Kombinatorika

1. Imamo število 1234232. Koliko različnih 7 mestnih števil lahko generiramo iz cifer danega števila?

$$\frac{7!}{3!\,2!} = 420$$

2. V zadrugi imajo 6 krav, 5 prašičev, 8 kokoši, kmet kupi 3 krave 2 prašiča in 4 kokoši. Na koliko različnih načinov lahko kmet izbere živali

$$\binom{6}{3}\binom{5}{2}\binom{8}{4} = 14\,000$$

3. Vržemo par poštenih igralnih kock. Kolikšna je verjetnost da je vsota enaka 6? Kolikšna je verjetnost, da je največje od obeh števil enako 4?

$$P(A) = \frac{5}{36}$$

$$P(B) = \frac{7}{36}$$

Narišemo vse kombinacije in preštejemo

4. V učilnici je n študentov, kolikšna je verjetnost, da bosta imela vsaj dva študenta na isti dan rojstni dan

$$P(n \geq 2) = 1 - P(n=1)$$ ← da ima le en na isti dan r.d.

$$= 1 - \frac{365 \cdot 364 \cdot \ldots (365-n+1)}{365^n}$$

$$= 1 - \frac{365!}{365^n (365-n)!}$$

Stirlingova formula $n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$

$$= 1 - \frac{\sqrt{\cancel{2\pi}\, 365}}{365^n \sqrt{\cancel{2\pi}(365-n)}} \frac{\left(\frac{365}{e}\right)^{365}}{\left(\frac{365-n}{e}\right)^{365-n}}$$

$$= 1 - \sqrt{\frac{365}{365-n}}\; e^{-n} \frac{1}{\left(1-\frac{n}{365}\right)^{365-n}}$$

$$= 1 - \frac{1}{\sqrt{1-\frac{n}{365}}}\; e^{-n} \frac{1}{\left(1-\frac{n}{365}\right)^{365-n}}$$

$$= 1 - e^{-n}\left(1-\frac{n}{365}\right)^{n-\frac{1}{2}-365}$$

$$= 11{,}7\,\%$$

Verjetnostne porazdelitve

1. Imejmo kumulativno porazdelitveno funk. naklj. spremen. X

$$F_X(x) = \begin{cases} 1-e^{-2x} & ; \ x \geq 0 \\ 0 & ; \ \text{sicer} \end{cases}$$

Določi
- gostoto verjetnosti
- verjetnost, da je $X > 2$
- verjetnost, da je $-3 < X \leq 4$

$$F_X(x) = P(X \leq x) = \int_{-\infty}^{x} f_X(x)\,dx$$

- $$f_X(x) = \frac{dF_X(x)}{dx} = \begin{cases} 2e^{-2x} & ; \ x > 0 \\ 0 & ; \ x < 0 \end{cases}$$

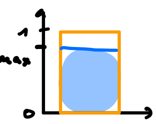
- $$P(X > 2) = \int_2^{\infty} f_X(x)\,dx = \int_2^{\infty} 2e^{-2x}\,dx = -e^{-2x}\Big|_2^{\infty} = 0 + e^{-4} = e^{-4} = 0{,}0183$$
$$= 1 - P(X \leq 2) = 1 - F(2) = 1 - (1 - e^{-4}) = e^{-4}$$

- $$P(-3 < X \leq 4) = \int_{-3}^{4} f_X(x)\,dx = \int_{-\infty}^{4} 2e^{-2x}\,dx = F_X(4) = 1 - e^{-8} = 0{,}999665$$
$$= F_X(4) - F_X(-3)$$

2. Količino tekočih odpadkov, ki jih kemična tovarna proizvede v enem tednu je porazdeljena kot:

$$f_X(x) = \begin{cases} 105x^4(1-x)^2 & ; \ 0 < x < 1 \\ 0 & ; \ \text{sicer} \end{cases}$$

relativna količina proizvedenih odpadkov



$x_{max} = ?$

kjer je $X = \frac{V}{10000\,l}$ tu je V volumen odpadkov. Kako velik rezervoar za odpadke mora imeti tovarna, da je verjetnost, da pride do prelivanja in onesnaženja okolja manjša od 5% ?

Normiranje:
$$\int_0^1 f(x)\,dx = \int_0^1 105x^4(1-x)^2\,dx = 105\int_0^1 x^4 - 2x^5 + x^6\,dx = 105\left(\frac{x^5}{5} - \frac{2x^6}{6} + \frac{x^7}{7}\right)\Big|_0^1 = 1$$

$$P(X \geq x_{max}) = 0{,}05 = 1 - P(X \leq x_{max}) = 1 - F(x_{max})$$

$$0{,}05 = 1 - \int_0^{x_{max}} 105x^4(1-x)^2\,dx = 1 - (21x^5 - 35x^6 + 15x^7)\Big|_0^{x_{max}}$$

$$21x_{max}^5 - 35x_{max}^6 + 15x_{max}^7 = 0{,}95$$

$\Downarrow$ bisekcija

$$x_{max} = 0{,}871 \Rightarrow V_{max} = 0{,}871 \cdot 10\,kL$$

Transformacije spremenljivk

(1) Imejmo gostoto verjetnosti zvezne naključne spremenljivke X, ki je podana kot:

$$f_X(x) = \begin{cases} c(x+\sqrt{x}) & ; \; 0<x<1 \\ 0 & ; \; \text{sicer} \end{cases}$$

Koliko je konstanta c in kakšno obliko ima verjetnostna gostota za spremenljivko $Y = \frac{1}{X}$?

(1) $$\int_{-\infty}^{\infty} f_X(x)\,dx = 1 = c\int_0^1 x+\sqrt{x}\,dx = c\left(\frac{x^2}{2} + \frac{x^{3/2}}{3/2}\right)\Big|_0^1 = c\left(\frac{1}{2}+\frac{2}{3}\right) = c\,\frac{7}{6}$$

$$c = \frac{6}{7}$$

$$f_Y(y) = ? \qquad Y = \frac{1}{X} \qquad x = \frac{1}{y}$$

V splošnem $\quad y = h(x) \qquad x = h^{-1}(y)$

$$f_Y(y) = f_X(h^{-1}(y))\left|\frac{dh^{-1}(y)}{dy}\right|$$

$$f_Y(y) = f_X\left(\frac{1}{y}\right)\left|-\frac{1}{y^2}\right|$$

$$f_Y(y) = \frac{6}{7}\left(\frac{1}{y}+\frac{1}{\sqrt{y}}\right)\frac{1}{y^2} = \frac{6}{7}\,\frac{1}{y^3}(1+\sqrt{y})$$

$$f_Y(y) = \begin{cases} \frac{6}{7}\frac{1}{y^3}(1+\sqrt{y}) & ; \; 1<y \\ 0 & ; \; \text{sicer} \end{cases}$$

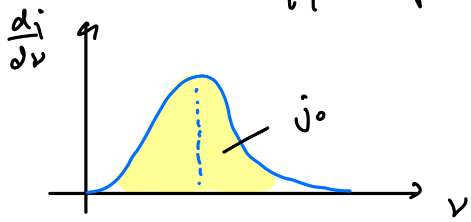
(2) $$F_Y(y) = P(Y\le y) = 1-P(Y>y) \overset{?}{=} 1-P(X<\overset{1/y}{x}) = 1-P\left(X<\frac{1}{y}\right) =$$
$$= 1-\left(P\left(X<\frac{1}{y}\right)+P\left(X=\frac{1}{y}\right)\right) = 1-P(X\le 1/y) = 1-F_X\left(\frac{1}{y}\right) =$$
$$= 1-\int_0^{1/y} f_X(x)\,dx = 1-\frac{6}{7}\left(\frac{x^2}{2}+\frac{2}{3}x^{3/2}\right)\Big|_0^{1/y} = 1-\frac{6}{7}\left(\frac{1}{2y^2}+\frac{2}{3}\,\frac{1}{y^{3/2}}\right)$$

$$f_Y(y) = \frac{dF_Y(y)}{dy} = \frac{6}{7}\left(\frac{1}{y^3}+\frac{1}{y^{5/2}}\right) = \frac{6}{7}\,\frac{1}{y^3}(1+\sqrt{y})$$

