Ponovitev integralov

Per partes

$$\int u\, dv = uv - \int v\, du$$

Določeni integrali

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)$$

Zamenjava spremenljivke

Splošna formula

$$\int_{\varphi(a)}^{\varphi(b)} f(u)\,du = \int_a^b f(\varphi(x))\,\varphi'\,dx$$

$$u = \varphi(x)$$
$$du = \varphi'(x)\,dx$$

$$\int_a^b f(x)\,dx = \int_\alpha^\beta f(\varphi(t))\,\dot{\varphi}(t)\,dt$$

$$x = \varphi(t) \qquad \varphi(\alpha) = a \qquad \varphi(\beta) = b$$
$$dx = \dot{\varphi}(t)\,dt$$

Najenostavnejši primer:

φ ... bijekcija

$$\int_a^b f(x)\,dx = \int_{\varphi^{-1}(a)}^{\varphi^{-1}(b)} f(\varphi(t))\,\dot{\varphi}(t)\,dt$$

1) Izračunaj nedoločene integrale

a) $$\int \left(x^2 + \frac{1}{x^2}\right)\sqrt{x}\,dx = \int x^{2,5} + x^{-1,5}\,dx = \frac{x^{3,5}}{3,5} + \frac{\sqrt{x}^{-1}}{-\frac{1}{2}} + C$$
$$= \frac{2}{7}x^{7/2} - \frac{2}{\sqrt{x}} + C$$

b) $$\int \frac{x^2+3}{3x+2}\,dx = \int \left(\frac{x}{3} - \frac{2}{9}\right) + \frac{31/9}{3x+2}\,dx = \frac{1}{6}x^2 - \frac{2}{9}x + \frac{31}{9}\frac{1}{3}\ln|3x+2| + C$$
$$= \frac{x^2}{6} - \frac{2x}{9} + \frac{31}{27}\ln|3x+2| + C$$

$$(x^2+3) : (3x+2) = \frac{x}{3} - \frac{2}{9}$$
$$-\left[\, x^2 + \frac{2}{3}x \right.$$
$$-\frac{2}{3}x + 3$$
$$-\left[\, -\frac{2}{3}x - \frac{4}{9} \right.$$
$$\frac{31}{9}$$

c) $\int (x^2+x)\underbrace{e^{-2x+7}}_{dv}dx$ with $u = x^2+x$: $= (x^2+x)\left(-\frac{1}{2}\right)e^{-2x+7} - \int -\frac{1}{2}e^{-2x+7}(2x+1)\,dx$

$du = (2x+1)dx \qquad v = -\frac{1}{2}e^{-2x+7}$

$$= -\frac{1}{2}(x^2+x)e^{-2x+7} + \int \underbrace{\left(x+\frac{1}{2}\right)}_{u} \underbrace{e^{-2x+7}dx}_{dv} =$$

$du = dx \qquad v = -\frac{1}{2}e^{-2x+7}$

$$= -\frac{1}{2}(x^2+x)e^{-2x+7} + \left(x+\frac{1}{2}\right)\left(-\frac{1}{2}e^{-2x+7}\right) - \int -\frac{1}{2}\underbrace{e^{-2x+7}dx}_{-\frac{1}{2}e^{-2x+7}} =$$

$$= -\frac{1}{2}e^{-2x+7}\left(x^2+x+x+\frac{1}{2}+\frac{1}{2}\right) = -\frac{1}{2}e^{-2x+7}(x+1)^2 + C$$

2) Izračunaj določene integrale

a) $$\int_0^{2\pi} \underbrace{\left|\sin x - \frac{1}{2}\right|}\,dx = \int_{\pi/6}^{5\pi/6}\left(\sin x - \frac{1}{2}\right)dx - \int_0^{\pi/6}\left(\sin x - \frac{1}{2}\right)dx - \int_{5\pi/6}^{2\pi}\left(\sin x - \frac{1}{2}\right)dx =$$

$$\begin{cases} \sin x - \frac{1}{2}\ ; & \sin x - \frac{1}{2} \geq 0 \qquad \sin x \geq \frac{1}{2} \Rightarrow x \in \left[\frac{\pi}{6}, \frac{5\pi}{6}\right] \\ -\sin x + \frac{1}{2}\ ; & \sin x - \frac{1}{2} < 0 \qquad \sin x < \frac{1}{2} \Rightarrow x \in \left(\frac{5\pi}{6}, \frac{\pi}{6}\right) \end{cases}$$

$$= \left(\frac{x}{2} + \cos x\right)\Big|_0^{\pi/6} + \left(-\cos x - \frac{x}{2}\right)\Big|_{\pi/6}^{5\pi/6} + \left(\frac{x}{2} + \cos x\right)\Big|_{5\pi/6}^{2\pi} =$$

$$= \frac{\pi}{12} + \frac{\sqrt{3}}{2} - \cancel{1} + \frac{\sqrt{3}}{2} - \frac{5\pi}{12} + \frac{\sqrt{3}}{2} + \frac{\pi}{12} + \pi + \cancel{1} - \frac{5\pi}{12} + \frac{\sqrt{3}}{2} = \frac{\pi}{3} + 2\sqrt{3}$$

b) $$\int_1^5 \frac{x\,dx}{\sqrt{2x-1}} = x\sqrt{(2x-1)}\Big|_1^5 - \int_1^5 (2x-1)^{1/2}dx$$

$u = x \qquad dv = \frac{dx}{\sqrt{2x-1}} = (2x-1)^{-\frac{1}{2}}dx$

$du = dx \qquad v = (2x-1)^{1/2}$

$$= 15 - 1 - \int_1^5 (2x-1)^{1/2}dx = 30 - 2 - \left(\frac{2}{3\cdot 2}(2x-1)^{3/2}\right)\Big|_1^5 = \frac{16}{3}$$

Alternativna metoda:

- $$\int_1^5 \frac{x\,dx}{\sqrt{2x-1}} = \int_1^9 \frac{\frac{u+1}{2}\cdot\frac{1}{2}du}{\sqrt{u}} = \frac{1}{4}\int_1^9 \left(u^{1/2} + u^{-1/2}\right)du = \ldots = \frac{16}{3}$$

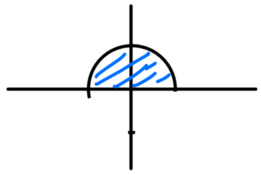
$u = 2x-1 \qquad x = \frac{u+1}{2}$

$du = 2dx$

- $$\int_1^5 \frac{x\,dx}{\sqrt{2x-1}} = \int_1^3 \frac{\frac{1+t^2}{2}}{t}\,t\,dt = \int_1^3 \frac{1}{2} + \frac{t^2}{2}\,dt = \ldots \frac{16}{3}$$

$t = \sqrt{2x-1} \qquad x = \frac{1+t^2}{2}$

$dx = t\,dt$

ⓒ $\displaystyle\int_{-1}^{1}\sqrt{1-x^2}\,dx =$ 

$$= \int_{-1}^{0}\sqrt{1-x^2}\,dx + \int_{0}^{1}\sqrt{1-x^2}\,dx$$

Splošna formula za zamenjavo spremenljivke

$$\int_a^b f(x)\,dx = \int_\alpha^\beta f(\varphi(t))\,\varphi'(t)\,dt$$

$x = \varphi(t)$

Integral razdelimo na intervale kjer ima φ inverz

$t = \sqrt{1-x^2}$

$x = -\sqrt{1-t^2}$ $\qquad\qquad$ $x = \sqrt{1-t^2}$

$dx = \dfrac{t\,dt}{\sqrt{1-t^2}}$ $\qquad\qquad$ $dx = \dfrac{-t\,dt}{\sqrt{1-t^2}}$

$$= \int_0^1 \frac{t\;t}{\sqrt{1-t^2}}\,dt + \int_1^0 t\,\frac{-t\,dt}{\sqrt{1-t^2}} = 2\int_0^1 \frac{t^2\,dt}{\sqrt{1-t^2}} = 2\int_0^{\pi/2}\frac{\sin^2\varphi\,\cos\varphi\,d\varphi}{|\cos\varphi|}$$

$t = \sin\varphi$

$dt = \cos\varphi\,d\varphi$

$$= 2\int_0^{\pi/2}\sin^2\varphi\,d\varphi = 2\int_0^{\pi/2}\frac{1-\cos 2\varphi}{2}\,d\varphi$$

$$= \varphi - \frac{\sin 2\varphi}{2}\Big|_0^{\pi/2} = \frac{\pi}{2}$$

ⓓ $$\int_0^2 \frac{(x^3-4x+2)\cos(x^4-8x^2+8x+3)}{x^4-8x^2+8x+3}\,dx = \int_{\varphi(0)}^{\varphi(2)}\frac{\cos\varphi(x)}{4\varphi(x)}\,\varphi'(x)\,dx$$

$t = \varphi(x) = x^4-8x^2+8x+3$

$dt = 4x^3-16x+8 = 4(x^3-4x+2)$

$$= \frac{1}{4}\int_3^3 \frac{\cos t}{t}\,dt = 0$$

ⓔ $$\int_0^{2\pi}\frac{dx}{16+9\cos^2 x} = 4\int_0^{\pi/2}\frac{dx}{16+9\cos^2 x} =$$

či pogledamo graf

Če je $\int R(\sin x, \cos x)\,dx$ $\quad t = \tan{x/2}$

$\int R(\sin^2 x, \cos^2 x)\,dx$ $\quad t = \tan x$

$t = \tan x$

$x = \arctan x$

$dx = \dfrac{1}{1+t^2}\,dt$

$\tan^2 x + 1 = \dfrac{1}{\cos^2 x}$

$$= 4\int_0^\infty \frac{\frac{1}{1+t^2}\,dt}{16+\frac{9}{1+t^2}} =$$

$$= 4\int_0^\infty\frac{dt}{25+16t^2} = \frac{4}{25}\int_0^\infty\frac{dt}{1+\left(\frac{4t}{5}\right)^2} = \frac{4}{25}\,\frac{5}{4}\arctan\frac{4}{5}t\,\Big|_0^\infty = \frac{\pi}{10}$$

(f) $\int_2^\infty \frac{dx}{x^2-1} = \frac12 \int_2^\infty \frac{1}{x-1} - \frac{1}{x+1}\, dx = \frac12 \left(\ln|x-1| - \ln|x+1|\right)\Big|_2^\infty =$

$\frac{1}{x^2-1} = \frac{1}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1} = \frac{Ax+A+Bx-B}{x^2-1}$ $A+B=0$, $A-B=1$; $A=\frac12$, $B=-\frac12$

$= \lim_{M\to\infty} \frac12 \left(\ln\left|\frac{x-1}{x+1}\right|\right)\Big|_2^M = \lim_{M\to\infty} \frac{\ln \overbrace{\frac{M-1}{M+1}}^{\to 1} - \ln\frac13}{2} = \frac12 \ln 3$

Integrali s parametrom

$$F(y) = \int_a^b f(x,y)\,dx \qquad y \in [c,d]$$

$f: [a,b]\times[c,d] \to \mathbb{R}$ zvezna sledi da je F zvezna in integ.

$$F(y) = \int_a^b f(x,y)\,dx$$

(1) F je zvezna za $y \in [c,d]$

(2) $\int_c^d F(y)\,dy = \int_c^d \int_a^b f(x,y)\,dx\,dy = \int_a^b \int_c^d f(x,y)\,dy\,dx$

(3) $\frac{\partial f}{\partial y}$ ... zvezna $F'(y) = \int_a^b \frac{\partial f}{\partial y}(x,y)\,dx$

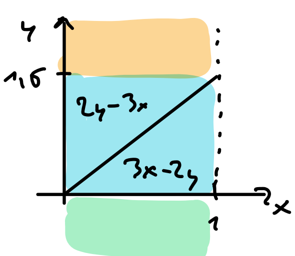
(4) $\frac{\partial f}{\partial y}$ ... zvezna $F(y) = \int_{\varphi(y)}^{\psi(y)} f(x,y)\,dx$ $\varphi, \psi: [c,d] \to [a,b]$ odvedljivi

$$F'(y) = f(\psi(y),y)\,\psi'(y) - f(\varphi(y),y)\,\varphi'(y) + \int_{\varphi(y)}^{\psi(y)} \frac{\partial f}{\partial y}(x,y)\,dx$$

(1) Izračunaj $F(y) = \int_0^1 |2y-3x|\,dx$ $|2y-3x| = \begin{cases} 2y-3x; & 2y-3x \ge 0 \quad y \ge \frac32 x \\ -2y+3x; & 2y-3x < 0 \quad y < \frac32 x \end{cases}$

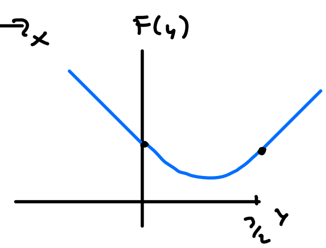
(i) $y \le 0$ $F(y) = \int_0^1 2y-3x\,dx$

$= 2xy - \frac32 x^2 \Big|_0^1 = \frac32 - 2y$



(ii) $y \in [0, 3/2]$

$F(y) = \int_0^{2/3 y} 2y-3x\,dx + \int_{2/3 y}^1 3x-2y\,dx = \frac43 y^2 - 2y + 3/2$



(iii) $y \ge 3/2$ $F(y) = \int_0^1 3x-2y\,dx = 2y - 3/2$

② $F(y) = \int_0^1 \frac{dy}{1+x^2y^2}$

a) Določi D_F
b) Ali je F zvezna
c) Izračunaj F

a) imenovalec je vedno $\geq 1 \Rightarrow y \in \mathbb{R}$

izberemo si $y \in [c,d]$

b) $f(x,y) = \frac{1}{1+x^2y^2}$ je zvezna na $[a,b]\times[c,d] \to \mathbb{R}$
je potem tudi F zvezna na $\mathbb{R}$

c) $F(y) = \int_0^1 \frac{dx}{1+x^2y^2} = \frac{1}{y}\arctan xy \Big|_0^1$ za $y \neq 0$

$= \frac{1}{y}\arctan y$ za $y=0$ $F(0) = 1$

$$F(y) = \begin{cases} 1 & ; \ y=0 \\ \frac{1}{y}\arctan y & ; \ y \neq 0 \end{cases}$$

poglejmo limito

$$\lim_{y\to 0} \frac{1}{y}\arctan y \overset{\text{L'Hosp.}}{=} \frac{\frac{1}{1+y^2}}{1} = 1 \ \checkmark$$

③ Izračunaj

$$\lim_{y\to\infty} \Bigg(\underbrace{\int_0^{\frac{\pi}{2}} \ln(y^2 - \sin^2\varphi)\,d\varphi - \pi \ln\left(y + \sqrt{y^2-1}\right)}_{F(y)} \Bigg) =$$

$$F(y) = \int_0^{\frac{\pi}{2}} \ln(y^2 - \sin^2\varphi)\,d\varphi - \pi \ln\left(y + \sqrt{y^2-1}\right) \cdot \frac{2}{\pi}\int_0^{\frac{\pi}{2}} d\varphi$$

$$F(y) = \int_0^{\pi} \ln(y^2 - \sin^2\varphi) - 2\ln\left(y + \sqrt{y^2-1}\right) d\varphi$$

$$F(y) = \int_0^{\pi} \ln \frac{y^2 - \sin^2\varphi}{\left(y + \sqrt{y^2-1}\right)^2}\, d\varphi$$

Nova spremenljivka $y = \frac{1}{t}$ $\quad y \to \infty, \ t \to 0$

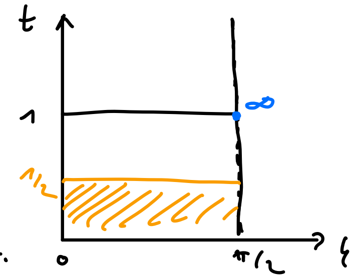
$$G(t) = \int_0^{\pi} \ln \frac{\frac{1}{t^2} - \sin^2\varphi \ |t^2}{\left(\frac{1}{t} + \sqrt{\frac{1}{t^2}-1}\right)^2 |t^2}\, d\varphi = \int_0^{\pi} \underbrace{\ln \frac{1 - t^2\sin^2\varphi}{\left(1 + \sqrt{1-t^2}\right)^2}}_{f(\varphi,t)}\, d\varphi$$

$t \in [0,1]$

$t = 1 \Rightarrow f(\varphi,1) = \ln\cos^2\varphi$

$f\left(\frac{\pi}{2},1\right) = \infty$

$\lim_{t\to 0} G(t)$



želimo poljuben zaprt int.

$f(\varphi,t)$ je zvezna na $[0,\frac{\pi}{2}]\times[0,\frac{1}{2}]$ in kar sledi da je $F(t)$ zvezna na za $t=[0,1/2)$ ker pomeni

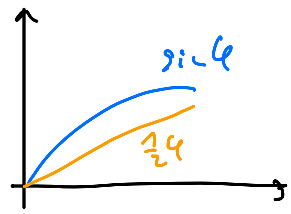
$$\lim_{t\to 0} G(t) = G(0) = \int_0^{\pi/2} \ln\frac{1}{4}\, d\varphi = -\pi\ln 2$$

če je zvezna

$$= \int_0^{\pi/2} \lim \ldots$$

④ $$\lim_{R\to\infty} \int_0^{\pi/2} e^{-R\sin\varphi}\, d\varphi$$

$$0 \le \int_0^{\pi/2} e^{-R\sin\varphi}\, d\varphi < \int_0^{\pi/2} e^{-\frac{R}{2}\varphi}\, d\varphi$$



$\frac{1}{2}\varphi < \sin\varphi$

$$= -\frac{2}{R}\, e^{-\frac{R}{2}\varphi}\Big|_0^{\pi/2} =$$

$$= -\frac{2}{R}\left(e^{-\frac{R\pi}{4}} - 1\right) \Longrightarrow \lim_{R\to\infty} = 0$$

Ker smo majoranto in minoranto zato je omejena vrednost enaka tudi 0.

⑤ Izračunaj $F'(y)$ kjer je

$$F(y) = \int_0^{\pi/2} \frac{\sin(xy)}{x}\, dx$$

Opomba: $\int \frac{\sin x}{x} dx$ se ne da nedoločeno int. ki bi imelo končen rezultat. Rešimo s Taylorjevo vrsto.

$$F'(y) = \frac{\partial}{\partial y}\int_0^{\pi/2} \frac{\sin(xy)}{x}\, dx =$$

$$= \int_0^{\pi/2} \frac{\partial}{\partial y}\frac{\sin(xy)}{x}\, dx =$$

$$= \int_0^{\pi/2} \frac{x\cos(xy)}{x}\, dx = \int_0^{\pi/2} \cos(xy)\, dx$$

$$= \frac{\sin(xy)}{y}\Big|_0^{\pi/2} = \frac{\sin(\frac{\pi}{2}y)}{y} \qquad y\neq 0$$

$y=0$ $$F'(0) = \int_0^{\pi/2} \cos 0\, dx = \frac{\pi}{2}$$

$$F' = \begin{cases} \frac{\pi}{2} & ; \; y=0 \\ \frac{\sin\frac{\pi}{2}y}{y} & ; \; y\neq 0 \end{cases}$$

Preverimo pravilnost korakov

① $F(y) = \int_0^{\pi/2} \frac{\sin xy}{x}\, dx$

$f(x,y) = \frac{\sin xy}{x}$ ... zvezna na $f:(0,\frac{\pi}{2}]\times[c,d]\to\mathbb{R}$

$$\lim_{(x,y)\to(0,y_0)} \frac{\sin xy}{x} = \lim_{(x,y)\to(0,y_0)} \frac{xy - \frac{(xy)^3}{3!} + \frac{(xy)^5}{5!} - \dots}{x} =$$

$$= \lim\left(y - \frac{x^2y^3}{3!} + \frac{x^4y^5}{5!} - \dots\right) \underset{\uparrow}{=} y_0$$

ker je izraz zvezen lahko kar vstavim x, y noter

$$\tilde f(x,y) = \begin{cases} y; & x=0 \quad y\in\mathbb{R} \\ \frac{\sin xy}{x}; & x\neq 0 \quad y\in\mathbb{R}\end{cases}$$

Zvezno smo razširili funkcijo na zaprt kvader

$\tilde f: [0,\pi/2]\times[c,d]\to\mathbb{R}$ ... zvezna

$\Rightarrow$ F je zvezna za vsak y

② Ali je $\frac{\partial \tilde f}{\partial y}$ zvezna funkcija $[0,\pi/2]\times[c,d]$

$$\frac{\partial f}{\partial y} = \begin{cases}1 & y=0\\ \cos xy & y\neq 0\end{cases} \Rightarrow \frac{\partial \tilde f}{\partial y} \text{ je zvezna}$$

Posplošeni integrali s parametrom

$f:[a,\infty)\times[c,d]$ -- zvezna

$$F(y) = \int_a^\infty f(x,y)\,dx$$

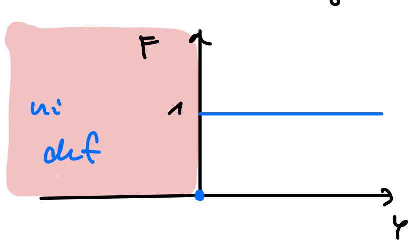
Primer

$$F(y) = \int_0^\infty y\, e^{-xy}\,dx$$

$F(0)=0$

$y<0$ divergentno

$y>0$ $\quad y\int_0^\infty e^{-xy}\,dx = -e^{-xy}\Big|_0^\infty = 1$



Def: Enakomerna konvergenca

$$F(y) = \int_a^\infty f(x,y)\,dx \qquad y \in [c,d]$$

Integral $\int_a^\infty f(x,y)\,dx$ konvergira enakomerno na $[c,d]$, če za $\forall \varepsilon > 0$ $\exists M_0$ $\forall M \geq M_0$ velja

$$\left| \int_M^\infty f(x,y)\,dx \right| < \varepsilon \qquad \text{za } \forall\, y \in [c,d]$$

Kaj velja

$$f : [a,\infty) \times [c,d] \to \mathbb{R} \quad \text{zvezna}$$

$$F(y) = \int_a^\infty f(x,y)\,dx \qquad y \in [c,d]$$

① $\int_a^\infty f(x,y)\,dx$ konv. enak. na $[c,d]$ $\Rightarrow$ F zvezna na $[c,d]$

② $\int_a^\infty f(x,y)\,dx$ konv. enak. na $[c,d]$ $\Rightarrow$ $\int_c^d \int_a^\infty f(x,y)\,dx\,dy = \int_a^\infty \int_c^d f(x,y)\,dy\,dx$

③ Če je $\int_a^\infty \frac{\partial f}{\partial y}(x,y)\,dx$ konv. enak. na $[c,d]$ $\Rightarrow$

$\frac{\partial f}{\partial y}$ zvezna funk. $\Longrightarrow F'(y) = \int_a^\infty \frac{\partial f}{\partial y}(x,y)\,dx$

④ Weierstrassov kriterij (za določanje enak. konv.)

$F(y) = \int_a^\infty f(x,y)\,dx$ Denimo, da imamo oceno $|f(x,y)| \leq W(x)$ za $\forall y \in [c,d]$

Če $\int_a^\infty W(x)\,dx$ konvergira $\Rightarrow$ $\int_a^\infty f(x,y)\,dx$ konv. enakomerno na $[c,d]$

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$$F(y) = \int_0^\infty \frac{y\,dx}{1+x^2y^2}$$

a) Za katere $y \in [0,\infty)$ integral konvergira?
b) Izračunaj $F(y)$
c) Ali integral konvergira enakomerno na $[0,1]$?
d) Ali integral konvergira enakomerno na $[c,d] \subset (0,\infty)$?
e) Ali integral konvergira enakomerno na $[c,\infty) \subset (0,\infty)$?
f) Ali integral konvergira enakomerno na $(0,\infty)$?

a) $y=0$ $\quad F(0)=0$ ✓

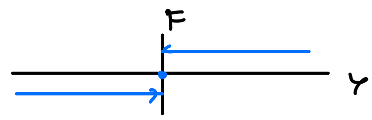
$y>0$ $\Longrightarrow$ $\int_0^\infty \frac{1}{x^2}\,dx$ konv. $\Longrightarrow \int_1^\infty \frac{y\,dx}{1+x^2y^2} \le \int_1^\infty \frac{dx}{x^2y}$ ... konv.

Ker je F liha konv. tudi za $y<0$

b) $y>0$

$$F(y) = y\int_0^\infty \frac{dx}{1+y^2x^2} = y\frac{1}{y}\arctan yx \Big|_0^\infty = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

Ker je F liha $\quad F(y) = -\frac{\pi}{2}$ za $y<0$



c) Če bi F konv. enak. na $[0,1]$ bi pomenilo, da bi bila F zvezna na $[0,1]$

Ne konv. enak.

$\Longrightarrow$ Vsak interval, ki vsebuje 0 ne konv. enak.

d) $[c,d] \subset (0,\infty)$

W. krit

$$\left|\frac{y}{1+x^2y^2}\right| \le \frac{d}{1+x^2c^2} = w(x) \qquad \int w(x) \text{ --- konv} \Rightarrow F \text{ enak. konv.}$$

e) $[c,\infty) \subset (0,\infty)$

$$\left|\frac{y}{1+x^2y^2}\right| \le \frac{?}{1+x^2c^2}$$

Weierstrassov kriterij
Ne moremo uporabiti iste ocene kot prej

Po def $\forall \varepsilon>0 \ \exists b_0 \ \forall b \ge b_0$ velja

$$\left|\int_b^\infty \frac{y\,dx}{1+y^2x^2}\right| < \varepsilon \qquad y \in [c,\infty)$$

$$\left|\int_b^\infty \frac{y\,dx}{1+y^2x^2}\right| = \left|\arctan xy\,\Big|_b^\infty\right| = \frac{\pi}{2} - \arctan by \le \frac{\pi}{2} - \arctan bc \quad \forall y \in [c,\infty)$$

$$\lim_{b\to\infty} \frac{\pi}{2} - \arctan bc = 0 \quad ✓ \; < \varepsilon$$

f) Ali $\int_0^\infty \frac{y\,dx}{1+x^2y^2}$ konv. enak. na $(0,\infty)$? **NE**

b poljubno velik, fiksen

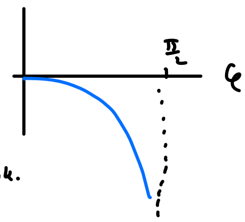
$$\left|\int_b^\infty \frac{y\,dx}{1+x^2y^2}\right| = \frac{\pi}{2} - \arctan by \xrightarrow{y\to 0} \frac{\pi}{2} \neq 0$$

7 Za $y \ge 1$ izračunaj

$$F(y) = \int_0^{\pi/2} \ln(y^2 - \sin^2\varphi)\,d\varphi$$

Območje integracije je $[0, \pi/2]$, ampak za $y=1$

$$F(1) = \int_0^{\pi/2} \ln(1-\sin^2\varphi)\,d\varphi = \int_0^{\pi/2} \ln(\cos^2\varphi)\,d\varphi$$



ni omejena

Če je $y > 1 \Rightarrow y^2 - \sin^2\varphi > 0 \Rightarrow \ln(y^2-\sin^2\varphi)$ zvezna funk. (omejena)

Sledi, da integral obstaja.

$y=1$: $$F(1) = \int_0^{\pi/2} \ln(\cos^2\varphi)\,d\varphi = 2\int_0^{\pi/2} \ln(\cos\varphi)\,d\varphi = 2\int_{\pi/2}^0 -\ln\left(\cos\left(\tfrac{\pi}{2}-u\right)\right)du =$$

$u = \frac{\pi}{2} - \varphi$
$du = -d\varphi$

$$= 2\int_0^{\pi/2} \ln(\sin u)\,du = 2\int_0^{\pi/2} \frac{\sqrt{u}\,\ln(\sin u)}{\sqrt{u}}\,du$$

$$\lim_{u\to 0} \sqrt{u}\,\ln(\sin u) = \lim_{u\to 0} \frac{\ln(\sin u)}{u^{-1/2}} \overset{\text{L'Hop}}{=} \lim_{u\to 0} \frac{\frac{1}{\sin u}\cos u}{-\frac{1}{2}u^{-3/2}} = -2\lim_{u\to 0}\cos u \lim_{u\to 0} \frac{u \cdot u^{1/2}}{\sin u} = 0$$

$\Rightarrow F(1)$ konvergira

Test konvergence
$\int_0^a \frac{g(x)}{x^\alpha}\,dx \quad \alpha < 1$
in je g omejena (zv.) na $[0,a]$ $\Rightarrow$ int. konv.

