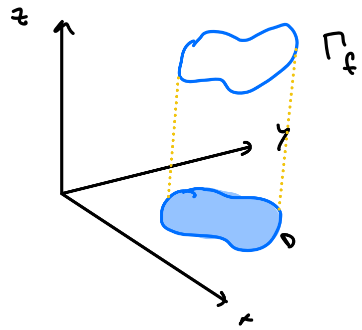
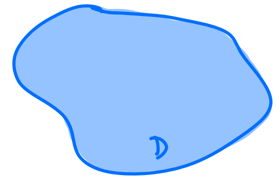
Ploskve (v $\mathbb{R}^3$)

Imejmo omejeno območje $D \subset \mathbb{R}^2$ in $f: D \to \mathbb{R}^3$.





Pričakujemo, da bomo graf
$\Gamma_f = \{(x, y, f(x,y)) \,;\, (x,y) \in D\}$
lahko rekli: ploskev

parametra

Def <u>Ploskev</u> v $\mathbb{R}^3$ je podana s parametrizacijo
$\vec{r} = \vec{r}(u, v) : D \to \mathbb{R}$,
kjer je D neko območje v $\mathbb{R}^2$ in $\vec{r} = (x, y, z)$ neka preslikava razreda (vsaj) C_1. Zahtevamo, da je $\vec{r}_u \times \vec{r}_v \neq 0$, tedaj pravimo, da je parametrizacija <u>regularna</u>. Pravimo, da sta u, v (krivočrtni) koordinati na ploskvi
$M = \{\vec{r}(u,v) \,;\, (u,v) \in D\}$.

$\frac{\partial \vec{r}}{\partial u}$

Komentar k pogoju $\vec{r}_u \times \vec{r}_v \neq 0$. Os oznakah $\vec{r} = \vec{r}(u,v) = (x, y, z)$ pogoj pomeni, da je

$$\begin{vmatrix} i & j & k \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix} \neq 0 \quad \text{povsod}$$

Recimo, da je v dani točki $\vec{r}(u,v)$ neničelna tretja komponenta

$$\begin{vmatrix} x_u & y_u \\ x_v & y_v \end{vmatrix} \neq 0$$

Tedaj za bližnje točke (slike $\vec{r}(u,v)$, torej za majhne $h, k \in \mathbb{R}$) velja

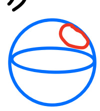
$$X = X(u+h, v+k) \doteq X(u,v) + X_u(u,v)\,h + X_v(u,v)\,k$$
$$Y = Y(u+h, v+k) \doteq Y(u,v) + Y_u(u,v)\,h + Y_v(u,v)\,k$$

linearen sistem, ki je enolično rešljiv če $\det \neq 0$

$$\begin{vmatrix} X_u & Y_u \\ X_v & Y_v \end{vmatrix} \neq 0$$

Hkrati je
$Z = Z(u+h, v+k)$ lokalno funkcija X, Y

Torej je ploskev lokalno podana kot graf,
$(X, Y, Z(X,Y))$

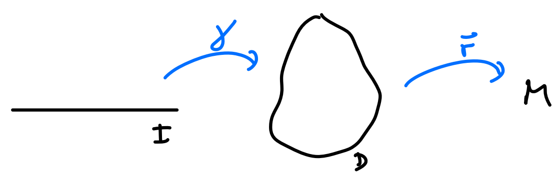


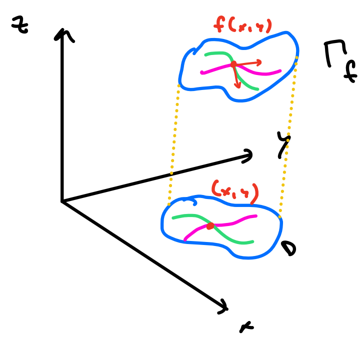
lokalno je graf

Tangentna ravnina

V območju D imejmo krivuljo $\gamma(t) = (\overset{u}{\alpha(t)}, \overset{v}{\beta(t)})$; $t \in I$

Slikajmo γ na ploskev
$\vec{r} \circ \gamma = \vec{r}(\alpha(t), \beta(t))$.





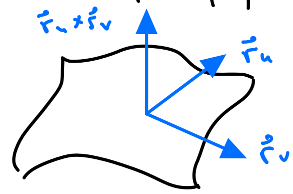
Velja:

$$\mathbb{R}'(t) = \vec{r}_u(\alpha(t), \beta(t)) \cdot \dot{\alpha}(t) + \vec{r}_v(\alpha(t), \beta(t)) \cdot \dot{\beta}(t)$$

To pomeni, da so tangentni vektorji na vse krivulje v M, ki grdo skozi točko $\vec{r}(u,v) \in M$ oblike
$a \cdot \vec{r}_u(u,v) + b\,\vec{r}_v(u,v)$ za neka $a, b \in \mathbb{R}$

Če torej tangentno ravnino na ploskev M v točki $m \in M$ def. kot unijo vseh tangentnih vektorjev na krivulji v M skozi točko m, premaknjene za m, tedaj je ta ravnina enaka

$$m + \underbrace{\text{Lin}}_{\text{linearna ogrinjača}}\{\vec r_u(u,v), \vec r_v(u,v)\}$$



Torej je normala tangentne ravnine enaka $(\vec r_u \times \vec r_v)(u,v)$.
Dobili smo enačbo tangentne ravnine

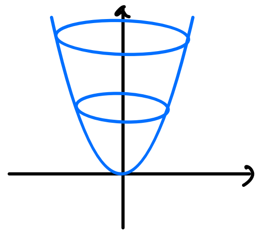
$$\langle (x,y,z) - \vec r(u,v), (\vec r_u \times \vec r_v)(u,v)\rangle = 0$$

Primer $f: D \to \mathbb{R}$ $D \subset \mathbb{R}^2$

$M = \Gamma_f$ $\vec r = (x, y, f(x,y))$

$\vec r_x \times \vec r_y = (-f_x, -f_y, 1)$

Primer Rotacijski paraboloid



M $z = x^2 + y^2 = f(x,y)$

v točki $\vec r(1,3) = (1, 3, 1^2 + 3^2)$

imamo tangentno ravnino z enačbo

$$\langle (x,y,z) - (1,3,10), (-2\cdot 1, -2\cdot 3, 1)\rangle = 0$$
$$-2(x-1) - 6(y-3) + z - 10 = 0$$
$$-2x + 2 - 6y + 18 + z - 10 = 0$$
$$-2x - 6y + z = -10$$
$$2x + 6y - z = 10$$

Implicitno podane ploskve

Alternativni opis ploskve:
za dano funkcijo $F: \mathbb{R}^3 \to \mathbb{R}$ def. ploskev kot nivojnico $\{F = 0\}$, to je, vse $(x,y,z) \in \mathbb{R}^3$ da:

$$F(x,y,z) = 0$$

Primer $\sin(xy) - e^{\sqrt{4y-z}} - 11\log(z^4 + x^6 + y^2) = 0$

Pogoj

$$\nabla F = (F_x, F_y, F_z) \neq 0$$

∇ "nabla" oz. gradient

Tangentna ravnina na tako (= implicitno) podano ploskev v točki $m \in M$
Za poljubno krivuljo $\vec r = \vec r(t) = (x(t), y(t), z(t))$ na ploskvi velja

$$F(x,y,z) = 0, \quad \text{oz.} \quad F(x(t), y(t), z(t)) = 0 \quad \Big/ \frac{d}{dt}$$

Pišemo $F = F(u,v,w)$. Dobimo

$$F_u(\vec r(t))\cdot \dot x(t) + F_v(\vec r(t))\,\dot y(t) + F_w(\vec r(t))\,\dot z(t) = 0 \quad \text{oz.}$$

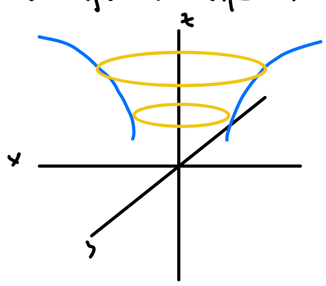
$$\langle \underbrace{(\nabla F)(\underbrace{\vec r(t)}_{m})}_{\text{Neodvisno od izbire krivulje (skozi točko } m\in M)}, \underbrace{\dot{\vec r}(t)}_{\text{leži v tang. ravnini}} \rangle_{\mathbb{R}^3} = 0 \qquad \dot{\vec r}(t) = (\dot x(t), \dot y(t), \dot z(t))$$

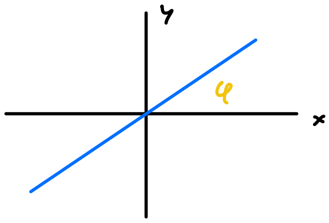
Normala na tang. ravnino je $(\nabla F)(\vec r)$, zato se enačba tang. ravn. glasi

$$\langle (x,y,z) - \vec r(t_0), (\nabla F)(\vec r(t_0)) \rangle = 0 \qquad m = \vec r(t_0)$$

Rotacijsko invariantne ploskve

Situacija: krivulja Γ v xz-ravnini zavrtimo okoli z osi:



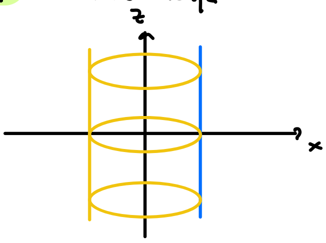


Torej, če je Γ dana z $(x(t), 0, z(t))$ $t \in I$, tedaj po rotaciji okoli z osi za kot φ dobimo

$$\begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ 0 \\ z(t) \end{bmatrix} = \begin{bmatrix} x(t)\cos\varphi \\ x(t)\sin\varphi \\ z(t) \end{bmatrix}$$

Za $(t,\varphi) \in I \times [0, 2\pi) = D$ je parametrizacija tako dobljene ploskve M.

