

Deformacijski tenzor

$\vec{u}(\vec{r})$... vektor premika

$$d\ell'^2 - d\ell^2 = 2u_{ij} dx_i dx_j$$

$$u_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_k}{\partial x_i} \frac{\partial u_k}{\partial x_j} \right)$$

zanemarimo

$$u_{ij} = \frac{1}{2} \left((\partial \vec{u})_{ij} + (\partial \vec{u})_{ji} \right) \quad \text{splošno}$$

Splošna deformacija

σ_{ij} ... napetostni tenzor

$$f_i = \frac{\partial}{\partial x_i} \sigma_{ij} \quad \text{površinska sila}$$

i-ta komponenta površinske

gostote sile na j-to komponento površine

$$F_i = \oint \sigma_{ij} dS_j$$

normala

Sprememba volumena v last. sist.

$$\frac{\Delta V}{V} = u_{kk}$$

Izotropni tlak $\sigma_{ij} = -p\delta_{ij}$

Hookov zakon

$$\sigma_{ij} = 2\mu u_{ij} + \lambda u_{kk} \delta_{ij}$$

$$u_{ij} = \frac{1}{2\mu} \left(\sigma_{ij} - \frac{\lambda}{(2\mu + 3\lambda)} \sigma_{kk} \delta_{ij} \right)$$

$$\nabla^2 \vec{u} = \vec{f} + \mu \nabla^2 \vec{u} + (\mu + \lambda) \nabla \nabla \cdot \vec{u}$$

Navierova enačba

$$\mu = \frac{E}{2(1+\sigma)} \quad \lambda = \frac{E\sigma}{(1-2\sigma)(1+\sigma)}$$

$$E = \frac{\mu(2\mu + 3\lambda)}{\mu + \lambda} \quad \sigma = \frac{1}{2(1 + \mu/\lambda)}$$

$$\sigma_{ij} = \frac{E}{1+\sigma} \left(u_{ij} + \frac{\sigma}{1-2\sigma} u_{kk} \delta_{ij} \right)$$

↪ $u_{kk} = \nabla \cdot \vec{u}$

$$u_{ij} = \frac{1}{E} \left((1+\sigma) \sigma_{ij} - \sigma \sigma_{kk} \delta_{ij} \right)$$

Navierova enačba

$$\nabla^2 \vec{u} = \vec{f} + \frac{E}{2(1+\sigma)} (\nabla^2 \vec{u} + \frac{1}{1-2\sigma} \nabla \nabla \cdot \vec{u})$$

telesno porazdeljena gostota sile

R.P.

• napetost na pov. je nič ker na njo ne deluje površinska sila
Nihanje: na stavki $u = u_0 e^{i\omega t}$

Tanka plošča

$u \dots$ odmik v z smeri

$$u \nabla^2 \nabla^2 u - p = 0$$

$$k = \frac{Eh^3}{12(1-\sigma^2)} \quad \text{površinska gostota sile (tlak)}$$

Za okroglo ploščo $u(r)$ [gib]

$$\nabla^4 u = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) \right) \right)$$

R.P.

to so vrata $u=0 \quad \frac{\partial u}{\partial n} = 0$ (na robu)

prislupek $u=0 \quad \frac{\partial^2 u}{\partial n^2} + \sigma \frac{\partial^2 u}{\partial r^2} = 0$

↪ $1/r$

Upogib palic

$u \dots$ dolžinska gostota zuv. sil

$$M = EI \left(-\frac{d^2 y}{dx^2}, \frac{d^2 x}{dx^2}, 0 \right) = EI (-\ddot{y}, \ddot{x}, 0)$$

$$F_x = -EI \ddot{x}_y + F_0 \ddot{x}_y$$

$$EI \ddot{x}_y^{(4)} - F_0 \ddot{x}_y - F_2 \ddot{x}_y - k_y \ddot{x}_y = 0$$

Krog $I = \frac{\pi R^4}{4}$

Kvader $I_1 = \frac{b^3 a}{12} \quad I_2 = \frac{a^3 b}{12}$

R.P.

