Opis gibanja enega delca

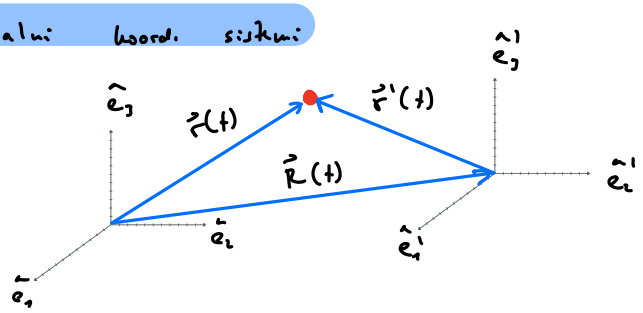
$$\vec{r}(t) = (x(t),\ y(t),\ z(t))$$
$$\vec{v}(t) = \frac{d\vec{r}}{dt} = \dot{\vec{r}} = (\dot{x}, \dot{y}, \dot{z})$$
$$\vec{a}(t) = \frac{d^2\vec{r}}{dt^2} = \ddot{\vec{r}} = (\ddot{x}, \ddot{y}, \ddot{z})$$

Obstaja 10 modelnih transformacij med inercialnimi sistemi:

- 3 rotacije $\vec{r}\,' = O\vec{r}$ $\quad O \ldots 3\times 3$ ortogonalna matrika transformacije
- 3 translacije $\vec{r}\,' = \vec{r} + \vec{R}$
- 3 premiki hitrosti
- 1 premik časa $t' = t + t_0$

Te transformacije predstavljajo grupo Galilejevih transformacij. Naravne enačbe so invariantne na te transformacije. To je poseben primer Poincarejeve grupe ($c \to \infty$).

Neinercialni koord. sistemi



Ortonormirana baza

$$\vec{e}_\alpha \cdot \vec{e}_\beta = \delta_{\alpha\beta} \qquad \alpha, \beta = 1,2,3 = x,y,z$$
$$\frac{d\vec{e}_\alpha}{dt} = 0$$
$$\vec{r} = \vec{R} + \vec{r}\,'$$

$$\vec{r} = \sum_{\alpha=1}^{3} x_\alpha \vec{e}_\alpha, \quad \vec{R} = \sum_{\alpha=1}^{3} X_\alpha \vec{e}_\alpha, \quad \vec{r}\,' = \sum_{\alpha=1}^{3} x'_\alpha \vec{e}\,'_\alpha$$

Izrazimo hitrost $\vec{v}\,'$

$$\vec{v} = \frac{d\vec{r}}{dt} = \underbrace{\frac{d\vec{R}}{dt}}_{\vec{v}_R(t)} + \frac{d\vec{r}\,'}{dt} = \sum_\alpha \dot{x}_\alpha \vec{e}_\alpha$$

$$\frac{d\vec{e}_\alpha'}{dt} = \sum_{\alpha=1}^{3} a_{\alpha\beta}\, \vec{e}\,'_\beta \qquad \Big|\ \vec{e}\,'_\gamma$$

$$\frac{de_\alpha'}{dt} = a_{\alpha\gamma} \qquad \text{matrika } a_{\alpha\gamma} \text{ je antisimetrična.}$$

$$a_{\alpha\gamma} = \frac{de_\alpha'}{dt} \cdot \vec{e}\,'_\gamma = \frac{d}{dt}\underbrace{(\vec{e}_\alpha' \cdot \vec{e}\,'_\gamma)}_{=0} - \vec{e}_\alpha' \cdot \frac{d\vec{e}\,'_\gamma}{dt} = -a_{\gamma\alpha}$$

$$a = \begin{bmatrix} 0 & a_{12} & a_{13} \\ -a_{12} & 0 & a_{23} \\ -a_{13} & -a_{23} & 0 \end{bmatrix} = \begin{bmatrix} 0 & \omega_3' & -\omega_2' \\ -\omega_3' & 0 & \omega_1' \\ \omega_2' & -\omega_1' & 0 \end{bmatrix}$$

$$\vec{\omega}\,' = \sum_\alpha \omega_\alpha' \vec{e}_\alpha \qquad \vec{\omega}\,' = (\omega_1', \omega_2', \omega_3')$$

$$\alpha = 1 \qquad \frac{d\vec{e}_1'}{dt} = \sum_\beta a_{1\beta} \vec{e}\,'_\beta = a_{12}\vec{e}\,'_2 + a_{13}\vec{e}\,'_3 = \omega_3' \vec{e}_2' - \omega_2' \vec{e}_3' = \vec{\omega}\,' \times \vec{e}_1$$

$$\boxed{\frac{d\vec{e}_\alpha'}{dt} = \vec{\omega}\,' \times \vec{e}_\alpha'}$$

$$\frac{d\vec{r}\,'}{dt} = \frac{d}{dt}\sum_\alpha x_\alpha' \vec{e}_\alpha' = \sum_\alpha \frac{dx'_\alpha}{dt}\vec{e}_\alpha' + \sum_\alpha x_\alpha' \frac{de_\alpha'}{dt} = \left(\frac{d\vec{r}\,'}{dt}\right)_{\text{neinerc. sistemu}} + \sum_\alpha x_\alpha' (\vec{\omega}\,' \times \vec{e}_\alpha')$$

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = \frac{d\vec{R}}{dt} + \frac{d\vec{r}\,'}{dt} = \frac{d\vec{R}}{dt} + \left(\frac{d\vec{r}\,'}{dt}\right)_{\text{neinerc. sistem.}} + \vec{\omega}\,' \times \vec{r}\,'$$

$$\vec{\omega} = \sum_\alpha \omega_\alpha' \vec{e}_\alpha'$$

$$\frac{d\vec{\omega}}{dt} = \sum_\alpha \dot{\omega}_\alpha' \vec{e}_\alpha' + \overbrace{\vec{\omega}' \times \vec{\omega}'}^{0} = \left(\frac{d\vec{\omega}'}{dt}\right)_{\text{mirnog. sistem}}$$

Izrazimo pospešek $\vec{a}'$

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{R}}{dt^2} + \frac{d^2\vec{r}'}{dt^2} = \sum_\alpha \ddot{x}_\alpha \vec{e}_\alpha$$

$$\frac{d^2\vec{r}'}{dt^2} = \frac{d}{dt}\left(\left(\frac{d\vec{r}'}{dt}\right)_{\text{mirnog.}} + \vec{\omega}' \times \vec{r}'\right) = \frac{d}{dt}\left(\sum_\alpha \dot{x}_\alpha' \vec{e}_\alpha' + \vec{\omega}' \times \sum_\alpha x_\alpha' \vec{e}_\alpha'\right) =$$

$$= \sum_\alpha \ddot{x}_\alpha' \vec{e}_\alpha' + \sum_\alpha \dot{x}_\alpha' \dot{\vec{e}}_\alpha' + \dot{\vec{\omega}}' \times \vec{r}' + \vec{\omega} \times \sum_\alpha \dot{x}_\alpha' \vec{e}_\alpha' + \vec{\omega}' \times \sum_\alpha x_\alpha \dot{\vec{e}}_\alpha' =$$

$$= \left(\frac{d^2\vec{r}}{dt^2}\right)_{\text{min.}} + \vec{\omega}' \times \sum_\alpha \dot{x}_\alpha' \vec{e}_\alpha' + \dot{\vec{\omega}}' \times \vec{r}' + \vec{\omega}' \times \vec{v}' + \vec{\omega}' \times (\vec{\omega}' \times \vec{r}')$$

$$\frac{d^2\vec{r}'}{dt^2} = \left(\frac{d^2\vec{r}}{dt^2}\right)_{\text{min.}} + 2\vec{\omega}' \times \underbrace{\left(\frac{d\vec{r}'}{dt}\right)_{\text{min.}}}_{\vec{v}'_{\text{min.}}} + \vec{\omega}' \times (\vec{\omega}' \times \vec{r}') + \dot{\vec{\omega}}' \times \vec{r}'$$

$$\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}(\vec{a} \cdot \vec{c}) - \vec{c}(\vec{a} \cdot \vec{b})$$

$$\vec{\omega} \times (\vec{\omega} \times \vec{r}) = \vec{\omega}(\vec{\omega} \cdot \vec{r}) - \omega^2 \vec{r}$$

Sistemska sila

$$\vec{F} = m\frac{d^2\vec{r}}{dt^2} = m\frac{d^2\vec{R}}{dt^2} + m\frac{d^2\vec{r}'}{dt^2}$$

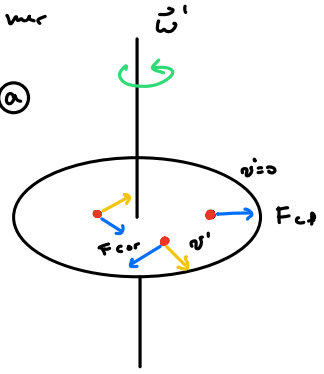
$$m\left(\frac{d^2\vec{r}'}{dt^2}\right)_{\text{min}} \overset{\text{def.}}{=} \vec{F}_{\text{sist}} + \vec{F}$$

$$\vec{a} = \vec{a}' + 2\vec{\omega}' \times \vec{v}' + \vec{\omega}' \times (\vec{\omega}' \times \vec{r}') + \dot{\vec{\omega}}' \times \vec{r}'$$

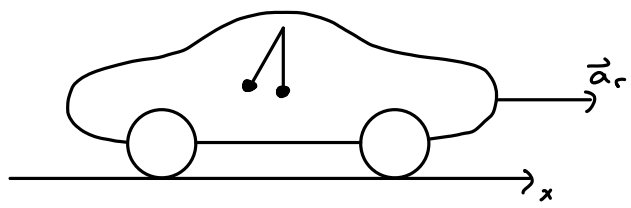
$$\vec{F}_{\text{sist}} = -m\frac{d^2\vec{R}}{dt^2} \underbrace{-2m\,\vec{\omega}' \times \vec{v}'_{\text{min}}}_{\text{Coriolisova sila}} \underbrace{- m\,\vec{\omega}' \times (\vec{\omega}' \times \vec{r}')}_{\underbrace{m\omega'^2\vec{r}'}_{\text{Centrifugalna sila}} - m(\vec{\omega}' \cdot \vec{r}')\vec{\omega}'} \underbrace{- m\left(\frac{d\vec{\omega}'}{dt}\right)_{\text{min}} \times \vec{r}'}$$

Primer

ⓐ



ⓑ Avto



Osnovni pojmi (1 delec)

Gibalna količina

$$\vec{p} = m\vec{v} \qquad \frac{d\vec{p}}{dt} = m\vec{a} = \vec{F}$$

Vrtilna količina

$$\vec{l} = \vec{r} \times \vec{p} = m\vec{r} \times \vec{v} \qquad \frac{d\vec{l}}{dt} = m\underbrace{\dot{\vec{r}} \times \vec{p}}_{0} + m\vec{r} \times \dot{\vec{p}} = \vec{r} \times \vec{F} = \vec{M} \quad \text{navor}$$

Energija

ⓐ kinetična energija $T = \frac{1}{2} m v^2$

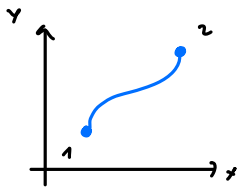
ⓑ delo sile

$$A = \int_{pot} \vec{F} \cdot d\vec{r} = \int_{t_1}^{t_2} \underbrace{\vec{F} \cdot \vec{v}}_{P \text{ moč}} dt = \int_{t_1}^{t_2} P(t)\, dt = \int_{t_1}^{t_2} m\vec{a} \cdot \vec{v}\, dt = \int_{t_1}^{t_2} m\dot{\vec{v}} \cdot \vec{v}\, dt =$$

$$= \frac{1}{2} \int_{t_1}^{t_2} m \frac{dv^2}{dt} dt = \frac{1}{2} m v^2(t_2) - \frac{1}{2} m v^2(t_1)$$

$$A = T(\vec{r}_2) - T(\vec{r}_1)$$

Konkreten primer



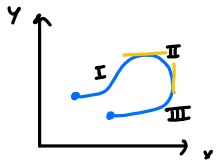
$$\vec{r}(s) = (x(s), y(s), z(s))$$

$$dr = \left(\frac{dx}{ds} ds, \frac{dy}{ds} ds, \frac{dz}{ds} ds\right)$$

Naj bo $s = x$

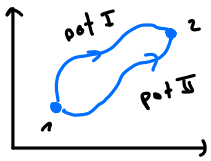
$$A = \int_{pot} \vec{F}\, d\vec{r} = \int_{pot} F_x\, dx + \int_{pot} F_y\, dy + \int_{pot} F_z\, dy =$$

$$A = \int ds = \int_{I} + \int_{II} + \int_{III} ds$$



ⓒ Potencialna energija

①



$$A_I = \int_{pot\,I} \vec{F}(\vec{r})\, d\vec{r} \qquad A_{II} = \int_{pot\,II} \vec{F}(\vec{r})\, d\vec{r}$$

Če velja $A_I = A_{II}$ $\quad \oint \vec{F} \cdot d\vec{r} = -A_I + A_{II} = 0$

② Naj bo $\vec{F}(\vec{r}) = -\nabla U = -\left(\frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}, \frac{\partial U}{\partial z}\right) \overset{def.}{=} -\frac{\partial U}{\partial \vec{r}}$

$$A = -\int_{pot} \nabla U\, d\vec{r} = -\left(U(\vec{r}_2) - U(\vec{r}_1)\right) = -U(\vec{r}_2) + U(\vec{r}_1)$$

ⓓ Celotna energija $E = T + U$

$$\Delta E = E_2 - E_1 = (T_2 + U_2) - (T_1 + U_1) = (T_2 - T_1) + (U_2 - U_1) = A - A = 0$$

ⓔ Konzervativne sile

① 1D $\quad U(x) = -\int_{x_0}^{x} F(s)\,ds \quad ; \quad -\frac{dU}{dx} = F(x)$

② 2D $\quad \vec{F}(x,y) = F_1(x,y)\,\vec{e}_1 + F_2(x,y)\,\vec{e}_2$

$$\vec{F} = -\nabla U = \underbrace{-\frac{\partial U}{\partial x}}_{F_1}\vec{e}_1 \underbrace{-\frac{\partial U}{\partial y}}_{F_2}\vec{e}_2$$

$$\frac{\partial F_1}{\partial y} = -\frac{\partial^2 U}{\partial x \partial y} \quad ; \quad \frac{\partial F_2}{\partial x} = -\frac{\partial^2 U}{\partial y \partial x}$$

Če je $\frac{\partial^2 U}{\partial x \partial y}$ zvezen $\Rightarrow$ $\boxed{\frac{\partial F_x}{\partial y} = \frac{\partial F_y}{\partial x}}$ Potrebni pogoj za konzervativnost.