(2) Imejmo telo, ki ga segrejemo na $T=6000K$. Takšno telo oddaja EM valovanje. Spektralno gostoto toka podaja Planckov zakon

$$\frac{dj}{d\nu} = \frac{2\pi}{c^2}\,\frac{h\nu^3}{e^{\frac{h\nu}{kT}}-1} \quad ; \quad \nu \in [0,\infty)$$

Določi porazdelitev svetlobnega toka v odvisnosti od ν. Primerjaj to porazdelitev po λ



velike črke ν

$$f_\nu(\nu) = \frac{1}{j_0}\,\frac{dj}{d\nu} \qquad j_0 = \sigma T^4$$

Stefanov zakon

$$j_0 = \int_0^\infty \frac{dj}{d\nu}\, d\nu = \frac{2\pi h}{c^2} \int_0^\infty \frac{\nu^3}{e^{\frac{h\nu}{kT}} - 1}\, d\nu = \frac{2\pi h}{c^2}\left(\frac{kT}{h}\right)^4 \underbrace{\int_0^\infty \frac{y^3}{e^y - 1}\, dy}_{\frac{\pi^4}{15}} = \underbrace{\frac{2\pi^5 k^4}{c^3 h^3 15}}_{\sigma} T^4$$

$$y = \frac{h\nu}{kT} \qquad dy = \frac{h}{kT}\, d\nu$$

$$f_\nu(\nu) = \frac{1}{\sigma T^4} \frac{2\pi}{c^2} \frac{h\nu^3}{e^{\frac{h\nu}{kT}} - 1}$$

velika črka λ

$$f_\lambda(\lambda) = ? \qquad \lambda = \frac{c}{\nu} \qquad \nu = \frac{c}{\lambda}$$

$$f_\lambda(\lambda) = f_\nu\left(\frac{c}{\lambda}\right) \left| \frac{d\frac{c}{\lambda}}{d\lambda} \right| = \frac{1}{\sigma T^4} \frac{2\pi}{c^2} \frac{h c^3}{\lambda^3 \left(e^{\frac{hc}{\lambda kT}} - 1\right)} \frac{c}{\lambda^2} = \frac{2\pi c^2 h}{\sigma T^4} \frac{1}{\lambda^5} \frac{1}{e^{\frac{hc}{\lambda kT}} - 1}$$

$$\nu_{max} \neq \frac{c}{\lambda_{max}}$$

$$y = \frac{h\nu}{kT}$$

$$\frac{df_\nu}{d\nu} = 0 = \frac{3y^2}{e^y - 1} - \frac{y^3 e^y}{(e^y - 1)^2} \implies 3 = \frac{y e^y}{e^y - 1}$$

$1 - e^{-y} = \frac{y}{3}$ Rešimo numerično

$y_{max} = 2{,}825$

$\nu_{max} = y \frac{kT}{h} = 3{,}576 \cdot 10^{14}\ Hz$

$\lambda' = \frac{c}{\nu_{max}} = 848\ nm$ (IR)

$$x = \frac{kT\lambda}{hc}$$

$$\frac{df_\lambda}{d\lambda} = 0 = -\frac{5}{x^6} \frac{1}{\left(e^{\frac{1}{x}} - 1\right)} + \frac{e^{\frac{1}{x}} \frac{1}{x^2}}{x^5 \left(e^{\frac{1}{x}} - 1\right)^2} \implies 5 = \frac{e^{\frac{1}{x}}}{x\left(e^{\frac{1}{x}} - 1\right)} \qquad z = \frac{1}{x}$$

nista enaki

$$1 - e^{-z} = \frac{z}{5}$$

$z \doteq 5 \implies x \doteq 0{,}2 \implies \lambda_{max} = 487\ nm$

Diskretne porazdelitve

1. Pošteno kocko vržemo 6x. Kolikšna je verjetnost, da je največja vržena cifra točno $X = x$, kjer je $x = 1, \dots, 6$

A_x ... dogodek, kjer je največje število X točno x

$P(A_x) = ?$ Zaloga vrednosti je $6^6 = 46656$ št. vseh dogodkov

$X = 1 \quad N = 1 \quad P(A_1) = \frac{1}{6^6} = 2{,}1 \cdot 10^{-6}$

$X = 2 \quad N = 6$

B_x ... dogodek, kjer je največja vrednost manjša ali enaka x, ni nujno, da vključuje x

$$B_x = B_{x-1} \cup A_x$$

$P(D_x) = P(D_{x-1}) + P(A_x)$

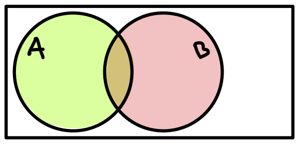
$$P(A_x) = P(B_x) - P(B_{x-1}) = \left(\frac{x}{6}\right)^6 - \left(\frac{x-1}{6}\right)^6$$

X	1	2	3	4	5	6
$P(A_x)$	$2\cdot10^{-5}$	$1\cdot10^{-3}$	$1\cdot10^{-2}$	$7\cdot10^{-2}$	0,25	0,6

Pogojna verjetnost

① 35% špansko
15% francosko
40% vsaj en jezik

Šola



A ... se uči špansko
B ... se uči francoščino

$P(A) = 0{,}35$
$P(B) = 0{,}15$
$P(A \cup B) = 0{,}40$

$P(B|A) = ?$

$$\boxed{P(B|A) = \frac{P(A \cap B)}{P(A)}}$$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$P(B|A) = \frac{P(A) + P(B) - P(A \cup B)}{P(A)} = \frac{0{,}35 + 0{,}15 - 0{,}40}{0{,}35} = 0{,}286$$

② Zbolelo je 2% ljudi. Dobim pozitiven rezultat. Kolikšna je verjetnost da sem res pozitiven, če ima test občutljivost 70% in specifičnost 95%.

Občutljivost: če sem poz ⇒ poz test
$P(V|B) = 0{,}70$

B = resnično sem okužen
V = test je pozitiven
N = test je negativen

Specifičnost: če sem neg ⇒ neg test
$P(N|\bar{B}) = 0{,}95$
$P(V|\bar{B}) = 1 - P(N|\bar{B})$ Fals poz.

Zanima nas $P(B|V) = ?$

$$P(B \cap V) = P(B|V)\,P(V) = P(V|B)\,P(B)$$

$$P(B|V) = \frac{P(B)\,P(V|B)}{P(V)}$$

$P(B) = 0{,}02$

Problemi:
- ni naključno izbran populacije
- 0,02 je napačna ker je ne poznamo dovolj natančno

$$P(V) = P(V \cap B) + P(V \cap \bar{B})$$
$$P(V) = P(V|B)\,P(B) + P(V|\bar{B})\,P(\bar{B})$$

$$P(B|V) = \frac{P(B)\,P(V|B)}{P(V|B)\,P(B) + (1 - P(N|B))(1 - P(B))}$$

$$= \frac{0{,}70 \cdot 0{,}02}{0{,}70 \cdot 0{,}02 + (1 - 0{,}95)(1 - 0{,}02)} = 0{,}22$$

③ Iz paketa 52 kart izbere S1 eno karto. S2 mora ugotoviti katero karto je izbral. S2 lahko S1 vpraša eno od dveh vprašanj:

- Ali je karta rdeča?
- Ali je karta pikov as?

S1 odgovori pošteno. Katero vprašanje naj vpraša, da bo imel večjo verjetnost da pravo izbere?

A = izberemo pravo karto

B = karta je rdeča $\quad P(B) = 0{,}5 \quad P(\bar{B}) = 0{,}5$ (ni B)

C = karta je pikov as $\quad P(C) = 1/52 \quad P(\bar{C}) = 51/52$

ⓐ Ali je karta rdeča?

$$\begin{aligned} P(A) &= \underbrace{P(A \cap B)} + \underbrace{P(A \cap \bar{B})} \\ &= P(A|B)P(B) + P(A|\bar{B})P(\bar{B}) \\ &= 1/26 \cdot \tfrac{1}{2} + 1/26 \cdot \tfrac{1}{2} \\ &= 1/26 \end{aligned}$$

ⓑ Ali je pikov as?

$$\begin{aligned} P(A) &= P(A \cap C) + P(A \cap \bar{C}) \\ &= P(A|C)P(C) + P(A|\bar{C})P(\bar{C}) \\ &= 1 \cdot 1/52 + 1/51 \cdot 51/52 \\ &= 1/26 \end{aligned}$$

Vseeno je kaj vprašamo.