vrata $x=0 \quad \dot{x}=0$

prosta $F=0 \quad M=0$

vrhlovo vrata $x=0 \quad M=0$

Struna $u = u_0 \sin kx e^{i\omega t}$

Rešitve DE

polinomi in racionalne funkcije (za sfere)

$$x^{(4)} = c \rightarrow \text{pol. 4. st.}$$

$$x^{(4)} + k^2 x = 0 \rightarrow A \sin kx + B \cos kx + Cx + D$$

$$x^{(4)} - k^4 x = 0 \rightarrow A \sinh kx + B \cosh kx + Cx + D$$

$$\ddot{x} - k^2 x = 0 \rightarrow \text{hiperbolični}$$

$$\frac{F}{s} = E \frac{\Delta \ell}{\ell}$$

$$\frac{\Delta \ell}{\ell} = \Delta T$$

$$E = \frac{\sigma_{xx}}{u_{xx}}$$

$$\sigma = -\frac{u_{xx}}{u_{xx}}$$

$$G = \frac{\sigma_{xy}}{2u_{xy}}$$

Matematične zveze

$$\epsilon_{ijk} \epsilon_{ijl} = 2\delta_{kl}$$

$$\epsilon_{ijk} \epsilon_{lmk} = \delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}$$

$$\epsilon_{ijk} \epsilon_{ijk} = 6$$

$$\nabla \nabla \cdot = \nabla^2 + \nabla \times \nabla \times$$

Desglav DE $\nabla \nabla \cdot \vec{u} + \mu \nabla^2 \vec{u} = 0$

Sfera Desglav $\frac{\partial^2 j_\ell}{\partial r^2} + \frac{2}{r} \frac{\partial j_\ell}{\partial r} + \left(1 - \frac{\ell(\ell+1)}{r^2}\right) j_\ell = 0$

$$-x^\ell j_{\ell+1} = (x^{-\ell} j_\ell)'$$

$$x^{\ell+1} j_{\ell-1} = (x^{\ell+1} j_\ell)'$$

$$f(x)H(x-x_0) = (F(x) - F(x_0))H(x-x_0) + C$$

Cilindrične koordinate

$$d^3r = r dr d\phi dz,$$

$$\nabla f = \frac{\partial f}{\partial r} \mathbf{e}_r + \frac{1}{r} \frac{\partial f}{\partial \phi} \mathbf{e}_\phi + \frac{\partial f}{\partial z} \mathbf{e}_z,$$

$$\nabla \cdot \vec{v} = \frac{1}{r} \frac{\partial (rv_r)}{\partial r} + \frac{1}{r} \frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z},$$

$$\nabla \times \vec{v} = \left(\frac{1}{r} \frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z} \right) \mathbf{e}_r + \left(\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r} \right) \mathbf{e}_\phi +$$

$$+ \frac{1}{r} \left(\frac{\partial (rv_\phi)}{\partial r} - \frac{\partial v_r}{\partial \phi} \right) \mathbf{e}_z,$$

$$\nabla^2 f = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial f}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2},$$

$$\nabla^2 \vec{v} = \left(\nabla^2 v_r - \frac{v_r}{r^2} - \frac{2}{r^2} \frac{\partial v_\phi}{\partial \phi} \right) \mathbf{e}_r +$$

$$+ \left(\nabla^2 v_\phi - \frac{v_\phi}{r^2} + \frac{2}{r^2} \frac{\partial v_r}{\partial \phi} \right) \mathbf{e}_\phi + \nabla^2 v_z \mathbf{e}_z,$$

