pi}\varepsilon_0\, 2}\frac{\cancel{2\pi} r'\sigma\, dr'}{r'^2+z^2}\,\frac{z}{\sqrt{r'^2+z^2}} \qquad \cos\theta = \frac{z}{\sqrt{r'^2+z^2}}$$

$$E = \frac{\sigma z}{2\varepsilon_0}\int_0^a \frac{r'\,dr'}{(\underbrace{r'^2+z^2}_{u})^{3/2}} \qquad du = 2r'dr'$$

$$E = \frac{\sigma z}{4\varepsilon_0}\int_{z^2}^{a^2+z^2} u^{-3/2}\,du$$

$$E = -\frac{\sigma z}{2\varepsilon_0}\, u^{-1/2}\Big|_{z^2}^{a^2+z^2}$$

$\boxed{(1+\varepsilon)^p = 1 + p\varepsilon + \binom{p}{2}\varepsilon^2 + \dots}$

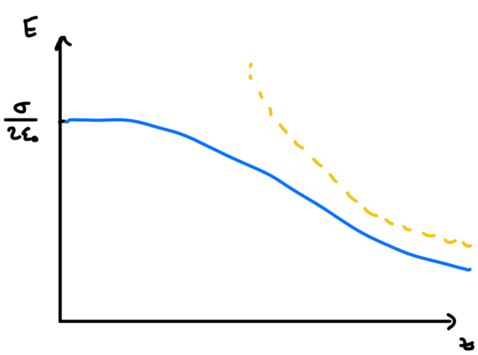
$$E = \frac{\sigma z}{2\varepsilon_0}\left(\frac{1}{z} - \frac{1}{\sqrt{a^2+z^2}}\right) = \frac{\sigma}{2\varepsilon_0}\left(1 - \frac{1}{\sqrt{(\frac{a}{z})^2+1}}\right)$$

Limitni primeri

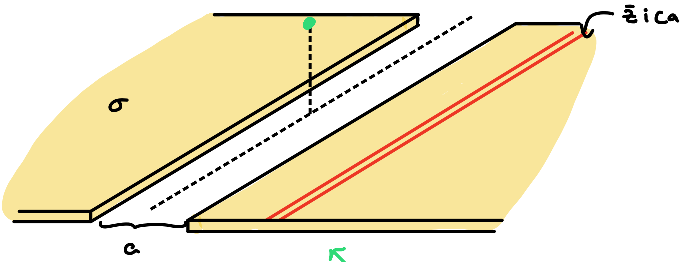
$\frac{a}{z} \ll 1$ $E = \frac{\sigma}{2\varepsilon_0}\left(1 - 1 + \frac{1}{2}\left(\frac{a}{z}\right)^2\right)$ $\frac{a}{z} \gg 1$ $E = \frac{\sigma}{2\varepsilon_0}$

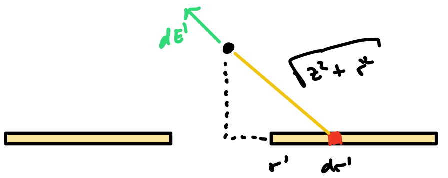
$E = \frac{\sigma a^2}{4\varepsilon_0 z^2} = \frac{e}{4\pi\varepsilon_0 z^2}$, $e = \sigma\pi a^2$

Neskončna ravna ploskev

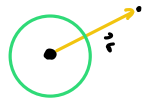


② El. polje ravn ploščic z režo





Podproblem: nabiti vodnik



$$e = \varepsilon_0 \oint \vec{E}\cdot d\vec{S}$$ Gaussov izrek

$$e = \varepsilon_0 E S = \varepsilon_0 E\, 2\pi r l$$

$$E = \frac{e}{2\pi\varepsilon_0 r l}$$

$$dE' = \frac{de}{2\pi\varepsilon_0 \sqrt{z^2+r^2}\, l} \qquad de = \sigma l\, dr'$$

$$dE' = \frac{\sigma}{2\pi\varepsilon_0}\frac{dr'}{\sqrt{z^2+r'^2}}$$

$$dE = \frac{\sigma}{2\pi\varepsilon_0}\frac{dr'}{\sqrt{z^2+r'^2}}\frac{z}{\sqrt{z^2+r'^2}}$$

$$E = 2\frac{\sigma z}{2\pi\varepsilon_0}\int_{a/2}^{\infty}\frac{dr'}{z^2+r'^2}$$

$$E = \frac{\sigma z}{\pi\varepsilon_0}\,\frac{1}{z}\arctan\frac{r'}{z}\Big|_{a/2}^{\infty}$$

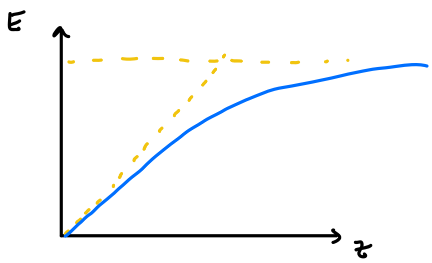
$$E = \frac{\sigma}{\pi\varepsilon_0}\left(\frac{\pi}{2} - \arctan\frac{a}{2z}\right)$$

Limiti

$z \gg a$ $\qquad E = \frac{\sigma}{2\varepsilon_0}$

$z \ll a$ $\qquad E = \frac{\sigma}{\pi\varepsilon_0}\left(\frac{\pi}{2} - \frac{\pi}{2} + \frac{2z}{a}\right) = \frac{\sigma 2 z}{\pi\varepsilon_0 a}$

$$\arctan\frac{1}{\varepsilon} \doteq \frac{\pi}{2} - \varepsilon + \dots$$



$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0}$$

Potencial (=skalarno polje)

$$\vec{E}(\vec{r}) = -\vec{\nabla}U(\vec{r})$$

Prostorninska gostota naboja

$$\rightarrow -\nabla^2 U(\vec{r}) = \frac{\rho(\vec{r})}{\varepsilon_0} \Rightarrow \boxed{\nabla^2 U(\vec{r}) = -\frac{\rho(\vec{r})}{\varepsilon_0}}$$

Laplaceov operator

Poissonova enačba - takšne enačbe tipično rešujemo, a so težke za rešit

En način, kako se tega lotimo je s Fourierovo transformacijo:

Fourierova transformacija

= razvoj po ravnih valovih : $e^{i\vec{k}\cdot\vec{r}}$

$$U(\vec{r}) = \underbrace{\int d^3\vec{k}}_{\text{"vsota"}} \cdot \underbrace{U(\vec{k})}_{\text{amplituda vala}} \cdot e^{i\vec{k}\vec{r}}$$

= ravni val = bazne funkcije

Fourierova Transformacija

Kako določimo amplitude valov? → Koeficiente razvoja po baznih funkcijah dobimo s skalarnim produktom z neko drugo bazno funkcijo. Skalarni produkt v takšnem ∞ ravnem prostoru je pa kar integral po prostoru:

$$\int d^3\vec{r}\cdot e^{-i\vec{k}'\vec{r}}$$ pomnožimo z obema stranema :

Sledi : $$\int d^3\vec{r}\cdot U(\vec{r})\, e^{-i\vec{k}'\vec{r}} = \iint d^3\vec{k}\, d^3\vec{r}\, U(\vec{k})\, e^{i(\vec{k}-\vec{k}')\vec{r}} = \int d^3\vec{k}\cdot U(\vec{k})\,(2\pi)^3\cdot\delta(\vec{k}-\vec{k}') =$$

→ Najprej bomo tole integrirali

$$= U(\vec{k}')\cdot(2\pi)^3$$

Ko bomo integrirali vse možne valove od $-\infty$ do ∞, bomo dobili večinoma 0, razen v izhodišču, ko je $\vec{k}=\vec{k}'$, kjer bo rezultat ∞. Funkcija, ki nam to opisuje je delta

$$\Rightarrow \boxed{U(\vec{k}') = \frac{1}{(2\pi)^3}\cdot\int d^3\vec{r}\; U(\vec{r})\cdot e^{-i\vec{k}\cdot\vec{r}}}$$

Amplitude

Inverzna Fourierova Transformacija

$$\vec{\nabla}e^{i\vec{k}\vec{r}} = e^{i\vec{k}\vec{r}}\cdot\vec{\nabla}(i\vec{k}\vec{r}) = e^{i\vec{k}\vec{r}}\cdot i\vec{k}$$

$$\boxed{FT} \longrightarrow \vec{\nabla}U(\vec{r}) = \int d^3\vec{k}\cdot U(\vec{k})\,\vec{\nabla}e^{i\vec{k}\cdot\vec{r}} = \int d^3\vec{k}\cdot i\vec{k}\,U(\vec{k})\cdot e^{i\vec{k}\cdot\vec{r}}$$

$$U(\vec{r}) \xrightarrow{FT} U(\vec{k}) \qquad \vec{\nabla} \xrightarrow{FT}$$

$$\vec{\nabla}U(\vec{r}) \xrightarrow{FT} i\vec{k}\,U(\vec{k}) \qquad \nabla^2 \xrightarrow{FT} -k^2 = (i\vec{k})\cdot(i\vec{k})$$

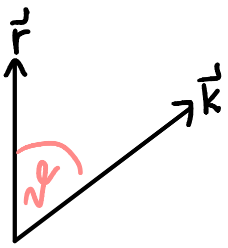
Razvoj $\delta(\vec r)$: $\frac{1}{(2\pi)^3}\int d^3\vec r\, \underbrace{\delta(\vec r)}_{\delta(\vec r-0)}\, e^{-i\vec k\vec r} = \frac{1}{(2\pi)^3}$

Delta funkcija se preslika v konstanto:

$$\delta(\vec r) \xrightarrow{FT} \frac{1}{(2\pi)^3}$$

(1) Poissonova enačba za točkast naboj

$\nabla^2 U(\vec r) = -\frac{\rho(\vec r)}{\varepsilon_0}$; $\rho(\vec r) = \underbrace{\delta(\vec r)}_{\text{3D delta fun } [1/m^3]}\cdot e$



$\Rightarrow \boxed{\nabla^2 U(\vec r) = -\frac{e}{\varepsilon_0}\cdot\delta(\vec r)}$ /FT   DE $\longrightarrow$ Algebrajska enačba

$+k^2 \cdot U(\vec k) = +\frac{e}{\varepsilon_0}\cdot\frac{1}{(2\pi)^3}$

$\underbrace{U(\vec k)}_{\text{Posamezna amplituda}} = \frac{e}{\varepsilon_0 k^2 (2\pi)^3}$

$$U(\vec r) = \int d^3\vec k\, U(\vec k)\, e^{i\vec k\cdot\vec r} = \frac{e}{\varepsilon_0(2\pi)^3}\int d^3\vec k\cdot\frac{1}{k^2}\cdot e^{i\vec k\vec r} = \frac{e}{\varepsilon_0(2\pi)^3}\int_0^{2\pi}\int_{-1}^{1}\int_0^{\infty}\frac{1}{\cancel{k^2}}\cdot e^{ikr\cos\vartheta}\cdot\cancel{k^2}\cdot\underbrace{\sin\vartheta\, d\vartheta}_{d(\cos\vartheta)}\, dk\, d\varphi$$

$$= 2\cancel{\pi}\cdot\frac{e}{\varepsilon_0(2\pi)^{\cancel{3}\,2}}\cdot\int_{-1}^{1}\int_0^{\infty} e^{iku}\, dk\, du = \frac{e}{\varepsilon_0(2\pi)^2}\int_{-\infty}^{\infty}\left(\frac{1}{ikr}e^{ikru}\right)\Big|_{-1}^{1} dk =$$

$$= \frac{e}{\varepsilon_0(2\pi)^2}\cdot\frac{2i}{ir}\int_0^{\infty}\frac{1}{k}\cdot\overbrace{\left(\frac{e^{ikr}-e^{-ikr}}{2i}\right)}^{\sin(kr)} dk = \frac{e}{\varepsilon_0 2\pi^2 r}\int_0^{\infty}\frac{\sin(kr)}{kr}\, d(kr) =$$

$$= \frac{e}{\varepsilon_0 2\pi^2 r}\underbrace{\int_0^{\infty}\frac{\sin u}{u}\, du}_{\pi/2} \Rightarrow \boxed{U(\vec r) = \frac{e}{4\pi\varepsilon_0 r}}$$ Točkast naboj

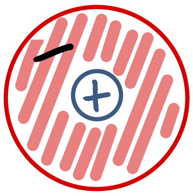
$$\nabla^2 U(\vec r) = -\frac{e}{\varepsilon_0}\delta(\vec r) \longrightarrow U(\vec r) = \frac{e}{4\pi\varepsilon_0|\vec r|}$$

$$\nabla^2 U(\vec r) = -\frac{\rho(\vec r)}{\varepsilon_0} \xrightarrow[\text{Splošno?}]{} U(\vec r) = \frac{1}{4\pi\varepsilon_0}\int\frac{d^3\vec r'\,\rho(\vec r')}{|\vec r-\vec r'|}$$

$\rho(\vec r)$ moramo razviti po δ-funkcijah: $\rho(\vec r) = \int d^3\vec r'\,\rho(\vec r')\,\delta(\vec r-\vec r')$

$$\Rightarrow \boxed{E(\vec r) = -\nabla U(\vec r) = \frac{1}{4\pi\varepsilon_0}\int\frac{d^3\vec r'\,\rho(\vec r')}{|\vec r-\vec r'|^2}\cdot\frac{\vec r-\vec r'}{|\vec r-\vec r'|}}$$

(2) Gostota naboja v vodikovem atomu



$$U(\vec{r})=\frac{1}{4\pi\varepsilon_0}\cdot\frac{e^{-\alpha r}}{r}\left(1+\frac{\alpha r}{2}\right);\quad \alpha=\frac{2}{a_B}$$ ← Bohrov radij

$$\rho(\vec{r})=?$$

$$\nabla^2 U(\vec{r})=-\frac{\rho(\vec{r})}{\varepsilon_0}$$

$$\nabla^2 U(\vec{r})=\frac{1}{r^2}\cdot\frac{\partial}{\partial r}\left(r^2\cdot\frac{\partial U}{\partial r}\right)+\underset{\vartheta}{\dots}+\underset{\varphi}{\dots}$$

→ Tole bomo uporabili za krogelne koordinate (mafija 1)

$$\Rightarrow \rho(\vec{r})=-\varepsilon_0\cdot\frac{1}{r^2}\cdot\frac{\partial}{\partial r}\left[r^2\cdot\frac{\partial}{\partial r}\left(\frac{e}{4\pi\varepsilon_0}\cdot\frac{e^{-\alpha r}}{r}\cdot\left(1+\frac{\alpha r}{2}\right)\right)\right]=$$

$$=-\frac{e}{4\pi r^2}\cdot\frac{\partial}{\partial r}\left(r^2\cdot\left(-\alpha e^{-\alpha r}\left[\frac{1}{r}+\frac{\alpha}{2}\right]+e^{-\alpha r}\cdot\left[-\frac{1}{r^2}\right]\right)\right)=$$

$$=+\frac{e}{4\pi r^2}\cdot\frac{\partial}{\partial r}\left[e^{-\alpha r}\left(+\alpha r+\frac{\alpha^2}{2}r^2+1\right)\right]=$$

$$=+\frac{e}{4\pi r^2}\cdot\left[+\alpha e^{-\alpha r}\left(\alpha r+\frac{\alpha^2}{2}r^2+1\right)+e^{-\alpha r}\left(\alpha+\alpha^2 r\right)\right]=$$

$$=+\frac{e}{4\pi r^2}\cdot e^{-\alpha r}\left[-\cancel{\alpha^2 r}-\frac{\alpha^3}{2}r^2-\cancel{\alpha}+\cancel{\alpha}+\cancel{\alpha^2 r}\right]=$$

$$=+\frac{e}{4\pi \cancel{r^2}}\cdot e^{-\alpha r}\left(-\frac{\alpha^3\cancel{r^2}}{2}\right)=\frac{\alpha^3 e}{8\pi}e^{-\alpha r}$$

$$\boxed{\rho(\vec{r})=-\frac{\alpha^3 e}{8\pi}e^{-\alpha r}}$$ → $\psi\propto e^{-\frac{\alpha r}{2}}$ √ 1. sferični harmonik