Konzervativne $\vec{F} = -\nabla U$

③ 3D analogno

Nekonzervativne sile

ⓐ linearni zakon upora

$$\vec{F} = -k\vec{v} \qquad k \geq 0$$

$$A = \int \vec{F}\cdot d\vec{r} = -k\int_{t_1}^{t_2} \vec{v}\cdot\vec{v}\,dt = -k\int v^2 dt \leq 0$$

ⓑ trenje

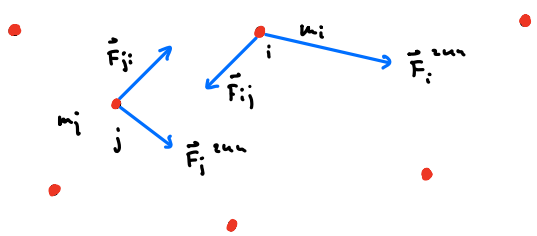
$$F_x = -k F_N \operatorname{sgn} v_x = -k|F_N|\frac{v}{|v|}$$

$$A \leq 0$$

ⓒ kvadratni zakon upora

$$\vec{F} = -k v^2 \frac{\vec{v}}{v}$$

Sistem N točkastih teles



Gibalna količina

$$m_1\vec{a}_1 = \vec{F}_1^{zun} + \vec{F}_{12} + \sum_{j\neq 1} F_{1j}$$

$$m_2\vec{a}_2 = \vec{F}_2^{zun} + \vec{F}_{21} + \sum_{j\neq 2} \vec{F}_{2j}$$

$$m_N\vec{a}_N = \dots$$

$$\sum_{i=1}^{N} m_i \vec{a}_i = \underbrace{\sum_{i=1}^{N} \vec{F}_i}_{\vec{F}_{zun}} + \underbrace{\vec{F}_{12} + \vec{F}_{21}}_{0 \ldots 3.N.Z.} + \sum_{\substack{i=1 \\ j \neq i}}^{N} \sum_{j=1}^{N} \vec{F}_{ij}$$

$$\sum_i \sum_{j<i} (F_{ij} + F_{ji}) = 0$$

$$m = \sum m_i$$

$$\vec{F}^{zun} = m \vec{a}^* = \sum_{i=1}^{N} m_i a_i$$

$$\vec{p} = \sum_{i=1}^{N} m_i \vec{v}_i = m \vec{v}^*$$

$$\vec{r}^* = \frac{\sum m_i \vec{r}_i}{\sum m_i} \qquad v^* = \dot{\vec{r}}^* = \frac{\sum m_i \vec{v}_i}{\sum m_i} = \frac{1}{m} \vec{p}$$

$$\vec{p} = m \vec{v}^*$$

$$\vec{a}^* = \frac{d}{dt} \vec{v}^*$$

$$\frac{d\vec{p}}{dt} = m \vec{a}^* \qquad \text{al:} \quad \frac{d\vec{p}}{dt} = m \vec{a}'$$

Vrtilna količina

celotna vrtilna

$$\vec{L} = \sum_i \vec{l}_i = \sum_i \vec{r}_i \times \vec{p}_i$$

$$\dot{\vec{L}} = \sum_i \frac{d}{dt} (\vec{r}_i \times \vec{p}_i) = \sum_i \dot{\vec{r}}_i \times m \vec{a}_i = \sum_i \vec{r}_i \times \vec{F}_i$$

$$\dot{\vec{L}} = \sum_i \vec{r}_i \times \vec{F}_i^z + \underbrace{\sum_{j \neq i} \vec{r}_i \times \vec{F}_{ij}} = \vec{M}^{zun} + \vec{M}^{not} = \vec{M}$$

$$\sum_{j \neq i} \vec{r}_i \times \vec{F}_{ij} = \sum_{j<i} (\vec{r}_i \times \vec{F}_{ij} + \vec{r}_j \times \vec{F}_{ji}) = \sum_{j<i} (\vec{r}_i - \vec{r}_j) \times \vec{F}_{ij} = \vec{M}^{not}$$

Energija sistema n točkastih teles

Delo

$$A = \sum_i \int_1^2 \vec{F}_i(\vec{r}_i) \, d\vec{r}_i \quad \text{poti:} = A^{zun} + A^{not}$$

$$A^{zun} = \sum_i \int_1^2 \vec{F}_i^z \, d\vec{r}_i \qquad A^{not} = \sum_{j \neq i} \int_1^2 F_{ij} \, d\vec{r}_i$$

$$A = \sum_i \int_{t_1}^{t_2} m_i \dot{\vec{v}}_i \cdot \dot{\vec{v}}_i \, dt = T(t_2) - T(t_1) = \Delta T \qquad T = \frac{1}{2} \sum_i m_i v_i^2 = \sum T_i$$

Kinetična energija

$$\vec{v}_i = \vec{v}^* + \vec{v}_i' \qquad T = \frac{1}{2} \sum_i m_i (\vec{v}^* + \vec{v}_i')(\vec{v}^* + \vec{v}_i') = \frac{1}{2} \sum_i m_i (v^{*2} + v_i'^2 + \underbrace{2 \vec{v}_i' \vec{v}^*}_{0})$$

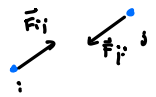
Potencialna energija (konservativne sile)

$$\vec{F}_i^{zun} = -\nabla_i V_i^z(\vec{r}_i) \qquad \nabla_i = \left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}, \frac{\partial}{\partial z_i}\right)$$

$$A^z = \sum_i \int_1^2 \vec{F}_i^z \cdot d\vec{r}_i = -V^z(t_2) + V^z(t_1) = -\Delta V^z$$

$$V^z = \sum_i V(\vec{r}_i)\Big|_t$$

Pogosto je V centralni potencial (energija)

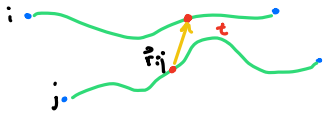


$$V_{ij} = V_{ij}(\vec{r}_{ij}) = V_{ij}(|\vec{r}_{ij}|)$$

$$\vec{r}_{ij} = \vec{r}_j - \vec{r}_i$$

$$A^{not} = \sum_{\substack{j \\ j \neq i}} \left(\int_1^2 F_{ij}\, d\vec{r}_i + \int_1^2 F_{ji}\, d\vec{r}_j \right)$$

$$= \sum_{\substack{j \\ j \neq i}} \int_1^2 \vec{F}_{ij} \cdot d\vec{r}_{ij}$$



$t \to t + dt$

$\vec{r}_{ij} \to \vec{r}_{ij} + d\vec{r}_{ij}$

Kdaj $A^{not} = 0$?

- $\vec{F} \perp d\vec{r}$
- $\vec{F} = 0$ neodvisni delci
- $d\vec{r} = 0$ toga telesa

$$\vec{F}_{ij} = -\nabla_{ij} V_{ij}(\vec{r}_{ij}) \qquad \nabla_{ij} = \left(\frac{\partial}{\partial x_{ij}}, \frac{\partial}{\partial y_{ij}}, \frac{\partial}{\partial z_{ij}}\right)$$

$$V = V(|\vec{r}|) = V(r) \qquad r = \sqrt{x^2+y^2+z^2}$$

$$\nabla V(r) = \left(\frac{\partial V}{\partial r}\frac{\partial r}{\partial x}, \frac{\partial V}{\partial r}\frac{\partial r}{\partial y}, \frac{\partial V}{\partial r}\frac{\partial r}{\partial z} \right) = \frac{\partial V}{\partial r} \nabla r \qquad \nabla r = \frac{\vec{r}}{r}$$

$$\nabla V(r) = \frac{\partial V}{\partial r}\frac{\vec{r}}{r}$$

$$\vec{F}_{ij} = \frac{\partial V_{ij}}{\partial r_{ij}} \frac{\vec{r}_{ij}}{r_{ij}}$$

$$A^{not} = -\sum_{\substack{j \\ j \neq i}} \int_1^2 \nabla_{ij} V_{ij}(r_{ij})\, d\vec{r}_{ij} = -\sum_{\substack{j \\ j \neq i}} \int_1^2 \left(\frac{\partial}{\partial x_i} V_{ij}\, dx_i + \dots \right) =$$

$$= -V^{not}(t_2) + V^{not}(t_1) = -\Delta V^{not}$$

$$V^{not} = \sum_{\substack{j \\ j \neq i}} V_{ij}(r_{ij}) = \sum_i \sum_j \frac{1}{2} V_{ij}(r_{ij}) = \sum_i \sum_j \frac{1}{2} V_{ji}(r_{ji})$$

$$E = W = T + V^z + V^n$$

$$\Delta E = 0 \quad \Leftarrow \quad \Delta T = A \qquad \Delta V = -A$$

Virialni teorem

$V(\vec{r}) = a r^n$ Število delcev se ne sme spreminjati.

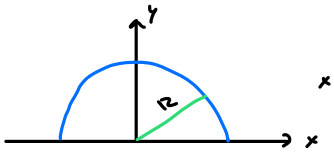
$$\boxed{2\bar{T} = n\bar{V}}$$

Lagrangev formalizem

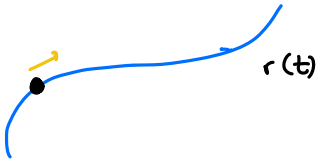
Newtonove enačbe povsem zadoščajo. Drugi formalizmi v določenih primerih olajšajo formulacijo in izračun.

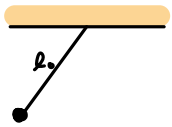
- vezi med koordinatami
- posplošene koordinate

Primeri:

- 

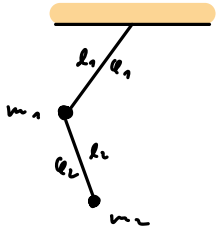
$x^2 + y^2 + z^2 = R^2$ vez

- 

- 

$x^2 + y^2 = l_0^2$ vez

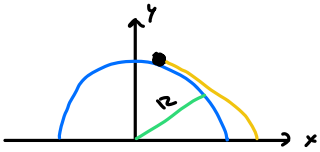
- dvojno nihalo



V splošnem so koordinate med seboj povezane z vezmi:

$$f(\vec{r}_1, \vec{r}_2, \dots \vec{r}_N, t) = 0 \qquad \text{holonomna vez}$$

Neholonomna vez: ne $\exists\, f = 0$



$$f = x^2 + y^2 + z^2 - R^2 \geq 0$$

Sistem holonomnih vezi

$$f_k(\vec{r}_1, \vec{r}_2, \dots \vec{r}_N, t) = 0 \qquad k = 1, \dots, K$$

Posplošene koordinate

- P delcev opišemo z $3N$ koordinatami
- Če imamo K vezi, ostane $n = 3N - K$ neodvisnih koordinat

Originalne koordinate izrazimo

$$\vec{r}_i = \vec{r}_i(q_1, q_2, \dots, q_n, t) \qquad i = 1, \dots, N$$

$$q_i = q_j(\vec{r}_1, \vec{r}_2, \dots, \vec{r}_N, t) \quad \text{in } K \text{ vezi}$$

$$j = 1, 2, \dots, n$$

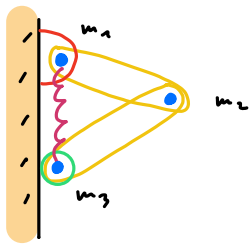
Primer: polarne koordinate

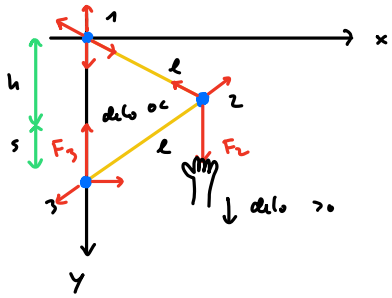
$$\vec{r} = (x, y) = r(\cos\varphi, \sin\varphi) \quad ; \quad r^2 = x^2 + y^2$$

$$q_1 = r, \quad q_2 = \varphi \qquad \frac{y}{x} = \tan\varphi$$

D'Alembertov princip

Začnemo z vajo





$\delta A = 0$ $\quad F_2, F_3$ $\quad$ ostale sile na shemi

$\vec{F}_i = \vec{F}_i^{\,aktivna} + \vec{F}_i^{\,vezi}$

$y_2 = l \sin\varphi \qquad y_3 = 2l \sin\varphi$

$\delta y_2 = l \cos\varphi \, d\varphi \qquad \delta y_3 = 2l \cos\varphi \, d\varphi$

$s = y_3 - h = 2l \sin\varphi - h$

$F_3 = ks$

$$\delta A = \sum \left(\vec{F}_i^{\,a} \, \delta \vec{r}_i + \underbrace{\vec{F}_i^{\,v} \, \delta \vec{r}_i}_{=0} \right)$$

D'Alembertov princip
Virtualno delo

$$= F_2 \, \delta y_2 - F_3 \, \delta y_3 = 0$$

$$F_2 \, l \cos\varphi \, \cancel{d\varphi} - h (2l \sin\varphi - h) \, 2l \cos\varphi \, \cancel{\delta\varphi} = 0$$

$$Q \, \delta\varphi = 0$$

posplošena sila $\qquad$ posplošena koordinata (neodvisna)

$$\rightarrow \quad \sin\varphi = \frac{F_2 + 2kh}{4kl} \qquad F_3 = \frac{F_2}{2}$$

Princip virtualnega dela

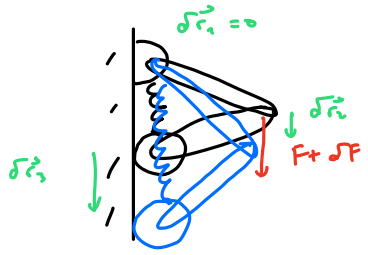
$i = 1, \dots, n$; 3N koordinat ; telesa povezana z vezmi

- Statika : $\dot{\vec{r}}_i = 0 \Rightarrow \ddot{\vec{r}} = 0 \Rightarrow \vec{F}_i = 0$

Virtualni premik : $\delta \vec{r}_i$, povezani

$\vec{F}_i \cdot \delta \vec{r}_i = 0 \qquad$ kar $\vec{F}_i = 0$

$\sum_{i=0}^{n} \vec{F}_i \cdot \delta \vec{r}_i = 0$



$\vec{F}_i = \vec{F}_i^{\,aktivna} + \vec{F}_i^{\,vezi}$

razdelimo na sile, ki opravljajo delo in sile vezi:

$$\underbrace{\sum_i \vec{F}_i^{\,a} \, \delta \vec{r}_i}_{A^a} + \underbrace{\sum_i \vec{F}_i^{\,v} \, \delta \vec{r}_i}_{A^v = 0} = 0$$

$\boxed{\sum_i \vec{F}_i^{\,a} \, \delta \vec{r}_i = 0}$ Pogoj statike ali princip virtualnega dela

$\delta A = 0$

• Dinamika

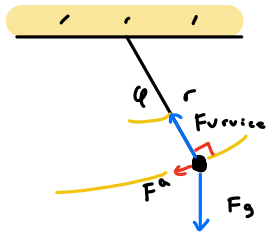
Bernoulli in D'Alembert sta postavila enačbo $\sum_i \vec{F}_i \delta \vec{r}_i = 0$

$$\vec{F}_i = m_i \ddot{\vec{r}}_i$$
$$\vec{F}_i - m_i \ddot{\vec{r}}_i = 0$$
$$(\vec{F}_i - m_i \ddot{\vec{r}}_i) \cdot \delta \vec{r}_i = 0$$
$$\sum_i (\vec{F}_i - m_i \ddot{\vec{r}}_i) \cdot \delta \vec{r}_i = 0$$

$$\sum_i (\vec{F}_i^a - m_i \ddot{\vec{r}}_i) \delta \vec{r}_i + \underbrace{\sum_i \vec{F}_i^v \delta \vec{r}_i}_{A^v = 0} = 0$$

$$\boxed{\sum_i (\vec{F}_i^a - m_i \ddot{\vec{r}}_i) \delta \vec{r}_i = 0}$$ D'Alembertov princip $\delta \vec{r}_i$ so med sabo odvisni

Primer



$q_1 = \varphi$

$q_2 = r$

vez: $r - R = 0$ (vrvica ni elastična / vzmet)

$s = r\varphi$

$ds = r\,d\varphi$; $d\varphi \sim dq_1$

$$(F^a - m a_T)\, ds$$
$$\left(-mg \sin\varphi - m R^2 \frac{d^2\varphi}{dt^2}\right) R\, d\varphi = 0$$

za $\forall\, \delta\varphi$

$\forall$, φ je neodvisna spremenljivka

$$\sum_i (\vec{F}_i^a - m_i \ddot{\vec{r}}_i) \delta \vec{r}_i = \sum_i \vec{F}_i^a \delta \vec{r}_i - \sum_i m_i \underbrace{\dot{\vec{v}}_i \cdot \vec{v}_i \delta t}_{\frac{1}{2}\frac{d}{dt} v_i^2}$$

$$= \boxed{\delta A - \delta T = 0}$$

Lagrangeova enačba

$$\sum_i (\vec{F}_i^a - m_i \ddot{\vec{r}}_i) \delta \vec{r}_i = 0$$ $\delta \vec{r}_i$ so odvisni

Vpeljemo posplošene koordinate q_j

$$\vec{r}_i = \vec{r}_i(q_1, q_2, \dots, q_n, t)$$ q ... so neodvisni

$$d\vec{r}_i = \sum_{j=1}^{n} \frac{\partial \vec{r}_i}{\partial q_j} dq_j + \frac{\partial \vec{r}_i}{\partial t} dt$$

$$\vec{v}_i = \dot{\vec{r}}_i = \sum_j \frac{\partial \vec{r}_i}{\partial q_j} \dot{q}_j + \frac{\partial \vec{r}_i}{\partial t}$$

Poglejmo os času t ; $\delta t = 0$

$$d\vec{r}_i = \sum_j \frac{\partial \vec{r}_i}{\partial q_j} dq_j$$

Statika

$$\delta A = \sum_i \vec{F}_i^a \, \delta \vec{r}_i = 0$$ zapišemo s q

$$\delta A = \sum_i \vec{F}_i^a \sum_j \frac{\partial \vec{r}_i}{\partial q_j} \delta q_j$$

$$\delta A = \sum_j \sum_i \vec{F}_i^a \frac{\partial \vec{r}_i}{\partial q_j} \delta q_j =$$

$$= \sum_j Q_j \, \delta q_j = 0 \qquad Q_j = \sum_i F_i^a \frac{\partial \vec{r}_i}{\partial q_j}$$ posplošena sila

$\Rightarrow Q_j = 0$ kar so koordinate neodvisne

Dinamika

$$\sum_i m_i \ddot{\vec{r}}_i \, \delta r_i = \sum_{i=1}^{N} m_i \ddot{\vec{r}}_i \sum_{j=1}^{n} \frac{\partial \vec{r}_i}{\partial q_j} \delta q_j =$$

$$= \sum_{i,j} \left(\frac{d}{dt} m_i \dot{\vec{r}}_i \frac{\partial \vec{r}_i}{\partial q_j} - m_i \dot{\vec{r}}_i \frac{d}{dt} \frac{\partial \vec{r}_i}{\partial q_j} \right) \delta q_j$$

$$\frac{\partial \vec{r}_i}{\partial t} = \vec{v}_i = \sum_j \frac{\partial \vec{r}_i}{\partial q_j} \dot{q}_j + \frac{\partial \vec{r}_i}{\partial t}$$

