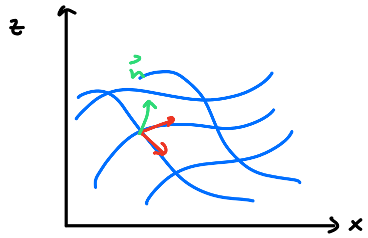
Metoda karakteristike

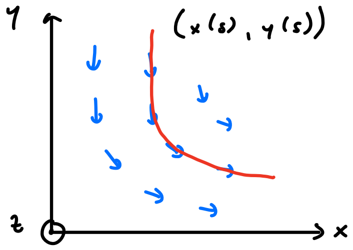
$$A(x,y,z)\frac{\partial z}{\partial x} + B(x,y,z)\frac{\partial z}{\partial y} = C(x,y,z) \quad ; \quad z = z(x,y)$$



$$\vec{D}_x = (1, 0, \underset{\frac{\partial z}{\partial x}}{\underset{\parallel}{z_x}}) \qquad \vec{D}_y = (0, 1, \underset{\frac{\partial z}{\partial y}}{\underset{\parallel}{z_y}})$$

$$\vec{n} = \vec{D}_x \times \vec{D}_y = \begin{pmatrix} 1 & 0 & z_x \\ 0 & 1 & z_y \end{pmatrix} = (-z_x, -z_y, 1)$$

Ekvivalentno: $\vec{n}\cdot\vec{F} = 0$, $\vec{F} = (A, B, C)$



Metoda karakteristike: vsedemo se na začetno točko in sledimo polju

$$\frac{dz}{ds} = \frac{\partial z}{\partial x}\underbrace{\frac{\partial x}{\partial s}}_{A} + \frac{\partial z}{\partial y}\underbrace{\frac{\partial y}{\partial s}}_{B} = \frac{\partial z}{\partial x}A + \frac{\partial z}{\partial y}B = C$$

1 Burgerjeva enačba

Iščemo $g(x,t)$, ki zadošča enačbi $\frac{\partial g}{\partial t} + g\frac{\partial g}{\partial x} = 0$

$\hookrightarrow A = 1, \quad B = g, \quad C = 0$

zač. pos. $g(x,0) = \begin{cases} a^2 - x^2 & ; |x| < a \\ 0 & ; |x| > a \end{cases}$

$x(s)$, $t(s)$

$$\frac{dg}{ds} = 0 \qquad \frac{dx}{ds} = g \qquad \frac{dt}{ds} = 1 \quad \Rightarrow \quad t = s + \overset{0}{\overset{\parallel}{D}}$$

(lahko $\neq 0$ vendar bi dobili le zamik v času)

$$g(x(s), t(s)) = g(x(0), t(0)) = g(x_0, 0) \quad \times$$

$$\hookrightarrow \frac{dx}{ds} = g(x_0, 0) \quad \Rightarrow \quad x = g(x_0,0)\,t + E \qquad [\,x \text{ pri } t=0 \text{ je } x_0 \Rightarrow E = x_0\,]$$

a) $|x_0| > a \quad \Rightarrow \quad x = x_0$

b) $|x_0| < a \quad \Rightarrow \quad x = (a^2 - x_0^2)t + x_0 \quad \rightarrow \quad x = a^2 t - x_0^2 t + x_0$

$$x_0^2 t - x_0 - a^2 t + x = 0$$

Rešimo kvadratno enačbo

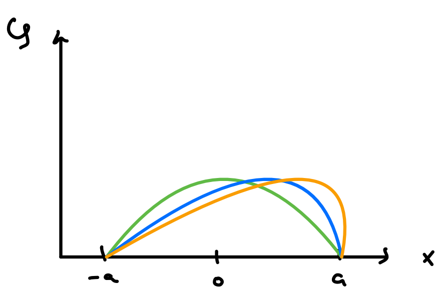
$$x_0 = \frac{1 \pm \sqrt{1 - 4t(x - a^2 t)}}{2t}$$

To damo v $\times$, ker ne vemo, koliko je bilo na začetku:

$$g(x,t) = g(x_0, 0) = a^2 - x_0^2 = a^2 - \left(\frac{1 \pm \sqrt{1 - 4t(x - a^2 t)}}{2t}\right)^2$$

Polna rešitev

$$g(x,t) = \begin{cases} a^2 - \left(\frac{1 \pm \sqrt{1 - 4t(x - a^2 t)}}{2t}\right)^2 & ; |x| < a \\ 0 & ; |x| > a \end{cases}$$



Klasifikacija PDE 2. reda

$A = A(x,y)$

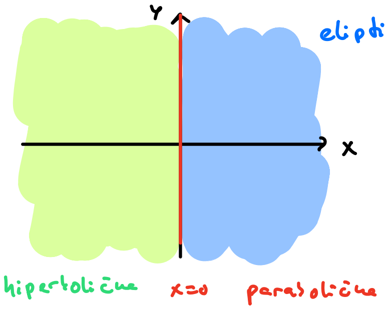
$A f_{xx} + B f_{xy} + C f_{yy} + D f_x + E f_y + F f = G$

Karakteristike $\quad \dfrac{dy}{dx} = \dfrac{1}{2A}\left(B \pm \sqrt{\underbrace{B^2 - 4AC}}\right)$

>0	Hiperbolična	$f_{yy} - f_{xx} = 0$	Val. en.
$=0$	Parabolična	$f_y - f_{xx} = 0$	dif. en.
<0	Eliptična	$f_{xx} + f_{yy} = 0$	Poissonova en.

② Klasificiraj en., poišči karakteristike in jo transformiraj v kanonično obliko

$$x\, u_{xx} + u_{yy} = x^2 \;* \quad \rightarrow A = x,\ B = 1,\ G = x^2 \qquad B^2 - 4AC = 0 - 4x \cdot 1 = -4x$$



Karakteristike:

$$\frac{dy}{dx} = \frac{1}{2x}\left(0 \pm \sqrt{-4x}\right) = \pm \frac{1}{\sqrt{-x}}$$

$$y = \int dy = \pm \int \frac{dx}{\sqrt{-x}} = \mp 2\sqrt{-x} + k$$

Vpeljimo novi neznanki $\quad \xi = y + 2\sqrt{-x}\,,\quad \eta = y - 2\sqrt{-x}$

$$u = u(\xi(x,y), \eta(x,y)) \qquad y = \frac{\eta + \xi}{2}$$

$$\xi - \eta = 4\sqrt{-x} \qquad x = -\left(\frac{\xi - \eta}{4}\right)^2$$

$$\Rightarrow \quad \frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial \xi}\right)\frac{\partial \xi}{\partial x} + \frac{\partial}{\partial x}\left(\frac{\partial \xi}{\partial x}\right)\frac{\partial u}{\partial \xi} + \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial \eta}\right)\frac{\partial \eta}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial^2 \eta}{\partial x^2} =$$

$$= \left(\frac{\partial^2 u}{\partial \xi^2}\frac{\partial \xi}{\partial x} + \frac{\partial^2 u}{\partial \eta \partial \xi}\frac{\partial \eta}{\partial x}\right)\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \xi}\frac{\partial^2 \xi}{\partial x^2} +$$

$$+ \left(\frac{\partial^2 u}{\partial \xi \partial \eta}\frac{\partial \xi}{\partial x} + \frac{\partial^2 u}{\partial \eta^2}\frac{\partial \eta}{\partial x}\right)\frac{\partial \eta}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial^2 \eta}{\partial x^2}$$

$$\frac{\partial \xi}{\partial x} = \frac{-1}{\sqrt{-x}} \qquad \frac{\partial \xi}{\partial y} = 1 \qquad \frac{\partial^2 \xi}{\partial x^2} = -\frac{1}{2x\sqrt{-x}}$$

$$\frac{\partial \eta}{\partial x} = \frac{1}{\sqrt{-x}} \qquad \frac{\partial \eta}{\partial y} = 1 \qquad \frac{\partial^2 \xi}{\partial y^2} = 0 \qquad \frac{\partial^2 \eta}{\partial y^2} = 0$$

$$\frac{\partial^2 \eta}{\partial x \partial y} = 0 \qquad \frac{\partial^2 \xi}{\partial x \partial y} = 0 \qquad \frac{\partial^2 \eta}{\partial x^2} = +\frac{1}{2x\sqrt{-x}}$$

Vstavimo vse v en. *

$$x\left[\left(\frac{\partial^2 u}{\partial \xi^2}\frac{-1}{\sqrt{-x}} + \frac{\partial^2 u}{\partial \eta \partial \xi}\frac{1}{\sqrt{-x}}\right)\frac{-1}{\sqrt{-x}} + \frac{\partial u}{\partial \xi}\frac{-1}{2x\sqrt{-x}} + \left(\frac{\partial^2 u}{\partial \xi \partial \eta}\frac{-1}{\sqrt{-x}} + \frac{\partial^2 u}{\partial \eta^2}\frac{1}{\sqrt{-x}}\right)\frac{1}{\sqrt{-x}} +\right.$$

$$\left. + \frac{\partial u}{\partial \eta}\frac{1}{2x\sqrt{-x}}\right] + \left[\left(\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta \partial \xi}\right) + \frac{\partial u}{\partial \xi}\cdot 0 + \left(\frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \eta^2}\right) + \frac{\partial u}{\partial \eta}\cdot 0\right] = x^2$$

$$\Rightarrow \quad 4\frac{\partial^2 u}{\partial \eta \partial \xi} - \frac{2}{\xi - \eta}\frac{\partial u}{\partial \xi} + \frac{2}{\xi - \eta}\frac{\partial u}{\partial \eta} = \left(\frac{\xi - \eta}{4}\right)^4$$

$\alpha = \frac{1}{2}(\xi + \eta) \qquad \beta = \frac{1}{2}(\xi - \eta)$

$\Rightarrow$ Na koncu dobimo

$$4\frac{\partial^2 u}{\partial\xi\partial\eta} = \frac{\partial^2 u}{\partial\alpha^2} - \frac{\partial^2 u}{\partial\beta^2}$$

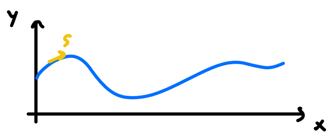
$$\frac{\partial u}{\partial\xi} = \frac{1}{2}\frac{\partial u}{\partial\alpha} + \frac{1}{2}\frac{\partial u}{\partial\beta}$$

$$\frac{\partial u}{\partial\eta} = \frac{1}{2}\frac{\partial u}{\partial\alpha} - \frac{1}{2}\frac{\partial u}{\partial\beta}$$

$$\Rightarrow \frac{\partial^2 u}{\partial\alpha^2} - \frac{\partial^2 u}{\partial\beta^2} - \frac{1}{2\beta}\left(\frac{\partial u}{\partial\alpha} + \frac{\partial u}{\partial\beta}\right) + \frac{1}{2\beta}\left(\frac{\partial u}{\partial\alpha} - \frac{\partial u}{\partial\beta}\right) = \left(\frac{\beta}{2}\right)^4$$

$$\Rightarrow \frac{\partial^2 u}{\partial\alpha^2} - \frac{\partial^2 u}{\partial\beta^2} - \frac{1}{\beta}\frac{\partial u}{\partial\beta} = \left(\frac{\beta}{2}\right)^4$$

Enačba strune

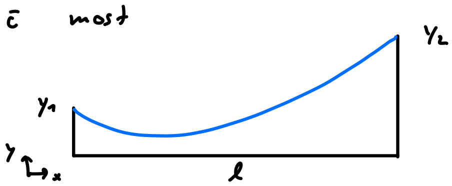


$$\rho\, x_{tt} = \frac{d}{ds}\left(F(s)\frac{dx}{ds}\right) + f^x$$
$$\rho\, y_{tt} = \frac{d}{ds}\left(F(s)\frac{dy}{ds}\right) + f^y$$

dolžinska gostota $\qquad$ gostota zun sil

$$ds^2 = dx^2 + dy^2$$

③ Viseč most



dolžina L, masa M, zanima nas stac. oblika

$$0 = \frac{d}{ds}\left(F\frac{dx}{ds}\right) + f^x$$
$$0 = \frac{d}{ds}\left(F\frac{dy}{ds}\right) + f^y$$