Primer Plašč valja

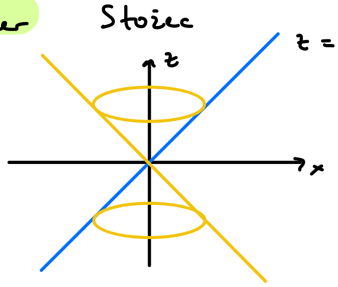


$\vec r(t) = (a, 0, t)$

Dobim

$$(t,\varphi) \to \begin{bmatrix} a\cos\varphi \\ a\sin\varphi \\ t \end{bmatrix}$$

Primer Stožec



$z = ax$; $a > 0$

$\vec r(t) = (t, 0, at)$

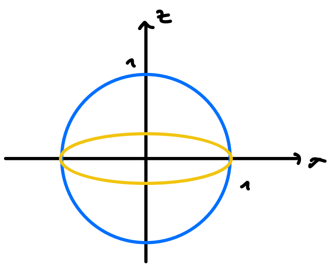
Dobim

$x = t\cos\varphi$

$y = t\sin\varphi$

$z = at$

Primer Sfera v $\mathbb{R}^3$



$\vec r(t) = (\cos(t), 0, \sin(t))$

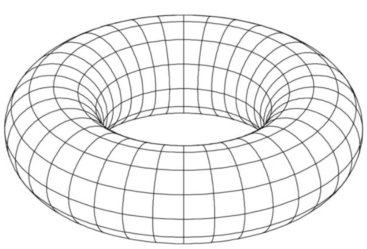
Dobim

$x = \cos t \cos\varphi$

$y = \cos t \sin\varphi$

$z = \sin t$

Primer torus



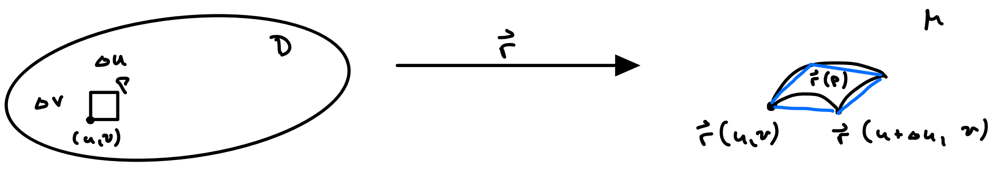
Dobim

$x = (R + \rho\cos t)\cos\varphi$

$y = (R + \rho\cos t)\sin\varphi$

$z = \rho\sin t$

Površina ploskve



$P(M) =$ vsota površin $\vec r(P)$ po vseh P

$|\vec r(P)| \doteq$ ploščina paralelograma

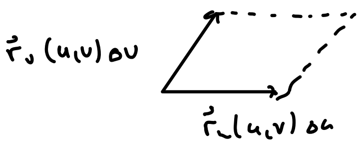
$\vec r(u, v+\Delta v) - \vec r(u,v) \doteq \vec r_v(u,v)\,\Delta v$

$\vec r(u+\Delta u, v) - \vec r(u,v)$

$= \vec r_u(u+\xi\Delta u, v)\,\Delta u \qquad \xi\in[0,1]$

Lagrange izrek $\quad \doteq \vec r_u(u,v)\,\Delta u$

Torej je ploščina paralelograma približno enaka ploščini paralelograma:



ki pa je enaka

$|(\Delta u\, \vec r_u(u,v)) \times (\Delta v\, \vec r_v(u,v))| = |\vec r_u \times \vec r_v|\,\Delta u\,\Delta v$

Dobili smo

„$|\vec r(P)|$" $\doteq |\vec r_u \times \vec r_v|\,\Delta v\,\Delta u$

Ta nerigorozni premislek uporabimo kot navdih za (eksaktno) definicijo površine ploskve M, podane s parametrizacijo $\vec r = \vec r(u,v)$; $(u,v)\in D$:

$$\boxed{P(\vec r(D)) = \iint_D |\vec r_u \times \vec r_v|\,du\,dv} \qquad (1)$$

Velja

kot med vektorjema

$$|\vec r_u \times \vec r_v| = |\vec r_u||\vec r_v| \sin\varphi = |\vec r_u||\vec r_v|\sqrt{1-\cos^2\varphi} = \sqrt{|\vec r_u|^2|\vec r_v|^2 - |\vec r_u|^2|\vec r_v|^2\cos^2\varphi}$$
$$= \sqrt{|\vec r_u|^2|\vec r_v|^2 - \langle \vec r_u, \vec r_v\rangle^2}$$
$$= \sqrt{EG - F^2}$$

kjer so

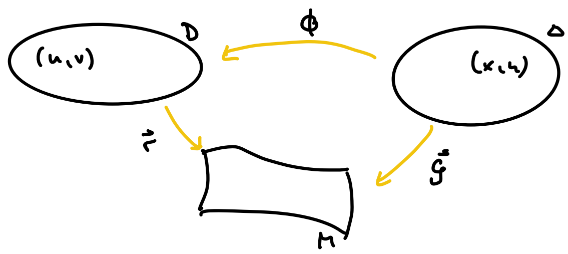
$E = \langle \vec r_u, \vec r_u\rangle = |\vec r_u|^2$

$G = \langle \vec r_v, \vec r_v\rangle = |\vec r_v|^2$

$F = \langle \vec r_u, \vec r_v\rangle$

Trditev Definicija (1) je dobra (to pomeni, da je neodvisna od parametrizacije)

Dokaz Imejmo parametrizaciji $\vec r$, $\vec\rho$ in bijekcijo Φ med njenima domenama



$\vec\rho = \vec r \circ \Phi$

$\Phi = (U,V)$, kjer sta $U = U(x,y)$ $V = V(x,y)$ funkciji na Δ.

Imamo $\vec\rho = \vec r(U,V)$, zato iz verižnega pravila sledi

$$\vec{\varphi}_x = \vec{r}_u(u,v)\, u_x + \vec{r}_v(u,v)\, v_x$$
$$\vec{\varphi}_y = \vec{r}_u(u,v)\, u_y + \vec{r}_v(u,v)\, v_y$$

$$\Rightarrow \quad \vec{\varphi}_x \times \vec{\varphi}_y = \underbrace{(u_x v_y - u_y v_x)}_{\det J\phi}(\vec{r}_u \times \vec{r}_v)\underbrace{(u,v)}_{\phi} \quad ②$$

izrek o zamenjavi koordinat v integralu za $(u,v) = \phi(x,y)$

$$\Rightarrow \quad \iint_D |\vec{r}_u \times \vec{r}_v|\, du\, dv = \iint_\Delta |(\vec{r}_u \times \vec{r}_v) \circ \phi| \cdot |\det J\phi|\, dx\, dy$$
$$\overset{②}{=} \iint_\Delta |\vec{\varphi}_x \times \vec{\varphi}_y|\, dx\, dy$$

$\square$

Primer $f: D \to \mathbb{R}$, $M = \Gamma_f$ $\quad (D \subset \mathbb{R}^2)$

$$\vec{r}(x,y) = (x, y, f(x,y))$$
$$\left.\begin{array}{l} \vec{r}_x = (1, 0, f_x) \\ \vec{r}_y = (0, 1, f_y) \end{array}\right\} \quad \begin{array}{l} E = 1 + f_x^2 \\ F = f_x f_y \\ G = 1 + f_y^2 \end{array} \qquad EG - F^2 = 1 + f_x^2 + f_y^2$$

$$P(\Gamma_f) = \iint_D \sqrt{1 + f_x^2 + f_y^2}\, dx\, dy$$

Primer $z = \frac{x^2+y^2}{2}$ za $(x,y) \in K(0,1) = D$ enotska krogla v $\mathbb{R}^2$

Tedaj $1 + z_x^2 + z_y^2 = 1 + x^2 + y^2$, zato je površina ploskve podana z

$$P = \iint_D \sqrt{1+x^2+y^2}\, dx\, dy \overset{\text{pol. koord.}}{=} \int_0^1 \int_0^{2\pi} \sqrt{1+r^2}\, r\, dr\, d\varphi = 2\pi \int_0^1 \sqrt{1+r^2}\, r\, dr$$

$$= 2\pi \left. \frac{(1+r^2)^{3/2}}{3/2} \right|_{r=0}^{r=1} = \frac{4\pi}{3}(2\sqrt{2} - 1)$$

Primer Sfera $S(0,a) \subset \mathbb{R}^3$. Gledamo jo kot nivojnico $\{r = a\}$ v sferičnih koordinatah. Torej jo parametriziramo z

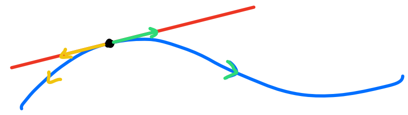
$$\vec{r}(\varphi, \theta) = a(\cos\varphi \sin\theta, \sin\varphi \sin\theta, \cos\theta) \qquad \varphi \in [0, 2\pi) \qquad \theta \in [0, \pi]$$

$$E = |\vec{r}_\varphi|^2 = (a \sin\theta)^2$$
$$F = \langle \vec{r}_\varphi, \vec{r}_\theta \rangle = 0 \qquad EG - F^2 = (a^2 \sin\theta)^2$$
$$G = |\vec{r}_\theta|^2 = a^2$$

$$P = \int_0^{2\pi} \int_0^{\pi} \sqrt{(a^2 \sin\theta)^2}\, d\theta\, d\varphi = \ldots = 4\pi a^2$$

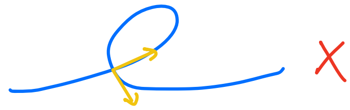
Orientacija

Smer krivulje / ploskve ?

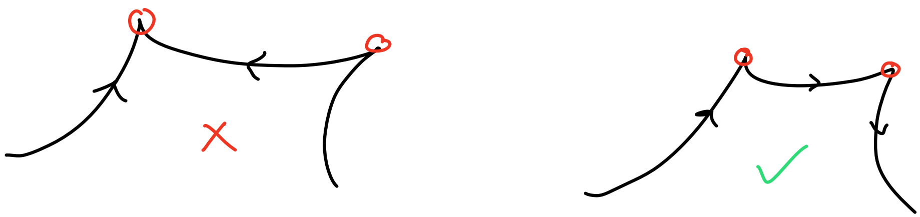


Def. Lokalna orientacija (gladke, neglobalne) krivulje Γ brez samopresečišč v točki $\gamma \in \Gamma$ je podana z izbiro enotskega tangentnega vektorja v točki γ.
Globalna orientacija je podana z zvezno izbiro enotskega tang. vek. po vseh $\gamma \in \Gamma$ (torej z zvezno izbiro lokalnih orientacij)

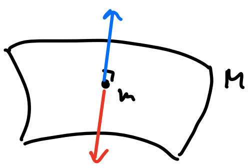
Obstoj Vsaka (gladka, regularna) [povezana] krivulja Γ brez samopresečišč ima globalno orientacijo, obstajata točno dve orientaciji



Def Orientacija odsekoma gladke neskljenjene krivulje $\Gamma_1 \cup \dots \cup \Gamma_n$ je določena tako, da je končna točka ($\vec{r}_j(1)$, če $\vec{r}_j : [0,1] \to \Gamma_j$) na Γ_j enaka začetni točki ($\vec{r}_{j+1}(0)$) na Γ_{j+1}. Pri tem so Γ_j ______ take, da je $\Gamma_{j-1} \cap \Gamma_j = \{\text{ena točka}\}$



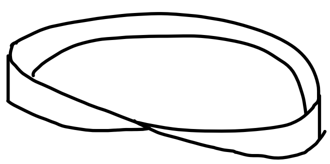
Def Lokalna orientacija (glad. regul.) ploskve M v točki $m \in M$ je izbira enotske normale na M v točki m.



Globalna orientacija ploskve M je zvezna izbira lokalnih orientacij, to je zvezna izbira enotskih normal po vseh $m \in M$

Obstoj Ni vsaka ploskev orientabilna

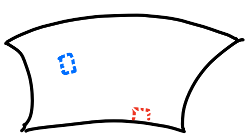
Primer Möbiusov trak



Komentar Če ima ploskev orientacijo, je le-ta ena od dveh obstoječih

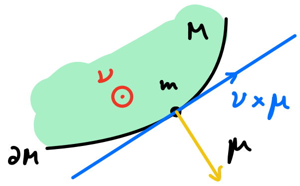
Inducirane orientacije

Def Naj bo M ploskev z robom; le-ta ni mišljen kot v smislu metričnih prostorov (notranjost množice, zunanjost, rob). Namesto natančne def. povemo idejo:
ploskev je lokalno podobna bodisi:



-  $= D(0,1) \subset \mathbb{R}^2$ ["notranja točka"]
-  $\subset \mathbb{R}^2$ ["robna točka"]

Def Naj bo M orientabilna ploskev z robom ∂M. Za $\forall m \in \partial M$ naj bo μ enotski vektor in tang. ravnine $T_m M$, pravokoten na $T_m \partial M$ (tang. premice na ∂M v točki m) in usmerjen <u>ven</u> iz M.