- $\frac{\partial \vec{v}_i}{\partial \dot{q}_j} = \frac{\partial \vec{r}_i}{\partial q_j}$; $\left(\frac{\partial \vec{r}_i}{\partial \dot{q}_j} = 0 \right)$
- $\frac{d}{dt} \frac{\partial \vec{r}_i}{\partial q_j} = \sum_{k=1}^{n} \frac{\partial^2 \vec{r}_i}{\partial q_j \partial q_k} \dot{q}_k + \frac{\partial^2 \vec{r}_i}{\partial q_j \partial t}$

$$= \frac{\partial}{\partial q_j} \sum_{k=1}^{n} \left(\frac{\partial \vec{r}_i}{\partial q_k} \dot{q}_k + \frac{\partial \vec{r}_i}{\partial t} \right) = \frac{\partial}{\partial q_j} \frac{d \vec{r}_i}{dt}$$

$$\sum_i m_i \ddot{\vec{r}}_i \, \delta \vec{r}_i = \sum_i \Big(\underbrace{\frac{d}{dt} m_i \dot{\vec{v}}_i \frac{\partial \vec{v}_i}{\partial \dot{q}_j}}_{\frac{\partial T_i}{\partial \dot{q}_j} = \frac{1}{2} \frac{\partial}{\partial \dot{q}_j} (m_i \vec{v}_i \cdot \vec{v}_i)} - \underbrace{m_i \vec{v}_i \frac{\partial \vec{v}_i}{\partial q_j}}_{\frac{\partial T_i}{\partial q_j}} \Big)$$

$$= \frac{d}{dt} \frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j}$$

$$\sum_j \Big(\underbrace{Q_j - \frac{d}{dt} \frac{\partial T}{\partial \dot{q}_j} + \frac{\partial T}{\partial q_j}}_{0} \Big) \delta q_j = 0$$ za $\forall \, \delta q_j$

q_j so neodvisni

$$\boxed{\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} = Q_j}$$

velja povsem splošno (tudi za nekonzervativne sile: trenje, upor)

Konzervativne sile

a ne pišemo več

$$\vec{F}_i = -\nabla_i V(\vec{r}_i)$$

$$Q_j = \sum_i \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} = -\sum_i \underbrace{\nabla_i V \cdot \frac{\partial \vec{r}_i}{\partial q_j}} = -\frac{\partial V}{\partial q_j}$$

$$\sum_i \frac{\partial V}{\partial x_i} \frac{\partial x_i}{\partial q_j} + \frac{\partial V}{\partial y_i} \frac{\partial y_i}{\partial q_j} + \frac{\partial V}{\partial z_i} \frac{\partial z_i}{\partial q_j} = \frac{\partial V}{\partial q_j}$$

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} = -\frac{\partial V}{\partial q_j}$$; $T - V = L$ Lagrangeova funkcija

$$L = L(q_i, \dot{q}_i, t) = T(q_j, \dot{q}_j) - V(q_j, t) \quad ; \quad (V(q_j, \dot{q}_j, t))$$

$$\boxed{\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0}$$ Lagrangeove enačbe $j = 1, 2, \dots, n$

Kaj na ek $V = V(q_j, \dot{q}_j)$ npr. Lorentzova sila

$$Q_j = -\frac{\partial V}{\partial q_j} + \frac{d}{dt}\frac{\partial V}{\partial \dot{q}_j}$$

$L = T - V$ → Lagrangeova enačba

$$\vec{a} = \vec{F} = e\vec{v} \times \vec{B} = e\dot{\vec{r}} \times \vec{B}$$

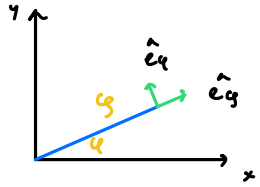
Primer

① $V(\vec{r})$ en delec z maso m

$$L = T - V = \frac{1}{2}mv^2 - V(\vec{r}) = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V(x, y, z) \qquad q_i = x, y, z$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = \frac{d}{dt}m\dot{x} - \frac{\partial V}{\partial x} = 0 \qquad m\ddot{x} = -\frac{\partial V}{\partial x} \qquad m\vec{a} = \vec{F}$$

② Ravninsko gibanje



ⓐ $\vec{r}_i = (x_i, y_i, z_i)$ $\qquad N = 1$

ⓑ $q_1 = \rho$ $\qquad n = 2$

$q_2 = \varphi$

ⓒ Vezi $k = 1$; $z = \text{konst.}$

$k + n = N$

$$\vec{r} = \rho(\cos\varphi, \sin\varphi, 0) = \rho\,\hat{e}_\rho \qquad \hat{e}_\rho \cdot \hat{e}_\varphi = 0 = \delta_{jj'}$$

$$\dot{\vec{r}} = \dot{\rho}(\cos\varphi, \sin\varphi, 0) + \rho(-\sin\varphi, \cos\varphi, 0)\dot{\varphi} = \dot{\rho}\,\hat{e}_\rho + \rho\dot{\varphi}\,\hat{e}_\varphi$$

$$\frac{\partial \dot{r}_i}{\partial \dot{q}_j} = \frac{\partial r_i}{\partial q_j} \qquad \frac{d}{dt}\frac{\partial \dot{r}_i}{\partial q_j} = \frac{\partial}{\partial q_j}\underbrace{\frac{d}{dt}\vec{r}_i}_{v_i}$$

$$\frac{\partial \dot{\vec{r}}}{\partial \dot{\rho}} = \frac{\partial \vec{r}}{\partial \rho} = \hat{e}_\rho \qquad \frac{\partial \dot{\vec{r}}}{\partial \dot{\varphi}} = \frac{\partial \vec{r}}{\partial \varphi}$$

$$\frac{d}{dt}\frac{\partial \vec{r}}{\partial \rho} = \frac{d}{dt}\hat{e}_\rho = \dot{\varphi}\,\hat{e}_\varphi = \frac{\partial \dot{\vec{r}}}{\partial \rho}$$

$$\frac{d}{dt}\frac{\partial \vec{r}}{\partial \varphi} = \frac{d}{dt}\rho\,\hat{e}_\varphi = \dot{\rho}\,\hat{e}_\varphi - \rho\dot{\varphi}\,\hat{e}_\rho = \frac{\partial \dot{\vec{r}}}{\partial \varphi}$$

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m(\dot{\rho}^2 + \rho^2\dot{\varphi}^2) \qquad V(\rho, \varphi), \quad L = T - V$$

$j = 1$ $\quad q_1 = \rho$,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_1} - \frac{\partial L}{\partial q_1} = 0$$

$$\frac{d}{dt}m\dot{\rho} - \left(m\rho\dot{\varphi} - \frac{\partial V}{\partial \rho}\right) = 0$$

$$m\ddot{\rho} = m\rho\dot{\varphi}^2 - \frac{\partial V}{\partial \rho}$$

$j = 2$ $\quad q_2 = \varphi$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\varphi}} - \frac{\partial L}{\partial \varphi} = 0$$

$$\frac{d}{dt}m\rho^2\dot{\varphi} - \left(-\frac{\partial V}{\partial \varphi}\right) = 0$$

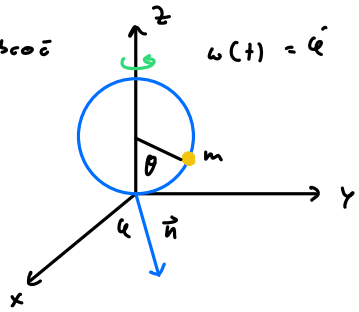
$$2m\rho\dot{\rho}\dot{\varphi} + m\rho^2\ddot{\varphi} = -\frac{\partial V}{\partial \varphi}$$

Naj V ne bo odvisen od φ, $V(\rho)$ $\Rightarrow$ $\frac{d}{dt}(\underbrace{m\rho^2\dot{\varphi}}_{\text{konst.}}) = 0$

- ohranitev vrtilne količine
- ohranjena količina $\quad m\rho^2\dot{\varphi} = l_0 \Rightarrow \dot{\varphi} = \frac{l_0}{m\rho^2}$

$$m\ddot{\rho} = m\rho\frac{l_0^2}{m^2\rho^4} - \frac{\partial V}{\partial \rho} = -\frac{\partial V_{eff}}{\partial \rho} \qquad V_{eff} = V(\rho) - \frac{l_0^2}{2m\rho^2}$$

③ Obroč



$\omega(t) = \dot{\varphi}$

$$\left.\begin{aligned} s &= R\theta \\ q_1 &= \varphi \\ q_2 &= \theta \end{aligned}\right\} \vec{r}(q)$$

$$\begin{aligned} x &= R\sin\theta\cos\varphi \\ y &= R\sin\theta\sin\varphi \\ z &= R - R\cos\theta \\ \dot{x} &= -R\dot{\varphi}\sin\theta\sin\varphi + R\dot{\theta}\cos\theta\cos\varphi \\ \dot{y} &= R\dot{\varphi}\sin\theta\cos\varphi + R\dot{\theta}\cos\theta\sin\varphi \\ \dot{z} &= R\dot{\theta}\sin\theta \end{aligned}$$

$$T = \frac{1}{2}mv^2 = \frac{1}{2}mR^2(\dot{\theta}^2 + \dot{\varphi}^2\sin^2\theta)$$

$$V = -mgR\cos\theta + V_0 = mgz$$

$$L = T - V$$

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\varphi}} - \underbrace{\frac{\partial L}{\partial\varphi}}_{0} = 0 \quad \Rightarrow \quad \frac{\partial L}{\partial\dot{\varphi}} = \text{konst.} = mR^2\sin\theta\,\dot{\varphi}$$

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}} - \frac{\partial L}{\partial\theta} = 0$$

$$\frac{d}{dt}(mR^2\dot{\theta}) + \frac{\partial V_{ef}}{\partial\theta} = 0 \qquad V_{ef} = mgR - mgR\theta - \frac{1}{2}mR^2\omega^2\sin^2\theta$$

$$mR^2\ddot{\theta} = -\frac{\partial V_{ef}}{\partial\theta}$$

Lastnosti Lagrangeove funkcije

- Lagrangeova funkcija je nedoločena do časovnega odvoda poljubne funkcije

$$L(\underline{q}, \dot{\underline{q}}, t) \qquad q_j = 1, 2, \dots, n$$

$$F(q_1, \dots, q_n, t) = F(\underline{q}, t)$$

$$L' = L + \frac{d}{dt}F$$

$$\frac{d}{dt}\frac{\partial L'}{\partial\dot{q}_j} - \frac{\partial L'}{\partial q_j} = \frac{d}{dt}\frac{\partial L}{\partial\dot{q}_j} - \frac{\partial L}{\partial q_j} + \underbrace{\frac{d}{dt}\frac{\partial}{\partial\dot{q}_j}\Big(\underbrace{\sum_i \frac{\partial F}{\partial q_i}\overset{\frac{dq_i}{dt}}{\overset{\parallel}{\dot{q}_i}} + \frac{\partial F}{\partial t}}_{\frac{dF}{dt}}\Big)}_{\frac{\partial F}{\partial q_j}} - \frac{\partial}{\partial q_j}\frac{d}{dt}F = 0$$

$$\Rightarrow \frac{d}{dt}\frac{\partial L'}{\partial\dot{q}_j} - \frac{\partial L'}{\partial q_j} = 0 \quad \Leftrightarrow \quad \frac{d}{dt}\frac{\partial L}{\partial\dot{q}_j} - \frac{\partial L}{\partial q_j} = 0$$

- Mehanska podobnost

$$L' = cL \qquad c \in \mathbb{R}$$

E-L za L' je enak za L.

- Naj bo potencialna energija homogena funkcija koordinat

$$\vec{r}_i \to \alpha\vec{r}_i \qquad V(\alpha\vec{r}_1, \alpha\vec{r}_2, \dots, \alpha\vec{r}_n) = \alpha^k V(\vec{r}_1, \vec{r}_2, \dots, \vec{r}_n) \qquad k \in \mathbb{R}$$

zamenjamo dolžinsko enoto

$t \to \beta t$ časovna enota

$$\left.\begin{aligned} \vec{v}_i &\to \frac{\alpha}{\beta}\vec{v}_i \\ T &\to \frac{\alpha^2}{\beta^2}T \\ V &\to \alpha^k V \end{aligned}\right\} \quad \text{Naj velja} \quad \left(\frac{\alpha}{\beta}\right)^2 = \alpha^k \qquad \beta = \alpha^{1-\frac{k}{2}}$$

$\Rightarrow L = T - V \rightarrow d^k L = CL \qquad \left(\frac{t'}{t}\right) = \left(\frac{\ell'}{\ell}\right)^{1-\frac{1}{2}k} \qquad \left(\frac{v'}{v}\right) = \left(\frac{\ell'}{\ell}\right)^{\frac{1}{2}k} \qquad \left(\frac{E'}{E}\right) = \left(\frac{\ell'}{\ell}\right)^{k}$

Trajektorije v novih enotah so si geometrijsko podobne

Primeri mehanske podobnosti

① $k=2$: harmonska sila $\quad V(x) \propto x^2$

$$\frac{t'}{t} = \left(\frac{\ell'}{\ell}\right)^{1-\frac{1}{2}2} = 1$$ nihajni čas je neodvisen od amplitude

② $k=1$ homogeno polje $\quad V(x) \propto x \quad$ npr. $V = mgh$

$$\frac{t'}{t} = \sqrt{\frac{\ell'}{\ell}}$$ prosti pad $\quad t = \sqrt{2gh}$

③ $k=-1$ gravitacija, elektrostatika $\quad V(x) \propto \frac{1}{x}$

$$\frac{t'}{t} = \left(\frac{\ell'}{\ell}\right)^{1-\frac{1}{2}(-1)} = \left(\frac{\ell'}{\ell}\right)^{3/2}$$

$$\left(\frac{t'}{t}\right)^2 = \left(\frac{\ell'}{\ell}\right)^3$$ Keplerjev zakon

④ Virialni teorem

$$2\langle T\rangle = k\langle V\rangle$$

$k=2 \qquad \langle T\rangle = \langle V\rangle$

$k=-1 \qquad 2\langle T\rangle = -\langle V\rangle$

$\langle V\rangle < 0$

:: Ciklične koordinate

Konstante gibanja ali ohranjene količine

$F(\underline{q}, \dot{\underline{q}}, t)$ če $\frac{dF}{dt} = \sum_i \frac{\partial F}{\partial q_i} d\dot{q}_i + \sum_i \frac{\partial F}{\partial \dot{q}_i} d\ddot{q}_i + \frac{\partial F}{\partial t} = 0 \Rightarrow F$ konst. gibanja

$x \Leftrightarrow m\dot{x}$

$q_j \Leftrightarrow p_i$ konjugirani (kanonični) spremenljivki

če $V \neq V(q_i)$, $T \neq T(q_i)$ za neki q_j

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} = \frac{\partial L}{\partial q_j} = 0 \Rightarrow p_j = \frac{\partial L}{\partial \dot{q}_j} = \text{konst.}$$ posplošeni moment

Primer $\quad L = \frac{1}{2}m\dot{x}^2 - V(x)$ ✗

$= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - V(x)$ ✓

$$\frac{\partial L}{\partial \dot{y}} = \frac{1}{2}\, 2m\dot{y} = 0$$

∴ Hamiltonova funkcija

$$H = \sum_j \frac{\partial L}{\partial \dot{q}_j}\dot{q}_j - L$$

po vseh posplošenih koord.

$$\frac{dH}{dt} = \sum_{j=1}^{n}\left(\frac{\partial L}{\partial \dot{q}_j}\ddot{q}_j + \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j}\right)\dot{q}_j - \frac{\partial L}{\partial q_j}\frac{\partial q_j}{\partial t} - \frac{\partial L}{\partial \dot{q}_j}\ddot{q}_j\right) - \frac{\partial L}{\partial t} = -\frac{\partial L}{\partial t}$$

kinetična energija

$$T = \sum_{i=1}^{N} \frac{1}{2} m_i \dot{\vec{r}}_i^{\,2} = \sum_{i=1}^{N} \frac{1}{2} m_i \sum_{j,k=1}^{n} \frac{\partial \vec{r}_i}{\partial q_j} \dot{q}_j \cdot \frac{\partial \vec{r}_i}{\partial q_k} \dot{q}_k = \sum_{j,k=1}^{n} \frac{1}{2} w_{jk}(q_1, \dots, q_n)\, \dot{q}_j \dot{q}_k$$

$$w_{jk} = \sum_{i=1}^{N} m_i \frac{\partial \vec{r}_i}{\partial q_j} \cdot \frac{\partial \vec{r}_i}{\partial q_k} = w_{kj}$$

! $$H = \sum_{j=1}^{n} \underbrace{\frac{\partial L}{\partial \dot{q}_j} \dot{q}_j}_{2T} - L = 2T - T + U = T + U = E = \text{konst.}$$

$L = T - U \quad U = U(q_1, \dots, q_n)$! $\quad T = T(q, \dot{q})$

::: Emmy Noether izrek

$q_i(t) \longrightarrow Q_i(s, t)$ $\qquad$ če $s \to 0$ $\quad q_j \to q_j$

Lagrangeova funkcija naj ne bo odvisna od s.