$f^x = 0 \qquad f^y = \frac{dF^y_{zun}}{ds} = -\frac{g\,dm}{ds} = -\frac{gM\,dx}{L\,ds}$

$$\frac{d}{ds}\left(F\frac{dx}{ds}\right) = 0 \qquad \Rightarrow F(s)\frac{dx}{ds} = F_0$$
$$\frac{d}{ds}\left(F\frac{dy}{ds}\right) - \frac{gM}{L}\frac{dx}{ds} = 0$$

$$\frac{d}{ds}\left(F_0\frac{ds}{dx}\frac{dy}{ds}\right) - \frac{gM}{L}\frac{dx}{ds} = 0$$

$$\frac{d}{ds}\frac{dy}{dx} - \frac{gM}{LF_0}\frac{dx}{ds} = 0$$

$$\cancel{\frac{dx}{ds}}\frac{d}{dx}\left(\frac{dy}{dx}\right) - \frac{gM}{LF_0}\cancel{\frac{dx}{ds}} = 0$$

$$\frac{d^2y}{dx^2} = \frac{gM}{LF_0}$$

$$\frac{dy}{dx} = \frac{gM}{\underbrace{LF_0}_{M/F_0}}x + C$$

$$y = \frac{gM}{2LF_0}x^2 + Cx + D \qquad F_0, C, D = ?$$

Rešimo pogoji: $y(0) = y_1$ $\quad y(l) = y_2$

$\hookrightarrow D = y_1$

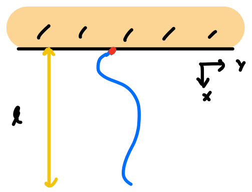
$\hookrightarrow y_2 = \frac{\mu l^2}{2F_0} + Cl + y_1$

$$L = \int ds = \int \sqrt{dx^2 + dy^2} = \int_0^l \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$L = \int_0^l \sqrt{1 + \left(\frac{\mu}{F_0}x + c\right)^2}\, dx$$

...

4 Majhno nihanje viseče strune



ρ gostota

l dolžina

$dy \ll 1$

$x \in [0, l]$

- Lastni nihajni načini?
- Časovni razvoj

Majhni odmiki $\begin{cases} ds^2 = dx^2 + dy^2 \\ ds = dx + O(dy^2) \end{cases}$

x: $$\rho \overset{0}{\overset{\shortparallel}{x_{tt}}} = \frac{d}{ds}\left(F(s)\frac{dx}{ds}\right) + f^x = \rho g$$

$$0 = \frac{d}{dx}F + \rho g$$

$$F = -\rho g x + C \qquad \text{R.P.} \qquad F(0) = \rho g l$$

$$\rho g l = C \quad \Rightarrow \quad F = \rho g (l - x)$$

y: $$\rho y_{tt} = \frac{\partial}{\partial x}\left(F\frac{\partial y}{\partial x}\right) + \overset{0}{\overset{\shortparallel}{f^y}}$$

$$\rho y_{tt} = \rho g \frac{\partial}{\partial x}\left((l-x)\frac{\partial y}{\partial x}\right)$$

$$\frac{1}{g} y_{tt} = -y_x + (l-x) y_{xx}$$

Separacija spremenljivk
$y(x,t) = R(x)\,T(t)$

$$\frac{1}{g} R\ddot{T} = -R'T + (l-x)R''T \qquad /: RT$$

$$\underbrace{\frac{1}{g}\frac{\ddot{T}}{T}}_{\text{konst.}} = \underbrace{-\frac{R'}{R} + (l-x)\frac{R''}{R}}_{\text{konst.}} = -\omega^2$$

$$\frac{1}{g}\frac{\ddot{T}}{T} = -\omega^2 \qquad\qquad -\frac{R'}{R} + (l-x)\frac{R''}{R} = -\omega^2$$

$$\ddot{T} = -\omega^2 g T \qquad\qquad (l-x)R'' - R' + \omega^2 R = 0$$

Besselova DE

$$z^2 \frac{d^2u}{dz^2} + z\frac{du}{dz} + (z^2 - n^2)u = 0$$

$$u(z) = A\, J_n(z) + B\, Y_n(z)$$

$$\underline{T = A\sin(\sqrt{g}\,\omega t) + B\cos(\sqrt{g}\,\omega t)}$$

! $z = \sqrt{1 - \frac{x}{l}}$

$$\frac{dR}{dx} = \frac{dR}{dz}\frac{dz}{dx} = \frac{dR}{dz}\frac{-\frac{1}{2l}}{\sqrt{1-\frac{x}{l}}} = -\frac{1}{2lz}\frac{dR}{dz}$$

$$\frac{d^2R}{dx^2} = \frac{dz}{dx}\frac{d}{dz}\left(-\frac{1}{2lz}\frac{dR}{dz}\right) = -\frac{1}{2lz}\left(\frac{1}{2lz^2}\frac{dR}{dz} - \frac{1}{2lz}\frac{d^2R}{dz^2}\right)$$

$$-\cancel{l} z^{\cancel{2}} \frac{1}{2\cancel{lz}}\left(\frac{1}{2lz^2}\frac{dR}{dz} - \frac{1}{2lz}\frac{d^2R}{dz^2}\right) + \frac{1}{2lz}\frac{dR}{dz} + \omega^2 R = 0$$

$$-\frac{1}{4lz}\frac{dR}{dz} + \frac{1}{4l}\frac{d^2R}{dz^2} + \frac{1}{2lz}\frac{dR}{dz} + \omega^2 R = 0$$

$$\frac{1}{4l}\frac{d^2R}{dz^2} + \frac{1}{4lz}\frac{dR}{dz} + \omega^2 R = 0 \qquad /\cdot z^2\, 4l$$

$$z^2\frac{d^2R}{dz^2} + z\frac{dR}{dz} + \omega^2 4l\, z^2 R = 0 \qquad z \to 2\omega\sqrt{l}\, z$$

$n = 0$

$$R = C J_0(2\omega\sqrt{l}\, z) + D Y_0(2\omega\sqrt{l}\, z)\ ; \quad z = \sqrt{1-\frac{x}{l}}$$

↳ neomejena $\Rightarrow D = 0$

$$Y_m = J_0(2\omega_m\sqrt{l-x})\left(A_m\sin(\sqrt{g}\,\omega t) + D_m\cos(\sqrt{g}\,\omega t)\right)$$

$$Y = \sum_{m=1}^{\infty} Y_m$$

Valovna enačba

$$\rho_L x_{tt} = \frac{d}{ds}\left(F(s)\frac{dx}{ds}\right)$$

$$\rho_L y_{tt} = \frac{d}{ds}\left(F(s)\frac{dy}{ds}\right) \Rightarrow \rho_L y_{tt} = F\frac{d^2y}{d^2x} \Rightarrow y_{tt} - c^2 y_{xx} = 0$$

$$c^2 = \frac{F}{\rho_L}$$

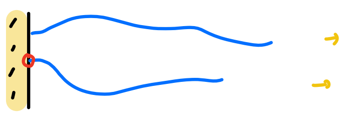
Če $dx \approx ds$ $\quad d_y \approx 1$

d'Alembertova rešitev (na neskončni struni)

$u(x,0)$, $u'(x,0)$

Rešitev $$u(x,t) = \frac{1}{2}\left(u(x-ct,0) + u(x+ct,0)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} u_t(x,0)\,dx$$

5 Polneskončna struna



ⓐ $u(0,t) = 0$

ⓑ $u'(0,t) = 0 \Rightarrow u(x,0) = \tilde{u}(x,0) + \tilde{u}(-x,0)$

ⓐ $$u(x,t) = \begin{cases} \tilde{u}(x,t) & x>0 \\ 0 & x<0 \end{cases}$$

$$u(x,0) = \tilde{u}(x,0) - \tilde{u}(-x,0) \qquad \text{liho prezrcalimo}$$

$$u_t(x,0) = \tilde{u}_t(x,0) - \tilde{u}_t(-x,0)$$

d'Alembert $$u(x,t) = \frac{1}{2}\Big(\tilde{u}(x-ct,0) - \tilde{u}(-x+ct,0) + \tilde{u}(x+ct,0) - \underbrace{\tilde{u}(-x-ct,0)}_{}\Big)$$
$$\overset{0}{\parallel}$$

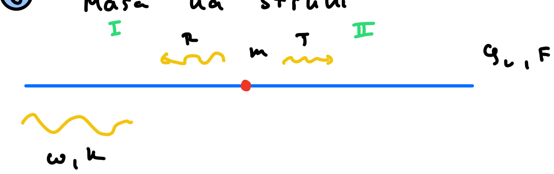
$$+ \frac{1}{2c}\int_{x-ct}^{x+ct}\tilde{u}_t(z,0)\,dz - \frac{1}{2c}\int_{-x-ct}^{-x+ct}\tilde{u}_t(z,0)\,dz$$

$$\int_0^{x+ct} - \int_0^{-x+ct} = \int_{-x+ct}^{x+ct}$$

Za $t>0$ $\quad u = 0$

$$u(x,t) = \begin{cases} \frac{1}{2}\left(-\tilde{u}(-x+ct,0) + \tilde{u}(x+ct)\right) + \frac{1}{2c}\int_{-x+ct}^{x+ct}\tilde{u}_t(z,0)\,dz & x<ct \\ \frac{1}{2}\left(\tilde{u}(x+ct,0) + \tilde{u}(x-ct,0)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct}\tilde{u}_t(z,0)\,dz & x>ct \end{cases}$$

6 Masa na struni

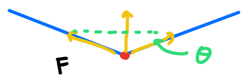


$$u_1 = e^{i(kx-\omega t)} + R\,e^{i(-kx-\omega t)} \qquad u_2 = T\,e^{i(kx-\omega t)}$$

RP $\bullet\ u_1(0,t) = u_2(0,t)$

$\bullet\ m\,\ddot{u}(t) = F\left(\frac{\partial u}{\partial x}\Big|_{0^+} - \frac{\partial u}{\partial x}\Big|_{0^-}\right)$

$u(t) = u_1(0,t) = u_2(0,t)$



$\sin\theta = \frac{F_y}{F} \approx \theta \qquad \tan\theta = \frac{\partial u}{\partial x} \approx \theta$

$\frac{\partial u}{\partial x} = \frac{F_y}{F}$

$e^{-i\omega t} + R\,e^{-i\omega t} = T\,e^{-i\omega t}$

$1 + R = T$

$u = T\,e^{-i\omega t}$

$Tm(-\omega^2)\,e^{-i\omega t} = F\left(Tik\,e^{-i\omega t} - ik\,e^{-i\omega t} + Rik\,e^{-i\omega t}\right)$

$-Tm\omega^2 = Fik\,(T - 1 + R)$

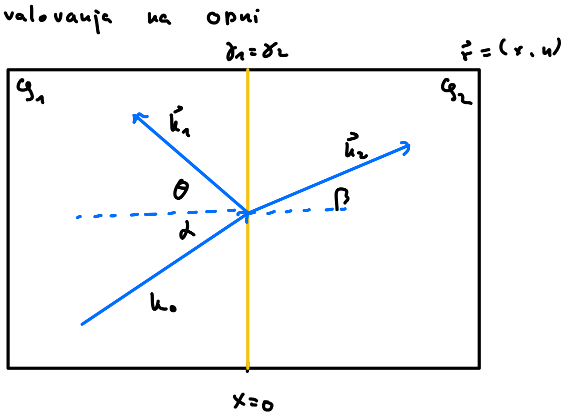
$0 = \frac{Fik}{m\omega^2}(T - 1 + T - 1) + T$

$\frac{2Fik}{m\omega^2} = T\left(2\frac{Fik}{m\omega^2} + 1\right) \qquad \omega = ck$