Dobili smo gostoto naboja v oblaku elektrona

$\propto |\psi|^2$

Kje pa je proton? → Pri odvajanju nismo upoštevali nezveznosti v izhodišču (torej jedru). $U(\vec{r})$ ima tam singularnost. Posebej si torej poglejmo še izhodišče:

$$U(\vec{r})\longrightarrow\frac{e}{4\pi\varepsilon_0}\cdot\frac{1}{r}(1+0)=\frac{1}{4\pi\varepsilon_0 r}$$ Točkast naboj

$$\Downarrow$$

$$\rho(\vec{r})=e\cdot\delta(\vec{r})$$ → Izhodišče nam da ta prispevek

Združimo skupaj, da dobimo celoten naboj:

$$\boxed{\rho(\vec{r})=\underbrace{-\frac{\alpha^3 e}{8\pi}e^{-\alpha r}}_{\text{elektron}}+\underbrace{e\cdot\delta(\vec{r})}_{\text{proton}}}$$

Poissonova enačba

$$\nabla^2 U(\vec r) = -\frac{\rho(\vec r)}{\varepsilon_0}$$

↳ potencial el polja

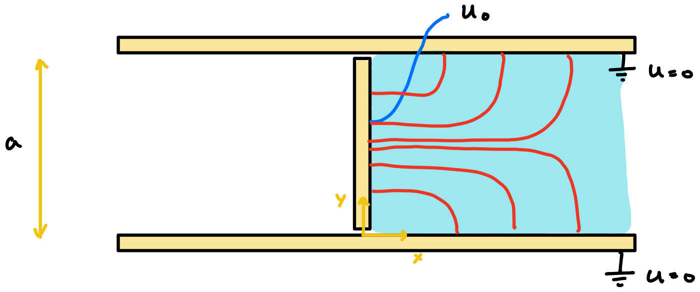
Lokalna enačba

$\rho(\vec r) = 0$ skoraj povsod

↳ $\nabla^2 U = 0$ Laplaceova enačba

kjer je $\rho(\vec r) \neq 0$ ⇒ robni pogoji

4 Prečni trak v ploščatem kondenzatorju



$U = ?$ znotraj kondenzatorja (desno)

$$\nabla^2 U(x,y) = 0$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) u = 0$$

poskusimo Nastavek $U(x,y) = X(x)Y(y)$

$$Y X'' + X Y'' = 0$$

$$-\frac{Y''}{Y} = \frac{X''}{X} = K^2$$

Robni pogoji

1. $U(0,y) = U_0$
2.3. $U(x,0) = U(x,a) = 0$
4. $U(\infty, y) < \infty$

$$Y'' = -K^2 Y \qquad X'' = K^2 X$$

$$Y = C\sin Ky + D\cos Ky \qquad X = Ae^{Kx} + Be^{-Kx}$$

RP 4. $A = 0$

2. $D \cdot B e^{-Kx} = 0 \Rightarrow D = 0$ $\qquad U = \underbrace{BC}_{B'} e^{-Kx} \sin Ky$

3. $B' \sin Ka \; e^{-Kx} = 0$ $\qquad B' \neq 0$ $\qquad Ka \overset{!}{=} \pi n \quad n = 1, 2, \dots$

$U_n = B' \sin\frac{n\pi}{a} y \; e^{-\frac{n\pi}{a}x}$ $\qquad K_n = \frac{\pi n}{a}$

$$U = \sum_n B_n' \sin\left(\frac{n\pi}{a} y\right) e^{-\frac{n\pi}{a}x}$$

1. $U(0,y) = U_0$

$$U_0 = \sum_n B_n' \underbrace{\sin\frac{n\pi}{a} y}_{\text{bazne funkcije}} \qquad \Big/ \int_0^a \sin\frac{m\pi}{a} y \, dy$$

$$L: \int_0^a U_0 \sin\frac{m\pi}{a} y \, dy = -U_0 \frac{a}{m\pi} \cos\frac{m\pi}{a} y \Big|_0^a = -U_0(\cos m\pi - 1)\frac{a}{m\pi}$$
$$= \frac{U_0 a}{m\pi}(1 - \cos m\pi) = \frac{U_0 a}{m\pi}(1 - (-1)^m)$$

$$D: \sum_n \int_0^a B_n' \sin\frac{n\pi}{a} y \sin\frac{m\pi}{a} y \, dy = \sum_n B_n' \delta_{mn} \int_0^a \sin^2\left(\frac{n\pi}{a} y\right) dy =$$

$$= \sum_n B_n' \delta_{mn} \frac{1}{2} a$$

$$\frac{U_0 a}{m\pi}(1 - (-1)^m) = \sum_n B_n' \delta_{mn} \frac{a}{2}$$

$$\frac{2U_0}{m\pi}(1 - (-1)^m) = B_m'$$

$$U(x,y) = \sum_n \frac{2U_0}{n\pi}(1 - (-1)^n) \sin\left(\frac{n\pi}{a} y\right) e^{-\frac{n\pi}{a}x}$$

① $x \gg a$ $\quad n=1$ $\quad U(x,y) = e^{-\frac{\pi}{a}x} \frac{2U_0}{\pi} 2 \sin\left(\frac{\pi}{a} y\right) = \frac{4U_0}{\pi} \sin\left(\frac{\pi}{a} y\right) e^{-\frac{\pi}{a}x}$

② $y = \frac{a}{2}$ $\quad E\left(0, \frac{a}{2}\right) = (E_x, 0)$

$$E_x = -\frac{\partial}{\partial x} U = -\sum_n \frac{2U_0}{n\pi}(1 - (-1)^n)\left(-\frac{n\pi}{a}\right) e^{-\frac{n\pi}{a}x} \sin\left(\frac{n\pi}{a} y\right)$$

$$E_x(0,\tfrac{a}{2}) = \sum_n \frac{2U_0}{a}(1-(-1)^n)\, e^{-\frac{n\pi}{a}x} \sin\left(\frac{n\pi}{2}\right)$$

$$= \frac{2U_0}{a}\sum_n (1-(-1)^n)(-1)^n \left(e^{-\frac{\pi x}{a}}\right)^n$$

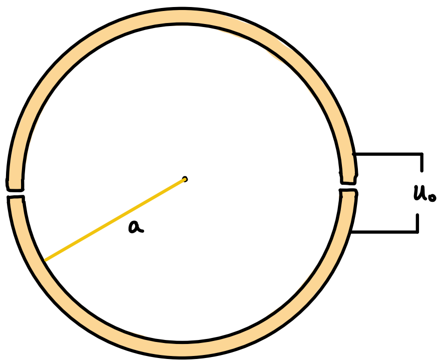
$$= \frac{2U_0}{a}\left(2\left(e^{-\frac{\pi x}{a}}\right)^1 + 0 - 2\left(e^{-\frac{\pi x}{a}}\right)^3 + o.t.\right)$$

$$= \frac{4U_0}{a} e^{-\frac{\pi x}{a}}\left(1 - e^{-\frac{2\pi x}{a}} + \left(e^{-\frac{2\pi x}{a}}\right)^2 - \dots\right)$$

$$= \frac{4U_0}{a} e^{-\frac{\pi x}{a}} \frac{1}{1+e^{-\frac{2\pi x}{a}}} = \frac{4U_0}{a}\frac{1}{e^{\frac{\pi x}{a}}+e^{-\frac{\pi x}{a}}} =$$

$$= \frac{4U_0}{2a}\frac{1}{\mathrm{ch}\frac{\pi x}{a}} = \frac{2U_0}{a}\frac{1}{\mathrm{ch}\frac{\pi x}{a}}$$

5 Prepolovljena prevodna cev



$\nabla^2 U(r,\varphi)=0$ $\qquad U = R(r)\,\Phi(\varphi)$

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}$$ Laplace v cilindričnih

$$\nabla^2 U = \frac{1}{r}\frac{\partial}{\partial r}(rR')\,\Phi + \frac{1}{r^2}R\,\Phi'' = 0 \qquad / : \Phi$$

$$\frac{1}{r}(R'+rR'') + \frac{R}{r^2}\frac{\Phi''}{\Phi}=0 \qquad / \cdot \frac{r^2}{R}$$

$$\frac{rR'+r^2R''}{R} = -\frac{\Phi''}{\Phi} = m^2$$

$\Phi''=-m^2\Phi$

$\Phi = A\sin m\varphi + B\cos m\varphi$, $m\in\mathbb{N}$

$m=0$: $\Phi''=0$, $\Phi = a\varphi+b$

$r^2R''+rR'+m^2R=0$ Bernoullijeva DE

$R = C\,r^m + D\,r^{-m}$, $m\in\mathbb{N}$

$m=0$

$r^2R''+rR'=0$

$rR''+R'=0$

$\frac{R''}{R'}=-\frac{1}{r}$

$(\ln R')' = -\frac{1}{r} \qquad /\int$

$\ln R' = -\int\frac{1}{r}$

$\ln R' = \ln c - \ln r$

$R' = \frac{c}{r}$

$R=\int\frac{c}{r}\,dr$

$R = c\ln r + d$

Rešitev $\nabla^2U=0$ v cilindričnih

$$U(r,\varphi) = \underbrace{\sum_{m=1}^\infty (A_m\sin m\varphi + B_m\cos m\varphi)(C_m r^m + D_m r^{-m})}_{\text{periodični del}} + \underbrace{(a\varphi+b)(c\ln r+d)}_{m=0}$$

$=0$ v našem primeru

Robni pogoji

1. $U(a,\varphi) = \begin{cases} U_0/2 & 0<\varphi<\pi \\ -U_0/2 & \pi<\varphi<2\pi\end{cases}$ = liha $\Rightarrow B_m=0$
2. $U(0,\varphi)<\infty \Rightarrow D=0$

$$U(r,\varphi)=\sum_{m=1}^\infty F_m \sin m\varphi\; r^m$$

$$\frac{U_0}{2} = \sum_{m=1}^\infty F_m \sin m\varphi\; a^m \qquad 0<\varphi<\pi \qquad /\cdot\int \sin n\varphi\, d\varphi$$

$$\int_0^{2\pi} U\sin n\varphi\, d\varphi = \sum_{m=1}^\infty F_m a^m \int_0^{2\pi}\sin n\varphi \sin m\varphi\, d\varphi$$

DS: $\sum_{m=1}^{\infty} F_m a^m \delta_{mn} \int_0^{2\pi} \sin^2 n\varphi \, d\varphi = F_n a^n \frac{2\pi}{2} = F_n a^n \pi$

LS: $\int_0^{\pi} \frac{u_0}{2} \sin n\varphi \, d\varphi - \int_{\pi}^{2\pi} \frac{u_0}{2} \sin n\varphi \, d\varphi = u_0 \int_0^{\pi} \sin n\varphi \, d\varphi = u_0 \frac{1}{n} (-\cos n\varphi)\Big|_0^{\pi} = \frac{u_0}{n}(-\cos(n\pi)+1)$

$$= \frac{u_0}{n}\left(1-(-1)^n\right)$$

$$\Rightarrow \quad F_n = \frac{u_0}{n a^n \pi}\left(1-(-1)^n\right)$$

$$\Rightarrow \quad u(r,\varphi) = \sum_{n=1}^{\infty} \frac{u_0}{n\pi}\left(1-(-1)^n\right) \sin n\varphi \left(\frac{r}{a}\right)^n$$

- $E(r,0) = ?$

$$E_\varphi = -\frac{1}{r}\frac{\partial}{\partial \varphi} u \Big|_{\varphi=0}$$

$$E_\varphi = -\frac{1}{r}\sum_{n=1}^{\infty} \frac{u_0}{n\pi}\left(1-(-1)^n\right)(-n) \cos n\varphi \left(\frac{r}{a}\right)^n$$

$$= -\frac{u_0}{\pi}\sum_{n=1}^{\infty} \left(1-(-1)^n\right) \overset{1}{\overset{\|}{\cos n\varphi}} \frac{r^{n-1}}{a^n} \Big|_{\varphi=0}$$

$$= -\frac{u_0}{\pi r}\sum_{n=1}^{\infty} \left(1-(-1)^n\right) \left(\frac{r}{a}\right)^n$$

$$= -\frac{u_0}{\pi r}\left(2\frac{r}{a} + 2\left(\frac{r}{a}\right)^3 + \dots\right)$$

$$= -\frac{2u_0}{\pi a}\left(1 + \left(\frac{r}{a}\right)^2 + \left(\frac{r}{a}\right)^4 + \dots\right)$$

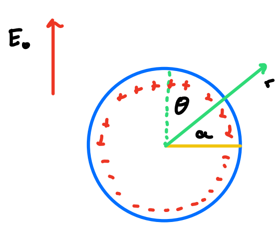
$$= -\frac{2u_0}{\pi a} \frac{1}{1-\left(\frac{r}{a}\right)^2}$$

- Navprečna ravnina

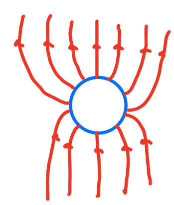
$$E\left(r,\tfrac{\pi}{2}\right) = -\frac{\partial}{\partial r} u \Big|_{\varphi=\frac{\pi}{2}} \qquad \text{DN}$$

Rezultat $\quad E_r = -\frac{2u_0}{\pi a} \frac{1}{1+\left(\frac{r}{a}\right)^2}$

⑥ Prevodna krogla v homogenem el. polju



- v krogli ni polja
- $u(r,\theta) = ?$ neodvisno od φ



Zunaj povsod velja $\nabla^2 u = 0$

Osno simetrične rešitve so:

$$\boxed{u(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + B_l r^{-(l+1)}\right) P_l(\cos\theta)}$$

$\hookrightarrow$ Legendrovi polinomi

$P_0(x) = 1$
$P_1(x) = x$
$P_2(x) = \frac{1}{2}(3x^2-1)$
$P_3(x) = \frac{1}{2}(5x^3-3x)$
$\vdots$

Ortogonalni

Robni pogoji:

RP1 $U(a,\theta) = 0$ potencial na robu kovine je konstanten

RP2 $U(r\to\infty,\theta) = \underbrace{-E_0 \underbrace{r\cos\theta}_{z}}_{\text{zunanje homogeno polje}} = -E_0 z$ $(u = Ex)$

RP2 $\overset{r\to\infty}{\sum_{l=0}^{\infty}} A_l r^l P_l(\cos\theta) = -E_0 r \underbrace{\cos\theta}_{r^1 P_1(\cos\theta)}$

$\Rightarrow A_1 = -E_0 \Rightarrow U(r,\theta) = -E_0 r\cos\theta + \sum_{l=0}^{\infty} B_l r^{-(l+1)} P_l(\cos\theta)$

RP1 $0 = -E_0 r\cos\theta + \sum_{l=0}^{\infty} B_l r^{-(l+1)} P_l(\cos\theta) \Big|_{r=a}$

$E_0 a\cos\theta = \sum_{l=0}^{\infty} B_l a^{-(l+1)} P_l(\cos\theta)$

$E_0 a P_1(\cos\theta) = B_1 a^{-2} P_1(\cos\theta)$

$B_1 = E_0 a^3$ $p_e = 4\pi\varepsilon_0 E_0 a^3$

$\Rightarrow U(r,\theta) = -E_0 r\cos\theta + E_0 a^3\cos\theta = \underbrace{E_0\cos\theta}\left(\underbrace{\frac{a^3}{r^2}} - \underbrace{r}\right)$

↳ naboji na krogli ↳ potencial zunanjega homogenega polja

Splošen robni pogoj

$\sum P_l(\cos\theta) = f(\cos\theta) \quad \Big| \int P_{l'}(\cos\theta)\, d(\cos\theta)$

Točkasti dipol $U_{DP}(\vec r) = \frac{\vec p_e\cdot\vec r}{4\pi\varepsilon_0 r^3}$

$\vec p_e$ dipolni moment

(b) Izračun $\vec p_e$ in $\sigma(\theta)$ na površini

$de = \sigma\, dS$

$\vec p_e = \int \vec r\, de = \int \vec r\, \rho(\vec r)\, d^3\vec r$

Gauss $de = \varepsilon_0 E\, dS \quad \Big| :dS$

$\frac{de}{dS} = \sigma = \varepsilon_0 E_\perp$

$= -\varepsilon_0 \nabla U$

$= -\varepsilon_0 \frac{\partial U}{\partial r}$

$= -\varepsilon_0 E_0\cos\theta\left(-\frac{2a^3}{r^3} - 1\right)\Big|_{r=a}$

$\sigma = 3\varepsilon_0 E_0 \cos\theta$

$\vec p_e = \int \vec r\, \sigma(\vec r)\, dS$

$p_{ez} = \int z\, \sigma(z)\, dS$

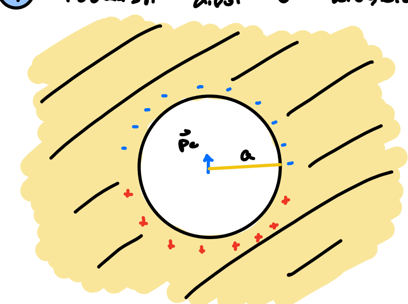
$p_{ez} = \int \underbrace{3\varepsilon_0 E_0 \cos\theta}\ \underbrace{a\cos\theta}\ \underbrace{a^2\, d\varphi\, d\cos\theta}$