$x \longrightarrow X = x + sa \qquad s \in \mathbb{R}$

enoparametrična transformacija koordinat

$$\frac{\partial L}{\partial s} = \frac{d}{ds} L(Q_i, \dot{Q}_i, t) \overset{\text{naj bo}}{=} 0$$

$$\frac{d}{ds} L = \sum_i \frac{\partial L}{\partial Q_i} \frac{\partial Q_i}{\partial s} + \sum_i \frac{\partial L}{\partial \dot{Q}_i} \frac{\partial \dot{Q}_i}{\partial s} \overset{s \to 0}{=} \sum_i \frac{\partial L}{\partial q_i} \frac{\partial Q_i}{\partial s}\Big|_{s=0} + \sum_i \frac{\partial L}{\partial \dot{q}_i} \frac{\partial \dot{Q}_i}{\partial s}\Big|_{s=0} =$$

$$= \sum_i \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} \frac{\partial Q_i}{\partial s}\Big|_{s=0} + \sum_i \frac{\partial L}{\partial \dot{q}_i} \frac{d}{dt} \frac{\partial Q_i}{\partial s}\Big|_{s=0} = \frac{d}{dt} \sum_i \underbrace{\frac{\partial L}{\partial \dot{q}_i}}_{p_i} \frac{\partial Q_i}{\partial s}\Big|_{s=0} = 0$$

$$\sum_i p_i \frac{\partial Q_i}{\partial s}\Big|_{s=0} = \text{konst.}$$

Primeri

1. $\vec{r}_i \longrightarrow \vec{Q}_i(s, t) = \vec{r}_i + s\vec{n}$

$$\sum_i \vec{p}_i \vec{n} = \left(\sum_i p_i\right) \vec{n} \quad \text{za } \forall \vec{n}$$

celotna gibalna količina $\qquad$ homogenost prostora

2. $\vec{Q}_i = \vec{r}_i + \varphi\, \vec{n} \times \vec{r}_i$

$$\sum_i \vec{p}_i \frac{\partial \vec{Q}_i}{\partial \varphi} = \sum_i \vec{p}_i \cdot (\vec{n} \times \vec{r}_i) = \left(\sum_i (\vec{r}_i \times \vec{p}_i)\right) \cdot \vec{n} = \text{konst.}$$

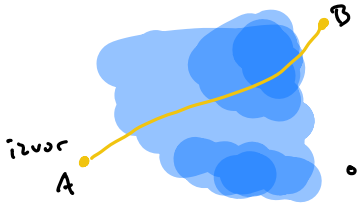
izotropnost prostora

3. L ni odvisna od časa

$t \to t + t_0$: $H = \text{konst}$ $\qquad$ homogenost glede časa

Hamiltonov princip minimalne akcije

Fermatov princip

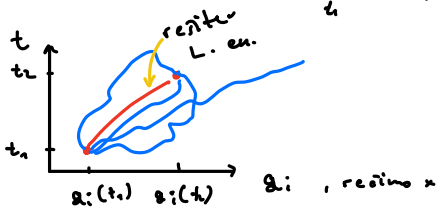


$$T = \int_A^B dt = \text{čas minimalen} = \int \frac{1}{c} \frac{c}{v} \frac{ds}{dt}\, dt = \frac{1}{c} \int n(s)\, ds$$

optično sredstvo $\qquad n(\vec{r}) = \frac{c_0}{c}$

Akcija – funkcional, preslika funkcije v $\mathbb{R}$

$$S = S[\{q_i\},\{\dot q_i\}] = \int_{t_1}^{t_2} L(\underline{q}, \dot{\underline{q}}, t)\, dt$$



$\underline{q}(t) = ?$

S je funkcija poti

Izpeljava Euler-Lagrangeevih enačb

$L = L(\underline{q}, \dot{\underline{q}}, t)$

$q_i(t) \to q_i(t) + \delta q_i(t)$

par parts $\to \frac{\partial L}{\partial \dot q_i}\delta q_i\Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{d}{dt}\frac{\partial L}{\partial \dot q_i}\delta q_i\, dt$

$$S = \int_{t_1}^{t_2} L\, dt \qquad \delta S = \int_{t_1}^{t_2}\Big(\sum_i \frac{\partial L}{\partial q_i}\delta q_i + \sum_i \frac{\partial L}{\partial \dot q_i}\delta \dot q_i\Big)dt =$$

$$= \sum_i \int_{t_1}^{t_2}\underbrace{\Big(\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot q_i}\Big)}_{\Rightarrow 0}\delta q_i\, dt = 0 \qquad \forall \delta q$$

Euler-Lagrangeova enačba

$$\frac{d}{dt}\frac{\partial L}{\partial \dot q_i} - \frac{\partial L}{\partial q_i} = 0$$

Hamiltonov princip

Spomnimo se

$$L' = L + \frac{d}{dt}F(\underline{q}, t)$$

$$S' = \int_{t_1}^{t_2} L'\, dt = \int_{t_1}^{t_2} L\, dt + \int_{t_1}^{t_2}\frac{d}{dt}F\, dt = S + F\Big|_{t_1}^{t_2}$$

$$\delta S' = \delta S + \delta F\Big|_{t_1}^{t_2}$$

$$\delta F\Big|_{t_1}^{t_2} = \sum_i \frac{\partial F}{\partial q_i}\delta q_i(t)\Big|_{t_1}^{t_2} = 0$$

$$\delta S' = \delta S$$

Enodimenzionalni primeri

$\underline{q}$ ena dimenzija, en delec

$q(t)$

$L = \frac{1}{2}\omega(q)\dot q^2 - V(q)$

E = konstanta

$E = H = T + V = \frac{1}{2}\omega(q)\dot q^2 + V(q)$

$$\dot q = \sqrt{\frac{2(E-V)}{\omega(q)}}$$

$$\int_{t_1}^{t_2} dt = \pm \int_{q_1}^{q_2} \sqrt{\frac{w(q)}{2(E-U(q))}}\, dq \quad \Rightarrow \quad q(t)$$

Primer

$q = x \qquad w(q) = m$

$E = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} k x^2$

$x(0) = x_0$

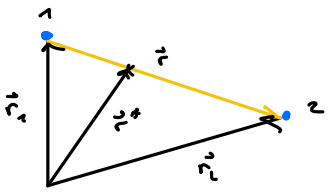
$\dot{x}(0) = 0$

$\omega^2 = \frac{k}{m}$

$$\omega \int_0^t dt = -\int_{x_0}^{x} \frac{dx}{\sqrt{x_0^2 - x^2}} = -\arcsin \frac{x(t)}{x_0} - \frac{\pi}{2}$$

$$x_0 \sin\left(\frac{\pi}{2} - \omega t\right) = x(t)$$

Problem dveh teles



$T = \frac{1}{2} m_1 |\dot{\vec{r}}_1|^2 + \frac{1}{2} m_2 |\dot{\vec{r}}_2|^2$

$M = m_1 + m_2 \qquad m = \frac{m_1 m_2}{M}$ reducirana masa

$\vec{r} = \vec{r}_2 - \vec{r}_1$

$\vec{r}_T = \frac{1}{M}(m_1 \vec{r}_1 + m_2 \vec{r}_2)$

$\vec{r}_1 = \vec{r}_T - \frac{m_2}{M} \vec{r}$

$\vec{r}_2 = \vec{r}_T + \frac{m_1}{M} \vec{r}$

$$T = \frac{1}{2} M |\dot{\vec{r}}_T|^2 + \frac{1}{2} m |\dot{\vec{r}}|^2$$

$$L = \frac{1}{2} M |\dot{\vec{r}}_T|^2 + \frac{1}{2} m |\dot{\vec{r}}|^2 - U(\vec{r}, \dot{\vec{r}})$$

Iščemo $\vec{r}_{1,2}(t)$; $\vec{r}_{1,2}(0)$, $\dot{\vec{r}}_{1,2}(0)$

Konstante gibanja

- $\vec{r}_T$ je ciklična $\Rightarrow \frac{d}{dt} \frac{\partial L}{\partial \dot{\vec{r}}_T} = 0 \Rightarrow M \dot{\vec{r}}_T = p_T = \text{konst.}$

  $\Rightarrow \vec{r}_T(t) = \vec{r}_T(0) + \dot{\vec{r}}_T(0)\, t$

- $L = \frac{1}{2} m |\dot{\vec{r}}|^2 - U(\vec{r}, \vec{r}')$

  $E = \frac{1}{2} m |\dot{\vec{r}}|^2 + U(\vec{r}, \vec{r}') = \text{konst}$

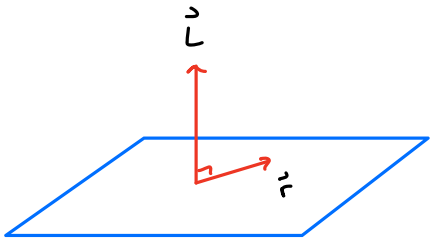
- Sile naj bo centralne

  $U(\vec{r}, \dot{\vec{r}}) = V(\vec{r}) = V(|\vec{r}|) = V(r)$

  $\vec{F} = -\nabla V = -\frac{dV}{dr} \frac{\vec{r}}{r}$

  $\dot{\vec{L}} = \vec{M} = \vec{r} \times \vec{F} = 0 \quad ; \quad \vec{L} = m \vec{r} \times \dot{\vec{r}} = \text{konst.}$

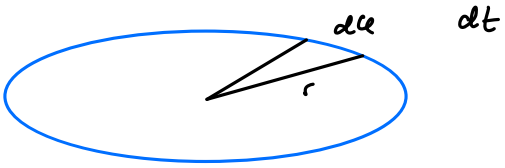
  $\vec{r} \cdot \vec{L} = \vec{r} \cdot (\vec{r} \times m \dot{\vec{r}}) = 0 \Rightarrow$ sistem se giblje po ravnini



V ravnini je ugodno odleti z r, φ

$$L = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\varphi}^2) - V(r)$$

$$p_\varphi = \frac{\partial L}{\partial \dot\varphi} = m r^2 \dot\varphi = \ell = \ell_0 = \text{konst.}$$

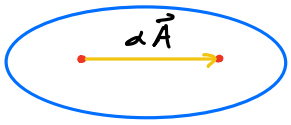


$$\Delta p = \frac{1}{2} r r \Delta\varphi = \frac{1}{2} r^2 \dot\varphi \Delta t \Rightarrow \frac{dp}{dt} = \frac{1}{2} r^2 \dot\varphi = \text{konst}$$

2. Keplerův zákon za $\forall\ V(r)$

Keplerův problém

- Konstanty pohybu: $\vec p_T$ (sys. kol.), E, $\vec\ell$ $(\vec r \cdot \vec\ell = 0)$
- $\vec A$ Laplace – Runge – Lenzův vektor



$$m\ddot{\vec r} = \vec F(\vec r) = f(r)\frac{\vec r}{r} \quad ; \quad f(r) = -\frac{\partial V}{\partial r}$$

$$\dot{\vec p} = m\ddot{\vec r}$$

$$\dot{\vec p} \times \vec L = \frac{m}{r} f(r)\left(\vec r \times (\vec r \times \dot{\vec r})\right) =$$

$$= \frac{m}{r} f(r)\left((\vec r \cdot \dot{\vec r})\vec r - \dot{\vec r} r^2\right)$$

$$\frac{d}{dt} r^2 = 2\vec r \dot{\vec r}$$

$$\frac{d}{dt}(\vec p \times \vec L) = \dot{\vec p} \times \vec L + \vec p \times \underset{0}{\dot{\vec L}} = -\frac{m f(r)}{r}\left(\frac{\dot{\vec r}}{r} - \frac{\dot{\vec r}\vec r}{r^2}\right) = \underbrace{-m f(r)\, r^2}_{\text{konst}} \frac{d}{dt}\left(\frac{\vec r}{r}\right)$$

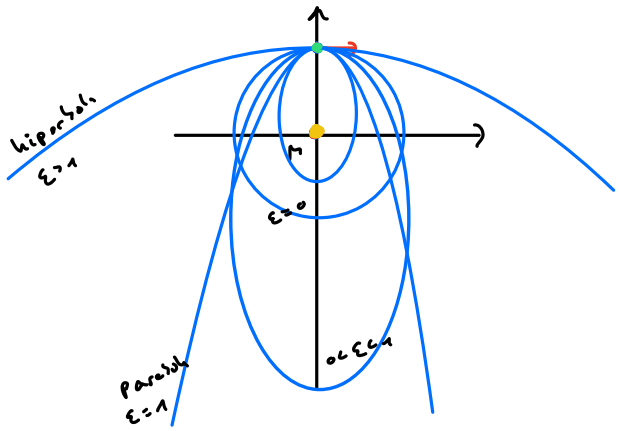
$$-f(r)\, r^2 = \text{konst} = k$$

$$\frac{d}{dt}\left(\vec p \times \vec L - mk\frac{\vec r}{r}\right) = 0 = \frac{d}{dt}\vec A$$

$$\boxed{\vec A = \vec p \times \vec L - mk\frac{\vec r}{r} = \text{konst}}$$

$$\vec A = \vec A_0 = \vec A(0) \qquad \vec p(t),\ \vec r(t) \qquad \vec p(0),\ \vec r(0)$$

$$\vec A \cdot \vec L = (\vec p \times \vec L)\cdot \vec L - \frac{mk}{r}\vec r \cdot \vec L = 0$$



$$\vec A \cdot \vec r = A_0 r \cos\varphi = \underbrace{\vec r \cdot (\vec p \times \vec L)}_{\vec L \cdot \underbrace{(\vec r \times \vec p)}_{\vec L}} - mkr = L^2 = \ell_0^2$$

$$r(A_0 \cos\varphi + mk) = \ell_0^2$$

$$r = \frac{\ell_0^2}{A_0 \cos\varphi + mk} = \frac{r_0}{1 + \varepsilon\cos\varphi} \qquad \varepsilon = \frac{A_0}{mk} \quad r_0 = \frac{\ell_0^2}{mk}$$

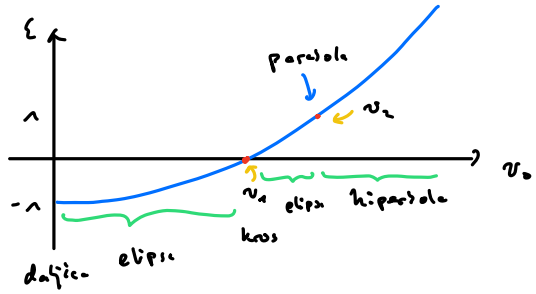
Ekscentričnost

Orbita při Keplerově mu problému

$$\vec A(0) = \vec A_0 = m\vec v \times (\vec r \times m\vec v) - mk\frac{\vec r}{r} \quad ; \quad v(0) = v_0 \qquad r(0) = R$$

$$\doteq |A_0| = m^2 R v_0^2 - m^2 M G \qquad k = mMG$$

$$\varepsilon = \frac{A_0}{mk} = \frac{m^2 R v_0^2 - m^2 M G}{m^2 M G} = \frac{R v_0^2}{MG} - 1$$



- $\varepsilon = -1$ prosti pad
- $\varepsilon = 0$ $\quad \frac{R v_0^2}{MG} = 1 \quad v_0 = v_1 = \sqrt{\frac{MG}{R}}$ prva kozmička brzina
- $\varepsilon = 1$ $\quad v_0 = v_2 = \sqrt{2\frac{MG}{R}}$ druga kozmička brzina
- $\varepsilon > 1$ $\quad v_0 > v_2$

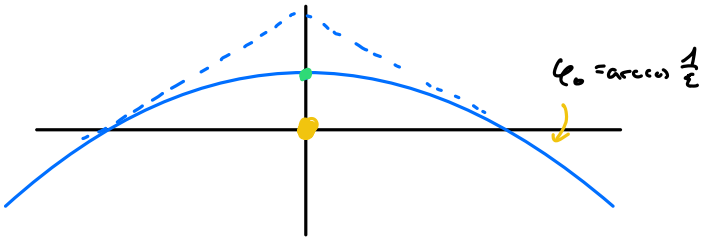
vezano

sipanje

$$r(\varphi) = \frac{r_0}{1+\varepsilon\cos\varphi}$$

$$1 + \varepsilon\cos\varphi = 0$$

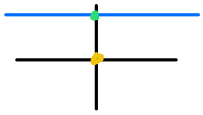
$$\cos\varphi_0 = \frac{1}{\varepsilon} < 1$$



- $\varepsilon \to \infty$ $\quad v_0 \to \infty$

$|A_0| \to \infty$

$\varphi_0 \to \frac{\pi}{2}$



Energija

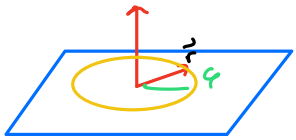
$$E = \frac{1}{2} m v^2(t) - \frac{mMG}{r(t)} = \text{konst} = E_0$$

$$E_0 = \frac{1}{2} m v_0^2 - \frac{mMG}{R}$$

Virialni teorem $\quad \langle T \rangle = -\frac{1}{2}\langle V \rangle$

Keplerjev problem II.