Opomba: Kaj pa če $y<1$? Naj bo $y = 1/2$

$$F(1/2) = \int_0^{\pi/2} \underbrace{\ln\left(\tfrac{1}{4} - \sin^2\varphi\right)}_{\text{Def. za } \varphi \in [0, \frac{\pi}{6}]} d\varphi = //$$

Izračunajmo še konkretno $F(1)$

$$F(1) = 2\int_0^{\pi/2} \ln(\cos\varphi)\,d\varphi = 2\int_0^{\pi/2} \ln(\sin\varphi)\,d\varphi = A$$

$$2A = 2\int_0^{\pi/2} \ln(\sin\varphi)\,d\varphi + 2\int_0^{\pi/2} \ln(\cos\varphi)\,d\varphi = 2\int_0^{\pi/2} \ln(\sin\varphi\cos\varphi)\,d\varphi =$$

$$= 2\int_0^{\pi/2} \ln\frac{\sin 2\varphi}{2}\,d\varphi = 2\int_0^{\pi/2} \ln(\sin 2\varphi) - 2\int_0^{\pi/2} \ln 2\,d\varphi =$$

$u = 2\varphi$
$du = 2\,d\varphi$
$d\varphi = \frac{1}{2}\,du$

simetrična na $[0,\pi]$

$$= \int_0^{\pi} \ln(\sin u)\,du - 2\frac{\pi}{2}\ln 2 = 2\int_0^{\pi/2} \ln(\sin u)\,du - \pi\ln 2 = A - \pi\ln 2 = 2A$$

$$A = -\pi\ln 2 \qquad F(1) = \int_0^{\pi/2} \ln(\cos\varphi)\,d\varphi = -\pi\ln 2$$

Ali je $F(y) = \int_0^{\pi/2} \ln(y^2-\sin^2\varphi)\,d\varphi$ zvezna na $y\in[1,\infty)$?

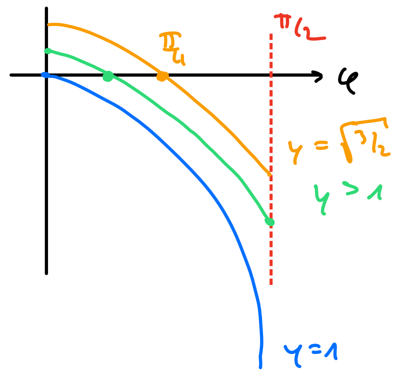
$y \geq 1$

$\ln(y^2-\sin^2\varphi) = f(x)$

Enak. konv.

$F(y) = \int_a^{\infty} f(x,y)\,dx \quad y\in[c,d]$

če enak. konv $\Rightarrow F$ zvez. na $[c,d]$



$f(x)=0$

$y^2 - \sin^2\varphi = 1$

$\varphi = \arcsin\left(\sqrt{y^2-1}\right)$

Ali $F(y) = \int_0^{\pi/2} \ln(y^2-\sin^2\varphi)\,d\varphi$ konv. enak. na $y\in\left[1, \frac{\sqrt3}{\sqrt2}\right]$

$\Updownarrow$

Če za $\forall \varepsilon>0\ \exists b_0$, da za $\forall b\geq b_0$ velja

$$\left|\int_b^{\pi/2} \ln(y^2-\sin^2\varphi)\,d\varphi\right| < \varepsilon \quad \text{za vse } y\in\left[1,\frac{\sqrt3}{\sqrt2}\right]$$

$(b_0 > \pi/4)$

max vrednost pri $y=1$ $\quad y\in\left[1,\frac{\sqrt3}{\sqrt2}\right]$

$$\left|\int_b^{\pi/2} \ln(y^2-\sin^2\varphi)\,d\varphi\right| \leq \int_b^{\pi/2}\left|\ln(y^2-\sin^2\varphi)\right|d\varphi \leq \int_b^{\pi/2}\left|\ln(1-\sin^2\varphi)\right|d\varphi =$$

$$= \int_b^{\pi/2}\left|\ln(\cos^2\varphi)\right|d\varphi < \varepsilon$$

$\left(\int_0^{\pi/2} \ln\cos^2\varphi\,d\varphi\right.$ konv.

pol ima pri $\varphi=\pi/2 \Rightarrow \varepsilon>0\ \exists b_0$ da $\forall b\geq b_0$

$\Rightarrow F(y)$ je zvezna na $\left[1,\frac{\sqrt3}{\sqrt2}\right]$. Kaj pa $\left[\frac{\sqrt3}{\sqrt2}, c\right]$, kjer je $c\in\mathbb{R}$ in $c>\frac{\sqrt3}{\sqrt2}$. To pa velja saj

$$F(y) = \int_0^{\pi/2} \underbrace{\ln(y^2-\sin^2\varphi)}\,d\varphi$$

$f(x,y): \left[0,\frac{\pi}{2}\right]\times\left[\frac{\sqrt3}{\sqrt2}, c\right] \to \mathbb{R}$

je zvezna na tem pravokotniku, saj nima več polov.

$\Rightarrow F$ zvezna za vse $y\geq 1$

$F(1) = -\pi\ln 2$

F je zvezna na $y\in[1,\infty)$

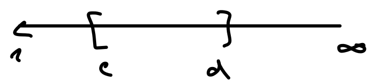
$$F(y) = \int_0^{\pi/2} \ln(y^2 - \sin^2\varphi)\, d\varphi$$

$$F'(y) = \int_0^{\pi/2} \frac{2y}{y^2 - \sin^2\varphi}\, d\varphi$$

$y=1$ $$\int_0^{\pi/2} \frac{2}{\cos^2\varphi}\, d\varphi = \int_0^{\pi/2} \frac{2}{\sin^2 u}\, du = \int^{\pi/2} \frac{2\,\frac{u^2}{\sin^2 u}}{u^2}\, du$$ $\frac{u^2}{\sin^2 u} \to 1$, $u^2 \to 0$ D.V.

$y>1$ $\quad \frac{2y}{y^2-\sin^2 y}$ je omejena

$y \in [c,d] \subset (1,\infty)$



$$g(\varphi, y) = \frac{2y}{y^2 - \sin^2\varphi} \quad \text{zvezna na } [0, \tfrac{\pi}{2}] \times [c,d] \to \mathbb{R}$$

$\Rightarrow \quad y \in [c,d] \subset (1,\infty)$

$$F'(y) = \int_0^{\pi/2} \frac{2y}{y^2 - \sin^2 y}\, d\varphi = \int_0^{\infty} \frac{2y}{y^2 - \frac{t^2}{1+t^2}} \frac{dt}{(1+t^2)} = \int_0^{\infty} \frac{2y\, dt}{y^2 + t^2y^2 - t^2} = \int_0^{\infty} \frac{2y\, dt}{y^2 + t^2(y^2-1)} =$$

$t = \tan\varphi$, $dt = \frac{d\varphi}{\cos^2\varphi}$, $c^2 = \frac{1}{1+t^2}$, $s^2 = 1 - \frac{1}{1+t^2} = \frac{t^2}{1+t^2}$, $d\varphi = \frac{dt}{1+t^2}$

$$= \frac{2y}{y^2} \int_0^{\infty} \frac{dt}{1 + \left(\frac{\sqrt{y^2-1}}{y} t\right)^2} = \frac{2}{y} \frac{y}{\sqrt{y^2-1}} \arctan\left(\frac{\sqrt{y^2-1}}{y} t\right)\Big|_0^{\infty}$$

$$= \frac{2}{\sqrt{y^2-1}} \left(\frac{\pi}{2} - 0\right) = \frac{\pi}{\sqrt{y^2-1}}$$

$$F'(y) = \frac{\pi}{\sqrt{y^2-1}} \qquad y>1$$

$$F(y) = \int \frac{\pi}{\sqrt{y^2-1}}\, dy = \pi \ln\left(y + \sqrt{y^2-1}\right) + C$$

$$F(y) = \pi \ln\left(y + \sqrt{y^2-1}\right) + C \qquad (y>1) \quad (\text{zvezna za } y \geq 1)$$

$$\int_0^{\pi/2} \ln(y^2 - \sin^2\varphi)\, d\varphi \qquad (\text{zvezna za } y \geq 1)$$

Pri $y=1$ $\quad F(1) = -\pi \ln 2$

$$\pi \cdot 0 + C = -\pi \ln 2$$

$$C = -\pi \ln 2$$

$$F(y) = \int_0^{\pi/2} \ln(y^2 - \sin^2\varphi)\, d\varphi = \pi \ln\left(\frac{y + \sqrt{y^2-1}}{2}\right)$$

8. Dan je integral s parametrom

$$F(y) = \int_0^\infty \frac{\arctan(xy)}{x(1+x^2)}\, dx$$

a) Za katere $y \in \mathbb{R}$ $F(y)$ konvergira
b) Izračunaj $F(y)$

- $y=0$ $\quad F(0) = \int_0^\infty 0\, dx = 0$ ✓ konvergira

$F(y)$ je liha funkcija torej lahko gledamo le $y>0$

- $y>0$

Konv. pri $x=0$

$$\int_0^1 \frac{\arctan xy}{x(1+x^2)}\, dx = \int_0^1 \frac{\frac{\arctan(xy)}{\sqrt{x}(1+x^2)}}{\sqrt{x}}\, dx$$

Omejenost pri $x=0$? ✓ $\Rightarrow$ int. konv.

$$\lim_{x\to 0} \frac{\arctan xy}{\sqrt{x}(1+x^2)} = \lim_{x\to 0} \frac{1}{1+x^2} \lim_{x\to 0} \frac{\arctan xy}{\sqrt{x}} \overset{L'Hop}{=} \lim_{x\to 0} \frac{\frac{1}{1+x^2y^2} y}{\frac{1}{2}x^{-\frac{1}{2}}} = \lim_{x\to 0} \frac{2y\sqrt{x}}{(1+y^2x^2)} = 0$$

Konv. pri $x=\infty$

$$(y>0) \quad \int_1^\infty \frac{\arctan xy}{x(1+x^2)} \leq \int_1^\infty \frac{\pi/2}{1+x^2} = \frac{\pi}{2} \arctan x \Big|_1^\infty = \frac{\pi}{2}\left(\frac{\pi}{2} - 1\right) \Rightarrow \text{konvergira} ✓$$

Ker je F liha konvergira tudi za $y<0$

F konv. za $y \in \mathbb{R}$

$$F'(y) \overset{?}{=} \int_0^\infty \frac{\frac{1}{1+x^2y^2}\cdot x}{x(1+x^2)}\, dx = \int_0^\infty \frac{dx}{(1+x^2)(1+x^2y^2)}$$

enak. konv. ?

Weierstrassov kriterij

$$\left| \frac{1}{(1+y^2x^2)(1+x^2)} \right| \leq W(x) \qquad y \in \mathbb{R}$$

$\int$ mora konvergirati

$$\frac{1}{(1+y^2x^2)(1+x^2)} \leq \frac{1}{1+x^2} \Rightarrow \int_0^\infty \frac{1}{1+x^2}\, dx = \frac{\pi}{2} \text{ konvergira} \Rightarrow F'(y) \text{ enakomerno konvergira na } y \in \mathbb{R}$$ ✓

$(1+y^2x^2) \geq 1$, $\quad y \in \mathbb{R}$

Izračunajmo $y>0$ nastavek za rac. funk.

$$\int \frac{dx}{(1+x^2)(1+x^2y^2)} = A \arctan(xy) + B \arctan x + C$$

odvajamo

$$\frac{1}{(1+x^2)(1+x^2y^2)} = \frac{Ay}{1+x^2y^2} + \frac{B}{1+x^2} = \frac{Ay(1+x^2) + B(1+x^2y^2)}{(1+x^2y^2)(1+x^2)} \Rightarrow$$

$$1 = Ay + Ax^2y + B + Bx^2y^2$$

$$1 = Ay + B \qquad 0 = Ay + By^2 \Rightarrow \quad A = \frac{y}{y^2-1}, \quad B = -\frac{1}{y^2-1}$$

$$\int \frac{dx}{(1+x^2)(1+x^2y^2)} = \frac{y}{y^2-1}\arctan(xy) - \frac{1}{y^2-1}\arctan x \Big|_0^\infty =$$

$(1 \neq y > 0)$ $\quad F'(y) = \frac{y}{y^2-1}\frac{\pi}{2} - \frac{1}{y^2-1}\frac{\pi}{2} = \underline{\frac{\pi}{2}\frac{1}{y+1}}$

$$F(y) = \int \frac{\pi}{2}\frac{1}{y+1}\,dy = \frac{\pi}{2}\ln|y+1| + C \qquad y>0$$

$$F(0)=0 = \frac{\pi}{2}\ln 1 + C \Rightarrow C=0$$

$$F(y) = \frac{\pi}{2}\ln(y+1) \qquad y \geq 0$$

Kaj pa $y<0$?
Že od začetka vemo, da je F liha funkcija $(F(-y) = -F(y))$

$$\underline{F(y) = \operatorname{sgn}(y)\frac{\pi}{2}\ln(|y|+1)} \qquad y \in \mathbb{R}$$

$$\operatorname{sgn}(y) = \begin{cases} 1; & y>0 \\ -1; & y<0 \end{cases}$$

Opomba Tipične naloge

(če so poli ali integrirani) drugače za zveznost

Enak. konv.

$$F(y) = \int_a^b f(x,y)\,dx \longrightarrow F'(y) = \int_a^b \frac{\partial f}{\partial y}(x,y)\,dx$$

$$\downarrow$$

$$F(y) = \int F'(y)\,dy + C$$

$$\downarrow$$

določimo konstanto C

pri določenem y da določimo C

9. Naj bo $y(x) = \int_0^\infty \frac{e^{-xz}}{1+z^2}\,dz$. Za $x>0$ izračunaj $y''+y$.

$x>0$ Ali $\int_0^\infty \frac{e^{-xz}}{1+z^2}\,dz$ konv?

$$\int_0^\infty \frac{\overbrace{e^{-xz}}^{\leq 1}}{\underbrace{1+z^2}_{>0}}\,dz \leq \int_0^\infty \frac{1}{1+z^2} = \arctan z \Big|_0^\infty = \frac{\pi}{2} \quad \text{konvergira}$$

$$y'(x) \overset{?}{=} \int_0^\infty \frac{e^{-xz}(-z)}{1+z^2}\,dz = -\int_0^\infty \frac{z e^{-xz}}{1+z^2}\,dz$$

Preverimo enakomerno konv.

$x \in [c,d] \subset (0,\infty)$

$$\left|\frac{z e^{-xz}}{1+z^2}\right| = \frac{z e^{-xz}}{1+z^2} \leq w(z) \qquad x \in [c,d]$$

$$\leq \frac{z e^{-cz}}{\underbrace{1+z^2}_{\geq 1}} \leq z e^{-cz} = w(z)$$

Weierstrassov kriterij

$$\int_0^\infty w(z) = \int_0^\infty z e^{-cz}\, dz \overset{\text{per partes}}{=} -\frac{z}{c} e^{-cz}\Big|_0^\infty + \frac{1}{c}\int_0^\infty e^{-cz}\, dz = -\frac{1}{c^2} e^{-cz}\Big|_0^\infty = \frac{1}{c^2}$$

$u = z \quad dv = e^{-cz} dz$

$du = dz \quad v = -\frac{1}{c} e^{-cz}$

konv.

$\Rightarrow$ Enakomerna konv. velja na poljubnem $x \in [c,d] \subset (0,\infty)$

$$y' \overset{?}{=} -\int_0^\infty \frac{z e^{-xz}(-z)}{1+z^2}\, dz = \int_0^\infty \frac{z^2 e^{-xz}}{1+z^2}\, dz$$

Preverimo enakomerno konvergenco (Weierstrassov kriterij)

$x \in [c,d] \subset (0,\infty)$

$$\left|\frac{z^2 e^{-xz}}{1+z^2}\right| = \frac{z^2 e^{-xz}}{1+z^2} \leq \frac{z^2 e^{-cz}}{1+z^2} \leq e^{-cz} = w(z) \qquad \forall x \in [c,d]$$

$\frac{z^2}{1+z^2} \leq 1$

$$\int_0^\infty e^{-cz} dz = -\frac{1}{c} e^{-cz}\Big|_0^\infty = \frac{1}{c} \quad \text{konv.}$$

$\Rightarrow$ Enakomerna konv. velja za $x \in [c,d] \subset (0,\infty)$

$$y + y'' = \int_0^\infty \frac{e^{-xz}}{1+z^2}\, dz + \int_0^\infty \frac{z^2 e^{-xz}}{1+z^2}\, dz = \int_0^\infty e^{-xz}\, dz = \frac{1}{x}$$

10 Naj bosta $a, b > 0$. Izračunaj integral

$$\int_0^1 \frac{x^b - x^a}{\ln x}\, dx$$

Zgleda kot da sta 2 parametra

Možen pristop $F(a,b) = \int_0^1 \frac{x^b - x^a}{\ln x} dx \to \frac{\partial F}{\partial a}, \frac{\partial F}{\partial b} \dots$?

Rešimo na drugačen način.

$$\int_0^1 \frac{x^b - x^a}{\ln x}\, dx \overset{!}{=} \int_0^1 \left(\frac{x^y}{\ln x}\Big|_a^b \right) dx = \int_0^1 \left(\int_a^b ?\, dy \right) dx = \qquad \frac{\partial}{\partial y}\frac{x^y}{\ln x} = \frac{x^y \ln x}{\ln x} = x^y$$

$$= \int_0^1\int_a^b x^y\, dy\, dx \overset{?}{=} \int_a^b \int_0^1 x^y\, dx\, dy = \int_a^b \frac{1}{y+1} x^{y+1}\Big|_0^1 dy = \int_a^b \frac{1}{y+1}\, dy = \ln\left(\frac{b+1}{a+1}\right)$$

zamenjava integralov

Spomnimo se zapri pravokotnik

- $f: [a,b] \times [c,d] \to \mathbb{R}$ zvezna
- $\int_c^d \int_a^b f = \int_a^b \int_c^d f$

V našem primeru imamo $a, b > 0$

$f(x,y) = x^y \qquad f: [0,1] \times [a,b] \to \mathbb{R}$ zvezna ?

$= e^{y \ln x} \to f: (0,1] \times [a,b] \to \mathbb{R}$ zvezna ✓

$y \in [a,b]$

$$\lim_{x \to 0} e^{y \ln x} = 0 = \quad \tilde{f}(x,y) = \begin{cases} 0 & ; x = 0 \\ e^{y \ln x} & ; x > 0 \end{cases}$$

Ali je $\tilde{f}$ zvezna na $[0,1]\times[a,b]$?

$\tilde{f}$ je zvezna če $\forall (x_0, y_0)$ velja

$$\lim_{(x,y)\to(x_0,y_0)} \tilde{f}(x,y) = \tilde{f}(x_0,y_0)$$

$x>0$ $\quad \tilde{f}(x,y) = e^{y\ln x}$

$\downarrow$

el. funkcija

$\downarrow$

je zvezna

Zanima nas samo še, ali je $\tilde{f}$ zvezna v točkah oblike $(0, y_0)$ $\quad y_0 \in [a,b]$

$$\lim_{(x,y)\to(0,y_0)} \tilde{f}(x,y) = 0$$

$$\quad\quad\quad\quad\quad\quad \overset{\shortparallel}{\tilde{f}(0,y_0)} \quad ✓$$

$$0 \le e^{y\ln x} \le e^{\overset{\nearrow y_0}{(y_0+\frac{1}{n})} \overset{\nearrow -\infty}{-\frac{1}{n}}} \to 0$$

$$n\to\infty$$

⑪ Dan je integral

$$\int_0^1 \frac{x^b - x^a}{\ln x}\,dx \quad , \text{ kjer sta } a,b>-1 \text{ . Izračunaj integral .}$$

Opomba: za $a,b>0$ smo izračunali v prejšnji nalogi

Poglejmo najprej, da integral konvergira tudi za $a,b>-1$

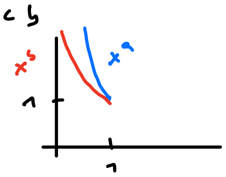
(i) Konvergenca pri $x=1$

poljubna

$$\int_{1/2}^1 \frac{x^b-x^a}{\ln x}\,dx =$$

$a, b \in (-1,0)$
$a<b$

$$\lim_{x\to1}\frac{x^b-x^a}{\ln x} \overset{\text{L'Hop}}{=} \lim_{x\to1}\frac{bx^{b-1}-ax^{a-1}}{\frac{1}{x}} = \lim_{x\to1} bx^b - ax^a = b-a$$



$\frac{x^b-x^a}{\ln x}$ je omejena pri $x=1$ $\to$ konvergira pri $x=1$

(ii) konvergenca pri $x=0$

$$\int_0^{1/2}\frac{x^b-x^a}{\ln x}\,dx = \int_0^{1/2}\frac{x^b}{\ln x}\,dx - \int_0^{1/2}\frac{x^a}{\ln x}\,dx$$

$$\int_0^{1/2}\frac{x^\alpha}{\ln x}\,dx \quad \text{Ali konvergira}$$

$\alpha>-1$

• $\alpha>0$: $\displaystyle\lim_{x\to0}\frac{x^\alpha}{\ln x} \overset{\text{L'Hop}}{=} \lim_{x\to0}\frac{\alpha x^{\alpha-1}}{1/x}$

$$=\lim_{x\to0}\alpha x^\alpha = 0 \quad ✓$$

• $\alpha=0$ $\displaystyle\lim_{x\to0}\frac{1}{\ln x} = 0$ ✓

$\alpha \in (-1, 0)$?

$\beta = -\alpha \Rightarrow \beta \in (0,1)$ omejena ?

$$\int_0^{1/2} \frac{dx}{x^\beta \ln x} = \int_0^{1/2} \frac{1/\ln x}{x^\beta}\, dx \qquad \lim_{x \to 0} \frac{1}{\ln x} = 0 \ ✓$$

$\beta < 1$

Rešimo do konca enako kot prej

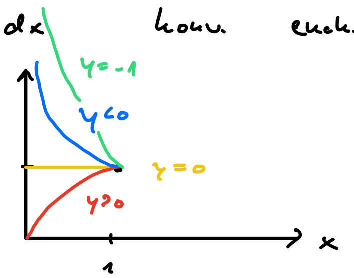
$$\int_0^1 \frac{x^b - x^a}{\ln x} = \int_0^1 \int_a^b x^y \, dy\, dx \overset{?}{=} \int_a^b \underbrace{\int_0^1 x^y \, dx}_{\text{posplošen integral za negativne } y}\, dy = \ln\left(\frac{b+1}{a+1}\right)$$

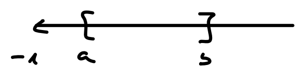
enakomerna konvergenca

$$\left(\int_c^d \underbrace{\int_a^\infty f(x,y)\, dx}_{\text{posplošen integral}}\, dy = \int_a^\infty \int_c^d f(x,y)\, dy\, dx \quad \text{lahko uredim če } \int_a^\infty f \text{ konv. enak. na } [c,d] \right)$$

Ali $\int_0^1 x^y\, dx$ konv. enak. na $y \in [a,b]$?

$a, b > -1$





$$|x^y| = x^y \le x^a \qquad y \in [a,b]$$

$\int_0^1 x^a\, dx$ ... konvergira saj $a > -1$

Torej po Weierstrassovem kriteriju

$$\int_0^1 x^y\, dx \le \int_0^1 x^a\, dx \Rightarrow \text{konv. enakomerno} \ ✓$$

Opomba: Fubini-Tonellijev izrek $\Rightarrow$ zamenjava vrstnega reda integracije tipično sploh ni problematična

$f(x,y)$ $\qquad \int_a^b \int_c^d |f(x,y)|\, dy\, dx$ obstaja

ali

$$\int_c^d \int_a^b |f(x,y)|\, dx\, dy \text{ obstaja}$$

$$\Downarrow$$

$$\int_a^b \int_c^d f = \int_c^d \int_a^b f$$

(12) Izračunaj (i) $\int_0^1 \int_0^1 \frac{y^2-x^2}{(x^2+y^2)^2}\,dx\,dy$

(ii) $\int_0^1 \int_0^1 \frac{y^2-x^2}{(x^2+y^2)^2}\,dy\,dx$

(i) $$\int_0^1 \int_0^1 \frac{y^2-x^2}{(x^2+y^2)^2}\,dx\,dy \overset{DN}{=} \int_0^1 \left(\frac{x}{x^2+y^2} \right)\Big|_0^1 dy = \int_0^1 \frac{dy}{1+y^2} = \arctan y \Big|_0^1 = \frac{\pi}{4}$$

(ii) $$\int_0^1 \int_0^1 \frac{y^2-x^2}{(x^2+y^2)^2}\,dy\,dx = \int_0^1 \frac{-y}{x^2+y^2}\Big|_0^1 = -\frac{\pi}{4}$$

V tem primeru je vrstni red pomemben. Razlog $f(x,y) = \frac{y^2-x^2}{(x^2+y^2)^2}$ ni zvezna.

Gama in beta funkciji

Gama $$\Gamma(s) = \int_0^\infty x^{s-1} e^{-x}\,dx \qquad s>0$$

$$\Gamma(1) = \Gamma(2) = 1 \qquad \Gamma(s+1) = s\,\Gamma(s)$$

$$\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi} \qquad \Gamma(n) = (n-1)!$$

$$\Gamma(s)\,\Gamma(1-s) = \frac{\pi}{\sin(\pi a)}$$

$$a \in (0,1)$$

Beta $$B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}\,dx \qquad a,b>0$$

$$B(a,b) = B(b,a)$$

$$B(a,b) = \frac{\Gamma(a)\,\Gamma(b)}{\Gamma(a+b)}$$

Alternativni obliki

$$B(a,b) = \int_0^\infty \frac{x^{a-1}}{(1+x)^{a+b}}\,dx$$

$$B(a,b) = 2\int_0^{\pi/2} \cos^{2a-1}x\,\sin^{2b-1}x\,dx$$

(13) Izračunaj s pomočjo Γ, B

$$\int_{-1}^1 \sqrt{1-x^2}\,dx = 2\int_0^1 \sqrt{1-x^2}\,dx = \int_0^1 \sqrt{1-t}\,\frac{1}{\sqrt{t}}\,dt = B\left(\tfrac{1}{2}, \tfrac{3}{2}\right) = \frac{\Gamma(\frac{1}{2})\,\Gamma(\frac{3}{2})}{\Gamma(2)}$$

$t = x^2$

$dt = 2x\,dx$

$dx = \frac{1}{2}\frac{1}{\sqrt{t}}\,dt$

$\Rightarrow a = \frac{1}{2} \quad b = \frac{3}{2}$

$a-1 = -\frac{1}{2} \quad b-1 = \frac{1}{2}$

$$= \frac{\sqrt{\pi}\ \frac{1}{2}\Gamma(\frac{1}{2})}{1} = \frac{\pi}{2}$$

Kako bi nalogo rešili direktno

$$\int_{-1}^{1}\sqrt{1-x^2}\,dx = 2\int_0^1\sqrt{1-x^2}\,dx = 2\int_0^{\pi/2}\cos^2 t\,dt = \frac{\pi}{2}$$

$x = \sin t$

$dx = \cos t\,dt$

14 Izračunaj

$$\int_0^\infty \frac{x^5}{1+x^{12}}\,dx = \frac{1}{12}\int_0^\infty \frac{t^{5/12}}{1+t}\,t^{-\frac{11}{12}}\,dt = \frac{1}{12}\int_0^\infty \frac{t^{-\frac{1}{2}}}{1+t}\,dt = \frac{1}{12}B\left(\frac{1}{2},\frac{1}{2}\right)$$

$t = x^{12}$

$dt = 12x^{11}\,dx$

$dx = \frac{1}{12}t^{-\frac{11}{12}}\,dt$

$-\frac{1}{2} = a-1 \Rightarrow a = \frac{1}{2}$

$1 = a+b \Rightarrow b = \frac{1}{2}$

$$= \frac{1}{12}\,\frac{\Gamma\left(\frac{1}{2}\right)^2}{\Gamma(1)} = \frac{\pi}{12}$$

Alt. rešitev

$t = x^6$

$\frac{1}{6}dt = x^5\,dx$

$$\frac{1}{6}\int_0^\infty \frac{dt}{1+t^2} = \frac{1}{6}\arctan t\Big|_0^\infty = \frac{\pi}{12}$$

Dvojni in trojni integral

Dvojni integral

$f: D \subset \mathbb{R}^2 \to \mathbb{R}$ (zvezna)

Kako izračunati

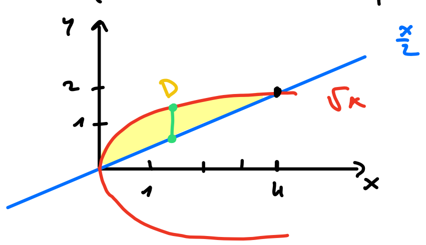
(i) $f: [a,b]\times[c,d] \to \mathbb{R}$ zvezna

$$\iint_{[a,b]\times[c,d]} f = \int_c^d\int_a^b f = \int_a^b\int_c^d f$$

(ii) $f: \underbrace{[a,b]\times[\varphi(x),\psi(x)]}_{D} \to \mathbb{R}$

$$\iint_D f = \int_a^b\int_{\varphi(x)}^{\psi(x)} f\,dy\,dx$$

1 Izračunaj $\iint_D (x^2+y)\,dx\,dy$ kjer je D območje, ki ga omejujeta krivulji $y = \frac{x}{2}$ in $x = y^2$



$\sqrt{x} = \frac{x}{2}$

$4x = x^2$

$x = 4$

$$\int_0^4\int_{\frac{x}{2}}^{\sqrt{x}} x^2+y\,dy\,dx = \int_0^4 x^2y+\frac{y^2}{2}\Big|_{\frac{x}{2}}^{\sqrt{x}}\,dx = \int_0^4 x^{\frac{5}{2}}+\frac{x}{2}-\frac{x^3}{2}-\frac{x^2}{8}\,dx$$

$$= \frac{2}{7}x^{\frac{7}{2}}+\frac{x^2}{4}-\frac{x^4}{8}-\frac{x^3}{24}\Big|_0^4 =$$