Če je ν izbrana zvezna družina enotskih normal na M, tedaj $\nu \times \mu : \partial M \to S^2 = \{\vec{r} \in \mathbb{R}^3 ; |\vec{r}| = 1\}$ določa orient. v ∂M, ki je <u>skladna</u> z orientacijo ν.

Opomba Vrstni red $(\nu \times \mu)$ je pomemben. "Rob orientiramo tako, da je ploskev na levi, če normala kaže gor."

Krivuljni integral

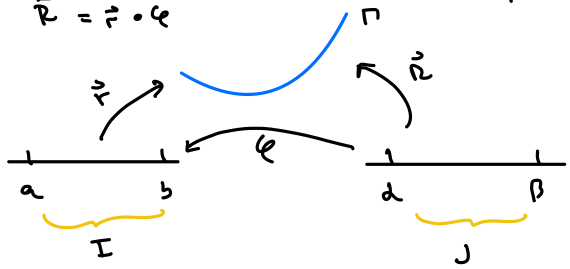
Naj bo $\vec{r} : I \to \mathbb{R}^3$ regul. param. neke krivulje Γ in $u : \Gamma \to \mathbb{R}$ zvezna.

Def Integral (skalarnega) polja u po Γ def. kot

$$\int_\Gamma u \, ds = \int_I u(\vec{r}(t)) |\dot{\vec{r}}(t)| \, dt$$

Trditev Ta def. je neodvisna od izbire parametrizacije $\vec{r}$.

Dokaz Naj bo $\vec{R}$ neka druga parametrizacija in φ bijekcija med domenama $\varphi = \vec{r}^{-1} \circ \vec{R}$ oz. $\vec{R} = \vec{r} \circ \varphi$.

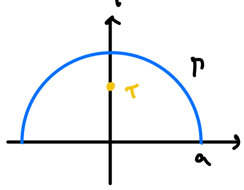


Dobimo $$\int_\alpha^\beta u(\vec{R}(s)) |\dot{\vec{R}}(s)| \, ds = \int_\alpha^\beta u(\vec{r}(\underbrace{\varphi(s)}_{w})) \cdot |\dot{\vec{r}}(\varphi(s))| \, |\dot{\varphi}(s)| \, ds =$$

$dw = \dot{\varphi}(s) \, ds$
$\dot{\varphi} > 0$ posebej ali
$\dot{\varphi} < 0$
(zaradi reg. param.)

$$= \overset{\dot{\varphi} > 0}{\int_a^b u(\vec{r}(w)) |\dot{\vec{r}}(w)| \, dw} = \overset{\dot{\varphi} < 0}{\int_b^a u(\vec{r}(w)) |\dot{\vec{r}}(w)| \, (-dw)}$$

Primer Γ naj bo homogena polkrožnica. Težišče $= (x_T, y_T) = ?$



$x_T = 0$ zaradi simetričnosti in homogenosti

$$y_T = \frac{1}{m(\Gamma)} \int_\Gamma y \rho \, ds = \frac{1}{\pi a \rho} \rho \int_0^\pi \underbrace{a \sin t}_{y} \, \underbrace{a}_{|\dot{\vec{r}}(t)|} \, dt = \frac{2a}{\pi}$$

$\vec{r}(t) = (a \cos t, a \sin t) \quad t \in [0, \pi]$

Def Naj bo še $\vec{F} : \Gamma \to \mathbb{R}^3$ zvezno. Integral vektorskega polja $\vec{F}$ po Γ def. kot

$$\int_\Gamma \vec{F} \, d\vec{r} = \int_I \langle \vec{F}(\vec{r}(t)), \dot{\vec{r}}(t) \rangle_{\mathbb{R}^3} \, dt$$

Komentar Če je $\vec{F} = (X, Y, Z)$ in $\vec{r} = (x, y, z)$, tedaj je

$$\langle \vec{F}, \dot{\vec{r}} \rangle = X\dot{x} + Y\dot{y} + Z\dot{z}$$

zato lahko pišemo $\langle \vec F, \dot{\vec r}\rangle dt = X dx + Y dy + Z dz$. Posledično tedaj lahko zapišemo

$$\int_\Gamma \vec F d\vec r = \int_\Gamma X dx + Y dy + Z dz$$

Trditev Če v definiciji $\int_\Gamma \langle F(\vec r), \dot{\vec r}\rangle$ nadomestimo $\vec r$ z drugo parametrizacijo $\vec R$ iste krivulje, tedaj je novi izraz, $\int_\Gamma \langle \vec F(\vec R), \dot{\vec R}\rangle$,

- enak, če $\vec R$ ohrani orientacijo Γ
- nasprotno enak, če jo spremeni

Dokaz DN

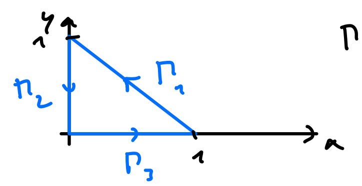
Primer $I = \int_\Gamma x\,dx + (x+y)\,dy + z^2 dz$ po Γ, param. z $\vec r(t) = (t, t^2, t^3)$; $t \in [0,1]$

$\hookrightarrow \vec F = (x, x+y, z^2)$

$$I = \int_0^1 \langle \vec F(\vec r(t)), \dot{\vec r}(t)\rangle dt = \int_0^1 \langle (t, t+t^3, t^6), (1, 2t, 3t^2)\rangle dt =$$

$$= \int_0^1 t + 2t^2 + 2t^4 + 3t^8 \, dt = \dots = \frac{19}{10}$$

Primer $\int_\Gamma (x^2 - y^2)\, dx dy$ po Γ



$\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$

$\Gamma_1 \quad t \to (t, 1-t)$

$\Gamma_2 \quad t \to (0, 1-t)$

$\Gamma_3 \quad t \to (t, 0)$

$$\Rightarrow \int_\Gamma = \int_{\Gamma_1} + \int_{\Gamma_2} + \int_{\Gamma_3}$$

$$\overset{DN}{=} \dots = -\frac{1}{3} + 0 + \frac{1}{3} = 0$$

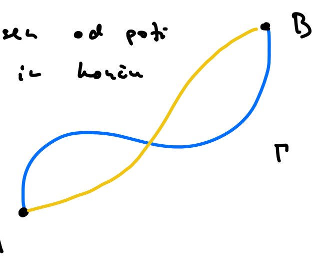
Def Polje $\vec F: \Omega \to \mathbb{R}^3$ je potencialno, če $\exists$ funkcija $u: \Omega \to \mathbb{R}$, za katero je $\vec F = \nabla u$. Funkciji u pravimo potencial (polja $\vec F$). ($\nabla = (\partial_x, \partial_y, \partial_z)$)

Torej tedaj za $\vec F = (X, Y, Z)$ velja $u_x = X$, $u_y = Y$, $u_z = Z$ za isto funkcijo u!

Trditev Naj bo Γ regularna krivulja med točkama $A, B \in \mathbb{R}^3$ in $\vec F = \nabla u$ (potencialno vek. polje). Tedaj je

$$\int_\Gamma \vec F d\vec r = u(B) - u(A).$$

$\int$ ni odvisen od poti le od zač. in konč. točk.



Komentar $n=1$: $\int_A^B u'(t)dt = u(B) - u(A)$

Dokaz

$$\int_\Gamma \nabla u \, d\vec r = \int_\Gamma \langle \nabla u \circ \vec r, \dot{\vec r}\rangle = \int_\alpha^\beta \langle (\nabla u)(\vec r(t)), \dot{\vec r}(t)\rangle_{\mathbb{R}^3} dt = \int_\alpha^\beta \frac{d}{dt}(u \circ \vec r)(t) dt =$$

$$= (u \circ \vec r)(\beta) - (u \circ \vec r)(\alpha) = u(B) - u(A)$$

Trditev Naj bo $\Omega \subset \mathbb{R}^3$ povezana odprta množica odp. množica in $\vec{F} : \Omega \to \mathbb{R}^3$ zvezno vektorsko polje. Naslednje trditve so ekvivalentne

1. Polje $\vec{F}$ je potencialno
2. Integral $\vec{F}$ po $\forall$ sklenjeni krivulji je enak 0
3. Za poljubna $A, B \in \Omega$ je integral $\vec{F}$ od A do B neodvisen od izbire poti med tema točkama.

Dokaz 1 $\Rightarrow$ 2

$$\oint_\Gamma \vec{F}\, d\vec{r} = u(A) - u(A) = 0$$

2 $\Rightarrow$ 3

Def. $\Gamma_1 \cup (-\Gamma_2) \overset{2}{\Longrightarrow} \int_\Gamma \vec{F}\, d\vec{r} = 0$

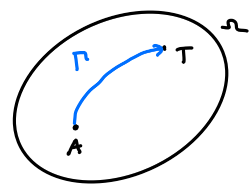
Hkrati pa je
$$\int_\Gamma = \int_{\Gamma_1 \cup (-\Gamma_2)} = \int_{\Gamma_1} + \int_{-\Gamma_2} = \int_{\Gamma_1} - \int_{\Gamma_2} = 0$$

3 $\Rightarrow$ 1 Fiksirajmo $A \in \Omega$. Če naj velja 1, je po trditvi

$$u(B) = u(A) + \int_\Gamma \vec{F}\, d\vec{r} \qquad \vec{F} = \nabla u \quad \Rightarrow \quad \vec{F} = \nabla(u + c)$$

za A, B, Γ kot zgoraj. To motivira naslednjo definicijo u:

Za polj: $T \in \Omega$ def:
$$u(T) = \int_\Gamma \vec{F}\, d\vec{r},$$

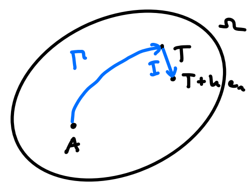


kjer je Γ neka (po 3 vseeno katera) pot od A do T.
Moramo videti $\nabla u = \vec{F}$.
Pišimo $F = (F_1, F_2, F_3)$. Dovolj je videti $u_x = F_1$ (enako za y, z)
To pomeni $\lim_{h \to 0} \frac{u(x+h, y, z) - u(x, y, z)}{h} = F_1(x, y, z)$ za $\forall (x, y, z) \in \Omega$

