Elastični valovi

Longitudinalni val: $\boldsymbol{\nabla}\times\mathbf{u}=0$

$c_l^2=\frac{E(1-\sigma)}{\rho(1+\sigma)(1-2\sigma)}=\frac{\lambda+2\mu}{\rho}=\frac{K+4\mu/3}{\rho}$

Transverzalni val: $\boldsymbol{\nabla}\cdot\mathbf{u}=0$

$c_t^2=\frac{E}{2\rho(1+\sigma)}=\frac{\mu}{\rho}<c_l^2$

$\ddot{\mathbf{u}}=-c_t^2\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\mathbf{u}+c_l^2\boldsymbol{\nabla}\boldsymbol{\nabla}\cdot\mathbf{u}$

$p_{ik}=2\rho c_t^2u_{ik}+\rho(c_l^2-2c_t^2)u_{ll}\delta_{ik}$

Odboj in lom:

$k_\parallel^{\text{vpadni}}=k_\parallel^{\text{odbiti}}=k_\parallel^{\text{lomljeni}}$

Hidrodinamika

Idealne tekočine

$\frac{\partial\rho}{\partial t}+\boldsymbol{\nabla}\cdot(\rho\mathbf{v})=0$ (kontinuitetna enačba)

Nestisljive tekočine ($v\ll c$): $\boldsymbol{\nabla}\cdot\mathbf{v}=0$

Brezvrtinčne tekočine: $\boldsymbol{\nabla}\times\mathbf{v}=0\implies\mathbf{v}=\boldsymbol{\nabla}\phi$

Nestisljive in brezvrtinčne tekočine: $\nabla^2\phi=0$

- $\rho\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}=-\boldsymbol{\nabla}p+\mathbf{f}^{(z)}$ (Eulerjeva enačba)

$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}=\frac{\partial\mathbf{v}}{\partial t}+(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v}$ (substancialni odvod)

$\frac{\partial g_i}{\partial t}+\frac{\partial\Pi_{ik}}{\partial x_k}=f_i$

$\Pi_{ik}=\rho v_iv_k+p\delta_{ik},\qquad g_i=\rho v_i$

$\mathrm{d}h=\mathrm{d}\left(\frac{H}{m}\right)=T\,\mathrm{d}s+\frac{\mathrm{d}p}{\rho}=\frac{\mathrm{d}p}{\rho}$ (izentropni tok)

$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}=-\boldsymbol{\nabla}h$

$\frac{\partial(\boldsymbol{\nabla}\times\mathbf{v})}{\partial t}-\boldsymbol{\nabla}\times(\mathbf{v}\times\boldsymbol{\nabla}\times\mathbf{v})=0$

Nestisljive tekočine:

$\frac{\partial\boldsymbol{\omega}}{\partial t}+(\mathbf{v}\cdot\boldsymbol{\nabla})\boldsymbol{\omega}=(\boldsymbol{\omega}\cdot\boldsymbol{\nabla})\mathbf{v}$ (Helmholtzova enačba)

$\boldsymbol{\omega}=\boldsymbol{\nabla}\times\mathbf{v}$

Bernoullijeva enačba $\frac{1}{2}\rho v^2+p=\text{konst}$

Tokovnica: $\frac{\mathrm{d}x}{v_x}=\frac{\mathrm{d}y}{v_y}=\frac{\mathrm{d}z}{v_z}$

Vrtinčnica: krivulja v smeri $\boldsymbol{\nabla}\times\mathbf{v}$

$\frac{v^2}{2}+h+gz=konst.$ na tokovnici in vrtinčnici, $\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}=0$

Nestisljive tekočine:

$\frac{v^2}{2}+\frac{p}{\rho}+gz=konst.$ na tokovnici in vrtinčnici, $\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}=0$

Brezvrtinčne tekočine:

$\frac{v^2}{2}+h+gz+\frac{\partial\phi}{\partial t}=konst.$ povsod

$\hookrightarrow p/\rho$

Cirkulacija

$\Gamma=\oint\mathbf{v}\cdot\mathrm{d}\mathbf{r}$

$\frac{\mathrm{d}\Gamma}{\mathrm{d}t}=0$ (Kelvinov izrek o ohranitvi cirkulacije)

$\mathbf{v}(\mathbf{r})=\frac{1}{4\pi}\int\frac{\boldsymbol{\omega}(\mathbf{r}')\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\,\mathrm{d}^3\mathbf{r}'=\frac{\Gamma}{4\pi}\int\frac{\mathrm{d}\mathbf{l}'\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}$

Dvodimenzionalni, nestisljiv in brezvrtinčni tok

$(v_x,v_y)=\left(\frac{\partial\phi}{\partial x},\frac{\partial\phi}{\partial y}\right)=\left(\frac{\partial\psi}{\partial y},-\frac{\partial\psi}{\partial x}\right)$