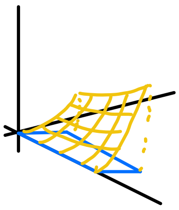
$$= \frac{2}{7}2^7+4-\frac{16\cdot 16}{8}-\frac{64}{24} = \dots = \frac{124}{21}$$

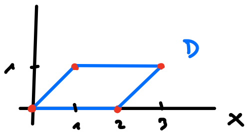
Izračunajmo z drugim vrstnim redom

$$\int_0^2\int_{y^2}^{2y} x^2+y\,dx\,dy = \int_0^2 \frac{x^3}{3}+yx\Big|_{y^2}^{2y} dx = \int_0^2 \frac{8y^3}{3}+2y^2-\frac{y^6}{3}-y^3\,dy$$

$$=\int_0^2 \frac{5y^3}{3}+2y^2-\frac{y^6}{3}\,dy = \dots = \frac{124}{21}$$

② Izračunaj $\iint_D xy\,dx\,dy$, kjer je D štirikotnik z oglišči $(0,0),(2,0),(3,1),(1,1)$





$x \in [0,3]$

$$\iint_D xy\,dx\,dy =$$

$$= \int_0^1\int_0^x xy\,dy\,dx + \int_1^2\int_0^1 xy\,dy\,dx + \int_2^3\int_{x-2}^1 xy\,dy\,dx$$

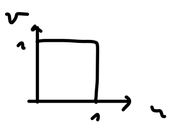
$$= \int_0^1 x\frac{y^2}{2}\Big|_0^x dx + \int_1^2 x\frac{y^2}{2}\Big|_0^1 dx + \int_2^3 x\frac{y^2}{2}\Big|_{x-2}^1 dx =$$

$$= \int_0^1 \frac{x^3}{2}dx + \int_1^2 \frac{x}{2}\,dx + \int_2^3 \frac{\overset{\text{Napaka}}{-x^3+4x^2-x}}{2}\,dx =$$

$$= \frac{x^4}{8}\Big|_0^1 + \frac{x^2}{4}\Big|_1^2 + \frac12\left(-\frac{x^4}{4}\overset{\text{Napaka}}{+}\frac43x^3-\frac{x^2}{2}\right)\Big|_2^3 = \dots = \frac53$$

Če gremo v obratnem vrstnem redu ne rabimo razpisati na več integralov saj sta meji $x=y+2$ in $x=y$ za $y\in[0,1]$

Lahko naredimo tudi z lin. preslikavo oz. novo spremenljivko



$\longrightarrow D$

$$\phi(u,v)=\begin{bmatrix}2&1\\0&1\end{bmatrix}\begin{bmatrix}u\\v\end{bmatrix}=(2u+v,\ v)$$

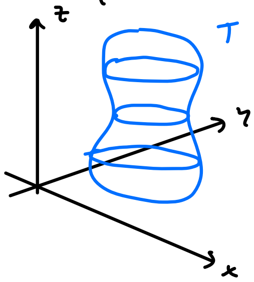
$$J\phi = \begin{bmatrix}2&1\\0&1\end{bmatrix} \to \det = 2$$

$$\iint_D xy\,dx\,dy = \iint_{[0,1]^2}(2u+v)v\;2\,du\,dv =$$

$$=\int_0^1\int_0^1 4uv+2v^2\,du\,dv = \dots = \frac53$$

Trojni integral

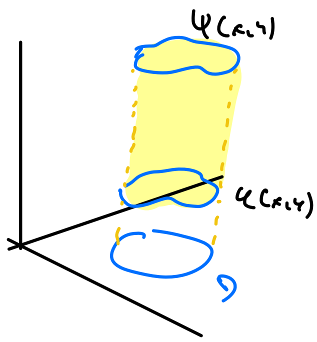
$f: T\to\mathbb{R}$



$$\iiint_T f(x,y,z)\,dx\,dy\,dz$$

‖ def.

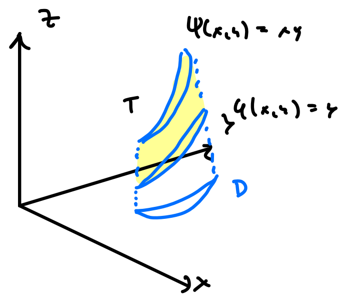
$$\lim_{\text{manjšam delitev}} \sum_{i=1}^n f(x_i,y_i,z_i)\,\Delta V_i$$

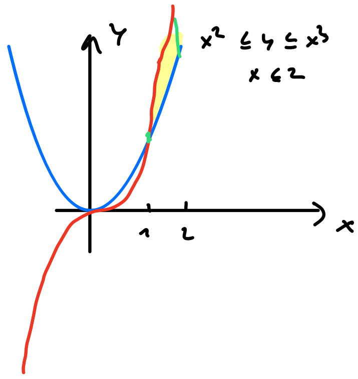


$$\iint_D \int_{\varphi(x,y)}^{\psi(x,y)} f\,dz\,dx\,dy$$

Če je f zvezna lahko menjamo vrstni red integracije

③ $\iiint_T \frac{1}{z}\,dx\,dy\,dz$, kjer je $T=\{(x,y,z)\in\mathbb{R}^3;\ x\le 2,\ x^2\le y\le x^3,\ y\le z\le xy\}$





$$\iiint_T \frac{1}{z}\,dx\,dy\,dz = \iint_D dx\,dy\int_y^{xy}\frac{1}{z}\,dz = \iint_D \ln xy-\ln y\,dx\,dy = \iint_D \ln x\,dx\,dy =$$

$$= \int_1^2\int_{x^2}^{x^3}\ln x\,dy\,dx = \int_1^2 \ln x\; y\Big|_{x^2}^{x^3}dx = \int_1^2 \ln x\,(x^3-x^2)\,dx \overset{\text{per partes}}{=}$$

$u=\ln x \qquad dv=(x^3-x^2)\,dx$

$du=\frac{dx}{x} \qquad v=\frac{x^4}{4}-\frac{x^3}{3}$

$$= \left(\frac{x^4}{4}-\frac{x^3}{3}\right)\ln x\Big|_1^2 - \int_1^2 \frac{x^3}{4}-\frac{x^2}{3}\,dx =$$

$$= \left(4-\frac{8}{3}\right)\ln 2 - \left(\frac{x^4}{16}-\frac{x^3}{9}\right)\Big|_1^2 = \frac{16}{3}\ln 2-\frac{23}{144} \doteq$$

④ Določi volumen telesa, ki ga določajo neenačbe

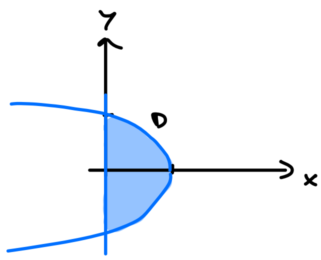
$x\ge 0$, $y^2\le z\le 1-x$.

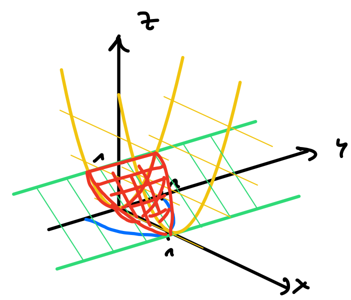
$\iiint_T dV = V$

$y^2\le 1-x$

$x\le 1-y^2$

$(x=1-y^2$





$$\iiint_T dx\,dy\,dz = \iint_D dx\,dy\int_{y^2}^{1-x}dz = \iint_D 1-x-y^2\,dx\,dy =$$

$$= \int_{-1}^1\int_0^{1-y^2} 1-x-y^2\,dx\,dy = \int_{-1}^1 \left(x-\frac{x^2}{2}-xy^2\right)\Big|_0^{1-y^2}dy =$$

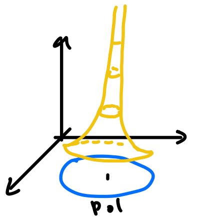
$$= \int_{-1}^1 1-y^2-\frac{(1-y^2)^2}{2}-(1-y^2)y^2\,dy = 2\int_0^1 \frac{y^4}{2}-y^2+\frac{1}{2}\,dy = \ldots = \frac{8}{15}$$

Posplošeni integrali

$f: D\to\mathbb{R}$ zvezna funkcija

↳ območje D neomejeno

↳ f neomejena



$$\iint_D f(x,y)\,dx\,dy = \lim_{i\to\infty}\iint_{D_i} f(x,y)\,dx\,dy$$

$f:\mathbb{R}^2\to\mathbb{R}$ denimo da obstaja

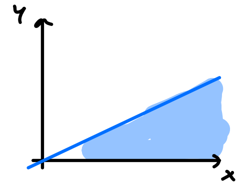
$$\iint_{\mathbb{R}^2}|f| \quad \text{ali} \quad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|f|\,dx\,dy \quad \text{ali} \quad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|f|\,dy\,dx$$

tedaj $$\iint_{\mathbb{R}^2} f = \iint_{-\infty}^{\infty} f\,dx\,dy = \iint_{-\infty}^{\infty} f\,dy\,dx$$

Fubinijev izrek

⑤ Izračunaj

$$\iint_D e^{-x}\,dx\,dy\ , \quad D=\{(x,y)\in\mathbb{R}^2;\ 0\le 2y\le x\}$$



neomejeno območje

• $\int_0^\infty\int_0^{x/2} e^{-x}\,dy\,dx = \int_0^\infty e^{-x}\,\frac{x}{2}\,dx \overset{\text{per part}}{=}$

$u=\frac{x}{2}$ $\quad e^{-x}dx=dv$

$du=\frac12 dx$ $\quad -e^{-x}=v$

$$= -\frac12 x e^{-x}\Big|_0^\infty + \int_0^\infty \frac12 e^{-x}\,dx = \frac12$$

• $\int_0^\infty dy\int_{2y}^\infty e^{-x}\,dx = \int_0^\infty -e^{-x}\Big|_{2y}^\infty dx = \int_0^\infty e^{-2y}\,dy = -\frac12 e^{-2y}\Big|_0^\infty = \frac12$

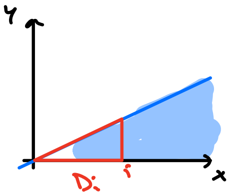
Po definiciji:

(ker e^{-x} pozitivna $\Rightarrow$ posplošeni int. lahko izračunamo s pomočjo izbire D_i

$D_1\subseteq D_2\subseteq D_3\subseteq\ldots\subseteq D$

$\bigcup_{i=1}^{\infty} D_i = D$

$$\iint_{D_i} e^{-x}\,dx\,dy = \int_0^i dx\int_0^{x/2} e^{-x}\,dy\,dx =$$

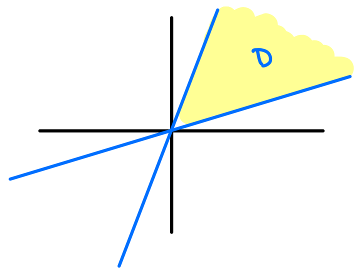


$$= \int_0^i \frac{x}{2}e^{-x}\,dx = -\frac{i}{2}e^{-i} - \frac12 e^{-i} + \frac12$$

$$\lim_{i\to\infty}\frac12\left(-ie^{-i}-e^{-i}+1\right)=\frac12$$

⑥ Izračunaj

$$\iint_D \frac{1}{xy}\,dx\,dy\ ; \quad D=\{(x,y)\in\mathbb{R}^2;\ y\le 2x \text{ in } x\le 2y\}$$

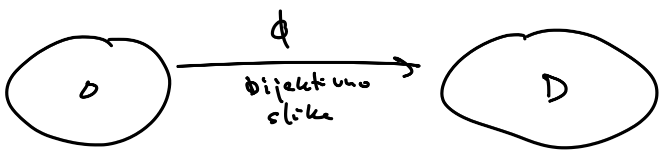


$f(x,y)>0$ na D

$$\iint_D \frac{1}{xy}\,dx\,dy = \int_0^\infty\int_{x/2}^{2x}\frac{1}{xy}\,dy\,dx =$$

$$=\int_0^\infty \frac1x\left(\ln y\Big|_{x/2}^{2x}\right)dx = \int_0^\infty \frac1x \ln\frac{2x\cdot 2}{x}\,dx = \int_0^\infty\frac{\ln 4}{x}\,dx = \ln 4\int_0^\infty\frac1x\,dx \ \ldots \text{divergira}$$

• Uvedba novih spremenljivk



$$J\phi = \begin{bmatrix} x_u & y_u\\ x_v & y_v\end{bmatrix}$$

$\phi: \Delta \to D$

$$\phi(u,v) = (x(u,v),\ y(u,v))$$

$$\iint_\Delta f(x(u,v),y(u,v))\,|\det J\phi|\,du\,dv = \iint_D f(x,y)\,dx\,dy$$

Pogosta zamenjava spremenljivk

- Polarne koordinate (r, φ)

$x = r\cos\varphi$

$y = r\sin\varphi$

$J\phi = r$

9. $$\iint_D \frac{xy}{(1+(x^2+y^2)^2)^2}\,dx\,dy \qquad D = \{(x,y)\in\mathbb{R}^2;\ x,y\geq 0,\ x^2+y^2\leq 4\}$$

$\varphi \in [0, \frac{\pi}{2}]$ $\qquad x = r\cos\varphi$

$r \in [0, 2]$ $\qquad y = r\sin\varphi$

$$\int_0^{\frac{\pi}{2}}\int_0^2 \frac{r^2\sin 2\varphi}{2(1+r^4)^2}\, r\, dr\, d\varphi =$$

$$\frac{1}{2}\int_0^{\pi/2}\sin 2\varphi \int_0^2 \frac{r^3}{(1+r^4)^2}\,dr = \frac{1}{2}\cdot\frac{1}{4}\int_0^{\pi/2}\sin 2\varphi\int_1^{17}\frac{dt}{t^2} = \frac{1}{8}\int_0^{\pi/2}\sin 2\varphi\,\left(-t^{-1}\right)\Big|_1^{17} d\varphi =$$

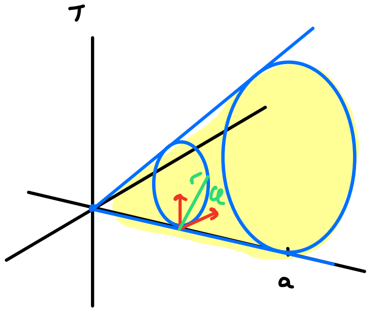
$t = 1 + r^4$

$dt = r^3 dr$

$$= \frac{1}{8}\cdot\frac{16}{17}\left(-\frac{1}{2}\cos 2\varphi\right)\Big|_0^{\pi/2} = \frac{2}{17}$$

10. Naj bo $a>0$

$$\iiint_T x\,dx\,dy\,dz \qquad T = \{(x,y,z)\in\mathbb{R}^3;\ x\in[0,a],\ y^2+z^2\leq 2xz\}$$



$y^2 + z^2 - 2xz \leq 0$

$y^2 + (z-x)^2 - x^2 \leq 0$

$y^2 + (z-x)^2 \leq x^2$

1. Uvedemo nove spremenljivke (x, r, φ)

$x = x$

$y = r\cos\varphi$

$z = r\sin\varphi$

$$J\phi = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -r\sin\varphi \\ 0 & \sin\varphi & r\cos\varphi \end{bmatrix} \Rightarrow |\det J\phi| = r$$

Kako teče spr.

$x \in [0, a]$ $\qquad \varphi \in [0, \pi]$ $\qquad r \in [0, 2x\sin\varphi]$

$y^2 + z^2 \leq 2xz$

$r^2\cos^2\varphi + r^2\sin^2\varphi \leq 2x\, r\sin\varphi$

$r^{\cancel{2}} \leq 2x\,\cancel{r}\sin\varphi$ $\qquad r \leq 2x\sin\varphi$

$$\iiint x\,dx\,dy\,dz = \int_0^a \int_0^\pi \int_0^{2x\sin\varphi} x\,r\,dr\,d\varphi\,dx =$$

$$= \int_0^a \int_0^\pi x \frac{r^2}{2}\Big|_0^{2x\sin\varphi} d\varphi\,dx =$$

$$= \int_0^a \int_0^\pi 2x^3 \sin^2\varphi\,d\varphi\,dx =$$

$$= \int_0^a \cancel{2} x^3\,dx \int_0^\pi \frac{1-\cos 2\varphi}{\cancel{2}}\,d\varphi = \frac{x^4}{4}\Big|_0^a \left(\varphi - \frac{1}{2}\sin 2\varphi\right)\Big|_0^\pi$$

$$= \frac{a^4}{4}\pi = \frac{\pi a^4}{4}$$

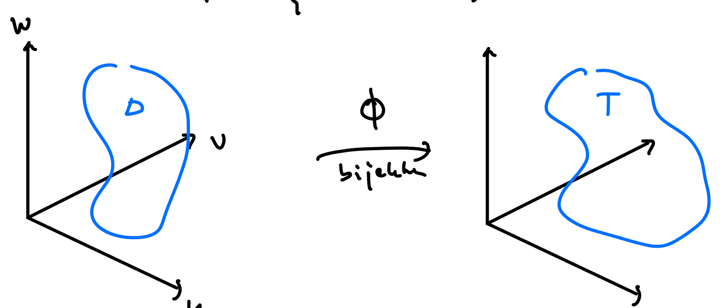
Alt. način

$x = x$ $\qquad x \in [0, a]$

$y = r\cos\varphi$ $\qquad \varphi \in [0, 2\pi)$

$z = r\sin\varphi + x$ $\qquad r \in [0, x]$

Uvedba novih spremenljivk v $\iiint$



$f: T \to \mathbb{R}$

$\phi: u, v, w \to x, y, z$

$$\iiint_D f(x(u,v,w), y(u,v,w), z(u,v,w))\,d$$

② Nove spremenljivke x, r, φ

$x = x$ $\qquad x \in [0, a)$

$y = r\cos\varphi$ $\qquad \varphi \in [0, 2\pi)$

$z = x + r\sin\varphi$ $\qquad r \in [0, x]$

$$\det J\phi = \left|\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -r\sin\varphi \\ 1 & \sin\varphi & r\cos\varphi \end{bmatrix}\right| = r$$

$$\iiint_T x\,dx\,dy\,dz = \int_0^a \int_0^{2\pi} \int_0^x x\,r\,dr\,d\varphi\,dx = \int_0^a \int_0^{2\pi} \left(\frac{x r^2}{2}\right)\Big|_0^x d\varphi\,dx = \int_0^a \int_0^{2\pi} \frac{x^3}{2}\,d\varphi\,dx =$$

$$= \int_0^a \pi x^3\,dx = \pi \frac{x^4}{4}\Big|_0^a = \frac{\pi a^4}{4}$$

③ Pogoj iz začetka

$$y^2 + (z-x)^2 = x^2$$

$$z = x \pm \sqrt{x^2 - y^2}$$

Če slečem T v tloris je to trikotnik

⋮

Posošti zamenjave spremenljivk v 3D

Cilindrične koordinate r, φ, z

$x = r\cos\varphi$

$y = r\sin\varphi$

$z = z$

$\det J\Phi = r$

Sferične koordinate r, φ, θ

$x = r\sin\theta\cos\varphi$

$y = r\sin\theta\sin\varphi$

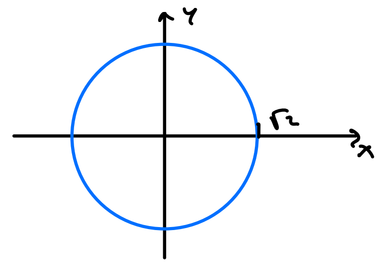
$z = r\cos\theta$

$\det J\Phi = r^2\sin\theta$

12 Izračunaj

$$\iint_D \sqrt{(x+1)^2 + (y+1)^2}\, dx\, dy$$

$$D = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \le 2\} \quad \text{krog}$$



- Polarne koordinate

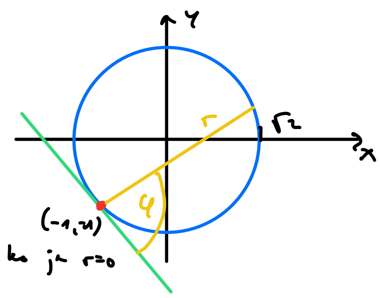
$x = r\cos\varphi \qquad y = r\sin\varphi$

$$\int_0^{2\pi}\int_0^{\sqrt{2}} \sqrt{(r\cos\varphi+1)^2 + (r\sin\varphi+1)^2}\, dr\, d\varphi =$$

$$= \int_0^{2\pi}\int_0^{\sqrt{2}} \sqrt{r^2 + 1 + 2r^2\sin\varphi\cos\varphi}\, dr\, d\varphi \qquad \text{težko izračunljivo}$$

- Varianta polarnih koordinat kjer se znebimo 1.

$x = r\cos\varphi - 1 \qquad y = r\sin\varphi - 1 \qquad \det J\varphi = r$



$\varphi \in\ ?$

$r \in\ ?$

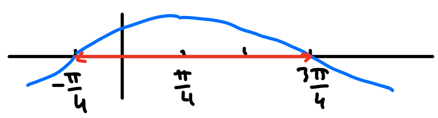
$$x^2 + y^2 \le 2$$

$$(r\cos\varphi - 1)^2 + (r\sin\varphi - 1)^2 \le 2$$

$$r^2\cos^2\varphi - 2r\cos\varphi + 1 + r^2\sin^2\varphi - 2r\sin\varphi + 1 \le 2$$

$$r^2 \le 2r(\sin\varphi + \cos\varphi)$$

$$r \le 2(\sin\varphi + \cos\varphi) = 2\sqrt{2}\sin\left(\varphi + \frac{\pi}{4}\right)$$



$$\varphi \in \left[-\frac{\pi}{4}, \frac{3\pi}{4}\right] \qquad r \in [0, 2(\sin\varphi + \cos\varphi)]$$

$$\int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} \int_0^{2(\sin\varphi+\cos\varphi)} r \cdot r\, dr\, d\varphi = \int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{\left(2\sqrt{2}\sin(\varphi+\frac{\pi}{4})\right)^3}{3}\, d\varphi = \frac{16\sqrt{2}}{3}\int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} \sin^3\left(\varphi + \frac{\pi}{4}\right) d\varphi =$$

$t = \varphi + \frac{\pi}{4} \qquad dt = d\varphi$

$$= \frac{16\sqrt{2}}{3}\ \underbrace{2\int_0^{\pi/2} \sin^3 t\, dt} = \frac{8\sqrt{2}}{3}\, B\left(2, \frac{1}{2}\right) = \frac{8\sqrt{2}}{3}\, \frac{\Gamma(2)\,\Gamma(1/2)}{\Gamma(5/2)}$$

$2a - 1 = 3$

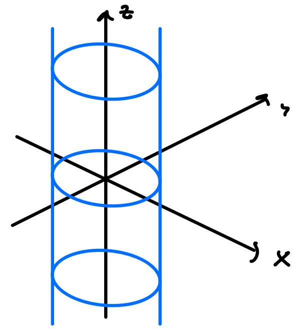
$2b - 1 = 0$

$$\Gamma(5/2) = \frac{3}{2}\Gamma\left(\frac{3}{2}\right) = \frac{3}{2}\cdot\frac{1}{2}\,\Gamma\left(\frac{1}{2}\right)$$

$$= \frac{8\sqrt{2}}{3}\, \frac{1 \cdot \sqrt{\pi}}{\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}} = \boxed{\frac{64\sqrt{2}}{9}}$$

(13) Izračunaj

$$\iiint_T \frac{x^2}{1+x^2z^2}\,dx\,dy\,dz \qquad T=\{(x,y,z)\in\mathbb{R}^3;\ x^2+y^2\le 3\}$$



- Cilindrične koordinate

$\varphi \in [0,2\pi)$

$r \in [0,\sqrt{3}]$

$z \in (-\infty,\infty)$

$$\int_0^{2\pi}\int_0^{\sqrt{3}}\int_{-\infty}^{\infty}\frac{r^2\cos^2\varphi}{1+r^2\cos^2\varphi\, z^2}\cdot r\,dr\,d\varphi\,dz =$$

$$= \int_0^{2\pi}\int_0^{\sqrt{3}} r^3\cos^2\varphi \int_{-\infty}^{\infty}\frac{1}{1+(r|\cos\varphi|z)^2}\,dz\,dr\,d\varphi =$$

$$= \int_0^{2\pi}\int_0^{\sqrt{3}} r^3\cos^2\varphi\,\frac{1}{r|\cos\varphi|}\,\arctan(r|\cos\varphi|z)\Big|_{-\infty}^{\infty}\,dr\,d\varphi =$$

$$= \int_0^{2\pi}\int_0^{\sqrt{3}} r^3\cos^2\varphi\,\frac{1}{r|\cos\varphi|}\,\pi\,dr\,d\varphi =$$

$$= \int_0^{2\pi}\int_0^{\sqrt{3}} r^2\pi|\cos\varphi|\,dr\,d\varphi = \int_0^{2\pi}\pi|\cos\varphi|\,\frac{r^3}{3}\Big|_0^{\sqrt{3}}\,d\varphi$$

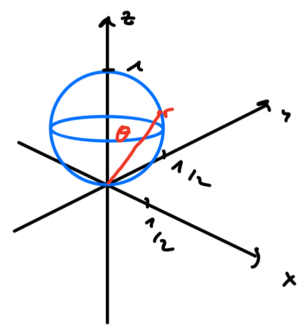
$$= \pi\sqrt{3}\int_0^{2\pi}|\cos\varphi|\,d\varphi = 4\pi\sqrt{3}\int_0^{\pi/4}\cos\varphi\,d\varphi = 4\pi\sqrt{3}\,\sin\varphi\Big|_0^{\pi/4} = \underline{\underline{4\pi\sqrt{3}}}$$

(14) Izračunaj

$$\iiint_T \sqrt{x^2+y^2+z^2}\,dx\,dy\,dz \qquad T:\ x^2+y^2+z^2\le z$$

$$x^2+y^2+z^2-z\le 0$$

$$x^2+y^2+\left(z-\tfrac{1}{2}\right)^2\le \tfrac{1}{4}$$



Sferične koordinate r,φ,θ

$\varphi\in[0,2\pi]$ $\qquad x = r\sin\theta\cos\varphi$

$\theta\in[0,\frac{\pi}{2}]$ $\qquad y = r\sin\theta\sin\varphi$

$r\in[0,\cos\theta]$ $\qquad z = r\cos\theta$

$$r^2\sin^2\theta\cos^2\varphi + r^2\sin^2\theta\sin^2\theta + r^2\cos^2\theta \le r\cos\theta$$

$$r\le\cos\theta$$

$$\int_0^{2\pi}d\varphi\int_0^{\pi/2}d\theta\int_0^{\cos\theta} r\; r^2\sin\theta\,dr = \int_0^{2\pi}d\varphi\int_0^{\pi/2}\sin\theta\,d\theta\;\frac{r^4}{4}\Big|_0^{\cos\theta} = \int_0^{2\pi}d\varphi\int_0^{\pi/2}\frac{1}{4}\sin\theta\cos^4\theta\,d\theta =$$

$$= \frac{\pi}{4}\,2\int_0^{\pi/2}\sin\theta\cos^4\theta\,d\theta = \frac{\pi}{4}B\left(1,\tfrac{5}{2}\right) = \frac{\pi}{4}\,\frac{\Gamma(1)\Gamma(5/2)}{\Gamma(7/2)} = \frac{\pi}{4}\,\frac{1\;\Gamma(5/2)}{5/2\;\Gamma(5/2)} = \underline{\underline{\frac{\pi}{10}}}$$

$2a-1=1 \quad a=1$

$2s-1=4 \quad s=\frac{5}{2}$

Uporaba $\iint$ in $\iiint$

① $\iint_D dx\,dy$ = ploščina lika D $\qquad \iiint_T dx\,dy\,dz$ = volumen

② $\rho(x,y)$ ... gostota

masa $= \iint_D \rho(x,y)\,dx\,dy$ $\qquad \iiint_T \rho(x,y,z)\,dx\,dy\,dz$ = masa