$T = \frac{1}{1 + \frac{m\omega^2}{2ikF}} = \frac{1}{1 - i\frac{k}{k_0}} \qquad \frac{m\omega^2}{2ikF} = \frac{mc^2k}{2iF} = \frac{k}{ik_0}$

$R = T - 1 = \frac{i\frac{k}{k_0}}{1 - i\frac{k}{k_0}} \overset{?}{=} \frac{-1}{1 + i\frac{k_0}{k}}$

7 Odboj valovanja na opni



Za opno

$z_{tt}(x,y,t) - c^2\nabla^2 z(x,y,t) = 0$

$c^2 = \frac{\gamma^2}{g}$ površinska napetost / površinska gostota

$z_1 = e^{i(\vec{k}_0\cdot\vec{r} - \omega t)} + R\,e^{i(\vec{k}_1\cdot\vec{r} - \omega t)}$

$z_2 = T\,e^{i(\vec{k}_2\cdot\vec{r} - \omega t)}$

Frekvenca enaka na obeh straneh

R.P. $\bullet\ z_1(0,t) = z_2(0,t)$

Vemo $F^y = \gamma^2\frac{\partial z}{\partial x}$

$\vec{k}_0 = k_0(\cos\alpha, \sin\alpha)$
$\vec{k}_1 = k_1(-\cos\theta, \sin\theta)$
$\vec{k}_2 = k_2(\cos\beta, \sin\beta)$

$\bullet\ F_1^y = F_2^y$

$\frac{\partial z_1}{\partial x}(0,y,t) = \frac{\partial z_2}{\partial x}(0,y,t)$

$\bullet\ e^{ik_0\sin\alpha\, y - \omega t} + R\,e^{ik_1\sin\theta\, y - \omega t} = T\,e^{ik_2\sin\beta\, y - \omega t}$

Faze in amplitude se morate ujemati

$k_0\sin\alpha = k_1\sin\theta = k_2\sin\beta \qquad 1 + R = T$

$\omega^2 = c^2\,\vec{k}\cdot\vec{k}$

$k_0 = k_1$ (ker c je enako) $\Rightarrow \alpha = \theta$

$k_0\sin\alpha = k_2\sin\beta \Rightarrow \frac{\omega}{c_1}\sin\alpha = \frac{\omega}{c_2}\sin\beta \Rightarrow \frac{\sin\alpha}{\sin\beta} = \frac{c_1}{c_2}$

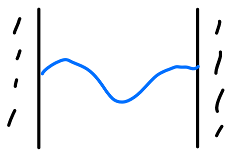
$$\bullet \quad k_0 \cos\alpha \, e^{i(k_0 y \sin\alpha - \omega t)} - R k_0 \cos\theta \, e^{i(k_0 y \sin\theta - \omega t)} = T k_2 \cos\beta \, e^{i(k_2 y \sin\beta - \omega t)}$$

$$k_0 \cos\alpha - R k_0 \cos\theta = T k_2 \cos\beta$$
$$k_0 (\cos\alpha - R\cos\theta) = (1+R) k_2 \cos\beta$$
$$k_0 \cos\alpha (1-R) = (1+R) k_2 \sqrt{1-\sin^2\beta} \qquad \sin^2\beta = \frac{c_2^2}{c_1^2}\sin^2\alpha$$
$$\cos\alpha (1-R) = (1+R)\frac{c_1}{c_2}\sqrt{1-\frac{c_2^2}{c_1^2}\sin^2\alpha}$$
$$1-R = (1+R)\underbrace{\frac{1}{\cos\alpha}\sqrt{\frac{c_1^2}{c_2^2}-\sin^2\alpha}}_{D}$$

$$1-R = D + RD$$
$$R(-D-1) = D-1$$
$$R = \frac{1-D}{1+D} \qquad T = \frac{2}{1+D}$$

DN na str. Síň je struna napeta s F, g_L

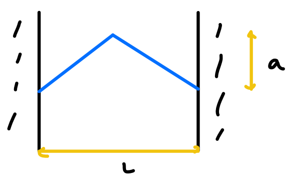
7



$$u_{tt} = c^2 u_{xx}$$
$$u(x,t) = \sum X_n T_n$$

$$T_n = A_n \cos\omega_n t + B_n \sin\omega_n t$$
$$X_n = C_n \cos k_n x + D_n \sin k_n x$$
$$\omega_n = c k_n$$

8



$$u(0,t) = u(L,t) = 0$$
$$u(x,0) = \begin{cases} \frac{2a}{L}x & ; \; x \le L/2 \\ -\frac{2a}{L}x + 2a & ; \; L/2 \le x \le L \end{cases}$$

$$\Rightarrow \quad C_n = 0 \qquad u(x,t) = \sum_n \sin k_n x (A_n \cos\omega_n t + B_n \sin\omega_n t)$$

$$k_n = \frac{n\pi}{L}$$
$$u(x,0) = \sum_n A_n \sin k_n x \qquad \Big/ \int \sin k_m x \, dx$$

$$\int_0^L u(x,0)\sin k_m x \, dx = A_m \frac{L}{2} \qquad \Rightarrow B_n = 0$$

$$\int_0^{L/2} \frac{2a}{L}x \sin k_m x\,dx + \int_{L/2}^{L}\left(2a - \frac{2a}{L}x\right)\sin k_m x \, dx = A_m \frac{L}{2}$$

$$\frac{2a}{L}\int_0^{L/2} x \sin k_m x \, dx + \frac{2a}{L}\int_{L/2}^{L}(L-x)\sin k_m x \, dx = A_m \frac{L}{2}$$

$$\int x \sin k_m x \, dx =$$
$$x = u \qquad \sin k_m x \, dx = dv$$
$$dx = du \qquad -\frac{1}{k_m}\cos k_m x = v$$
$$= -\frac{x}{k_m}\cos k_m x + \frac{1}{k_m^2}\sin k_m x$$

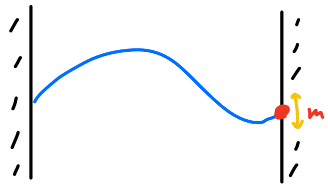
$$\frac{2a}{L k_m}\left(-x\cos k_m x + \frac{1}{k_m}\sin k_m\right)\Big|_0^{L/2} + \frac{2a}{L}L\frac{1}{k_m}\cos k_m x\Big|_{L/2}^{L}$$
$$+ \frac{2a}{L}\left(\frac{x}{k_m}\cos k_m x - \frac{1}{k_m^2}\sin k_m x\right)\Big|_{L/2}^{L} = A_m \frac{L}{2}$$

$$\vdots$$

$$A_m = \begin{cases} 0 & ; \; m \text{ sod} \\ \frac{8a}{\pi^2 m^2}(-1)^{\frac{m-1}{2}} \end{cases}$$

9 Struna z maso

Lastne frekvence / načini ?



Robni pogoji:

$u(0,t) = 0 \quad \Rightarrow \quad C_n = 0$

$$m \frac{d^2u}{dt^2}(L,t) = -F \frac{du}{dx}$$

$F_s = -F \frac{du}{dx}$

F napetost v struni

$$u(x,t) = \sum X_n T_n$$

$$T_n = A_n \cos \omega_n t + B_n \sin \omega_n t$$

$$X_n = C_n \cos k_n x + D_n \sin k_n x$$

$$u(x,t) = \sum_n \sin k_n x \,(A_n \cos \omega_n t + B_n \sin \omega_n t)$$

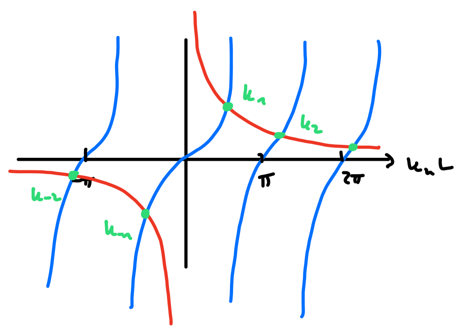
$$u_{tt} = \sum_n \sin k_n L\,(-A_n \omega_n^2 \cos \omega_n t - B \omega_n^2 \sin \omega_n t)$$

$$u_x = \sum k_n \cos k_n L \,(A_n \cos \omega_n t + B_n \sin \omega_n t)$$

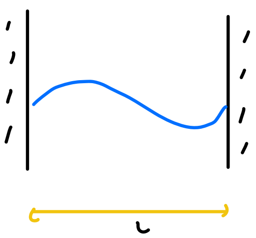
$$\sum_n \sin k_n L\,(-A_n \omega_n^2 \cos \omega_n t - B \omega_n^2 \sin \omega_n t) = -\frac{F}{m} \sum k_n \cos k_n L \,(A_n \cos \omega_n t + B_n \sin \omega_n t)$$

$\forall n$:
$$\omega_n^2 (A_n \cos \omega_n t + B_n \sin \omega_n t) = \frac{F}{m} k_n \frac{\cos k_n L}{\sin k_n L} (A_n \cos \omega_n t + B_n \sin \omega_n t)$$

$$k_n \tan k_n L = \underbrace{\frac{F}{mc^2}}_{k_0} \qquad \tan k_n L = \frac{k_0}{k_n}$$



10 Struna z zunanjo silo



$$u_{tt} = c^2 u_{xx} + \underbrace{f(x,t)}_{\text{zunanja sila}}$$

$$f(x,t) = f_0 \cos(\omega t)$$

$C_n = 0$

$$X_n(x) = \sqrt{\frac{2}{L}} \sin k_n x \qquad k_n = \frac{n\pi}{L}$$

$$f(x,t) = \sum_n X_n(x) f_n(t) \qquad / \int X_m \, dx$$

$$\int_0^L f(x,t) X_m \, dx = f_m(t)$$

$$\int_0^L f_0 \cos \omega t \sqrt{\frac{2}{L}} \sin k_m x \, dx = f_m(t)$$

$$f_m(t) = f_0 \frac{\sqrt{2L}}{n\pi} \left(1 - (-1)^n\right) \cos \omega t$$

$$f = \sum_n \frac{2f_0}{n\pi}\left(1-(-1)^n\right) \cos \omega t \, \sin k_n x$$

$$u = \sum_n \sin k_n x \left(A_n \cos \omega_n t + B_n \sin \omega_n t\right)$$

$$u_{tt} = c^2 u_{xx} + f$$

$$\sqrt{\tfrac{2}{L}} \sum \sin k_n x \left(-A_n \omega_n^2 \cos \omega_n t - D_n \omega_n^2 \sin \omega_n t\right) = \sum_n \left(-k_n^2 \sin k_n x\right)\left(A_n \cos \omega_n t + B_n \sin \omega_n t\right) c^2 \sqrt{\tfrac{2}{L}}$$

$$+ \sum_n \frac{2f_0}{n\pi}\left(1-(-1)^n\right) \cos \omega t \, \sin k_n x$$

$$A_n \cos \omega_n t + D_n \sin \omega_n t = \underbrace{A_n \cos \omega_n t + B_n \sin \omega_n t}_{u_n} - \frac{2f_0}{\omega_n^2 n\pi} \sqrt{\tfrac{2}{L}} \left(1-(-1)^n\right) \cos \omega t$$

$$\ddot{u}_n(t) = u_n(t) + f_n \cos(\omega t)$$

$$u = \underbrace{\left(A_n \cos \omega_n t + D_n \sin \omega_n t\right)}_{u^{Hom}} + u^{part}$$

$$u^p = C_n \sin \omega t + D_n \cos \omega t$$

$$-C_n \omega^2 \sin \omega t - D_n \omega^2 \cos \omega t = -\omega_n^2 \left(C_n \sin \omega t + D_n \cos \omega t\right) + f_n \cos(\omega t)$$

$$\left(C_n \sin \omega t + D_n \cos \omega t\right)\left(\omega_n^2 - \omega^2\right) = f_n \cos(\omega t)$$

$$C_n = 0 \qquad D_n = \frac{f_n}{\omega_n^2 - \omega^2} \qquad \text{Ce} \quad \omega^2 \neq \omega_n^2$$