Vzemimo $T = (x, y, z) \in \Omega$ in $h \in \mathbb{R}$ majhen. Tedaj za $e_1 = (1, 0, 0)$ velja

$$u(T + he_1) - u(T) = \left(\int_{\Gamma_{T+he_1}} \vec{F}\, d\vec{r} - \int_{\Gamma_T} \vec{F}\, d\vec{r} \right) = \int_I \vec{F}\, d\vec{r}$$

izb. $\Gamma_T \cup \{\text{daljica } I\}$



Parametrizacija za daljico I:
$$\vec{r}(t) = (x, y, z) + t(1, 0, 0) \qquad t \in [0, h]$$
$$= (x + t, y, z)$$

Sledi
$$\frac{u(T + he_1) - u(T)}{h} = \frac{1}{h} \int_0^h \langle \vec{F}(x+t, y, z), (1, 0, 0) \rangle\, dt =$$
$$= \frac{1}{h} \int_0^h F_1(x+t, y, z)\, dt = \langle F_1(x + \cdot, y, z) \rangle_{[0,h]}$$

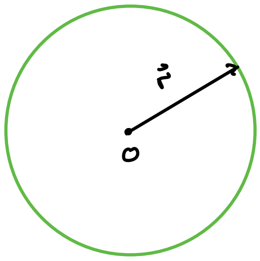
povprečje funkcije $t \to F_1(x+t, y, z)$ na $[0, h]$

$\xrightarrow{h \to 0} F_1(x, y, z)$ zaradi zveznosti F_1. Res, če je $\varphi : [0, \infty) \to \mathbb{R}$ zvezna
tedaj $\langle \varphi \rangle_{[0,h]} - \varphi(0) = \frac{1}{h} \int_0^h \varphi(t)\, dt - \varphi(0) = \int_0^h \frac{\varphi(t) - \varphi(0)}{h}\, dt \overset{s = t/h}{=}$
$= \int_0^1 (\varphi(hs) - \varphi(0))\, ds \leq \max_{\tau \in [0,h]} |\varphi(\tau) - \varphi(0)| \xrightarrow{h \to 0} 0$

$\square$

Primer Gravitacijski potencial

Delec z maso m se giblje vzdolž poti Γ po površju Zemlje. Položaj označimo z $\vec{r}$, kjer je izhodišče postavljeno v središče Zemlje. Na delec deluje gravitacijska sila:



$$\vec{F}_g = \vec{F}_g(\vec{r}) = -G\frac{Mm}{r^2}\frac{\vec{r}}{r}$$

Vidimo: $\vec{F}_g = \nabla\left(\frac{GMm}{r}\right)$

$$\partial_x\left(\frac{1}{r}\right) = \partial_x \frac{1}{(x^2+y^2+z^2)^{1/2}} = -\frac{1}{2}(\dots)^{-3/2}\,2x = -\frac{x}{r^3}$$

$$\Rightarrow \nabla\left(\frac{1}{r}\right) = \left(\partial_x \frac{1}{r}, \partial_y \frac{1}{r}, \partial_z \frac{1}{r}\right) = -\frac{(x,y,z)}{r^3}$$

Delo, ki ga sila F_g opravi pri tem, je enako

$$A = \int_\Gamma \vec{F}\,d\vec{r} = \int_\Gamma \nabla\left(\frac{GMm}{r}\right)d\vec{r} = \frac{GMm}{r_1} - \frac{GMm}{r_0}$$

kjer sta $r_0 = |\vec{r}_0|$ in $r_1 = |\vec{r}_1|$ višini delčeve pozicije za zač. in konca

Za $\Delta r = r_1 - r_0$ dobimo

$$A = GMm\left(\frac{1}{r_1} - \frac{1}{r_0}\right) = GMm\frac{-\Delta r}{r_0(r_0+\Delta r)} \doteq -\frac{GM}{R^2}m\Delta r = -gm\Delta r$$

Kaj pomeni predpostavka o gibanju po površju Zemlje $= r_1 \approx r_0$ oz. $|\Delta r| \ll r_0$

Izrek Delo rezultante (= vsote vseh) sil, ki delujejo na delec = sprem. kin. energije

Celotna en. = kin. en. + pot. en. $\Rightarrow$ delo = − sprem. pot. en.

$$A = -\Delta U = U(r_0) - U(r_1)$$

Sledi, gravitacijska potencialna energija U def kot $U(r) = -\frac{GMm}{r}$

Če vzamemo $r_1 = \infty$, vidimo, da je $U(r)$ delo, ki je potrebno, da delec spravimo v ∞.

Alternativni model: delovanje gravitacijske sile Sonca na posamezne planete v vesolju

Opomba Če je polje $F = (U,V) \in C^1$ potencialno, torej če za neki $u \in C^2$ velja $U = u_x$ in $V = u_y$, tedaj je seveda tudi $U_y = V_x$ $(= u_{xy})$. Ni pa $\boxed{U_y = V_x}$ zadosten pogoj za potencialnost (U,V):

npr. $U = -\frac{y}{x^2+y^2}$ in $V = \frac{x}{x^2+y^2}$.

$$U_y = -\frac{x^2+y^2-y2y}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2} = V_x = \frac{x^2+y^2-x2x}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

Toda $\int_{\partial K(0,1)} (U,V)\,d\vec{r} = \dots \neq 0$, zato po trditvi polje (U,V) ni potencialno.

Ploskovni integrali

Def Naj bo $M \subset \mathbb{R}^3$ ploskev in $f: M \to \mathbb{R}$ zvezna. Ploskovni integral skalarnega polja f def. s predpisom:

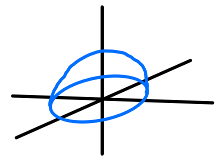
$$\boxed{\int_M f\,dS = \iint_D f(\vec{r}(u,v))\,|\vec{r}_u \times \vec{r}_v|\,du\,dv,}$$

kjer je $\vec{r}: D \to M$ (poljubna) regularna parametrizacija za M.

Opomba ① za $f \geq 1$ dobimo površino ploskve
② definicija je neodvisna od izbire parametrizacije DN

Primer Naj bo M (zgornja) polovica homogene sfere v $\mathbb{R}^3$ z radijem a. Iščemo njeno težišče $T = (x_T, y_T, z_T)$. Za središče sfere vzamemo $(0,0,0)$. Sledi $x_T = y_T = 0$

$$z_T = \frac{1}{m(M)} \iint_M z \,\rho \, dS = \frac{1}{\frac{1}{2} P(\text{sfere})\, \rho}\, \rho \iint_M z\, dS = \frac{1}{2\pi a^2} \iint_M z\, dS$$



M je nivojnica $\{r = a\}$ v sferičnih koordinatah

Param. za M: $\vec{r} = \vec{r}(\varphi, \theta) = (x, y, z)$

$x = a \sin\theta \cos\varphi \qquad \varphi \in [0, 2\pi)$

$y = a \sin\theta \sin\varphi \qquad \theta \in [0, \frac{\pi}{2})$

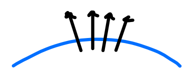
$z = a \cos\theta$

$$z_T = \iint_M z\, dS = \int_0^{2\pi} \int_0^{\pi/2} a \cos\theta \, |\vec{r}_\varphi \times \vec{r}_\theta| \, d\theta \, d\varphi = \dots = \pi a^3$$

$$z_T = \frac{a}{2}$$

Def. Naj bo M ploskev z orientacijo $\vec{N}$ (tj. polje enotskih normal na M). Ploskovni integral (2. vrste) vekt. polja $\vec{F} : M \to \mathbb{R}^3$ je def s predpisom:

$$\boxed{\int_M \vec{F}\, d\vec{S} = \underbrace{\int_M \langle \vec{F}, \vec{N} \rangle \, dS}}$$



Ploskovni integral skalarnega polja

Enoličnost: predznak integrala je odvisen od izbire (ene izmed dveh) smeri orientacije $\vec{N}$.

Opomba: če je $\vec{r} = \vec{r}(u, v) : D \to M$ neka (regul.) parametrizacija za M, lahko vzamemo
$\vec{N} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$ in sledi

$$\int_M \vec{F}\, d\vec{S} = \iint_D \langle \vec{F}, \vec{N} \rangle |\vec{r}_u \times \vec{r}_v| \, du\, dv = \iint_D \langle \vec{F}(\vec{r}(u,v)), \vec{r}_u \times \vec{r}_v \rangle \, du\, dv$$

Operacije na vekt. in skal. poljih v $\mathbb{R}^3$

Imejmo $\Omega \subset \mathbb{R}^3$

Skalarno polje na Ω je funkcija $\Omega \to \mathbb{R}$

Vektorsko polje na Ω je funkcija $\Omega \to \mathbb{R}^3$

Temeljni diferencialni operatorji na (skal. oz. vekt.) poljih

Gradient $\quad \nabla u = \operatorname{grad} u = (u_x, u_y, u_z) \qquad$ skal. polje $\to$ vekt. polje

Divergenca $\quad \operatorname{div}(u, v, w) = u_x + v_y + w_z \qquad$ vekt. polje $\to$ skal. polje

alternativna oznaka $\quad \operatorname{div} \vec{F} = \nabla \cdot \vec{F} \qquad \underbrace{(\partial_x, \partial_y, \partial_z)}_{\nabla} \cdot (u, v, w)$

Rotor $\quad \operatorname{rot}(u, v, w) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ u & v & w \end{vmatrix} = (w_y - v_z, \; u_z - w_x, \; v_x - u_y)$

$$= \nabla \times \vec{F}$$

Lastnosti: (a) $\operatorname{div} = -\nabla^*$ v smislu, da je $\int_\Omega \langle \nabla u, \vec F\rangle_{\mathbb R^3} = -\int_\Omega u\cdot \operatorname{div}\vec F$ za $u,\vec F \in C_c^1(\Omega)$

$-\operatorname{div}$ je adjungirani operator za ∇ glede na skal. produkt funkcij, podan z $\underbrace{\int_\Omega \langle F, G\rangle}$

$\langle F, G\rangle_{L^2}$

Za matrike: $A\in\mathbb R^{3,3}$, je A^* def. z
$\langle Ax, y\rangle_{\mathbb R^3} = \langle x, A^* y\rangle_{\mathbb R^3}$

Hočemo
$\langle \nabla u, F\rangle_{L^2} = \langle u, \nabla^* F\rangle_{L^2}$

$\Rightarrow \nabla^* = -\operatorname{div}$

(b) $\operatorname{rot}^* = \operatorname{rot}$

(c) $\operatorname{rot}\circ\operatorname{grad} = 0$

(d) $\operatorname{div}\circ\operatorname{rot} = 0$

(e) $\operatorname{div}\circ\operatorname{grad} = -\nabla^*\nabla = \Delta = \partial_{xx}^2 + \partial_{yy}^2 + \partial_{zz}^2$ Laplaceov operator

(f) $\nabla(fg) = f\nabla g + g\nabla f$

(g) $\operatorname{div}(u\vec F) = u \operatorname{div}\vec F + \langle \vec F, \nabla u\rangle$

(h) $\operatorname{div}(\vec F\times\vec G) = \langle \vec G, \operatorname{rot}\vec F\rangle - \langle \vec F, \operatorname{rot}\vec G\rangle$

(i) $\operatorname{rot}(u\vec F) = u\operatorname{rot}\vec F + (\nabla u)\times\vec F$