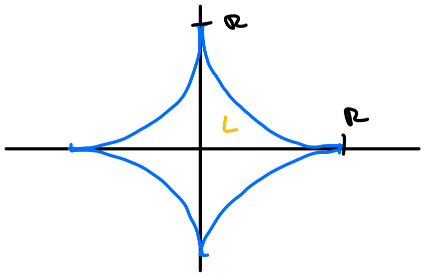
③ Težišče – masno središče

$x^T = \frac{1}{\text{masa}}\iint_D x\,\rho(x,y)\,dx\,dy$ podobno za y $\qquad x_T = \frac{1}{\text{masa}}\iiint_T x\,\rho(x,y,z)\,dx\,dy\,dz$

④ Vztrajnostni moment okoli z osi

$J_z = \iint_D (x^2+y^2)\,\rho(x,y)\,dx\,dy$ razdalja do osi na kvadrat $\qquad J_z = \iiint_T (x^2+y^2)\,\rho(x,y,z)\,dx\,dy\,dz$

18 Dan naj bo lik, ki ga omejuje astroida $x^{2/3}+y^{2/3}=R^{2/3}$. Lik naj ima konst. gostoto 1. Izračunaj maso lika in vztrajnostni moment okoli osi z.



$$\text{masa} = \iint_D dx\,dy = 4\iint_L dx\,dy \qquad y=\left(R^{2/3}-x^{2/3}\right)^{\frac{3}{2}}$$

$$= 4\int_0^R dx \int_0^{(R^{2/3}-x^{2/3})^{3/2}} dy = 4\int_0^R \left(R^{2/3}-x^{2/3}\right)^{3/2} dx =$$

$$= 4\int_0^R \left(R^{2/3}\left(1-\left(\tfrac{x}{R}\right)^{2/3}\right)\right)^{3/2} dx = 4R\int_0^R \left(1-\left(\tfrac{x}{R}\right)^{2/3}\right)^{3/2} dx$$

$t=\left(\frac{x}{R}\right)^{2/3}$

$x = R\,t^{3/2}$

$dx = R\,\frac{3}{2}\,t^{1/2}\,dt$

$$= 4R\int_0^1 (1-t)^{3/2} R\,\tfrac{3}{2}\,t^{1/2}\,dt = 6R^2\int_0^1 (1-t)^{3/2}\,t^{1/2}\,dt = \qquad b-1=\tfrac{3}{2} \quad a-1=\tfrac{1}{2}$$

$$= 6R^2\,B\left(\tfrac{3}{2},\tfrac{5}{2}\right) = 6R^2\,\frac{\Gamma(\frac{3}{2})\Gamma(\frac{5}{2})}{\Gamma(4)} =$$

$$= 6R^2\,\frac{\frac{1}{2}\sqrt{\pi}\;\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}{6} = \underline{\underline{\frac{3\pi R^2}{8}}}$$

$$J_z = \iint_D (x^2+y^2)\,dx\,dy = 4\iint_L (x^2+y^2)\,dx\,dy = 4\int_0^R \int_0^{(R^{2/3}-x^{2/3})^{3/2}} x^2+y^2\,dy\,dx =$$

$$= 4\int_0^R x^2y + \frac{y^3}{3}\Big|_0^{(\dots)} dx = 4\int_0^R x^2\left(R^{2/3}-x^{2/3}\right)^{3/2} + \frac{1}{3}\left(R^{2/3}-x^{2/3}\right)^{9/2} dx =$$

$$= 4\int_0^R x^2\left(R^{2/3}\left(1-\left(\tfrac{x}{R}\right)^{2/3}\right)\right)^{3/2} + \frac{1}{3}\left(R^{2/3}\left(1-\left(\tfrac{x}{R}\right)^{2/3}\right)\right)^{9/2} dx$$

$$= 4\int_0^R x^2 R\left(1-\left(\tfrac{x}{R}\right)^{\frac{2}{3}}\right)^{\frac{3}{2}} + \frac{1}{3}R^3\left(1-\left(\tfrac{x}{R}\right)^{2/3}\right)^{9/2} dx$$

$t = \left(\frac{x}{R}\right)^{2/3}$

$x = R\,t^{3/2}$

$dx = R\,\frac{3}{2}\,t^{1/2}\,dt$

$$= 4\int_0^1 \left(R\,R^2\,t^3(1-t)^{3/2} + \frac{1}{3}R^3(1-t)^{9/2}\right) R\,\tfrac{3}{2}\,t^{1/2}\,dt =$$

$$= 4\,\tfrac{3}{2}\,R^4\left(\int_0^1 t^{7/2}(1-t)^{3/2}\,dt + \frac{1}{3}\int_0^1 t^{1/2}(1-t)^{9/2}\,dt\right) =$$

$a-1=7/2 \quad b-1=3/2 \qquad a-1=\frac{1}{2} \quad b-1=\frac{9}{2}$

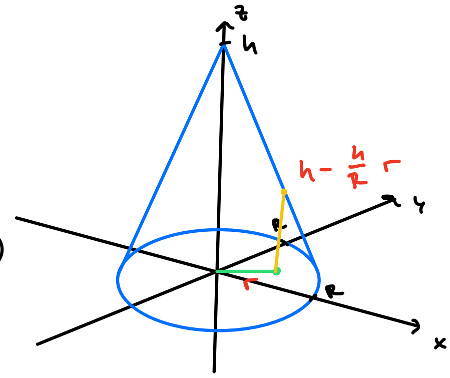
$$= 6R^4\left(B\left(\tfrac{9}{2},\tfrac{5}{2}\right) + \tfrac{1}{3}B\left(\tfrac{3}{2},\tfrac{11}{2}\right)\right) =$$

$$= 6R^4\left(\frac{\Gamma(\frac{9}{2})\Gamma(\frac{5}{2})}{\Gamma(7)} + \frac{1}{3}\,\frac{\Gamma(\frac{3}{2})\Gamma(\frac{11}{2})}{\Gamma(7)}\right)$$

$$= \frac{6R^4}{6!}\left(\tfrac{7}{2}\,\tfrac{5}{2}\,\tfrac{3}{2}\,\tfrac{1}{2}\sqrt{\pi}\;\tfrac{3}{2}\,\tfrac{1}{2}\sqrt{\pi} + \tfrac{1}{3}\,\tfrac{1}{2}\sqrt{\pi}\;\tfrac{9}{2}\,\tfrac{7}{2}\,\tfrac{5}{2}\,\tfrac{3}{2}\,\tfrac{1}{2}\sqrt{\pi}\right)$$

$$= \frac{R^4\pi}{5!\,2^6}\left(7\cdot5\cdot3^2 + 7\cdot5\cdot3^2\right) = \frac{R^4\pi\;7\cdot5\cdot3^2\cdot2}{5!\;2^6} = \frac{21\pi R^4}{2^8}$$

19 Dan je pokončen stožec, pri katerem je gostota premo sorazmerna z višino. Izračunaj težišče in vztrajnostni moment okoli z-osi.



$$\rho(x,y,z) = cz \qquad (c>0,\ c=\text{konst.})$$

Cilindrične koordinate

① $\varphi \in [0, 2\pi)$

$r \in [0, R]$

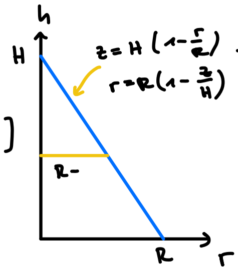
$z \in [0, H(1-\frac{r}{R})]$

Alternativna meja

② $\varphi \in [0, 2\pi)$

$z \in [0, H]$

$r \in [0, R(1-\frac{z}{H})]$



$$\text{masa} = \int_0^{2\pi} d\varphi \int_0^R dr \int_0^{H(1-\frac{r}{R})} cz\, r\, dz =$$

$$= 2\pi \int_0^R c\frac{r}{2}\left(H(1-\frac{r}{R})\right)^2 dr = \pi H^2 c \int_0^R r\left(1-\frac{r}{R}\right)^2 dr = \pi H^2 \int_0^R r - \frac{2r^2}{R} + \frac{r^3}{R^2}\, dr$$

$$= \pi H^2 c\left(\frac{R^2}{2} - \frac{2R^3}{3R} + \frac{R^4}{4R^2}\right) = \pi H^2 R^2 c\left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right) = \frac{\pi H^2 R^2 c}{12}$$

težišče $x_T, y_T = 0$

$$z_T = \frac{1}{m}\iiint_T z\rho\, dx\,dy\,dz = \frac{1}{m}\iiint_T z^2 c\, dV = \frac{c}{m}\int_0^{2\pi} d\varphi \int_0^R \int_0^{H(1-\frac{r}{R})} z^2 r\, dz\, dr =$$

$$= \frac{2\pi c}{m}\int_0^R \frac{r}{3}\left(H(1-\frac{r}{R})\right)^3 dr = \frac{2\pi c H^3}{3m}\int_0^R r\left(1-\frac{r}{R}\right)^3 dr = \qquad t = \frac{r}{R} \quad dr = R\,dt$$

$$= \frac{2\pi c H^3 R^2}{3m}\int_0^1 t(1-t)^3 dt = \frac{2\pi c H^3 R^2}{3m} B(2,4) = \frac{2\pi c H^3 R^2}{3m}\frac{\Gamma(2)\Gamma(4)}{\Gamma(6)} = \frac{\pi c H^3 R^2}{30m} =$$

$a-1 = 1 \quad b-1 = 3$

$$= \frac{2H}{5}$$

vztrajnostni moment

$$J_z = \iiint_T (x^2+y^2)\, dV = \int_0^{2\pi} d\varphi \int_0^R \int_0^{H(1-\frac{r}{R})} r^2 r c z\, dz\, dr = 2\pi c \int_0^R r^3 \frac{1}{2}\left(H(1-\frac{r}{R})\right)^2 dr =$$

$t = \frac{r}{R} \quad dr = R\,dt$

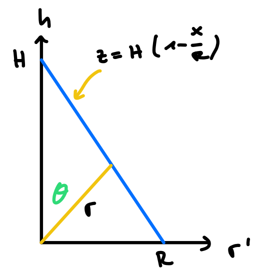
$$= \pi c H^2 R^3 \int_0^1 t^3 (1-t)^2 R\, dt = \pi c H^2 R^4\, B(4,3) = \pi c H^2 R^4 \frac{3!\,2!}{6!} = \frac{c\pi H^2 R^4}{60}$$

Nastavimo v sferičnih koordinatah

$x = r\sin\theta\cos\varphi \qquad \varphi = [0, 2\pi)$

$y = r\sin\theta\sin\varphi \qquad \theta = [0, \frac{\pi}{2}]$

$z = r\cos\theta \qquad r = \left[0, \frac{H}{\cos\theta + \frac{H}{R}\sin\theta}\right]$



$r' = \sqrt{x^2+y^2}$

$$\text{masa} = \iiint_T cz\, dV = \int_0^{2\pi} d\varphi \int_0^{\pi/2} d\theta \int_0^{\frac{H}{\cos\theta + \frac{H}{R}\sin\theta}} c r\cos\theta\, r^2 \sin\theta\, dr$$

$\cos\theta\; r = z$

$\cos\theta\; r = H\left(1 - \frac{\sqrt{x^2+y^2}}{R}\right)$

$\cos\theta\; r = H\left(1 - \frac{r\sin\theta}{R}\right)$

$= H - \frac{H}{R} r\sin\theta$

$r\left(\cos\theta + \frac{H}{R}\sin\theta\right) = H$

$$r = \frac{H}{\cos\theta + \frac{H}{R}\sin\theta}$$

Krivulje in ploskve

Ⓣ **Def.** Pot je preslikava $\vec{r} : I \to \mathbb{R}^3$; $t \in I$ $(I = [a,b])$, $\vec{r}(t) \in \mathbb{R}^3$,
$\vec{r}(t) = (x(t), y(t), z(t))$
Krivulja je pa slika poti $\vec{r}$

Pravimo, da je pot $\vec{r}$ parametrizacija krivulje.

Primer:

① Parametriziraj krivuljo $K = \{ (x,y) \in \mathbb{R}^2 ; x^2 + y^2 = 1 \}$

$x = \cos t$ $\qquad$ $\vec{r} : I \to \mathbb{R}^2$

$y = \sin t$ $\qquad$ $t \to \vec{r}(t)$

$t \mapsto (\cos t, \sin t)$ $\qquad$ $\text{im}(t) = K$

$\vec{r}_1 : [0, 2\pi) \to \mathbb{R}^2$

$\vec{r}_2 : \mathbb{R} \to \mathbb{R}^2$ $\qquad$ $\vec{r}_2$ ni injektivna

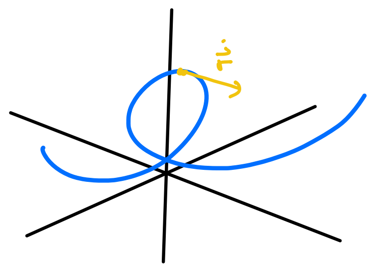
$\vec{r}_2(t) = (\cos t, \sin t)$

$\vec{r}_3(t) = (\cos t^2, \sin t^2)$

Odvod

$\vec{r}(t) = (x(t), y(t), z(t))$

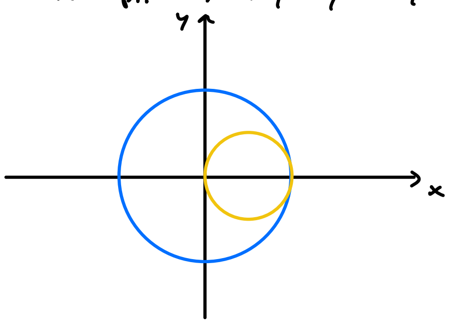
$\dot{\vec{r}}(t) = (\dot{x}(t), \dot{y}(t), \dot{z}(t))$

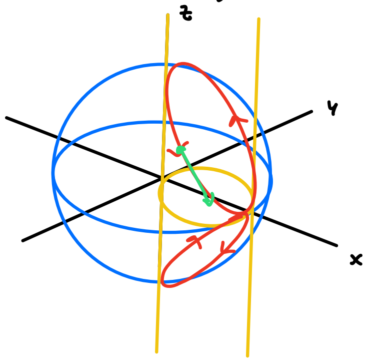


① Parametriziraj krivuljo dano z

- $x^2 + y^2 + z^2 = 4$
- $(x-1)^2 + y^2 = 1$

in pri $x = 1$, $y < 0$, $z > 0$ zapiši enačbo tangente na krivuljo.





$x = \cos t + 1$

$y = \sin t$

$z = \pm\sqrt{4 - x^2 - y^2} = \pm\sqrt{4 - \sin^2 t - \cos^2 t - 2\cos t - 1}$

$= \pm\sqrt{2 - 2\cos t} = \pm\sqrt{2(1-\cos t)} = \pm\sqrt{2(\sin^2 t/2 + \cos^2 t/2 - \cos^2 t/2 + \sin^2 t/2)} = \pm 2\sin t/2$

$\vec{r}(t) = (\cos t + 1, \sin t, 2\sin t/2)$ $\qquad$ $t \in [0, 4\pi]$

ker zamik zato ni več ne kaj označimo

$\dot{\vec{r}}(t) = (-\sin t, \cos t, \cos t/2) = (1, 0, -\frac{\sqrt{2}}{2})$

$x = 1$ $\qquad$ $\cos t = 0$ $\qquad$ $t = \frac{\pi}{2} + \pi k$ $\Rightarrow$ $t = \frac{3\pi}{2}$

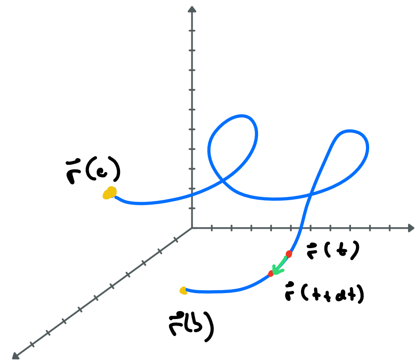
$T(1, -1, \sqrt{2})$

Tang. premica $\qquad$ $\vec{q} = (1, -1, \sqrt{2}) + s(1, 0, -\frac{\sqrt{2}}{2})$

T Dolžina poti (krivulje)

$$\vec{r} : [a,b] \to \mathbb{R}^3$$

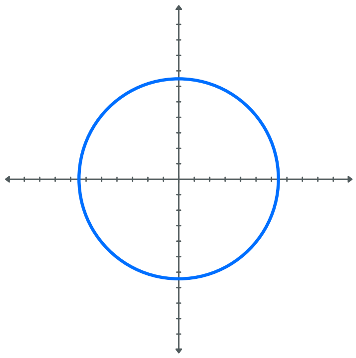
$$\boxed{L = \int_a^b |\dot{\vec{r}}(t)|\, dt}$$



$$\vec{r}(t+dt) - \vec{r}(t) = \dot{\vec{r}}(t)\, dt$$

Ali se to ujema z dolžino krivulje

$\vec{r}(t) = (\cos t, \sin t) \qquad t \in [0, 4\pi]$



$$\dot{\vec{r}} = (-\sin t, \cos t)$$

$$|\dot{\vec{r}}| = 1$$

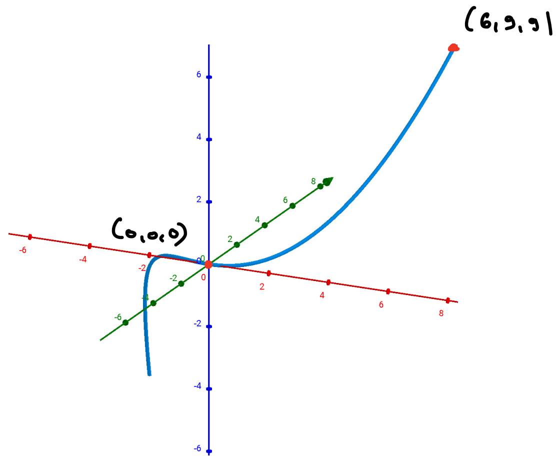
$$L = \int_0^{4\pi} 1\, dt = 4\pi = 2\times \text{ obseg krožnice}$$

$\Rightarrow$ Če $\vec{r} : [a,b] \to \mathbb{R}^3$ injektivna je dolžina poti enaka dolžini krivulje.

2 Dana je krivulja K parametrizirana z

$$\vec{r}(t) = (2t, t^2, t^3/3) \qquad t \in [0,3]$$

Določi dolžino krivulje K.



$$\dot{\vec{r}} = (2, 2t, t^2)$$

$$|\dot{\vec{r}}| = \sqrt{4 + 4t^2 + t^4} = \sqrt{(2+t^2)^2} = 2 + t^2$$

$$L = \int_0^3 2 + t^2\, dt = 2t + \frac{t^3}{3}\Big|_0^3 = 15$$

T Naravna parametrizacija krivulje

$$\vec{r} : [a,b] \to \mathbb{R}^3$$
$$t \mapsto \vec{r}(t)$$

pravimo, da je t naravni parameter, če je

$$|\dot{\vec{r}}(t)| = 1$$

$$\Downarrow$$

tipično namesto t pišemo s.

$$\vec{r}(s) : [0, a] \to \mathbb{R}^3$$
$$|\vec{r}\,'(s)| = 1$$

Dolžina krivulje $\quad L(a) = \int_0^a 1\, ds = a$

Tipično $\quad \vec{r}: I \to \mathbb{R}^3$, $|\dot{\vec{r}}(t)| \neq 1$ $\xrightarrow[\text{reparametrizirati}]{\text{Hočemo naravno}}$ Predpostavka $\dot{\vec{r}}(t) \neq 0$

$$s = \int_a^t |\dot{\vec{r}}(\tau)| d\tau \quad \Rightarrow \quad s(t)$$

poljubna spodnja meja

$$\dot{s} = |\dot{\vec{r}}(t)| \neq 0 \qquad \underset{>0}{} \qquad \text{injektivna} \Downarrow \qquad t(s)$$

$$t' = \frac{1}{|\dot{\vec{r}}(t)|}$$

Reparametrizacija $\quad \vec{\rho}(s) = \vec{r}(t(s))$

Ali je s naravni parameter

$$\vec{\rho}\,' = \dot{\vec{r}}\, t' = \dot{\vec{r}} \frac{1}{|\dot{\vec{r}}|} = \text{enotski vektor} \Rightarrow |\vec{\rho}\,'| = 1$$

• Primer $\quad \vec{r}(t) = (3\cos t, 3\sin t) \qquad t \in [0, 2\pi]$

$$\dot{\vec{r}}(t) = (-3\sin t, 3\cos t)$$

$$|\dot{\vec{r}}(t)| = 3$$

$\hookrightarrow t$ ni naravni parameter

$$s(t) = \int_0^t 3 dt = 3t \quad \Rightarrow s = 3t \quad \Rightarrow \quad t = \frac{1}{3}s$$

$$\vec{\rho}(t) = \left(3\cos\frac{s}{3}, 3\sin\frac{s}{3}\right) \quad \text{Naravna parametrizacija}$$

$$s \in [0, 6\pi]$$

Opomba: Vzamemo kakšno drugo spodnjo mejo

$$s = \int_{\pi/2}^t 3 d\tau = 3\left(t - \frac{\pi}{2}\right) \quad \Rightarrow t - \frac{\pi}{2} = \frac{s}{3} \qquad t = \frac{s}{3} + \frac{\pi}{2}$$

$$\vec{\rho}(s) = \left(3\cos\left(\frac{\pi}{2} + \frac{s}{3}\right), 3\sin\left(\frac{\pi}{2} + \frac{s}{3}\right)\right)$$

Spodnja meja je poljubna.

3 $\quad \vec{r}(t) = (2t, t^2, t^3/3)$ Izračunaj w'', kjer je $w = x^2 + 4y^2 + 9z^2$ in s črtico (') je označen odvod po naravnem parametru.

Poiščimo naravno parametrizacijo

$$\dot{\vec{r}}(t) = (2, 2t, t^2) \quad \Rightarrow \quad s = \int_0^t 2 + t^2 \, dt = 2t + \frac{t^3}{3} \qquad s = 2t + \frac{t^3}{3}$$

$$|\dot{\vec{r}}(t)| = 2 + t^2$$

Nalogo bomo rešili brez $\quad t(s)$

$$w = x^2 + 4y^2 + 9z^2$$

$$w = 4t^2 + 4t^4 + t^6$$

$$w' = 8t\,t' + 16t^3 t' + 6t^5 t'$$

$$w'' = t'(8t' + 48t^2 t' + 30t^4 t') + (8t + 16t^3 + 6t^5)t''$$

$$s = 2t + \frac{t^3}{3}$$

$$1 = 2t' + t^2 t' \quad \Rightarrow \quad t' = \frac{1}{2+t^2}$$

$$0 = 2t'' + t' 2t t' + t'' t^2 \quad \Rightarrow \quad t'' = \frac{-2tt'^2}{2+t^2} = \frac{-2t}{(2+t^2)^3}$$

$$\omega'' = \frac{(2+t^2)(8+48t^2+30t^4) - 2t(8t+16t^3+6t^5)}{(2+t^2)^3}$$

$$= \frac{16 + 96t^2 + 60t^4 + 8t^2 + 48t^4 + 30t^6 - 16t^2 - 32t^4 - 12t^6}{(2+t^2)^3}$$

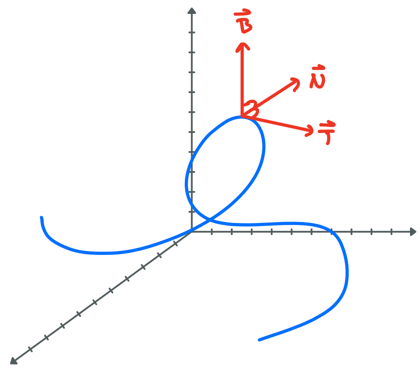
$$= \frac{16 + 88t^2 + 76t^4 + 18t^6}{(2+t^2)^3}$$

T Frenetova baza

$\vec{r}(s)$ ... naravna param. krivulje

$$\boxed{\begin{aligned} \vec{T} &= \vec{r}\,' \\ \vec{N} &= \vec{r}\,''/|\vec{r}\,''| = \vec{T}\,'/|\vec{T}\,'| \\ \vec{B} &= \vec{T} \times \vec{N} \end{aligned}}$$

$|\vec{r}\,''| = \varkappa \neq 0$



Fleksijska ukrivljenost

Torzijska ukrivljenost

$$\boxed{\begin{aligned} \varkappa &= |\vec{r}\,''| \\ \tau &= \frac{\vec{r}\,' \cdot (\vec{r}\,'' \times \vec{r}\,''')}{|\vec{r}\,''|^2} \end{aligned}}$$

$R = \frac{1}{\varkappa}$ ... radij ukrivljenosti

Frenetove formule

$$\boxed{\begin{aligned} \vec{T}\,' &= \varkappa \vec{N} \\ \vec{N}\,' &= \tau \vec{B} - \varkappa \vec{T} \\ \vec{B}\,' &= -\tau \vec{N} \end{aligned}}$$

Za naravno param.

Izrek

Dani $f(s) > 0$, $g(s)$

$\Downarrow$

$\exists\ \vec{r}(s)$

$\varkappa(s) = f(s)$

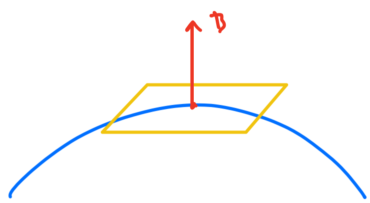
$\tau(s) = g(s)$

obstaja natanko ena, do rotacij in translacij natančno.

$\vec{r}(t)$ ni naravna param. $\Rightarrow$ $\vec{T}, \vec{N}, \vec{B}, \varkappa, \tau$

$$\boxed{\begin{aligned} \vec{T} &= \frac{\dot{\vec{r}}}{|\dot{\vec{r}}|} \qquad \vec{B} = \frac{\dot{\vec{r}} \times \ddot{\vec{r}}}{|\dot{\vec{r}} \times \ddot{\vec{r}}|} \qquad \vec{N} = \vec{B} \times \vec{T} \\ \varkappa &= \frac{|\dot{\vec{r}} \times \ddot{\vec{r}}|}{|\dot{\vec{r}}|^3} \qquad \tau = \frac{\dot{\vec{r}} \cdot (\ddot{\vec{r}} \times \dddot{\vec{r}})}{|\dot{\vec{r}} \times \ddot{\vec{r}}|^2} \end{aligned}}$$

Pritisnjena ravnina $\perp \vec{B}$



Binormala ravnina $\perp \vec{N}$

④ Dana je krivulja s parametrizacijo

$$\vec r(t) = \left(\frac{t^4}{4}, \frac{t^3}{3}, \frac{t^2}{2}\right) \qquad t \ge 0$$

ⓐ Določi $\vec T, \vec N, \vec B, \varkappa, \tau$ za $\forall t \in \mathbb{R}$

ⓑ Določi enačbo binormalne in pritisnjene ravnine v točki $t=1$

ⓐ

$$\vec r(t) = \left(\frac{t^4}{4}, \frac{t^3}{3}, \frac{t^2}{2}\right)$$
$$\dot{\vec r}(t) = (t^3, t^2, t)$$
$$\ddot{\vec r}(t) = (3t^2, 2t, 1)$$
$$\dddot{\vec r}(t) = (6t, 2, 0)$$
$$\dot{\vec r} \times \ddot{\vec r} = (-t^2, 2t^3, -t^4)$$
$$\ddot{\vec r} \times \dddot{\vec r} = (-2, 6t, -6t^2)$$
$$|\dot{\vec r}| = \sqrt{t^2+t^4+t^6} = t\sqrt{t^4+t^2+1}$$

$$\vec T = \frac{(t^3, t^2, t)}{t\sqrt{t^4+t^2+1}} = \frac{(t^2, t, 1)}{\sqrt{t^4+t^2+1}}$$

$$\vec B = \frac{(-t^2, 2t^3, -t^4)}{\sqrt{t^4+4t^6+t^8}} = \frac{(-1, 2t, -t^2)}{\sqrt{1+4t^2+t^4}}$$

$$\vec N = \vec B \times \vec T = \frac{(t^3+2t, 1-t^4, -2t^3-t)}{\sqrt{t^8+5t^6+6t^4+5t^2+1}}$$

$$\varkappa = \frac{|\dot{\vec r}\times\ddot{\vec r}|}{|\dot{\vec r}|^3} = \frac{\sqrt{t^4+4t^6+t^8}}{t^3\sqrt{t^4+t^2+1}^3} = \frac{\sqrt{1+4t^2+t^4}}{t\sqrt{1+t^2+t^4}^3}$$

$$\tau = \frac{\dot{\vec r}\cdot(\ddot{\vec r}\times\dddot{\vec r})}{|\dot{\vec r}\times\ddot{\vec r}|} = \frac{(t^3, t^2, t)\cdot(-2, 6t, -6t^2)}{t^4+4t^6+t^8} = \frac{-2t^3+6t^3-6t^3}{t^4(1+4t^2+t^4)}$$
$$= \frac{-2}{t(1+4t^2+t^4)}$$

ⓑ

$$\vec r(1) = \left(\frac{1}{4}, \frac{1}{3}, \frac{1}{2}\right)$$
$$\vec T(1) = \frac{1}{\sqrt3}(1, 1, 1)$$
$$\vec B(1) = \frac{1}{\sqrt6}(-1, 2, -1)$$
$$\vec N(1) = \frac{1}{3\sqrt2}(3, 0, -3) = \frac{1}{\sqrt2}(1, 0, -1)$$

Prit. ravnina gre skozi $\vec r(1)$ in je $\perp$ na $\vec B$

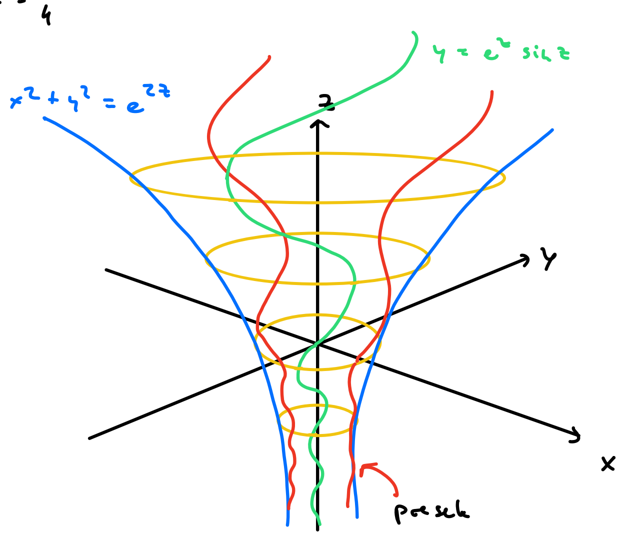
$$-x + 2y - z = -\frac14 + \frac23 - \frac12 = -\frac{1}{12}$$

Binormalna ravnina gre skozi $\vec r(1)$ in je $\perp$ na $\vec N$

$$x - z = \frac14 - \frac12 = -\frac14 \qquad z - x = \frac14$$

⑤ Parametriziraj krivuljo dano s pogoji
$y = e^z \sin z \qquad x^2 + y^2 = e^{2z}$

ter pri $z=0$ $(x>0)$ določi $\varkappa$ in τ.