$$u_{part} = \frac{f_n}{\omega_n^2 - \omega^2} \cos \omega t$$

Ce $\omega_n = \omega$

$$u_{part} = t\left(C_n' \sin \omega t + D_n' \cos \omega t\right)$$

$$u_t = C_n' \sin \omega t + D_n' \cos \omega t + t\left(+C_n' \omega \cos \omega t - D_n' \omega \sin \omega t\right)$$

$$u_{tt} = C_n' \omega \cos \omega t - D_n' \omega \sin \omega t + C_n' \omega \cos \omega t - D_n' \omega \sin \omega t + t\left(-C_n' \omega^2 \sin \omega t - D_n' \omega^2 \cos \omega t\right)$$

$$\ddot{u} = -c^2 u + f$$

$$2C_n' \omega \cos \omega t - 2D_n' \omega \sin \omega t + t\left(-C_n' \omega^2 \cancel{\sin \omega t} - D_n' \omega^2 \cos \omega t\right) = -\omega_n^2 t \left(C_n' \cancel{\sin \omega t} + D_n' \cos \omega t\right) + f_n \cos \omega t$$

$$2C_n' \omega \cos \omega t - 2D_n' \omega \sin \omega t = f_n \cos \omega t$$

$$D_n' = 0 \qquad C_n' = \frac{f_n}{2\omega}$$

$$\Rightarrow \quad u_{part} = \frac{f_n}{2\omega} t \sin \omega t$$

$$u(x,t) = \sum_n \sqrt{\tfrac{2}{L}} \sin k_n x \left(A_n \cos \omega_n t + D_n \sin \omega_n t + \begin{cases} \dfrac{f_n}{\omega_n^2 - \omega^2} \cos \omega t & \omega_n \neq \omega \\ \dfrac{f_n}{2\omega} t \sin \omega t & \omega_n = \omega \end{cases} \right)$$

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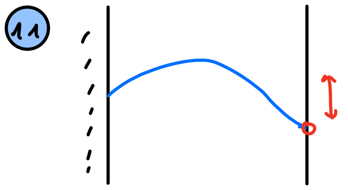


$u(0,t)=0$

$u(L,t)=U_0 \cos \omega t$ vzbujamo

$$u_{tt} = c^2 u_{xx}$$

$$\hat{u}(x,t) = u(x,t) - \frac{x}{L} U_0 \cos \omega t$$ trik

$\hat{u}(0,t)=0$

$\hat{u}(L,t)=0$

$$\hat{u}_{tt} - \frac{x}{L} U_0 \omega^2 \cos \omega t = c^2 \hat{u}_{xx}$$

$$\hat{u}_{tt} = c^2 \hat{u}_{xx} + \frac{x}{L} U_0 \omega^2 \cos \omega t$$

⋮ postopek enak kot pri prejšnji nalogi

T Difuzijska enačba

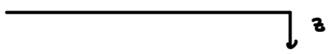
izvir toplote (volumska gostota)

$$\frac{\partial T}{\partial t} = D \nabla^2 T + \frac{q}{\rho c_p} \qquad D = \frac{\lambda}{\rho c_p}$$

12 Toplotni valovi

$T_z = T_0 e^{i\omega t}$

$$\frac{\partial T}{\partial t} = D \nabla^2 T \qquad \text{R.P.} \quad T(0,t) = T_0 e^{i\omega t}$$



Nastavek $T(z,t) = f(z)\, e^{i\omega t}$

$T(z,t) = ?$

$$f(z)\, i\omega\, e^{i\omega t} = D f''\, e^{i\omega t}$$

$$f'' - \underbrace{\frac{i\omega}{D}}_{\omega_c^2} f = 0$$

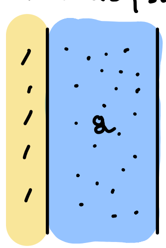
$$f = A \sin \omega_c z + B \cos \omega_c z = A e^{\omega_c z} + B e^{-\omega_c z}$$

R.P. $A + B = T_0$ $\qquad A=0 \qquad f(z\to\infty) < \infty$

$$\Rightarrow \quad T = T_0\, e^{i\omega t}\, e^{-\sqrt{\frac{i\omega}{D}}\, z} \qquad \sqrt{i} = \frac{1}{\sqrt{2}}(1+i)$$

$$T = T_0\, \mathrm{Re}\left(e^{i\omega t - \sqrt{\frac{\omega}{2D}}(1+i) z}\right) = T_0\, e^{-\sqrt{\frac{\omega}{2D}}\, z} \cos\left(\omega t - \sqrt{\frac{\omega}{2D}}\, z\right)$$

13 Radiacijsko oklajanje



T_z

$x=0$ $\qquad a$

$$\frac{\partial T}{\partial t} = D \nabla^2 T + \frac{q}{\rho c_p}$$

izolacija

R.P. ① $\left.\frac{\partial T}{\partial x}\right|_{x=0} = 0$ ② $j = \sigma T^4\big|_{x=a} - \sigma T_e^4 = -\lambda \left.\frac{\partial T}{\partial x}\right|_{x=a}$

okolica peče

$T(x) = ?$

$$\frac{\partial^2 T}{\partial x^2} = -\frac{q}{\rho c_p D}$$

RP 1 $\quad 0 = 0 + C \Rightarrow C = 0$

zanemarimo konvekcijo, upoštevamo sevanje črnega telesa

$$\frac{\partial T}{\partial x} = -\frac{q x}{\rho c_p D} + C$$

RP 2 $\quad -\lambda \frac{\partial T}{\partial x} = \left.\frac{q \lambda x}{\rho c_p D}\right|_{x=a} = \sigma\left(T^4\big|_{x=a} - T_e^4\right)$

$$T = -\frac{q x^2}{2\rho c_p D} + C x + E \quad \Rightarrow \quad E = \pm \sqrt[4]{\frac{q \lambda a}{\sigma \rho c_p D} + T_e^4} + \frac{q a^2}{2\rho c_p D}$$

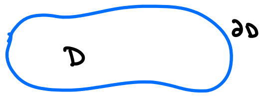
$$\Rightarrow \quad T = -\frac{q}{2\rho c_p D} x^2 \pm \sqrt[4]{\frac{q \lambda a}{\sigma \rho c_p D} + T_e^4} + \frac{q a^2}{2 \rho c_p D}$$

ⓣ Greenova funkcija za difuzijsko enačbo

$$\mathcal{L}(u(\vec r,t)) = f(\vec r,t)$$
$$\mathcal{L} = \frac{\partial}{\partial t} - D\nabla^2 \qquad \Rightarrow \qquad \left(\frac{\partial}{\partial t} - D\nabla^2\right)u = f \qquad \text{Dif. en.}$$

$$\mathcal{L}\,\underbrace{G(\vec r, t-\tau)}_{\text{Greenova funkcija}} = \delta(\vec r-\vec r_0)\,\delta(t-\tau)$$

Splošna rešitev



$$u(\vec r,t) = \int_{-\infty}^{t} d\tau \int_D d^3\vec r_0\; G(\vec r,\vec r_0, t-\tau)\, f(\vec r_0,\tau)$$

nehomogenost

začetni pogoj

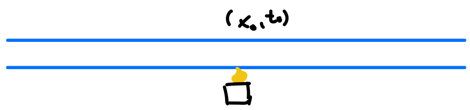
$$+ \int_D d^3\vec r_0\, u_0(\vec r_0)\, G(\vec r,\vec r_0,t)$$
$$- \int_{-\infty}^{t} d\tau \int_{\partial D} g(\vec r_0,\tau)\,\frac{\partial G}{\partial \hat n}(\vec r,\vec r_0,t-\tau)\,dS$$

robni pogoj

Greenova funkcija za difuzijo v 1D prostoru

$$G(x,x_0,t) = \frac{1}{\sqrt{4\pi D t}}\, e^{-(x-x_0)^2/4Dt}$$

⑭ Točkovni segrevalec na palici



$T(x,t) = ?$

Od časa t_0 začnemo segrevati palico v x_0.

$$\frac{\partial T}{\partial t} = D\nabla^2 T + f(x,t) \qquad f(x,t) = h\,\delta(x-x_0)\,H(t-t_0)$$

$$T(x,t) = \int_{-\infty}^{t} d\tau \int_{-\infty}^{\infty} dx' \frac{1}{\sqrt{4\pi D(t-\tau)}}\, e^{-(x-x')^2/4D(t-\tau)}\; h\,\delta(x-x_0)\,H(\tau-t_0)$$

$$= \int_{t_0}^{t} \frac{h}{\sqrt{4\pi D(t-\tau)}}\, e^{-\frac{(x-x_0)^2}{4D(t-\tau)}}\, d\tau \qquad \text{Nastavek } z = \frac{x-x_0}{\sqrt{4D(t-\tau)}}$$

$$\overset{x>x_0}{=} \int_{z_0}^{\infty} \frac{z h}{\sqrt{\pi}}\, e^{-z^2}\, \frac{dz}{z^3}\,\frac{(x-x_0)^2}{D\;\;2} \qquad dz = \frac{x-x_0}{\sqrt{4D}\,(t-\tau)^{3/2}}\left(-\tfrac{1}{2}\right)(-d\tau)$$

$$= \frac{(x-x_0)^2 h}{2D\sqrt{\pi}} \int_{z_0}^{\infty} \frac{e^{-z^2}}{z^2}\,dz = \qquad dz = \frac{x-x_0}{4\sqrt{D}\,(t-\tau)^{3/2}}\,d\tau$$

Per partes $\quad dv = \frac{dz}{z^2} \quad v = -\frac{1}{z}$
$\quad u = e^{-z^2} \quad du = -2z\,e^{-z^2}dz$

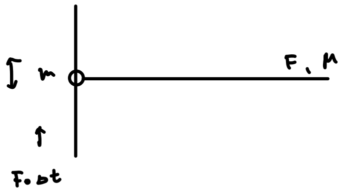
$$\sqrt{\pi}\,\mathrm{erf}(x) = 2\int_0^x e^{-z^2}dz$$

$$= \frac{(x-x_0)^2 h}{2D\sqrt{\pi}}\left(-\frac{1}{z}e^{-z^2}\Big|_{z_0}^{\infty} - 2\int_{z_0}^{\infty} e^{-z^2}dz\right)$$

$$= \frac{(x-x_0)^2 h}{2D\sqrt{\pi}}\left(\frac{e^{-z_0^2}}{z_0} - \sqrt{\pi}\,(\mathrm{erf}(\infty) - \mathrm{erf}(z_0))\right)$$

$$= \frac{h(x-x_0)^2}{2D\sqrt{\pi}}\left(\frac{e^{-z_0^2}}{z_0} - \sqrt{\pi}\,(1-\mathrm{erf}(z_0))\right) \qquad z_0 = \frac{x-x_0}{4\sqrt{D}\,(t-t_0)^{3/2}}$$

(u) Kolokvijska naloga



$$c^2 u_{xx} = u_{tt} \qquad c^2 = \frac{F}{\mu}$$

Z.P. $\quad u(x,0) = 0$

$$u_t(x,0) = \begin{cases} \frac{F_0 \delta t}{m} = v_0 & x = 0 \\ 0 & x > 0 \end{cases}$$

R.P. $\quad m \ddot{u}(0,t) = F \frac{\partial u}{\partial x}\Big|_{x=0}(t)$

$$u(x,t) = f(x-ct) + g(x+ct)$$

$x > 0$

- $0 = f(x) + g(x)$
- $-c f'(x) + c g'(x) = 0$

$$f'(x) - g'(x) = 0$$

$$-f(x) = -g(x) = c$$

$$u(x,t) = \begin{cases} f(x-ct) + c & x-ct \le 0 \\ 0 & x-ct > 0 \end{cases}$$

V R.P.