$V(r)$ je splošno

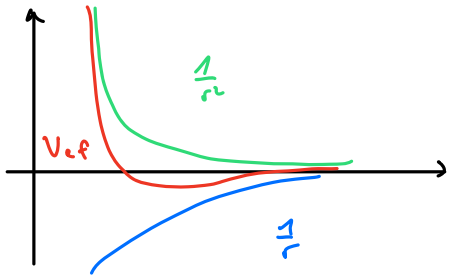


$$L = \frac{1}{2} m (\dot{r}^2 + r^2\dot{\varphi}^2) - V(r)$$

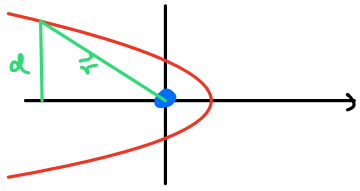
$$p_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = m r^2 \dot{\varphi} = l = l_0 = \text{konst.}$$

$$\dot{\varphi} = \frac{l_0}{m r^2}$$

$$V_{ef}(r) = V(r) + \underbrace{\frac{l_0^2}{2 m r^2}}_{\text{centrifugalni}}$$



$V_{eff} = \frac{1}{2} m v^2$ ← tangentna hitrost



$l = m v d$

$v = \frac{l}{md} \Rightarrow V_{eff} = \frac{l^2}{2md^2}$

$\dot{r} = 0$
$v = r\dot{\varphi}$

$E = \frac{1}{2} m \dot{r}^2 + V_{eff}(r) \qquad V_{eff} = V + \frac{l^2}{2mr^2}$

$\dot{r} = \frac{dr}{dt} = \pm\sqrt{\frac{2}{m}}\sqrt{E - V_{eff}(r)}$

$\int_{t_0}^{t} dt = \pm\sqrt{\frac{m}{2}} \int_{r_0}^{r(t)} \frac{dr}{\sqrt{E - V_{eff}(r)}} \Rightarrow t = f(r) \Rightarrow r(t) = f^{-1}(t)$

Vemo da $mr^2\dot{\varphi} = l$

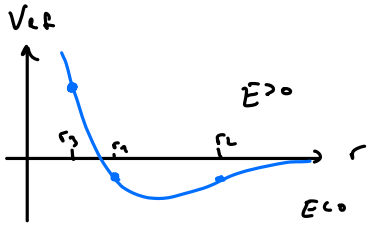
$\dot{\varphi} = \frac{d\varphi}{dt} = \frac{l}{mr^2}$ oz. $dt = \frac{mr^2}{l} d\varphi$

zato je $\int_{\varphi_0}^{\varphi} d\varphi = \pm\sqrt{\frac{l}{2m}} \int_{r_0}^{r(t)} \frac{l\,dr}{r^2\sqrt{E - V_{eff}}} \Rightarrow \varphi = g(r) \Rightarrow r(\varphi) = g^{-1}(\varphi)$

Klasifikacija orbit

konst E, $l \neq 0$

① $V(r) = -\frac{|\alpha|}{r}$

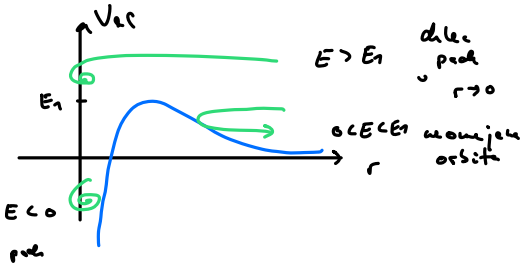


hiperbola prost delec

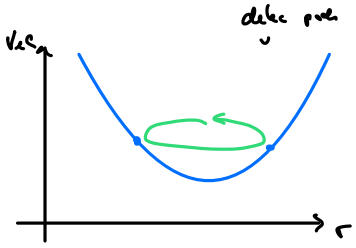
elipsa vezan delec

② $V(r) = -\frac{|\alpha|}{r^\beta}$

- $0 < \beta < 2$ podobno kot pri $\beta = 1$
- $\beta > 2$ $r \to V_{eff} < 0$



③ $V(r) = |\alpha| r^\beta$ $\beta > 0$



pri $\beta = 2$ orbite elipse

Rešljivi primeri

$\beta = -2, -1, 2$ $\qquad V = \alpha r^\beta$

$\beta = -6, -4, -2, 1, 4, 6$ $\qquad$ eliptični integrali

Binetova zveza

$$\frac{1}{r} = u(\varphi)$$

$$m\ddot{r} - m r^2 \dot{\varphi}^2 = f(r) \quad \text{in} \quad \dot{\varphi} = \frac{\ell}{m r^2}$$

$$\Rightarrow \quad \frac{d^2}{d\varphi^2} u + u = F(u) \qquad F(u) = -\frac{m}{\ell^2} \frac{1}{u^2} f\left(\frac{1}{u}\right)$$

Ze vključujev problem $\quad f\left(\frac{1}{u}\right) = -\alpha u^2$

$$\Rightarrow \quad \frac{d^2}{d\varphi^2} u + u = \frac{\alpha m}{\ell^2}$$

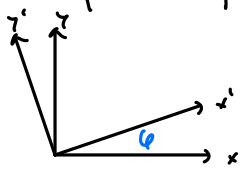
$$u = A\cos\varphi + B\sin\varphi + C = \frac{1}{r(\varphi)}$$

Eulerjevi koti

Togo telo $\quad |\vec{r}_i - \vec{r}_j| = \text{konst.}$

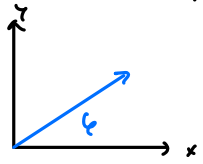
Rotacija koordinatnega sistema (telesa)

ⓐ Pasivna rotacija (rotacija sistema)



$$\begin{bmatrix} \cos\varphi \\ -\sin\varphi \end{bmatrix} = \begin{bmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

ⓑ Aktivna rotacija (vrtenje telesa)

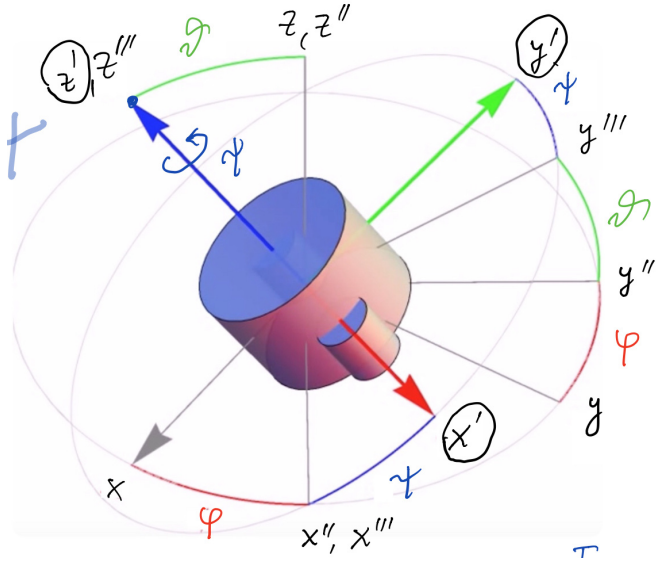


$$\begin{bmatrix} \cos\varphi \\ \sin\varphi \end{bmatrix} = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Rotacija je ortogonalna transformacija

Nekaka last. determinant

$$\det AB = \det A \det B$$
$$\det R^T = \det R$$
$$\det R^T R = \det R^T \det R = \det I = 1 \quad \Rightarrow \quad \det R = \pm 1$$
$$\det R^T = \det R^{-1} = \frac{1}{\det R}$$



Precesija rotacija okoli z za φ

$$T(\varphi) = \begin{bmatrix} c & s & 0 \\ -s & c & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Nutacija rotacija okoli x'' za θ

$$U(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c & s \\ 0 & -s & c \end{bmatrix}$$

Rotacija rotacija okoli z'' za ψ

$$V(\psi) = \begin{bmatrix} c & s & 0 \\ -s & c & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Eulerjev izrek o rotacijah

Naj bo R rotacijska matrika, ki transformira $\vec{r} \to \vec{r}'$, $R\vec{r} = \vec{r}'$

Eulerjev izrek: za $\forall R$ $\exists$ $\vec{u} \neq 0$ $\ni$ $R\vec{u} = \vec{u}$, torej $\exists$ lastni vektor z lastno vrednostjo 1.

Dokaz

$$R^{-1} = R^T \quad \text{oz} \quad RR^T = R^TR = I$$

Determinanta je ± 1:

$$1 = \det RR^T = \det R^T \det R = (\det R)^2 \Rightarrow \det R = \pm 1$$

Če je $\det R = -1$ je to nepravilna rotacija, sestavljena iz refleksije in rotacije $\vec{r} \to -\vec{r}$

Pokažimo torej zlahka, da $\exists$ $\vec{u}$,

$$R\vec{u} = \vec{u}$$

$$(R - I)\vec{u} = 0 \Rightarrow \det(R - I) = 0$$

$$\det(-A) = \det\left[\begin{bmatrix} -1 & & \\ & -1 & \\ & & -1 \end{bmatrix} A\right] = \det\begin{bmatrix} -1 & & \\ & -1 & \\ & & -1 \end{bmatrix} \det A = -\det A$$

torej $\det R^{-1} = 1$

in $\det(R - I) = \det(R - I)^T = \det(R^T - I) = \det(R^{-1} - R^{-1}R) =$

$$= \det R^{-1}(I - R) = \det R^{-1} \det(-(R - I)) = -\det(R - I)$$

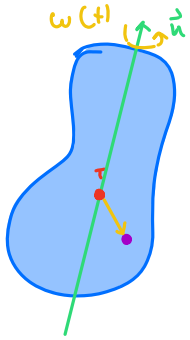
$$\Rightarrow \det(R - I) = 0$$

Torej $\det(R - \lambda I) = 0 \Rightarrow \lambda = 1$

$$\Rightarrow \quad (R - I)\vec{u} = 0 \quad \Leftrightarrow \quad R\vec{u} = \vec{u}$$

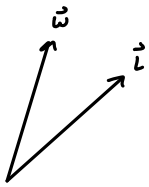
Dokazali smo, da je vsaka ortogonalna matrika rotacija

Togo telo z eno nepremično točko



$$\vec{\omega}(t) = \omega(t)\,\vec{u}(t)$$

$$\frac{d\vec{r}}{dt} = \left(\frac{d\vec{r}}{dt}\right)_{\text{rein.}} + \vec{\omega}\times\vec{r}$$



$$\vec{v}(t) = \vec{\omega}\times\vec{r} = \vec{\omega}(t)\times\vec{r}(t)$$

$$\vec{L} = \sum_{i=1}^{N}\vec{l}_i = \sum_{i=1}^{N} m_i\,(\vec{r}_i\times\vec{v}_i) = \sum_{i=1}^{N} m_i\,(\vec{r}_i\times(\vec{\omega}\times\vec{r}_i)) = \underline{J}\,\vec{\omega}$$

$$\vec{L} = \sum_{i=1}^{N} m_i\left((\vec{r}_i\cdot\vec{r}_i)\vec{\omega} - (\vec{r}_i\cdot\vec{\omega})\,\vec{r}_i\right) = \underline{J}\,\vec{\omega}$$

vztrajnostni moment

$$\rho(\vec{r}) = \lim_{\Delta V\to 0}\frac{\Delta m}{\Delta V} = \frac{dm}{dV}$$

$$M = \sum_{i=1}^{N} m_i \to \int \rho\, d^3r$$

$$J_{\alpha\beta} = \int \rho(\vec{r})\left(r^2\delta_{\alpha\beta} - x_\alpha x_\beta\right) d^3r$$

tenzor

$$\underline{J} = \int \rho(\vec{r})\begin{bmatrix} y^2+z^2 & -xy & -xz \\ -yx & x^2+z^2 & -yz \\ -zx & -zy & x^2+y^2 \end{bmatrix} d^3r$$

$$\vec{L}' = R\vec{L} \quad \text{in} \quad \vec{\omega}' = R\vec{\omega}$$

$$\vec{L} = \underline{J}\,\vec{\omega} \quad \Rightarrow \quad \vec{L}' = \underline{J}'\,\vec{\omega}' \;;\quad \vec{L}' = R\vec{L} = R\underline{J}\,\vec{\omega} = R\underline{J}\cdot R^{-1}R\vec{\omega}$$

$$= R\underline{J}R^{-1}\,\vec{\omega}' = \underline{J}'\vec{\omega}'$$

Lastne vrednosti vektorjev

$$\underline{J}\,\hat{e}_\alpha = \lambda_\alpha\,\hat{e}_\alpha \qquad \lambda_\alpha = J_{1,2,3} \text{ ali } J_{x,y,z}$$

$$\lambda_\alpha = J_{\alpha\alpha} \geq 0$$

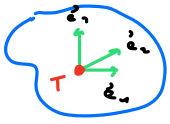
Če $J_\alpha \neq J_\beta \Rightarrow \vec{e}_\alpha\cdot\vec{e}_\beta = 0$;

Energija vrtečega se togega telesa

$V = 0$

$$T = \frac{1}{2}\sum_{i=1}^{N} m_i v_i^2 = \frac{1}{2}\sum_{i=1}^{N} m_i\,(\vec{\omega}\times\vec{r}_i)\cdot\vec{v}_i =$$

$$= \frac{1}{2}\sum_{i=1}^{N} m_i\,\vec{\omega}\cdot(\vec{r}_i\times\vec{v}_i) = \frac{1}{2}\vec{\omega}\cdot\vec{L} = \frac{1}{2}\vec{\omega}\cdot\underline{J}\cdot\vec{\omega}$$



V lastnem sistemu

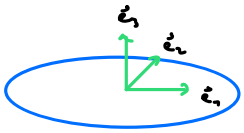
$$\underline{J} = \begin{bmatrix} J_1 & & \\ & J_2 & \\ & & J_3 \end{bmatrix} \qquad \vec{\omega} = \sum_{\alpha=1}^{3}\omega_\alpha\,\vec{e}_\alpha$$

$$\underline{J}\,\vec{\omega} = \sum_{i=1}^{N} J_{\alpha\alpha}\,\omega_\alpha\,\vec{e}_\alpha$$

$$\frac{1}{2}\vec{\omega}\cdot \underline{J}\vec{\omega} = \frac{1}{2}\sum_{i=1}^{3} J_{\alpha\alpha}\omega_\alpha^2 \geq 0 \qquad \forall \omega$$

$$\Rightarrow \quad J_{\alpha\alpha} \geq 0 \quad \text{pozitivno definitna}$$

Frisbee



Enačbe gibanja

v inercialnem sistemu $\quad \frac{d\vec{L}}{dt} = \left(\frac{d\vec{L}}{dt}\right)_{\text{rein.}} + \vec{\omega}\times\vec{L} = \vec{M}$

$$\vec{L} = \underline{J}\vec{\omega} = \sum_{\alpha=1}^{3} J_{\alpha\alpha}\omega_\alpha \vec{e}_\alpha = \begin{bmatrix} J_1\omega_1 \\ J_2\omega_2 \\ J_3\omega_3 \end{bmatrix}$$

v lastnem sistemu vrtenjna za telesa

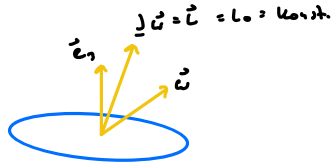
$$\left(\frac{d\vec{L}}{dt}\right)_{\text{rein.}} = \sum_{\alpha=1}^{3} J_{\alpha\alpha}\dot{\omega}_\alpha \vec{e}_\alpha$$

$$\vec{\omega}\times\vec{L} = \begin{bmatrix}\omega_1\\ \omega_2\\ \omega_3\end{bmatrix} \times \begin{bmatrix} J_1\omega_1 \\ J_2\omega_2 \\ J_3\omega_3\end{bmatrix} = \left((J_3-J_2)\omega_2\omega_3,\ (J_1-J_3)\omega_1\omega_3,\ (J_2-J_1)\omega_1\omega_2\right)$$

$$\left.\begin{aligned} J_1\dot{\omega}_1 - (J_2-J_3)\omega_2\omega_3 &= M_1 \\ J_2\dot{\omega}_2 - (J_3-J_1)\omega_1\omega_3 &= M_2 \\ J_3\dot{\omega}_3 - (J_1-J_2)\omega_1\omega_2 &= M_3 \end{aligned}\right\} \text{Eulerjeve enačbe}$$

- Najenostavnejši primer: $\vec{M} = 0$

$$\frac{d\vec{L}}{dt} = 0 \quad \Rightarrow \quad \vec{L} = \text{konst}$$



napaj me gre

- Osno simetrično telo

ⓐ $J_1 = J_2 = J_3 = J$ krogla, kocka

$\Rightarrow \dot{\omega}_1 = 0 \quad \Rightarrow \quad \vec{\omega} = \vec{\omega}_0 = \text{konst} \qquad \vec{L} = \underline{J}\vec{\omega} = J\vec{\omega}$ ni trescija