Naj bo $z = t$, dobimo

$y = e^t \sin t$

$x^2 + e^{2t}\sin^2 t = e^{2t}$

$x^2 = e^{2t}\cos^2 t$

$x = \pm e^t \cos t$

$\vec r(t) = (\pm e^t \cos t, e^t \sin t, t)$

krivulja iz dveh delov

Za nadaljevanje vzemimo $\vec r(t) = (e^t \cos t, e^t \sin t, t)$ pri $t=0$

$$\dot{\vec r}(t) = (e^t\cos t - e^t \sin t, e^t \sin t + e^t \sin t, 1) = (1, 1, 1)$$
$$\ddot{\vec r}(t) = (-2e^t \sin t, 2e^t \cos t, 0) = (0, 2, 0)$$
$$\dddot{\vec r}(t) = (-2e^t \sin t - 2e^t \cos t, 2e^t \cos t - 2e^t \sin t, 0) = (-2, 2, 0)$$

$$\varkappa = \frac{|(-2, 0, 2)|}{\sqrt3^3} = \frac{2\sqrt2}{3\sqrt3}$$

$$\tau = \frac{(1,1,1)\cdot(0,0,4)}{|(-2,0,2)|^2} = \frac12$$

T Ploskve

Podajanje ploskev:

1. $f: D \subseteq \mathbb{R}^2 \to \mathbb{R}$ graf funkcije f

2. $f: \mathbb{R}^3 \to \mathbb{R}$ $\quad P = \{(x,y,z);\ f(x,y,z) = 0\}$

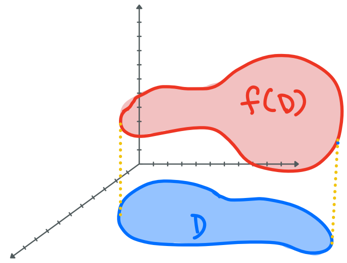
   primer $\quad x^2 + y^2 + z^2 - 1 = 0$

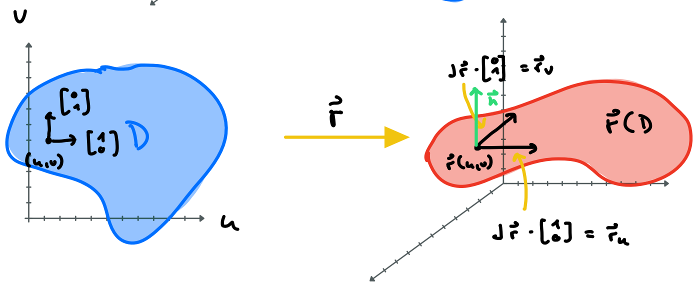
3. Parametrično podane ploskve

   $\vec{r}: D \subseteq \mathbb{R}^2 \to \mathbb{R}^3$

   $(u,v) \mapsto (x(u,v), y(u,v), z(u,v))$

   parametrizacija ploskve



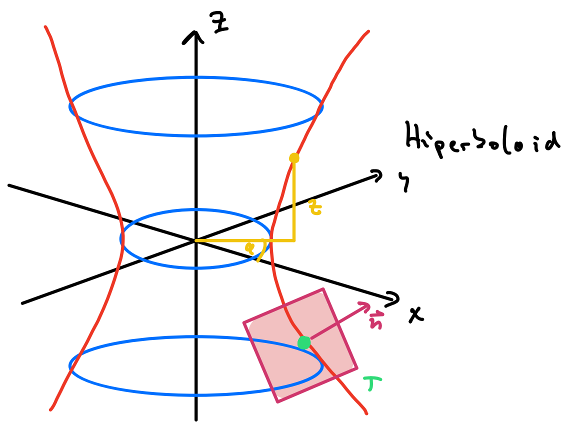


$$\vec{r}(u,v) = \begin{bmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{bmatrix}$$

$$J\vec{r} = \begin{bmatrix} x_u & x_v \\ y_u & y_v \\ z_u & z_v \end{bmatrix}$$

Naj bo $\vec{r}$ parametrizacija $\Rightarrow$ v $\forall$ točki $\vec{r}(u,v)$ imamo dva tangentna vektorja $\frac{\partial \vec{r}}{\partial u}$, $\frac{\partial \vec{r}}{\partial v}$ in enotski normalni vektor $\vec{n} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$

6 Parametriziraj ploskev $x^2 + y^2 - z^2 = 1$ in v točki $T(3,4,z)$, kjer je $z<0$, določi normalni vektor in tangentno ravnino.



$$\vec{r}(\varphi, z) = \left(\sqrt{1+z^2}\cos\varphi, \sqrt{1+z^2}\sin\varphi, z\right)$$

$$z \in (-\infty, \infty), \quad \varphi \in [0, 2\pi)$$

ali: pr.: $\vec{r}(\varphi, r) = (r\cos\varphi, r\sin\varphi, \sqrt{1-r^2})$

Točka $T(3,4,z)$, $z<0$

$$x^2 + y^2 - z^2 = 1$$
$$9 + 16 - z^2 = 1$$
$$z = \pm\sqrt{24} \quad \Rightarrow T(3,4,-2\sqrt{6}) = \vec{r}(\varphi_0, z_0)$$

$$\Rightarrow z_0 = -2\sqrt{6}$$
$$3 = \sqrt{1+(-2\sqrt{6})^2}\cos\varphi_0 \Rightarrow \frac{3}{5} = \cos\varphi_0 \Rightarrow \varphi_0 = \operatorname{arctan}\frac{4}{3}$$
$$4 = \sqrt{1+(-2\sqrt{6})^2}\sin\varphi_0 \Rightarrow \frac{4}{5} = \sin\varphi_0$$

$$\vec{n} = \vec{r}_\varphi \times \vec{r}_z = \left(-\sqrt{1+z^2}\sin\varphi, \sqrt{1+z^2}\cos\varphi, 0\right) \times \left(\cos\varphi\frac{z}{\sqrt{1+z^2}}, \frac{z}{\sqrt{1+z^2}}\sin\varphi, 1\right)$$
$$= \left(-5\cdot\frac{4}{5}, 5\cdot\frac{3}{5}, 0\right) \times \left(\frac{3}{5}\left(-\frac{2\sqrt{6}}{5}\right), -\frac{2\sqrt{6}}{5}\cdot\frac{4}{5}, 1\right)$$
$$= (-4, 3, 0) \times \left(-\frac{6\sqrt{6}}{25}, -\frac{8\sqrt{6}}{25}, 1\right)$$

7 Dana krivulja K $\quad \vec{r}(t) = (3t, 3t^2, 2t^3)$

a) Parametriziraj ploskev, ki jo sestavljajo premice, ki grejo skozi točke krivulje K v smeri normalnega vektorja $\vec{N}$.

b) Poišči evoluto te krivulje, tj. krivuljo, ki jo sestavljajo središča pritisnjenih krožnic na K.

a) $\vec{r}(t) = (3t, 3t^2, 2t^3)$

$\dot{\vec{r}}(t) = (3, 6t, 6t^2)$

$\ddot{\vec{r}}(t) = (0, 6, 12t)$

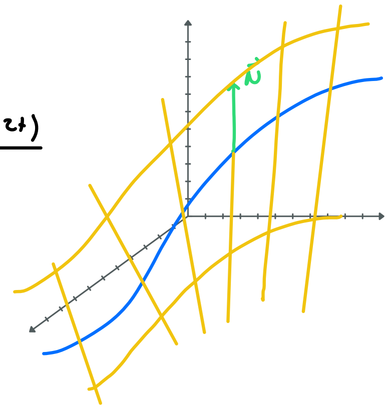
$$\vec{T} = \frac{\dot{\vec{r}}}{|\dot{\vec{r}}|} = \frac{(3, 6t, 6t^2)}{\sqrt{9 + 36t^2 + 36t^4}} = \frac{(1, 2t, 2t^2)}{2t^2+1}$$

$$\vec{B} = \frac{(36t^2, -36t, 18)}{18\sqrt{4t^4+4t^2+1}} = \frac{(2t^2, -2t, 1)}{2t^2+1}$$

$$\vec{N} = \vec{B} \times \vec{T} = \frac{1}{(2t^2+1)^2}\left(-4t^3-2t,\ 1-4t^4,\ 4t^3+2t\right) = \frac{(-2t,\ 1-2t^2,\ 2t)}{2t^2+1}$$

Param. $\vec{\varphi}(t,s) = \vec{r}(t) + s\vec{N}(t)$

$$= \begin{bmatrix} 3t + s\frac{-2t}{2t^2+1} \\ 3t^2 + s\frac{1-2t^2}{2t^2+1} \\ 2t^3 + s\frac{2t}{2t^2+1} \end{bmatrix} \qquad t, s \in \mathbb{R}$$

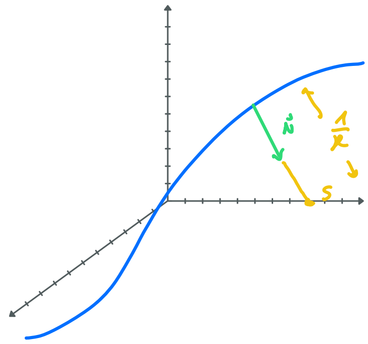


b) Evoluta

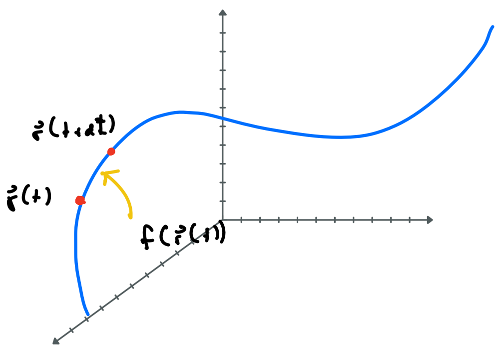
$$S(t) = \vec{r}(t) + \frac{1}{\varkappa(t)}\vec{N}(t)$$

$$\varkappa(t) = \frac{|\dot{\vec{r}} \times \ddot{\vec{r}}|}{|\dot{\vec{r}}|^3} = \frac{18(2t^2+1)}{3^3(2t^2+1)^3} = \frac{2}{3}\frac{1}{(2t^2+1)^2}$$

$$S(t) = \begin{bmatrix} 3t + \frac{3}{2}(2t^2+1)(-2t) \\ 3t^2 + \frac{3}{2}(2t^2+1)(1-2t^2) \\ 2t^3 + \frac{3}{2}(2t^2+1)(2t) \end{bmatrix}$$



Krivuljni integrali skalarne funkcije



f ... kao duljinska gustoća krivulje

$I = \int_K f\,ds$ = oznaka za kr. int. skal. funkcije
= masa krivulje

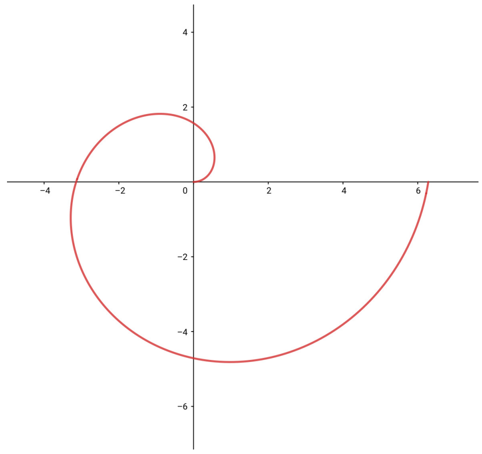
$$dM = |\vec{r}(t+dt) - \vec{r}(t)|\ f(\vec{r}(t)) =$$
$$= |\vec{r}(t) + \dot{\vec{r}}(t)dt - \vec{r}(t)|\ f(\vec{r}(t))$$
$$= f(\vec{r}(t))\,|\dot{\vec{r}}(t)|\,dt$$

$$\int_K f\,ds = \int_a^b f(\vec{r}(t))\,|\dot{\vec{r}}(t)|\,dt$$

parametriziramo
$\vec{r}(t)$

8 Izračunaj

$\int_K \sqrt{x^2+y^2+z^2}\,ds$, gdje je K prvi zavoj Arhimedove spirale
$\vec{r}(t) = (t\cos t,\ t\sin t,\ 0)$ $\quad t \in [0, 2\pi]$



$$f(\vec{r}(t)) = \sqrt{t^2\cos^2 t + t^2\sin^2 t + 0} = t$$
$$\dot{\vec{r}}(t) = (\cos t - t\sin t,\ \sin t + t\cos t,\ 0)$$
$$|\dot{\vec{r}}(t)| = \sqrt{\cos^2 t - 2t\sin t\cos t + t^2\sin^2 t + \sin^2 t + 2t\sin t\cos t + t^2\cos^2 t + 0}$$
$$= \sqrt{1+t^2}$$

$$\int_K f\,ds = \int_0^{2\pi} t\sqrt{1+t^2}\,dt = \int_1^{1+4\pi^2} \frac{1}{2}\sqrt{u}\,du = \frac{1}{3}u^{\frac{3}{2}}\Big|_1^{1+4\pi^2}$$

$1+t^2 = u$
$2t\,dt = du$

$$= \frac{1}{3}\left((1+4\pi^2)^{3/2} - 1\right)$$

(T) Uporaba krivuljnega integrala skalarne funkcije

$\rho(x,y,z)$ ... (dolžinska) gostota

$$\int_K \rho \, ds = \text{masa krivulje}$$

$$\rho = 1 \Rightarrow \int_K ds = \int_a^b |\dot{\vec r}|\, dt = \text{dolžina krivulje}$$

$(\bar x, \bar y, \bar z)$ ... masno središče krivulje K

$$\bar x = \frac{\int_K x \rho \, ds}{\int_K ds} \quad \dots \quad \bar y, \bar z$$

Vztrajnostni moment

$$J_z = \int_K (x^2+y^2)\, \rho \, ds$$

(9) Izračunaj J_x krivulje $\vec r(t) = (3t, 3t^2, 2t^3)$ $t \in [0,1]$

$\rho(x,y,z) = \frac{y}{x+3z}$

$\dot{\vec r}(t) = (3, 6t, 6t^2) \Rightarrow |\dot{\vec r}| = 6t^2+3$

$$J_x = \int_K (y^2+z^2)\, \rho \, ds = \int_0^1 \left((3t^2)^2 + (2t^3)^2\right) \frac{3t^2}{3t+6t^3} |\dot{\vec r}| \, dt$$

$$= \int_0^1 (9t^4 + 4t^6) \frac{3t^2}{3t+6t^3} (6t^2+3)\, dt$$

$$= \int_0^1 27t^5 + 12t^7 \, dt = \frac{27}{6} + \frac{12}{8} = \frac{9}{2} + \frac{3}{2} = 6$$

(9a) Dana krivulja $\vec r(t) = (3t, 3t^2, 2t^3)$ $t \in [0,1]$

$\rho(x,y,z) = \frac{y}{x+3z}$

$\dot{\vec r} = (3, 6t, 6t^2)$

$|\dot{\vec r}| = 6t^2+3$

Določi masno središče krivulje.

$$\text{masa} = \int_K \rho \, ds$$

masno središče $\bar x = \frac{1}{\text{masa}} \int_K x \rho \, ds$ $\qquad \bar y = \frac{1}{m} \int_K y \rho \, ds$ $\qquad \bar z = \frac{1}{m} \int_K z \rho \, ds$

$$\text{masa} = \int_0^1 \frac{3t^2}{3t+6t^3} (6t^2+3)\, dt = \int_0^1 3t = \frac{3}{2} t^2 \Big|_0^1 = \frac{3}{2}$$

$x:$ $\int_0^1 3t \frac{3t^2}{3t+6t^3} (6t^2+3)\, dt = \int_0^1 9t^2 \, dt = 3$ $\qquad \bar x = \frac{3}{\frac{3}{2}} = 2$

$y:$ $\int_0^1 3t^2 \frac{3t^2}{3t+6t^3} (6t^2+3)\, dt = \int_0^1 9t^3 \, dt = \frac{9}{4}$ $\qquad \bar y = \frac{\frac{9}{4}}{\frac{3}{2}} = \frac{3}{2}$

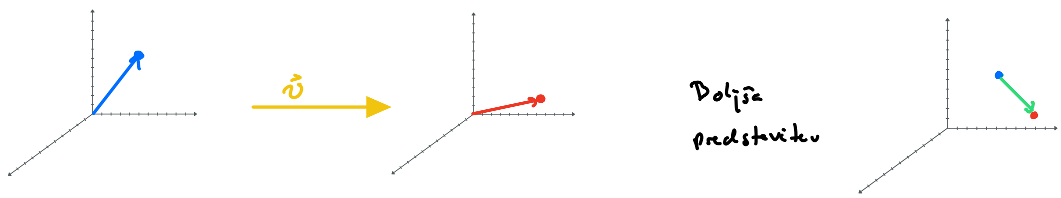
$z:$ $\int_0^1 2t^3 \frac{3t^2}{3t+6t^3} (6t^2+3)\, dt = \int_0^1 6t^4 \, dt = \frac{6}{5}$ $\qquad \bar z = \frac{4}{5}$

$$\bar r = \left(2, \frac{3}{2}, \frac{4}{5}\right)$$

T Krivuljni integral vektorskega polja

Vek. polje $\vec v : \mathbb{R}^3 \to \mathbb{R}^3$

$(x, y, z) \to (P(x,y,z), Q(x,y,z), R(x,y,z))$



① K, $\vec r : [a, b] \to \mathbb{R}^3$

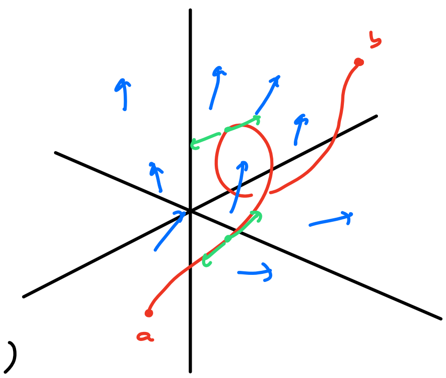
$\vec r : t \longmapsto \vec r(t)$

② $\vec v = (P, Q, R)$

③ Potrebno je izbrati še orientacijo krivulje

$(\sigma : K \to \mathbb{R}^3$

$\sigma(p) = \vec T$ ... tangentni vekt. na K $\quad |\vec T| = 1)$



$\Rightarrow$ orientacija = usmerjenost krivulje

Oznaka za krivuljni integral vekt. polja $\vec v$ po K v smeri dane orientacije

(primer $\int_K \vec F d\vec s = A$ po poti K)

$$\int_K \vec v \cdot d\vec s = \int_K \vec v \cdot \vec T \, ds = \int_a^b \vec v(\vec r(t)) \frac{\dot{\vec r}}{|\dot{\vec r}|} |\dot{\vec r}| \, ds$$

Naj bo $\vec T = \dfrac{\dot{\vec r}}{|\dot{\vec r}|}$

$$\boxed{\int_K \vec v \cdot d\vec s = \int_a^b \vec v(\vec r(t)) \cdot \dot{\vec r}(t) \, dt}$$

Če se orientacija ujema z naraščajočim parametrom t.

Če je parametrizacija taka, da kaže v nasprotni smeri orientacije

$$\int_K \vec v \cdot d\vec s = -\int_a^b \vec v(\vec r(t)) \dot{\vec r}(t) \, dt$$

10

$\vec v = (y, -z, x)$

K_1: $\vec r_1(t) = (t, t, t)$ $\quad t \in [0, 1]$

K_2: $\vec r_2(t) = (t, t^2, t^3)$ $\quad t \in [0, 1]$

K_1, K_2 orientirana v smeri naraščajočega t.

Izračunaj

$$\int_{K_1} \vec v \cdot d\vec s = \int_0^1 (t, -t, t) \cdot (1, 1, 1) \, dt = \int_0^1 t - t + t \, dt = \frac{1}{2}$$

$$\int_{K_2} \vec v \cdot d\vec s = \int_0^1 (t^2, -t^3, t) \cdot (1, 2t, 3t^2) \, dt = \int_0^1 t^2 - 2t^4 + 3t^3 \, dt = \frac{1}{3} - \frac{2}{5} + \frac{3}{4} = \frac{41}{60}$$

11 $\vec{v} = (2xz, z^2, 2yz + x^2)$

K_1: $\vec{r}_1 = (t, t, t)$

K_2: $\vec{r}_2 = (t, t^2, t^3)$ $\quad t \in [0,1]$

$$\int_{K_1} \vec{v} \cdot d\vec{s} = \int_0^1 (2t^2, t^2, 3t^2)(1,1,1)\,dt = \int_0^1 6t^2\,dt = 2$$

$$\int_{K_2} \vec{v} \cdot d\vec{s} = \int_0^1 (2t^4, t^6, 2t^5 + t^2)(1, 2t, 3t^2)\,dt = \int_0^1 2t^4 + 2t^7 + 6t^7 + 3t^4\,dt =$$
$$= \frac{2}{5} + \frac{2}{8} + \frac{6}{8} + \frac{3}{5} = 2$$

Videli bomo da je $\vec{v}$ $\overset{\text{pot.}}{\swarrow}$ vekt. polje

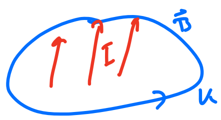
T $\vec{v} : \mathbb{R}^3 \to \mathbb{R}^3$ je potencialno vekt. polje, če $\exists\, f : \mathbb{R}^3 \to \mathbb{R}$, da je

$$\vec{v} = \operatorname{grad} f = \nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

primer pot. polja

($f(x,y,z) = x^2 + y^2 + z \;\to\; \operatorname{grad} f = (2x, 2y, 1)$)

Opomba: krivuljni integral vekt. polja



$\int_K \vec{B} \cdot d\vec{s} =$ količina toka ki jo obdaja

Naj bo $\vec{v}$ pot. polje, $\vec{v} = \operatorname{grad} f$ in K krivulja od točke $A \in \mathbb{R}^3$ do $B \in \mathbb{R}^3$.

$$\int_K \vec{v} \cdot d\vec{s} = \int_a^b \vec{v}(\vec{r}(t))\,\dot{\vec{r}}(t)\,dt = \int_a^b \operatorname{grad} f(\vec{r}(t))\,\dot{\vec{r}}(t)\,dt =$$
$$= \int_a^b \frac{\partial f}{\partial x}\dot{x} + \frac{\partial f}{\partial y}\dot{y} + \frac{\partial f}{\partial z}\dot{z}\,dt = \int_a^b \frac{d}{dt}\big(f(\vec{r}(t))\big)\,dt = f(\vec{r}(b)) - f(\vec{r}(a))$$

$$\boxed{\int_K \operatorname{grad} f \cdot d\vec{s} = f(B) - f(A)}$$

Rešimo nalogo $\vec{v} = (2xz, z^2, 2yz + x^2)$

Ali je $\vec{v}$ pot. polje?

Ali $\exists\, f$ da $\vec{v} = \operatorname{grad} f$?

f more biti takšna, da velja

$$\frac{\partial f}{\partial x} = 2xz \qquad \frac{\partial f}{\partial y} = z^2 \qquad \frac{\partial f}{\partial z} = 2yz + x^2$$

$$f = x^2 z + C \qquad f = yz^2 + C \qquad f = yz^2 + zx^2 + C$$

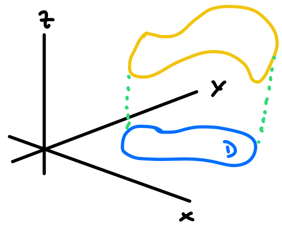
$$\Rightarrow \quad f = x^2 z + yz^2 + C$$

Kako lahko vnaprej presodimo, da je ta sistem rešljiv? $\quad \operatorname{rot} \vec{v} = 0$

Posledice: $\vec{v} = \text{grad}\, f$ in K sklenjena krivulja

$$\oint_K \vec{v} \cdot d\vec{s} = 0$$

Ploskve v $\mathbb{R}^3$



① $f : D \subseteq \mathbb{R}^2 \to \mathbb{R}$ graf
$(x,y) \to f(x,y)$
Eksplicitno dana ploskev

② $f : \mathbb{R}^3 \to \mathbb{R}$
$P = \{ (x,y,z) \in \mathbb{R}^3 \mid f(x,y,z) = 0 \}$
Implicitno podana ploskev

③ $\vec{r} : D \subseteq \mathbb{R}^2 \to \mathbb{R}^3$
$(u,v) \longmapsto \vec{r}(u,v) = (x(u,v), y(u,v), z(u,v))$
Parametrično dana ploskev
($\vec{r}$ parametrizacija ploskve

Preostanek teorije na strani 33.