④ Vržemo dve 5-strani kocki. Njune izida naj bosta p, q. Naj bodo dogodki:

- A: $p + q \le 5$
- B: $|p - q| \ge 1$
- C: $p + q \le 4$

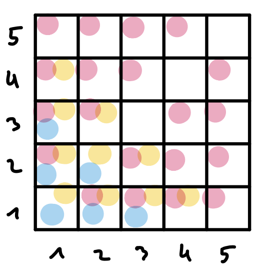
Ali so dogodki odvisni ali neodvisni?

Ⓣ Dogodka sta neodvisna

$P(A|B) = P(A) \qquad P(B|A) = P(B)$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$\begin{aligned} P(A \cap B) &= P(A|B)P(B) \\ &= P(A)P(B) \end{aligned}$$ za neodvisna dogodka.



$P(A) = \frac{10}{25} = \frac{2}{5}$

$P(B) = \frac{20}{25} = \frac{4}{5}$

$P(C) = \frac{6}{25}$

ⓐ $P(A \cap B) = \frac{8}{25} \overset{?}{=} P(A)P(B) = \frac{8}{25}$ ✓

$P(A \cap C) = 6/25 \overset{?}{=} \frac{2}{5} \cdot \frac{6}{25} = \frac{12}{125}$ ✗

$P(B \cap C) = 4/25 \overset{?}{=} \frac{4}{5} \cdot \frac{6}{25} = \frac{24}{125}$ ✗

A, B sta neodvisna

A, C sta odvisna

B, C sta odvisna

ⓑ $P(A|B) = \frac{8}{20} = \frac{2}{5} \overset{?}{=} P(A) = \frac{2}{5}$ ✓

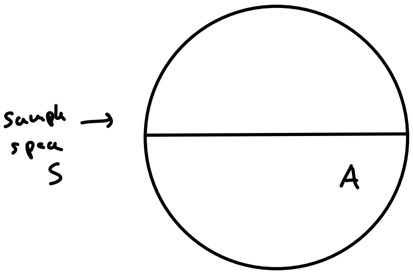
$P(B|A) = \frac{8}{10} \overset{?}{=} P(B) = \frac{4}{5}$ ✓

Razlaga, zakaj A in B neodvisna, B in C pa neodvisna?

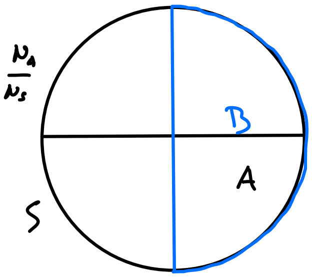
$P(A) = \frac{10}{25} \xrightarrow[-20\%]{-20\%} P(A|B) = \frac{8}{20}$ $\qquad$ $P(C) = \frac{6}{25} \xrightarrow[-20\%]{-33\%} P(C|B) = \frac{4}{20}$

oboje zmanjšali za 20%

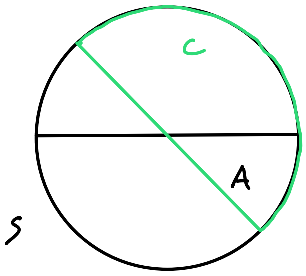
Pri C se pa zmanjšata za različno.



$P(A) = \frac{1}{2} = \frac{N_A}{N_S}$



$P(A|B) = \frac{1/2\, N_a}{1/2\, N_s} = \frac{N_a}{N_s}$



$P(A|C) = \frac{1/4\, N_a}{1/2\, N_s} = \frac{1}{2} \frac{N_a}{N_s}$

5 Imejmo 2D zvezno porazd. naklj. sprem.

$$f_{X,Y}(x,y) = \begin{cases} C x y^2 ; & 0 < x, y < 1 \\ 0 ; & \text{sicer} \end{cases}$$

Določi C

Izračunaj verjetnost, da je $Y > \frac{1}{2}$ in verjetnost da je $Y > \frac{1}{2}$ ob pogoju da je $x \doteq 1/2$.

$C = ?$

$P(Y > \frac{1}{2}) = ?$

$P(Y > 1/2 \mid X = 1/2)$

ⓐ $\int_0^1 \int_0^1 f_{XY}(x,y)\, dx\, dy = \int_0^1 \int_0^1 C x y^2\, dx\, dy = C \frac{1}{2} \frac{1}{3} = \frac{C}{6} = 1$

$C = 6$

ⓑ $P(Y > \frac{1}{2}) = \int_0^1 \int_{1/2}^1 f_{X,Y}(x,y)\, dx\, dy = \int_{1/2}^1 \int_0^1 6 x y^2\, dx\, dy = C \frac{1}{2} \frac{1}{3} (1^3 - (\frac{1}{2})^3) = \frac{7}{8}$

ⓒ $P(Y > \frac{1}{2} \mid x \approx 1/2) = \int_{1/2}^1 f_{Y|X}(Y|X)\, dy = \int_{1/2}^1 \frac{6 x y^2}{2x}\, dx = 1 - (\frac{1}{2})^3 = \frac{7}{8}$

Pogojna verjetnostna gostota

$f_{Y|X}(Y|X) = \frac{f_{XY}(X,Y)}{f_X(X)}$ $\quad f_X(x) = \int_0^1 f_{X,Y}(x,y)\, dy = \int_0^1 6 x y^2 dy = 2x$

Ker $P(Y > 1/2) = P(Y > 1/2 \mid X \approx 1/2)$ sta X, Y neodvisni.

6 Skupna verjet. gostota sprem. X in Y je:

$$f_{X,Y}(x,y) = \begin{cases} x + y ; & 0 < x, y < 1 \\ 0 ; & \text{sicer} \end{cases}$$

Kolikšna je verjetnost, da bo imela enačba (po sprem. ξ):

$$X \xi^2 + Y \xi + 1 = 0$$

dva realni rešitvi?

$D = Y^2 - 4X \geq 0$

$$P\left(X \leq \frac{Y^2}{4}\right) = \int_{y=0}^{1} \int_{0}^{\frac{y^2}{4}} f_{x,y}(x,y)\, dx\, dy = \int_0^1 \int_0^{\frac{y^2}{4}} x + y \, dx\, dy =$$

$$= \int_0^1 \frac{x^2}{2} + yx \Big|_0^{\frac{y^2}{4}} dy = \int_0^1 \frac{y^4}{32} + \frac{y^3}{4}\, dy = \frac{1}{5\cdot 32} + \frac{1}{16} = \frac{11}{160} \doteq 0{,}0687$$

6 Kombinirana verjet. gostota za sprem. X, Y je:

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{2}(x+y)e^{-(x+y)} & ; \ x, y > 0 \\ 0 & ; \ \text{sicer} \end{cases}$$

Kakšna je verjet. gostota za sprem. $U = X + Y$

izmislimo si eno čim bolj preprosto

$X, Y \to U \ \underbrace{V}$

DN $V = X$

rezultat je isti

$$g_{UV}(u,v) = f_{X,Y}(x(u,v), y(u,v)) \left| \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \right|$$

$U = X + Y, \quad V \overset{?}{=} X - Y \quad$ (DN $V = X$)

$X = \frac{1}{2}(u+v), \quad Y = \frac{1}{2}(u - v)$

$$g_{u,v}(u,v) = f_{X,Y}\left(\tfrac{1}{2}(u+v), \tfrac{1}{2}(u-v)\right) \left| \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} \right|$$

$$= \frac{1}{2} \frac{1}{2} u e^{-u} = \frac{1}{4} u e^{-u} \neq f_u(u)$$

$$f_u(u) = \int_{?}^{?} g_{u,v}(u,v)\, dv$$

Vemo $0 < X, Y < \infty$

Na osi u, je potem $v = 0$

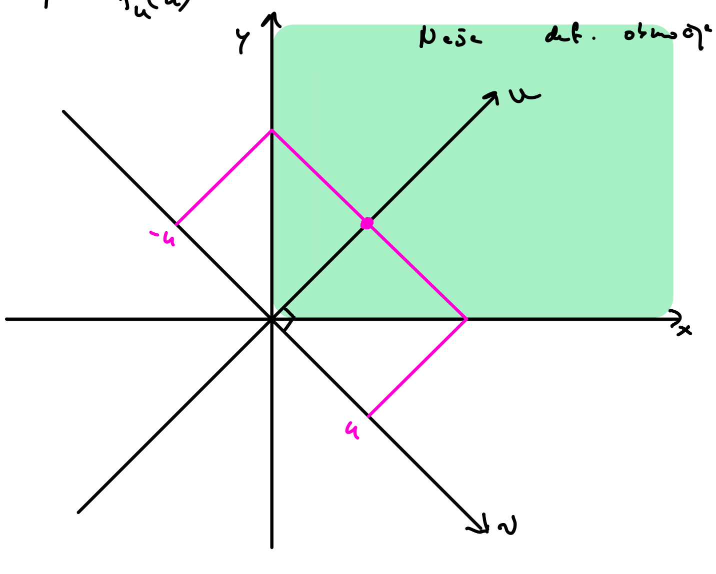
$u = x + y$

$0 = x - y \Rightarrow x = y$

Os V je pri $u = 0$

$V = x - y$

$0 = x + y \Rightarrow x = -y$



Za dani u je v na intervalu $[-u, u]$

Zato sledi:

$$f_u(u) = \int_{-u}^{u} g_{uv}(u,v)\, dv = \int_{-u}^{u} \frac{1}{4} u e^{-u}\, dv = \frac{1}{2} u^2 e^{-u}$$