$p_{ez} = 2\pi\cdot 3\varepsilon_0 E_0 a^3 \int_{-1}^{+1} \cos^2\theta\, d\cos\theta$

$= 6\pi\varepsilon_0 E_0 a^3 \left.\frac{x^3}{3}\right|_{-1}^{+1}$

$= 4\pi\varepsilon_0 E_0 a^3$

(7) Točkasti dipol v krogelni votlini



$U(r,\theta) = ?$ znotraj

$U(r,\theta) = \sum_{l=0}^{\infty} \left(A_l r^l + B_l r^{-(l+1)}\right) P_l(\cos\theta)$

RP1 $U(a,\theta) = 0$

RP2 $U(r\to 0,\theta) = \frac{p_e\cos\theta}{4\pi\varepsilon_0 r^2}$ $U_{dip} = \frac{\vec p_e\cdot\vec r}{4\pi\varepsilon_0 r^3}$

RP2 $r\to 0$ $U(r,\theta) = \sum_{l=0}^{\infty} B_l r^{-(l+1)} P_l(\cos\theta) = \frac{p_e\cos\theta}{4\pi\varepsilon_0 r^2}$

$l=1$ $B_1 \frac{1}{r^2}\cos\theta = \frac{p_e}{4\pi\varepsilon_0}\frac{1}{r^2}\cos\theta \Rightarrow B_1 = \frac{p_e}{4\pi\varepsilon_0}$ $B_l = 0$ $l\neq 1$

$$U(r,\theta) = \frac{p_e \cos\theta}{4\pi\varepsilon_0 r^2} + \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$$

$r=a$

$$U(a,\theta) = \frac{p_e \cos\theta}{4\pi\varepsilon_0 a^2} + \sum_{l=0}^{\infty} A_l a^l P_l(\cos\theta) = 0$$

$$\sum_{l=0}^{\infty} A_l a^l P_l(\cos\theta) = -\frac{p_e}{4\pi\varepsilon_0}\frac{1}{a^2}\cos\theta$$

$$l=1 \qquad A_1 a \cos\theta = -\frac{p_e}{4\pi\varepsilon_0}\frac{1}{a^2}\cos\theta$$

$$A_1 = -\frac{p_e}{4\pi\varepsilon_0 a^3}$$

$$\Rightarrow U(r,\theta) = \frac{p_e}{4\pi\varepsilon_0}\cos\theta\left(\frac{1}{r^2} - \frac{r}{a^3}\right)$$

točkasti dipol   pot. nabojev na površini

= homogeno polje

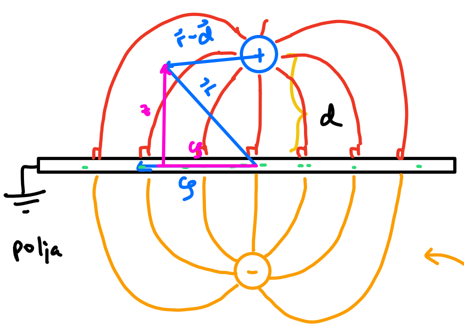
kot vektor

$$\sigma = \frac{de}{dS} = -\varepsilon_0 E_\perp \qquad \vec{E}$$

$$\sigma = -\varepsilon_0\left(-\frac{\partial U}{\partial r}\right) = \frac{p_e}{4\pi}\cos\theta\left(-\frac{2}{r^3} - \frac{1}{a^3}\right)\Big|_{r=a}$$

$$\sigma = -\frac{3p_e}{4\pi a^3}\cos\theta$$

8 Točkasti naboj nad prevodno ploščo



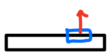
Ni polja

Ali plošča ali negativen naboj

$U(\vec{r}) = ?$

$\sigma_{ind}(\rho) = ?$

$\sigma_{ind} = ?$



Polje dveh točkastih nabojev

$$U(\vec{r}) = \frac{e_0}{4\pi\varepsilon_0|\vec{r}-\vec{d}|} - \frac{e_0}{4\pi\varepsilon_0|\vec{r}+\vec{d}|}$$

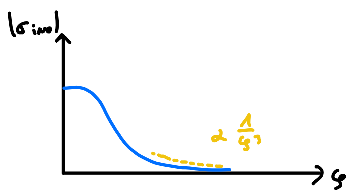
$$U(\rho,z) = \frac{e_0}{4\pi\varepsilon_0}\left(\frac{1}{\sqrt{\rho^2+(z-d)^2}} - \frac{1}{\sqrt{\rho^2+(z+d)^2}}\right)$$

$$\sigma_{IND} = \varepsilon_0 E_\perp = \varepsilon_0\left(-\frac{\partial U}{\partial z}\right)$$

$$\sigma_{IND} = \frac{-e_0}{4\pi}\left(-\cancel{\tfrac{1}{2}}\frac{\cancel{2}(z-d)}{(\rho^2+(z-d)^2)^{3/2}} + \cancel{\tfrac{1}{2}}\frac{\cancel{2}(z+d)}{(\rho^2+(z+d)^2)^{3/2}}\right)\Big|_{z=0}$$

$$\sigma_{IND} = \frac{e_0}{4\pi}\left(\frac{-d}{(\rho^2+d^2)^{3/2}} - \frac{d}{(\rho^2+d^2)^{3/2}}\right)$$

$$\sigma_{IND}(\rho) = \frac{-e_0 d}{2\pi(\rho^2+d^2)^{3/2}}$$



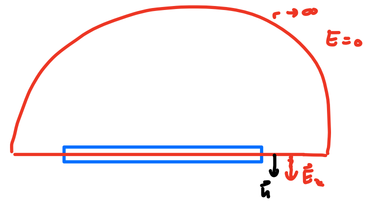
$$\sigma_{IND} = \int_0^\infty \frac{-e_0 d}{2\pi(\rho^2+d^2)^{3/2}}\rho\, d\rho \int_0^{2\pi} d\varphi$$

$$= -\frac{e_0 d}{2}\int_{d^2}^{\infty} u^{-3/2}\, du \qquad u = \rho^2+d^2, \quad du = 2\rho\, d\rho$$

$$= -\frac{e_0 d}{2}\frac{1}{\sqrt{u}}(-2)\Big|_{d^2}^{\infty}$$

$$= e_0 d\left(0 - \frac{1}{d}\right) = \underline{-e_0}$$

Električna sila



celotno polje

$$\vec F_e = \varepsilon_0 \oint \left(\vec E(\vec E\cdot\vec n) - \frac{1}{2}E^2\vec n\right) dS$$

po ploskvi

Pomembno! izbira ploskve

$$E_z = \frac{-e_0 d}{2\pi\varepsilon_0(\rho^2+d^2)^{3/2}} \qquad \vec n = (0,0,1), \quad \vec E = (0,0,E_z)$$

$\vec E \parallel \vec n$ $\qquad \vec F_e = \varepsilon_0 \oint \frac{1}{2}E^2\vec n\, dS$

$$\vec F_e = \varepsilon_0 \frac{e_0^2 d^2}{4\pi^2\varepsilon_0^2}\oint \frac{1}{(\rho^2+d^2)^3}\,\frac{1}{2}\,dS\,\vec n$$

$$\vec F_e = \frac{e_0^2 d^2}{8\pi^2\varepsilon_0}\int_0^\infty\int_0^{2\pi}\frac{1}{(\rho^2+d^2)^3}\,\rho\, d\rho\, d\varphi$$

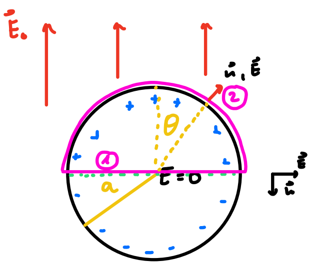
$u = \rho^2+d^2$
$du = 2\rho\, d\rho$

$$\vec F_e = \frac{e_0^2 d^2}{4\pi\varepsilon_0}\int_{d^2}^\infty u^{-3}\,\frac{du}{2}\,\vec n$$

$$\vec F_e = \frac{e_0^2 d^2}{4\pi\varepsilon_0}\,\frac{1}{2}\,\frac{1}{u^2}\left(-\frac{1}{2}\right)\Big|_{d^2}^{\infty}$$

$$\vec F_e = \frac{e_0^2}{16\pi\varepsilon_0 d^2}\,\vec n$$

9 Sila na polovico prevodne krogle



$\vec F_e = ?$

↑ od polja
↓ od ⊖ naboju

$$U(r,\theta) = -E_0 r\cos\theta + \frac{E_0 a^3}{r^2}\cos\theta$$

1 $\quad \vec E = 0 \qquad \vec F_{e_1} = 0$

2 $\quad \vec n \parallel \vec E \;\to\; \vec E\cdot\vec n = E \qquad \Rightarrow \quad \vec F_e = \varepsilon_0\oint\frac{1}{2}E^2\vec n\, dS = \frac{\varepsilon_0}{2}\int_2 E^2\vec n\, dS$

$$E_r = -\frac{\partial}{\partial r}U\Big|_{r=a} = E_0\cos\theta\left(1+2\frac{a^3}{r^3}\right) = 3E_0\cos\theta$$

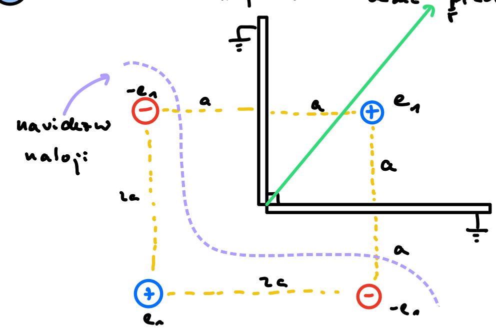
$$\vec n(\varphi,\theta) = (\cos\varphi\sin\theta,\ \sin\varphi\sin\theta,\ \cos\theta)$$

$$dS = a^2\sin\theta\, d\theta\, d\varphi$$

$$\vec F_e = \frac{\varepsilon_0}{2}\int_0^{\pi/2}\int_0^{2\pi} 9E_0^2\cos^2\theta\,(\underset{=0}{\cos\varphi\sin\theta},\ \underset{=0}{\sin\varphi\sin\theta},\ \cos\theta)\, a^2\sin\theta\, d\theta\, d\varphi$$

$$F_{e_z} = \frac{\varepsilon_0}{2}\int_0^1 2\pi\; 9E_0^2\cos^3\theta\; a^2\; d\cos\theta = 9\varepsilon_0 E_0^2 a^2\pi\int_0^1 x^3 dx = 9\varepsilon_0 E_0^2 a^2\,\frac{x^4}{4}\Big|_0^1 = \frac{9\pi}{4}\varepsilon_0 E_0^2 a^2$$

⑩ Točkasti naboj med dvema pravokotnima ploščama



$U(\vec{r}) = ?$ daleč stran $r \gg a$

↳ multipolni razvoj

Dobimo kvadrupol

Multipolni razvoj

$$U(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\Big(\underbrace{\frac{e}{r}}_{\text{monopol}} + \underbrace{\frac{1}{r^3}\sum_{i=1}^{3} p_i r_i}_{\text{dipol}\ (\vec{p}\cdot\vec{r})} + \underbrace{\frac{1}{r^5}\sum_{i=1}^{3}\sum_{j=1}^{3} Q_{ij} r_i r_j}_{\text{kvadrupol}} + \dots\Big)$$

Monopolni moment $e = \int \rho(\vec{r}')\, d^3\vec{r}'$ skalar

Dipolni moment $p_i = \int r_i' \rho(\vec{r}')\, d^3\vec{r}'$ vektor

Kvadrupolni moment $Q_{ij} = \int (3 r_i' r_j' - \delta_{ij} r'^2)\, \rho(\vec{r}')\, d^3\vec{r}'$ tenzor

diskretna verzija $Q_{ij} = \sum_n \left(3 (r_i)_n (r_j)_n - \delta_{ij} r_n^2\right) e_n$

$\mathrm{tr}\, Q \overset{\text{mora veljati to}}{=} 0$

Naš primer

$\sum e = 0$ ni monopolnega momenta

$\uparrow \downarrow = 0$ ni dipolnega momenta

$\vec{r}' = (\pm a, \pm a)$

$$Q_{xx} = (3aa - 2a^2)\, e + (3(-a)(a) - 2a^2)(-e) + (3(-a)(-a) - 2a^2)\, e + (3\, a\, a - 2a^2)(-e) = 0$$

$\Rightarrow Q_{yy} = 0 \quad Q_{zz} = 0$

$\Rightarrow Q_{xz} = Q_{yz} = 0$

$$Q_{xy} = (3aa\, e + 3a(-a)(-e))\cdot 2 = (a^2 e + 5a^2 e)\, 2 = 12a^2 e$$

$$Q = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} 12a^2 e$$

$$4\pi\varepsilon_0 U(\vec{r}) = \frac{1}{r^5}\sum_{i=1}^{3}\sum_{j=1}^{3} Q_{ij} r_i r_j = \frac{1}{r^5}\vec{r}^T Q \vec{r} = \frac{12a^2 e}{r^5}(x, y, z)\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{12a^2 e}{r^5}\, 2xy$$

$$U(\vec{r}) = \frac{6a^2 e_1}{\pi\varepsilon_0}\, \frac{xy}{r^5}$$

(T) Magnetostatika

$\nabla\cdot\vec B = 0$ $\qquad$ $\vec B = \nabla\times\vec A$

vektorski potencial

$\nabla\times\vec B = \mu_0 \vec j$

gostota el. toka

$\nabla\times(\nabla\times\vec A) = \mu_0\vec j$

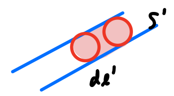
$\vec\nabla(\underbrace{\vec\nabla\cdot\vec A}_{=0}) - \nabla^2\vec A = \mu_0\vec j$

(si izberemo tak potencial) $\Rightarrow$ $\nabla^2\vec A = -\mu_0\vec j$ $\qquad$ Poissonova enačba

$$\nabla^2 u = -\frac{\rho}{\varepsilon_0} \rightarrow u(\vec r) = \frac{1}{4\pi\varepsilon_0}\int\frac{\rho(\vec r')\,d^3\vec r'}{|\vec r-\vec r'|}$$

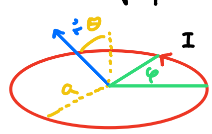
$$\nabla^2 A = -\mu_0\vec j \longrightarrow \vec A(\vec r) = \frac{\mu_0}{4\pi}\int\frac{\vec j(\vec r')\,d^3\vec r'}{|\vec r-\vec r'|}$$

Za vodnike (diskretna porazdelitev) $\quad \vec j(\vec r')\,d^3\vec r' = \vec j(\vec r)\,S'\,dl' = I\vec t\,dl' = I\,d\vec l'$



$$\vec A(\vec r) = \frac{\mu_0 I}{4\pi}\int\frac{d\vec l'}{|\vec r-\vec r'|}$$

(11) Magnetno polje krožne tokovne zanke



$\vec A(\vec r) = ?$ $\quad$ daleč stran $\quad r \gg a \quad \frac{a}{r} \ll 1$

$$d\vec l' = \begin{bmatrix}-\sin\varphi\\ \cos\varphi\\ 0\end{bmatrix} a\,d\varphi$$

naj bo $\vec r$ med x osjo

$$\vec r' = \begin{bmatrix}-\cos\varphi\\ -\sin\varphi\\ 0\end{bmatrix} a \qquad \vec r = \begin{bmatrix}\sin\theta\\ 0\\ \cos\theta\end{bmatrix} r$$

$$\vec A(\vec r) = \frac{\mu_0 I}{4\pi}\int\frac{d\vec l'}{|\vec r-\vec r'|}$$

$$\vec A(\vec r) = \frac{\mu_0 I}{4\pi}\int_0^{2\pi}\frac{(-\sin\varphi, \cos\varphi, 0)\,a\,d\varphi}{\sqrt{(\sin\theta\, r+\cos\varphi\, a)^2+\sin^2\varphi\, a^2+\cos^2\theta\, r^2}}$$