Koeficienti 1. fundamentalne forme

$$E = \vec{r}_u \cdot \vec{r}_u \qquad F = \vec{r}_u \cdot \vec{r}_v \qquad G = \vec{r}_v \cdot \vec{r}_v$$

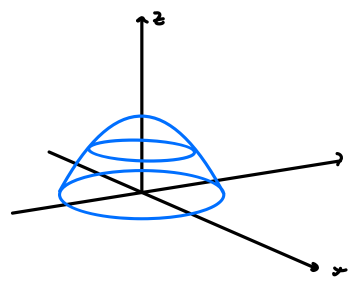
Opomba $|\vec{r}_u| = \sqrt{E}$ $\quad |\vec{r}_v| = \sqrt{G}$

$$\cos\varphi = \frac{F}{\sqrt{EG}}$$

Površina ploskve $\quad P = \iint_D \sqrt{EG - F^2}\, du\, dv$

$$dP = |\vec{r}_u du \times \vec{r}_v dv| = |\vec{r}_u \times \vec{r}_v| dv\, du = \sqrt{EG - F^2}\, du\, dv$$

13 Izračunaj površino dela ploskve $x^2 + y^2 + z = 1$, ki leži nad xy ravnino.



Paraboloid

$z = 1 - x^2 - y^2 \qquad x^2 + y^2 \leq 1$

- $\vec{r}(u,v) = (u, v, 1 - u^2 - v^2)$
  $(u,v) \in \{ u^2 + v^2 \leq 1 \}$

- $\vec{r}(\varphi, r) = (r\cos\varphi, r\sin\varphi, 1 - r^2)$
  $\varphi \in [0, 2\pi) \qquad r \in [0, 1]$

$\vec{r}_\varphi = (-r\sin\varphi, r\cos\varphi)$
$\vec{r}_r = (\cos\varphi, \sin\varphi, -2r)$
$E = r^2 \qquad F = 0 \qquad G = 1 + 4r^2$

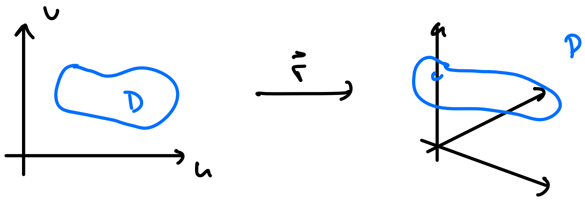
$$P = \int_0^{2\pi} d\varphi \int_0^1 \sqrt{r^2(1+4r^2) - 0^2}\, dr = 2\pi \int_0^1 r\sqrt{1+4r^2}\, dr =$$

$$= \frac{2\pi}{8} \int_1^5 \sqrt{u}\, du = \frac{\pi}{4} \frac{2}{3} t^{3/2} \Big|_1^5 = \frac{\pi}{6} (5^{3/2} - 1)$$

$u = 1 + 4r^2$
$du = 8r\, dr$

$$= \frac{\pi}{6} (5\sqrt{5} - 1)$$

T Ploskovni integral skalarnega polja (funkcije)



$f: \mathbb{R}^3 \to \mathbb{R}$

$$\iint_P f\,dS = \iint_D f(\vec r(u,v))\sqrt{EG-F^2}\,du\,dv$$

motivacija f ... gostota $\to$ $\iint_P f\,dS$ = masa

Uporaba

ρ ... gostota materiala P $\qquad \iint_P \rho\,dS$ = masa P = m

$$\bar x = \frac{1}{m}\iint_P x\,\rho\,dS \qquad \bar y = \frac{1}{m}\iint_P y\,\rho\,dS \qquad \bar z = \frac{1}{m}\iint_P z\,dS$$

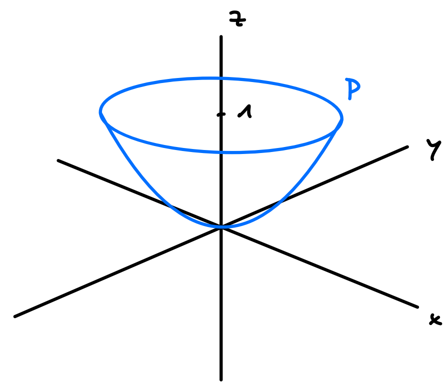
Vztrajnostni moment

$$J_z = \iint_P (x^2+y^2)\,\rho\,dS$$

14 Izračunaj m, $\bar x, \bar y, \bar z$ in J_z za ploskev P

$$P = \{(x,y,z)\in\mathbb{R}^3,\ z = x^2+y^2 \text{ in } z \le 1\}$$

$$\rho(x,y,z) = x^2+y^2$$



$$\vec r(\varphi, r) = (r\cos\varphi,\ r\sin\varphi,\ r^2)$$
$\varphi \in [0, 2\pi)$
$r \in [0, 1]$

$\vec r_\varphi = (-r\sin\varphi,\ r\cos\varphi,\ 0)$ $\qquad E = \vec r_\varphi \cdot \vec r_\varphi = r^2$
$\vec r_r = (\cos\varphi,\ \sin\varphi,\ 2r)$ $\qquad F = \vec r_\varphi \vec r_r = 0$
$\qquad G = \vec r_r \cdot \vec r_r = 1 + 4r^2$

$$\sqrt{EG-F^2} = \sqrt{r^2(1+4r^2)} = r\sqrt{1+4r^2}$$

$$m = \iint_P \rho\,dS = \int_0^{2\pi}\int_0^1 r^2\, r\sqrt{1+4r^2}\,dr\,d\varphi = 2\pi\int_0^1 r^3\sqrt{1+4r^2}\,dr =$$

$u = 4r^2 \to 1$ $\quad r = \frac{\sqrt u}{2}$
$du = 8r\,dr$

$$= 2\pi\int_0^4 \frac{1}{8}u^{3/2}\sqrt{1+u}\ \frac{1}{4}u^{-\frac12}\,du = \frac{\pi}{16}\int_0^4 u\sqrt{1+u}\,du$$

$$= \frac{\pi}{16}\int_1^5 (t-1)\sqrt t\,dt = \frac{\pi}{16}\int_1^5 (t^{3/2} - t^{1/2})\,dt = \dots =$$

$\bar x = \bar y = 0$

$$\bar z = \frac{1}{m}\int_0^{2\pi} d\varphi \int_0^1 r^2\, r^2\, r\sqrt{1+4r^2}\,dr = \frac{2\pi}{m}\int_0^1 r^5\sqrt{1+4r^2}\,dr$$

$2r = \operatorname{sh} t \qquad r = \frac{\operatorname{sh} t}{2}$

$2\,dr = \frac12 \operatorname{ch} t\, dt$

$$= \frac{2\pi}{m}\int_0^{\operatorname{arsh}(2)} \frac12\, \frac{\operatorname{sh}^5 t}{32}\, \operatorname{ch}t\,\operatorname{ch}t\,dt = \frac{2\pi}{64m}\int_0^{\operatorname{arsh}2}\operatorname{sh}^5t\,\operatorname{ch}^2t\,dt =$$

$$= \frac{\pi}{32m}\int_0^{\operatorname{arsh}2}\operatorname{sh}t\,(\operatorname{ch}^2t-1)^2\operatorname{ch}^2t\,dt =$$

$u = \operatorname{ch} t \qquad \operatorname{ch}(\operatorname{arsh}(2)) =$

$du = \operatorname{sh} t\, dt$

$$= \frac{\pi}{32}\int_1^{\sqrt5}(u^4-2u^2+1)\,u^2\,du$$

$$= \frac{\pi}{32}\int_1^{\sqrt5} u^6-2u^4+u^2\,du = \dots$$

$$J_z = \int_0^{2\pi} d\varphi\int_0^1 r^2\, r^2\, r\sqrt{1+4r^2}\,dr = \bar z \text{ masa}$$

$$J_x = \int_0^{2\pi}d\varphi\int_0^1 (r^2\sin^2\varphi + r^4)\, r^2\, r\sqrt{1+4r^2}\,dr =$$

$$= \int_0^{2\pi}d\varphi\int_0^1 r^5\sin^2\varphi\sqrt{1+4r^2} + r^7\sqrt{1+4r^2}\,dr =$$

$t = 1+4r^2 \qquad r^2 = \frac{t-1}{4}$

$dt = 8r\,dr \qquad r\,dr = \frac{dt}{8}$

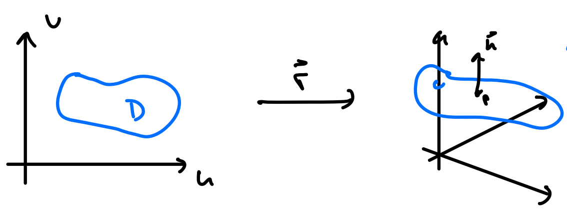
$$= \int_0^{2\pi}d\varphi\int_1^5 \left(\frac{t-1}{4}\right)^2\sin^2\varphi\sqrt t + \left(\frac{t-1}{4}\right)^3\sqrt t\,\frac{dt}{8} =$$

$$= \frac18\int_0^{2\pi}\sin^2\varphi\, d\varphi\int_1^5\left(\frac{t-1}{4}\right)^2\sqrt t\, dt + \int_0^{2\pi}\frac18\int_1^5\left(\frac{t-1}{4}\right)^3\sqrt t\, dt$$

$$= \frac18\int_0^{2\pi}\frac12-\frac{\cos2\varphi}{2}\,d\varphi\int_1^5\frac1{16}\left(t^{5/2}-2t^{3/2}+t^{1/2}\right)dt + \frac18\, 2\pi\int_1^5\frac{t^{7/2}-3t^{5/2}+3^{3/2}-t^{1/2}}{64}\,dt =$$

$$= \frac1{16}\left(\varphi - \frac12\sin2\varphi\right)\Big|_0^{2\pi}\frac1{16}\left(\frac27t^{7/2}-\frac45t^{5/2}+\frac23t^{7/2}\right)\Big|_1^5 + \frac{2\pi}{8\cdot64}\left(\frac29t^{9/2}-\frac67t^{7/2}+\frac65t^{5/2}-\frac23t^{3/2}\right)\Big|_1^5 = \dots$$

(T) Ploskovni integral vektorskega polja



tok tekočine

$\vec v : \mathbb{R}^3 \to \mathbb{R}^3$ vektorsko polje

Orientacija ploskve P $\Rightarrow$ zvezna preslikava $\vec n : P \to \mathbb{R}^3$

$p \longmapsto \vec n(p)$ enotski vektor k na ploskev P.

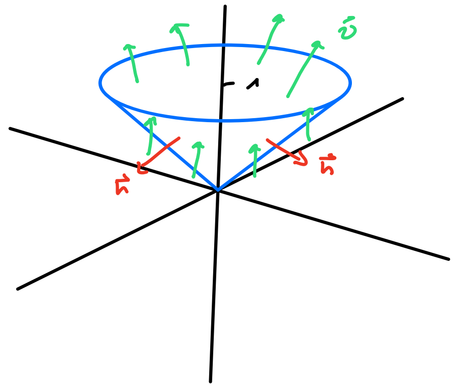
$$\vec n = \frac{\vec r_u\times\vec r_v}{|\vec r_u\times\vec r_v|}$$

$$\iint_P \vec v\cdot d\vec S = \iint_P \vec v\cdot\vec n\,dS = \iint_D \vec v(\vec r(u,v))\cdot\frac{\vec r_u\times\vec r_v}{|\vec r_u\times\vec r_v|}\sqrt{EG-F^2}\,du\,dv$$

$$\iint_P\vec v\cdot d\vec S = \iint_D \vec v(\vec r(u,v))\cdot(\vec r_u\times\vec r_v)\,du\,dv$$

(16) $P = \{(x,y,z) \in \mathbb{R}^3 ; z^2 = x^2+y^2 ; 0 \le z \le 1\}$, orientiramo v "smeri navzdol"

$v = (x, y, 2z)$



Parametrizacija

$\vec{r}(\varphi, r) = (r\cos\varphi, r\sin\varphi, r)$

$r \in [0,1] \qquad \varphi \in [0, 2\pi)$

$\vec{r}_\varphi = (-r\sin\varphi, r\cos\varphi, 0)$

$\vec{r}_r = (\cos\varphi, \sin\varphi, 1)$

$\vec{r}_\varphi \times \vec{r}_r = (r\cos\varphi, r\sin\varphi, -r)$ kaže dol torej se ujema z izbrano orientacijo

$|\vec{r}_\varphi \times \vec{r}_r| = \sqrt{r^2+r^2} = r\sqrt{2}$

$$\iint_P \vec{v}\cdot d\vec{S} = \int_0^{2\pi}\int_0^1 (r\cos\varphi, r\sin\varphi, 2r)\cdot(r\cos\varphi, r\sin\varphi, -r)\, dr\, d\varphi =$$

$$= \int_0^{2\pi}\int_0^1 r^2\cos^2\varphi + r^2\sin^2\varphi - 2r^2 \, dr\, d\varphi = \int_0^{2\pi}\int_0^1 -r^2\, dr\, d\varphi = 2\pi \left. -\frac{r^3}{3}\right|_0^1 = -\frac{2\pi}{3}$$

(T) Gaussov izrek

$$\oiint_{\partial T} \vec{v}\cdot d\vec{S} = \iiint_T \operatorname{div}\vec{v} \cdot dx\,dy\,dz$$

Stokesova formula

$$\oint_{\partial P} \vec{v}\cdot d\vec{s} = \iint_P \operatorname{rot}\vec{v}\cdot d\vec{S}$$

Operacije z vektorskimi polji:

$F(\mathbb{R}^3) = \{ f : \mathbb{R}^3 \to \mathbb{R}$, gladki (∞-krat zvezno parc. odv. po vseh spremen.)

$\mathcal{V}(\mathbb{R}^3) = \{ \vec{v} : \mathbb{R}^3 \to \mathbb{R}^3$, gladka

Gradient $\quad \nabla = \operatorname{grad} : F(\mathbb{R}^3) \to \mathcal{V}(\mathbb{R}^3)$

$$f \to \operatorname{grad} f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

Rotor $\quad \nabla \times = \operatorname{rot} : \mathcal{V}(\mathbb{R}^3) \to \mathcal{V}(\mathbb{R}^3)$

$$\vec{v} = (P, Q, R) \to \operatorname{rot}\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{vmatrix} = (R_y - Q_z, P_z - R_x, Q_x - P_y)$$

Divergenca $\quad \nabla\cdot = \operatorname{div} : \mathcal{V}(\mathbb{R}^3) \to F(\mathbb{R}^3)$

$$\vec{v} = (P, Q, R) \to \operatorname{div}\vec{v} = P_x + Q_y + R_z$$

$$F \xrightarrow{\operatorname{grad}} \mathcal{V} \xrightarrow{\operatorname{rot}} \mathcal{V} \xrightarrow{\operatorname{div}} F$$

$\operatorname{rot}(\operatorname{grad} f) = (0,0,0) \qquad \operatorname{div}(\operatorname{rot}\vec{v}) = 0$

Obratno: $\quad \operatorname{rot}\vec{v} = 0 \Leftrightarrow \vec{v} = \operatorname{grad} f$

$\operatorname{div}\vec{v} = 0 \Leftrightarrow \vec{v} = \operatorname{rot}\vec{A}$

Velja le kadar so $\vec{v}$ definirani na celemu prostoru ali na območjih brez lukenj.

18 $\vec{v} = (x^2 z \,,\, -x y^2 z \,,\, 3 y z^2)$

$$\operatorname{div} \vec{v} = \frac{\partial}{\partial x}(x^2 z) + \frac{\partial}{\partial y}(-x y^2 z) + \frac{\partial}{\partial z}(y z^2 3) = 2xz - 2xyz + 6yz \neq 0$$

20 $\vec{v} = (x z^3 \,,\, -2x^2 y z \,,\, 2 y z^4)$

$$\operatorname{rot} \vec{v} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x z^3 & -2x^2 y z & 2 y z^4 \end{vmatrix} = (2z^4 + 2x^2 y \,,\, 3xz^2 - 0 \,,\, -4xyz - 0)$$

21 Poišči $a, b \in \mathbb{R}$ da bo

$$\vec{v} = (2x^a \sin z \,,\, 3y^b \sin z \,,\, (x^{a+1} + y^{b+1}) \cos z)$$

potencialno in poišči potencial

$\vec{v}$ je potencialno $\Leftrightarrow$ $\vec{v} = \operatorname{grad} f$

Če je $\vec{v} = \operatorname{grad} f \Rightarrow \operatorname{rot} \vec{v} = 0$

$$\operatorname{rot} \vec{v} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ 2x^a \sin z & 3y^b \sin z & (x^{a+1} + y^{b+1}) \cos z \end{vmatrix}$$

$$= ((b+1) y^b \cos z - 3y^b \cos z \,,\, \cos z\, 2x^a - (a+1) x^a \cos z \,,\, 0) = (0, 0, 0)$$

$(b+1) y^b \cos z = 3y^b \cos z$ $\qquad$ $2x^a \cos z = (a+1) x^a \cos z$

$b+1 = 3 \quad b = 2$ $\qquad$ $2 = a+1 \quad a = 1$

$\vec{v} = (2x \sin z \,,\, 3y^2 \sin z \,,\, (x^2 + y^3) \cos z) \Rightarrow \operatorname{rot} \vec{v} = 0$

$\vec{v} : \mathbb{R}^3 \to \mathbb{R}^3$

$\Downarrow$

$\exists f : \mathbb{R}^3 \to \mathbb{R}$

$\vec{v} = \operatorname{grad} f$

Poiščimo f

$$\frac{\partial f}{\partial x} = 2x \sin z \qquad \frac{\partial f}{\partial y} = 3y^2 \sin z \qquad \frac{\partial f}{\partial z} = (x^2 + y^3) \cos z$$

$$f = \frac{2x^2}{2} \sin z + C(z, y) \qquad f = \frac{3y^3}{3} \sin z + C(x, z) \qquad f = (x^2 + y^3) \sin z + C(x, y)$$

$$f(x, y, z) = (x^2 + y^3) \sin z + C \qquad C \in \mathbb{R}$$

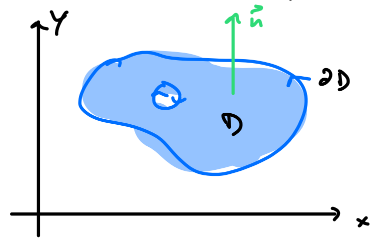
Primer: ko $\operatorname{rot} \vec{v} \neq 0$ a vseeno iščemo f, ki $\vec{v} = \operatorname{grad} f$

$\vec{v} = (2y, x, 2z)$

$$\frac{\partial f}{\partial x} = 2y \qquad \frac{\partial f}{\partial y} = x \qquad \frac{\partial f}{\partial z} = 2z$$

$$f = 2yx \neq f = xy \qquad f = z^2$$

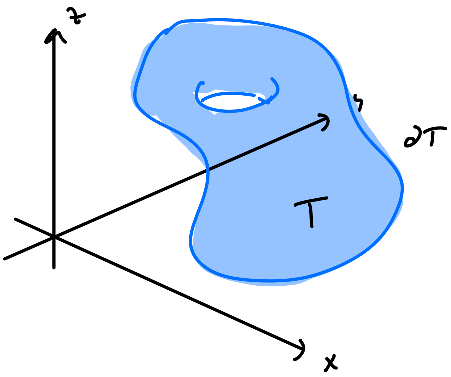
Greenova formula, Gaussov in Stokesov izrek



$\vec v = (P, Q)$

Greenova formula

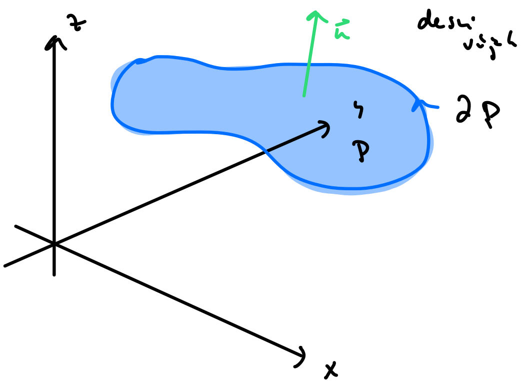
$$\oint_{\partial D} \vec v\, d\vec s = \iint_D (Q_x - P_y)\, dx\, dy$$



$\vec v = (P, Q, R)$

Gaussov izrek

$$\oiint_{\partial T} \vec v \cdot d\vec S = \iiint_T \operatorname{div} \vec v\; dx\, dy\, dz$$



Stokesov izrek

$$\oint_{\partial P} \vec v\, d\vec s = \iint_P \operatorname{rot} \vec v\; d\vec S$$

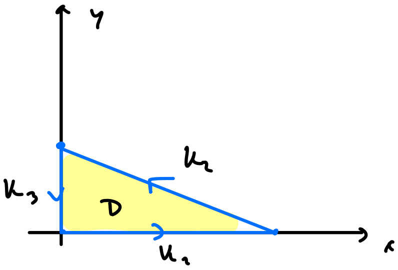
$$j\left(\begin{smallmatrix}\text{končna}\\ \text{točka}\end{smallmatrix}\right) - j\left(\begin{smallmatrix}\text{začetna}\\ \text{točka}\end{smallmatrix}\right) = \int_K \operatorname{grad} f\; d\vec s$$

$$\oint_{\partial P} \vec v\, d\vec s = \iint_P \operatorname{rot} \vec v\; d\vec S$$

$$\oiint_{\partial T} \vec v \cdot d\vec S = \iiint_T \operatorname{div} \vec v\; dx\, dy\, dz$$

Posplošitve osnovnega izreka analize

22. Krivulja K naj bo rob trikotnika z oglišči $(0,0)$, $(2,0)$, $(0,1)$ (poz. orientirana). Dano naj bo polje $\vec{v} = (-\frac{y}{2}, \frac{x}{2})$. Izračunaj $\int_K \vec{v}\cdot d\vec{s}$ direktno in s pomočjo Greenove formule.



$K_1: \vec{r}_1(t) = (t,0) \quad t\in[0,2]$
$K_2: \vec{r}_2(t) = (2,0) + t(-2,1) = (2-2t, t) \quad t\in(0,1)$
$K_3: \vec{r}_3(t) = (0, 1-t) \quad t\in[0,1]$

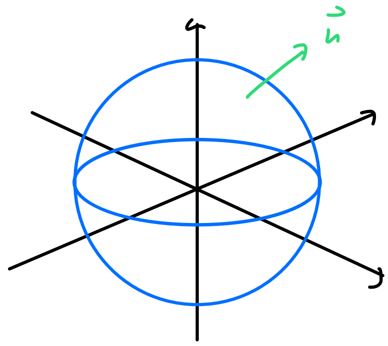
$$\int_K \vec{v}\cdot d\vec{s} = \int_{K_1} \vec{v}\cdot d\vec{s} + \int_{K_2} \vec{v}\cdot d\vec{s} + \int_{K_3} \vec{v}\cdot d\vec{s} = \int_0^2 (0,\tfrac{t}{2})(1,0)\,dt + \int_0^1 (-\tfrac{t}{2}, \tfrac{2-2t}{2})(-2,1)\,dt +$$
$$+ \int_0^1 (-\tfrac{1-t}{2}, 0)(0,-1)\,dt = \int_0^1 t + 1 - t\,dt = 1$$

Po Greenovi formuli:

$$\int_K \vec{v}\cdot d\vec{s} = \iint_D (\tfrac{1}{2} - (-\tfrac{1}{2}))\,dx\,dy = \iint_D dx\,dy = \text{ploščina}(D) = 1$$

23. $\vec{v} = (x(y^2+z^2), y(x^2+z^2), z(x^2+y^2))$ in standardna enotska sfera S^2, orientirana z zunanjo normalo. Izračunaj

$$\oint\!\!\oint_{S^2} \vec{v}\cdot d\vec{S} = \underset{\text{enotska krogla}}{\iiint_B} \operatorname{div}\vec{v}\,dx\,dy\,dz$$

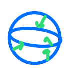


$$= \iiint_B (y^2+z^2+x^2+z^2+x^2+y^2)\,dx\,dy\,dz = 2\iiint_B (x^2+y^2+z^2)\,dV =$$

$$\overset{\text{sferične koord.}}{=} 2\int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \int_0^1 dr\, r^2 r^2 \sin\theta = 2\cdot 2\pi \cdot \frac{1}{5}\cdot 2 = \frac{8\pi}{5}$$

Direkten izračun

$S^2: \vec{r}(\varphi,\theta) = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta) \qquad \varphi\in(0,2\pi) \qquad \theta\in[0,\pi)$
$\vec{r}_\varphi = (-\sin\theta\sin\varphi, \sin\theta\cos\varphi, 0)$
$\vec{r}_\theta = (\cos\theta\cos\varphi, \cos\theta\sin\varphi, -\sin\theta)$
$\vec{r}_\varphi\times\vec{r}_\theta = (-\sin^2\theta\cos\varphi, -\sin^2\theta\sin\varphi, -\sin\theta\cos\theta)$



$$\iint_P \vec{v}\cdot d\vec{S} = +\iint_{(u,v)\in D} \vec{v}(\vec{r}(u,v))(\vec{r}_u\times\vec{r}_v)$$

če $\vec{n}\uparrow\uparrow\vec{r}_u\times\vec{r}_v$

$$\iint_{S^2} \vec{v}\cdot d\vec{S} = -\iint_D \big(\sin\theta\cos\varphi(\sin^2\theta\sin^2\varphi+\cos^2\theta), \sin\theta\sin\varphi(\cos^2\theta\cos^2\varphi+\cos^2\theta), \cos\theta\sin^2\varphi\big)\cdot$$
$$(-\sin^2\theta\cos\varphi, -\sin^2\theta\sin\varphi, -\sin\theta\cos\theta)\,d\varphi\,d\theta$$

$$= \int_0^{2\pi}\!\!\int_0^{\pi} \sin^3\theta\cos^2\theta(\sin^2\theta\sin^2\varphi+\cos^2\theta) + \sin^3\theta\sin^2\varphi(\sin^2\theta\cos^2\varphi+\cos^2\theta) + \sin^3\theta\cos^2\theta\,d\varphi\,d\theta$$

Napaka

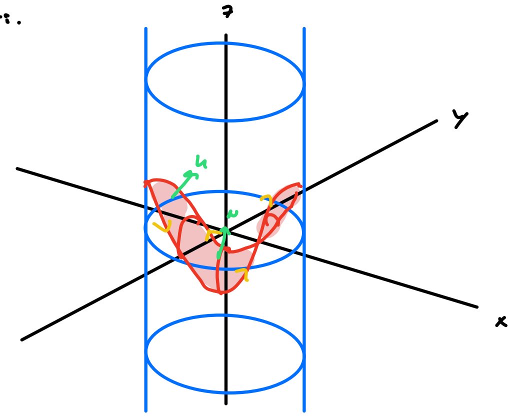
$$= \int_0^{2\pi}\int_0^{\pi} 2\sin^3\theta\cos^2\theta + \int_0^{2\pi}\sin^2\varphi\,d\varphi\int_0^{\pi}\sin^5\theta\,d\theta =$$

$$= 2\pi\cdot 2\cdot 2\int_0^{\pi/2}\sin^3\theta\cos^2\theta\,d\theta + 4\int_0^{\pi/2}\sin^2\varphi\,d\varphi\cdot 2\int_0^{\pi/2}\sin^5\theta\,d\theta =$$

$$= 4\pi B(2,\tfrac{3}{2}) + 2B(\tfrac{3}{2},\tfrac{1}{2})\cdot B(3,\tfrac{1}{2}) = 4\pi\frac{4}{15} + \pi\frac{16}{15} = \frac{32\pi}{15}$$

23 Dano vekt. polje $v=\left(e^{x^2}+xy,\ z+\sin^5(y^5),\ e^{z^2}+2x+y+\cos z\right)$

Izračunaj
$$\oint_K \vec{v}\cdot d\vec{s}\ ,$$
kjer je K presek ploskev $x^2+y^2=1$ in $z=xy$ orientiran pa je v pozitivni smeri gledano z vrha z-osi.



$$\oint_K \vec{v}\, d\vec{s} = \iint_P \mathrm{rot}\,\vec{v}\, d\vec{S}$$

P je taka, da je K njen rob

en možen kandidat (del graf funkcije $z=xy$)

Orientacija P ?
normala kaže gor

Parametrizacija ploskve

$\vec{r}(\varphi, r) = (r\cos\varphi,\ r\sin\varphi,\ r^2\sin\varphi\cos\varphi)$ $\quad \varphi\in[0,2\pi)$ $\quad r\in[0,1]$

$\vec{r}_\varphi = (-r\sin\varphi,\ r\cos\varphi,\ r^2(\cos^2\varphi-\sin^2\varphi))$

$\vec{r}_r = (\cos\varphi,\ \sin\varphi,\ 2r\sin\varphi\cos\varphi)$

$\vec{r}_\varphi\times\vec{r}_r = \big(2r^2\sin\varphi\cos^2\varphi - r^2\sin(\cos^2\varphi-\sin^2\varphi),$
$r^2\cos\varphi(\cos^2\varphi-\sin^2\varphi)+2r^2\sin^2\varphi\cos\varphi,$
$-r\sin^2\varphi - r\cos^2\varphi\big) = (r^2\sin\varphi,\ r^2\cos\varphi,\ -r)$

kaže dol, v nasprotni smeri od orientacije ploskve.