ⓑ $J_1 = J_2 \neq J_3$ frisbee

$$J_3\dot{\omega}_3 = 0 \quad \Rightarrow \quad \omega_3 = \omega_0 = \text{konst}$$

$$J_1\dot{\omega}_1 - (J_1-J_3)\omega_2\omega_0 = 0$$

$$J_1\dot{\omega}_2 - (J_3-J_1)\omega_1\omega_0 = 0$$

$$J_1\dot{\omega}_1 = (J_1-J_3)\omega_2\omega_0$$

$$J_1\dot{\omega}_2 = (J_3-J_1)\omega_1\omega_0$$

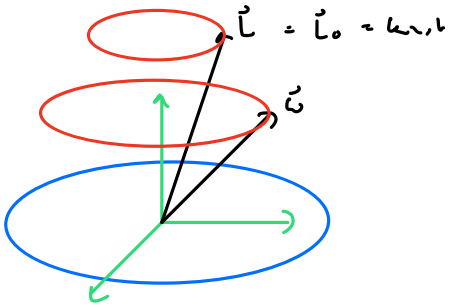
$$\frac{\dot{\omega}_1}{\dot{\omega}_2} = -\frac{\omega_2}{\omega_1} \quad \Rightarrow \quad \begin{aligned}\dot{\omega}_1 &= -\Omega\,\omega_2 \\ \dot{\omega}_2 &= \Omega\,\omega_1\end{aligned}$$

$$\Rightarrow \quad \ddot{\omega}_1 + \Omega^2\omega_1 = 0 \qquad \Omega = \frac{J_3-J_1}{J_1}\omega_0$$

wobble, trescija

$$\Rightarrow \quad \begin{aligned}\omega_1(t) &= A\cos(\Omega t+\delta) \\ \omega_2(t) &= A\sin(\Omega t+\delta)\end{aligned} \qquad \omega_1^2+\omega_2^2 = A^2$$

$$\vec{\omega} = \begin{pmatrix}\omega_1(t)\\ \omega_2(t)\\ \omega_0\end{pmatrix}; \qquad |\vec{\omega}|^2 = A^2+\omega_0^2$$



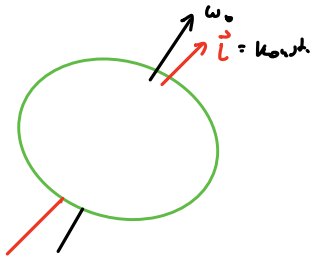
$$J_3 = \frac{1}{2} m R^2$$

$$J_1 = J_2 = \dots = \frac{1}{4} m R^2 \qquad d \ll R$$

$$\Omega = \frac{J_3 - J_1}{J_1}\omega_0 = \frac{\frac{1}{2} - \frac{1}{4}}{\frac{1}{4}}\omega_0 = \omega_0 \qquad \text{v lastnem sistemu}$$

$2\omega_0$ v mirujočem sistemu

ⓒ Tresenje Zemlje

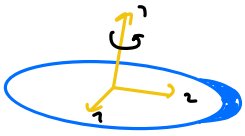


$$J_3 \neq J_1 = J_2$$

$$\frac{J_3 - J_1}{J_1} = 0{,}00327 \; ; \quad \omega = \frac{2\pi}{\text{dan}} \quad \Rightarrow \quad \Omega = \frac{2\pi}{300\,\text{dni}}$$

Eksperimentalno $\Omega_0 = \frac{2\pi}{433\,\text{dni}}$

Efekt Dzanibekova – stabilnost vrtenke



$$J_1 \neq J_2 \neq J_3$$

$$J_1 \dot{\omega}_1 - (J_2 - J_3)\,\omega_2 \omega_3 = 0$$
$$J_2 \dot{\omega}_2 - (J_3 - J_1)\,\omega_3 \omega_1 = 0$$
$$J_3 \dot{\omega}_3 - (J_1 - J_2)\,\omega_1 \omega_2 = 0$$

naj se vrti okoli tretje osi

$\omega_1 = \omega_2 = 0 \qquad \omega_3 = \Omega$

$\vec{\omega} \to \vec{\omega} + \vec{\eta} \qquad |\vec{\eta}| \ll |\vec{\omega}_0|$ frekvenco

$$\omega_1 = \eta_1 \qquad \omega_2 = \eta_2 \qquad \omega_3 = \Omega + \eta_3$$

$$J_1 \dot{\eta}_1 - (J_2 - J_3)\,\eta_2(\Omega + \eta_3) = 0$$
$$J_2 \dot{\eta}_2 - (J_3 - J_1)\,\eta_1(\Omega + \eta_3) = 0$$
$$J_3(\dot{\Omega} + \dot{\eta}_3) - (J_1 - J_2)\,\eta_1 \eta_2 = 0$$

$\dot{\Omega} = 0 \qquad \eta_i \eta_j$ majhno

$$J_1 \dot{\eta}_1 - (J_2 - J_3)\,\eta_2 \Omega = 0$$
$$J_2 \dot{\eta}_2 - (J_3 - J_1)\,\eta_1 \Omega = 0$$
$$J_3 \dot{\eta}_3 = 0 \quad \Rightarrow \quad \dot{\eta}_3 = \text{konst.}$$

$$J_1 \ddot{\eta}_1 = \frac{(J_2 - J_3)(J_3 - J_1)}{J_2}\Omega^2 \eta_1$$

$$\ddot{\eta}_1 - c\,\eta_1 = 0 \qquad c = \frac{(J_2-J_3)(J_3-J_1)}{J_1 J_2}\,\Omega^2$$

Rešitve

- $c<0$; $c = -\omega_c^2$

$$\eta_1 = A\cos(\sqrt{|c|}\,t + \delta)$$

$$\eta_2 = \frac{J_1}{\Omega(J_2-J_3)}\,\dot{\eta}_1 = B\sin(\sqrt{|c|}\,t + \delta)$$

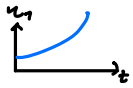
$$\frac{J_1}{\Omega(J_2-J_3)}\,\frac{\Omega}{\sqrt{J_1 J_2}}\,\sqrt{(J_2-J_3)(J_3-J_1)} = \sqrt{\frac{J_1}{J_2}\,\frac{J_3-J_1}{J_2-J_3}} = -\frac{B}{A}$$

- $c>0$ $\qquad \eta_1 = A_1 e^{\pm\sqrt{c}\,t}$

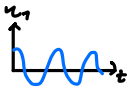
$$(J_2-J_3)(J_3-J_1)$$

Pojem Dzanibekova

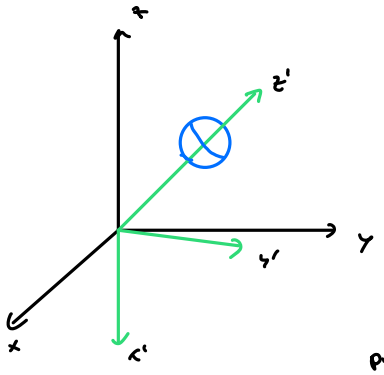
(i) $J_1 < J_3 < J_2$ ali $J_2 < J_3 < J_1$ je nestabilno



(ii) $J_3 \lessgtr (J_1, J_2)$

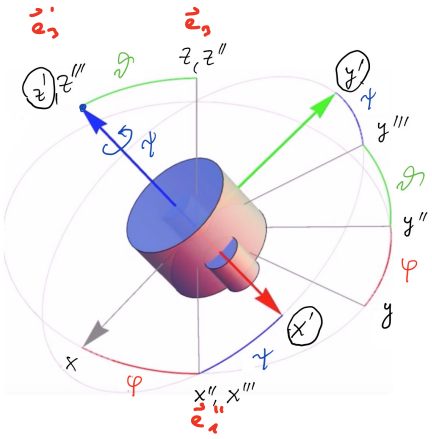


Vrtavka



(φ, θ, ψ) Eulerjevi koti

v odvisnosti od časa



$$\vec{\omega} = \underbrace{\dot{\varphi}\,\vec{e}_3}_{\text{precesija}} + \underbrace{\dot{\theta}\,\vec{e}_1''}_{\text{nutacija}} + \underbrace{\dot{\psi}\,\vec{e}_3'}_{\text{rotacija}}$$

$$\vec{e}_3 = \begin{pmatrix}0\\0\\1\end{pmatrix} \qquad \vec{e}_1'' = \begin{pmatrix}1\\0\\0\end{pmatrix}$$

$$\begin{pmatrix}\cos\varphi & \sin\varphi & 0\\ -\sin\varphi & \cos\varphi & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\1\end{pmatrix} \qquad \begin{pmatrix}1 & 0 & 0\\ 0 & \cos\theta & \sin\theta\\ 0 & -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix}$$

$$\begin{pmatrix}1 & 0 & 0\\ 0 & \cos\theta & \sin\theta\\ 0 & -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\\sin\theta\\\cos\theta\end{pmatrix} \qquad \begin{pmatrix}\cos\psi & \sin\psi & 0\\ -\sin\psi & \cos\psi & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}\cos\psi\\-\sin\psi\\0\end{pmatrix}$$ v bazi x', y', z'

$$\begin{pmatrix}\cos\psi & \sin\psi & 0\\ -\sin\psi & \cos\psi & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}0\\\sin\theta\\\cos\theta\end{pmatrix} = \begin{pmatrix}\sin\psi\sin\theta\\\cos\psi\sin\theta\\\cos\theta\end{pmatrix}$$ v bazi x', y', z'

$$\vec{e}_3' = \vec{e}_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$\vec{\omega} = \dot{\varphi}\,\vec{e}_3 + \dot{\theta}\,\vec{e}_1'' + \dot{\psi}\,\vec{e}_3'$$

$$\vec{\omega} = \begin{pmatrix}\sin\theta\sin\psi\,\dot{\varphi} + \cos\psi\,\dot{\theta}\\ \sin\theta\cos\psi\,\dot{\varphi} - \sin\psi\,\dot{\theta}\\ \cos\theta\,\dot{\varphi} + \dot{\psi}\end{pmatrix}$$

2 manjka

5.5.

Zadatki

$$\vec{\omega} = \begin{pmatrix} \sin\theta \sin\psi\, \dot\varphi + \cos\psi\, \dot\theta \\ \sin\theta \cos\psi\, \dot\varphi - \sin\psi\, \dot\theta \\ \cos\theta\, \dot\varphi + \dot\psi \end{pmatrix}$$

$V = mgl\cos\theta$

$L = T - V$

$T = \frac{1}{2} J_1 (\omega_1^2 + \omega_2^2) + \frac{1}{2} J_3 \omega_3^2 \qquad J_1 = J_2$

$$\omega_1^2 + \omega_2^2 = \sin^2\theta \sin^2\psi\, \dot\varphi^2 + \sin^2\theta \cos^2\psi\, \dot\varphi^2 + (\cos^2\psi + \sin^2\psi)\dot\theta + 2(\sin\theta \sin\psi \cos\psi - \sin\theta \cos\psi \sin\psi)\, \dot\varphi \dot\theta =$$

$$= \sin^2\theta\, \dot\varphi^2 + \dot\theta^2$$

$$\omega_3^2 = (\cos\theta\, \dot\varphi + \dot\psi)^2$$

$$L = \frac{1}{2} J_1 (\sin^2\theta\, \dot\varphi^2 + \dot\theta^2) + \frac{1}{2} J_3 (\cos\theta\, \dot\varphi + \dot\psi)^2 - mgl\cos\theta$$

$L = L(\theta, \dot\varphi, \dot\theta, \dot\psi) \qquad \varphi, \psi$ sta ciklična

Konstante gibanja

- Energija $\quad E = T + V = \text{konst.}$
- $p_\psi = \frac{\partial L}{\partial \dot\psi} = J_3 (\cos\theta\, \dot\varphi + \dot\psi) = \text{konst.} = J_3 \omega_3 := J_1 a \qquad a = \frac{J_3}{J_1}\omega_3 \quad J_1 = J_2, \; J_3$
- $p_\varphi = J_1 \sin^2\theta\, \dot\varphi + J_3 (\cos\theta\, \dot\varphi + \dot\psi)\cos\theta = \text{konst}$

$$= (J_1 \sin^2\theta + J_3 \cos^2\theta)\dot\varphi + J_3 \dot\psi \cos\theta := J_1 b$$

$$J_3 \dot\psi = J_1 a - J_3 \cos\theta\, \dot\varphi \quad \longrightarrow p_\varphi$$

$$(J_1 \sin^2\theta + J_3 \cos^2\theta)\dot\varphi + J_1 a \cos\theta - J_3 \cos^2\theta\, \dot\varphi = J_1 b$$

$$\dot\varphi = \frac{J_1 b - J_1 a \cos\theta}{J_1 \sin^2\theta} = \frac{b - a\cos\theta}{\sin^2\theta}$$

$$E = \frac{1}{2} J_1 (\sin^2\theta\, \dot\varphi^2 + \dot\theta^2) + \frac{1}{2} J_3 (\cos\theta\, \dot\varphi + \dot\psi)^2 + mgl\cos\theta$$

$$E = \frac{1}{2} J_1 \left(\frac{(b - a\cos\theta)^2}{\sin^2\theta} + \dot\theta^2 \right) + \frac{1}{2} J_3 \omega_3^2 + mgl\cos\theta = \text{konst.}$$

$$= \tilde{E}(\theta, \dot\theta) + \frac{1}{2} J_3 \omega_3^2$$

$$\tilde{E}_0 = \tilde{E} = \frac{1}{2} J_1 \dot\theta^2 + \tilde{V}(\theta) \quad ; \quad \tilde{V}(\theta) = \frac{1}{2} \frac{(b - a\cos\theta)^2}{\sin^2\theta} + mgl\cos\theta$$

$$\frac{d\theta}{dt} = \pm\sqrt{\frac{2}{J_1}(\tilde{E}_0 - \tilde{V})}$$

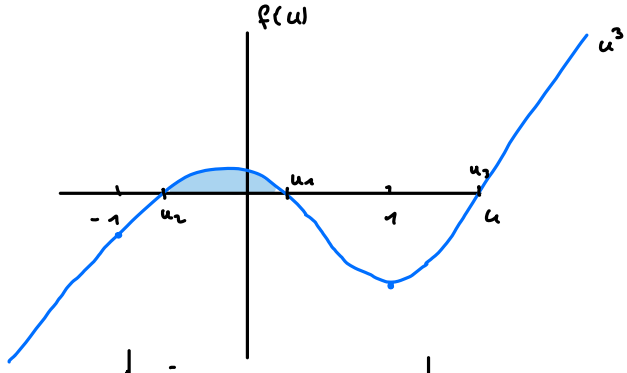
$$\frac{d\theta}{\sqrt{\tilde{E}_0 - \tilde{V}}} = \pm\sqrt{\frac{2}{J_1}}\, dt$$

$u = \cos\theta \qquad du = -\sin\theta\, d\theta$

$$f(u) = (1-u^2)(\alpha - \beta u) - (s - au)^2 \qquad \alpha = \frac{2E_0}{J_1} \qquad \beta = \frac{2mgl}{J_1}$$

$$t = \int_{u(0)}^{u(t)} \frac{du}{\sqrt{f(u)}}$$

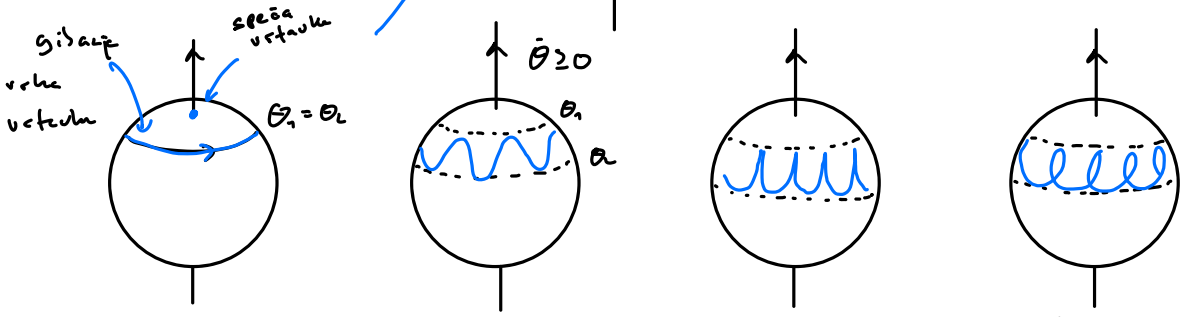
Risanje $f(u)$



Limite

$u \to \infty \qquad f \to \beta u^3$

$f(\pm 1) = -(s \mp a)^2 \le 0$



$\dot\theta = 0$ — $\omega_p = \dot\varphi = \text{konst}$ — Enakomerna precesija

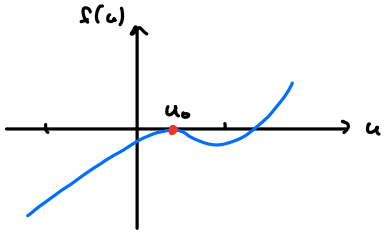
$\dot\varphi(0) = 0 \quad \dot\theta(0) = 0$ — Nutacija