Posebne verjetnostne porazdelitve

① Muha del dneva leti, del čepi. Čas ko leti je enakomerno porazdeljen med 0 in 30s. Kolikšna je verjetnost, da muha leti več kot 20s, ob pogoju, da leti vsaj 10 s.

Dogodki:

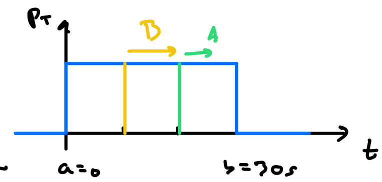
A = "Muha leti vsaj 20s"

B = "Muha leti vsaj 10s"

$P(A|B) = ?$

Verjetnostna gostota

$$p_T(t) = \begin{cases} c & ; a < t < b \\ 0 & ; \text{sicer} \end{cases}$$



Z normalizacijo dobimo $c = \frac{1}{b-a}$

$$P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{\int_{20}^{30} p_T\,dt}{\int_{10}^{30} p_T\,dt} = \frac{10}{20} = \frac{1}{2}$$

② Imejmo plin CO_2 pri $T=300K$. Uporabi Maxwellovo porazdelitev in izračunaj, kolikšen del molekul ima hitrost med 200 m/s in 250 m/s? Pri kateri hitrosti ima porazdelitev maksimum.

$v_1 = 200\ \mathrm{m/s}$

$v_2 = 250\ \mathrm{m/s}$

$P(v_1 < v < v_2) = ?$

$v_{max} = ?$

$$f_v(v) = 4\pi\left(\frac{m}{2\pi k_B T}\right)^{\frac{3}{2}} v^2 e^{-\frac{mv^2}{2k_BT}} \qquad \alpha = \frac{m}{2k_BT}$$

$$P(v_1 < v < v_2) = \int_{v_1}^{v_2} f_v(v)\,dv$$

$$= \sqrt{\frac{16\alpha^3}{\pi}} \int_{v_1}^{v_2} v^2 e^{-\alpha v^2} dv = \sqrt{\frac{\cancel{16}^{\,4}\alpha^{\cancel{3}}}{\pi}}\left(-\frac{v}{2\cancel{\alpha}} e^{-\alpha v^2}\Big|_{v_1}^{v_2} + \frac{1}{\cancel{2\alpha}}\int_{v_1}^{v_2} e^{-\alpha v^2} dv\right) =$$

Per partes

$u = v$

$dw = v e^{-\alpha v^2}$

$w = -\frac{1}{2\alpha} e^{-\alpha v^2}$

$$= \sqrt{\frac{4\alpha}{\pi}}\left(-v e^{-\alpha v^2}\Big|_{v_1}^{v_2} + \underbrace{\int_{v_1}^{v_2} e^{-\alpha v^2} dv}\right) = \qquad \text{nova spr.} \quad x = \sqrt{\alpha}\,v$$

$$Erf(\tilde{x}) = \frac{2}{\sqrt{\pi}}\int_0^{\tilde{x}} e^{-x^2} dx$$

$$= \sqrt{\frac{4}{\pi}}\left(-x e^{-x^2}\Big|_{x_1}^{x_2} + \int_{x_1}^{x_2} e^{-x^2} dx\right) =$$

$$= -\frac{2}{\sqrt{\pi}} x e^{-x^2}\Big|_{x_1}^{x_2} + Erf(x_2) - Erf(x_1) =$$

$$\overset{\text{tabele}}{=} \dots = 0{,}0971$$

$\alpha = \frac{M}{2RT} = 8{,}83\cdot 10^{-6}\ \frac{s^2}{m^2}$

$x_1 = \sqrt{\alpha}\,v_1 \qquad x_2 = \sqrt{\alpha}\,v_2$

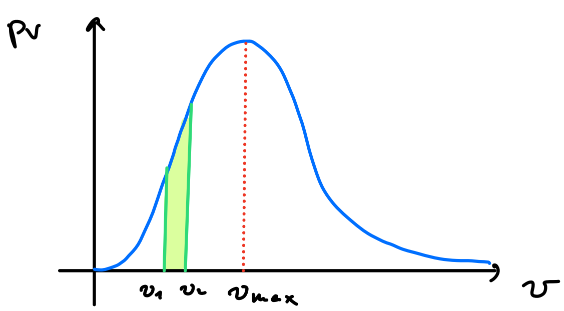
$x_1 = 0{,}5941 \qquad x_2 = 0{,}743$

$v_{max} = ?$

$$\frac{df_v}{dv} = \sqrt{\frac{16\alpha^3}{\pi}}\left(2v e^{-\alpha v^2} + v^2(-\alpha 2v) e^{-\alpha v^2}\right) = 0$$

$$\sqrt{\frac{16\alpha^3}{\pi}}\, 2v\, e^{-\alpha v^2}(1 - \alpha v^2) = 0$$

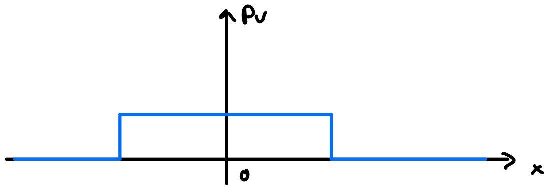
$$v = 0 \qquad v_{max} = \sqrt{\frac{1}{\alpha}} = \sqrt{\frac{2k_BT}{m}} = \sqrt{\frac{2RT}{M}} = 336{,}7\ \frac{m}{s}$$



Momenti

1 Določi stand. deviacijo sprem. X, ki ji ustreza verjetnostna gostota:

$$p_x = \begin{cases} \frac{1}{a} & ; -\frac{a}{2} \le x \le \frac{a}{2} \\ 0 & ; \text{sicer} \end{cases}$$



$\mathrm{var}[X] = ?$

$$\bar{X} = E[x] = \int_{-\frac{a}{2}}^{\frac{a}{2}} x\, p_x \, dx = 0$$

$$\sigma_x^2 = \mathrm{var}[X] = E[x^2] - \underbrace{E[x]}_{0}{}^2 = \int_{-\frac{a}{2}}^{\frac{a}{2}} (x - \underbrace{\bar{x}}_{0})^2 p_x \, dx = E[x^2]$$

$$E[x^2] = \int_{-\frac{a}{2}}^{\frac{a}{2}} x^2 p_x \, dx = \frac{1}{a} \frac{x^3}{3} \Big|_{-\frac{a}{2}}^{\frac{a}{2}} = \frac{1}{a}\, 2 \frac{a^3}{2^3 \cdot 3} = \frac{a^2}{12} \qquad \sigma_x = \frac{a}{\sqrt{12}}$$