$r \gg a$

$$\vec A(\vec r) = \frac{\mu_0 I}{4\pi}\int_0^{2\pi}\frac{a(-\sin\varphi, \cos\varphi, 0)\,d\varphi}{r\sqrt{1-2\frac{a}{r}\sin\theta\cos\varphi+\left(\frac{a}{r}\right)^2}} \quad \to 0$$

$$\vec A(\vec r) = \frac{\mu_0 I a}{4\pi r}\int_0^{2\pi}\frac{(-\sin\varphi, \cos\varphi, 0)\,d\varphi}{\sqrt{1-2\underbrace{\frac{a}{r}}_{\ll 1}\sin\theta\cos\varphi}}$$

$$\frac{1}{\sqrt{1-\varepsilon}} = (1-\varepsilon)^{-\frac{1}{2}} = 1-\frac{1}{2}(-\varepsilon) = 1+\frac{\varepsilon}{2}$$

$$\vec A(\vec r) = \frac{\mu_0 I a}{4\pi r}\int_0^{2\pi}\left(1+\frac{a}{r}\sin\theta\cos\varphi\right)(-\sin\varphi, \cos\varphi, 0)\,d\varphi$$

$$\vec A(\vec r) = \frac{\mu_0 I a}{4\pi r}\begin{bmatrix}0 + 0\\ 0 + \int_0^{2\pi}\frac{a}{r}\sin\theta\cos^2\varphi\,d\varphi\\ 0\end{bmatrix}$$

$$\int_0^{2\pi}\cos^2\varphi\,d\varphi = \frac{1}{2}\cdot\underbrace{2\pi}_{\text{dolžina intervala}}$$

$$\vec A(\vec r) = \frac{\mu_0 I a^2}{4\pi r^2}\sin\theta\,\hat e_y\,\pi$$

$$\vec A(\vec r) = \frac{\mu_0 I a^2}{4 r^2}\sin\theta\,\hat e_y \qquad\qquad \sin\theta\,\hat e_y = \hat e_z\times\frac{\vec r}{r}$$

$$\vec A(\vec r) = \frac{\mu_0 I a^2}{4 r^2}\,\hat e_z\times\frac{\vec r}{r}$$

magnetni moment zank

$p_m = IS = I\pi a^2$

kaže v $\hat e_z$ smer $\qquad \hat e_z\, p_m = \vec p_m$

$$\vec A(\vec r) = \frac{\mu_0}{4\pi}\frac{\vec p_m\times\vec r}{r^3}$$

dipolni člen (točkasti dipol)

za vodnik

$$\vec A(\vec r) = \frac{\mu_0 I}{4\pi}\int\frac{d\vec l'}{|\vec r-\vec r'|}$$

$$\vec A(\vec r) = \frac{\mu_0}{4\pi}\int\frac{\vec j(\vec r')\,d^3\vec r'}{|\vec r-\vec r'|}$$

$$\vec A_{dip} = \frac{\mu_0}{4\pi}\frac{\vec p_m\times\vec r}{r^3}$$

$$u_{dip} = \frac{1}{4\pi\varepsilon_0}\frac{\vec p_e\cdot\vec r}{r^3}$$

$\vec p_m = I\vec S$ $\quad$ krožna zanka

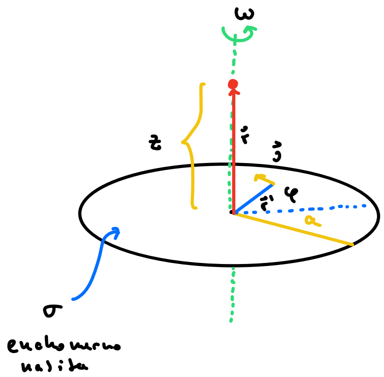
za vodnik

$$\vec p_m = \frac{I}{2}\int\vec r'\times d\vec l'$$

splošno

$$\vec p_m = \frac{1}{2}\int\vec r'\times\vec j(\vec r')\,d^3\vec r'$$

12 Magnetno polje nabite vrteče se okrogle plošče na osi



Biot - Savart

$$\vec B = \frac{\mu_0}{4\pi}\int \frac{\vec j' \times (\vec r - \vec r')\, d^3 r'}{|\vec r - \vec r'|^3}$$

$\vec r = (0, 0, z)$

$\vec r' = r'(\cos\varphi, \sin\varphi, 0)$

$\vec j = j(r')(-\sin\varphi', \cos\varphi', 0)$ $\qquad j(\vec r')\, dS' = dI = \frac{de}{t_0} = \sigma\, dr' \frac{2\pi r'}{2\pi}\omega = \sigma\omega r' dr'$

$$\vec B(\vec r) = \frac{\mu_0}{4\pi}\int \frac{j(\vec r)}{(r'^2+z^2)^{3/2}} \begin{bmatrix} -\sin\varphi' \\ \cos\varphi' \\ 0 \end{bmatrix} \times \begin{bmatrix} -r'\cos\varphi' \\ -r'\sin\varphi' \\ z \end{bmatrix} dS'\, r' d\varphi'$$

$$= \frac{\mu_0 \omega \sigma}{4\pi} \int_0^{2\pi}\int_0^a \begin{bmatrix} z\cos\varphi' \\ z\sin\varphi' \\ r' \end{bmatrix} \frac{1}{(r'^2+z^2)^{3/2}}\, r'^2 dr'\, d\varphi'$$

$$= \hat e_z \frac{\mu_0 \sigma \omega 2\pi}{4\pi} \underbrace{\int_0^a \frac{r'^3}{(r'^2+z^2)^{3/2}} dr'}_{I} = \dots$$

$u = r'^2 + z^2$
$du = 2r' dr'$

$$I = \int_{z^2}^{a^2+z^2} \frac{du\,(u - z^2)}{2u^{3/2}} = \frac{1}{2}\int_{z^2}^{a^2+z^2} u^{-1/2} - z^2 u^{-3/2}\, du = \frac{1}{2}\left(2\sqrt{u} + 2z^2 \frac{1}{\sqrt{u}}\right)\Big|_{z^2}^{a^2+z^2} =$$

$$= \sqrt{z^2+a^2} + \frac{z^2}{\sqrt{z^2+a^2}} - \sqrt{z^2} - \frac{z^2}{\sqrt{z^2}} = \frac{2z^2+a^2}{\sqrt{z^2+a^2}} - 2z$$

$$\dots = \hat e_z \frac{\mu_0 \sigma\omega}{2}\underbrace{\left(\frac{2z^2+a^2}{\sqrt{a^2+z^2}} - 2z\right)}_{\sigma}$$

Razvoj za $a \ll z$ $\qquad (1+\varepsilon)^p = 1 + p\varepsilon + \frac{p(p-1)}{2}\varepsilon^2 + \dots$

$$\sigma = \left(2z + \frac{a^2}{z}\right)\left(1 - \frac{1}{2}\left(\frac{a}{z}\right)^2 + \frac{3}{8}\left(\frac{a}{z}\right)^4\right) - 2z =$$

$$= 2z - \frac{a^2}{z} + \frac{3}{4}\frac{a^4}{z^3} + \frac{a^2}{z} - \frac{1}{2}\frac{a^4}{z^3} + \frac{3}{8}\frac{a^6}{z^5} - 2z =$$

$$= \frac{1}{4}\frac{a^4}{z^3} + \cancel{\frac{3}{8}\frac{a^6}{z^5}}$$

ker smo v limiti

$$\Rightarrow \quad \vec B(\vec r) = \hat e_z \frac{\mu_0 \sigma\omega}{8}\frac{a^4}{z^3}$$

$$\boxed{\vec B_{dip} = -\frac{\mu_0}{4\pi}\frac{\vec p_m r^2 - 3(\vec p_m \cdot \vec r)\vec r}{r^5}}$$

Za naš primer $\quad \vec B_{dip} = \frac{2\mu_0}{4\pi}\frac{\vec p_m \cdot \vec r^2}{r^5} = \frac{\mu_0}{2\pi}\frac{\vec p_m}{r^3}$

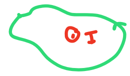
$$\vec p_m = \int \vec S\, dI = \int_0^a \pi r'^2\, \sigma\omega r' dr' = \pi\sigma\omega\int_0^a r'^3 dr' = \frac{\pi\sigma\omega}{4}a^4$$

$$B_{dip} = \frac{\mu_0}{2\pi}\frac{\pi\sigma\omega}{4}\frac{a^4}{r^3} = \frac{\mu_0\sigma\omega}{8}\frac{a^4}{r^3}$$

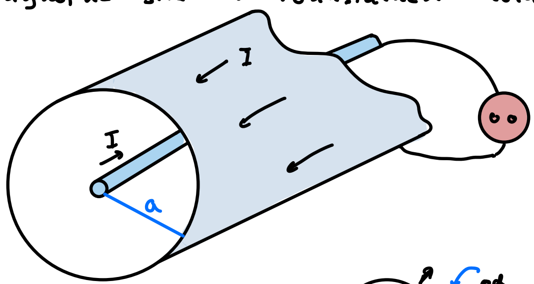
DN $\quad \vec p_m$ spirala  $\quad r = \alpha\varphi \quad \varphi \in [0, 4\pi]$

(T) $\vec F_m = \frac{1}{\mu_0}\oint\left(\vec B(\vec B\cdot\vec n) - \frac{1}{2}B^2\vec n\right)dS$

Ampere $\oint \vec B\, d\vec l = \mu_0 I$



(13) Magnetne sile v koaksialnem kablu



I, a

napetost plašča kabla $\frac{F}{l} = ?$

$\vec B(\vec r) = ?$

- $r > a$

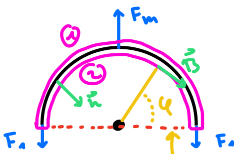
Ampere $B\,2\pi r = \mu_0(I - I) = 0$

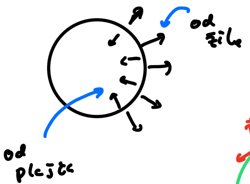
$B = 0$

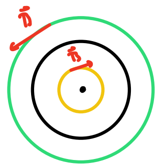
- $r < a$

Ampere $B\,2\pi r = \mu_0 I$

$\vec B = \frac{\mu_0 I}{2\pi r}\hat e_\varphi$







$F_m = 2F_1$

zaradi singularnosti ne vzamemo

Sila (integral po •)

(1) $F_{m_1} = 0$ ker $B = 0$

(2) $\vec B = \frac{\mu_0 I}{2\pi a}(\sin\varphi, -\cos\varphi, 0)$

$\vec n = (-\cos\varphi, -\sin\varphi, 0)$

$\vec B\cdot\vec n = 0$

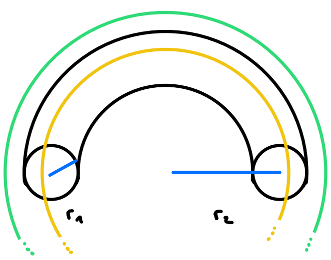
$$\vec F_{m_2} = \frac{1}{\mu_0}\int_2 -\frac{1}{2}\frac{\mu_0^2 I^2}{4\pi^2 a^2}(-\cos\varphi, -\sin\varphi, 0)\underbrace{l\,a\,d\varphi}_{dS}$$

$$= \frac{\mu_0 I^2 l}{8\pi^2 a}\hat e_y\int_0^\pi \sin\varphi\, d\varphi = \frac{\mu_0 I^2 l}{4\pi^2 a}\hat e_y$$

$\vec F_1 = \frac{\vec F_m}{2}$ $\qquad \frac{\vec F_1}{l} = \frac{\mu_0 I^2}{8\pi^2 a}\hat e_y$

Kaže navzgor, ga vleče narazen

(14) Magnetna sila v toroidni tuljavi



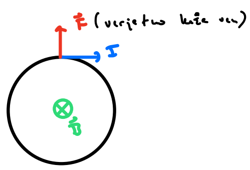
$r_2 \gg r_1$, $N \gg 1$

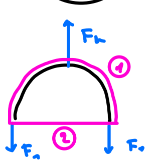
$F_1 = ?$ Sila napetosti posameznega ovoja

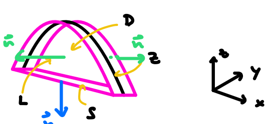
Ampere:

- znotraj $B\,2\pi r = \mu_0 N I$ $\qquad \vec B = \frac{\mu_0 N I}{2\pi r}\hat e_\varphi \overset{r_2 \gg r_1}{=} \frac{\mu_0 N I}{2\pi r_2}$
- zunaj $B\,2\pi r = \mu(NI - NI) = 0$ $\qquad B = 0$

En ovoj







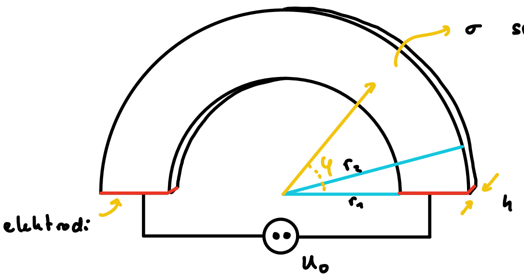
- Z (zgoraj) $B = 0 \Rightarrow F = 0$
- L, D $\quad B$ je enak, $\vec n$ sta nasprotna, torej $F = 0$
- S (spodaj) $\vec n = -\hat e_z$ $\quad \vec B = \frac{\mu_0 N I}{2\pi r_2}\hat e_y$ $\quad \vec n\cdot\vec B = 0$

$$\vec F_m = \frac{1}{\mu_0}\left(-\frac{1}{2}\right)\left(\frac{\mu_0 N I}{2\pi r_2}\right)^2(-\hat e_z)\int_S dS = \frac{\hat e_z}{2\mu_0}\left(\frac{\mu_0 N I}{2\pi r_2}\right)^2 2r_1\frac{2\pi r_2}{N}$$

$$\vec F_m = \hat e_z\frac{\mu_0 N I^2}{2\pi}\frac{r_1}{r_2}$$

$$F_1 = \frac{\mu_0 N I^2}{4\pi}\frac{r_1}{r_2}$$

15 Upor prevodne ploščice



T $\vec{j} = \sigma \vec{E}$ Ohmov zakon

$\nabla \vec{j} + \frac{\partial \rho}{\partial t} = 0$ kontinuitetna enačba

$$\nabla \vec{j} + \underbrace{\frac{\partial \rho}{\partial t}}_{0} = 0 \quad \text{v stacionarnih razmerah}$$

$$\nabla j = \sigma \nabla E$$

$$0 = \sigma(-\nabla^2 U)$$

$$\nabla^2 U = 0 \qquad \text{Laplaceova enačba}$$

Rešitev v cilindričnih koord. $U(r,\varphi) = \underbrace{\sum_{m=0}^{\infty} (A_m \cos m\varphi + B_m \sin m\varphi)}_{\text{ne zdijo se primerni}} (C_m r^m + D_m r^{-m}) + (a\varphi + b)(c \ln r + d)$

RP1 $U(r, 0) = 0$

RP2 $U(r, \pi) = -U_0$

RP3 $\vec{j} \cdot \hat{e}_r \Big|_{\substack{r=r_1 \\ r=r_2}} = 0$

$$j = -\sigma(\nabla U)_r = -\sigma \frac{\partial U}{\partial r} = 0$$

$$\Rightarrow \frac{\partial U}{\partial r}\Big|_{\substack{r_1 \\ r_2}} = 0$$

RP3 $\Rightarrow A_m = D_m = 0$

$$U(r,\varphi) = (a\varphi + b)(c \ln r + d)$$

$$\frac{\partial U}{\partial r}\Big|_{r_1} = (a\varphi + b)\frac{c}{r}$$

$$0 \overset{?}{=} (a\varphi + b)\frac{c}{r_1}$$

$$0 = (a\varphi + b)\frac{c}{r_2} \qquad \Rightarrow c = 0$$

$$U(r,\varphi) = a\varphi + b$$

RP1 $b = 0$

RP2 $-U_0 = \pi a \qquad a = -\frac{U_0}{\pi}$

$$\Rightarrow U(r,\varphi) = -\frac{U_0}{\pi}\varphi$$

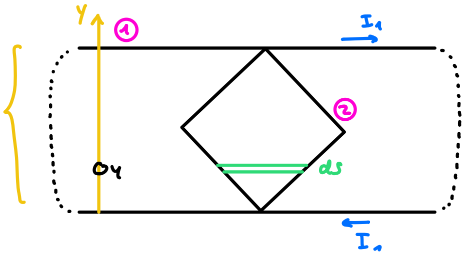
$$I = \int j_\varphi \, dS = \int -\sigma(\nabla U)_\varphi dS = \int -\sigma \frac{1}{r}\frac{\partial U}{\partial \varphi} dS =$$

$$= \frac{\sigma U_0}{\pi} \int \frac{1}{r} dS = \frac{\sigma U_0}{\pi} \int_0^h \int_{r_1}^{r_2} \frac{1}{r} \, dz \, dr = \frac{\sigma U_0 h}{\pi} \ln \frac{r_2}{r_1}$$

celoten upor $R = \frac{U}{I} = \frac{U_0}{\frac{\sigma U_0 h}{\pi} \ln \frac{r_2}{r_1}} = \frac{\pi}{\sigma h \ln r_2/r_1}$