$v=\left(e^{x^2}+xy,\ z+\sin^5(y^5),\ e^{z^2}+2x+y+\cos z\right)$

$$\mathrm{rot}\ v = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ e^{x^2}+xy & z+\sin^5(y^5) & e^{z^2}+2x+y+\cos z \end{vmatrix} = (1-1,\ 0-2,\ 0-x) = (0,-2,-x)$$

$$\oint_K \vec{v}\cdot d\vec{s} = \iint_P \mathrm{rot}\,\vec{v}\, d\vec{S} = -\int_0^{2\pi}\int_0^1 (0,-2,-r\cos\varphi)\cdot(r^2\sin\varphi,\ r^2\cos\varphi,\ -r)\, d\varphi\, dr =$$
$$= \int_0^{2\pi}\int_0^1 2r^2\cos\varphi - r^2\cos\varphi = \int_0^1 r^2\,dr\int_0^{2\pi}\cos\varphi\, d\varphi = 0$$

Diferencialne enačbe

(navadne, ne parcialne)

$y(x) = ?$ $\qquad F(x, y, y', \dots, y^{(n)}) = 0$

Po tipih diferencialnih enačb

(1) Dif. en. z ločljivimi spremenljivkami

$$g(y)\,y' = f(x) \Rightarrow \int g(y)\,dy = \int f(x)\,dx \Rightarrow G(y) = F(x) + C$$

Rešitev enačbe zapišemo v implicitni obliki

(1) Poišči rešitev

$$y^2 = xyy' + 9 \qquad y(1) = -2$$

$$y^2 - 9 = xyy'$$

$$\frac{yy'}{y^2-9} = \frac{1}{x}$$

$$\int \frac{y}{y^2-9}\,dy = \int \frac{dx}{x}$$

$$\frac{1}{2}\ln|y^2-9| = \ln|x| + c$$

$$\ln|y^2-9| = 2\ln|x| + c$$

$$y^2 - 9 = e^{2\ln|x|}e^c$$

$$y^2 = De^{\ln|x|^2} + 9$$

$$y = \pm\sqrt{Dx^2+9} \qquad D \in \mathbb{R}$$

$$y(1) = -2 \qquad -2 = -\sqrt{D\cdot 1 + 9}$$

$$4 = D + 9$$

$$D = -5$$

Končna rešitev $\quad y = -\sqrt{9-5x^2}$

(2) Homogene dif. en.

$$y' = f\left(\frac{y}{x}\right)$$

uvedemo novo funkcijo

$$z(x) = \frac{y(x)}{x} \qquad y = xz$$

$$y' = z + xz'$$

$$z + xz' = f(z)$$

$$xz' = f(z) - z$$

$$\frac{z'}{f(z)-z} = \frac{1}{x}$$

$\Downarrow$

rešimo $z(x, c)$

$\Downarrow$

$$y(x) = x\,z(x, c)$$

② Poišči rešitev

$$xy' - y = x\tan\left(\frac{y}{x}\right) \qquad y(3) = 8\pi$$

$$y' = \tan\frac{y}{x} + \frac{y}{x} \qquad z = \frac{y}{x} \qquad y = xz$$

$$z + xz' = \tan z + z$$

$$\frac{z'}{\tan z} = \frac{1}{x}$$

$$\int \frac{dz}{\tan z} = \ln|x| + C$$

$$\int \frac{\cos z \, dz}{\sin z} = \ln|x| + C$$

$$\ln|\sin z| = \ln|x| + C$$

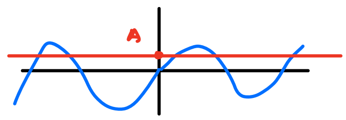
$$|\sin z| = e^{C}|x|$$

$$\sin z = \pm e^{C} x$$

$$\sin z = D x$$

$$\sin\frac{y}{x} = D x$$

$\sin x = A$



$$x = \arcsin A + 2k\pi \qquad k \in \mathbb{Z}$$

$$x = \pi - \arcsin A + 2k\pi$$

$$y = x(\arcsin Dx + 2k\pi) \qquad k \in \mathbb{Z}$$

$$y = x(\pi - \arcsin Dx + 2k\pi) \qquad D \in \mathbb{R}$$

$$y = x(\arcsin Dx + k\pi)$$

$$y(3) = 8\pi$$

$$8\pi = 3(\underbrace{\arcsin 3D}_{[-\frac{\pi}{2}, \frac{\pi}{2}]} + k\pi)$$

$k=0$: $\quad 8\pi = 3\frac{\pi}{2}$

$k=1$: $\quad 8\pi = 3\left(\frac{\pi}{2} + \pi\right)$

$k=2$: $\quad 8\pi = 3\left(\frac{\pi}{2} + 2\pi\right)$

$k=3$: $\quad 8\pi = 3(\arcsin 3D + 3\pi)$

$$\arcsin 3D = -\frac{\pi}{3}$$

$$3D = -\frac{\sqrt{3}}{2}$$

$$D = -\frac{\sqrt{3}}{6} \qquad y = x\left(\arcsin\left(-\frac{\sqrt{3}}{6}x\right) + 9\pi\right)$$

(7) Lin. dif. en.

$$p(x)\,y' + g(x)\,y = r(x)$$

(i) Rešimo homogeno enačbo

$$py' + gy = 0$$

$$\frac{y'}{y} = -\frac{g}{p} \rightarrow \text{Integriramo} \Rightarrow y = De^{-\int \frac{g}{p}dx} = D\,\varphi(x)$$

(ii) Partikularna rešitev s pomočjo variacije konstante

$$y_p = D(x)\,\varphi(x) \rightarrow D(x) = ? \rightarrow py' + gy = r$$

$$p(D'\varphi + D\varphi') + gD\varphi = r$$

$$pD'\varphi + D\underbrace{(p\varphi' + g\varphi)}_{0} = r \qquad D'(x) = \frac{r(x)}{p(x)\,\varphi(x)} \xrightarrow{\int} D(x)$$

Splošna rešitev $y = D\,\varphi(x) + D(x)\varphi(x)$

(8) $xy' - 3y = x$

(i) $xy' - 3y = 0$

$$\frac{y'}{y} = \frac{3}{x} \qquad \ln|y| = 3\ln|x| + C$$

$$|y| = e^C e^{3\ln|x|}$$

$$y = \pm e^C x^3$$

$$y = Dx^3 \qquad y_H = Dx^3 \quad D \in \mathbb{R}$$

(ii) $y_p = D(x)\,x^3 \rightarrow xy' - 3y = x$

$$x(D'x^3 + D\,3x^2) - 3Dx^3 = x$$

$$D'x^4 + 3Dx^3 - 3Dx^3 = x$$

$$D' = x^{-3} \qquad D = \int x^{-3}dx = -\frac{1}{2}x^{-2} + \cancel{C}$$

je že v homogenem delu

$$y_p = x^3 \cdot \left(-\frac{1}{2}x^{-2}\right) = -\frac{1}{2}x$$

$$y = Dx^3 - \frac{x}{2} \qquad \text{splošna rešitev}$$

(4) Bernoullijeva dif. en.

$$p(x)\,y' + g(x)\,y = r(x)\,y^{\alpha} \qquad \alpha \in \mathbb{R}$$

↳ $\alpha = 1 \Rightarrow$ homogena lin. d.e.

↳ $\alpha = 0 \Rightarrow$ lin. d.e.

$\downarrow : y^{\alpha}$

$$p(x)\,y'\,y^{-\alpha} + g(x)\,y^{1-\alpha} = r(x)$$

uvedemo novo funkcijo $z = y^{1-\alpha}$ $\qquad z' = (1-\alpha)\,y^{-\alpha}\,y'$

$$\frac{p(x)}{(1-\alpha)}\,z' + g(x)\,z = r(x) \quad \Rightarrow \text{ lin. d.e. za } z$$

$\Downarrow$

poiščemo splošno rešitev $z(x, c)$

$$\Rightarrow y = z^{\frac{1}{1-\alpha}}$$

(9)

$$3y' + 2y = (1+3e^{x})\,y^{4} \qquad /: y^{4}$$

$$3y'y^{-4} + 2y^{-3} = (1+3e^{x}) \qquad y^{-3} = z \qquad z' = -3y^{-4}y'$$

$$-z' + 2z = 1 + 3e^{x} \qquad \text{lin d.e.}$$

(i) Homogena rešitev

$$z' = 2z \qquad \frac{z'}{z} = 2$$

$$\frac{dz}{z} = 2\,dx$$

$$\ln|z| = 2x + C$$

$$z_H = De^{2x}$$

(ii) Partikularna rešitev

$$z_p = D(x)\,e^{2x} \longrightarrow -z' + 2z = 1 + 3e^{x}$$

$$-(D'e^{2x} + 2De^{2x}) + 2De^{2x} = 1 + 3e^{x}$$

$$-D'e^{2x} = 1 + 3e^{x}$$

$$D' = -e^{-2x} - 3e^{-x}$$

$$D(x) = \frac{1}{2}e^{-2x} + 3e^{-x} \qquad \text{ni treba konstante}$$

$$z_p = \left(\frac{1}{2}e^{-2x} + 3e^{-x}\right)e^{2x} = \frac{1}{2} + 3e^{x}$$

Splošna rešitev $\quad z = De^{2x} + \frac{1}{2} + 3e^{x} \qquad D \in \mathbb{R}$

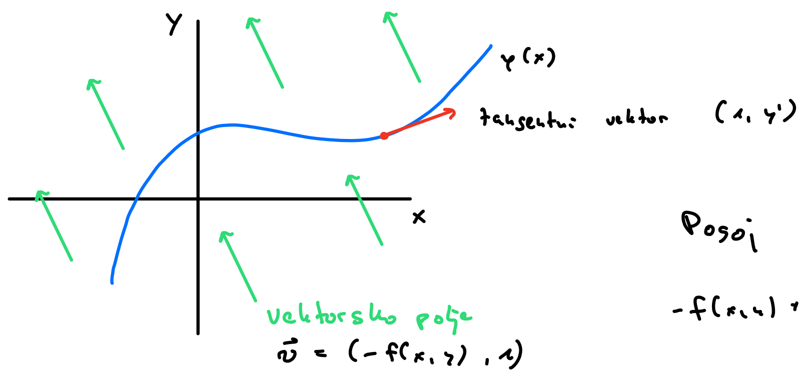
$$y = z^{-\frac{1}{3}}$$

$$y = \frac{1}{\sqrt[3]{De^{2x} + \frac{1}{2} + 3e^{x}}} \qquad D \in \mathbb{R}$$

(5) Eksaktne D.E.

Tipične DE

$$y' = f(x,y) \Rightarrow -f(x,y) + y' = 0$$



Pogoj

$$-f(x,y) + y' = 0 \iff \vec{v} \cdot (1, y') = 0$$

Torej DE lahko interpretiramo na naslednji (geometrijski) način: Dano je vektorsko polje $\vec{v}$ ⇒ iščemo krivuljo, ki je v vsaki točki pravokotna na $\vec{v}$.

⇒ $\vec{v} = (P(x,y), Q(x,y))$ iščemo krivulje, ki je $\perp$ na $\vec{v}$.

$$(P, Q)\,(1, y') = 0 \qquad P + Q y' = 0$$

$$y' = \frac{dy}{dx} \qquad P\,dx + Q\,dy = 0 \qquad \text{Eksaktna dif. en.}$$

Ali obstaja kakšen eksakten način, da rešimo to enačbo?

Spomnimo se, da če imamo $u(x,y)$ ⇒ grad u je $\perp$ na nivojnico $u(x,y) = C$

$(P, Q) = \text{grad}\, u$ ⇒ rešitev d.e. $P\,dx + Q\,dy = 0$ ravno nivojnice (implicitno)

$$u(x,y) = C$$

Če velja $(P, Q) = \text{grad}\, u \Rightarrow \text{rot}(P,Q) = 0$

$$\text{rot}\,(P, Q, 0) = 0$$

$$\text{rot}\,(P,Q) = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ P & Q & 0 \end{vmatrix} = (0, 0, Q_x - P_y)$$

Pogoj ⇒ $Q_x = P_y$ ⇒ $(P,Q) = \text{grad}\, u$

$$\Downarrow$$
$$u$$
$$\Downarrow$$
$$u(x,y) = C$$

Opomba: kaj storiti, če $Q_x \neq P_y$ (kasneje)

(12)

$$\underbrace{(2x^3 - y)}_{P}\,dx + \underbrace{(-x-y)}_{Q}\,dy = 0$$

$$Q_x = P_y$$

$$-1 = -1 \quad ✓ \quad \Rightarrow \exists u;\ \text{grad}\, u = (P,Q)$$

Veljati mora

$$\frac{\partial u}{\partial x} = 2x^3 - y \qquad \frac{\partial u}{\partial y} = -x - y$$

$$u = \frac{1}{2}x^4 - yx + C(y) \qquad u = -xy - \frac{y^2}{2} + c(x)$$

$$u = -xy + \frac{1}{2}x^4 - \frac{y^2}{2} + c \qquad c \in \mathbb{R}$$

$\Rightarrow$ rešitev DE je poljubna nivojnica funkcije u

$$\frac{1}{2}x^4 - xy - \frac{y^2}{2} = C_1 \qquad C_1 \in \mathbb{R}$$

Rešitev je zapisana v implicitni obliki

(T) Integrirajoči množitelj

$$P(x,y)\,dx + Q(x,y)\,dy = 0$$

Če $Q_x = P_y$

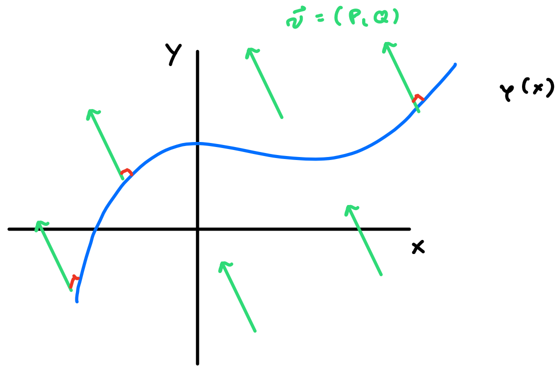
$\Downarrow$

$\exists u \quad \text{grad}\, u = (P,Q)$

$\Downarrow$

rešitev $u(x,y) = C_1$

Če $Q_x \neq P_y$



Naj bo $\eta(x,y)$ poljubna neničelna funkcija

$\vec{v}$ pomnožimo z η

$$\eta(x,y)(P,Q) = (\eta P, \eta Q)$$

Smer $(\eta P, \eta Q)$ se ohrani, le dolžina se spremenijo

$\Rightarrow$ rešitve d.e. $\eta P\,dx + \eta Q\,dy = 0$ so enake kot za $P\,dx + Q\,dy = 0$

$\Downarrow$

Če $P_y \neq Q_x$

$\Downarrow$

poskusimo najti $\eta(x,y)$ ← integrirajoči množitelj

Rešitev $u(x,y) = C \Leftarrow \exists u,\ \text{grad}\, u = (\eta P, \eta Q) \Leftarrow \frac{\partial}{\partial y}(\eta P) = \frac{\partial}{\partial x}(\eta Q)$

13 $\left(3\ln(x+y) + \frac{x}{x+y}\right)dx + \left(\frac{x}{x+y}\right)dy = 0$

pri čemer je int. množitelj

$$\eta(x,y) = x^p \quad (p \text{ neka potenca})$$

$P = 3\ln(x+y) + \frac{x}{x+y}$ $\qquad$ $Q = \frac{x}{x+y}$

$P_y = \frac{3}{x+y} - \frac{x}{(x+y)^2} = \frac{2x+3y}{(x+y)^2}$ $\qquad$ $Q_x = \frac{x+y-x}{(x+y)^2} = \frac{y}{(x+y)^2}$

$(\eta P)_y = \eta_y P + \eta P_y$ $\qquad$ $(\eta Q)_x = \eta_x Q + \eta Q_x$

$$= x^p \frac{2x+3y}{(x+y)^2} \quad = \quad = p x^{p-1} \frac{x}{x+y} + x^p \frac{y}{(x+y)^2}$$

$$x^p \frac{2x+3y}{(x+y)^2} = x^p \frac{p(x+y)}{(x+y)^2} + x^p \frac{y}{(x+y)^2}$$

$$2x + 3y = p(x+y) + y$$

$$p = 2 \Rightarrow \eta = x^2$$

Torej $\exists\, u(x,y)$ $\qquad$ $\operatorname{grad} u = (x^2 P, x^2 Q)$

$\frac{\partial u}{\partial x} = x^2 3\ln(x+y) + \frac{x^3}{x+y}$ $\qquad$ $\frac{\partial u}{\partial y} = \frac{x^3}{x+y}$

$= \frac{d}{dx}\left(x^3 \ln(x+y)\right)$ $\qquad$ $u = \int \frac{x^3}{x+y}\, dy = x^3 \ln|x+y| + c$

$u = x^3 \ln|x+y| + c$ $\qquad$ Lahko bi po tem u odvajali po x in primerjali z desno stranjo pri $\frac{\partial u}{\partial x}$.

Torej $u(x,y) = x^3 \ln(x+y) + C \quad C \in \mathbb{R}$

Splošna rešitev $\quad x^3 \ln(x+y) = C \quad C \in \mathbb{R}$

$$y = e^{\frac{C}{x^3}} - x$$

Opomba $\quad P\,dx + Q\,dy = 0$

$$P + Q y' = 0$$

$$y' = -\frac{P(x,y)}{Q(x,y)}$$

Linearni sistemi DE s konstantnimi koeficienti

n funkcij (t neodvisna sprem.)

$$x_1(t), x_2(t), \dots x_n(t)$$

$$\dot{x}_1 = f_1(t, x_1, \dots, x_n)$$
$$\dot{x}_2 = f_2(t, x_1, \dots, x_n) \qquad \text{Splošen sistem DE}$$
$$\vdots$$
$$\dot{x}_n = f_n(t, x_1, \dots, x_n)$$

$$\downarrow$$

$$\dot{x}_1 = a_{11} x_1 + \dots + a_{1n} x_n + f_1(t)$$
$$\dot{x}_2 = a_{21} x_1 + \dots + a_{2n} x_n + f_2(t) \qquad \text{Linearni sistem DE}$$
$$\vdots$$
$$\dot{x}_n = a_{n1} x_1 + \dots + a_{nn} x_n + f_n(t)$$
$$\|$$
$$\dot{X} = AX + f(t) \qquad \dot{X} = \begin{bmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_n \end{bmatrix} \quad X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \quad A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \quad f(t) = \begin{bmatrix} f_1(t) \\ \vdots \\ f_n(t) \end{bmatrix}$$

Postopek reševanja

① Poiščemo splošno rešitev homogenega sistema $\dot{X} = AX$

$\rightarrow$ rešitev oblike $\quad x(t) = e^{tA} c \qquad c = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$

$$e^{tA} = I + tA + \frac{1}{2!} t^2 A^2 + \dots$$

$$\Rightarrow \quad x_H = \Theta(t)\, c \qquad \Theta = n \times n \text{ matrika}$$

$$\dot{x}_H = \Theta x_H$$

$$\dot{\Theta} c = A \Theta c$$

② Poiščemo partikularno rešitev (uporabimo variacijo konstant)

$$x_p = \Theta(t)\, c(t) \qquad c(t) = \begin{vmatrix} c_1 \\ c_n \end{vmatrix}(t) = ?$$

$$\dot{x}_p = A x_p + f(t)$$

$$\dot{\Theta} c + \Theta \dot{c} = A \Theta c + f$$

$$\Theta \dot{c} = f$$

$$\dot{c} = \Theta^{-1}(t) f(t) \xrightarrow{\int dt} c(t)$$

Splošna rešitev $\quad x = \Theta(t)\, c + \Theta(t)\, c(t)$

$$\downarrow$$
$$c \in \mathbb{R}^n$$

Reševanje $\dot{x} = A x$

① $\dot{x} = A x$ λ lastna vrednost mat. A, v lastni vektor A za λ

$$x = C e^{\lambda t} v$$

$$C \lambda e^{\lambda t} v = A C e^{\lambda t} v = C e^{\lambda t} A v = C e^{\lambda t} \lambda v$$

Postopek

$A \to$ dobimo lastne vrednosti

$\det(A - \lambda I) = 0$

$$\begin{matrix} \lambda_1, \dots & \lambda_n \\ \downarrow & \downarrow \\ v_1, \dots, & v_n \end{matrix}$$

Rešitev $\dot{x} = A x$

$C_i \in \mathbb{R}$

$$x = C_1 e^{\lambda_1 t} v_1 + C_2 e^{\lambda_2 t} v_2 + \dots + C_n e^{\lambda_n t} v_n$$

$$= \begin{bmatrix} | & & | \\ e^{\lambda_1 t} v_1 & \dots & e^{\lambda_n t} v_n \\ | & & | \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \Theta(t)\, c$$

(22)

$$\dot{x} = 2x + 3y$$
$$\dot{y} = \tfrac{1}{3} x + 2y$$

$$\dot{X} = \begin{bmatrix} 2 & 3 \\ \frac{1}{3} & 2 \end{bmatrix} X$$

$$\begin{vmatrix} 2-\lambda & 3 \\ \frac{1}{3} & 2-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)^2 - 1 = (2-\lambda-1)(2-\lambda+1) = (\lambda-1)(\lambda-3) = 0$$

$\lambda_1 = 1$

$$(A - \lambda_1 I) v_1 = 0$$

$$\begin{bmatrix} 1 & 3 \\ \frac{1}{3} & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = 0$$

$$1\alpha + 3\beta = 0 \qquad \alpha = -3\beta$$
$$\tfrac{1}{3}\alpha + 1\beta = 0 \qquad \beta = \beta$$

$$v_1 = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

$\lambda_2 = 3$

$$\begin{bmatrix} -1 & 3 \\ \frac{1}{3} & -1 \end{bmatrix} \overset{v_2}{\begin{bmatrix} \alpha \\ \beta \end{bmatrix}} = 0$$

$$-\alpha + 3\beta = 0 \qquad \alpha = 3\beta$$
$$\tfrac{1}{3}\alpha - \beta = 0$$

$$v_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$C_1, C_2 \in \mathbb{R}$

Splošna rešitev $x = C_1 e^{t} \begin{bmatrix} -3 \\ 1 \end{bmatrix} + C_2 e^{3t} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} -3e^{t} & 3e^{3t} \\ e^{t} & e^{3t} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$

Opomba: lahko se zgodi $\lambda \in \mathbb{C}$

$$\det(A-\lambda I)=0 \qquad \lambda = a+ib \qquad \bar\lambda = a-ib$$

$$Av = \lambda v \qquad A\bar v = \bar\lambda \bar v$$

$$x = C_1 \underbrace{e^{\lambda t} v} + C_2 \underbrace{e^{\bar\lambda t} \bar v} + \dots$$

kompleksne funkcije $\Rightarrow$ rešitev bi radi v realni obliki

$\Downarrow$

$$x = C_1 \operatorname{Re}(e^{\lambda t} v) + C_2 \operatorname{Im}(e^{\lambda t} v) + \dots$$

(27)

$$\begin{bmatrix} \dot x \\ \dot y \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{vmatrix} 4-\lambda & -3 \\ 3 & 4-\lambda \end{vmatrix} = 0$$

$$\lambda^2 - 8\lambda + 25 = 0$$

$$\lambda_{1,2} = 4 \pm 3i$$

$\lambda_1 = 4+3i$

$$\begin{bmatrix} -3i & -3 \\ 3 & -3i \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = 0$$

$-i\alpha = \beta$

$$v_1 = \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

$\lambda_2 = 4-3i$

$$\begin{bmatrix} 3i & -3 \\ 3 & 3i \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = 0$$

$i\alpha = \beta$

$$v_2 = \begin{bmatrix} 1 \\ i \end{bmatrix}$$

$$x = C_1 e^{(4+3i)t}\begin{bmatrix} 1 \\ -i \end{bmatrix} + C_2 e^{(4-3i)t}\begin{bmatrix} 1 \\ i \end{bmatrix}$$

$$x = C_1 \operatorname{Re}\left(e^{(4+3i)t}\begin{bmatrix} 1 \\ -i \end{bmatrix}\right) + C_2 \operatorname{Im}\left(e^{(4+3i)t}\begin{bmatrix} 1 \\ -i \end{bmatrix}\right)$$

$$e^{(4+3i)t}\begin{bmatrix} 1 \\ -i \end{bmatrix} = e^{4t}(\cos 3t + i \sin 3t)\begin{bmatrix} 1 \\ -i \end{bmatrix} = \begin{bmatrix} e^{4t}\cos 3t + i e^{4t} \sin 3t \\ e^{4t} \sin 3t - i e^{4t} \cos 3t \end{bmatrix} = \begin{bmatrix} e^{4t}\cos 3t \\ e^{4t}\sin 3t \end{bmatrix} + i \begin{bmatrix} e^{4t}\sin 3t \\ -e^{4t}\cos 3t \end{bmatrix}$$

$$x = C_1 \begin{bmatrix} e^{4t}\cos 3t \\ e^{4t}\sin 3t \end{bmatrix} + C_2 \begin{bmatrix} e^{4t}\sin 3t \\ -e^{4t}\cos 3t \end{bmatrix} \qquad C_1, C_2 \in \mathbb{R}$$

(T)

$A_{n \times n} \to \lambda_1, \dots, \lambda_n$

$\downarrow$

λ je k-kratna lastna vred., ampak, da za to last. vred. nimamo k lin. neodv. last. vekt.

$\Downarrow$

Potrebno je določiti korenske vektorje

Splošen postopek določitve korenskih vektorjev

$\dot{x} = Ax \qquad A \quad n \times n$ matrika

① Določitev last. vred.

$$\det(A - \lambda I) = 0 \rightarrow \lambda_1, \lambda_2, \dots$$

② Naj bo λ neka lastna vrednost

$$0 < \underbrace{\dim(\ker(A - \lambda I))}_{d_1} < \underbrace{\dim(\ker(A-\lambda I)^2)}_{d_2} < \dots < \underbrace{\dim(\ker(A-\lambda I)^k)}_{\boxed{d_k}} = d_{k+1} = \dots$$

→ linearno neodvisnih lastnih vektorjev je d_1
→ $k \times k$ je dimenzija največje Jordanove kletke

Cilj: Najti lastne in korenske vektorje:

$$A = P \, J(A) \, P^{-1} \qquad P = \begin{bmatrix} | & | & \cdots & | \end{bmatrix}$$

last. in korenski vektorji

$$J(A) = \begin{bmatrix} \boxed{\begin{matrix} \lambda_1 & 1 & 0 \\ & \ddots & \ddots \\ 0 & & \lambda_1 \end{matrix}} & \\ & \boxed{\begin{matrix} \lambda & 1 & \\ & \ddots & \\ & & \lambda \end{matrix}} \end{bmatrix}$$

Dimenzije kletk uredimo po vrsti

$\underset{k}{n_1} \geq n_2 \geq n_3 \dots$

(n_1) $v_1^{(1)} \in \ker(A - \lambda I)^k - \ker(A - \lambda I)^{k-1}$

$$v_1^{(2)} = (A - \lambda I)\, v_1^{(1)}$$
$$v_1^{(3)} = (A - \lambda I)\, v_1^{(2)}$$
$$\vdots$$
$$v_1^{(k)} = (A - \lambda I)\, v_1^{(k-1)} = \text{lastni vektor}$$

(n_2) $v_2^{(1)} \in \ker(A - \lambda I)^{n_2} - \ker(A - \lambda I)^{n_2 - 1}$ + Lin. neodv. od vseh prej

$$v_2^{(2)} = (A - \lambda I)\, v_2^{(1)}$$
$$\vdots$$
$$v_2^{(n_2)} = (A - \lambda I)\, v_2^{(n_2-1)} = \text{last. vektor}$$

itd.

Tvorimo matriko P

$$P = \begin{bmatrix} | & | & & | & | & & | & \\ v_1^{(n_1)} & v_1^{(n_1-1)} & \cdots & v_1^{(1)} & v_2^{(n_2)} & \cdots & v_2^{(1)} & \cdots \\ | & | & & | & | & & | & \end{bmatrix} \qquad A = \begin{bmatrix} \boxed{\begin{matrix} \lambda & 1 & \\ & \ddots & 1 \\ & & \lambda \end{matrix}} & & 0 \\ 0 & \boxed{\begin{matrix} \lambda & 1 \\ & \lambda \end{matrix}} & \\ & & \ddots \end{bmatrix}$$

Začnemo z $\dot{x} = Ax$ → A dobimo l. vred., korenske in lastne vektorje

Splošna rešitev

λ ... last. vred.