2 V sferično simetričnem protonu je električni naboj porazdeljen kot:

$$f_e(r) = \frac{a^3}{8\pi} e^{-ar} \quad ; \quad a = 4{,}27 \, \frac{1}{fm}$$

kjer je R odmik od središča nukleona. Določi nabojni radij protona, ki je def. kot

$$r_E = \sqrt{E[R^2]}$$

Normalizacija $\iiint \frac{a^3}{8\pi} e^{-ar} \sin\theta \, d\varphi \, d\theta \, dr = 4\pi \int_0^\infty \frac{a^3}{8\pi} e^{-ar} r^2 \, dr \overset{x = ar}{=} \frac{1}{2} \int_0^\infty x^2 e^{-x} dx = \frac{1}{2} \Gamma(3) = 1$

$$E[R^2] = \iint d\Omega \int_0^\infty r^2 f_R(r) r^2 dr \qquad \int_0^{2\pi} d\varphi \int_0^\pi \sin\theta \, d\theta \int_0^\infty r^2 f_e \, r^2 dr = 4\pi \int_0^\infty r^4 f_e \, dr =$$

$$= \frac{a^2}{a^2} \frac{a^3}{2} \int_0^\infty r^2 e^{-ar} r^2 \, dr \overset{x = ar}{=} \frac{1}{2a^2} \int_0^\infty x^4 e^{-x} dx = \frac{1}{2a^2} \Gamma(5) = \frac{12}{a^2}$$

$$r_E = \sqrt{\frac{12}{a^2}} = \frac{1}{a} 2\sqrt{3} = 0{,}811 \text{ fm}$$

3 Preveri normalizacijo Cauchyjeve porazdelitve

$$f_X(x) = \frac{1}{\pi} \frac{1}{1+x^2} \quad ; \quad -\infty < x < \infty$$

in določi njena prva dva momenta

- $I = \int_{-\infty}^{\infty} \frac{1}{\pi} \frac{1}{1+x^2} dx = \frac{1}{\pi} \arctan x \Big|_{-\infty}^{\infty} = \frac{1}{\pi} \left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) = 1$ ✓

Alternativno $x = \tan\varphi = \frac{\sin\varphi}{\cos\varphi}$ $\quad dx = \frac{\cos^2\varphi + \sin^2\varphi}{\cos^2\varphi} d\varphi = 1 + \tan^2\varphi \, d\varphi$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{\pi} \frac{1 + \tan^2\varphi}{1 + \tan^2\varphi} d\varphi = \frac{1}{\pi} \pi = 1$$

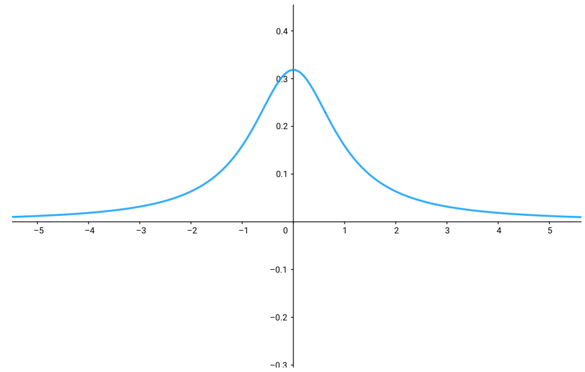
- $\bar{X} = E[X] = \int_{-\infty}^{\infty} x\, f_X(x)\, dx = \int_{-\infty}^{\infty} \frac{x}{1+x^2}\, dx = \dots$

$u = x^2$

$\frac{du}{2} = x\, dx$ V končnih mejah a do b $\frac{1}{2}\int_{a^2}^{b^2} \frac{du}{1+u} = \frac{1}{2} \ln|1+u| \Big|_{a^2}^{b^2} = \frac{1}{2} \ln|1+x^2| = \frac{1}{2} \ln\left|\frac{1+b^2}{1+a^2}\right|$

$\dots \frac{1}{\pi} \lim_{b\to\infty} \lim_{a\to -\infty} \left(\frac{1}{2} \ln|1+b^2| - \frac{1}{2} \ln|1+a^2|\right) = \frac{1}{2\pi} \lim_{b\to\infty} (-\infty) \Rightarrow$ ne obstaja

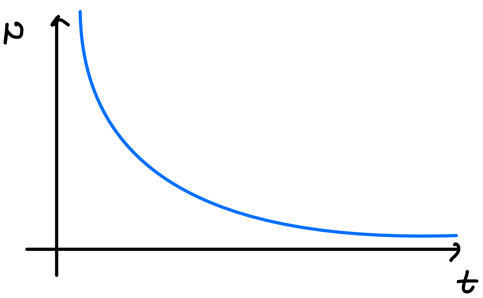
Tudi če vzamemo poljuben $a \in \mathbb{R}$, je $\int_{-\infty}^{a} = \int_{a}^{\infty} = \infty$ zato $a = E[X]$ ni def.



- $E[X^2] = \int_{-\infty}^{\infty} \frac{1}{\pi} \frac{x^2}{1+x^2}\, dx = \frac{1}{\pi} \int_{-\infty}^{\infty} 1 - \frac{1}{1+x^2}\, dx = \frac{1}{\pi} \underbrace{\int_{-\infty}^{\infty} dx}_{\text{divergira}} - \frac{1}{\pi} \underbrace{\int_{-\infty}^{\infty} \frac{1}{1+x^2}\, dx}_{1} = \infty$

Variance ne obstaja

4 Imamo ampulo z radioaktivnim jodom ^{131}I. Prvi dan pomerimo njeno aktivnost. Meritev traja 10 s in izmerimo 1000 razpadov/s. Enako meritev ponovimo čez 10 dni in izmerimo 40 razpadov/s. Kolikšen je razpolovni čas ^{131}I. Za pripadajočo porazdelitev časov razpada T, določi $E[T]$ in $\mathrm{Var}[T]$



Zakaj eksponentna porazdelitev T:

$\frac{dN}{dt} = -\frac{1}{\tau} N$

$N(t) = N_0 e^{-\frac{t}{\tau}}$

N_0: število radioaktivnih jeder v ampuli ob nekem času

Kaj merimo:

$R = -\frac{\Delta N(t)}{\Delta t} \xrightarrow{\Delta t \to 0} -\frac{dN}{dt}$

"rate" = "št. razpadov na časovno enoto"

$R = -\frac{d}{dt} N_0 e^{-\frac{t}{\tau}} = \underbrace{\frac{N_0}{\tau}}_{R_0(t=0)} e^{-\frac{t}{\tau}}$

$R(t=0) = R_0 = \frac{1000 \text{ sunkov}}{10 \text{ s}} = 100 \frac{1}{s}$

$R(t_1 = 10 \text{ dni}) = 40 \frac{1}{s}$

Razpolovni čas

$\frac{N_0}{2} = N_0 e^{-\frac{t_{1/2}}{\tau}}$

$t_{1/2} = \tau \ln 2$

$t_{1/2} = 7{,}56$ dni

$R = R_0 e^{-\frac{t_1}{\tau}}$

$-\ln \frac{R}{R_0} = \frac{t}{\tau}$ $\tau = \frac{t_1}{\ln R_0/R} = 10{,}9$ dni

Razpadni čas

Naprej računamo prihodnji č (Poisson)

Izračunajmo momente

$$p_T(t) = \frac{1}{\tau} e^{-\frac{t}{\tau}} \qquad \tau > 0$$

$$N_0 - N(t') = N_0 \left(1 - e^{-\frac{t'}{\tau}}\right) \quad /: N_0$$

$$\frac{N_0 - N(t')}{N_0} = 1 - e^{-\frac{t'}{\tau}} = P_T(T \leq t') \overset{\text{odvajamo}}{\Rightarrow} p_T(t) = \frac{1}{\tau} e^{-\frac{t}{\tau}}$$

$$E[T] = \int_0^\infty t \, \frac{1}{\tau} e^{-\frac{t}{\tau}} dt = \tau \int_0^\infty x e^{-x} dx = \tau \, \Gamma(2) = \tau \, 1! = \tau$$

$$\mathrm{var}[T] = E[T^2] - E[T]^2 = \int_0^\infty t^2 \frac{1}{\tau} e^{-\frac{t}{\tau}} dt - \tau^2 = 2!\tau^2 - \tau^2 = \tau^2$$

5) Imejmo dvorazsežno zvezno porazdelitev sluč. sprem. X, Y:

$$f_{X,Y}(x,y) = \begin{cases} 6xy^2 & ; \ 0 < x,y < 1 \\ 0 & ; \ \text{sicer} \end{cases}$$

Dobi $E[Y]$, $\mathrm{Var}[Y]$ ter $E[Y|X]$, $\mathrm{Var}[Y|X]$

$$f_Y(y) = \int_0^1 6xy^2 \, dx = \frac{6y^2}{2} = 3y^2$$

$$E[Y] = \int_0^1 y \, 3y^2 \, dy = \frac{3}{4} \qquad \mathrm{Var}[Y] = E[Y^2] - E[Y]^2 = \int_0^1 y^2 \, 3y^2 \, dy - \left(\frac{3}{4}\right)^2 = \frac{3}{5} - \frac{9}{16} = \frac{3}{80}$$