Sferične koordinate

$$d^3r = r^2 dr d(\cos \theta) d\phi,$$

$$\nabla f = \frac{\partial f}{\partial r} \mathbf{e}_r + \frac{1}{r} \frac{\partial f}{\partial \theta} \mathbf{e}_\theta + \frac{1}{r \sin \theta} \frac{\partial f}{\partial \phi} \mathbf{e}_\phi,$$

$$\nabla \cdot \vec{v} = \frac{1}{r^2} \frac{\partial (r^2 v_r)}{\partial r} + \frac{1}{r \sin \theta} \left(\frac{\partial (\sin \theta v_\theta)}{\partial \theta} + \frac{\partial v_\phi}{\partial \phi} \right),$$

$$\nabla \times \vec{v} = \frac{1}{r \sin \theta} \left(\frac{\partial (\sin \theta v_\phi)}{\partial \theta} - \frac{\partial v_\theta}{\partial \phi} \right) \mathbf{e}_r + \frac{1}{r} \left[\left(\frac{1}{\sin \theta} \frac{\partial v_r}{\partial \phi} - \frac{\partial (rv_\phi)}{\partial r} \right) \mathbf{e}_\theta + \left(\frac{\partial (rv_\theta)}{\partial r} - \frac{\partial v_r}{\partial \theta} \right) \mathbf{e}_\phi \right],$$

$$\nabla^2 f = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial f}{\partial r} \right) + \frac{1}{r^2} \left[\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial f}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2 f}{\partial \phi^2} \right].$$

Elastomehanika

Kinematika deformacije

$\mathbf{u}(\mathbf{r}) = \mathbf{r}'(\mathbf{r}) - \mathbf{r}$ (vektor premika)
 $u_{ik} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} + \frac{\partial u_l}{\partial x_i} \frac{\partial u_l}{\partial x_k} \right) \approx \frac{1}{2} \left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} \right)$
(deformacijski tenzor)

$d\mathbf{r}'^2 = d\mathbf{r}^2 + 2u_{ik} dx_i dx_k$
 $\frac{dV'}{dV} = \text{tr } \mathbf{u} = u_{ii}$

$$\begin{bmatrix} u_{rr} & u_{r\vartheta} & u_{r\varphi} \\ u_{\vartheta r} & u_{\vartheta\vartheta} & u_{\vartheta\varphi} \\ u_{\varphi r} & u_{\varphi\vartheta} & u_{\varphi\varphi} \end{bmatrix} = \begin{bmatrix} \frac{\partial u_r}{\partial r} & \frac{1}{2} \left(\frac{\partial u_\vartheta}{\partial r} - \frac{u_\vartheta}{r} + \frac{1}{r} \frac{\partial u_r}{\partial \vartheta} \right) & \frac{1}{2} \left(\frac{\partial u_\varphi}{\partial r} - \frac{u_\varphi}{r} + \frac{1}{r \sin \vartheta} \frac{\partial u_r}{\partial \varphi} \right) \\ \frac{1}{r} \frac{\partial u_r}{\partial \vartheta} + \frac{u_r}{r} & \frac{1}{2} \left(\frac{\partial u_\vartheta}{\partial \vartheta} - \frac{u_\vartheta}{r \tan \vartheta} + \frac{1}{r \sin \vartheta} \frac{\partial u_\vartheta}{\partial \varphi} \right) & \frac{1}{2} \left(\frac{\partial u_\varphi}{\partial \vartheta} - \frac{u_\varphi}{r \tan \vartheta} + \frac{1}{r \sin \vartheta} \frac{\partial u_\varphi}{\partial \varphi} \right) \\ \frac{1}{r \sin \vartheta} \frac{\partial u_r}{\partial \varphi} + \frac{u_r}{r} & \frac{1}{r \sin \vartheta} \frac{\partial u_\vartheta}{\partial \varphi} + \frac{u_\vartheta}{r} & \frac{1}{r \sin \vartheta} \frac{\partial u_\varphi}{\partial \varphi} + \frac{u_\varphi}{r} \end{bmatrix}$$
$$\begin{bmatrix} u_{rr} & u_{r\varphi} & u_{rz} \\ u_{\varphi r} & u_{\varphi\varphi} & u_{\varphi z} \\ u_{zr} & u_{z\varphi} & u_{zz} \end{bmatrix} = \begin{bmatrix} \frac{\partial u_r}{\partial r} & \frac{1}{2} \left(\frac{\partial u_\varphi}{\partial r} - \frac{u_\varphi}{r} + \frac{1}{r} \frac{\partial u_r}{\partial \varphi} \right) & \frac{1}{2} \left(\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z} \right) \\ \frac{1}{r} \frac{\partial u_r}{\partial \varphi} + \frac{u_r}{r} & \frac{1}{2} \left(\frac{\partial u_\varphi}{\partial \varphi} - \frac{u_\varphi}{r} + \frac{1}{r} \frac{\partial u_r}{\partial \varphi} \right) & \frac{1}{2} \left(\frac{\partial u_z}{\partial \varphi} + \frac{\partial u_\varphi}{\partial z} \right) \\ \frac{1}{2} \left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} \right) & \frac{1}{2} \left(\frac{\partial u_\varphi}{\partial z} + \frac{\partial u_z}{\partial \varphi} \right) & \frac{\partial u_z}{\partial z} \end{bmatrix}$$