(f)–(i): Izpeljanke Leibnizovega pravila

Dokaz (b) $A,B,C,X,Y,Z \in C_c^1(\Omega)$ $\Omega =$  $= [0,1]^3$

$\int_\Omega \langle \operatorname{rot}(A,B,C), (X,Y,Z)\rangle = \int_\Omega \langle (C_y - B_z, A_z - C_x, B_x - A_y), (X,Y,Z)\rangle =$

$= \int_\Omega (C_y X + B_z X + A_z Y - C_x Y + B_x Z - A_y Z) =$

per partes

$= \int_\Omega (-C X_y + B X_z - A Y_z + C Y_x - B Z_x + A Z_y) =$

$= \int_\Omega \langle (A,B,C), (Z_y - Y_z, X_z - Z_x, Y_x - X_y)\rangle = \int_\Omega \langle (A,B,C), \operatorname{rot}(X,Y,Z)\rangle$

$\Rightarrow \operatorname{rot}^* = \operatorname{rot}$ v smislu skal. prod. (oz. adjungiranja) v $L^2(\Omega\to\mathbb R^3)$:
$\langle \operatorname{rot}\vec F, \vec G\rangle_{L^2} = \langle \vec F, \operatorname{rot}\vec G\rangle_{L^2}$

Če sta $\vec N, \vec M$ vekt. polji na Ω, def. $\langle \vec M, \vec N\rangle_{L^2} = \int_\Omega \langle \vec M, \vec N\rangle dV$

Kaj je L^2: $L^2(J) = \{ f: J\to\mathbb R \,;\, \|f\|_{L^2} = \left(\int_J |f(x)|^2 dx\right)^{1/2} \}$ $J\in[a,b]$

Integracija po delih; Gaussov in Stokesov izrek

Osnovni izrek analize: $f:[a,b]\to\mathbb R$ $\int_a^b f'(x)\,dx = f(b) - f(a)$
$\approx \int_{\partial[a,b]} f$

Po vzoru na to zvezo želimo nekaj podobnega v višjih dimenzijah:

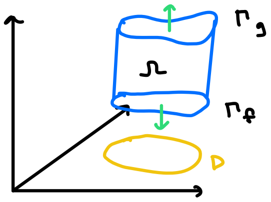
$\int_D (\text{diferencialni operator})\vec F = \int_D \vec F$

Izrek Gaussov izrek

Naj bo Ω odp. omejena množica v $\mathbb R^3$ z (odsekoma) gladkim robom ($\partial\Omega$ je (odsekoma) gladka ploskev z orientacijo $\vec N$, ki kaže ven iz Ω; pravimo ji zunanja normala). $\vec F$ naj bo C^1 vekt. polje v okolici $\bar\Omega = \Omega\cup\partial\Omega$. Tedaj je:

$$\boxed{\iint_{\partial\Omega} \vec F\, d\vec S = \iiint_\Omega \operatorname{div}\vec F\, dV}$$

Dokaz za primer, ko Ω lahko zapišemo kot območje med dvema grafoma, za vse tri koordinatne ravnine:

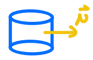


Pišimo $\vec F = (X, Y, Z)$. Tedaj je $\operatorname{div}\vec F = X_x + Y_y + Z_z$.
Označimo z $\vec N = (N_1, N_2, N_3)$ zunanjo enotsko normalo na $\partial\Omega$.
Dokazujemo:

$$\iint_{\partial\Omega} (XN_1 + YN_2 + ZN_3)\,dS = \iiint_\Omega (X_x + Y_y + Z_z)\,dV$$

Dovolj je dokazati: $\overset{\text{leva stran}}{\iint_{\partial\Omega} \underset{\text{skal. f.}}{ZN_3}\,dS} = \overset{\text{desna stran}}{\iiint_\Omega Z_z\,dV}$

Naj bo $\Omega = \{(x,y,z) \in \mathbb{R}^3;\ (x,y) \in D \text{ in } f(x,y) < z < g(x,y)\}$

Leva stran $= \iint_{\Gamma_f} ZN_3 + \iint_{\Gamma_g} ZN_3 + \iint_{\text{navpični del}} ZN_3 = \dots$ → npr. plašč valja  $N_3 = 0$

Param. grafa $\vec r(x, y, f(x,y))$; $(x,y) \in D$ $\quad \vec r_x \times \vec r_y = (-f_x, -f_y, 1)$

$$\dots \iint_D z(x,y,f(x,y)) \cdot \frac{-1}{\sqrt{f_x^2+f_y^2+1}} \cdot \overbrace{\sqrt{f_x^2+f_y^2+1}}^{|\vec r_x \times \vec r_y|}\,dx\,dy + (\text{enako za } g \text{, le brez minusa}) =$$

$$= \iint_D \big(z(x,y,f(x,y)) - z(x,y,g(x,y))\big)\,dx\,dy$$

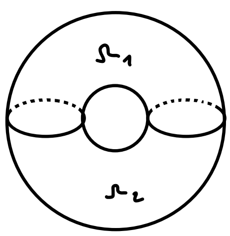
Desna stran $\iiint_\Omega z_z\,dV = \iint_D \left(\int_{f(x,y)}^{g(x,y)} z_z\,dz\right) dx\,dy = \iint_D z(x,y,g(x,y)) - z(x,y,f(x,y))\,dx\,dy$

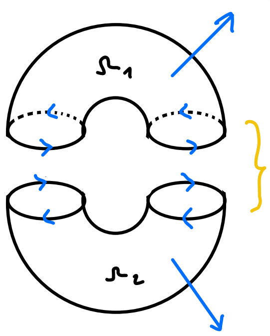
$\square$

„Sestavljanje" območij

Torus $\qquad \Omega = \Omega_1 \cup \Omega_2 \qquad$ čeprav $\partial\Omega_1 + \partial\Omega_2 \neq \partial\Omega$

$$\iiint_\Omega \operatorname{div}\vec F\,dV = \iiint_{\Omega_1} + \iiint_{\Omega_2} \overset{\text{Gaussov izrek}}{=} \iint_{\partial\Omega_1} + \iint_{\partial\Omega_2} = \iint_{\partial\Omega}$$





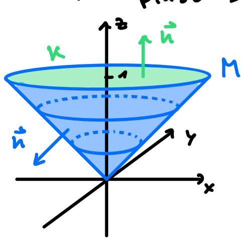
podvojeni integrali se izničijo zaradi nasprotnih orientacij

Primer $\Omega = K(0,1)$ en. krogla v $\mathbb{R}^3$, $\vec F = (x,y,z)$ $\quad (\vec F = \mathrm{id})$

$$\iint_{\partial\Omega} \vec F\,d\vec S = \iiint \operatorname{div}\vec F\,dV = \iiint_\Omega 1+1+1\,dV = 3V(\Omega) = 4\pi$$

enotska sfera

Primer $\vec F = (y^2, z^3, x^2)$
$M =$ plašč stožca $\{(x,y,z) \in \mathbb{R}^3;\ \underbrace{x^2+y^2 \le z^2}\ \text{in}\ 0 \le z \le 1\}$
v cilindričnih koordinatah $r \le z \le 1$



Ugotovim $\operatorname{div}\vec F = 0$

Če želimo upor. Gaussov izrek, izberemo za Ω najpreprostejše telo, za katero velja M je del $\partial\Omega$. Ta Ω je stožec. Naj bo K njegova zgornja ploskev (krog) orientiran navzven. Tedaj je $\partial\Omega = M \cup K$

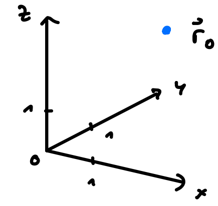
Sledi:

$$I = \iint_{\partial\Omega} \vec F\, d\vec S = \iiint_{\Omega} \underbrace{\operatorname{div} \vec F}_{0}\, dV = 0$$

Zaradi $\partial\Omega = M \cup K$

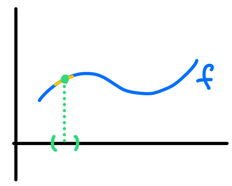
$$I = \iint_M + \iint_K = 0 \quad \Rightarrow \quad \iint_M \vec F\, d\vec S = -\iint_K \vec F\, d\vec S = \dots = -\frac{\pi}{4}$$

Opombe Operator divergence je neodvisen od izbire ortonormirane baze (ONB) v $\mathbb{R}^3$. Če je $\vec F = (u, v, w)$ vekt. polje def. v okolici točke $\vec r_0 \in \mathbb{R}^3$, je



$$(\operatorname{div} \vec F)(\vec r_0) = \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right)(\vec r_0)$$

$$\underbrace{(\operatorname{div} \vec F)(\vec r_0)}_{\substack{\text{zvezna funkcija} \\ \text{v okolici } \vec r_0}} = \lim_{\varepsilon \to 0} \underbrace{\langle \operatorname{div} \vec F \rangle_{K^3(\vec r_0, \varepsilon)}}_{\substack{\text{povprečje } \operatorname{div} \vec F \text{ po} \\ \text{krogli } K^3(\vec r_0, \varepsilon)}} =$$



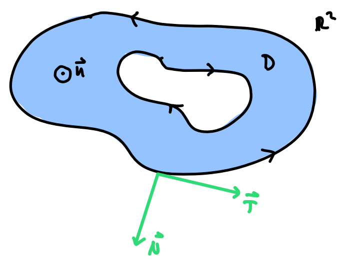
$$= \lim_{\varepsilon \to 0} \frac{1}{V(K^3(\vec r_0, \varepsilon))} \iiint_{K^3} \operatorname{div} \vec F \overset{\substack{\text{Gaussov} \\ \text{izrek}}}{=} \frac{3}{4\pi} \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \iint_{\underbrace{S^2(\vec r_0, \varepsilon)}_{\text{neodvisen od ONB koordinat v } \mathbb{R}^3}} \vec F\, d\vec S$$

$\leftarrow$ sfera

Izrek Greenova formula

Predpostavke:

- $D \subset \mathbb{R}^2$ z (odsekoma) gladkim robom (∂D je končna unija (odsekoma) gladkih krivulj orientirana skladno z normalo $(0, 0, +1)$ na D)
- $\vec F = (X, Y)$ naj bo C^1 vektorsko polje na okolici $\bar D$



Sledi da:

$$\int_{\partial D} X\,dx + Y\,dy = \iint_D (Y_x - X_y)\, dx\, dy$$

Komentar „dokaz" Greenove formule

Greenova formula je reformulacija „dvodimenzionalne različice Gaussovega izreka."

$$\iint_{\partial D} \langle \vec G, \vec N \rangle\, ds = \iint_D \operatorname{div} \vec G\, dS \quad \text{(ki jo privzamemo kot dejstvo)}$$

Če bi namesto $\vec N$ imeli tangento $\vec T$, bi leva stran bila krivuljni integral $\vec G$ po (orientirani) krivulji ∂D. Zato poiščemo še polje $\vec H$, za katero je $\langle \vec G, \vec N \rangle = \langle \vec H, \vec T \rangle$. Ker je $\vec T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \vec N$ (rotacija za $\pi/2$ v ravnini, R), zato je $\vec N = R^{-1} \vec T$, zato

$$\langle \vec G, \vec N \rangle = \langle \vec G, R^{-1} \vec T \rangle = \langle \vec G, R^T \vec T \rangle \overset{\text{adjungiranje}}{=} \langle R \vec G, \vec T \rangle = \langle \vec H, \vec T \rangle$$