16 Magnetna indukcija v okvirju

a



$L_{12} = ?$ (vzajemna) medsebojna induktivnost

T medsebojna ind. lastna ind.

$$\Phi_2 = L_{21} I_1 + L_{22} I_2$$

$$L_{12} = L_{12}$$

$$L_{12} = \frac{\mu_0}{4\pi} \oint\oint \frac{d\vec{l}_1 \cdot d\vec{l}_2}{|\vec{r}_1 - \vec{r}_2|}$$

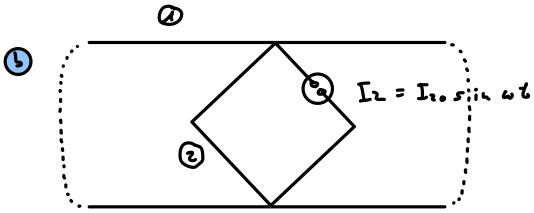
$$B(y) = \frac{\mu_0 I_1}{2\pi y} + \frac{\mu_0 I_1}{2\pi (d-y)} = \frac{\mu_0 I_1}{2\pi}\left(\frac{1}{y} + \frac{1}{d-y}\right)$$

$$\Phi_2 = \int B \, dS = \frac{\mu_0 I_1}{2\pi} \int \left(\frac{1}{y} + \frac{1}{d-y}\right) x(y) \, dy \qquad x(y) = \begin{cases} 2y & y < d/2 \\ d - 2y & y > d/2 \end{cases}$$

$$= 2 \frac{\mu_0 I_1}{2\pi} \int_0^{d/2} \left(\frac{1}{y} + \frac{1}{d-y}\right) 2y \, dy = \frac{2\mu_0 I_1}{\pi} \int_0^{d/2} 1 + \frac{y}{d-y} \, dy = \frac{2\mu_0 I_1}{\pi} \int_0^{d/2} \frac{d}{d-y} \, dy = -\frac{2\mu_0 I d}{\pi} \ln \frac{d - d/2}{d} =$$

$$= \frac{2 \ln 2}{\pi} \mu_0 I_1 d$$

$$L_{12} = \frac{\Phi_2}{I_1} = \frac{2 \ln 2}{\pi} \mu_0 d$$



$I_2 = I_{20} \sin \omega t$

$I_2 \to B_2 \to \Phi_1 \to U_{i1} \to I_1$

Razmerje amplitud $\frac{I_{10}}{I_{20}} = ?$

(T) En tokokrog

$U = RI + L\dot{I} + \frac{e}{C}$, $\quad L\dot I = \dot\phi$

2 sklopljena tokokroga

$U_1 = R_1 I_1 + \frac{e_1}{C_1} + L_{11}\dot{I}_1 + L_{12}\dot{I}_2$

$U_2 = \dots$

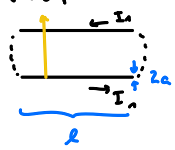
$R_1 = 0 \qquad C_1 = \infty \qquad U_1 = 0$

$0 = L_{11}\dot{I}_1 + L_{12}\dot{I}_2$

$I_2 = I_{20}\sin\omega t \Rightarrow I_1 = I_{10}\sin\omega t$

$$\frac{\dot I_1}{\dot I_2} = \frac{I_{10}\cos\omega t\,\omega}{I_{20}\cos\omega t\,\omega} = \frac{I_{10}}{I_{20}} = \left|-\frac{L_{12}}{L_{11}}\right|$$

Podpreskva



$L_{11} = \frac{\phi_1}{I_1}$

$$\phi_1 = \int B\,ds = \frac{\mu_0 I_1}{2\pi}\int_a^{d-a}\left(\frac{1}{y}+\frac{1}{d-y}\right)\ell\,dy = \frac{\mu_0 I_1 \ell}{\pi}\int_a^{d/2}\left(\frac{1}{y}+\frac{1}{d-y}\right)dy =$$

$$= \frac{\mu_0 I_1 \ell}{\pi}\left(\ln\frac{d}{2a} - \ln\frac{a}{2(d-a)}\right) = \frac{\mu_0 I_1 \ell}{\pi}\ln\frac{d}{2a}\frac{2(d-a)}{d}$$

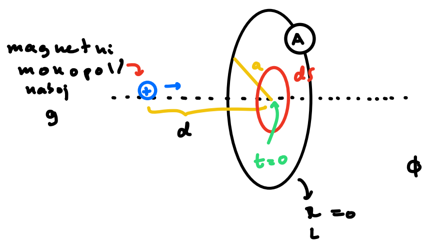
$$= \frac{\mu_0 I_1 \ell}{\pi}\ln\left(\frac{d}{a}-1\right) \Rightarrow L_{11} = \frac{\mu_0 \ell}{\pi}\ln\left(\frac{d}{a}-1\right)$$

$$\frac{I_{10}}{I_{20}} = \frac{\frac{2\ln 2}{\cancel{\pi}}\cancel{\mu_0} d}{\frac{\cancel{\mu_0}\ell}{\cancel{\pi}}\ln\left(\frac{d}{a}-1\right)} = \frac{2\ln 2}{\ln\left(\frac{d}{a}-1\right)}\frac{d}{\ell}$$

Običajno $d \gg a \qquad \ell \gg d$

$$\frac{I_{10}}{I_{20}} = \frac{2\ln 2}{\underbrace{\frac{\ell}{d}}_{\gg 0}\underbrace{\ln\frac{d}{a}}_{\gg 0}} \Rightarrow I_{10} \ll I_{20}$$

(17) Cabrerov eksperiment



$\phi_m = ?$

$$\vec B = \frac{\mu_0}{4\pi}\frac{g}{r^2}\frac{\vec r}{r} \qquad r = \sqrt{d^2+\rho^2} \qquad B_\perp = \frac{d}{r}B$$

$$\phi_m = \int \vec B\,d\vec S = \int B_\perp dS = \int \frac{\mu_0 g}{4\pi}\frac{1}{d^2+\rho^2}\frac{d}{\sqrt{d^2+\rho^2}}\rho\,d\rho\,d\varphi \quad (2\pi)$$

$$= \frac{\mu_0 g d\, 2\pi}{4\pi}\int_0^a \frac{\rho}{(d^2+\rho^2)^{3/2}}d\rho \qquad u = d^2+\rho^2,\ du = 2\rho\,d\rho$$

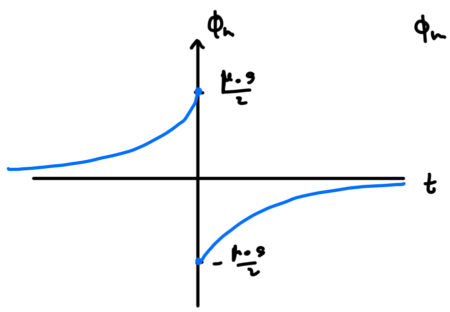
$$= \frac{\mu_0 g d}{4}\int_{d^2}^{d^2+a^2} u^{-3/2}du = \frac{\mu_0 g d}{4}(-2)\left(\frac{1}{\sqrt{d^2+a^2}}-\frac{1}{d}\right) =$$

$$= \frac{\mu_0 g}{2}\left(1-\frac{1}{\sqrt{1+\left(\frac{a}{d}\right)^2}}\right)$$

Levo $d = -vt$

Desno $d = vt$ in $\phi \to -\phi$

$$\phi_m(t) = \begin{cases} \frac{\mu_0 g}{2}\left(1+\frac{vt}{\sqrt{v^2t^2+a^2}}\right) & t<0 \\ -\frac{\mu_0 g}{2}\left(1-\frac{vt}{\sqrt{v^2t^2+a^2}}\right) & t>0 \end{cases}$$



$U_i = ?$

$$\nabla\times\vec E = -\frac{\partial\vec B}{\partial t}\underbrace{-\mu_0\vec j_m}_{\text{zaradi monopolov}} \qquad \iint d\vec S$$

$$\underbrace{\int(\nabla\times\vec E)\,d\vec S}_{\oint \vec E\,d\vec\ell\ =\ U} = -\frac{\partial}{\partial t}\int\vec B\,d\vec S - \mu_0\underbrace{\int\vec j_m\,d\vec S}_{I_m}$$

Faradayev zakon za monopole

$$U = -\frac{\partial\phi_m}{\partial t} - \mu_0 I_m$$

tok skozi zanko

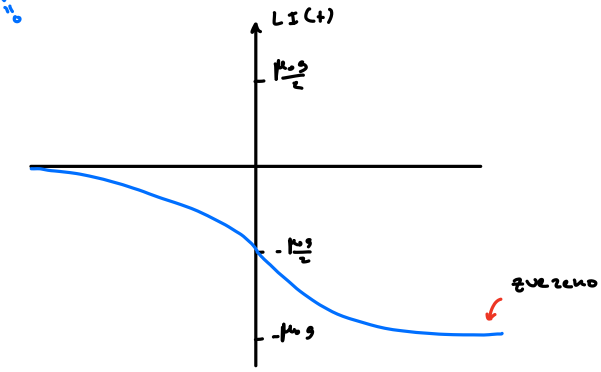
$I_m = \mu_0 \delta(t)$

$U = -\dot\phi - g\mu_0\delta(t)$ $\qquad U = L\dot I$

$L\dot I = -\dot\phi - g\mu_0\delta(t) \qquad \Big| \int_{-\infty}^{t} dt$

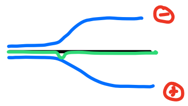
$-(\phi(t) - \underbrace{\phi(-\infty)}_{0}) - \mu_0 g \underbrace{\int_{-\infty}^{t} \delta(t)\,dt}_{H(t)} = L\,(I(t) - \underbrace{I(-\infty)}_{0})$

$L I(t) = -\phi(t) - \mu_0 g\, H(t)$

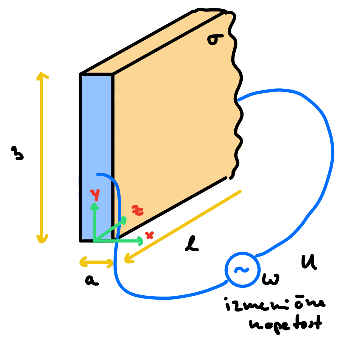


Če si skozi letel dipol

$\Rightarrow \quad L I_+ + L I_- = 0$



(18) Kožni pojav v vodniku v obliki traku



U napetost na robu traku

trak $b \gg a \qquad l \gg b$

(a) Impedanca $Z(\omega) = ?$

(b) Visoke frekvence $\frac{R}{R_0} = ?$ $\qquad R_0 \ldots$ statični upor

$\nabla \times (\nabla \times \vec E) = -\frac{\partial}{\partial t} \nabla \times \vec B$

$\nabla (\underbrace{\nabla \cdot \vec E}_{0}) - \nabla^2 \vec E = -\frac{\partial}{\partial t} \mu_0 \sigma \vec E$

$\nabla^2 \vec E = \mu_0 \sigma \frac{\partial \vec E}{\partial t}$ $\qquad$ Difuzijska enačba

(T) $\nabla \cdot \vec E = 0$ (ni nabojev) $\qquad \nabla \cdot \vec B = 0$

$\nabla \times \vec E = -\frac{\partial \vec B}{\partial t} \qquad \nabla \times \vec B = \mu_0 \vec j = \mu_0 \sigma \vec E$

$\vec E(\vec r, t) = \vec E(\vec r)\, e^{i\omega t}$ $\qquad$ pričakujemo

$\nabla^2 \vec E(\vec r) - \underbrace{i\omega\mu_0\sigma}_{k^2} \vec E(\vec r) = 0$

$\nabla^2 \vec E - k^2 \vec E = 0$

$\vec E = \vec E(x)\, \hat e_z$ $\qquad$ Odvodi v drugih smereh sta zelo majhni ker so dolžine zelo dolge.

$\frac{\partial^2 E_z}{\partial x^2} - k^2 E_z = 0$

$E_z = A \operatorname{ch} kx + B \operatorname{sh} kx$

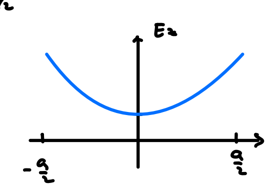
$B = 0$ zaradi simetrije

RP $\quad U(\frac{a}{2}) = U_0$

$U_0 = E_z l \Big|_{x = a/2}$

$\frac{U_0}{l} = A \operatorname{ch} k\frac{a}{2} \qquad A = \frac{U_0}{l \operatorname{ch} k a/2}$

$\vec E(\vec r, t) = \frac{U_0}{l} \frac{\operatorname{ch} kx}{\operatorname{ch} k a/2}\, \hat e_z\, e^{i\omega t}$



$$Z = \frac{U_0}{I}$$

$I = ?$ $\qquad I = \int j\,dS = \int \sigma E_z\,dS = 2\int_0^{a/2} \sigma\, bA\,\mathrm{ch}\,kx\,dx = 2\sigma b A \frac{1}{k}\,\mathrm{sh}\,kx \Big|_0^{a/2} =$

$$= 2\sigma b A \frac{1}{k}\,\mathrm{sh}\,k\,a/2 = \frac{2\sigma b}{k}\,\frac{U_0}{l\,\mathrm{ch}\,ka/2}\,\mathrm{sh}\,ka/2 = \frac{2\sigma b U_0}{kl}\,\mathrm{th}(ka/2)$$

$$Z = \frac{kl}{2b\sigma}\,\frac{1}{\mathrm{th}(ka/2)}$$

$$R_0 = \frac{l}{\sigma\, ab}$$

$$Z = \frac{kR_0 a}{2\,\mathrm{th}(ka/2)}$$

$$\frac{Z}{R_0} = \frac{ka/2}{\mathrm{th}\,ka/2}$$

Limiti • $\omega \ll 1$ $\qquad k = \sqrt{i\sigma\mu_0\omega}$ $\qquad ka \ll 1$

$$Z = R_0\,\frac{\frac{ka}{2}}{\frac{ka}{2}} = R_0$$

• $\omega \gg 1$ $\qquad ka \gg 1$ $\qquad \mathrm{th}\,\frac{ka}{2} = 1$

$$Z = R_0\,\frac{ka}{2} = R_0\,\frac{1+i}{\sqrt{2}}\sqrt{\sigma\mu_0\omega}\,\frac{a}{2}$$

$$R = \mathrm{Re}\,Z$$

$$\frac{R}{R_0} = \frac{a\sqrt{\sigma\mu_0\omega}}{2\sqrt{2}}$$ velik ω velik R

(T) Kontinuitetna enačba za energijo EMP:

$\frac{\partial w}{\partial t} + \nabla\cdot\vec{P} + \vec{j}\cdot\vec{E} = 0 \qquad /\int dV \;\rightarrow\; \frac{\partial W}{\partial t} + \int P\,dS + \int \vec{j}\cdot\vec{E}\,dV = 0$ Integralska oblika

ohmske izgube

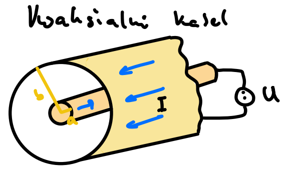
$$w = \frac{dW}{dV} = \frac{dW_m}{dV} + \frac{dW_e}{dV}$$

$\vec{P} = \frac{1}{\mu_0}\vec{E}\times\vec{B}$ Poyntingov vektor (gostota energijskega toka)

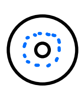
$\int \vec{P}\cdot d\vec{S}$ = energijski tok