Mehanična napetost

Predpostavimo, da sile delujejo le preko stične ploskve.

$F_i = \int f_i(\mathbf{r}) d^3\mathbf{r} + \int f_i^{(z)}(\mathbf{r}) d^3\mathbf{r}$
 $\int_V f_i(\mathbf{r}) d^3\mathbf{r} = \oint_{\partial V} p_{ik} dA_k$ (def. napetostnega tenzorja)
 $f_i = \frac{\partial p_{ik}}{\partial x_k}$

Iz poljubnega napetostnega tenzorja lahko vedno tvorimo enakovreden napetostni tenzor, ki je simetričen.

$\rho \ddot{u}_i = f_i^{(z)} + \frac{\partial p_{ik}}{\partial x_k}$ (Cauchyeva enačba)

$\delta W = \int f_i \delta u_i d^3\mathbf{r} = - \int p_{ik} \delta u_{ik} d^3\mathbf{r}$
 $d\mathbf{f} = d \left(\frac{f}{V} \right) = -s dT + d\mathbf{w} = -s dT + p_{ik} du_{ik}$
 $p_{ik} = \left(\frac{\partial f}{\partial u_{ik}} \right)_T$

Hookov zakon

$f = f_0 + \frac{1}{2} u_{ik} C_{iklm} u_{lm}$
 $p_{ik} = C_{iklm} u_{lm}$

Izotropno sredstvo

$p_{ik} = \lambda u_{ll} \delta_{ik} + 2\mu u_{ik} = \frac{E}{1+\sigma} \left(u_{ik} + \frac{\sigma}{1-2\sigma} u_{ll} \delta_{ik} \right)$
 $u_{ik} = \frac{1}{2\mu} \left(p_{ik} - \frac{\lambda}{3\lambda+2\mu} p_{ll} \delta_{ik} \right) = \frac{1}{E} [(1+\sigma)p_{ik} - \sigma p_{ll} \delta_{ik}]$

$\lambda = K - \frac{2\mu}{3} = \frac{\sigma E}{(1+\sigma)(1-2\sigma)}$

$\mu = \frac{E}{2(1+\sigma)}$
 $\sigma = \frac{\lambda}{2(\lambda+\mu)}$
 $E = \frac{\mu(3\lambda+2\mu)}{\mu+\lambda}$
 $K = \frac{E}{3(1-2\sigma)}$

Prosta energija

$f = f_0 + \frac{1}{2} u_{ll}^2 + \mu u_{ik} u_{ki} = f_0 + \mu \left(u_{ik} - \frac{1}{3} u_{ll} \delta_{ik} \right)^2 + \frac{K}{2} u_{ll}^2 =$
 $= f_0 + \frac{E}{2} \left[\frac{\sigma}{(1+\sigma)(1-2\sigma)} u_{ll}^2 + \frac{1}{1+\sigma} u_{ik} u_{ki} \right]$