Če je $\vec G = (u, v)$ je $\vec H = (-v, u)$

Sledi:
$$\int_{\partial D} \langle \vec G, \vec N \rangle\, ds = \int_{\partial D} \langle \vec H, \vec T \rangle\, ds = \int_{\partial D} \vec H\, d\vec r = \int_{\partial D} -v\, dx + u\, dy$$

Hkrati je
$$\iint_D \operatorname{div} \vec G\, dS = \iint_D (u_x + v_y)\, dx\, dy \quad ''$$

Sedaj za $(u, v) = (Y, -X)$ sledi Greenova formula $\square$

Izrek Stokesov izrek

Predpostavke:

- M omejena, odsekoma gladka orientirana ploskev v $\mathbb{R}^3$ z odsekoma gladkim robom (∂M je končna unija odsekoma gladkih krivulj orientiranih koherentno/skladno na M)
- $\vec F$ vekt. polje (gladko), def. v okolici M

Sledi:

$$\int_{\partial M} \vec F\, d\vec r = \iint_M \operatorname{rot} \vec F\, d\vec S \qquad (\text{podobno Greenovi formuli})$$

Dokaz skica dokaza

Obravnavamo primer, ko je, za neko gladko funkcijo $f: D \subset \mathbb{R}^2 \to \mathbb{R}$,

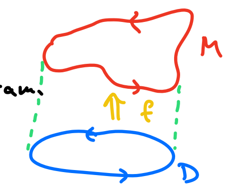
$$M = \operatorname{graf} f = \{\underbrace{(x, y, f(x,y))}_{\text{Parametrizacija za } M \ \ \vec r(x,y)} ;\ (x,y) \in D \subset \mathbb{R}^2\}$$

Pripadajoča normala je: $\vec N = \dfrac{\vec r_x \times \vec r_y}{|\vec r_x \times \vec r_y|} = \dfrac{(-f_x, -f_y, 1)}{\sqrt{f_x^2 + f_y^2 + 1}}$

Pišimo še $\vec F = (X, Y, Z)$; $X, Y, Z : M \to \mathbb{R}$ so gladke funkcije in $X = X(u,v,w)$, enako za Y, Z

Sedaj je: $\displaystyle\int_{\partial M} \vec F\, d\vec r = \int_{\partial M} X\,dx + Y\,dy + Z\,dz =$

Parametrizacija za ∂M je porojena z zožitvijo param. $\vec r = \vec r(x,y)$ na ∂D.



$$= \int_{\partial D} X(\vec r)\,dx + Y(\vec r)\,dy + Z(\vec r)(f_x\,dx + f_y\,dy)$$

$$= \int_{\partial D} \underbrace{(X(\vec r) + Z(\vec r) f_x)}_{\partial/\partial y}dx + \underbrace{(Y(\vec r) + Z(\vec r) f_y)}_{\partial/\partial x}dy \qquad \vec r = \vec r(x, y, f(x,y))$$

Greenova formula in posredno odvajanje

$$= \iint_D \Big[\Big(Y_u(\vec r) \underbrace{\frac{\partial x}{\partial x}}_{=1} + Y_v(\vec r) \underbrace{\frac{\partial y}{\partial x}}_{=0} + Y_w(\vec r) \underbrace{\frac{\partial f(x,y)}{\partial x}}_{=f_x} + (Z_u + \cancel{Z_w f_x}) f_y + \cancel{Z \underbrace{\frac{\partial f_y}{\partial x}}_{f_{xy}}} \Big)$$

$$- \Big(X_v + X_w f_y + (Z_v + \cancel{Z_w f_y}) f_x + \cancel{Z \underbrace{\frac{\partial f_x}{\partial y}}_{=f_{xy}}} \Big)\Big]\, dx\,dy$$

$$= \iint_D \big(-f_x (Z_v - Y_w) - f_y (X_w - Z_u) + (Y_u - X_v) \big)\, dx\,dy$$

$$= \iint_D \langle (\operatorname{rot} \vec F)(\vec r), \vec N \rangle \cdot \sqrt{1 + f_x^2 + f_y^2}\, dx\,dy$$

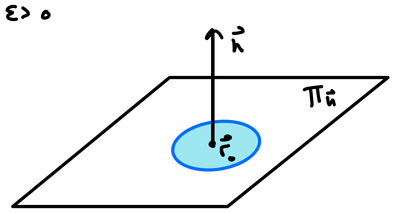
$$= \iint_M \operatorname{rot} \vec F\, d\vec S$$

$\square$

Opombe Tudi rot je, tako kot div, neodv. od izbire ONB v $\mathbb{R}^3$:

Naj bo $\vec r_0 \in \mathbb{R}^3$ in $\vec F$ vekt. polje, def. na okolici $\vec r_0$. Vzemimo enotski vektor $\vec n \in \mathbb{R}^3$ in označimo s $\Pi_{\vec n}$ ravnino v $\mathbb{R}^3$ skozi točko $\vec r_0$ in z normalo $\vec n$. Označimo še:

- $K^2_{\vec n}(\vec r_0, \varepsilon)$ ... krog v $\Pi_{\vec n}$ s sred. v $\vec r_0$ in polmerom $\varepsilon > 0$
- $S^1_{\vec n}(\vec r_0, \varepsilon)$ ... krožnica



Tedaj je: $\displaystyle\langle (\operatorname{rot} \vec F)(\vec r_0), \vec n \rangle = \lim_{\varepsilon \to 0} \frac{1}{\pi \varepsilon^2} \int_{K^2_{\vec n}(\vec r_0, \varepsilon)} \langle \operatorname{rot} \vec F, \vec n \rangle\, dS =$

Stokesov izrek $\longrightarrow$ $\displaystyle = \frac{1}{\pi} \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \int_{S^1_{\vec n}(\vec r_0, \varepsilon)} \vec F\, d\vec r$ $\longleftarrow$ Neodvisno od ONB

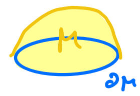
V def. rot nastopajo koordinatni odvodi $\partial_x\ \partial_y\ \partial_z$

$\vec v = (\operatorname{rot} \vec F)(\vec r_0)$, $\langle \vec v, \vec n \rangle$ je za vse $\vec n$ neodv. od izbire ONB $\Rightarrow$ $\vec v$ neodvisen od ONB

Primer Naj bo M zgornja polovica enotske sfere v $\mathbb{R}^3$, torej graf funkcije $z=\sqrt{1-x^2-y^2}$.
Za $\vec F(x,y,z) = (3yz, -xz, xy)$ izračunajmo

$$I = \iint_M \operatorname{rot}\vec F \, d\vec S$$

na dva načina:



① Direktno $\operatorname{rot}\vec F = (2x, 2y, -4z)$

$\langle \operatorname{rot}\vec F, \vec n\rangle = 2x^2+2y^2-4z^2$

$\vec n = (x,y,z)$ na enotski sferi

$$I = \iint_M (2x^2+2y^2-4z^2)\, dS =$$

$$= 2\iint_M \left(x^2+y^2-2(1-(x^2+y^2))\right) dS = 2\underbrace{\iint_M 3(x^2+y^2)\, dS}_{J} - 4\underbrace{\text{površina}(M)}_{\frac{1}{2}4\pi 1^2} = 2J + 8\pi$$

$$J = 3\int_0^{2\pi}\int_0^1 r^2 \frac{r}{\sqrt{1-r^2}}\, dr\, d\varphi = 3\cdot 2\pi \int_0^1 \frac{r^3}{\sqrt{1-r^2}}\, dr = \dots$$

Polarne koordinate $(x, y, \sqrt{1-x^2-y^2})$

$\vec R(r,\varphi) = (r\cos\varphi, r\sin\varphi, \sqrt{1-r^2})$

$|\vec R_r \times \vec R_\varphi| = \dots = \frac{r}{\sqrt{1-r^2}}$

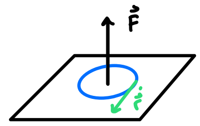
$$I = \dots = 0$$

② S pomočjo Stokesovega izreka

$$I = \iint_M \operatorname{rot}\vec F\, d\vec S = \int_{\partial M} \vec F\, d\vec r = \dots = 0$$

Na ∂M je $z=0$, zato je tam
$\vec F(x,y,0) = (0,0,xy)$

Tangenta je del ravnine $z=0$, zato je tam
$\langle \vec F, \dot{\vec r}\rangle = 0$



Smerni odvod

Spomnimo se: če je
- $\vec v \in \mathbb{R}^3$
- u zvezna realna f. na okolici točke $\vec v$
- $\vec a \in \mathbb{R}^3$, $|\vec a| = 1$,

tedaj je smerni odvod funkcije u v točki $\vec v$ in v smeri $\vec a$ def. kot

$$\frac{\partial u}{\partial \vec a}(\vec v) = \lim_{h\to 0}\frac{u(\vec v + h\vec a) - u(\vec v)}{h}, \quad \text{če } \lim \exists$$

Velja:

$$\boxed{\frac{\partial u}{\partial \vec a} = \langle \nabla u, \vec a\rangle}$$

$$\nabla u = (u_x, u_y, u_z) = \left(\frac{\partial u}{\partial e_1}, \frac{\partial u}{\partial e_2}, \frac{\partial u}{\partial e_3}\right)$$

Izrek Greenovi identiteti

Naj bo
- $\Omega \subset \mathbb{R}^3$ odprta množica z odsekoma gladkim robom
- u, v skal. polji, def. in gladki na neki okolici $\bar\Omega$.

Tedaj je:

1. G.I. $$\iint_{\partial\Omega} u \frac{\partial v}{\partial \vec n}\, dS = \iiint_\Omega \left(u\overset{\nabla^2}{\Delta} v + \langle \nabla u, \nabla v\rangle\right) dV$$

enotska zunanja normala na $\partial\Omega$

2. G.I. $$\iint_{\partial\Omega}\left(u\frac{\partial v}{\partial \vec n} - v\frac{\partial u}{\partial \vec n}\right) dS = \iiint_\Omega (u\Delta v - v\Delta u)\, dV$$

Dokaz Def $\vec{F} = u \nabla v$ in uporabimo Gaussovo formulo:

$$\iint_{\partial\Omega} \vec{F}\, d\vec{S} \overset{\text{Gaussova for.}}{=} \iiint_{\Omega} \operatorname{div} \vec{F}\, dV = \iiint_{\Omega} (u \Delta v + \langle \nabla u, \nabla v\rangle)\, dV$$

Lastnosti
$\operatorname{div}(u\vec{F}) = u \operatorname{div} \vec{F} + \langle \vec{F}, \nabla u\rangle$
$\operatorname{div} \circ \operatorname{grad} = \Delta$

$$\iint_{\partial\Omega} \vec{F}\, d\vec{S} \overset{\text{def. plosk. int. vek. polja}}{=} \iint_{\partial\Omega} u \langle \nabla v, \vec{n}\rangle\, dS = \iint_{\partial\Omega} u \frac{\partial v}{\partial \vec{n}}\, dS$$

S tem smo dokazali 1. G. identiteto. Dokaz 2. G. I. sledi iz 1., z zamenjavo vlog u in v in odštevanjem.