(19) Energijski tok v dveh vodnikih

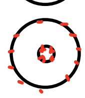
(a) Koaksialni kabel $\qquad \int \vec{P}\cdot d\vec{S} = ?$



$$\vec{P} = \frac{1}{\mu_0}\vec{E}\times\vec{B}$$



$B\,2\pi r = I\mu_0 \qquad \vec{B} = \frac{I\mu_0}{2\pi r}\,\hat{e}_\varphi$

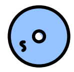


$\frac{e}{\varepsilon_0} = ES = E\,2\pi r l$

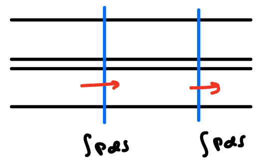
$$\vec{E} = \frac{e}{2\pi\varepsilon_0 r l}\,\hat{e}_r$$

$e = ?$ $\qquad U = \int_a^b E\,dr = \int_a^b \frac{e}{2\pi\varepsilon_0 r l}\,dr = \frac{e}{2\pi\varepsilon_0 l}\ln\frac{b}{a} \qquad e = \frac{U\,2\pi\varepsilon_0 l}{\ln b/a} \;\Rightarrow\; E = \frac{U}{r\ln b/a}$

$$\vec P = \frac{1}{\mu_0}\frac{I\mu_0}{2\pi r}\frac{U}{r\ln b/a}\,\hat e_r \times \hat e_\varphi = \frac{IU}{2\pi r^2 \ln b/a}\,\hat e_z$$ vzdolž vodnika



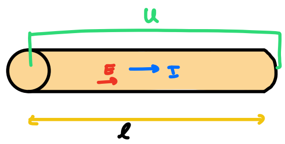
$$\int \vec P\, d\vec S = \int_0^{2\pi}\int_a^b \frac{IU}{2\pi r^2 \ln b/a}\, r\,dr\,d\varphi = \frac{IU}{2\pi \ln b/a}\,2\pi \int_a^b \frac{dr}{r} = IU$$



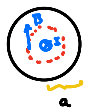
ohmskih izgub ni

$\frac{\partial W}{\partial t} = 0$ stacionarnost ⇒ Velja kontinuiteta energije

(b) Valjast vodnik z uporom

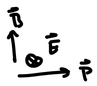


$$E = \frac{U}{l} = \frac{RI}{l}$$



Ampere $B\,2\pi r = \frac{\pi r^2}{\pi a^2} I \mu_0$ $\qquad B = \frac{I\mu_0}{2\pi a^2}\, r$ znotraj

$B = \frac{I\mu_0}{2\pi r}$ zunaj (kot na žici)



$$P = \frac{1}{\mu_0}\frac{I\mu_0}{2\pi a^2}\, r\, \frac{RI}{l} = \frac{RI^2 r}{2\pi a^2 l}$$

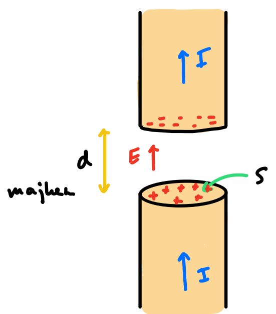
Cel vodnik: S ... plašč

$$\int_{\text{plašč}} \vec P \cdot d\vec S = -\frac{RI^2 r}{2\pi a^2 l}\Big|_{r=a} 2\pi a l = -RI^2$$

Ohmske izgube IR^2

⇒ kont. energije drži

(20) Prekinjeni vodnik



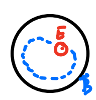
$\int \vec P\, d\vec S = ?$ (v špranji)

$E = \frac{\sigma}{\varepsilon_0} = \frac{e}{\varepsilon_0 S} = \frac{It}{\varepsilon_0 S}$ (ploščati kondenzator)

Ker se E spreminja ⇒ $B \neq 0$

$$\nabla\times\vec B = \underset{\substack{\overset{\parallel}{0}\ \text{ni toka}\\ \text{v špranji}}}{\mu_0 \vec j} + \mu_0\varepsilon_0 \frac{\partial \vec E}{\partial t}$$

$$\nabla\times\vec B = \mu_0\varepsilon_0 \frac{I}{\varepsilon_0 S} = \frac{\mu_0 I}{S}\,\hat e_z \quad / \int dS$$



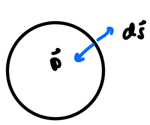
v tangentni smeri

$$\int \nabla\times\vec B\, dS = \oint \vec B\, d\vec l = \frac{\mu_0 I}{S}\int \hat e_z\, d\vec S = \frac{\mu_0 I \pi r^2}{S}$$

$$B\,2\pi r = \frac{\mu_0 I\, \pi r^2}{S}$$

$$\vec B = \frac{\mu_0 I r}{2S}\,\hat e_\varphi$$

$$\vec P = \frac{1}{\mu_0}\vec E\times\vec B = \frac{1}{\mu_0}\frac{It}{\varepsilon_0 S}\frac{\mu_0 I r}{2S} = \frac{I^2}{2\varepsilon_0 S^2}\, r t$$



$$\int \vec P\, d\vec S = -\frac{I^2}{2\varepsilon_0 S^2}\, r t\; 2\pi r d\,\Big|_{r=a} = -\frac{I^2 d}{\varepsilon_0 S}\, t$$

$$\oint \vec{P}\cdot d\vec{S} + \frac{\partial W}{\partial t} + \int \vec{j}\vec{E}\,dV = 0$$

$$-\frac{I^2 d}{\varepsilon_0 S}t + \frac{\partial W}{\partial t} = 0$$

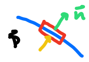
$$W = W_e + W_m = \frac{1}{2}\varepsilon_0 E^2 Sd + \frac{1}{2\mu_0}B^2 Sd$$
$$= \frac{\varepsilon_0}{2}\frac{I^2 t^2}{\varepsilon_0^2 S^2}Sd + \dots$$
$$= \frac{I^2 d}{2\varepsilon_0 S}t^2 + \underbrace{\dots}_{\text{ni odvisen od časa}}$$

$$\frac{\partial W}{\partial t} = \frac{\partial W_e}{\partial t} = \frac{I^2 d}{\varepsilon_0 S}t \quad ✓$$

(T) Snov v električnem polju

vezani naboji: $\rho_v = -\nabla\vec{P}$

vsi naboji: $\rho = \nabla\varepsilon_0\vec{E}$



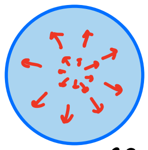
$$\rho_v = -\nabla\vec{P} \quad / \int dV$$

$$\underbrace{\int \rho_v\, dV}_{e_v} = -\oint \vec{P}\,d\vec{S} = -PS\Big|^{\text{zunaj}}_{\text{znotraj}}$$

$$\Rightarrow \sigma_v = \vec{P}\cdot\vec{n}$$ normala površine

(21) Radialno polarizirana krogla

$\vec{P} = k\vec{r}$ polarizacija



spontana polarizacija

ρ_v prostorninska gostota vezanega naboja

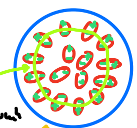
σ_v površinska gostota vezanega naboja

e_v vezan naboj

$\vec{E}(\vec{r}) = ?$

$$\rho_v = -\nabla\vec{P} = -k\nabla\vec{r} = -3k$$

$$\sigma_v = \vec{P}\cdot\vec{n} = k\vec{r}\cdot\vec{e}_r = kr\big|_{r=a} = ka$$



več nasprotnih

na površini pozitivno

$$e_v = \underbrace{\int_0^a -3k\, r^2 4\pi dr}_{\text{volumen}} + \underbrace{4\pi a^2 ka}_{\text{na površini}}$$

$$= -12\pi k\frac{a^3}{3} + 4\pi k a^3 = 0$$

ker so sami dipoli, se pokrajšajo

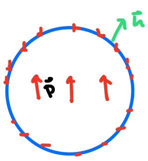
$$e(\vec{r}) = -3k\frac{4\pi r^3}{3} = \varepsilon_0 \vec{E}(\vec{r})\, 4\pi r^2$$

$$\Rightarrow \vec{E}(\vec{r}) = -\frac{k}{\varepsilon_0}\vec{r} = -\frac{\vec{P}}{\varepsilon_0} \qquad r < a$$

$$\vec{E}(\vec{r}) = 0 \qquad r > a$$

(22) Prerezana polarizirana krogla

a, P



(a) $u(r,\theta) = ?$

$$\rho_v = -\nabla\vec{P} = 0$$

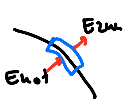
$$\sigma_v = \vec{P}\cdot\vec{n} = P\cos\theta = P_1(\cos\theta)\, P$$

$$u(r,\theta) = \left(A_1 r + B_1\frac{1}{r^2}\right)P_1(\cos\theta) = \begin{cases} A_1 r\, P_1(\cos\theta) & r \le a \\ B_1\frac{1}{r^2}P_1(\cos\theta) & r > a\end{cases}$$

RP 1 zveznost potenciala na robu

$$u_{not}(a,\theta) = u_{zun}(a,\theta) \Rightarrow A_1 a = B_1\frac{1}{a^2} \quad\Rightarrow\quad A_1 = \frac{B_1}{a^3}$$

RP 2



$e = \sigma_v S$

$$\sigma_v S = \varepsilon_0 S(E_{zun} - E_{not})$$

$$E_{zun} - E_{not} = \frac{\sigma_v}{\varepsilon_0}$$

$$E_{not} = -\nabla_r U_{not} = -A \cos\theta$$

$$E_{zun} = -\nabla_r U_{zun} = \frac{B}{2}\frac{1}{r^3}\cos\theta$$

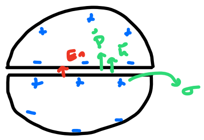
$$\frac{B}{2}\frac{1}{a^3} - A = \frac{P}{\varepsilon_0}$$

$$\frac{B}{2a^3} - \frac{B}{a^3} = \frac{P}{\varepsilon_0} \qquad B = \frac{P a^3}{\varepsilon_0 3} \qquad A = \frac{P}{3\varepsilon_0}$$

$$U(r,\theta) = \frac{P}{3\varepsilon_0}\begin{cases} \overbrace{r\cos\theta}^{z} & r<a \\ \frac{a^3}{r^2}\cos\theta & r>a \end{cases}$$ homogeno polje

$$\vec{E}_{not} = -\frac{\partial V}{\partial z} e_z = -\frac{P}{3\varepsilon_0} e_z = -\frac{\vec{P}}{3\varepsilon_0}$$ $\downarrow \; \downarrow \; \downarrow \vec{E}$

ⓑ Polje v špranji



$$\vec{E}_1 = \underbrace{-\frac{\vec{P}}{3\varepsilon}}_{\text{naboji na obodu krogle}} + \underbrace{\frac{\sigma}{\varepsilon_0}}_{\text{naboji na perezu}} \hat{e}_0 = \frac{2\vec{P}}{3\varepsilon_0}$$

ploščati kondenzator

$$\sigma = \vec{P}\cdot\hat{n} = P$$

ⓣ Dielektriki

Spontana polarizacija ni

$\vec{E} \to \vec{P}$

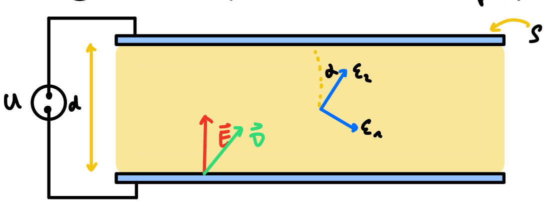
$\rho = \rho_v + \rho_z$ zunanji naboji

$\rho_z = \rho - \rho_v = \nabla(\varepsilon_0\vec{E} - \vec{P}) = \nabla\vec{D}$ $\qquad e_z = DS$ $\qquad \vec{D} = \varepsilon_0\vec{E} - \vec{P}$

$\sigma_v = \vec{P}\cdot\hat{n} \qquad \rho_v = \nabla\vec{P}$ $\qquad$ ↳ gostota el. polja

Za dielektrike velja $\vec{D} \propto \vec{E} \qquad \vec{D} = \varepsilon\varepsilon_0\vec{E}$

ⓩ Anizotropni elektrik v ploščatem kondenzatorju



$$C = \frac{e_0}{U}$$ (e_0 → $\vec{D}$, U → $\vec{E}$)

$$\vec{D} \nparallel \vec{E}$$

$$\vec{D} = \varepsilon_0 \underline{\varepsilon}\vec{E}$$

↓

$$\underline{\varepsilon}' = \begin{bmatrix} \varepsilon_1 & 0 \\ 0 & \varepsilon_2 \end{bmatrix}$$ lastni koord. sistem

$$\underline{\varepsilon} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}\begin{bmatrix} \varepsilon_1 & 0 \\ 0 & \varepsilon_2 \end{bmatrix}\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} = \begin{bmatrix} \varepsilon_1\cos^2\alpha + \varepsilon_2\sin^2\alpha & (\varepsilon_2-\varepsilon_1)\cos\alpha\sin\alpha \\ (\varepsilon_2-\varepsilon_1)\cos\alpha\sin\alpha & \varepsilon_1\sin^2\alpha + \varepsilon_2\cos^2\alpha \end{bmatrix}$$

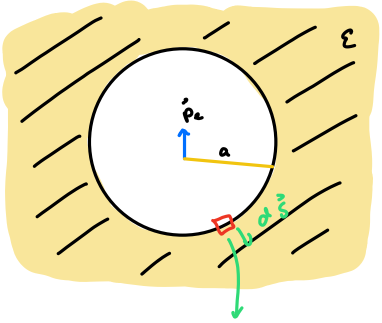
$$\begin{bmatrix} D_x \\ D_y \end{bmatrix} = \varepsilon_0 \underset{\frac{U}{d}}{E}\, \underline{\varepsilon}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$ $\qquad D_y \to \frac{e_0}{S}$

$$y: \quad \frac{e_0}{S} = \varepsilon_0\frac{U}{d}(\varepsilon_1\sin^2\alpha + \varepsilon_2\cos^2\alpha)$$

$$C = \frac{e_0}{U} = \varepsilon_0\frac{S}{d}(\varepsilon_1\sin^2\alpha + \varepsilon_2\cos^2\alpha)$$

(24) Točkasti dipol v krogelni votlini dielektrika

$a,\ p_e,\ \varepsilon$



$\nabla D = \rho - \rho_v = \rho_z$

$= 0 \big|_{r=a}$

$u(r,\theta) = ?$

$\vec{p}_e\,' = ?$

$$u_{dip} = \frac{p_e}{4\pi\varepsilon_0}\,\frac{\cos\theta}{r^2} \quad \text{"} P_1(\cos\theta) \Downarrow \text{ samo } l=1$$

$\nabla u = 0$ RP $r \to 0$

$$B_1 = \frac{p_e}{4\pi\varepsilon_0}$$

$$u(r,\theta) = \begin{cases} (A_1 r + B_1 r^{-2})\cos\theta & r<a \\ (\underbrace{A_1}_{=0 \text{ divergira}} r + B_1 r^{-2})\cos\theta & r>a \end{cases}$$

RP 2: $u(a^+) = u(a^-)$

$$A_1 a + \frac{p_e}{4\pi\varepsilon_0}\frac{1}{a^2} = B_1\frac{1}{a^2} \Rightarrow A_1 = \frac{B_1}{a^3} - \frac{p_e}{4\pi\varepsilon_0 a^3}$$

RP3 $\nabla\vec{D} = 0 \quad \Big| \int dV$

$\oint \vec{D}\, d\vec{S} = 0$

$D_\perp^{not} = D_\perp^{zun}$

$\hookrightarrow \frac{\partial}{\partial r}$

$$-\varepsilon_0\, \partial_r u\big|_{r=a^-} = -\varepsilon_0\varepsilon\, \partial_r u\big|_{r=a^+}$$

$$A_1 - \frac{2p_e}{4\pi\varepsilon_0}\frac{1}{a^3} = -2\varepsilon B_1 \frac{1}{a^3}$$

$$\frac{B_1}{a^3} - \frac{p_e}{4\pi\varepsilon_0 a^3} - \frac{2p_e}{4\pi\varepsilon_0}\frac{1}{a^3} = -2\varepsilon B_1\frac{1}{a^3}$$

$$-\frac{3p_e}{4\pi\varepsilon_0 a^3} = -(2\varepsilon+1)\frac{B_1}{a^3}$$

$$B_1 = \frac{p_e}{4\pi\varepsilon_0}\,\frac{3}{2\varepsilon+1} \qquad A_1 = \frac{1}{a^3}\,\frac{p_e}{4\pi\varepsilon_0}\,\frac{2-2\varepsilon}{2\varepsilon+1}$$

$$u(r,\theta) = \begin{cases} \left(\frac{1}{a^3}\frac{p_e}{4\pi\varepsilon_0}\frac{2-2\varepsilon}{2\varepsilon+1}\, r + \frac{p_e}{4\pi\varepsilon_0}\frac{1}{r^2}\right)\cos\theta & r<a \\ \frac{p_e}{4\pi\varepsilon_0}\,\frac{3}{2\varepsilon+1}\,\frac{1}{r^2}\cos\theta & r>a \end{cases}$$

homogeno polje ↑ točkasti dipol

$$u(r,\theta) = \begin{cases} \frac{p_e}{4\pi\varepsilon_0}\left(\frac{2-2\varepsilon}{2\varepsilon+1}\frac{r}{a^3} + \frac{1}{r^2}\right)\cos\theta & r<a \\ \underbrace{\frac{p_e}{4\pi\varepsilon_0}\,\frac{3}{2\varepsilon+1}\,\frac{1}{r^2}\cos\theta}_{\text{polje točkastega dipola}} & r>a \end{cases}$$