$\rho \ddot{\mathbf{u}} = \mathbf{f}^{(z)} + (\lambda + \mu) \nabla \nabla \cdot \mathbf{u} + \mu \nabla^2 \mathbf{u} =$ (Navierova enačba)
 $= \mathbf{f}^{(z)} + \frac{E}{2(1+\sigma)} \left[\nabla^2 \mathbf{u} + \frac{1}{1-2\sigma} \nabla \nabla \cdot \mathbf{u} \right]$

Plošče

Nevaltarne ploskev je ploskev v materialu, ki ne doživi ne raztega ne skrčeka.

Mongeova reprezentacija: $z \zeta(x, y)$ podamo premik nevtralne ploskve vzdolž osi z .

Zaradi tankosti je p enak po celi snovi.

Zunanje sile so mnogo manjše od notranjih napetosti \Rightarrow

$p_{xz} = p_{yz} = p_{zz} = 0.$
 $\mathbf{u} = \begin{bmatrix} -z \frac{\partial^2 \zeta}{\partial x^2} & -z \frac{\partial^2 \zeta}{\partial x \partial y} & 0 \\ & -z \frac{\partial^2 \zeta}{\partial y^2} & 0 \\ \frac{\sigma}{1-\sigma} z \left(\frac{\partial^2 \zeta}{\partial x^2} + \frac{\partial^2 \zeta}{\partial y^2} \right) \end{bmatrix}$

$f = f_0 + \frac{E z^2}{2(1-\sigma^2)} \left\{ \left(\frac{\partial^2 \zeta}{\partial x^2} + \frac{\partial^2 \zeta}{\partial y^2} \right)^2 + 2(1-\sigma) \left[\left(\frac{\partial^2 \zeta}{\partial x \partial y} \right)^2 - \frac{\partial^2 \zeta}{\partial x^2} \frac{\partial^2 \zeta}{\partial y^2} \right] \right\}$

$D = \frac{E h^3}{12(1-\sigma^2)}$
 $D \nabla^2 \nabla^2 z = 0$ (enačba ravnovesja)

Robni pogoj prislonjene plošče: $\frac{\partial^2 \zeta}{\partial n^2} + \sigma \frac{\partial \varphi}{\partial l} \frac{\partial \zeta}{\partial n} = 0$

Palice

Torzija

$\tau = \frac{d\varphi}{dz}$
 $\mathbf{u} = \tau(-yz, xz, \psi(x, y))$

$u_{xz} = \frac{\tau}{2} \left(-y + \frac{\partial \psi}{\partial x} \right)$
 $u_{yz} = \frac{\tau}{2} \left(x + \frac{\partial \psi}{\partial y} \right)$
 $p_{xz} = \mu \tau \left(-y + \frac{\partial \psi}{\partial x} \right) = 2\mu \tau \frac{\partial \chi}{\partial x}$
 $p_{yz} = \mu \tau \left(x + \frac{\partial \psi}{\partial y} \right) = -2\mu \tau \frac{\partial \chi}{\partial y}$
Vsi ostali u_{ij}, p_{ij} so enaki nič.

Ravnovesje

$\nabla^2 \psi = 0$
 $\nabla^2 \chi = -1$
Prost robni pogoj ($\mathbf{f} = 0$) vzdolž roba preseka: $\chi|_{\text{rob}} = \text{konst.}$
 $M_z = -2\mu \tau \int \mathbf{r} \cdot \nabla \chi dS = G \tau$

$C = 4\mu \int (\nabla \chi)^2 d\mathbf{x} d\mathbf{y}$
 $F = \frac{1}{2} \int C \tau^2 dz$

Upogib

$u_{zz} = \frac{x}{R}$
 $u_{xx} = u_{yy} = -\sigma \frac{x}{R}$
 $p_{zz} = E \frac{x}{R}$