$\psi=konst.$ vzdolž tokovnice

$\Phi_V=\int_{\mathbf{r}_1}^{\mathbf{r}_2}\mathbf{v}\cdot\mathrm{d}\hat{\mathbf{n}}=\psi(\mathbf{r}_2)-\psi(\mathbf{r}_1)$

$\nabla^2\psi=0$

Splošno ($\boldsymbol{\nabla}\times\mathbf{v}\neq0$): $\frac{\partial\nabla^2\psi}{\partial t}+\frac{\partial\psi}{\partial y}\frac{\partial\nabla^2\psi}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial\nabla^2\psi}{\partial y}=0$

$z=x+iy$

$w(z)=\phi(x,y)+i\psi(x,y)$

$\frac{\mathrm{d}w}{\mathrm{d}z}=\frac{\mathrm{d}w}{\mathrm{d}x}=\frac{1}{i}\frac{\mathrm{d}w}{\mathrm{d}y}=v_x-iv_y$

Viskozne tekočine

$p_{ik}=-p\delta_{ik}+p'_{ik}$

$p'_{ik}=\eta\left(\frac{\partial v_i}{\partial x_k}+\frac{\partial v_k}{\partial x_i}-\frac{2}{3}\delta_{ik}\frac{\partial v_l}{\partial x_l}\right)+\zeta\delta_{ik}\frac{\partial v_l}{\partial x_l}$

- $\rho\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}=-\boldsymbol{\nabla}p+\eta\nabla^2\mathbf{v}+\left(\frac{\eta}{3}+\zeta\right)\boldsymbol{\nabla}\boldsymbol{\nabla}\cdot\mathbf{v}$ (Navier-Stokes)

Nestisljive tekočine:

$\rho\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}=-\boldsymbol{\nabla}p+\eta\nabla^2\mathbf{v}$

$\nu=\frac{\eta}{\rho}$

$\boldsymbol{\nabla}\times\frac{\partial\mathbf{v}}{\partial t}=\boldsymbol{\nabla}\times(\mathbf{v}\times\boldsymbol{\nabla}\times\mathbf{v})+\eta\boldsymbol{\nabla}\times\nabla^2\mathbf{v}$

$\frac{\partial\boldsymbol{\omega}}{\partial t}+(\mathbf{v}\cdot\boldsymbol{\nabla})\boldsymbol{\omega}=(\boldsymbol{\omega}\cdot\boldsymbol{\nabla})\mathbf{v}+\frac{\eta}{\rho}\nabla^2\boldsymbol{\omega}$

$\rho\frac{\partial v_k}{\partial x_i}\frac{\partial v_i}{\partial x_k}=-\frac{\partial^2p}{\partial x_i^2}$

Disipacija viskoznosti

$\frac{\mathrm{d}E_{\text{disipirana}}}{\mathrm{d}t}=-\int p'_{ik}\frac{1}{2}\left(\frac{\partial v_i}{\partial x_k}+\frac{\partial v_k}{\partial x_i}\right)\mathrm{d}V=$

$=-\frac{1}{2}\eta\int\left(\frac{\partial v_i}{\partial x_k}+\frac{\partial v_k}{\partial x_i}\right)^2\mathrm{d}V$

Hidrodinamična podobnost

$\mathrm{Re}=\frac{ul}{\nu}\sim\frac{|\rho(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v}|}{|\eta\nabla^2\mathbf{v}|}$ (Reynoldsovo število)

$\mathbf{v}=u\mathbf{f}\left(\frac{\mathbf{r}}{l},\mathrm{Re}\right)$ (hitrostna polja z enakim Re lahko preslikamo eno na drugo)

Stokesov približek ($\mathrm{Re}\ll1$ in $\boldsymbol{\nabla}\cdot\mathbf{v}=0$):

$\rho\frac{\partial\mathbf{v}}{\partial t}=-\boldsymbol{\nabla}p+\eta\nabla^2\mathbf{v}$

Obtekanje krogle v Stokesovem približku

$\mathbf{v}=\mathbf{u}-\frac{3R}{4r}[\mathbf{u}+(\hat{\mathbf{n}}\cdot\mathbf{u})\hat{\mathbf{n}}]-\frac{R^3}{4r^3}[\mathbf{u}-3(\hat{\mathbf{n}}\cdot\mathbf{u})\hat{\mathbf{n}}]$

$p=p_0-\eta\frac{3R}{2r^2}\mathbf{u}\cdot\hat{\mathbf{n}}$

$F=6\pi\eta Ru$

Mejna plast (Prandtlove enačbe)