$v_1^{(k)}, v_1^{(k-1)}, \dots, v_1^{(1)}$

last. vekt.

$$x = C_1 e^{\lambda t} v_1^{(k)} + C_2 e^{\lambda t}\left(t v_1^{(k)} + v_1^{(k-1)}\right) + \dots + C_k e^{\lambda t}\left(\frac{t^{k-1}}{(k-1)!} v_1^{(k)} + \dots + t v_1^{(2)} + v_1^{(1)}\right)$$

$+ \dots$

(del rešitve, ki se nanaša na preostale lastne in korenske vektorje

Opomba: za majhne matrike

v last. vekt. matrike A ⇒ korenski vektor u je rešitev sistema $(A-\lambda I)u = v$

24 $$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ -1 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Lastne vrednosti

$$\det(A - \lambda I) = \begin{vmatrix} -2-\lambda & +1 \\ -1 & -4-\lambda \end{vmatrix} = (-2-\lambda)(-4-\lambda) + 1 = \lambda^2 + 6\lambda + 9 = (\lambda+3)^2$$

$\lambda_{1,2} = -3$ (dvojna lastna vrednost)

(Opomba: Poglejmo koliko lin. neodv. last. vektorjev pripada $\lambda = -3$

$(A - \lambda I)v = 0$

$$\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} v = 0 \qquad \alpha = -\beta \qquad v = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Rešitev sistema

$$x = C_1 e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$ ← ni splošna rešitev)

Iskanje korenskih vektorjev

$\lambda = -3$ $\quad A - \lambda I = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$ ⇒ rang $= 1$ $\quad$ ker $= n - 1 = 1$

$$0 < \underbrace{\dim(\ker(A-\lambda I))}_{d_1 = 1} < \underbrace{\dim(\ker(A-\lambda I)^2)}_{d_2 = 2} = \dots = 2$$

$$(A-\lambda I)^2 = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$ ⇒ rang $= 0$ $\quad$ dim $= 2$

Poiščemo korenske vektorje

$$v_1^{(1)} \in \ker\overset{\text{vsi vekt.}}{(A-\lambda I)^2} - \ker(A-\lambda I)$$

korenski vektor $\quad v_1^{(1)} = \begin{bmatrix}1\\0\end{bmatrix}$ ✓ $\qquad \begin{bmatrix}1&1\\-1&-1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1\\-1\end{bmatrix} \neq 0$ ✓

$$v_1^{(2)} = \begin{bmatrix}1&1\\-1&-1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1\\-1\end{bmatrix} = \text{last. vekt.}$$

Rešitev sistema

$$x = C_1 e^{-3t}\begin{bmatrix}1\\-1\end{bmatrix} + C_2 e^{-3t}\left(t\begin{bmatrix}1\\-1\end{bmatrix} + \begin{bmatrix}1\\0\end{bmatrix}\right)$$

$$x = C_1\begin{bmatrix}e^{-3t}\\-e^{-3t}\end{bmatrix} + C_2\begin{bmatrix}e^{-3t}(1+t)\\-te^{-3t}\end{bmatrix} \qquad C_1, C_2 \in \mathbb{R}$$

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$$\dot{x} = \begin{vmatrix}1&2&1\\-8&-9&-3\\12&12&3\end{vmatrix} x$$

$$\det(A-\lambda I) = \begin{vmatrix}1-\lambda&2&1\\-8&-9-\lambda&-3\\12&12&3-\lambda\end{vmatrix} = (1-\lambda)\big((3-\lambda)(-9-\lambda)+36\big) - 2\big(-8(3-\lambda)+36\big)$$
$$+ \big(-8\cdot 12 + (9+\lambda)12\big)$$

$$= (1-\lambda)(\lambda^2+6\lambda+9) - 2(8\lambda+12) + 108 - 96 + 12\lambda =$$
$$= -\lambda^3 - 6\lambda^2 - 9\lambda + \lambda^2 + 6\lambda + 9 - 16\lambda - 24 + 108 - 96 + 12\lambda =$$
$$= -\lambda^3 - 5\lambda^2 - 7\lambda - 3 = -(\lambda+1)^2(\lambda+3) = 0$$

$$\lambda_1 = -3 \qquad \lambda_{2,3} = -1$$

$\lambda_1 = -3$

$$\begin{bmatrix}4&2&1\\-8&-6&-3\\12&12&6\end{bmatrix} \sim \begin{bmatrix}4&2&1\\0&-2&-1\\0&6&3\end{bmatrix} \sim \begin{bmatrix}4&2&1\\0&2&1\\0&0&0\end{bmatrix} \sim \begin{bmatrix}4&0&0\\0&2&1\\0&0&0\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = 0$$

$$4x = 0 \qquad z = -2y \qquad v = \begin{bmatrix}0\\1\\-2\end{bmatrix}$$
$$2y + z = 0$$

$$\lambda_{2,3} = -1$$

$$A - \lambda I = \begin{bmatrix}2&2&1\\-8&-8&-3\\12&12&4\end{bmatrix} \sim \begin{bmatrix}2&2&1\\0&0&1\\0&0&0\end{bmatrix} \Rightarrow \text{rang } 2 \Rightarrow \dim = 1$$

$$0 < \underbrace{\dim(\ker(A-\lambda I))}_{d_1 = 1} < \underbrace{\dim(\ker(A-\lambda I)^2)}_{d_2 = 2} \quad ✓$$

$$(A-\lambda I)^2 = \begin{bmatrix} 2 & 2 & 1 \\ -8 & -8 & -3 \\ 12 & 12 & 4 \end{bmatrix}\begin{bmatrix} 2 & 2 & 1 \\ -8 & -8 & -3 \\ 12 & 12 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 12 & 12 & 4 \\ -24 & -24 & -8 \end{bmatrix} \qquad \text{rang} = 1 \qquad \text{dim} = 2$$

$$v_1^{(1)} \in \ker(A-\lambda I)^2 - \ker(A-\lambda I)$$

$$v_1^{(1)} = \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}$$

preveri!

Poskusimo (ugibamo (ponavadi $x=1$, $y=0$, $z=?$)
Ali rešimo sistem enačb.

$$v_1^{(2)} = \begin{bmatrix} 2 & 2 & 1 \\ -8 & -8 & -3 \\ 12 & 12 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

Splošna rešitev

$$x = C_1 e^{-3t}\begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} + C_2 e^{-t}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + C_3 e^{-t}\left(t\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} \right)$$

$$C_1, C_2, C_3 \in \mathbb{R}$$

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$$\dot{x} = \begin{bmatrix} -2 & 1 & 3 \\ 0 & -2 & 5 \\ 0 & 0 & -2 \end{bmatrix} x$$

$$\det(A-\lambda I) \Rightarrow \quad \lambda_{1,2,3} = -2 = \lambda$$

$$A-\lambda I = \begin{bmatrix} 0 & 1 & 3 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{rang} = 2 \qquad \text{dim} = 1$$

$$0 < \underbrace{\dim(\ker(A+2I))}_{d_1 = 1} < \underbrace{\dim(\ker(A+2I)^2)}_{d_2 = 2} < \underbrace{\dim(\ker(A+2I)^3)}_{d_3 = 3}$$

$$(A+2I)^2 = \begin{bmatrix} 0 & 0 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{rang} = 1 \qquad \text{dim} = 2$$

$$(A+2I)^3 = \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} \qquad \text{rang} = 0 \qquad \text{dim} = 3$$

$v_1^{(1)} \in \ker((A+2I)^3) - \ker(A+2I)^2$

$$v_1^{(1)} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$v_1^{(2)} = (A+2I)\, v_1^{(1)} = \begin{bmatrix} 0 & 1 & 3 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 0 \end{bmatrix}$$

$$v_1^{(3)} = \begin{bmatrix} 0 & 1 & 3 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}$$

$$x = C_1 e^{-2t} \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix} + C_2 e^{-2t} \left(t \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 3 \\ 5 \\ 0 \end{bmatrix} \right) + C_3 e^{-2t} \left(\frac{t^2}{2} \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 3 \\ 5 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right)$$

T Nehomogen sistem enačb

$$\dot{x} = Ax + f(t)$$

1. $\dot{x} = Ax$ Poiščemo splošno rešitev homog. $x = C_1 e^{\lambda_1 t} v_1 + \dots + C_n e^{\lambda_n t} v_n$

$$x = \begin{bmatrix} | & & | \\ e^{\lambda_1 t} v_1 & \cdots & e^{\lambda_n t} v_n \\ | & & | \end{bmatrix} \begin{bmatrix} C_1 \\ \vdots \\ C_n \end{bmatrix}$$

$$x_H = \Theta(t)\, C$$

2. $x_p = \Theta(t)\, C(t)$ $\qquad C(t) = ?$

$$\dot{x}_p = A x_p + f(t)$$

$$\cancel{\dot{\Theta} C} + \Theta \dot{C} = \cancel{A \Theta C} + f(t)$$

$$\dot{C} = \Theta^{-1} f(t)$$

$$C = \int \Theta^{-1} f(t)\, dt$$

Spl. rešitev $\qquad x = \Theta(t)\, C + \Theta(t)\, C_1(t)$

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$$\dot{x} = \begin{bmatrix} -2 & 1 \\ -1 & -4 \end{bmatrix} x + \begin{bmatrix} e^t \\ 0 \end{bmatrix}$$

a)

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ -1 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

Lastne vrednosti

$$\det(A - \lambda I) = \begin{vmatrix} -2-\lambda & +1 \\ -1 & -4-\lambda \end{vmatrix} = (-2-\lambda)(-4-\lambda) + 1 = \lambda^2 + 6\lambda + 9 = (\lambda+3)^2$$

$\lambda_{1,2} = -3$ (dvokratna lastna vrednost)

Iskanje korenskih vektorjev

$\lambda = -3$ $\qquad A - \lambda I = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \Rightarrow$ rang $= 1$ $\quad$ ker $= n - 1 = 1$

$$0 < \underbrace{\dim(\ker(A-\lambda I))}_{d_1 = 1} < \underbrace{\dim(\ker(A-\lambda I)^2)}_{d_2 = 2} = \dots \quad 2$$

$$(A-\lambda I)^2 = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow \text{rang} = 0 \quad \dim = 2$$

Poiščemo korenske vektorje

$$v_1^{(1)} \in \overset{\text{vsi vekt.}}{\ker (A-\lambda I)^2} - \ker(A - \lambda I)$$

korenski vektor $\quad v_1^{(1)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ ✓

$$v_1^{(2)} = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \text{last. vekt.}$$

Rešitev sistema

$$x = C_1 e^{-3t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + C_2 e^{-3t}\left(t\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$$

$$x = C_1\begin{bmatrix} e^{-3t} \\ -e^{-3t} \end{bmatrix} + C_2\begin{bmatrix} e^{-3t}(1+t) \\ -te^{-3t} \end{bmatrix} \qquad C_1, C_2 \in \mathbb{R}$$

$$x_H = \begin{bmatrix} e^{-3t} & e^{-3t}(1+t) \\ -e^{-3t} & e^{-3t}(-t) \end{bmatrix}\begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$$

b)

$$C = \int \Theta^{-1} f(t)\, dt$$

$$\left[\begin{array}{cc|cc} e^{-3t} & e^{-3t}(1+t) & 1 & 0 \\ -e^{-3t} & e^{-3t}(-t) & 0 & 1 \end{array}\right] = \left[\begin{array}{cc|cc} 1 & 1+t & e^{3t} & 0 \\ -1 & -t & 0 & e^{3t} \end{array}\right] = \left[\begin{array}{cc|cc} 1 & 1+t & e^{3t} & 0 \\ 0 & 1 & e^{3t} & e^{3t} \end{array}\right]$$

$$\Theta^{-1}(t) = \begin{bmatrix} e^{3t}(1-t) & -te^{3t} \\ e^{3t} & e^{3t} \end{bmatrix} = e^{3t}\begin{bmatrix} 1-t & -t \\ 1 & 1 \end{bmatrix}$$

Na spodaj

$$\begin{bmatrix} \dot c_1 \\ \dot c_2 \end{bmatrix} = e^{3t} \begin{bmatrix} -t & -(1+t) \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{-t} \\ 0 \end{bmatrix} = \begin{bmatrix} -te^{2t} \\ e^{2t} \end{bmatrix}$$

$$\dot c_1 = -t e^{2t} \qquad c_1 = -\int t e^{2t}\, dt \overset{\text{per partes}}{=} -\frac{t}{2} e^{2t} - \frac{1}{4} e^{2t} \cancel{+C}$$

$$c_1 = -\frac{e^{2t}}{4}(2t+1)$$

$$\dot c_2 = e^{2t} \qquad c_2 = \frac{1}{2} e^{2t}$$

$$x_p = \begin{bmatrix} e^{-3t} & e^{-3t}(1+t) \\ -e^{-3t} & e^{-3t}(-t) \end{bmatrix} \begin{bmatrix} e^{2t}\left(\frac{1}{4} - t/2\right) \\ \frac{1}{2} e^{2t} \end{bmatrix} = \begin{bmatrix} e^{-t}\left(\frac{1}{4} - t/2\right) + e^{-t}\left(\frac{t}{2} + \frac{1}{2}\right) \\ -e^{-t}\left(\frac{1}{4} - \frac{t}{2}\right) - \frac{t}{2} e^{-t} \end{bmatrix}$$

$$x_p = \begin{bmatrix} \frac{3}{4} e^{-t} \\ -\frac{1}{4} e^{-t} \end{bmatrix}$$

Splošna rešitev

$$x = \begin{bmatrix} e^{-3t} & e^{-3t}(1+t) \\ -e^{-3t} & e^{-3t}(-t) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \begin{bmatrix} \frac{3}{4} e^{-t} \\ -\frac{1}{4} e^{-t} \end{bmatrix}$$

Lin. dif. en. višjega reda s konst. koeficienti

$y(x)$

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y = g(x)$$

$a_1, \dots, a_n \in \mathbb{R}$
$g(x)$ neka funkcija

Po stopnjah

(1) Poiščemo splošno rešitev homogene enačbe

$$a_n y^{(n)} + \dots a_1 y' + a_0 y = 0$$

Uporabimo nastavek $y = e^{\lambda x}$

$$a_n \lambda^n e^{\lambda x} + a_{n-1} \lambda^{n-1} e^{\lambda x} + \dots a_1 \lambda e^{\lambda x} + a_0 e^{\lambda x} = 0$$

Dobimo t.i. karakteristični polinom

$$a_n \lambda^n + \dots + a_1 \lambda + a_0 = 0$$

↳ ničle $\lambda_1, \dots, \lambda_n$

Splošna rešitev homogene enačbe $y = C_1 e^{\lambda_1 x} + \dots + C_n e^{\lambda_n x}$ (λ_i med seboj različne)

V primeru, da je λ k-kratna ničla karakt. polinoma

$$y = \underbrace{C_1 e^{\lambda x} + C_2 x e^{\lambda x} + C_3 x^2 e^{\lambda x} + \dots + C_k x^{k-1} e^{\lambda x}}_{\text{od k-kratne ničle}} + C_{k+1} e^{\lambda_1 x} + \dots$$

V primeru, da je $\lambda = a+ib \in \mathbb{C} \Rightarrow \bar{\lambda} = a-ib$

$$y = C_1 e^{(a+ib)x} + C_2 e^{(a-ib)x} + C_3 e^{\lambda_3 x} + \dots$$

$\Downarrow$

Izrazimo z realnimi f. $e^{(a+ib)} = e^{ax} e^{ibx} = e^{ax}(\cos bx + i \sin bx)$

Spl. rešitev

$$y = C_1 e^{ax} \cos bx + C_2 e^{ax} \sin bx + C_3 e^{\lambda_3 x} + \dots$$

(2) Poiščemo partikularno rešitev

$$y_H = C_1 y_1(x) + C_2 y_2(x) + \dots + C_n y_n(x)$$

Variacija konst.

$$y_p = C_1(x) y_1(x) + \dots + C_n(x) y_n(x)$$

$C_1(x) \dots C_n(x)$ hočemo, da je y_p rešitev prvotne enačbe

$$\begin{vmatrix} y_1 & y_2 & \dots & y_n \\ y_1' & y_2' & \dots & y_n' \\ \vdots & & & \\ y_1^{(n-1)} & \dots & & y_n^{(n-1)} \end{vmatrix} \begin{vmatrix} C_1' \\ \vdots \\ \\ C_n' \end{vmatrix} = \begin{vmatrix} 0 \\ \vdots \\ 0 \\ \frac{g(x)}{a_n} \end{vmatrix}$$

$\int C_1'(x) \Rightarrow C_1$

$\vdots$

Op: Na vajah bomo y_p poiskali z nastavkom $g(x) = P(x) e^{\lambda x}$

polinom

Reševanje homogenega dela

(1) Poišči splošno rešitev

(a) $y'' - y' - 2y = 0$

$y = e^{\lambda x}$

$\lambda^2 e^{\lambda x} - \lambda e^{\lambda x} - 2 e^{\lambda x} = 0$

$\lambda^2 - \lambda - 2 = 0$

$(\lambda + 1)(\lambda - 2) = 0$

$\lambda_1 = -1 \quad \lambda_2 = 2$

$x = C_1 e^{-x} + C_2 e^{2x}$

(b) $y'' + 2y' = 0$

$y = e^{\lambda x}$

$\lambda^2 + 2\lambda = 0$

$\lambda(\lambda + 2) = 0$

$\lambda = 0 \quad \lambda = -2$

$y = C_1 + C_2 e^{-2x}$

(c) $y'' + 6y' + 9y = 0$

$y = e^{\lambda x}$

$\lambda^2 + 6\lambda + 9 = 0$

$(\lambda + 3)^2 = 0$

$\lambda_{1,2} = -3$

$y = C_1 e^{-3x} + C_2 x e^{-3x}$

(d) $y''' + 3y'' + 3y' + y = 0$

$\lambda^3 + 3\lambda^2 + 3\lambda + 1 = 0$

$(\lambda + 1)^3 = 0$

$\lambda_{1,2,3} = -1$

$y = C_1 e^{-x} + C_2 x e^{-x} + C_3 x^2 e^{-x}$

(e) $y''' - 2y'' - 4y' + 8y = 0$

$\lambda^3 - 2\lambda^2 - 4\lambda + 8 = 0$

$\lambda^2(\lambda - 2) - 4(\lambda - 2) = 0$

$(\lambda - 2)(\lambda - 2)(\lambda + 2) = 0$

$\lambda_{1,2} = 2 \qquad \lambda_3 = -2$

$y = C_1 e^{2x} + C_2 x e^{2x} + C_3 e^{-2x}$

(f) $y'' + 2y' + 5y = 0$

$\lambda^2 + 2\lambda + 5 = 0$

$$\lambda_{1,2} = \frac{-2 \pm \sqrt{4 - 4 \cdot 5}}{2} = \frac{-2 \pm i4}{2} = -1 \pm 2i = \begin{cases} -1 + 2i \\ -1 - 2i \end{cases}$$

$y = C_1 e^{-x} \cos 2x + C_2 e^{-x} \sin 2x$

(g) $y'''' + 6y'' + 9y = 0$

$\lambda^4 + 6\lambda^2 + 9 = 0$

$(\lambda^2 + 3)^2 = 0$

$\lambda^2 = -3$

$\lambda = \pm i\sqrt{3} \qquad \lambda_{1,2} = i\sqrt{3} \qquad \lambda_{3,4} = -i\sqrt{3}$

$$y = C_1 e^{i\sqrt{3}x} + C_2 x e^{i\sqrt{3}x} + C_3 e^{-i\sqrt{3}x} + C_4 x e^{-i\sqrt{3}x}$$

$$= C_1 \cos\sqrt{3}x + C_2 \sin\sqrt{3}x + x C_3 \cos\sqrt{3}x + x C_4 \sin\sqrt{3}x$$

Euler - Cauchyjeva enačba (homogena)

$$a_n x^n y^{(n)} + a_{n-1} x^{n-1} y^{(n-1)} + \dots + a_1 x y' + a_0 y = 0$$

$$a_n, \dots, a_0 \in \mathbb{R}$$

Nastavek $y = x^\lambda$

Dobimo karak. polinom.

$$a_n \lambda(\lambda-1)\cdots(\lambda-n+1) + \dots + a_0 = 0$$

$$\lambda_1, \lambda_2, \dots \lambda_n$$

Spl. rešitev

① Če so vse λ_i med seboj različne

$$y = C_1 x^{\lambda_1} + \dots + C_n x^{\lambda_n}$$

② Če je λ k-kratna ničla

$$y = C_1 x^\lambda + C_2 (\ln x) x^\lambda + \dots C_k (\ln x)^{k-1} x^\lambda + \dots$$

③ $\lambda = a + ib$

$$y = C_1 x^{a+ib} + C_2 x^{a-ib}$$

$$y = C_1 x^a \cos(b \ln x) + C_2 x^a \sin(b \ln x)$$

②

ⓐ $2x^2 y'' - 7x y' + 9y = 0$

$$y = x^\lambda$$

$$2x^2 \lambda(\lambda-1) x^{(\lambda-2)} - 7x \lambda x^{\lambda-1} + 9x^\lambda = 0$$

$$2\lambda(\lambda-1) - 7\lambda + 9 = 0$$

$$2\lambda^2 - 9\lambda + 9 = 0$$

$$\lambda_1 = 3 \qquad \lambda_2 = 3/2$$

$$y = C_1 x^3 + C_2 x^{3/2}$$

ⓑ $x^2 y'' - 3x y' + 4y = 0$

$$y = x^\lambda$$

$$x^2 \lambda(\lambda-1) x^{\lambda-2} - 3x \lambda x^{\lambda-1} + 4x^\lambda = 0$$

$$\lambda^2 - 4\lambda + 4 = 0$$

$$(\lambda - 2)^2 = 0$$

$$\lambda_{1,2} = 2$$

$$y = C_1 x^2 + C_2 (\ln x) x^2$$

Reševanje partikularnega dela

Reševanje DE oblike

$$a_n y^{(n)} + \dots + a_1 y' + a_0 y = P(x) e^{\lambda x} \qquad P(x) \dots \text{polinom}$$

① Rešimo homogeno enačbo

$$a_n y^{(n)} + \dots + a_1 y' + a_0 y = 0$$
$$y = e^{\lambda x}$$

$$\Rightarrow \quad y_H = C_1 e^{\lambda_1 x} + \dots + C_n e^{\lambda_n x}$$

② Poiščimo y_p

$g(x) = P(x) e^{\lambda x} \longrightarrow \lambda$ k-kratna ničla karakterističnega polinoma

$$y_p = Q(x)\, x^k e^{\lambda x}$$

↳ • polinom z neznanimi koef.
• st Q = st P

(če λ ni ničla karakt. polinoma
$y_p = Q(x)\, e^{\lambda x}$)

9

$$y'' - y' - 2y = x e^{-2x}$$

ⓐ homog. en.

$$y'' - y' - 2y = 0$$
$$y = e^{\lambda x}$$
$$\lambda^2 - \lambda - 2 = 0$$
$$\lambda_1 = -1 \qquad \lambda_2 = 2$$

$$y_H = c_1 e^{-x} + c_2 e^{2x}$$

ⓑ part. reš.

-2 ni ničla karakt. polinoma

$$y_p = (Ax + B) e^{-2x} \qquad A, B = ?$$

$$y' = A e^{-2x} - 2(Ax + B) e^{-2x}$$
$$y'' = -2A e^{-2x} - 2A e^{-2x} + 4(Ax+B) e^{-2x}$$

$$-4A e^{-2x} + 4(Ax+B) e^{-2x} - A e^{-2x} + 2(Ax+B) e^{-2x} - 2(Ax+B) e^{-2x} = x e^{-2x}$$

$$-5A + 4(Ax + B) = x$$
$$4B - 5A + 4Ax = x$$
$$4B = 5A \qquad 4A = 1 \qquad A = \frac{1}{4}$$
$$B = \frac{5}{4} A = \frac{5}{16}$$

$$y_p = \left(\frac{x}{4} + \frac{5}{16}\right) e^{-2x}$$
$$y = c_1 e^{-x} + c_2 e^{2x} + \left(\frac{x}{4} + \frac{5}{16}\right) e^{-2x}$$

(4) $y'' - y' - 2y = \cos x$

Iz naloge 3 vemo $\quad y_H = C_1 e^{-x} + C_2 e^{2x}$

Spomnimo se

$$e^{ix} = \cos x + i \sin x \Rightarrow \cos x = \operatorname{Re} e^{ix}$$

Vzemimo enačbo

$$y'' - y' - 2y = e^{ix}$$

$$\Downarrow$$

$$\tilde{y}_p \Rightarrow y_p = \operatorname{Re}(\tilde{y}_p)$$

$$\tilde{y}_p = Ae^{ix} \qquad A = ?$$

$$\tilde{y}_p'' - \tilde{y}_p' - 2\tilde{y}_p = e^{ix}$$

$$A i^2 e^{ix} - Ai e^{ix} - 2Ae^{ix} = e^{ix}$$

$$-A - Ai - 2A = 1$$

$$3A + Ai = -1$$

$$A(3+i) = -1$$

$$A = \frac{1}{-3-i} = \frac{-3+i}{10}$$

$$\tilde{y}_p = \frac{-3+i}{10} e^{ix}$$

$$y_p = \operatorname{Re} \tilde{y}_p = \operatorname{Re}\left(\frac{-3+i}{10}(\cos x + i \sin x)\right) = -\frac{3}{10}\cos x - \frac{1}{10}\sin x$$

$$y = C_1 e^{-x} + C_2 e^{2x} - \frac{3}{10}\cos x - \frac{1}{10}\sin x$$

(5) $y'' + 6y' + 9y = e^{-3x}$

Homog.

$$\lambda^2 + 6\lambda + 9 = 0$$

$\lambda_{1,2} = -3 \qquad$ dvojna ničla

$$y_H = C_1 e^{-3x} + C_2 x e^{-3x}$$

Part. rešitev

$$\tilde{y}_p = A x^2 e^{-3x}$$

$$\tilde{y}_p' = A(2x - 3x^2) e^{-3x}$$

$$\tilde{y}_p'' = A\big((2-6x) - 3(2x - 3x^2)\big) e^{-3x}$$

$$A\big((2-6x) - 3(2x-3x^2)\big) e^{-3x} + 6A(2x - 3x^2) e^{-3x} + 9Ax^2 e^{-3x} = e^{-3x}$$

$$\frac{1}{A} = 2 - 6x - 6x + 9x^2 + 12x - 18x^2 + 9x^2$$

$$\frac{1}{A} = 2 \qquad A = \frac{1}{2}$$

$$y_p = \frac{1}{2} x^2 e^{-3x}$$

$$y = C_1 e^{-3x} + C_2 x e^{-3x} + \frac{1}{2} x^2 e^{-3x}$$

Variacijski račun

$y : [a, b] \to \mathbb{R}$ $\qquad$ $y(a) = A$, $y(b) = B$

Iščemo tako funkcijo y, pri kateri doseže $I(y) = \int_a^b L(x, y, y')\,dx$ ekstremno vrednost.

y zadošča Euler-Lagrangevi enačbi

$$\frac{\partial L}{\partial y} = \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) \qquad \begin{matrix} y(a) = A \\ y(b) = B \end{matrix} \quad \Rightarrow y(x)$$

① Določi funkcijo $y : [1, 2] \to \mathbb{R}$, $y(1) = 4$, $y(2) = 1$, pri kateri doseže

$$I = \int_1^2 \underbrace{(x^2 y'^2 + 2y^2)}_{L}\, dx$$ ekstremno vrednost

$$\frac{\partial L}{\partial y} = \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right)$$

$$4y = \frac{d}{dx}(2x^2 y')$$

$$4y = 4x y' + 2x^2 y''$$

$$2x^2 y'' + 4x y' - 4y = 0$$

Homogena Euler-Cauchyjeva en.

$y = x^\lambda$

$$2x^2 \lambda(\lambda - 1) x^{\lambda - 2} + 4x \lambda x^{\lambda - 1} - 4 x^{\lambda - 1} = 0$$

$$2\lambda^2 - 2\lambda + 4\lambda - 4 = 0$$

$$\lambda^2 + \lambda - 2 = 0$$

$$(\lambda + 1)(\lambda - 2) = 0$$

$$\lambda_1 = 1 \qquad \lambda_2 = -2$$

$$y = C_1 x^{-2} + C_2 x$$

$$y(1) = 4 \qquad y(2) = 1$$

$$4 = C_1 + C_2 \qquad 1 = \frac{C_1}{4} + 2C_2$$

$$\Rightarrow C_1 = 4 \qquad C_2 = 0 \qquad\qquad y = \frac{4}{x^2}$$

Tipične naloge

① $y: [a, b] \to \mathbb{R}$ $\quad y(a) = A \quad y(b) = B$

$$\int_a^b L(x, y, y')\,dx = \text{ekstrem.}$$

$$\frac{\partial L}{\partial y} = \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) \to \text{rešimo}$$

② Pogoji kakšnega od robnih pogojev ni (lahko tudi oba)

Dinamični robni pogoji

$$L_{y'}(a, y(a), y'(a)) = 0$$
$$L_{y'}(b, y(b), y'(b)) = 0$$

② $y: [0, 1] \to \mathbb{R}$ $\quad y(1) = 0$

$$I(y) = \int_0^1 (y^2 + y'^2 + 2ye^x)\,dx$$

$$L(x, y, y') = y^2 + y'^2 + 2ye^x$$

$$L_y = \frac{d}{dx} L_{y'}$$

$$2y + 2e^x = \frac{d}{dx}(2y') = 2y''$$

$$y'' - y = e^x \quad \text{Euler-Lagrange en.}$$

① $y'' - y = 0$

$$y = e^{\lambda x}$$
$$\lambda^2 - 1 = 0$$
$$\lambda = \pm 1$$
$$y_H = C_1 e^x + C_2 e^{-x}$$

② Part. $\quad y_p = A x e^x$

$$y'_p = A e^x + x e^x A$$
$$y''_p = Ae^x + e^x A + x e^x A$$

$$A(2e^x + e^x x - xe^x) = e^x$$
$$A = \frac{1}{2}$$

Splošna rešitev $\quad y = C_1 e^x + C_2 e^{-x} + \frac{1}{2} x e^x$

Pogoj $\quad y(1) = 0$

$$0 = C_1 e + C_2 e^{-1} + \frac{1}{2} e$$

V krajišču $x=0$ imamo poleg [illegible] robni pogoj

$$L_{y'}(0, y(0), y'(0)) = 0$$

$$2y' = 0$$

$$y' = 0 \qquad = \qquad y' = C_1 e^x - C_2 e^{-x} + \frac{1}{2} e^x + \frac{1}{2} x e^x$$

$$0 = C_1 - C_2 + \frac{1}{2} + \frac{1}{2} \cdot 0 \qquad\qquad 0 = C_1 e + C_2 e^{-1} + \frac{1}{2} e$$

$$0 = C_1 - C_2 + \frac{1}{2} \qquad / \cdot e$$
$$0 = C_1 e + C_2 e^{-1} + \frac{1}{2} e$$
$$0 = 0 + 2C_2 e^{-1} + 0$$
$$C_2 = 0 \qquad C_1 = -\frac{1}{2}$$

Končna rešitev $y = -\frac{1}{2} e^x + \frac{1}{2} x e^x = \frac{e^x}{2}(x-1)$

Op: Beltramijeva identiteta

$$\int_a^b L(x, y, y')\, dx$$

v posebnem

$$\int_a^b L(y, y')\, dx$$

$$\Downarrow$$

$$\frac{\partial L}{\partial y} = \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) \qquad \text{2. reda}$$

$\Downarrow$ ker x ne nastopa se da zintegrirati

$$L - y' \frac{\partial L}{\partial y'} = C \qquad C \ldots \text{konst.} \qquad \text{1. reda}$$