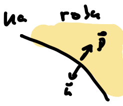
$$\to p_e' = \frac{3p_e}{2\varepsilon+1}$$

$\varepsilon = 1$: $p_e' = p_e$ ni dielektrika

DN $\sigma_v(\theta)$ Gauss: $\frac{\sigma_v}{\varepsilon_0} = E_{zun}^{\perp} - E_{not}^{\perp}\big|_{r=a}$

Alternativno $\sigma_v = \vec{P}\cdot\vec{n}$

na robu

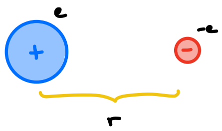


$$\varepsilon\varepsilon_0 \vec{E} = \vec{D} = \varepsilon_0\vec{E} + \vec{P}$$

$$\vec{P} = \varepsilon_0(\varepsilon-1)\vec{E}$$

(25) Dielektrična konstanta plazme

↳ plin ionov in e^-



dipol

(a) $\vec{E}(t) = \vec{E}_0 e^{-i\omega t}$

$\vec{r}(t) = ?$

$\downarrow$

$\vec{p}_e$

$\downarrow$

$\vec{P} = \varepsilon_0(\varepsilon - 1)\vec{E}$

$\downarrow$

$\varepsilon(\omega)$

$\vec{F}_e = -e_0 \vec{E} = m\ddot{\vec{r}}$

$-e_0 E_0 e^{-i\omega t} = m \frac{d^2\vec{r}}{dt^2}$

$\vec{r} = \vec{r}_0 e^{-i\omega t}$

$-e_0 \vec{E}_0 e^{-i\omega t} = -m\omega^2 e^{-i\omega t} \vec{r}_0$

$\vec{r}_0 = \frac{e_0 \vec{E}_0}{m\omega^2} \qquad \vec{r} = \frac{e_0 \vec{E}_0}{m\omega^2} e^{-i\omega t}$

$\vec{p}_e = -e_0 \vec{r} = -\frac{e_0^2 \vec{E}_0}{m\omega^2} e^{-i\omega t}$

$\vec{P}_0 = n \vec{p}_{e0}$

↳ številska gostota e^-

$-n \frac{e_0^2}{m\omega^2} \vec{E}_0 = \varepsilon_0 (\varepsilon - 1) \vec{E}_0$

$\varepsilon(\omega) = 1 - \frac{n e_0^2}{m\omega^2 \varepsilon_0} = 1 - \frac{\omega_p^2}{\omega^2} \Rightarrow \omega > \omega_p$

$\omega_p^2 = \frac{n e_0^2}{m\varepsilon_0}$

plazemska frekvenca

(b) Disperzijska relacija $\quad \omega(k) = ?$

↳ valovni vektor

$\vec{E} = \vec{E}_0 e^{i(kz - \omega t)}$ ravni val vzdolž z po snovi

$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad / \nabla \times$

$\nabla \times \vec{B} = \mu_0 \varepsilon_0 \varepsilon \frac{\partial \vec{E}}{\partial t}$

$\nabla \times (\nabla \times E) = -\frac{\partial}{\partial t}\left(\varepsilon_0 \varepsilon \mu_0 \frac{\partial \vec{E}}{\partial t}\right)$

$\nabla(\underbrace{\nabla \vec{E}}_{=0}) - \nabla^2 \vec{E} = -\varepsilon_0 \varepsilon \mu_0 \frac{\partial^2 \vec{E}}{\partial t^2}$

$\nabla^2 \vec{E} - \underbrace{\varepsilon_0 \mu_0}_{\frac{1}{c_0^2}} \varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} = 0$ valovna enačba

$\frac{\partial^2 \vec{E}}{\partial z^2} - \varepsilon_0 \mu_0 \varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} = 0$

$\vec{E}_0 (-k^2) e^{i(kz - \omega t)} = \varepsilon \varepsilon_0 \mu_0 (-\omega^2) e^{i(kz - \omega t)} \vec{E}_0$

$k^2 = \varepsilon \varepsilon_0 \mu_0 \omega^2$

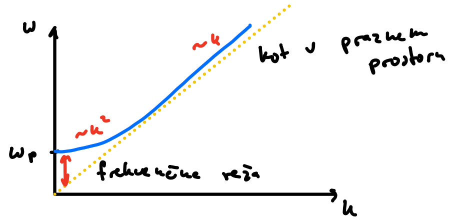
$k = \frac{\sqrt{\varepsilon}}{c_0} \omega \qquad \boxed{\omega = k \frac{c_0}{\sqrt{\varepsilon}}}$

splošno v dielektriku

$\omega = k c_0 \frac{1}{\sqrt{1 - \frac{\omega_p^2}{\omega^2}}}$

$\omega^2 \left(1 - \frac{\omega_p^2}{\omega^2}\right) = k^2 c_0^2$

$\omega = \sqrt{k^2 c_0^2 + \omega_p^2}$



$(1+\varepsilon)^n = 1 + n\varepsilon$

velik k: $\omega = k c_0$

majhen k: $\omega = \omega_p \left(1 + \frac{1}{2}\left(\frac{k c_0}{\omega_p}\right)^2\right)$

ⓒ Fazna in grupna hitrost

$$v_F = \frac{\omega}{k} = \sqrt{c_0^2 + \left(\frac{\omega_p}{k}\right)^2} > c_0$$

$$v_G = \frac{\partial \omega}{\partial k} = \frac{1}{2} \frac{2 c_0^2 k}{\sqrt{k^2 c_0^2 + \omega_p^2}} = c_0 \frac{1}{\sqrt{1 + \frac{\omega_p^2}{k^2 c_0^2}}} < c_0 \quad \checkmark$$

$$v_F \, v_G = c_0^2$$

V frekvenčni reži $\omega < \omega_p$ $\quad \varepsilon < 0 \Rightarrow \varepsilon = -\varepsilon' \quad \sqrt{\varepsilon} = i\sqrt{\varepsilon'}$

$$\omega = \frac{k c_0}{i \sqrt{\varepsilon'}}$$

$$\vec{E}(z,t) = \vec{E}_0 \, e^{-\frac{\omega\sqrt{\varepsilon'}}{c_0} z} \, e^{-i\omega t}$$

eksponentno pojemanje

$\Rightarrow$ tako polje se delno absorbira

večina pa odbije

Ⓣ EM valovanje v omejeni geometriji (valovni vodnik)

$$\nabla^2 \vec{E} = \frac{1}{c_0^2} \frac{\partial^2 \vec{E}}{\partial t^2} \qquad \text{Valovna enačba}$$

Nastavek $\vec{E}(\vec{r},t) = \vec{E}(\vec{\varsigma}) \, e^{i(kz - \omega t)}$ širjenje vzdolž z osi

$\vec{\varsigma}$ – koordinate x, y ob $r_\perp$

$$\frac{\partial}{\partial t} = -i\omega$$

$$\frac{\partial}{\partial z} = ik \quad \Rightarrow \quad \nabla^2 = \nabla_\perp^2 + \frac{\partial^2}{\partial z^2} = \nabla_\perp^2 - k^2$$

$$\Rightarrow \left(\nabla_\perp^2 - k^2 + \frac{\omega^2}{c_0^2}\right) \vec{E}(\vec{\varsigma}) = 0$$

$$\left(\nabla_\perp^2 + \left(\frac{\omega^2}{c_0^2} - k^2\right)\right) \vec{E}(\vec{\varsigma}) = 0 \qquad \text{tri komponente } \vec{E}$$

$$\vec{H}(\vec{\varsigma}) = 0 \qquad \text{tri komponente } \vec{H}$$

ⓐ Pokaži, da lahko E_x, E_y, H_x, H_y izrazimo z E_z, H_z.

$$\nabla \times \vec{E} = -\mu_0 \frac{\partial \vec{H}}{\partial t} \qquad \nabla \times \vec{H} = \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

$$\begin{bmatrix} \frac{\partial}{\partial y} E_z - ik E_y \\ ik E_x - \frac{\partial}{\partial x} E_z \\ \frac{\partial}{\partial x} E_y - \frac{\partial}{\partial y} E_x \end{bmatrix} = \mu_0 i\omega \begin{bmatrix} H_x \\ H_y \\ H_z \end{bmatrix} \qquad \begin{bmatrix} \frac{\partial}{\partial y} H_z - ik H_y \\ ik H_x - \frac{\partial}{\partial x} H_z \\ \frac{\partial}{\partial x} H_y - \frac{\partial}{\partial y} H_x \end{bmatrix} = -\varepsilon_0 i\omega \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}$$

- $$ik E_x - \mu_0 i\omega H_y = \frac{\partial}{\partial x} E_z$$
  $$\varepsilon i\omega E_x - ik H_y = -\frac{\partial}{\partial y} H_z \quad \Big/ \frac{ik}{\varepsilon i\omega} \quad \Big] -$$

$$\left(-\mu_0 i\omega + \frac{(ik)^2}{\varepsilon i\omega}\right) H_y = \frac{\partial}{\partial x} E_z + \frac{ik}{\varepsilon i\omega} \frac{\partial}{\partial y} H_z$$

$$H_y = \frac{1}{\left(i\frac{k^2}{\varepsilon\omega} - \mu_0\omega i\right)} \left(\frac{\partial}{\partial x} E_z + \frac{k}{\varepsilon\omega} \frac{\partial}{\partial y} H_z\right)$$

$$E_x = \frac{1}{\varepsilon i\omega}\left(-\frac{\partial}{\partial y} H_z + ik H_y\right) = \frac{1}{i\omega\varepsilon}\left(-\frac{\partial}{\partial y} H_z + \frac{k}{\frac{k^2}{\varepsilon\omega} - \mu_0\omega}\left(\frac{\partial}{\partial x} E_z + \frac{k}{\varepsilon\omega} \frac{\partial}{\partial y} H_z\right)\right)$$

$$= \left(\frac{i}{\varepsilon\omega} - \frac{i\left(\frac{k}{\omega\varepsilon}\right)^2}{\frac{k^2}{\omega\varepsilon} - \mu_0\omega}\right) \frac{\partial}{\partial y} H_z - \frac{\frac{ik}{\omega\varepsilon}}{\frac{k^2}{\omega\varepsilon} - \mu_0\omega} \frac{\partial}{\partial x} E_z$$

$$= i\left[\left(\frac{1}{\varepsilon\omega} - \frac{1}{\omega\varepsilon - \frac{\mu_0\omega^2\varepsilon}{k^2}}\right)\frac{\partial}{\partial y} H_z - \frac{1}{k - \frac{\mu_0\omega^2\varepsilon}{k}}\frac{\partial}{\partial x} E_z\right] \qquad c_0 = \frac{1}{\varepsilon\mu_0}$$

Podobno tudi za ostale

$$H_x = \frac{i}{k^2 - \frac{\omega^2}{c_0^2}}\left(\omega\varepsilon_0 \frac{\partial}{\partial y} E_z - k\frac{\partial}{\partial x} H_z\right)$$

$$E_y = \frac{i}{k^2 - \frac{\omega^2}{c_0^2}}\left(\omega\mu_0 \frac{\partial}{\partial x} H_z - k\frac{\partial}{\partial y} E_z\right)$$

$$E_x = \frac{i}{k^2 - \frac{\omega^2}{c_0^2}}\left(-\omega\mu_0 \frac{\partial}{\partial y} H_z - k\frac{\partial}{\partial x} E_z\right)$$

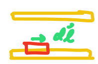
$$H_y = \frac{i}{k^2 - \frac{\omega^2}{c_0^2}}\left(-\omega\varepsilon_0 \frac{\partial}{\partial x} E_z - k\frac{\partial}{\partial y} H_z\right)$$

Robni pogoji:

$\nabla \vec{E} \to \rho$ //

$\nabla \times \vec{E} = -\mu_0 \frac{\partial \vec{H}}{\partial t}$ $\quad \Big/ \int d\vec{S}$

$\oint \vec{E}\, d\vec{l} = 0$



$\to$ $E_\parallel \big|_b = 0$

$\nabla \times \vec{H} \to \vec{j}$ / ne poznamo toka

$\nabla \cdot \vec{H} = 0$

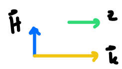
$\oint \vec{H}\, d\vec{S} = 0$

$\to$ $H_\perp \big|_b = 0$

E_z, H_z izračunamo, razcepimo na dve komponenti

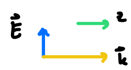
(1) $H_z = 0$, $E_z \neq 0$

Transverzalni magnetni način TM



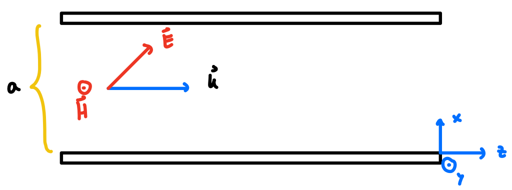
(2) $E_z = 0$, $H_z \neq 0$

Transverzalni električni način TE



Poljubna rešitev je lin. kombinacija (1) in (2)

(26) Valovni vodnik iz vzporednih prevodnih plošč



Vzdolž y se nič ne spreminja $\Rightarrow \frac{\partial}{\partial y} = 0$

(a) TM način, $H_z = 0$, $E_z \neq 0$

$$H_x = \frac{i}{k^2 - \frac{\omega^2}{c_0^2}}\left(\omega\varepsilon_0 \underbrace{\frac{\partial}{\partial y}}_{=0} E_z - k\frac{\partial}{\partial x} \underbrace{H_z}_{=0}\right) = 0$$

$$E_y = \frac{i}{k^2 - \frac{\omega^2}{c_0^2}}\left(\omega\mu_0 \frac{\partial}{\partial x} \underbrace{H_z}_{=0} - k\underbrace{\frac{\partial}{\partial y}}_{=0} E_z\right) = 0$$

$$E_x = \frac{i}{k^2 - \frac{\omega^2}{c_0^2}}\left(-\omega\mu_0 \frac{\partial}{\partial y} \underbrace{H_z}_{=0} - k\frac{\partial}{\partial x} E_z\right) \neq 0$$

$$H_y = \frac{i}{k^2 - \frac{\omega^2}{c_0^2}}\left(-\omega\varepsilon_0 \frac{\partial}{\partial x} E_z - k\frac{\partial}{\partial y} \underbrace{H_z}_{=0}\right) \neq 0$$

$$\Rightarrow \vec{H} = (0, H_y, 0) \qquad \vec{E} = (\vec{E}_x, 0, \vec{E}_z)$$

Iščemo E_z $\qquad \Big(\underbrace{\nabla_\perp^2}_{\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2},\ \frac{\partial^2}{\partial y^2}=0} + \underbrace{\left(\frac{\omega^2}{c_0^2} - k^2\right)}_{K^2}\Big)\vec{E}(\vec{r}) = 0$

$$\left(\frac{\partial^2}{\partial x^2} + K^2\right) E_z(x) = 0$$

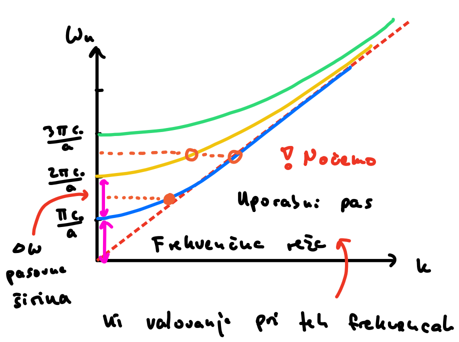
RP: $E_\parallel = E_z\big|_b = E_y\big|_b = 0$