$$m c^2 f''(-ct) = F f'(-ct) \qquad \xi = \frac{mc^2}{F}$$

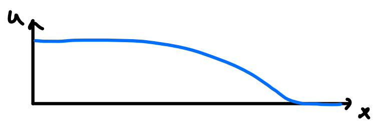
$$f(-ct) = A e^{-ct/\xi} + B$$

$$\Rightarrow \quad u(x,t) \underset{x-ct \le 0}{=} A e^{x-ct/\xi} + B + c$$

$$0 = A + B + c$$

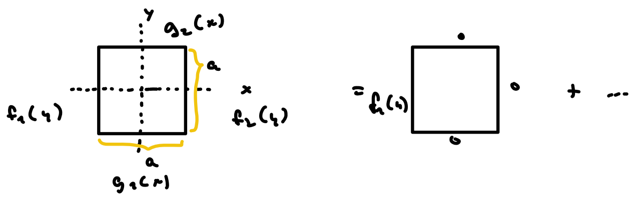
$$v_0 = -A \frac{c}{\xi}$$

$$u(x,t) = \begin{cases} \frac{v_0}{c} \xi \left(1 - e^{x-ct/\xi}\right) & x-ct \le 0 \\ 0 & x-ct > 0 \end{cases}$$

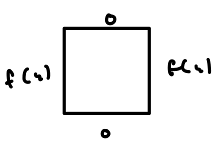


(T) Laplaceova enačba

$$\nabla^2 \varphi = 0$$



(15)



$$\nabla^2 \varphi = \frac{\partial^2}{\partial x^2} \varphi + \frac{\partial^2}{\partial x^2} \varphi = 0$$

$$\varphi\left(x, y \pm \tfrac{a}{2}\right) = 0 \qquad \varphi = X(x)\, Y(y)$$

$$\varphi\left(x \pm \tfrac{a}{2}, y\right) = f(y)$$

$$\underbrace{\frac{X''}{X}}_{m^2} + \underbrace{\frac{Y''}{Y}}_{-m^2} = 0$$

$$f(y) = \underbrace{\frac{f(y) + f(-y)}{2}}_{\text{sod}} + \underbrace{\frac{f(y) - f(-y)}{2}}_{\text{lih}}$$

V našem primeru f sod $\qquad Y'' + m^2 Y = 0 \qquad Y = \sum_n A_n \sin m_n y + B_n \cos m_n y$

$$\Rightarrow A_n = 0$$

$$y\left(-\tfrac{a}{2}\right) = y\left(\tfrac{a}{2}\right) = \sum_n D_n \cos m_n \tfrac{a}{2} = 0$$

$$m_n \frac{a}{2} = (n+1/2)\pi$$

$$m_n = \frac{2\pi}{a}(n+1/2)$$

$$X'' - m^2 X = 0$$

$$X = \sum_n C_n \operatorname{sh}\frac{2\pi}{a}(n+1/2)x + D_n \operatorname{ch}\frac{2\pi}{a}(n+1/2)x$$

ker $f(\frac{a}{2}) = f(-\frac{a}{2})$ sodost $\Rightarrow C_n = 0$

$$\Rightarrow \varphi(x,y) = \sum_{n=0}^{\infty} A_n \cos\frac{2\pi}{a}(n+1/2)y \operatorname{ch}\frac{2\pi}{a}(n+1/2)x$$

$$\varphi\left(\tfrac{a}{2}, y\right) = \sum_{n=0}^{\infty} A_n \cos\frac{2\pi}{a}(n+1/2)y \operatorname{ch}(\pi(n+1/2)) = f(y) \quad \Big| \int_{-a/2}^{a/2} \cos\frac{2\pi}{a}(l+1/2)y\, dy$$

$$\int_{-a/2}^{a/2} f(y)\cos\left(\frac{2\pi}{a}(l+1/2)y\right) dy = \sum_n A_n \operatorname{ch}(\pi(n+1/2))\,\delta_{nl}\,\frac{a}{2}$$

$\hookrightarrow \int_{-a/2}^{a/2} \cos^2(\ ) = \frac{a}{2}$

$$A_l = \frac{\int_{-a/2}^{a/2} f(y)\cos\left(\frac{2\pi}{a}(l+1/2)y\right) dy}{\frac{a}{2}\operatorname{ch}(\pi(l+1/2))}$$

(T) Hidrodinamika - Navier-Stokesova enačba

$$\rho\left(\frac{D\vec v}{Dt}\right) = \rho\vec f - \nabla p + \eta\nabla^2\vec v + (\zeta + \eta/3)\nabla(\nabla\vec v)$$ Ohranitev gibalne količine

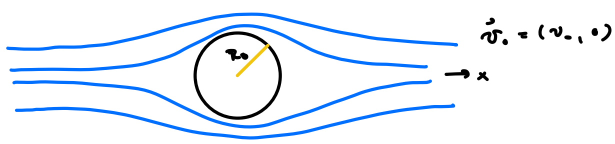
gostota sil; tlak; viskoznost; volumska viskoznost

$$\frac{D\rho}{Dt} = -\rho\nabla\vec v$$ Ohranitev mase

$$\frac{Dy}{Dt} = \frac{\partial y}{\partial t} + (\vec v\nabla)y$$

Nestisljiva tekočina $\nabla\vec v = 0$

(16) Steber v toku



$\vec v(x,y) = ?$

Predpostavke:

- stacionarno stanje $\frac{\partial\vec v}{\partial t} = 0$
- nestisljive tekočine $\nabla\cdot\vec v = 0$
- brezvrtinčna tekočina $\nabla\times\vec v = 0$

$\Downarrow$

konzervativno polje $\vec v = -\nabla\varphi$

tokovni potencial

$$\Rightarrow 0 = \nabla\cdot\vec v = -\nabla^2\varphi$$

V cilindričnih koordinatah

$$\vec v(r,\varphi) = v_r(r,\varphi)\hat e_r + v_\varphi(r,\varphi)\hat e_\varphi$$

R.P. $v_r(R_0,\varphi) = 0 \rightarrow \frac{\partial\phi}{\partial r}(R_0,\varphi) = 0$ $\qquad \nabla f = \frac{\partial f}{\partial r}\hat e_r + \frac{1}{r}\frac{\partial f}{\partial\varphi}\hat e_\varphi$

$\vec v(r\to\infty,\varphi) = v_0\hat e_x$ $\qquad \phi(x,y) = -v_0 x$

$\phi(r,\varphi) = -v_0 r\cos\varphi$

Splošna rešitev $\nabla^2 \varphi = 0$ v cilindričnih koordinatah

$$\phi(r,\varphi) = (A_0 + \underset{\substack{\shortparallel\\0}}{B_0} \ln r)(a_0 + \underset{\substack{\shortparallel\\0}}{b_0} \varphi) + \sum_{n=1}^{\infty} \left(A_n r^n + B_n \frac{1}{r^n}\right)(\underset{\substack{\shortparallel\\0}}{C_n} \sin n\varphi + D_n \cos n\varphi)$$

za naše RP

$$\phi(r\to\infty,\varphi) = \sum_{n=1}^{\infty} A_n r^n D_n \cos n\varphi = -v_0 r \cos\varphi$$

$$A_n = 0,\ n \neq 1 \qquad A_1 r D_1 \cos\varphi = -v_0 r \cos\varphi$$

$$\phi(r,\varphi) = -v_0 \cos\varphi\, r + \sum_{n=1}^{\infty} \tilde{D}_n \cos n\varphi \frac{1}{r^n}$$

$$\left.\frac{\partial\phi}{\partial r}\right|_{R_0} = 0 = -v_0 \cos\varphi + \sum \tilde{D}_n \cos n\varphi\,(-n) \left.\frac{1}{r^{n+1}}\right|_{R_0}$$

$$\tilde{D}_1 = \ = -v_0 R_0^2$$

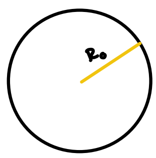
$$\Rightarrow \quad \phi(r,\varphi) = -v_0 \cos\varphi\, r - v_0 R_0^2 \cos\varphi \frac{1}{r}$$

$$\phi(r,\varphi) = -v_0 \cos\varphi \left(r + \frac{R_0^2}{r}\right)$$

$$\vec{v} = -\nabla\phi = -\frac{\partial\phi}{\partial r}\hat{e}_r - \frac{1}{r}\frac{\partial\phi}{\partial\varphi}\hat{e}_\varphi$$

$$= v_0\left(1 - \frac{R_0^2}{r^2}\right)\cos\varphi\,\hat{e}_r - v_0\left(1 + \frac{R_0^2}{r^2}\right)\sin\varphi\,\hat{e}_\varphi$$

17 Kako se obesi opna posode pod lastno težo



$$\nabla^2 u + \frac{p}{\gamma} = \frac{1}{c^2}\underset{\substack{\shortparallel\\0}}{\frac{\partial^2 u}{\partial t^2}}$$

stacionarno stanje

$$p = \frac{dF_g}{dS} = -\underbrace{\frac{dm}{dS}}_{\mu} g = -\mu g$$

površinska gostota

$$\nabla^2 u = \frac{\mu g}{\gamma}$$

Cilindrične koordinate

$$u(r,\varphi) \to u(r)$$

zaradi simetrije

RP $\quad u(R_0,\varphi) = 0$

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) = \frac{\mu g}{\gamma}$$

$$\frac{1}{r}\frac{d}{dr}(r u') = \frac{\mu g}{\gamma}$$

$$\int d\left(r\frac{du}{dr}\right) = \int r \frac{\mu g}{\gamma} dr$$

$$r\frac{du}{dr} = \frac{r^2}{2}\frac{\mu g}{\gamma} + C$$

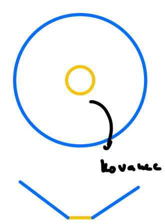
$$u = \int \frac{r}{2}\frac{\mu g}{\gamma} + \frac{C}{r}\, dr$$

$$u = \frac{r^2}{4}\frac{\mu g}{\gamma} + C\ln r + D$$

R.P. $\quad C = 0$ ne fizikalno ker

$$D = -\frac{R_0^2 \mu g}{4\gamma}$$

$$u = \frac{\mu g}{4\gamma}(r^2 - R_0^2)$$



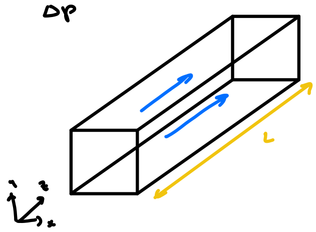
$$\frac{dF}{dr} = \gamma \frac{\partial u}{\partial r}$$

Dodatni primer

18 Pretok skozi cev

Q ... volumski pretok skozi cev ?