$v_x\frac{\partial v_x}{\partial x}+v_z\frac{\partial v_x}{\partial z}-\nu\frac{\partial^2v_x}{\partial z^2}=u\frac{\mathrm{d}u}{\mathrm{d}x}$

$\frac{\partial v_x}{\partial x}+\frac{\partial v_z}{\partial z}=0$

$v_x(z=0)=v_z(z=0)=0$

$v_x(z\to\infty)=u,\quad v_z(z\to\infty)=0$

Dinamični vzgon (izrek Kutta-Žukovski)

$F=\rho u\Gamma l$

Kartezične koordinat

$\nabla f=\left(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y},\frac{\partial f}{\partial z}\right)$

$\nabla\cdot\vec{v}=\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}+\frac{\partial v_z}{\partial z}$

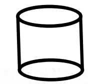
$\vec{v}=\frac{Q}{2\pi r}\hat{e}_r$

Izvir $W(z)=\frac{Q}{2\pi}\ln z$

$\nabla\times\vec{v}=0$ razen v $\vec{0}$.

Vrtinec $W(z)=-i\frac{\Gamma}{2\pi}\ln z$ $\quad\vec{v}=\frac{\Gamma}{2\pi r}\hat{e}_\varphi$ $\quad\vec{\omega}=\Gamma\delta^2(\vec{r})\hat{e}_z$

Cilindrične koordinate



$\mathrm{d}^3r=r\,\mathrm{d}r\,\mathrm{d}\phi\,\mathrm{d}z,$

$$\nabla f=\frac{\partial f}{\partial r}\mathbf{e}_r+\frac{1}{r}\frac{\partial f}{\partial\phi}\mathbf{e}_\phi+\frac{\partial f}{\partial z}\mathbf{e}_z,$$

$$\nabla\cdot\vec{\mathbf{v}}=\frac{1}{r}\frac{\partial(rv_r)}{\partial r}+\frac{1}{r}\frac{\partial v_\phi}{\partial\phi}+\frac{\partial v_z}{\partial z},$$

$$\nabla\times\vec{\mathbf{v}}=\left(\frac{1}{r}\frac{\partial v_z}{\partial\phi}-\frac{\partial v_\phi}{\partial z}\right)\mathbf{e}_r+\left(\frac{\partial v_r}{\partial z}-\frac{\partial v_z}{\partial r}\right)\mathbf{e}_\phi+\frac{1}{r}\left(\frac{\partial(rv_\phi)}{\partial r}-\frac{\partial v_r}{\partial\phi}\right)\mathbf{e}_z,$$

$$\nabla^2f=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2f}{\partial\phi^2}+\frac{\partial^2f}{\partial z^2},$$

$$\nabla^2\vec{\mathbf{v}}=\left(\nabla^2v_r-\frac{v_r}{r^2}-\frac{2}{r^2}\frac{\partial v_\phi}{\partial\phi}\right)\mathbf{e}_r+\left(\nabla^2v_\phi-\frac{v_\phi}{r^2}+\frac{2}{r^2}\frac{\partial v_r}{\partial\phi}\right)\mathbf{e}_\phi+\nabla^2v_z\,\mathbf{e}_z,$$

Sferične koordinate

$\mathrm{d}^3r=r^2\,\mathrm{d}r\,\mathrm{d}(\cos\theta)\,\mathrm{d}\phi,$

$$\nabla f=\frac{\partial f}{\partial r}\mathbf{e}_r+\frac{1}{r}\frac{\partial f}{\partial\theta}\mathbf{e}_\theta+\frac{1}{r\sin\theta}\frac{\partial f}{\partial\phi}\mathbf{e}_\phi,$$

$$\nabla\cdot\vec{\mathbf{v}}=\frac{1}{r^2}\frac{\partial(r^2v_r)}{\partial r}+\frac{1}{r\sin\theta}\left(\frac{\partial(\sin\theta\,v_\theta)}{\partial\theta}+\frac{\partial v_\phi}{\partial\phi}\right),$$

$$\nabla\times\vec{\mathbf{v}}=\frac{1}{r\sin\theta}\left(\frac{\partial(\sin\theta\,v_\phi)}{\partial\theta}-\frac{\partial v_\theta}{\partial\phi}\right)\mathbf{e}_r+\frac{1}{r}\left[\left(\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi}-\frac{\partial(rv_\phi)}{\partial r}\right)\mathbf{e}_\theta+\left(\frac{\partial(rv_\theta)}{\partial r}-\frac{\partial v_r}{\partial\theta}\right)\mathbf{e}_\phi\right],$$

$$\nabla^2f=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right)+\frac{1}{r^2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2f}{\partial\phi^2}\right].$$