$\dot\varphi(0) = 0 \quad \dot\theta(0) = 0$ — Spuščam vrtavko

$\dot\varphi(0) < 0$

<u>Enakomerna precesija</u>

$\dot\theta = 0 \qquad \dot\varphi = \text{konst.}$



$$f(u_0) = (1-u_0^2)(\alpha - \beta u_0) - (s - a u_0)^2 = 0$$

$$\frac{df}{du_0} = 0 = -2u_0(\alpha - \beta u_0) - \beta(1-u_0^2) + 2a(s - a u_0)$$

$$\dot\varphi = \frac{s - au}{1-u^2}$$

$$0 = f(u_0) = (1-u_0^2)(\alpha - \beta u_0) - (1-u_0^2)^2 \dot\varphi^2$$

Naj bo $u_0 \neq \pm 1$

$$\frac{df}{du_0} = 0 = -2u_0\dot\varphi^2(1-u_0^2) - \beta(1-u_0^2) + 2a\dot\varphi(1-u_0^2)$$

$$\frac{\beta}{2} = a\dot\varphi - \dot\varphi^2 u_0 \qquad u_0 \to \dot\varphi$$

$$\frac{mgl}{J_1} = \dot\varphi \frac{J_3\omega_3}{J_1} - \dot\varphi u_0$$

$$A\dot\varphi^2 + B\dot\varphi + C = 0$$

$-J_1\cos\theta_0 \qquad J_3\omega_3 \qquad -mgl$

Kdaj imamo rešitev

$$B^2 - 4AC \geq 0$$

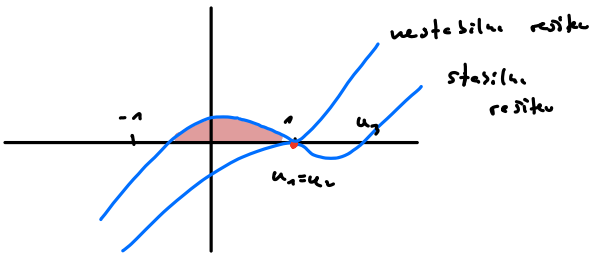
$$J_3^2 \omega_3^2 \geq 4 J_1 mgl \cos\theta$$ Vrtavka mora biti dovolj hitra $\omega_3 \geq \tilde{\omega}_3$

Specia vrtavka

$\theta = 0$ oz $u = 1$

$\dot{\theta} = 0$ $f(u) = 0$

$a = b$ $\alpha = \beta$



$$f(u) = (1-u^2)\,\alpha\,(1-u) - a^2(1-u)^2 = 0$$

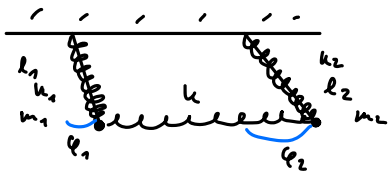
$$u_1 = u_2 = 1$$

$$u_3: \quad u_3 = \frac{a^2}{\alpha} - 1 \geq 1 \quad : \quad a^2 \geq 2\alpha$$

$$\ldots \quad T \geq 2\frac{J_1}{J_3} V$$

$$\frac{1}{2} J_3 \omega_3^2 \geq 2 \frac{J_1}{J_3} mgl$$

Majhna nihanja



$$T = \frac{1}{2} m_1 (\dot{l}_1^2 + l_1^2 \dot{\varphi}_1^2) + \frac{1}{2} m_2 (\dot{l}_2^2 + l_2^2 \dot{\varphi}_2^2)$$

$$V = m_1 g l_1 (1 - \cos\varphi_1) + m_2 g l_2 (1 - \cos\varphi_2) + \frac{1}{2} k_1 (l_1 - l)^2 + \frac{1}{2} k_2 (l_2 - l)^2 + \frac{1}{2} k_3 (\ldots)^2$$

$$L = T - V = \ldots \quad f(\underline{q}, \dot{\underline{q}}, t)$$

① mirovne lega (ravnovesja)

② majhne amplitude (okoli ravnovesja)

$$L = T - V = \frac{1}{2} \sum_{ij} \omega_{ij}(\underline{q})\, \dot{q}_i \dot{q}_j - V(\underline{q})$$

$\omega_{ij} = \omega_{ji}$ matrika

① Stabilne lege

$$q_i(t) = q_i^0 \quad ; \quad \dot{q}_i(t) = 0 \Big|_{t=0}$$

$$-\frac{\partial V}{\partial q_i}\Big|_{\underline{q}^0} = 0 \quad ; \quad \underline{q}^0 = (q_1^0, q_2^0, \ldots, q_n^0)$$

$V(\underline{q}) \geq V(\underline{q}^0)$ dovolj je da najdemo lokalni minimum

$$V(\underline{q}) = V(\underline{q}^0) + \underbrace{\sum_i \frac{\partial V}{\partial q_i}\Big|_{\underline{q}_i^0}}_{0} \eta_i + \underbrace{\frac{1}{2}\sum_{ij} \frac{\partial^2 V}{\partial q_i \partial q_j}\Big|_{\underline{q}_i^0}}_{>0} \eta_i \eta_j \dots$$ Taylorjev razvoj

stabilna lega

$$\eta_i = q_i - q_i^0$$

$$w_{ij}(\underline{q}) = w_{ij}(\underline{q}^0) + \underbrace{\sum_l \frac{\partial w_{ij}}{\partial q_l} \eta_l}_{} + \dots$$

$$w_{ij}\, \dot{q}_i\, \dot{q}_j \longrightarrow \underbrace{\eta_l\, \dot{\eta}_i\, \dot{\eta}_j}_{O(\eta^3)}$$ zanemarimo

$\dot{q}_i \to \dot{\eta}_i$, $\dot{q}_j \to \dot{\eta}_j$

Matriki

$$T_{ij} = w_{ij}(\underline{q}^0) \qquad V_{ij} = \frac{\partial^2 V}{\partial q_i \partial q_j}$$

$$\tilde{L} = \frac{1}{2}\sum_{ij}\left(T_{ij}\dot{\eta}_i\dot{\eta}_j + V_{ij}\eta_i\eta_j\right)$$

$L = T - V$

$\tilde{L} = T - (V - V_0)$

Primer: $L = \frac{1}{2}\overset{T_{ij}}{m}\dot{x}^2 - \frac{1}{2}\overset{V_{ij}}{k}x^2$

$$\frac{d}{dt}\frac{\partial \tilde{L}}{\partial \dot{\eta}_i} - \frac{\partial \tilde{L}}{\partial \eta_i} = 0 \qquad i = 1, 2, \dots, n$$

sklopljen sistem linearnih enačb

Notacija

- vektorji: $\underline{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$ stolpec

  $\underline{a}^T = (a_1\ a_2 \dots a_n)$ vrstica

- matrika $\underline{\underline{A}} = \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ \vdots & & \ddots & \vdots \\ A_{n1} & & \dots & A_{nn} \end{bmatrix}$ $\qquad \underline{\underline{A}}^T$

- delovanje matrike na vektor

  $(\underline{\underline{A}}\,\underline{a})_i = \sum_{j=1}^{n} A_{ij} a_j$ rezultat je stolpec

  $\underline{a}^T \underline{\underline{A}} = (a_1\ a_2 \dots a_n)\begin{bmatrix} A_{11} & \dots \\ \vdots & \ddots \\ & & A_{nn} \end{bmatrix} = \sum_j \underline{a}_j^T \underline{\underline{A}}_{ij}$ rezultat je vrstica

- $\underline{\underline{a}}\,\underline{u} = \underline{b}$

$$\sum_j T_{ij}\ddot{\eta}_j + \sum_j V_{ij}\eta_j = 0 \qquad \text{za } i = 1, 2, \dots, n$$

$$\underline{\eta} = \begin{pmatrix} \eta_1 \\ \vdots \\ \eta_n \end{pmatrix}$$

$$\tilde{L} = \frac{1}{2}\left(\dot{\underline{\eta}}^T \underline{\underline{T}}\,\dot{\underline{\eta}} - \eta^T \underline{\underline{V}}\,\eta\right)$$

Normalne koordinate

$$\tilde{L} = \frac{1}{2} \dot{\underline{y}} \underline{\underline{T}} \dot{\underline{y}} - \frac{1}{2} \underline{y} \underline{\underline{V}} \underline{y}$$

$$y_i(t) = \sum_k \alpha_k(t) \, \underline{a}_{ki} = \sum_k \underline{a}_{ki} \, \alpha_k(t)$$

$$\underline{y} = \underline{\underline{A}} \, \underline{\alpha} \qquad \text{in} \qquad \underline{y}^T = \underline{\alpha}^T \underline{\underline{A}}^T$$

$$\tilde{L} = \frac{1}{2} \dot{\underline{\alpha}}^T \underline{\underline{A}}^T \underline{\underline{T}} \underline{\underline{A}} \, \dot{\underline{\alpha}} - \frac{1}{2} \underline{\alpha}^T \underline{\underline{A}}^T \underline{\underline{V}} \underline{\underline{A}} \, \underline{\alpha} = \frac{1}{2} \dot{\underline{\alpha}}^T \underline{\alpha} - \frac{1}{2} \underline{\alpha}^T \underline{\underline{\Lambda}} \, \underline{\alpha}$$

$$\tilde{L} = \frac{1}{2} \sum_k \left(\dot{\alpha}_k^2 - \omega_k^2 \alpha_k^2 \right)$$

$$\tilde{H} = \frac{1}{2} \sum_k \left(\dot{\alpha}_k^2 + \omega_k^2 \alpha_k^2 \right) = H_k$$

Lastna nihanja

$$y_i(t) = \alpha a_i e^{i\omega t} \qquad \left(e^{\pm |\omega| t} \right)$$

Lažilna lega

$$\sum_j V_{ij} a_j - \omega^2 \sum_j T_{ij} a_j = 0$$

$$\underline{\underline{V}}\, \underline{a} = \omega^2 \underline{\underline{T}}\, \underline{a} = \lambda \underline{\underline{T}}\, \underline{a}$$

Posplošena diagonalizacija

$$\lambda = \omega^2 \geq 0 \qquad \omega \in \mathbb{R}$$

Poseben primer $\underline{\underline{T}} = T \underline{\underline{I}}$

$$(\underline{\underline{V}} - \lambda \underline{\underline{T}})\, \underline{a} = 0 \rightarrow (\underline{\underline{V}} - \underbrace{\lambda T}_{\tilde{\lambda}} \underline{\underline{I}})\, \underline{a} = 0$$

$$\forall k \qquad (\underline{\underline{V}} - \tilde{\lambda}_k)\, \underline{a}_k = 0$$

$$\forall l \qquad \underline{a}_l^T (\underline{\underline{V}} - \tilde{\lambda}_l) = 0$$

$$\underline{\underline{A}} = (\underbrace{\underline{a}_1}_{\text{stolpec}}, \underline{a}_2, \dots \underline{a}_n) = \begin{pmatrix} a_{11} & \cdots \\ \vdots & \end{pmatrix}$$

$$\underline{\underline{A}}^T \underline{\underline{V}}\, \underline{\underline{A}} - \underline{\underline{A}}^T \underline{\underline{A}}\, \underline{\underline{\tilde{\Lambda}}} = 0 \qquad \underline{\underline{\tilde{\Lambda}}} = \begin{pmatrix} \tilde{\lambda}_1 & & \\ & \ddots & \\ & & \tilde{\lambda}_n \end{pmatrix}$$

$$A^T A = A A^T = I \Rightarrow A^{-1} = A^T$$

$$A^T \underline{\underline{V}}\, A = \tilde{\Lambda}$$

prostorska sum diagonalizirala

Splošnejši primer

$$\underline{\underline{V}}\, \underline{a} = \lambda \underline{\underline{T}}\, \underline{a} \qquad \underline{\underline{T}} \text{ ni diag.}$$

$$\underline{\underline{V}}\, \underline{a}_k = \lambda_k \underline{\underline{T}}\, \underline{a}_k \qquad | \underline{a}_l^T$$

$$\underline{a}_l^T \underline{\underline{V}} = \lambda_l \underline{a}_l^T \underline{\underline{T}} \qquad | \underline{a}_k$$

$$\left.\begin{aligned} \underline{a}_l^T \underline{\underline{V}}\, \underline{a}_k &= \lambda_k \underline{a}_l^T \underline{\underline{T}}\, a_k \\ \underline{a}_l^T \underline{\underline{V}}\, \underline{a}_k &= \lambda_l \underline{a}_l^T \underline{\underline{T}}\, a_k \end{aligned}\right\} -$$

$$0 = (\lambda_k - \lambda_l)\, \underline{a}_l^T \underline{\underline{T}}\, \underline{a}_k$$

$$\text{Če } \lambda_k \neq \lambda_l \Rightarrow \underline{a}_l^T \underline{\underline{T}}\, \underline{a}_k = 0$$

$$\underline{\underline{A}}^T \underline{\underline{T}}\, \underline{\underline{A}} = \underline{\underline{I}}$$

$$\underline{\underline{A}}^T \underline{\underline{V}}\, \underline{\underline{A}} = \underline{\underline{A}}^T \underline{\underline{T}}\, \underline{\underline{A}}\, \underline{\underline{\Lambda}} = \underline{\underline{\Lambda}}$$

Hamiltonov formalizem

$L = T - V \qquad L = L(q_i, \dot{q}_i, t)$

$p_i = \frac{\partial L}{\partial \dot{q}_i}$

$\frac{d}{dt} \underbrace{\frac{\partial L}{\partial \dot{q}_i}}_{p_i} - \frac{\partial L}{\partial q_i} = 0$

$\frac{dp_i}{dt} = \dot{p}_i = \frac{\partial L}{\partial q_i}$

$H(q_i, p_i, t) \overset{\text{def.}}{=} \sum_i p_i \dot{q}_i - L = \sum_i \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i - L = T + V$

$q_i \to q_i + dq_i \qquad p_i \to p_i + dp_i$

$dH = \sum_i (\dot{q}_i dp_i + p_i d\dot{q}_i) - \sum_i \left(\frac{\partial L}{\partial q_i} dq_i + \frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i\right) - \frac{\partial L}{\partial t} dt =$

$= \sum_i (\dot{q}_i dp_i - \dot{p}_i dq_i) - \frac{\partial L}{\partial t} dt$

$H(q, p, t)$

$dH = \sum_i \left(\frac{\partial H}{\partial q_i} dq_i + \frac{\partial H}{\partial p_i} dp_i\right) + \frac{\partial H}{\partial t} dt$

$$\boxed{\dot{q}_i = \frac{\partial H}{\partial p_i} \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i} \qquad \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}}$$

$i = 1, \dots, N$ Hamiltonove enačbe

Ham. en. povezuje q in p

$\frac{dH}{dt} = \sum_i \underbrace{\left(\frac{\partial H}{\partial q_i} \dot{q}_i + \frac{\partial H}{\partial p_i} \dot{p}_i\right)}_{=0} + \frac{\partial H}{\partial t} = \frac{\partial H}{\partial t}$

Primeri

① En delec v 1D

$T = \frac{1}{2} m \dot{x}^2 \qquad V = V(x, t)$

$L = T - V = \frac{1}{2} m \dot{x}^2 - V(x, t) \qquad L = L(x, \dot{x}, t)$

E-L : $0 = m\ddot{x} + \frac{\partial V}{\partial x} \Rightarrow F = ma = -\frac{\partial U}{\partial x}$

$H = p\dot{x} - L$

$p = \frac{\partial L}{\partial \dot{x}} = m\dot{x}$ $\qquad$ $\dot{x}$ moramo izraziti s p

$H = m\dot{x}^2 - \frac{1}{2} m \dot{x}^2 + V = \frac{1}{2} m \dot{x}^2 + V(x, t) = \frac{p^2}{2m} + V$

$H = \frac{p^2}{2m} + V = H(x, p, t)$

H enačba

$\dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m}$ ✓ $\qquad \dot{p} = -\frac{\partial H}{\partial x} = -\frac{\partial U}{\partial x}$ ✓ $\qquad \dot{H} = \frac{\partial H}{\partial t} = \frac{\partial U}{\partial t}$

② Delec v 3D

$T = \frac{1}{2} m \dot{\vec{r}}^2 \qquad V = V(\vec{r}, t) \qquad L = T - V$

$$m\ddot{\vec{r}} + \nabla V = 0$$

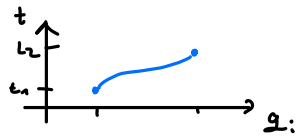
$$H = T + V = \frac{\vec{p}^2}{2m} + V$$

$$\dot{\vec{r}} = \frac{\vec{p}}{m} \qquad \dot{\vec{p}} = -\nabla V$$

Izpeljava Hamiltonovih enačb iz Hamiltonovega principa

Akcija

$$S = \int_{t_1}^{t_2} L(q_i, \dot{q}_i, t)\, dt$$



$$\delta S = \int_{t_1}^{t_2} \delta\left(\sum_i (p_i \dot{q}_i) - H\right) dt = \int_{t_1}^{t_2} \sum_i \left(\delta p_i\, \dot{q}_i + p_i\, \delta \dot{q}_i - \frac{\partial H}{\partial q_i}\delta q_i - \frac{\partial H}{\partial p_i}\delta p_i\right) dt$$

$$= \int_{t_1}^{t_2} \sum_i \left(\left(\dot{q}_i - \frac{\partial H}{\partial p_i}\right)\delta p_i + \left(-\dot{p}_i - \frac{\partial H}{\partial q_i}\right)\delta q_i\right) dt \qquad \forall\, \delta p_i, \delta q_i$$