Δp



$$\rho \left(\frac{D\vec{v}}{Dt}\right) = \underset{\substack{\| \\ 0 \text{ sil ni}}}{\rho \vec{f}} - \nabla p + \eta \nabla^2 \vec{v} + (\zeta + \eta/3)\nabla(\underset{\substack{\| \\ 0}}{\nabla \cdot \vec{v}})$$

$\eta \neq 0$ je viskoznost; $\nabla\cdot\vec{v} = 0$ nestisljivost

$$\frac{D\vec{v}}{Dt} = \frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v}, \qquad \frac{\partial \vec{v}}{\partial t} = 0 \text{ stacionarno stanje}$$

$$\nabla p = \begin{bmatrix} 0 \\ 0 \\ \Delta p / L \end{bmatrix}$$

$$\rho (\vec{v}\cdot\nabla)\vec{v} = -\frac{\Delta p}{L}\hat{e}_z + \eta \nabla^2 \vec{v}$$

Simetrija: $\quad v = (0, 0, v(x,y))$

RP $\quad v(\text{rob}) = 0 \quad$ zaradi viskoznosti

$\Rightarrow v(0,y) = v(a,y) = v(x,0) = v(x,a) = 0$

$$\rho\, \underbrace{v_z \frac{\partial}{\partial z}(0, 0, v(x,y))}_{0} = -\frac{\Delta p}{L}\hat{e}_z + \eta \nabla^2 (0, 0, v(x,y))$$

$$\nabla^2 \begin{bmatrix} 0 \\ 0 \\ v_z \end{bmatrix} = \frac{\Delta p}{L\eta}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$z: \quad \frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} = \frac{\Delta p}{L\eta} \qquad Z(x,y) = X(x)\,Y(y)$$

Iščem lastne vrednosti in vekt. $\quad \nabla^2 Z = -\lambda^2 Z$

$$X''Y + Y''X = -\lambda^2 XY = -(\lambda_x^2 + \lambda_y^2) XY$$

$$X'' + \lambda_x^2 X = 0 \qquad X = A \sin \lambda_x x + \underset{\substack{\| \\ 0}}{B} \cos \lambda_x x$$

RP. $v(0,y) = 0$, $v(a,y) = 0 \qquad a\lambda_x = n\pi \qquad \lambda_x = \frac{n\pi}{a}$

$\Rightarrow X_n = A_n \sin \frac{n\pi}{a} x$

$$Y'' + \lambda_y^2 Y = 0 \quad \Rightarrow \quad Y_m = C_m \sin \frac{m\pi}{a} y$$

$$\Rightarrow Z = \sum_{n,m} \tilde{C}_{nm} \sin\frac{n\pi}{a}x \sin\frac{m\pi}{a}x$$

$$\nabla^2 Z = \frac{\partial^2 Z}{\partial x^2} + \frac{\partial^2 Z}{\partial y^2} + \underset{\substack{\| \\ 0}}{\frac{\partial^2 Z}{\partial z^2}} = \sum_{nm} \tilde{C}_{nm} \underbrace{\left(-\left(\frac{n\pi}{a}\right)^2 - \left(\frac{m\pi}{a}\right)^2\right)}_{\left(-\frac{\pi^2}{a^2}(n^2+m^2)\right)} \sin\frac{n\pi}{a}x \sin\frac{m\pi}{a}y$$

Rešujemo $\quad \nabla^2 Z = \frac{\Delta p}{L\eta} \qquad \Big| \int_0^a \sin\frac{n'\pi}{a}x\,dx \int_0^a \sin\frac{m'\pi}{a}y\,dy$

$$\sum_{nm} \tilde{C}_{nm} \left(-\frac{\pi^2}{a^2}\right)(n^2+m^2)\left(\frac{a}{2}\right)^2 \delta_{nn'}\delta_{mm'} = \frac{\Delta p}{L\eta}\int_0^a \sin\frac{n'\pi}{a}x\,dx \int_0^a \sin\frac{m'\pi}{a}y\,dy$$

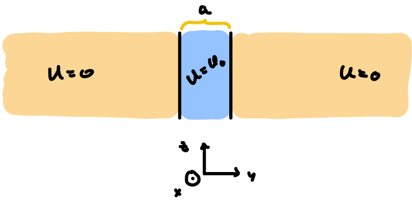
$$-\tilde{C}_{nm}\frac{\pi^2}{4}(n^2+m^2) = \frac{\Delta p}{L\eta}\,\frac{a}{n\pi}(1-\cos n\pi)\,\frac{a}{m\pi}(1-\cos m\pi)$$

$$\tilde{C}_{nm} = -\frac{\Delta p}{L\eta}\,\frac{4a^2}{\pi^4\, nm}\,\frac{1}{n^2+m^2}\,(1-(-1)^n)(1-(-1)^m)$$

$$\Rightarrow v_z(x,y) = -\frac{4a^2\Delta p}{\pi^4 L\eta}\sum_{n,m}\frac{(1-(-1)^n)(1-(-1)^m)}{nm(n^2+m^2)}\sin\frac{n\pi}{a}x \sin\frac{m\pi}{a}x$$

$$Q = \iint_0^a dx\,dy\, v_z(x,y) \Rightarrow Q \propto \frac{a^4\Delta p}{L\eta}$$

23 Elektrostatik



$U(x,y,z) = ?$

$\nabla\cdot\vec{E} = \frac{\rho}{\varepsilon} = 0$

$\nabla\times\vec{E} = -\frac{\partial B}{\partial t} = 0 \Rightarrow E = -\nabla U$

$\Rightarrow \nabla^2 U = 0$

R.P.

$$U(x=0,y,z) = \begin{cases} 0 ; & |y| \geq \frac{a}{2} \\ U_0 ; & |y| < a/2 \end{cases}$$

$U(x \to \infty) < \infty$

$U(x,y) = X(x)\,Y(y)$

$X''Y + Y''X = 0$

$$\frac{X''}{X} = -\frac{Y''}{Y} = k^2$$

$X'' - k^2 X = 0$ $\qquad$ $Y'' + k^2 Y = 0$

$X = Ae^{kx} + Be^{-kx}$ $\qquad$ $Y = Ce^{iky} + De^{-iky}$

$x > 0$ $\qquad$ simetrična rešitev, soda f.

RP2 $\quad A = 0$ $\qquad$ $Y(y) = Y(-y) \Rightarrow C = D$

$X_k = B_k e^{-|k|x}$

$$U(x,y) = \int_{-\infty}^{\infty} D(k)\, e^{-|k|x}\left(e^{iky} + e^{-iky}\right) dk$$

Če $k \to -k \Rightarrow D(k) = D(-k) \Rightarrow D(k)e^{iky} + D(-k)e^{-iky} = C(k)e^{iky}$

$/\cdot \int_{-\infty}^{\infty} e^{-ik'y}\,dy$ $\qquad$ $U(x=0,y) = \int_{-\infty}^{\infty} C(k)\, e^{iky}\, dk = \begin{cases} 0 & |y| \geq \frac{a}{2} \\ U_0 & |y| < \frac{a}{2} \end{cases}$ $\qquad$ $\int_{-\infty}^{\infty} e^{iky}\, dy = 2\pi\delta(k)$

$$= \iint_{-\infty}^{\infty} C(k)\, e^{i(k-k')y}\, dk\, dy$$

$$= \int C(k)\,\delta(k-k')\, 2\pi\, dk = C(k')\, 2\pi$$

$$\int_{-a/2}^{a/2} e^{-ik'y}\, U_0\, dy = \frac{U_0}{-ik'}\left(e^{-ik'a/2} - e^{ik'a/2}\right) = \frac{2U_0}{k'}\sin\left(k'\frac{a}{2}\right)$$

$$\Rightarrow C(k) = \frac{U_0 \sin\left(k\frac{a}{2}\right)}{\pi k}$$

$$\Rightarrow U(x,y) = \int_{-\infty}^{\infty} \frac{U_0 \sin k\frac{a}{2}}{k\pi}\, e^{-|k|x + iky}\, dk$$

$$= \int_{-\infty}^{\infty} \frac{U_0 \sin k\frac{a}{2}}{k\pi}\, e^{-|k|x}\left(\cos ky + \underbrace{i\sin ky}_{=0\ \text{liha}}\right) dk =$$

$$= 2\frac{U_0}{\pi}\int_0^{\infty} \frac{\sin k\frac{a}{2}}{k}\cos ky\; e^{-kx}\, dk$$

$$I(z) = \frac{2U_0}{\pi}\int_0^{\infty} \sin\frac{ka}{2}\,\cos ky\; e^{-kz}\, dk$$

$$U(x,y) = \int_x^{\infty} I(z)\, dz \xrightarrow{\frac{d}{dx}} -I(x) = \frac{dU}{dx}$$

$$\sin u \cos v = \frac{1}{2}\left(\sin(u+v) - \sin(u-v)\right)$$

$$\int_0^{\infty} \sin ks\; e^{-kz}\, dk = \frac{s}{s^2+z^2}$$

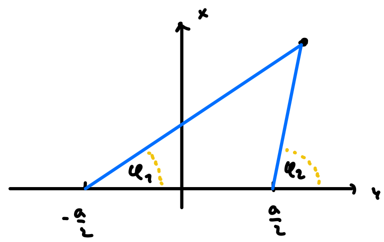
$$I(z) = \frac{2u_0}{\pi} \int_0^\infty \frac{1}{2}\left(\sin k\left(\frac{a}{2}+y\right) - \sin k\left(\frac{a}{2}-y\right)\right) e^{-kz}\, dk$$

$$= \frac{u_0}{\pi}\left(\frac{\frac{a}{2}+y}{(\frac{a}{2}+y)^2+z^2} - \frac{\frac{a}{2}-y}{(\frac{a}{2}-y)^2+z^2} \right)$$

$$U(x,y) = -\int I\, dz = \frac{u_0}{\pi}\left(\arctan \frac{z}{\frac{a}{2}+y} - \arctan \frac{z}{\frac{a}{2}-y} \right)\Bigg|_{z=x}^{\to\infty} =$$

$$= \frac{u_0}{\pi}\left(\frac{\pi}{2} - \arctan \frac{x}{\frac{a}{2}+y} - \frac{\pi}{2} + \arctan \frac{x}{\frac{a}{2}-y} \right)$$

$$u(x,y) = \frac{u_0}{\pi}\left(\arctan \frac{x}{\frac{a}{2}-y} - \arctan \frac{x}{\frac{a}{2}+y} \right)$$



$$U(\varphi_1, \varphi_2) = -\frac{u_0}{2}(\varphi_1 + \varphi_2)$$

Nastavek $\quad u(x,y) = \int_{-\infty}^{\infty} c(k)\, e^{-|k|x}\, e^{iky}\, dk$

(T) Helmholzova enačba

$$\nabla^2 f + k^2 f = 0$$

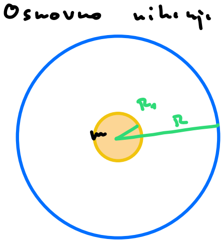
$$u_{tt} = c^2 \nabla^2 u$$
$$u = T(t) R(\vec{r})$$
$$\frac{T''}{T} = c^2 \frac{\nabla^2 R}{R} = -\omega^2$$

$$T = A e^{i\omega t} + B e^{-i\omega t}$$
$$\nabla^2 R + k^2 R = 0$$

Rešitev v cilindričnih koordinatah

$$f(\varphi, r) = \sum_{m=0}^{\infty} \left(A_m J_m(kr) + B_m Y_m(kr)\right)\left(C_m \cos m\varphi + D_m \sin m\varphi\right)$$

(20) Osnovno nihanje opne z maso



$\omega_0 = ?$

Za majhne nihanje velja enačba

$$c^2 \nabla^2 u = u_{tt} \qquad c^2 = \frac{\gamma}{\mu}$$

$$u(r, \varphi, t) = f(r, \varphi)\, e^{-i\omega t}$$

Osnovni način $\quad f(r,\varphi) \to f(r)$

RP $\quad u(R, \varphi) = 0$

$$m\, u_{tt} = 2\pi R_1 \gamma \frac{du}{dr}\Big|_{r=R_1}$$

(+ navori če bi bili)

Na stiku $\quad \frac{dF}{dl} = \gamma \frac{du}{dr}$

$$c^2 \nabla^2 u = u_{tt} \qquad \nabla^2 f_r = \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial f}{\partial r}\right)$$

$$c^2 \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial f}{\partial r}\right) e^{-i\omega t} = f\,(-\omega^2)\, e^{-i\omega t} \qquad \omega = kc$$

$$\underbrace{\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial f}{\partial r}}_{\nabla^2} + k^2 f = 0$$

Neodvisno od φ k za $n=0$ $\qquad f(r) = A\, J_0(kr) + B\, Y_0(kr)$

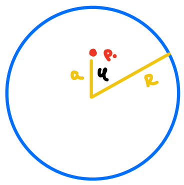
RP $\qquad 0 = A\, J_0(kR) + B\, Y_0(kR) \qquad B = -A \frac{J_0(kR)}{Y_0(kR)}$

$$M(-\omega^2)\left(A\, J_0(kR_1) + B\, Y_0(kR_1)\right) e^{-i\omega t} = 2\pi R_1 \gamma\, e^{-i\omega t} \left(A k\, J_0'(kR_1) + D k\, Y_0'(kR_1)\right)$$

$$\cancel{A}\left(J_0(kR_1) - \frac{J_0(kR)}{Y_0(kR)} Y_0(kR_1) \right) = -\frac{2\pi R_1 \gamma}{\mu k c^2} \cancel{A} \left(J_0'(kR_1) - \frac{J_0(kR)}{Y_0(kR)} Y_0'(kR_1) \right)$$