Posledica Za $u = 1$ dobimo

$$\iint_{\partial\Omega} \frac{\partial v}{\partial \vec{n}}\, dS = \iiint_{\Omega} \Delta v\, dV$$

Standardni dif. operatorji v polj. ortog. koordinatah

$\partial_x, \partial_y, \partial_z$

$\nabla = (\partial_x, \partial_y, \partial_z)$

$\operatorname{div}(U, V, W) = \partial_x U + \partial_y V + \partial_z W \qquad \nabla \cdot \vec{v}$

$$\operatorname{rot}(U, V, W) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ U & V & W \end{vmatrix} \qquad \nabla \times \vec{v}$$

$\Delta = \partial_x^2 + \partial_y^2 + \partial_z^2 \qquad \nabla^2 \cdot \vec{v}$

Imejmo še en nabor koordinat (cilindrične, sferične, ...)
$u = u(x, y, z)$ izrazimo v teh koordinatah (npr $u = x^2 + y^2 + z^2 = r^2$)
Kako zgornje dif. op izrazimo z dif. op. v novih koordinatah?

Vzemimo koordinate $\xi = (\xi_1, \xi_2, \xi_3)$ v $\mathbb{R}^3$. Naj bo $\vec{r} = \vec{r}(\xi)$ parametrizacija prostora $\mathbb{R}^3$ (torej izražava kartezičnih koordinat s ξ_1, ξ_2, ξ_3)
Predpostavimo, da so ξ ortogonalne: za $\forall$ nabor ξ ter

$$\vec{r}_j = \vec{r}_j(\xi) = \frac{\partial \vec{r}_j}{\partial \xi_j}$$

zahtevamo $\langle \vec{r}_j, \vec{r}_k \rangle_{\mathbb{R}^3} = 0$ za $j \neq k$

Definiramo Laméjeve koeficiente

$$H_j = \sqrt{\langle \vec{r}_j, \vec{r}_j\rangle} = |\vec{r}_j| \quad \text{za } j = 1, 2, 3$$

ter $H = H_1 H_2 H_3$. Označimo

$$\vec{u}_j = \frac{\vec{r}_j}{|\vec{r}_j|}$$

Poudarimo, da so $\vec{r}_j, H_j, \vec{u}_j$ funkcije ξ.
Naj bodo $x = (x_1, x_2, x_3)$ stand. kart. koord. v $\mathbb{R}^3$.
Naj bo $u = u(x): \mathbb{R}^3 \to \mathbb{R}$ gladka funkcija in def. njeno izražavo U v koord. ξ, torej:

$$u(x) = U(\xi), \quad \text{oz.}$$
$$U(\xi) = u(\vec{r}(\xi)).$$

Izračuna ∇u v kartez. koord. z ∇U v koord. ξ.

Trditev Velja

$$(\nabla_x u)(\vec{r}) = \left\langle \nabla_\xi U, \begin{bmatrix} \vec{\eta}_1 / H_1 \\ \vec{\eta}_2 / H_2 \\ \vec{\eta}_3 / H_3 \end{bmatrix} \right\rangle .$$

gledе na kartez. koord. — glede na koord. ξ

Dokaz Iz def. $U = u \circ \vec{r}$ dobimo s pomočjo verižnega pravila. $\vec{r} = (r_1, r_2, r_3)$

$$\frac{\partial U}{\partial \xi_j}(\xi) = \frac{\partial u}{\partial x}(\vec{r}(\xi))\frac{\partial r_1}{\partial \xi_j} + \frac{\partial u}{\partial y}(\vec{r}(\xi))\frac{\partial r_2}{\partial \xi_j} + \frac{\partial u}{\partial z}(\vec{r}(\xi))\frac{\partial r_3}{\partial \xi_j}$$

$$= \left\langle (\nabla_x u)(\vec{r}(\xi)), \frac{\partial \vec{r}}{\partial \xi_i}(\xi) \right\rangle_{\mathbb{R}^3}$$

$$= \left\langle (\nabla_x u)(\vec{r}(\xi)), (H_i \vec{\eta}_i)(\xi) \right\rangle_{\mathbb{R}^3}$$

$$= H_i(\xi) \left\langle (\nabla_x u)(\vec{r}(\xi)), \vec{\eta}_i(\xi) \right\rangle_{\mathbb{R}^3}$$

Poenostavljen zapis

$$\frac{\partial U}{\partial \xi_j} = H_j \langle (\nabla_x u) \circ \vec{r}, \vec{\eta}_i \rangle \qquad \text{za vse } j \qquad (1)$$

Privzetek: za $\forall \xi$ so $\{\vec{\eta}_1(\xi), \vec{\eta}_2(\xi), \vec{\eta}_3(\xi)\}$ ortonormirana baza vekt. prostora $\mathbb{R}^3$, zato je

$$\vec{v} = \sum_{j=1}^{3} \langle \vec{v}, \vec{\eta}_i \rangle \vec{\eta}_i \qquad \forall \vec{v} \in \mathbb{R}^3$$

Torej za $\vec{v} = (\nabla_x u) \circ \vec{r}$ iz (1) dobimo

$$(\nabla_x u)(\vec{r}) = \sum_{j=1}^{3} \langle (\nabla_x u) \circ \vec{r}, \vec{\eta}_i \rangle \vec{\eta}_i \overset{(1)}{=} \sum_{j=1}^{3} \frac{1}{H_j} \frac{\partial U}{\partial \xi_i} \vec{\eta}_i$$

oz.

$$(\nabla_x u)(\vec{r}) = \sum_{j=1}^{3} \frac{\partial U}{\partial \xi_j} \frac{\vec{\eta}_i}{H_i} = \left\langle \nabla_\xi U, \begin{bmatrix} \vec{\eta}_1 / H_1 \\ \vec{\eta}_2 / H_2 \\ \vec{\eta}_3 / H_3 \end{bmatrix} \right\rangle .$$

$\square$

Izrek Imamo

$$\Delta u = \frac{1}{H} \sum_{j=1}^{3} \frac{\partial}{\partial \xi_i} \left(\frac{H}{H_j^2} \frac{\partial U}{\partial \xi_i} \right) \circ \vec{R} ,$$

kjer je $\vec{R} = \vec{r}^{-1}$ (lokalni inverz): $x = \vec{r}(\xi) \iff \xi = \vec{R}(x)$.

Dokaz DN ali na spletni učilnici

Primer Cilindrične koordinate

$$\xi = (\xi_1, \xi_2, \xi_3) = (\rho, \varphi, z) \qquad \rho \in [0,\infty), \quad \varphi \in [0, 2\pi), \quad z \in \mathbb{R}$$

in

$$(x, y, z) = \vec{r}(\rho, \varphi, z) = (\rho\cos\varphi, \rho\sin\varphi, z), \quad \text{zato je}$$

$$\vec{r}_1 = \frac{\partial \vec{r}}{\partial \rho} = (\cos\varphi, \sin\varphi, 0)$$
$$\vec{r}_2 = \frac{\partial \vec{r}}{\partial \varphi} = (-\rho\sin\varphi, \rho\cos\varphi, 0)$$
$$\vec{r}_3 = \frac{\partial \vec{r}}{\partial z} = (0, 0, 1)$$

Preverimo ortogonalnost

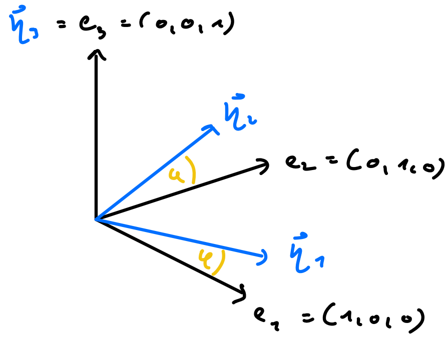
$$\langle \vec{r}_1, \vec{r}_2 \rangle = -\rho\cos\varphi\sin\varphi + \rho\sin\varphi\cos\varphi = 0$$
$$\langle \vec{r}_2, \vec{r}_3 \rangle = 0$$
$$\langle \vec{r}_1, \vec{r}_3 \rangle = 0$$

Lamejevi koeficienti

$$H_1 = |\vec{r}_1| = 1$$
$$H_2 = |\vec{r}_2| = \rho$$
$$H_3 = |\vec{r}_3| = 1$$

Posledično:

$$\vec{\eta}_1 = \frac{\vec{r}_1}{H_1} = (\cos\varphi, \sin\varphi, 0)$$
$$\vec{\eta}_2 = (-\sin\varphi, \cos\varphi, 0)$$
$$\vec{\eta}_3 = (0, 0, 1)$$



Po trditvi, za $u(x,y,z) = U(\rho, \varphi, z)$

$$\nabla_x u(\vec{r}) = \frac{\partial U}{\partial \xi_1}\frac{\vec{\eta}_1}{H_1} + \frac{\partial U}{\partial \xi_2}\frac{\vec{\eta}_2}{H_2} + \frac{\partial U}{\partial \xi_3}\frac{\vec{\eta}_3}{H_3}$$

$$\boxed{= \frac{\partial U}{\partial \rho}(\cos\varphi, \sin\varphi, 0) + \frac{\partial U}{\partial \varphi}\frac{(-\sin\varphi, \cos\varphi, 0)}{\rho} + \frac{\partial U}{\partial z}(0, 0, 1)}$$

Po izreku

$$(\Delta u)(x) = \frac{1}{H}\sum_{j=1}^{3} \frac{\partial}{\partial \xi_j}\left(\frac{H}{H_j^2}\frac{\partial U}{\partial \xi_j}\right) \circ \vec{R}$$

$$= \frac{1}{1\cdot\rho\cdot 1}\left(\frac{\partial}{\partial \rho}\left(\rho\frac{\partial U}{\partial \rho}\right) + \frac{\partial}{\partial \varphi}\left(\frac{1}{\rho}\frac{\partial U}{\partial \varphi}\right) + \frac{\partial}{\partial z}\left(\rho\frac{\partial U}{\partial z}\right)\right) \circ \vec{R}$$

$$= \left(\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial U}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 U}{\partial \varphi^2} + \frac{\partial^2 U}{\partial z^2}\right) \circ \vec{R}$$

$$\boxed{(\Delta u)\circ\vec{r} = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial U}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 U}{\partial \varphi^2} + \frac{\partial^2 U}{\partial z^2}}$$

Če je $u = u(x,y,z) = x^2 + y^2 + z^2$, izračunaj ∇u, Δu na dva načina

- $\nabla u \overset{\text{direktno}}{=} (2x, 2y, 2z)$ $\qquad u(x,y,z) = U(\rho, \varphi, z) = \rho^2 + z^2$