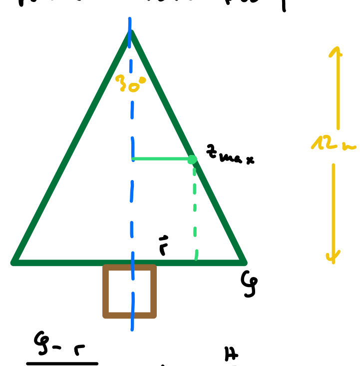
Pogojni momenti

$$E[Y|X] = \int_0^1 y \, f_{Y|X}(y) \, dy = \int_0^1 y \, 3y^2 = \frac{3}{4}$$

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{6xy^2}{\int_0^1 6xy^2 \, dy} = \frac{6xy^2}{2x} = 3y^2$$

$$\mathrm{Var}[Y|X] = E[Y^2|X] - E[Y|X]^2 = \int_0^1 y^2 \, 3y^2 \, dy - \left(\frac{3}{4}\right)^2 = \frac{3}{5} - \left(\frac{3}{4}\right)^2 = \frac{3}{80}$$

6) Na trgu stoji jelka z višino 12m in obliko stožca s kotom v vrhu 30°. Iglice so na začetku enakomerno porazdeljene po volumnu. Kakšna je porazdelitev iglic pod jelko? Kakšen je $\bar{R}$, σ_R



Porazdelitev po volumnu

$$p_{R\varphi z}(r, \varphi, z) = \frac{1}{V} \qquad \text{enakomerna porazdelitev}$$

$$p_R(r) = \int_0^{H - \frac{H}{\rho} r} \int_0^{2\pi} p_{R\varphi z} \, d\varphi \, dz$$

$$p_R = \frac{2\pi}{V} \int_0^{H - \frac{H}{\rho} r} dz = \frac{2\pi}{V}\left(H - \frac{H}{\rho} r\right) = \frac{2\pi H}{V}\left(1 - \frac{r}{\rho}\right)$$

$$z_{max} = \frac{\rho - r}{\tan\frac{\alpha}{2}} = H - \frac{H}{\rho} r$$

Poiščimo volumen (normalizacija)

$$\frac{2\pi H}{V}\int_0^{\rho}\left(1-\frac{r}{\rho}\right) r\,dr = \frac{2\pi H}{V}\left(\frac{r^2}{2}-\frac{r^3}{3\rho}\right)\Bigg|_0^{\rho} = \frac{1}{V}\,2\pi H\left(\frac{\rho^2}{2}-\frac{\rho^2}{3}\right) = 1$$

$$V = \frac{\pi H \rho^2}{3}$$

$$p_R(r) = \frac{6}{\rho^2}\left(1-\frac{r}{\rho}\right) \qquad \rho = H\tan\frac{\lambda}{2}$$

Momenti

$$E[R] = \bar{R} = \int_0^{\rho}\frac{6}{\rho^2}\left(1-\frac{r}{\rho}\right) r\,\overset{\text{jakobijan}}{r}\,dr = \frac{6}{\rho^2}\left(\frac{\rho^3}{3}-\frac{\rho^4}{4\rho}\right) = \frac{\rho}{2}$$

$$E[R^2] = \int_0^{\rho}\frac{6}{\rho^2}\left(1-\frac{r}{\rho}\right) r^2\, r\,dr = \frac{6}{\rho^2}\left(\frac{\rho^4}{4}-\frac{\rho^5}{5\rho}\right) = \frac{3}{10}\rho^2$$

$$\mathrm{Var}[R] = \frac{3}{10}\rho^2 - \left(\frac{\rho}{2}\right)^2 = \frac{\rho^2}{20}$$

$$\sigma_R = \frac{\rho}{\sqrt{20}}$$

D: skretne porazdelitve

1. Zavarovalnica v primeru poškodbe izplača 100.000 €. Letna premija je 25€. Povprečno izplača zavar. odškodnino 100 zavarovancem. Zavar. ima 1M zavarovancev. Verjetnost da izplača več kot 15 M €

$\lambda = 100$ $\qquad X =$ " št. izplačanih odškodnin "

$P(X > 150) = ?$

Verjetnost da posameznik potrebuje odškodnino

$$p = \frac{\lambda}{N} = \frac{100}{10^6} = 10^{-4}$$

Verjetnost, da so odškodnino uveljavlo n ljudi : (binomska porazdelitev)

$$B(n;\lambda,N) = \binom{N}{n} p^n (1-p)^{N-n}$$

$$= \binom{N}{n}\left(\frac{\lambda}{N}\right)^n\left(1-\frac{\lambda}{N}\right)^{N-n}$$

Ker je p majhen in v limiti $N\to\infty$, $\lambda = p\cdot N$ ostaja končen potem lahko

$$B(n;\lambda,N) \to P(n;\lambda) = \frac{\lambda^n}{n!}e^{-\lambda}$$

Mi računamo $P(X > 150)$

$$P(X>150) = \sum_{n=151}^{\infty}\frac{\lambda^n}{n!}e^{-\lambda} = \sum_{n=151}^{\infty}\frac{100^n}{n!}e^{-100} =$$

$$= \sum_{n=151}^{\infty}\frac{100^n e^n}{n^n\sqrt{2\pi n}}e^{-100} = \sum_{n=151}^{\infty}\underbrace{\left(\frac{100}{n}\right)^n e^{n-100}\frac{1}{\sqrt{2\pi n}}}_{F(n)}$$

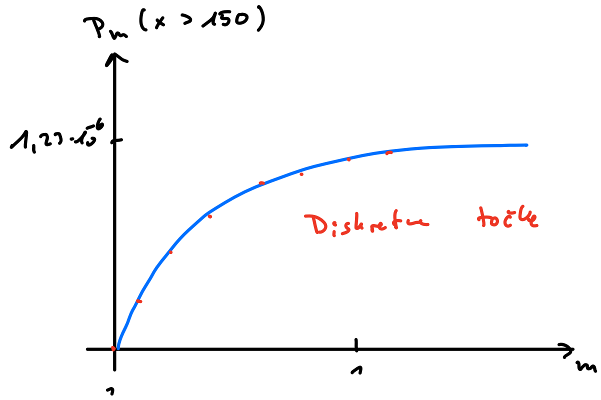
Stirlingova formula

$$n! = n\ln n - n$$

$$n! \doteq \left(\frac{n}{e}\right)^n\sqrt{2\pi n}\left(1+\frac{1}{12n}+\dots\right)$$

$$P_m(X>150) = \sum_{n=1}^{m} F(\hat{n}+150) \qquad m\to\infty$$

m	$P_m(x > 150)$
1	$4,7 \cdot 10^{-7}$
2	$7,2 \cdot 10^{-7}$
3	$9,0 \cdot 10^{-7}$
4	$1,02 \cdot 10^{-6}$
5	$1,1 \cdot 10^{-6}$
6	$1,15 \cdot 10^{-6}$
⋮	
10	$1,22 \cdot 10^{-6}$
⋮	
20	$1,23 \cdot 10^{-6}$



(2) Določi pričakovano vrednost diskretne naklj. spre. X, $x \in \mathbb{N}_0$, ki jo opisuje verjet. funk.

$$f_X(x) = (1-p)\,p^x \qquad 0 \le p < 1$$

$$E[x] = \sum_{x=0}^{\infty} x\, f_X(x) = \sum_{x=0}^{\infty} x(1-p)\,p^x = (1-p)\sum_{x=0}^{\infty} x p^x = (1-p)\sum_{x=0}^{\infty} p\, x\, p^{x-1} =$$

$$= (1-p)\,p \sum_{x=0}^{\infty} \frac{d}{dp} p^x = (1-p)\,p\,\frac{d}{dp}\sum_{x=0}^{\infty} p^x = (1-p)\,p\,\frac{d}{dp}\frac{1}{1-p} = \frac{(1-p)\,p}{(1-p)^2} = \frac{p}{1-p}$$