Rešimo numerično

(21) Boben



ob $t=0$ z momentom p_0 sunemo po bobnu, $u(r,\varphi,t)=?$

$$\frac{\gamma}{\mu} \nabla^2 u = u_{tt}$$

RP $\quad u(R,\varphi,t)=0$

ZP $\quad u(r,\varphi,0)=0$

$\dot u(r,\varphi,0) = \frac{c_0}{r}\delta(r-a)\delta(\varphi) = \frac{p_0}{\sigma r}\delta(r-a)\delta(\varphi)$

zaradi δ

$$\sigma \int \dot u \, r\, dr\, d\varphi = p_0 \Big|_{t=0}$$

$$\sigma \int \frac{c_0}{r}\delta(r-a)\delta(\varphi)\, r\, d\varphi\, dr = p_0$$

$$\sigma c_0 = p_0 \qquad c_0 = p_0/\sigma$$

$$u(r,\varphi,t) = T(t)\, R(r,\varphi)$$

$$c^2\, T \nabla^2 R = \ddot T R$$

$$c^2 \frac{\nabla^2 R}{R} = \frac{\ddot T}{T} = -\omega^2$$

$$\nabla^2 R + k^2 R = 0 \qquad \ddot T = -\omega^2 T \qquad T = A \sin \omega t + B \cos \omega t$$

$$R = \sum_{n=0}^{\infty} \left(A_n J_n(kr) + B_n Y_n(kr)\right)\left(C_n \cos n\varphi + D_n \sin n\varphi\right)$$

$B_n = 0 \quad$ Y_0 ima pol v ničli

$D_n = 0 \quad$ ker $\varphi \to -\varphi$ simetrija

$$R = \sum_{n=0}^{\infty} E_n J_n(kr) \cos n\varphi$$

$$R(R,\varphi) = 0 = \sum_{n=0}^{\infty} E_n \underbrace{J_n(kR)}_{=0} \cos n\varphi$$

$$k_{np} = \frac{\xi_{np}}{R} \quad \text{p-ta ničla } J_n$$

$$R(r,\varphi) = \sum_{n,p} E_{np} J_n\left(\frac{\xi_{np}}{R} r\right) \cos n\varphi$$

$$u(r,\varphi,t) = \sum_{n,p} E_{np} J_n\left(\frac{\xi_{np}}{R} r\right) \cos n\varphi \left(\sin \omega_{np} t + A_{np} \cos \omega_{np} t\right)$$

$$u(r,\varphi,0) = 0 \Rightarrow A_{np} = 0$$

$$\boxed{\int_0^1 x\, J_m(x\,\xi_{mp})\, J_m(x\,\xi_{mp'})\, dx = \frac{\delta_{pp'}}{2}\, J_{m+1}^2(\xi_{mp})}$$

$$\dot{u}(r,\varphi,0) = \sum_{mp} E_{mp}\, J_m\!\left(\frac{\xi_{mp}}{R} r\right) \cos m\varphi \;\; \omega_{mp} \underbrace{\cos\omega_{mp} 0}_{1} = \frac{p_0}{\sigma r}\,\delta(r-a)\,\delta(\varphi)$$

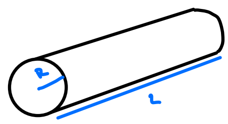
$$/\cdot \int_0^R r J_n\!\left(\frac{\xi_{np'}}{R} r\right) dr \int_0^{2\pi} \cos n\varphi$$

$$\frac{p_0}{\sigma}\,\frac{a\, J_n\!\left(\frac{\xi_{np}}{R} a\right)}{a} = \sum_{mp} \pi\, E_{mp}\;\; \frac{\delta_{pp'}}{2}\, J_{m+1}^2(\xi_{mp})\; \delta_{mn}$$

$$\frac{p_0}{\sigma}\, J_n\!\left(\frac{\xi_{np}}{R} a\right) = \pi\, E_{n,p}\, \tfrac{1}{2} J_{n+1}^2(\xi_{np})\; R^2\; \omega_{np}$$

$$E_{np} = \frac{2p_0}{\sigma \pi R^2 \omega_{np}}\;\frac{J_n\!\left(\frac{\xi_{np}}{R} a\right)}{J_{n+1}^2(\xi_{np})}$$

(22) Pretok skozi okroglo cev



$\eta,\ \varrho$

a) Δp

b) $\Delta p\, e^{i\omega t}$

nestisljiva $\nabla\cdot\vec{v} = 0$

$$\varrho\left(\underbrace{\frac{\partial\vec{v}}{\partial t}}_{\substack{0\\ \text{stac. stanje}}} + \underbrace{(\vec{v}\cdot\nabla)\vec{v}}_{0}\right) = -\nabla p + \eta\nabla^2\vec{v} + \underbrace{\varrho\vec{f}}_{0}$$

Simetrija $\quad \vec{v} = \vec{v}(r,t) = (0,0,v_z(r,t)) = \hat{e}_z\, v_z(r,t)$

$$\nabla p = \left(0,0,\frac{\Delta p(t)}{L}\right)$$

RP $\quad v_z(R,\varphi,t) = 0$

(a)

$$\frac{\Delta p(t)}{L} = \eta\, \nabla^2 v_z(r)$$

$$\frac{\Delta p}{L\eta} = \frac{1}{r}\frac{\partial}{\partial r}\, r\, \frac{\partial v_z}{\partial r}$$

$$\int \frac{\Delta p}{L\eta}\, r\, dr = r\,\frac{\partial v_z}{\partial r}$$

$$\int \frac{\Delta p}{2L\eta}\, r + \frac{C}{r}\, dr = v_z$$

ker divergira $r\to 0$

$$v_z = \frac{\Delta p}{4L\eta} r^2 + \underset{\substack{\shortparallel\\0}}{C}\ln r + D$$

R.P. $\quad 0 = \frac{\Delta p R^2}{4L\eta} + D \qquad D = -\frac{\Delta p\, R^2}{4L\eta}$

$$v_z = \frac{\Delta p}{4L\eta}\left(r^2 - R^2\right)$$

(b)

$$\varrho\,\frac{\partial v_z}{\partial t} = -\frac{\Delta p}{L} e^{i\omega t} + \eta\nabla^2 v_z$$

$$v_z = R(r)\, e^{i\omega t}$$

$$i\omega\varrho\, R\, e^{i\omega t} = -\frac{\Delta p}{L} e^{i\omega t} + \eta\, e^{i\omega t}\, \nabla^2 R$$

$$\nabla^2 R - \frac{i\omega\varrho}{\eta} R = \frac{\Delta p}{L\eta} \qquad k^2 = -\frac{i\omega\varrho}{\eta}$$

$$\nabla^2 R + k^2 R = \frac{\Delta p}{L\eta}$$

Splošna rešitev $\quad R = R_{hom} + R_{part}$

Homogena rešitev

$$\nabla^2 R_H + h^2 R_H = 0$$

Ker R ni funk. $\varphi \to k$ pri $m=0$ ostane

$$R_H(r) = A J_0(hr) + B\, Y_0(hr)$$

$B = 0$ divergira v $r \to 0$

Partikularna rešitev

$$R_p = C$$

$$0 + h^2 C = \frac{\Delta p}{L\eta} \qquad C = \frac{\Delta p}{L\eta h^2}$$

$$\Rightarrow \quad R = A J_0(hr) + \frac{\Delta p}{L\eta h^2} = A J_0(hr) + \frac{\Delta p\, i}{\rho \omega L}$$

RP $\quad R(R,t) = 0$

$$0 = A J_0(hR) + \frac{\Delta p\, i}{\rho \omega L}$$

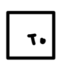
$$A = -\frac{\Delta p\, i}{\rho \omega L\, J_0(hR)}$$

$$v_z(r,t) = \mathrm{Re}\left(e^{i\omega t}\, \frac{\Delta p\, i}{\rho \omega L}\left(1 - \frac{J_0(hr)}{J_0(hR)}\right)\right) \qquad h^2 = -\frac{i\omega\rho}{\eta}$$

$J_0(ax) = 1 - \frac{a^2x^2}{4} + O(x^4)$ za majhne k oz. ω dobimo enako rešitev kot v (c)

(k)

(a)



T_v

$T(x,y,z,t) = ?$

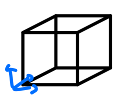
$$D\nabla^2 T = \frac{dT}{dt} \qquad T|_{\partial D} = T_v$$

$$\tilde{T} \to T - T_v \qquad \tilde{T}|_{\partial D} = 0$$

$$T = X\, Y\, Z\, \tau(t)$$

$$\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = \underbrace{\frac{1}{D}\frac{\tau'}{\tau} = -\lambda^2} = -\lambda_x^2 - \lambda_y^2 - \lambda_z^2$$

$$\tau(t) = A e^{-t\lambda^2 D}$$



$$\frac{X''}{X} = -\lambda_x^2 \qquad X = B\sin\lambda_x x + C\cos\lambda_x x, \quad C = 0$$

$$\sin\lambda_x a = 0 \qquad \lambda_x a = n\pi$$

$$\lambda_x = \frac{n\pi}{a}$$

$$\Rightarrow \quad T = \sum_{nml} C_{nml} \sin\frac{n\pi}{a}x \sin\frac{m\pi}{a}y \sin\frac{l\pi}{a}z\; e^{-\frac{\pi^2}{a^2}(n^2+m^2+l^2)Dt}$$

$$T(t=0) = T_0 - T_v\Big|_D = \sum_{nml} C_{nml} \sin\frac{n\pi}{a}x \sin\frac{m\pi}{a}y \sin\frac{l\pi}{a}z \qquad \Big/ \int_0^a \sin\frac{n'\pi}{a}x\, dx$$

$\vdots$

$$C_{nml} = (T_0 - T_v)\left(\frac{2}{a}\right)^3\left(\frac{a}{\pi}\right)^3 \frac{1}{nml}\left(1-(-1)^n\right)\left(1-(-1)^m\right)\left(1-(-1)^l\right)$$

$$T = T_0 + C_{111} \sin\frac{\pi}{a}x \sin\frac{\pi}{a}y \sin\frac{\pi}{a}z \, e^{-\lambda t D (\frac{\pi}{a})^2}$$ razvoj

(b) $-\lambda \frac{\partial T}{\partial z}\Big| = j$ $\quad \nabla^2 T = 0$

$T_{00} = T_0 \to 0$ $\quad T = X Y Z$

$$\frac{X''}{X} + \frac{Y''}{Y} = -\frac{Z''}{Z} = -\lambda_n^2 - \lambda_m^2 = -(\frac{\pi}{a})^2(n^2+m^2)$$

$$X = \sin\frac{n\pi}{a}x \qquad Y = \sin\frac{m\pi}{a}y$$

$$Z'' = Z(\frac{\pi}{a})^2(n^2+m^2)$$

$$Z = A_{nm}\, sh\, \frac{\pi}{a}\sqrt{n^2+m^2}\, z + \underbrace{D_{nm}}_{=0}\, ch\, \frac{\pi}{a}\sqrt{n^2+m^2}\, z$$

ker je na enem robu nič

$$T(x,y,z) = \sum_{n,m} C_{nm} \sin\frac{n\pi}{a}x \sin\frac{m\pi}{a}y \; sh\, \frac{\pi}{a}\sqrt{n^2+m^2}\, z$$

$$-\lambda \frac{\partial T}{\partial z}\Big|_a = j$$

$$-\frac{j}{\lambda} = \sum_{n,m} C_{n,m} \frac{\pi}{a}\sqrt{n^2+m^2} \sin\frac{n\pi}{a}x \sin\frac{m\pi}{a}y \; ch(\frac{\pi}{a}\sqrt{n^2+m^2}\,a)$$