  $\overset{\text{formula}}{=} 2\rho(\cos\varphi, \sin\varphi, 0) + 0\cdot\frac{1}{\rho}(-\sin\varphi, \cos\varphi, 0) + 2z(0,0,1)$

  $= 2(\rho\cos\varphi, \rho\sin\varphi, z)$

- $\Delta u = u_{xx} + u_{yy} + u_{zz} = 6$

  $= \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho\, 2\cdot\rho) + \frac{1}{\rho^2}\cdot 0 + 2$

  $= 4 + 2 = 6$

Primer Sferične koordinate

Imamo $\xi = (\xi_1, \xi_2, \xi_3) = (\rho, \varphi, \theta)$ za $\rho \in [0,\infty)$, $\varphi \in [0, 2\pi)$, $\theta \in [0,\pi]$

in $(x, y, z) = \vec{r}(\rho, \varphi, \theta) = (\rho \cos\varphi \sin\theta, \rho \sin\varphi \sin\theta, \rho \cos\theta)$

zato je

$$\vec{r}_1 = \frac{\partial \vec{r}}{\partial \rho} = (\cos\varphi \sin\theta, \sin\varphi \sin\theta, \cos\theta)$$
$$\vec{r}_2 = \frac{\partial \vec{r}}{\partial \varphi} = \rho \sin\theta \, (-\sin\varphi, \cos\varphi, 0)$$
$$\vec{r}_3 = \frac{\partial \vec{r}}{\partial \theta} = \rho \, (\cos\varphi \cos\theta, \sin\varphi \cos\theta, -\sin\theta)$$

Ortogonalnost $\{\vec{r}_1, \vec{r}_2, \vec{r}_3\}$: ✓

Lamejevi koef. $H_1 = 1$ $H_2 = \rho \sin\theta$ $H_3 = \rho$

Sledi
$$\vec{\eta}_1 = (\cos\varphi \sin\theta, \sin\varphi \sin\theta, \cos\theta)$$
$$\vec{\eta}_2 = (-\sin\varphi, \cos\varphi, 0)$$
$$\vec{\eta}_3 = (\cos\varphi \cos\theta, \sin\varphi \cos\theta, -\sin\theta)$$

$$(\nabla_x u)(\vec{r}) = \sum \frac{\partial u}{\partial \xi_i} \frac{\vec{\eta}_i}{H_i}$$
$$= \frac{\partial u}{\partial \rho} (\cos\varphi \sin\theta, \sin\varphi \sin\theta, \cos\theta) + \frac{\partial u}{\partial \varphi} \frac{1}{\rho \sin\theta} (-\sin\varphi, \cos\varphi, 0) + \frac{\partial u}{\partial \theta} \frac{1}{\rho} (\cos\varphi \cos\theta, \sin\varphi \cos\theta, -\sin\theta)$$

$$= 0 + \frac{1}{\rho \sin\theta} (-\cos\varphi, -\sin\varphi, 0) + \frac{1}{\rho} (-\cos\varphi \sin\theta, -\sin\varphi \sin\theta, -\cos\theta)$$

$$(\Delta u)(\vec{r}) = \frac{1}{\rho^2 \sin\theta} \left(\frac{\partial}{\partial \rho} \left(\rho^2 \sin\theta \frac{\partial u}{\partial \rho} \right) + \frac{\partial}{\partial \varphi} \left(\frac{1}{\sin\theta} \frac{\partial u}{\partial \varphi} \right) + \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial u}{\partial \theta} \right) \right)$$
$$= \frac{1}{\rho^2} \frac{\partial}{\partial \rho} \rho^2 \frac{\partial u}{\partial \rho} + \frac{1}{\rho^2 \sin^2\theta} \frac{\partial^2 u}{\partial \varphi^2} + \frac{1}{\rho^2 \sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial u}{\partial \theta} \right)$$

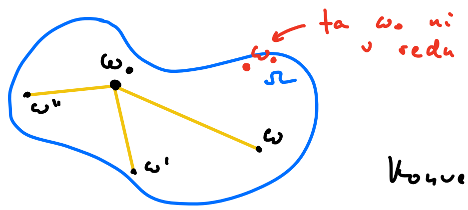
Čemu te formule?

Večasih je izraz za u bistveno bolj preprost od u. Če želimo izračunati Laplaceov operator naše funkcije, je pogosto bolj enostavno uporabiti na u izračun Δ v novih koord., kot pa računati Δu v kartez. koord. in potem rezultat še pretvarjati v nove koord.

Vemo
- $F = \nabla u \Rightarrow \operatorname{rot} F = 0$
- obrat $\Leftarrow$ ne velja nujno

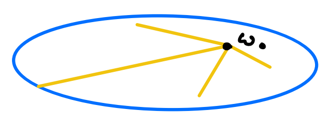
Kdaj (os katerih pogojih) velja obrat?

Def. Območje $\Omega \subseteq \mathbb{R}^3$ je zvezdasto, če $\exists\, \omega_0 \in \Omega$ da za $\forall\, \omega \in \Omega$ je daljica $[\omega_0, \omega] = \{(1-t)\omega_0 + t\omega \,;\, t \in [0,1]\}$ cela vsebovana v Ω.



Konveksna množica je zvezdasto območje, pri katerem je vsak element „dober" za ω_0

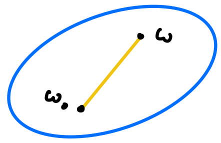
Konveksna $\Rightarrow$ zvezdasto
$\not\Leftarrow$



Izrek Naj bo Ω zvezdasto območje v $\mathbb{R}^3$, F glad. vekt. polje na Ω da $\operatorname{rot} F = 0$. Tedaj je F potencialno.

Dokaz Po def zvezdastega območja $\exists\, \omega_0 \in \Omega$ da $[\omega_0, \omega] \subset \Omega \quad \forall\, \omega \in \Omega$.

Def $\quad u(\omega) = \int_{[\omega_0,\omega]} \vec{F} \cdot d\vec{s}$



Če je F potencialno je integral odvisen le od končne in začetne točke

Za $\quad \omega = (x, y, z)$
$\quad\;\;\; \omega_0 = (x_0, y_0, z_0)$ je parametrizacija

$$\omega - \omega_0 = (x - x_0, y - y_0, z - z_0)$$

daljica $[\omega_0, \omega]$ podana z

$$\vec{r}(t) = (1-t)\,\omega_0 + t\omega = \begin{bmatrix} (1-t)x_0 + tx \\ (1-t)y_0 + ty \\ (1-t)z_0 + tz \end{bmatrix} = \begin{bmatrix} r_1(t) \\ r_2(t) \\ r_3(t) \end{bmatrix}$$

Sledi $\dot{\vec{r}}(t) = \omega - \omega_0$, zato je

$$u(\omega) = \int_0^1 \langle F(\vec{r}(t)), \overbrace{\omega - \omega_0}^{\dot{\vec{r}}(t)} \rangle_{\mathbb{R}^3}\, dt$$

Izračunajmo $(\nabla u)(\omega) = (\partial_x u, \partial_y u, \partial_z u)(\omega)$. Dovolj je izračunati $(\partial_x u)(\omega)$; ostale člene enako. Velja

$$(\partial_x u)(\omega) = \int_0^1 \frac{\partial}{\partial x} \langle F(\vec{r}(t)), \omega - \omega_0 \rangle\, dt =$$

$$= \int_0^1 \langle \frac{\partial}{\partial x} F(\vec{r}(t)), \omega - \omega_0 \rangle + \langle F(\vec{r}(t)), \overbrace{\frac{\partial}{\partial x}(\omega - \omega_0)}^{(1,0,0)} \rangle\, dt$$

Pišimo $\quad \vec{F} = (X, Y, Z)$ ter
$\quad X = X(a, b, c)$
$\quad Y = Y(a, b, c)$
$\quad Z = Z(a, b, c)$

Sledi

$$\frac{\partial}{\partial x} F(\vec{r}(t)) = \frac{\partial}{\partial x} (X(\vec{r}), Y(\vec{r}), Z(\vec{r}))$$

$$= (\partial_x X(\vec{r}), \partial_x Y(\vec{r}), \partial_x Z(\vec{r}))$$

$$\partial_x X(\vec{r}) = (\partial_a X)(\vec{r}) \cdot \underset{t}{\underset{\parallel}{\partial_x (r_1(t))}} + (\partial_b X)(\vec{r})\, \underset{0}{\underset{\parallel}{\partial_x (r_2(t))}} + (\partial_c X)(\vec{r})\, \underset{0}{\underset{\parallel}{\partial_x (r_3(t))}}$$

$$= t\, (\partial_a X)(\vec{r})$$

Podobno za ostale člene

$$\Rightarrow \frac{\partial}{\partial x} F(\vec{r}(t)) = t\left((\partial_a X)(\vec{r}), (\partial_a Y)(\vec{r}), (\partial_a Z)(\vec{r}) \right)$$

$$\operatorname{rot}(X, Y, Z) = 0 \qquad = t\left((\partial_a X)(\vec{r}), (\partial_b X)(\vec{r}), (\partial_c X)(\vec{r}) \right)$$

$$= \begin{vmatrix} i & j & k \\ \partial_a & \partial_b & \partial_c \\ X & Y & Z \end{vmatrix} \qquad = t\,(\tilde{\nabla} X)(\vec{r}), \quad \text{kjer je } \tilde{\nabla} = (\partial_a, \partial_b, \partial_c)$$

$$= (\underbrace{Z_b - Y_c}_{0}, \underbrace{X_c - Z_a}_{0}, \underbrace{Y_a - X_b}_{0}) = 0$$

Dobili smo

$$(\partial_x u)(\omega) = \int_0^1 \big(\langle t\, \underbrace{(\tilde\nabla X)(\vec r), \omega - \omega_0 \rangle}_{\frac{\partial}{\partial t}(X(\vec r(t)))} + X(\vec r)\big)\, dt$$

$$= \int_0^1 t \frac{\partial}{\partial t} X + X \, dt$$

$$= \int_0^1 \frac{\partial}{\partial t} t X \, dt$$

$$= t\, X(\vec r(t)) \Big|_0^1$$

$$= X(\vec r(1)) - 0 \cdot X(\vec r(0))$$

$$= X(\omega)$$

Torej je $\partial_x u = X$. Podobno $\partial_y u = Y$, $\partial_z u = Z$.
To je $\nabla u = F$.

$\square$

Formule Krivuljni in ploskovni integral skal. in vekt. polj

	skalarno (f)	vektorsko $(\vec F)$
krivulja $\vec r = \vec r(t)$, $t \in J \subset \mathbb{R}$	$\int_J f(\vec r(t)) \lvert \dot{\vec r}(t) \rvert \, dt$	$\int_J \langle \vec F(\vec r(t)), \dot{\vec r}(t) \rangle \, dt$
ploskev $\vec r = \vec r(u,v)$, $(u,v) \in D \subset \mathbb{R}^2$	$\int_D f(\vec r(u,v)) \lvert \dot{\vec r}_u \times \dot{\vec r}_v \rvert \, du\, dv$	$\int_D \langle \vec F(\vec r(u,v)), \dot{\vec r}_u \times \dot{\vec r}_v \rangle \, du\, dv$