$E_z = A \sin Kx + B\cos Kx$ $\qquad E_z(0) = E_z(a) = 0$

$B = 0$

$0 = \sin Ka \qquad Ka = n\pi \qquad K = \frac{n\pi}{a} \qquad E_z = \sum_n a_n \sin\frac{n\pi}{a}x$

$n \in \mathbb{N}$



$$K = \frac{n\pi}{a} = \frac{\omega^2}{c_0^2} - k^2 \quad \Rightarrow \quad \omega = \sqrt{\underbrace{\left(\frac{n\pi c_0}{a}\right)^2}_{\omega_n^2} + k^2c_0^2} \quad \text{Disperzijska relacija}$$

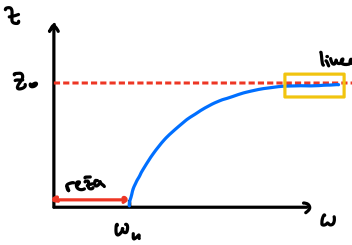
b TM način → Transverzalna impedanca vodnika

$$\boxed{Z(\omega) \overset{def}{=} \frac{E_\perp}{H_\parallel}} = \frac{E_x}{H_y} = \frac{-\omega\mu_0 \frac{\partial H_z}{\partial y} - k\frac{\partial E_z}{\partial x}}{-\omega\varepsilon_0 \frac{\partial E_z}{\partial x} - k\frac{\partial H_z}{\partial y}} = \frac{k}{\omega\varepsilon_0} = \frac{\sqrt{\frac{\omega^2}{c_0^2} - \frac{n\pi}{a}}}{\omega\varepsilon_0} = \sqrt{\frac{1}{c_0^2\varepsilon_0^2} - \frac{n\pi}{a\varepsilon_0^2}\frac{1}{\omega^2}}$$

$H_z = 0$

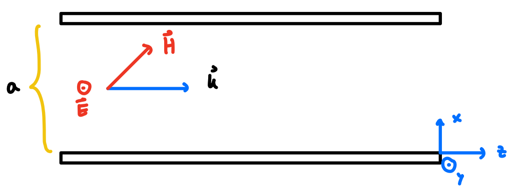
$$= \frac{1}{\varepsilon_0 c_0}\sqrt{1-\left(\frac{\omega_n}{\omega}\right)^2} = \sqrt{\frac{\mu_0}{\varepsilon_0}}\sqrt{1-\left(\frac{\omega_n}{\omega}\right)^2} = Z_0\sqrt{1-\left(\frac{\omega_n}{\omega}\right)^2}$$

$Z_0 = 376\,\Omega$ impedanca vakuuma



linearna $\omega(k)$ kot v vakuumu

c TE način



$E_z = 0$ $\quad H_z \neq 0$

$E_x = 0$ $\quad H_x \neq 0$

$E_y \neq 0$ $\quad H_y = 0$

Iščemo H_z

$$\Big(\nabla_\perp^2 + \underbrace{\Big(\frac{\omega^2}{c_0^2} - k^2\Big)}_{K^2}\Big)\vec H(\vec r) = 0$$

$$\Big(\frac{\partial^2}{\partial x^2} + K^2\Big)H_z(x) = 0$$

RP: $H_\perp = 0 = H_x\big|_b$

$H_x(0) = H_x(a) = 0$

$H_z = A\sin Kx + B\cos Kx$

$A = 0$ RP

$0 = -BK\sin Ka$

$Ka = n\pi$

$H_z = B\cos\frac{n\pi}{a}x$

$H_x \propto \frac{\partial H_z}{\partial x} \Rightarrow \frac{\partial H_z}{\partial x}(0) = 0$

(a)

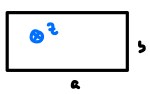
normalni odvod = 0

velja splošno za TE

d TE način – transverzalna impedanca

$$\boxed{Z(\omega) = \frac{E_\parallel}{H_\perp}} = \frac{E_y}{H_x} = \dots = \frac{Z_0}{\sqrt{1-\left(\frac{\omega_n}{\omega}\right)^2}}$$

e Vodnik s pravokotnim presekom



$$\Big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + K^2\Big)\underbrace{E_z(x,y)}_{X(x)\,Y(y)} = 0$$

$$X''Y + XY'' + K^2XY = 0 \qquad /:XY$$

$$\underbrace{\frac{X''}{X}}_{-K_x^2} + \underbrace{\frac{Y''}{Y}}_{-K_y^2} + K^2 = 0 \qquad K^2 = K_x^2 + K_y^2$$

$$X'' + K_x^2X = 0 \quad\Rightarrow\quad K_x = \frac{n\pi}{a}$$

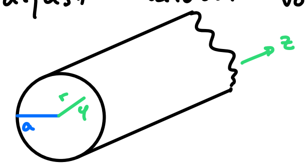
$$Y'' + K_y^2Y = 0 \quad\Rightarrow\quad K_y = \frac{m\pi}{b}$$

$$K^2 = K_x^2 + K_y^2 \Rightarrow \frac{\omega^2}{c_0^2} - k^2 = \pi^2\left(\left(\frac{n}{a}\right)^2 + \left(\frac{m}{b}\right)^2\right) \qquad k = \sqrt{\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2 + c_0^2k^2}$$

TM $n, m = 1, 2, 3, \dots$

TE eden izmed n, m je lahko 0

27 Valjast valovni vodnik



$$\left(\nabla_\perp^2 + \underbrace{\left(\frac{\omega^2}{c_0^2} - k^2\right)}_{K^2}\right)\begin{Bmatrix} E_z \\ H_z \end{Bmatrix} = 0$$

RP $\quad E_{\parallel}\big|_\partial = 0 \qquad H_\perp\big|_\partial = 0$

$\boxed{TM}$ način $\quad H_z = 0 \quad E_z \neq 0 \quad , \; E_z(r,\varphi) = ?$

$$\nabla_\perp^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2} \qquad \text{u valjni geometriji}$$

$$E_z(r,\varphi) = R(r)\,\phi(\varphi)$$

$$\frac{1}{r}(rR')'\phi + \frac{1}{r^2}\phi'' R + K^2 R\phi = 0$$

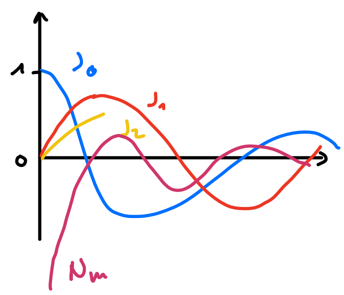
$$\left(\frac{1}{r}R' + R''\right)\phi + \frac{1}{r^2}\phi'' R + K^2 R\phi = 0 \qquad /:R\phi \quad \cdot r^2$$

$$\frac{R''}{R}r^2 + \frac{R'}{R}r + r^2K^2 = -\frac{\phi''}{\phi} = m^2$$

$$\phi'' + m^2\phi = 0 \qquad\qquad r^2R'' + rR' + (K^2r^2 - m^2)R = 0 \quad \text{Besselova DE}$$

$$\phi(\varphi) = A_m \sin(m\varphi - \varphi_m) \qquad\qquad R(r) = A_m J_m(Kr) + B_m N_m(Kr)$$

Besselova f. $\qquad$ Neumannova f.



Splošna rešitev

$$E_z(r,\varphi) = \sum_{m=0}^{\infty} A_m J_m(Kr)\sin(m\varphi - \varphi_m)$$

RP $\quad E_{\parallel}\big|_\partial = E_z\big|_\partial = E_z(r=a) = 0$

$$E_z(r,\varphi) = \sum_{m=0}^{\infty} A_m J_m(Ka)\sin(m\varphi - \varphi_m) = 0$$

$J_m(Ka) = 0 \qquad$ ničle Besselovih f.

n \ m	J_0	J_1	J_2
1	2,40	3,83	5,14
2	5,52	7,02	⋮
3	⋮	⋮	

$$\xi_{m,n} = Ka$$

$$K = \frac{\xi_{m,n}}{a}$$

$$K^2 = \frac{\omega^2}{c_0^2} - k^2 = \frac{\xi_{mn}^2}{a^2}$$

$$\omega = c_0\sqrt{k^2 + \frac{\xi_{mn}^2}{a^2}}$$

$\boxed{TE}$ način

$$H_z(r,\varphi) = \sum_{m=0}^{\infty} A_m J_m(Kr)\sin(m\varphi - \varphi_m)$$

RP $\quad H_r(r=a,\varphi) = 0$

$$\Downarrow$$

$$\frac{\partial H_z}{\partial r}(r=a,\varphi) = 0$$

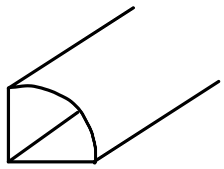
$$\frac{\partial H_z}{\partial r}(r=a,\varphi) = 0 = \sum_{m=0}^{\infty} A_m k J_m'(ka)\sin(m\varphi-\varphi_m)$$

$J_m'(ka) = 0$ ničle odvodov Besslovih funkcij

$$ka = \xi_{mn}'$$

$n \backslash m$	J_0'	J_1'	J_2'
1	3,83	1,84	3,05
2	7,02	5,33	6,71
3	⋮	⋮	⋮

$$\omega = c_0\sqrt{k^2+\frac{\xi_{mn}'^2}{a^2}}$$



Delni valjni vodnik

Enaka splošna rešitev, RP se spremenijo

TM
$$E_z(r,\varphi=0)=0 \Rightarrow \varphi_m = 0$$
$$E_z(r,\varphi=\tfrac{\pi}{2})=0 \Rightarrow \sin m\tfrac{\pi}{2}=0 \qquad m = \underset{\text{trivialna rešitev}}{\cancel{0}}, 2, 4, \dots$$
$$E_z(r=a,\varphi)=0 \Rightarrow \xi_{mn} = ka$$

TE
$$H_r(r,\varphi=0)=0 \sim \frac{\partial H_z}{\partial \varphi} \qquad H_\perp \qquad \Rightarrow \cos m\varphi$$
$$H_r(r,\varphi=\tfrac{\pi}{2})=0 \sim \frac{\partial H_z}{\partial \varphi} \Rightarrow \sin m\tfrac{\pi}{2}=0 \qquad m = 0, 2, 4, \dots$$
$$H_r(r=a,\varphi)=0 \Rightarrow ka = \xi_{mn}'$$

(28) TEM valovanje v valovnem vodniku

↓

TE + TM obe polji sta transverzalni $E_z=0$, $H_z=0$ (npr. svetloba v vakuumu)

(a) Pokaži
$$\left.\begin{aligned}\nabla\times\vec{E} &= i\vec{k}\times\vec{E}\\ \nabla\times\vec{H} &= i\vec{k}\times\vec{H}\end{aligned}\right\} \rightarrow \text{disperzijska relacija } \omega = c_0 k$$

$$\vec{E}(\vec{r},t) = \vec{E}(\vec{\rho})\,e^{i(kz-\omega t)}$$

$$\nabla\times\vec{E} = \underbrace{\left(\nabla\times\vec{E}(\vec{\rho})\right)}\,e^{i(kz-\omega t)} + \underbrace{\nabla e^{i(kz-\omega t)}}\times\vec{E}(\vec{\rho}) = i\vec{k}\times\vec{E}$$

$$\underbrace{\begin{bmatrix}\partial_x\\ \partial_y\\ \partial_z\end{bmatrix}\times\begin{bmatrix}E_x\\ E_y\\ 0\end{bmatrix} = \begin{bmatrix}-\partial_z E_y(\rho)\\ \partial_z E_x(\rho)\\ \partial_x E_y-\partial_y E_x\end{bmatrix} = \begin{bmatrix}0\\ 0\\ \dots\end{bmatrix}}_{[\nabla\times\vec{E}(\vec{r},t)]_z} \qquad \underbrace{\underbrace{\hat{e}_z\, ik}_{\vec{k}}\, \underbrace{e^{i(kz-\omega t)}\,\vec{E}(\vec{\rho})}_{\vec{E}(\vec{r},t)}}_{i\vec{k}\times\vec{E}(\vec{r},t)}$$

$$= -\mu_0\left(\frac{\partial \vec{H}}{\partial t}\right)_z = 0$$

Podobno dokažemo tudi drugo enačbo

$$i\vec{k}\times\vec{E} = \nabla\times\vec{E} = i\mu_0\omega\vec{H} \Rightarrow \vec{H} = \frac{\vec{k}}{\mu_0\omega}\times\vec{E}$$
$$i\vec{k}\times\vec{H} = \nabla\times\vec{H} = \varepsilon_0\frac{\partial\vec{E}}{\partial t} = -i\omega\varepsilon_0\vec{E} \Rightarrow \vec{E} = -\frac{\vec{k}}{\omega\varepsilon_0}\times\vec{H}$$

$$\Rightarrow \vec{k}\times\vec{H} = \vec{k}\times\left(\frac{\vec{k}}{\mu_0\omega}\times\vec{E}\right) = -\omega\varepsilon_0\vec{E}$$

$$\frac{\vec{k}}{\mu_0\omega}\underbrace{\vec{k}\cdot\vec{E}}_{\substack{0\\ k\perp E}} - \frac{k^2}{\mu_0\omega}\vec{E} = -\omega\varepsilon_0\vec{E} \Rightarrow k^2 = \omega^2\varepsilon_0\mu_0 \Rightarrow \omega = kc_0$$

(b) $$\Big(\nabla_\perp^2 + \underbrace{\Big(\frac{\omega^2}{c_0^2} - k^2\Big)}_{0} \Big) \left\{ \begin{matrix} \vec E \\ \vec H \end{matrix} \right\} = 0$$

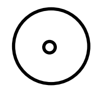
TEM $\quad \nabla_\perp^2 \left\{ \begin{matrix} \vec E(\vec\rho) \\ \vec H(\vec\rho) \end{matrix} \right\} = 0$ $\qquad$ Isto kot da rešujemo statični problem

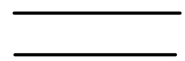
$$\nabla_\perp^2 E_\parallel = 0 \qquad \text{RP} \qquad E_\parallel \big|_\partial = 0$$

Za preprosto ploskev $\Rightarrow E_\parallel = 0$

Ploskev / presek z luknjami $\Rightarrow E_\parallel$ lahko $\neq 0$ $\qquad$ npr koaksialni vodnik $\Rightarrow$ TEM način obstaja

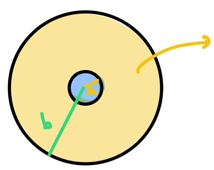
ali: vzporedni ravni plošči





premo reta
TEM način

(29) Koaksialni valovni vodnik s plazmo, TEM način



$$\varepsilon(\omega) = 1 - \frac{\omega_p^2}{\omega^2}$$

$\omega(k) = ?$ $\qquad Z(\omega) = ?$

$$\omega = c k$$

$$\omega = \frac{1}{\sqrt{\varepsilon \varepsilon_0 \mu_0}} k = \frac{c_0 k}{\sqrt{\varepsilon}} = \frac{c_0 k}{\sqrt{1 - \omega_p^2/\omega^2}}$$

$$\omega^2 - \omega_p^2 = (c_0 k)^2$$

Enako kot v neskončni fazi $\qquad \omega = \sqrt{c_0^2 k^2 + \omega_p^2}$

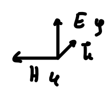
$$Z = \frac{U}{I}$$

Glej malo še z magnetno silo

$$Z = \frac{E_y}{H_\varphi} \frac{\ln b/a}{2\pi}$$

$$I = 2\pi r H_\varphi \qquad U = E_y r \ln\frac{b}{a}$$

$$\vec E = -\frac{\vec k}{\omega \varepsilon_0 \varepsilon} \times \vec H$$



$$E_y = \frac{k H_\varphi}{\omega \varepsilon_0 \varepsilon}$$

$$Z = \frac{\ln b/a}{2\pi} \frac{k}{\omega \varepsilon_0} = \frac{\ln b/a}{2\pi \omega \varepsilon_0 \varepsilon} \frac{1}{c_0} \sqrt{\omega^2 - \omega_p^2} = \frac{\ln b/a}{2\pi\varepsilon} \sqrt{\frac{\mu_0}{\varepsilon_0}} \sqrt{1 - \omega_p^2/\omega^2} = \frac{\ln b/a}{2\pi\varepsilon} Z_0 \sqrt{1 - \omega_p^2/\omega^2}$$

$$= \frac{\ln b/a}{2\pi} Z_0 \frac{1}{\sqrt{1 - \omega_p^2/\omega^2}}$$