Per partes

$$\int_{t_1}^{t_2} p_i\, \delta \dot{q}_i = \underbrace{p\, \delta q_i \Big|_{t_1}^{t_2}}_{0} - \int_{t_1}^{t_2} \dot{p}_i\, \delta q_i\, dt$$

$\delta q_i \big|_{t_1, t_2} = 0$ za δp_i takšna, posoja ne potrebujemo

$$\forall\, \delta p_i, \delta q_i : \quad \delta S = 0 \Rightarrow \underline{\qquad} = 0$$

$$\Rightarrow \boxed{\dot{q}_i = \frac{\partial H}{\partial p_i} \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}}$$

Nabit delec v magnetnem polju: Lagrangeev in Hamiltonov formalizem

$$\vec{F} = e(\vec{E} + \vec{v} \times \vec{B})$$

$$F_i = -\frac{\partial V}{\partial q_i} + \frac{d}{dt}\frac{\partial V}{\partial \dot{q}_i} \quad ; \quad \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0$$

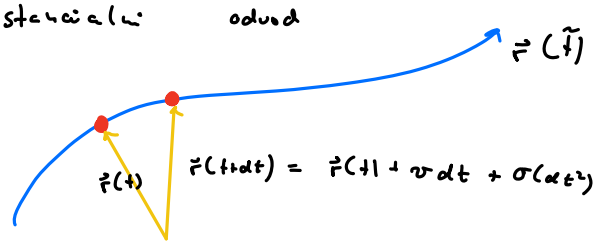
$\nabla \cdot \vec{B} = 0 \Rightarrow \vec{B} = \nabla \times \vec{A}$ $\quad$ $\vec{A}$ je vektorski potencial

$\vec{E} = -\nabla \phi - \frac{\partial \vec{A}}{\partial t}$ $\quad$ ϕ el. potencial

$$\nabla \times \vec{E} = -\frac{dB}{dt}$$

$$\vec{F} = e\left(-\nabla\phi - \frac{\partial \vec{A}}{\partial t} + v \times (\nabla \times \vec{A})\right)$$

Substancialni odvod



$$\vec{r}(t+dt) = \vec{r}(t) + v\, dt + \mathcal{O}(dt^2)$$

$$\vec{A}(\vec{r}, t)$$

$$dA_x = A_x(\vec{r} + \vec{v}\, dt, t + dt) - A_x(\vec{r}, t)$$

$$= \frac{\partial A_x}{\partial x}\underbrace{dx}_{v_x dt} + \frac{\partial A_x}{\partial y}\underbrace{dy}_{v_y dt} + \frac{\partial A_x}{\partial z}\underbrace{dz}_{v_z dt} + \frac{\partial A_x}{\partial t}\, dt$$

$$= \left(v_x \frac{\partial A_x}{\partial x} + v_y \frac{\partial A_x}{\partial y} + v_z \frac{\partial A_x}{\partial z}\right) dt + \frac{\partial A_x}{\partial t}\, dt$$

$$\frac{dA_x}{dt} = \frac{\partial A_x}{\partial t} + \vec{v}\cdot\nabla A_x$$

$$\boxed{\frac{d\vec{A}}{dt} = \frac{\partial \vec{A}}{\partial t} + (\vec{v}\cdot\nabla)\vec{A}}$$ Substancialna odvod

$$\vec{v}\times(\nabla\times\vec{A}) = \nabla(\vec{v}\cdot\vec{A}) - (\vec{v}\cdot\nabla)\vec{A}$$

x: Leva stran $v_y(\nabla\times\vec{A})_z - v_z(\nabla\times\vec{A})_y = v_y\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) - v_z\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)$ ✓ 11

Desna stran $\frac{\partial}{\partial x}(v_x A_x + v_y A_y + v_z A_z) - \left(v_x\frac{\partial}{\partial x} + v_y\frac{\partial}{\partial y} + v_z\frac{\partial}{\partial z}\right)A_x$

$$\vec{F} = e\left(-\nabla\phi - \frac{\partial\vec{A}}{\partial t} + \vec{v}\times(\nabla\times\vec{A})\right)$$

$$\frac{d\vec{A}}{dt} = \frac{\partial\vec{A}}{\partial t} + \nabla(\vec{v}\vec{A}) - \vec{v}\times(\nabla\times\vec{A})$$

$$\vec{F} = e\left(-\nabla(\phi - \vec{v}\cdot\vec{A}) - \frac{d\vec{A}}{dt}\right) =$$
$$= e\left(-\nabla(\phi - \vec{v}\cdot\vec{A}) + \frac{d}{dt}\left(-\nabla_{\vec{v}}(\vec{v}\cdot\vec{A})\right)\right)$$

$\hookrightarrow \left(\frac{\partial}{\partial v_x}, \frac{\partial}{\partial v_y}, \frac{\partial}{\partial v_z}\right)$

Def
$$U = e\phi - e\vec{v}\cdot\vec{A} = U(\vec{r}, \vec{v}, t)$$

$$\vec{F} = -\nabla_{\vec{r}}U + \frac{d}{dt}\nabla_{\vec{v}}U$$

$$\boxed{L = T - U = \frac{1}{2}mv^2 - e\phi + e\vec{v}\cdot\vec{A}}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0 \Rightarrow m\ddot{x} = -e\frac{\partial}{\partial x}\phi + e(\vec{v}\times\vec{B})_x$$

$H = ?$

$H(\vec{r}, \vec{p}, t)$

$p_i = \frac{\partial L}{\partial \dot{q}_i}$

$$\boxed{\vec{p} = \nabla_{\vec{v}}L = m\vec{v} + e\vec{A}}$$

$$H = \vec{p}\cdot\vec{v} - L = mv^2 + e\vec{v}\cdot\vec{A} - \frac{1}{2}mv^2 + e\phi - e\vec{v}\cdot\vec{A} = \frac{1}{2}mv^2 + e\phi$$

Ker je $e\vec{v}\times\vec{B} \perp \vec{v}$ je dela $= 0$
zato ni nobenega magnetnega dela

$v \to p$

$$\boxed{H = \frac{(\vec{p} - e\vec{A})^2}{2m} + e\phi}$$

$$\dot{\vec{r}} = \frac{\partial H}{\partial \vec{p}} \qquad \dot{\vec{p}} = -\frac{\partial H}{\partial \vec{r}} \quad \Big\} \quad m\ddot{\vec{r}} = \vec{F}$$

Poissonovi oklepaji

$f(\underline{q}, \underline{p}, t)$; $\underline{q}_i(t)$, $\underline{p}_i(t) = \frac{\partial L}{\partial \dot{q}_i}$

odvod po času

$$\frac{df}{dt} = \sum_i \left(\frac{\partial f}{\partial q_i} \dot{q}_i + \frac{\partial f}{\partial p_i} \dot{p}_i \right) + \frac{\partial f}{\partial t} = \sum_i \left(\frac{\partial f}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial H}{\partial q_i} \right) + \frac{\partial f}{\partial t} =$$

$$\overset{def.}{=} \{ f, H \} + \frac{\partial f}{\partial t}$$

$$\boxed{\{ f, g \} = \sum_i \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right)}$$ Poissonov oklepaj

$$\frac{df}{dt} = \{ f, H \} + \frac{\partial f}{\partial t}$$

če f ni direktno odvisna od t sledi $\frac{df}{dt} = \{ f, H \}$

če je $\{ f, H \} = 0 \Rightarrow \frac{df}{dt} = 0$ $f = \text{konst}$

$\{ H, H \} = 0$

Lastnosti:

1. Linearnost $\{ f, \lambda g + \mu h \} = \lambda \{ f, g \} + \mu \{ f, h \}$
2. Antisimetričnost $\{ f, g \} = - \{ g, f \}$
3. Produkt $\{ f, g \cdot h \} = \sum_i \left(\frac{\partial f}{\partial q_i} \frac{\partial gh}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial gh}{\partial q_i} \right) = \sum_i \left(\frac{\partial f}{\partial q_i} \left(h \frac{\partial g}{\partial p_i} + g \frac{\partial h}{\partial p_i} \right) - \frac{\partial f}{\partial p_i} \left(h \frac{\partial g}{\partial q_i} + g \frac{\partial h}{\partial q_i} \right) \right) =$
$= g \{ f, h \} + \{ f, g \} h$
4. Jacobijeva identiteta (ciklična enost)
$\{ f, \{ g, h \} \} + \{ g, \{ h, f \} \} + \{ h, \{ f, g \} \} = 0$

Primeri

1. $\{ q_i, q_j \} = 0$
2. $\{ p_i, p_j \} = 0$
3. $\{ q_i, p_j \} = \sum_l \left(\frac{\partial q_i}{\partial q_l} \frac{\partial p_j}{\partial p_l} - \overbrace{\frac{\partial q_i}{\partial p_l} \frac{\partial p_j}{\partial q_l}}^{=0} \right) = \sum_l \delta_{il} \delta_{jl} = \sum_l \delta_{ij}$
$\{ q_i, p_j \} = \delta_{ij}$ kanonično spremenljivki q_i in p_j
4. $\dot{q}_i = \{ q_i, H \}$ $\dot{p}_i = \{ p_i, H \}$
5. Vrtilna količina

$\vec{L} = \vec{r} \times \vec{p}$ $x_i \to q_i$ $i = 1,2,3 \Leftrightarrow x, y, z$

$l_x = y p_z - z p_y$

$l_y = z p_x - x p_z$

$l_z = x p_y - y p_x$

$$\{ l_x, l_y \} = \frac{\partial l_x}{\partial x} \frac{\partial l_y}{\partial p_x} + \frac{\partial l_x}{\partial y} \frac{\partial l_y}{\partial p_y} + \frac{\partial l_x}{\partial z} \frac{\partial l_y}{\partial p_z} - \frac{\partial l_x}{\partial p_x} \frac{\partial l_y}{\partial x} - \frac{\partial l_x}{\partial p_y} \frac{\partial l_y}{\partial y} - \frac{\partial l_x}{\partial p_z} \frac{\partial l_y}{\partial z}$$

$$= 0 + 0 + (-p_y)(-x) - 0 - 0 - (y) p_x = x p_y - y p_x = l_z$$

$\{ l_i, l_j \} = \varepsilon_{ijk} l_k$

x	y	1	z
y	z	1	x
z	x	1	y

$$\{ l^2, l_x \} = \{ l_x^2 + l_y^2 + l_z^2, l_x \} = 0 + \{ l_y^2, l_x \} + \{ l_z^2, l_x \}$$
$$= 0 + 2 l_y \{ l_y, l_x \} + 2 l_z \{ l_z, l_x \} = -2 l_y l_z + 2 l_z l_y = 0$$

6 Laplace Runge Lenzov vektor

$$\vec{A} = \vec{p} \times \vec{L} - k m \frac{\vec{r}}{r}$$

$$\{ l_i , A_j \} = \varepsilon_{ijk} A_k$$

7 $$\{ l_i , q_j \} = \varepsilon_{ijk} q_k$$

$$\{ l_i , p_j \} = \varepsilon_{ijk} p_k$$

$$\{ \vec{p} , \vec{n} \cdot \vec{l} \} = \vec{n} \times \vec{p}$$

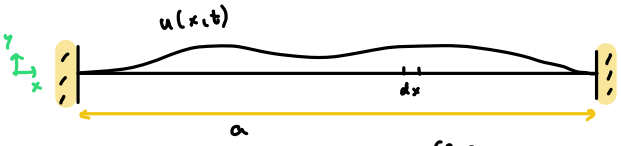
8 $$H = \frac{p^2}{2m} - k \frac{m}{r}$$

$$\{ \vec{A} , H \} = 0$$

$$\{ \vec{l} , H \} = 0 \qquad \text{Konstanta gibanja}$$

$$\{ l , H \} = 0$$

Lagrangeev formalizem za zvezno sredstvo



$$dm = \rho S_0 dx$$

Kinetična energija $W_k = \frac{1}{2} dm \left(\frac{\partial u}{\partial t} \right)^2$ $\qquad u_x = \frac{\partial u}{\partial x} \qquad u_t = \frac{\partial u}{\partial t}$

Potencialna energija (napetostna) $\qquad \frac{F}{S_0} = E_y \frac{\Delta l}{l}$

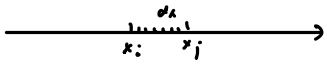
longitudinalno gibanje

$$\frac{\Delta l}{l} = \frac{u}{a} = \frac{F}{S_0 E_y} \qquad k = \frac{S_0 E_y}{a} \qquad m = \rho S_0 a$$

$$\frac{k a^2}{m} = \frac{S_0 E_y a^2}{\rho S_0 a^2} = \frac{E_y}{\rho} = c^2$$

$$W_{pr} = \frac{1}{2} k u_x^2$$

Lagrangeev formalizem



$$u(x_i, t) \rightarrow u(x, t) \qquad \frac{dm}{S_0 dx} = \rho$$

$$\mathcal{L} = \frac{1}{2} \rho u_t^2 - \frac{1}{2} k u_x^2 \qquad \text{Lagrangeova gostota}$$

$$L = \int \mathcal{L} dV = S_0 \int_0^X \mathcal{L}(u, u_x, u_t, x, t) dx$$

koordinate $\sim q_i$ $\quad$ novo $\quad$ hitrost $\sim \dot{q}_i$ $\quad$ novo

Iščemo enačbe gibanja za $q_i \rightarrow u(x,t)$

Akcija

$$S = \int_{t_1}^{t_2} L dt = \overset{\text{plošina}}{S_0} \int_{t_1}^{t_2} \int_0^X \mathcal{L} dx dt$$

$$\mathcal{L} = \mathcal{L}(u, u_x, u_t, t)$$

$u(x,t)$ rešitev, ki jo iščemo

$$u(x,t,\alpha) = u(x,t) + \alpha\, \eta(x,t)$$ poljubna variacija

$$\forall x \quad \eta(x,t_1) = \eta(x,t_2) = 0$$
$$\forall t \quad \eta(0,t) = \eta(X,t) = 0$$

$$\delta S = S_0 \int_{t_1}^{t_2}\int_0^X \left(\frac{\partial \mathcal{L}}{\partial u}\frac{\partial u}{\partial \alpha} + \frac{\partial \mathcal{L}}{\partial u_x}\frac{\partial u_x}{\partial \alpha} + \frac{\partial \mathcal{L}}{\partial u_t}\frac{\partial u_t}{\partial \alpha} \right) \delta\alpha\, dx\, dt$$

$$= \delta\alpha\, S_0 \int_{t_1}^{t_2}\int_0^X \left(\frac{\partial \mathcal{L}}{\partial u}\eta + \frac{\partial \mathcal{L}}{\partial u_x}\frac{\partial \eta}{\partial x} + \frac{\partial \mathcal{L}}{\partial u_t}\underbrace{\frac{\partial u_t}{\partial t}}_{\dot\eta} \right) dx\, dt$$

Velja

$$\int_{t_1}^{t_2} \frac{\partial \mathcal{L}}{\partial u_t}\frac{\partial \eta}{\partial t}\, dt = \eta \underbrace{\frac{\partial \mathcal{L}}{\partial u_t}}_{0}\Bigg|_{t_1}^{t_2} - \int_{t_1}^{t_2} \eta \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial u_t}\right) dt$$

$$\int_0^X \frac{\partial \mathcal{L}}{\partial u_x}\frac{\partial \eta}{\partial x}\, dx = \eta \underbrace{\frac{\partial \mathcal{L}}{\partial u_x}}_{0}\Bigg|_0^X - \int_0^X \eta \frac{d}{dx}\left(\frac{\partial \mathcal{L}}{\partial u_x}\right) dx$$

$$\delta S = \delta\alpha\, S_0 \int_{t_1}^{t_2}\int_0^X \underbrace{\left[\frac{\partial \mathcal{L}}{\partial u} - \frac{d}{dx}\left(\frac{\partial \mathcal{L}}{\partial u_x}\right) - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial u_t}\right)\right]}_{0} \eta\, dx\, dt$$

$$\forall \eta(x,t)$$

$$\boxed{\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial u_t} + \frac{d}{dx}\frac{\partial \mathcal{L}}{\partial u_x} - \frac{\partial \mathcal{L}}{\partial u}}$$ E.L. enačba za zvezno sredstvo

Nazaj na primer strune

$$\frac{d}{dt}\rho u_t - \frac{d}{dx} E_Y u_x = \rho u_{tt} - E_Y u_{xx} = 0$$

$$\frac{1}{c^2} u_{tt} = u_{xx}$$

Energija?

Poskusimo še: impulz?

$$\Pi = \frac{\partial \mathcal{L}}{\partial u_t} = \Pi(x,t)$$
$$H = S_0 \int_0^X (\Pi u_x - \mathcal{L})\, dx = S_0 \int_0^X \mathcal{H}\, dx$$