DN: Izračunaj $Var[x]$:

$$Var[x] = E[x^2] - E[x]^2$$

$$E[x^2] = \sum_{x=0}^{\infty} x^2 (1-p)\,p^x = (1-p)\,p^2 \sum_{x=0}^{\infty} x^2 p^{x-2} = (1-p)\,p^2 \sum_{x=0}^{\infty} \frac{d^2}{dp^2} p^x \;\ldots \qquad ?$$

(3) Naloga nadaljevanje od naloge z jedrom

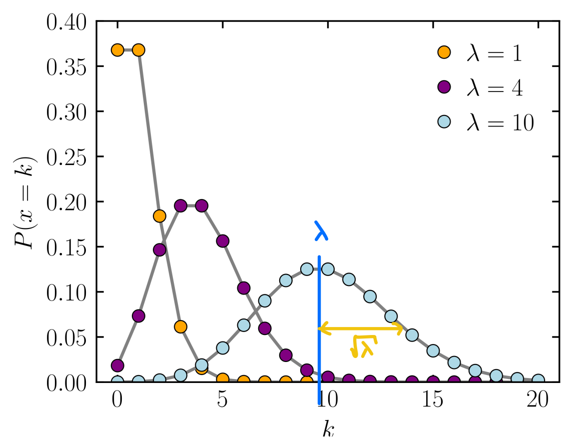
Izmerili smo $t_{1/2} = 10,$ dni

Dejanska vrednost $t_{1/2} = 11,$ dni

Meritve so porazdeljene Poissonsko.

$$p(m;\lambda) = \frac{\lambda^m}{m!}e^{-\lambda}$$

št sunkov



$E[M] = \lambda$

$Var[M] = \lambda$

$\sigma = \sqrt{\lambda}$

Povezava z našimi meritvami:

$m_0 = 1000 \pm \sqrt{1000} = 1000 \pm 32$

$m_1 = 400 \pm \sqrt{400} = 400 \pm 20$

Naredili smo približek, saj ne poznamo pravega λ.

$$m \sim \lambda$$

Iz napake meritev do napake rezultata (τ)

$$\sigma_\tau = d\tau \le \left|\frac{\partial \tau}{\partial R_0}\right| \overset{\sigma_{R_0}}{\overset{\shortparallel}{dR_0}} + \left|\frac{\partial \tau}{\partial R_1}\right| \overset{\sigma_{R_1}}{\overset{\shortparallel}{dR_1}} \quad (+ \text{ covarianca, ki je tu ni})$$

$$= \left|\frac{-t_1}{\ln^2(R_0/R_1)} \frac{R_1}{R_0} \frac{1}{R_1}\right| \sigma_{R_0} + \left|\frac{-t_1}{\ln^2\frac{R_0}{R_1}} \frac{R_1}{R_0} \frac{R_0}{R_1^2}\right| \sigma_{R_1} = \qquad \left(\tau = \frac{t_1}{\ln\frac{R_0}{R_1}}\right)$$

$$= \frac{1}{t_1}\left|\frac{t_1^2}{\ln^2(R_0/R_1)} \frac{1}{R_0}\right| \sigma_{R_0} + \frac{1}{t_1}\left|\frac{t_1^2}{\ln^2(R_0/R_1)} \frac{1}{R_1}\right| \sigma_{R_1}$$

$$\sigma_\tau = \frac{\tau^2}{t_1}\left(\left|\frac{\sigma_{R_0}}{R_0}\right| + \left|\frac{\sigma_{R_1}}{R_1}\right|\right)$$

$$\frac{\sigma_\tau}{\tau} \le \frac{\tau}{t_1}\left(\left|\frac{\sigma_{R_0}}{R_0}\right| + \left|\frac{\sigma_{R_1}}{R_1}\right|\right) = \frac{\tau}{t_1}\left(\frac{1}{\sqrt{n_0}} + \frac{1}{\sqrt{n_1}}\right)$$

$$\frac{\sigma_{R_0}}{R_0} = \frac{\frac{\sqrt{n_0}}{\Delta t}}{\frac{n_0}{\Delta t}} = \frac{\sqrt{n_0}}{n_0} = \frac{1}{\sqrt{n_0}}$$

$$\sigma_\tau \le \frac{\tau^2}{t_1}\left(\frac{1}{\sqrt{n_0}} + \frac{1}{\sqrt{n_1}}\right) = 0{,}95 \text{ dni}$$

$$\tau = 10{,}9 \text{ dni} \pm 0{,}95 \text{ dni}$$

Mi vemo še več: meritve so neodvisne ⇒ lahko seštejemo variance ⇒ boljša ocena

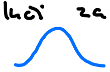
$$\sigma_\tau = \frac{\tau^2}{t_1}\sqrt{\left|\frac{1}{\sqrt{n_0}}\right|^2 + \left|\frac{1}{\sqrt{n_1}}\right|^2} = 0{,}7 \text{ dni}$$

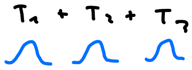
Konvolucija

1. Nek tip žarnic ima povprečno življenjsko dobo 1500 ur in std. dev. 150 ur. Tri žarnice so povezane tako, da ko ena žarnica crkne začne delovati naslednja. Življenjski časi so porazdeljeni normalno. Kolikšna je verjetnost, da bo luč delovala
   - a) vsaj $\tau = 5000$ ur
   - b) največ $\tau = 4200$ ur

T_i = čas delovanja i-te žarnice

T = čas delovanja luči za izhod v sili

$T = T_1 + T_2 + T_3 =$  ⇒ konvolucija (ker so +)



Ugotovimo, da je porazdelitev nakl. spr. T konvolucija porazdelitev T_1, T_2, T_3. Ker so T_1, T_2, T_3 porazdel. normalno velja

$$\mu = \langle T\rangle = \langle T_1\rangle + \langle T_2\rangle + \langle T_3\rangle = 4500 \text{ ur}$$

$$\sigma^2 = \operatorname{var} T = \operatorname{var} T_1 + \operatorname{var} T_2 + \operatorname{var} T_3 = 260 \text{ ur}$$

$$P(T \ge \tau) = \int_\tau^\infty f_T(t)\,dt = \frac{1}{\sqrt{2\pi}\,\sigma}\int_\tau^\infty e^{-\frac{(t-\mu)^2}{2\sigma^2}}\,dt \overset{u=\frac{t-\mu}{\sqrt{2}\sigma}}{=} \frac{1}{\sqrt{2\pi}\,\sigma}\sqrt{2}\,\sigma\int_{\frac{\tau-\mu}{\sqrt{2}\sigma}}^\infty e^{-u^2}\,du$$

$$P(T \geq \tau) = \frac{1}{\sqrt{\pi}} \frac{\sqrt{\pi}}{2} \operatorname{Erf}(u) \Big|_{\frac{\tau-\mu}{\sqrt{2}\sigma}}^{\infty} = \frac{1}{2}\left(1 - \operatorname{Erf}\Big(\underbrace{\frac{\tau-\mu}{\sqrt{2}\sigma}}_{1{,}36}\Big)\right)$$

$$= \frac{1}{2}(1 - 0{,}946) = 0{,}0272 = 2{,}72\%$$

(b) Verjetnost da bi svetil največ $\tau = 4200\,s$

$$P(T \leq \tau) = \int_0^{\tau} f_T(t)\,dt = \frac{1}{2}\left(\operatorname{Erf} \underbrace{\frac{\tau-\mu}{\sqrt{2}\sigma}}_{-0{,}8158} - \operatorname{Erf} \underbrace{\frac{0-\mu}{\sqrt{2}\sigma}}_{\approx -7 \;\to\; -1}\right)$$

$$= \frac{1}{2}(0{,}754 + 1) =$$

(2) Imejmo curek ionov, katerih energije T so porazdeljene kot:

$$f_T(t) = \lambda e^{-\lambda t}$$

kjer je T kin. en. ionov, λ karakteristična konst. pospeševalnika. Energijsko porazdelitev e^- želimo izmeriti z detektorjem, ki ima energijsko ločljivost σ. Izračunaj kakšno porazdelitev ionov bomo izmerili in kako se izmerjena povprečna energija iona razlikuje od prave.

Prava pričakovana vrednost T

$$\langle T \rangle = \int f_T(t)\, t\, dt = \int_0^{\infty} \lambda e^{-\lambda t} t \, dt = \frac{1}{\lambda} \int_0^{\infty} x e^{-x} dx = \frac{1}{\lambda}\Gamma(2) = \frac{1}{\lambda}$$

(to je resnica)

Kar mi zmerimo je pa: $z = T + x$

prava energija (T), „šum", ki ga prinese detektor (x)

Nas zanima kako se $E[z]$ razlikuje od $E[T]$?