Idealne tekočine

Eulerjeva enačba

$$\rho\left(\frac{\partial \vec v}{\partial t} + (\vec v\cdot\nabla)\vec v\right) = -\nabla p + \vec f$$

$\vec f$ ... volumska gostota zun. sile,
npr. $\vec f = \rho \vec g$

Kontinuitetna enačba

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\vec v) = 0$$

$$\frac{\partial g_i}{\partial t} + \frac{\partial}{\partial x_j}\Pi_{ij} = f_i$$

$$\Pi_{ij} = \rho v_i v_j + p\delta_{ij}$$

$$0 = \frac{\partial G_i}{\partial t} = \int dV \frac{\partial g_i}{\partial t} = -\int dV \frac{\partial}{\partial x_j}\Pi_{ij}$$

$$= -\oint dS_j \Pi_{ij}$$

$$\frac{p}{\rho} + \frac{1}{2}v^2 + gh = \text{konst.}$$

Brezvrtinčni, stacionarni tok

Vrtinec

$$w = -i\frac{\Gamma}{2\pi}\ln z \qquad \vec v = \frac{\Gamma}{2\pi r}\hat e_\varphi$$

$$\vec\omega = \Gamma\delta^2(\vec r)\,\hat e_z$$

Izvor

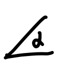
$$w = \frac{Q}{2\pi}\ln z \qquad \vec v = \frac{Q}{2\pi r}\hat e_r$$

Enačba tokovnic

$$\psi = \text{konst.}$$

Homo. hitrostno polje

$$w(z) = v_0 z$$



$$z = z'^{\pi/\alpha}$$

$$w'(z') = w_0(z(z'))$$

$$z(z') = z' + \frac{a^2}{z'}$$

Viskozne tekočine

$$\rho\left(\frac{\partial \vec v}{\partial t} + (\vec v\cdot\nabla)\vec v\right) = -\nabla p + \nabla\cdot\sigma^v + f$$

$$\sigma^v_{ij} = \underline{2\eta\,(v_{ij}} - \frac{1}{3}v_{kk}\delta_{ij}) + \zeta v_{kk}\delta_{ij}$$

za stacionarno polje

$$v_{ij} = \frac{1}{2}\left(\partial_j v_i + \partial_i v_j\right)$$

$$f_i = \frac{\partial}{\partial x_j}\sigma_{ij}$$

$$F_i = \oint \sigma_{ij}\,dS_j$$

normala

$$\rho\left(\frac{\partial \vec v}{\partial t} + (\vec v\cdot\nabla)\vec v\right) = -\nabla p + \eta\nabla^2\vec v + \left(\zeta + \frac{1}{3}\eta\right)\nabla\nabla\cdot\vec v + \vec f$$

$$Re = \frac{\rho v l}{\eta} = \frac{|\rho(\vec v\cdot\nabla)\vec v|}{|\eta\nabla^2 v|}$$

$$St = \frac{l}{\tau v} = \left|\frac{\partial\vec v}{\partial t}\right|\cdot\frac{1}{|(\vec v\cdot\nabla)\vec v|}$$

Helmholtzova en. za vrtinčnost

$$\vec\omega = \nabla\times\vec v$$

$$\frac{\partial\vec\omega}{\partial t} + (\vec v\cdot\nabla)\vec\omega = (\vec\omega\cdot\nabla)\vec v + \frac{\eta}{\rho}\nabla^2\vec\omega$$

Cilindrične koord.

$$v_{rr} = \frac{\partial v_r}{\partial r}$$

$$v_{\varphi\varphi} = \frac{\partial v_\varphi}{r\partial\varphi} + \frac{v_r}{r}$$

$$v_{r\varphi} = \frac{1}{2}\left(\frac{\partial v_\varphi}{\partial r} - \frac{v_\varphi}{r} + \frac{\partial v_r}{r\partial\varphi}\right)$$

$$\vec v = v(r)\,\hat e_\varphi$$

$$(\vec v\cdot\nabla)\vec v = -\frac{v^2}{r}\hat e_r$$

Difuzijska enačba

$$\frac{\partial u}{\partial t} - D\frac{\partial^2 u}{\partial x^2} = \delta(x)\delta(t)$$

$$u = \frac{1}{\sqrt{4\pi D t}}\,e^{-x^2/4Dt}$$

Matematične zveze

$$\nabla\nabla\cdot = \nabla^2 + \nabla\times\nabla\times$$

Disipacija, navor

$$\frac{dF_\varphi}{l} = \sigma^v_{\varphi r}\,R\,d\varphi$$

$$\int\frac{dM}{l} = \int\sigma^v_{\varphi r}\,R^2\,d\varphi$$

$$\frac{P}{l} = \frac{M\omega}{l}$$