Vsi ostali u_{ij}, p_{ij} so enaki nič.
 $\mathbf{u} = \left(-\frac{1}{2R} [x^2 + \sigma(x^2 - y^2)], -\frac{\sigma xy}{R}, \frac{xz}{R} \right)$

$I = \int x^2 d\mathbf{x} d\mathbf{y}$
 $F = \frac{1}{2} \int E I \frac{1}{R^2} dz$

Globalna teorija palic

$\hat{\mathbf{t}}(l) = \frac{\partial \mathbf{r}(l)}{\partial l}$ (tangenta)
 $\hat{\mathbf{n}}(l) = R \frac{\partial \hat{\mathbf{t}}(l)}{\partial l}$ (normala)
 $\frac{1}{R} = \hat{\mathbf{n}} \cdot \frac{\partial \hat{\mathbf{t}}}{\partial l} = \left| \frac{\partial \hat{\mathbf{t}}}{\partial l} \right|$
 $\hat{\mathbf{t}} \parallel \hat{\mathbf{e}}_\zeta, \quad \hat{\mathbf{t}} \perp \hat{\mathbf{e}}_\xi, \hat{\mathbf{e}}_\eta$

$\frac{\partial \hat{\mathbf{t}}}{\partial l} = \Omega \times \hat{\mathbf{t}}$
 $I_{ij} = \iint (r^2 \delta_{ij} - r_i r_j) d\mathbf{x} \eta d\mathbf{x} \xi, \quad \mathbf{r} = (x_\eta, x_\xi)$
 $F = \frac{1}{2} \int \left[E(I_\eta \Omega_\eta^2 + 2I_\eta \xi \Omega_\eta \Omega_\xi + I_\xi \xi \Omega_\xi^2) + C \Omega_\xi^2 \right] dl$

Ravnovesje

$\frac{d\mathbf{F}}{dl} = -\mathbf{K}$
 $\frac{d\mathbf{M}}{dl} + \hat{\mathbf{t}} \times \mathbf{F} = 0$

Samo upogib (tudi velike deformacije): $E I \frac{d\mathbf{r}}{dl} \times \frac{d^3 \mathbf{r}}{dl^3} = \mathbf{F} \times \frac{d\mathbf{r}}{dl}$

Majhen upogib pravokoten na os z

Pišemo vektorje samo v xy ravnini. $\mathbf{t} \doteq (\dot{x}, \dot{y}, 1)$

$\mathbf{M} = E I \left(-\frac{d^2 y}{dz^2}, \frac{d^2 x}{dz^2} \right)$

$\mathbf{F} = -E I \frac{d^3 \mathbf{r}}{dz^3} + F_z \frac{d\mathbf{r}}{dz}$
 $E I \frac{d^4 \mathbf{r}}{dz^4} - F_z \frac{d^3 \mathbf{r}}{dz^3} - \frac{dF_z}{dz} \frac{d\mathbf{r}}{dz} - \mathbf{K} = 0 = -\mathcal{L} \frac{\partial^4 \mathbf{u}}{\partial z^4}$

Elastični valovi

Longitudinalni val: $\nabla \times \mathbf{u} = 0$
 $c_l^2 = \frac{E(1-\sigma)}{\rho(1+\sigma)(1-2\sigma)} = \frac{\lambda+2\mu}{\rho} = \frac{K+4\mu/3}{\rho}$
Transverzalni val: $\nabla \cdot \mathbf{u} = 0$
 $c_t^2 = \frac{E}{2\rho(1+\sigma)} = \frac{\mu}{\rho} < c_l^2$

$\ddot{\mathbf{u}} = -c_l^2 \nabla \times \nabla \times \mathbf{u} + c_t^2 \nabla \nabla \cdot \mathbf{u}$
 $p_{ik} = 2\rho c_t^2 u_{ik} + \rho(c_l^2 - 2c_t^2) u_{ll} \delta_{ik}$

Odboj in lom:

$k_{\parallel}^{\text{vpadni}} = k_{\parallel}^{\text{odbiti}} = k_{\parallel}^{\text{lomljeni}}$