V ta namen najprej določimo $f_z(z)$.

$$f_z(z) = f_T(t) * f_x(x) = \int f_T(\underbrace{z-x}_{T}) f_x(x)\,dx = \dots$$

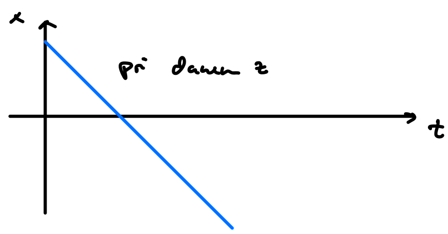
Šum detektorja je porazdeljen normalno

$$f_x(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{x^2}{2\sigma^2}}$$

Meje $x = z - t$

$t = 0: \quad x = z$

$t = \infty: \quad x = -\infty$



$$\dots = \int_{-\infty}^{z} f_T(z-x) f_x(x)\,dx =$$

$$= \int_{-\infty}^{z} \lambda e^{-\lambda(z-x)} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{x^2}{2\sigma^2}} dx = \lambda e^{-\lambda z} \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{z} e^{\lambda x - \frac{x^2}{2\sigma^2}} dx = \dots$$

Eksponent $-\lambda(z-x) - \frac{x^2}{2\sigma^2} = \frac{1}{2\sigma^2}\left(-2\sigma^2\lambda z + 2\sigma^2\lambda x - x^2\right) = -\lambda z - \frac{1}{2\sigma^2}(x - \sigma^2\lambda)^2 + \frac{\sigma^4\lambda^2}{2\sigma^2}$

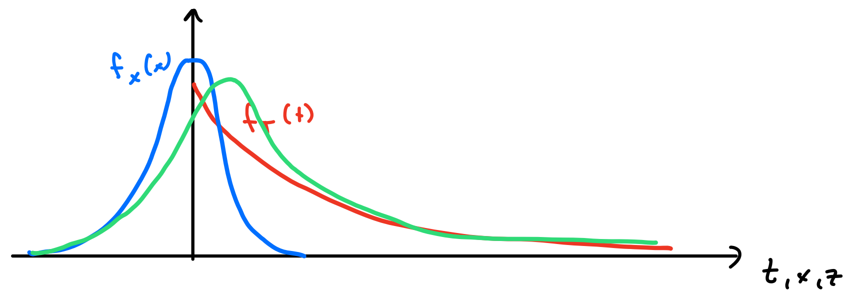
$$\dots = \frac{\lambda}{\sqrt{2\pi}\,\sigma}\, e^{-\lambda t + \frac{\sigma^2\lambda^2}{2}} \int_{-\infty}^{z} e^{-\frac{1}{2\sigma^2}(x-\sigma^2\lambda)^2}\, dx = \left| \begin{array}{l} \frac{(x-\sigma^2\lambda)^2}{2\sigma^2} = u^2 \\ \frac{x-\sigma^2\lambda}{\sqrt{2}\,\sigma} = u \end{array} \right| =$$

$$dx = \sqrt{2}\,\sigma\, du$$

$$= \frac{\lambda}{\sqrt{2\pi}\,\sigma}\, e^{-\lambda t + \frac{\sigma^2\lambda^2}{2}}\, \sqrt{2}\,\sigma \underbrace{\int_{-\infty}^{\frac{z-\sigma^2\lambda}{\sqrt{2}\sigma}} e^{-u^2}\, du}_{\frac{\sqrt{\pi}}{2} Erf(\,)} =$$

$$= \frac{\lambda}{\sqrt{\pi}}\, e^{-\lambda t + \frac{\sigma^2\lambda^2}{2}}\, \frac{\sqrt{\pi}}{2}\left(Erf\left(\frac{z-\sigma^2\lambda}{\sqrt{2}\,\sigma}\right) - \overset{-1}{Erf(-\infty)} \right) = \frac{\lambda}{2}\, e^{-\lambda t + \frac{\sigma^2\lambda^2}{2}} \left(1 + Erf\left(\frac{z-\sigma^2\lambda}{\sqrt{2}\,\sigma}\right)\right)$$

$z \in (-\infty, \infty)$



Iščemo povprečno izmerjeno energijo

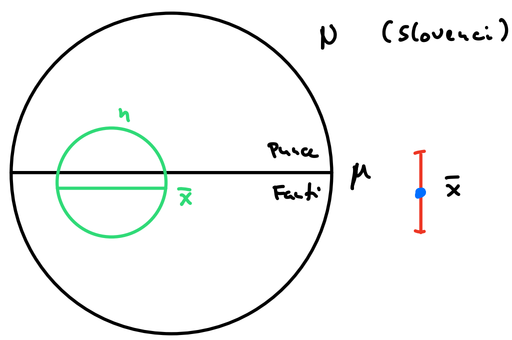
$$E[z] = \int_{-\infty}^{\infty} f_z(z)\, z\, dz = \int_{-\infty}^{\infty} z\, \frac{\lambda}{2}\, e^{-\lambda t + \frac{\sigma^2\lambda^2}{2}} \left(1 + Erf\left(\frac{z-\sigma^2\lambda}{\sqrt{2}\,\sigma}\right)\right) dz \overset{\text{Mathematica}}{=} \dots = \frac{1}{\lambda}$$

Odg: na našo srečo je pričakovana energ. za teh posameznih enaka.

Statistika

1. Prvi dan leta se je rodilo 100 otrok od tega 55 deklic in 45 dečkov. Določite interval zaupanja, na katerem lahko z 95% verjetnostjo pričakujemo pravo razmerje med deklicami in dečki.

$n = 100$ $n = 55$



$X =$ "rojena je deklica (punčka)"

$$\{x_i\}_{i=1,\dots,N} = \{1, 0, 0, 1, 1, 0, 1, \dots\}$$

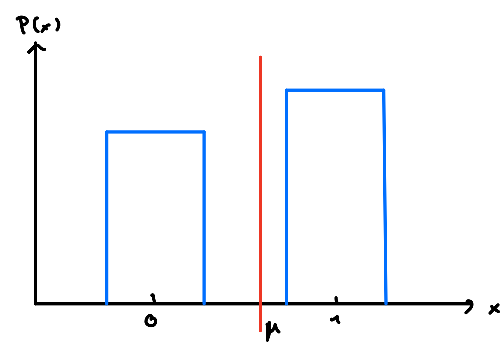
$$\mu = \frac{1}{N}\sum x_i = E[X]$$

Če poznamo μ, kolikšna je verjetnost, da je rojen otrok punčka ali fantek:

$$P(x; \mu) = \mu^x (1-\mu)^{1-x} \quad ; \quad x = \begin{cases} 1 & \text{Punčka} \\ 0 & \text{Fantek} \end{cases}$$

Bernoullijeva porazdelitev

$$E[x] = \mu \qquad Var[x] = \mu(1-\mu)$$



Težave: Mi lastnosti populacije ne poznamo. Poznamo le naš vzorec. Iz te vzorec lahko tvorimo neke statistike.

$$\{x_i\}_{i=1,\dots,n} = \{1,0,1,0,0,1,1,1,\dots\}_{100}$$

$$\bar{X} = \frac{1}{n}\sum_i x_i = \frac{55}{100} = 0{,}35 \quad \text{Vzorčno povprečje}$$

$$S_x^2 = \left(\begin{smallmatrix}\text{po Bernoullijevi}\\ \text{porazdelitvi}\end{smallmatrix}\right) = \bar{x}(1-\bar{x}) \quad \text{Vzorčna varianca}$$

- Iz vzorčnih statistik želimo sklepati na lastnosti populacije. Zanima nas μ.
- Zanima nas, kako blizu je $\bar{x}$ pravemu povprečju μ. Zato vpeljemo novo statistiko:

$$T = \frac{\bar{x}-\mu}{S_x^2}\sqrt{n}$$

Če je x porazdeljen normalno je potem T porazdeljen po Studentovi porazdelitvi s parametrom $\nu = n-1$.

Delamo se, da je naša porazdelitev normalna (čeprav je Bernoullijeva)

Nadaljevanje v zapiskih asistenta.