$/\int_0^a \sin\frac{n'\pi}{a}x\,dx$ ...

$$-\frac{(1-(-1)^n)(1-(-1)^m)}{nm}\frac{j}{\lambda}(\frac{a}{\pi})^2 = C_{n,m}\frac{\pi}{a}\sqrt{n^2+m^2}\,(\frac{a}{2})^2\, ch(\frac{\pi}{a}\sqrt{n^2+m^2}\,a)$$

(J) Helmholzova enačba v sferičnih koordinatah

$$\nabla^2 f + \lambda f = 0 \qquad f = R(r)\, Y_\ell^m(\theta,\varphi)$$

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \frac{\partial f}{\partial r}) - \frac{L^2}{r^2} f \qquad L^2 = -\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial}{\partial\theta}) - \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}$$

$$L^2 Y_\ell^m(\theta,\varphi) = \ell(\ell+1) Y_\ell^m(\theta,\varphi)$$

$$Y_\ell^m = \sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}\, P_\ell^m(\cos\theta)\, e^{im\varphi}$$

$$\int_0^{2\pi} d\varphi \int_{-1}^{1} d\cos\theta \; Y_{\ell'}^{m'} Y_\ell^{m*} = \delta_{\ell\ell'}\delta_{mm'}$$

r: $$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial R}{\partial r}) - \frac{\ell(\ell+1)}{r^2}R + \lambda R = 0$$

$$R(r) = A\, j_\ell(\sqrt{\lambda}\, r) + B\, y_\ell(\sqrt{\lambda}\, r)$$

sferične Besslove funkcije — sferične Neumannove funkcije

$$j_\ell(r) = \sqrt{\frac{\pi}{2r}}\, J_{\ell+1/2}(r) \qquad y_\ell(r) = \sqrt{\frac{\pi}{2r}}\, Y_{\ell+1/2}(r)$$

$$j_0 = \frac{\sin x}{x} \qquad y_0 = -\frac{\cos x}{x}$$

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$$

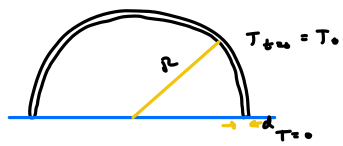
Ortogonalne z utežjo r^2.

$$\int_{-1}^{1} P_n(\cos\theta) P_m(\cos\theta)\, d\cos\theta = \frac{2}{2n+1}\delta_{nm}$$

Če $\lambda = 0$ $\quad \nabla^2 f = 0 \quad R(r) = A_\ell r^\ell + B_\ell \frac{1}{r^{\ell+1}}$

$P_0 = 1 \quad P_1 = x \quad P_2 = \frac{1}{2}(3x^2-1)$

(23) Polkroglna lupina



$d \ll R$ $\quad T(\vec{r}, t) = ?$

$$D\nabla^2 T = \frac{\partial T}{\partial t}$$

$T = T(\theta)$ ker je ozka
$= \Theta(\theta)\,\tau(t)$ in osno simetrično

$$\frac{\nabla^2\Theta}{\Theta} = \frac{1}{D}\frac{\tau'}{\tau} = -\lambda$$

$$\nabla^2\Theta + \lambda\Theta = 0 \qquad \tau = A e^{-D\lambda t}$$

$$\Theta(\theta) \sim P_l^0(\cos\theta)$$

$$T = \sum_{l=0}^{\infty} A_l\, P_l^0(\cos\theta)\, e^{-D\frac{l(l+1)}{R^2}t}$$

R.P. $\quad T(\pm\frac{\pi}{2}, t) = 0$

$$0 = \sum_l A_l\, P_l(0)\, e^{-D\frac{l(l+1)}{R^2}t}$$

Iščemo ničle $P_l(0)$

Vsi lihi členi imajo še ničlo v 0.

$\Rightarrow$ Vsi sodi členi so enaki 0.

Z.P. $\quad T(\theta, 0) = T_0$

$$T_0 = \sum_{\text{lihi}} A_l\, P_l(\cos\theta) \qquad \Big/ \int_0^1 P_{l'}(\cos\theta)\, d\cos\theta$$

$$T_0 \int_0^1 P_{l'}(\cos\theta)\, d\cos\theta = \sum_{\text{lihi}} \int_0^1 A_l\, P_l(\cos\theta)\, P_{l'}(\cos\theta)\, d\cos\theta$$

lihe $\quad$ lihe $\Rightarrow$ sodo $\Rightarrow \int_0^1 = \frac{1}{2}\int_{-1}^1$

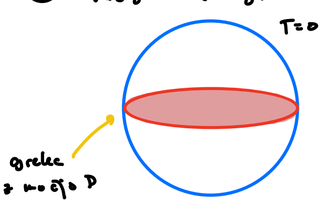
$$T_0 \int_0^1 P_l(\cos\theta)\, d\cos\theta = \frac{2}{2l+1}\,\frac{1}{2}A_l$$

Če nas zanima po daljšem času je le $l=1$

$$T_0 \int_0^1 x\, dx = \frac{A_1}{3}$$

$$\frac{3}{2}T_0 = A_1 \qquad T = \frac{3T_0}{2}\cos\theta\, e^{-\frac{2D}{R^2}t}$$

(24) Krogla z grelcem



stacionarni $T = ?$

$$D\nabla^2 T + \frac{q}{\rho c_p} = 0$$

$$\nabla^2 T = -\frac{q(\vec{r})}{\rho c_p D}$$

$$q(\vec{r}) = q_0\,\delta(z) = q_0\,\frac{\delta(r\cos\theta)}{r^2} =$$

$$= q_0\,\frac{\delta(\cos\theta)}{r}$$

$$P = \int \rho\, dV = \rho_0 \int_0^{2\pi}\int_0^{\pi}\int_0^{R} \frac{r^2 dr\, d\varphi}{r}\, d(\cos\theta)\, \delta(\cos\theta)$$

$$= \rho_0 \frac{R^2}{2} 2\pi \cdot 1$$

$$\rho_0 = \frac{P}{\pi R^2}$$

Simetrija $\quad T = T(r, \theta) \quad \Rightarrow \quad m = 0$

$$T = \sum_l (A_l j_l \sqrt{\lambda} r + \underset{\substack{\uparrow \\ 0}}{B_l}\, y_l \sqrt{\lambda} r)\; P_l(\cos\theta)$$

divergira v $r \to 0$

R.P. $\quad T(r = R, \theta) = 0$

$$0 = \sum_l A_l \underbrace{j_l \sqrt{\lambda} R}_{=0} P_l(\cos\theta)$$

$$\xi_{l,n} = \sqrt{\lambda} R \quad \Rightarrow \quad \lambda = \frac{\xi_{l,n}^2}{R^2}$$

n-ta ničla

$$\Rightarrow \quad T = \sum_{l,n} A_l\, j_l\left(\xi_{l,n} \frac{r}{R}\right) P_l(\cos\theta)$$

Moramo še zadostiti prvotni enačbi $\quad \nabla^2 T = -\frac{\rho(\vec{r})}{D \rho c_p}$

$$\nabla^2 T = -\lambda T = -\sum_{l,n} \frac{\xi_{l,n}^2}{R^2} A_{l,n}\, j_l\left(\xi_{l,n} \frac{r}{R}\right) P_l(\cos\theta) = -\frac{\rho_0\, \delta(\cos\theta)}{D \rho c_p\, r}$$

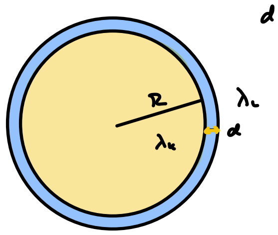
$$\Big/ \int_{-1}^{1} P_{l'}(\cos\theta)\, d\cos\theta \quad \int_0^R j_{l'}\left(\frac{\xi_{l'n'}}{R} r\right) r^2 dr$$

$$A_{ln} \frac{\xi_{ln}^2}{R^2} \frac{2}{2l+1} \underbrace{\int_0^R j_l\left(\frac{\xi_{nl}}{R} r\right) j_{l'}\left(\frac{\xi_{n'l'}}{R} r\right) r^2 dr}_{\propto\, \delta_{nn'}\, \delta_{ll'}} = \frac{\rho_0}{D \rho c_p} \underbrace{\int_{-1}^{1} \delta(\cos\theta) P_l(\cos\theta) d\cos\theta}_{P_l(0)} \int_0^R j_l\left(\frac{\xi_{ln}}{R} r\right) r\, dr$$

$$A_{l,n} = \frac{P(2l+1)}{\lambda_{l,n} R^2 \pi^2 D \rho c_p} \frac{P_l(0) \int_0^R r\, j_l\left(\frac{\xi_{nl}}{R} r\right) dr}{2 \int_0^R r^2 j_l^2\left(\frac{\xi_{nl}}{R} r\right) dr}$$

25 Krogla z lakom

$d \ll R \qquad T$ po dolgem času



$T_0 = 0$

$$T(\vec{r}, t) \to T(r, t) \qquad t \gg 1$$

R.P. $\quad j\Big|_{r=R^+} = \lambda_L \frac{(T(R,t) - 0)}{d} \qquad$ tok skozi lak

$$\parallel$$

$$j\Big|_{r=R^-} = -\lambda_k \frac{\partial T}{\partial r}\Big|_{r=R}$$

$$T(R,t) = -\frac{\lambda_k d}{\lambda_L} \frac{\partial T}{\partial r}\Big|_{r=R}$$

$$\frac{\partial T}{\partial t} = D \nabla^2 T \qquad T = \tau(t)\, f(r)$$

$$\frac{1}{D}\frac{\tau'}{\tau} = \frac{\nabla^2 f}{f} = -k^2$$

$\nabla^2 f + k^2 f = 0$ $\qquad\qquad$ $\tau' = -k^2 D \tau$

$f = A\, j_0(kr) + \underset{\overset{\shortparallel}{0}}{B}\, y_0(kr)$ $\qquad\qquad$ $\tau = e^{-k^2 D t}$

divergira v $r \to 0$

$$T(r,t) = A\, j_0(kr)\, e^{-k^2 D t} \qquad\qquad j_0(x) = \frac{\sin x}{x}$$

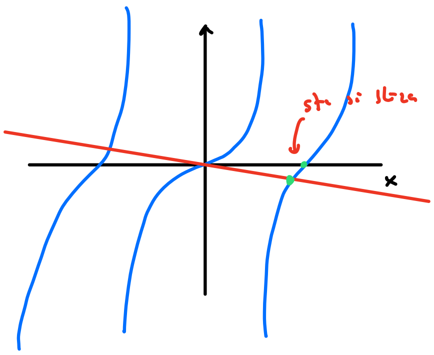
R.P. $\qquad A \left.\frac{\sin kr}{\cancel{kr}}\right|_R e^{-k^2 D t} = -\frac{\lambda_k d}{\lambda_L} \frac{1}{\cancel{k}} \left(\frac{kr\cos kr - \sin kr}{r^{\cancel{2}}} \right)\Big|_{r=R} e^{-k^2 D t} A$

$$\sin kR = -\frac{\lambda_k d}{\lambda_R} \left(kR \cos kR - \sin kR \right)$$

$$\tan kR \left(1 - \frac{\lambda_k}{\lambda_L} \frac{d}{R} \right) = -\frac{\lambda_k}{\lambda_L} \frac{d}{R} kR$$

$$\tan kR = \frac{1}{-\frac{\lambda_L}{\lambda_k} \frac{R}{d} \frac{1}{kR} + \frac{1}{kR}}$$

$$\tan kR = \frac{1}{1 - \underbrace{\frac{\lambda_L}{\lambda_K} \frac{R}{d}}_{\frac{1}{\alpha}}} kR \qquad\qquad d \ll 1$$



$$\left(1 - \frac{1}{\alpha}\right) \tan x = x \qquad\qquad x = kR$$

majhen naklon $\Rightarrow k$

(T) Greenove funkcije

$\mathcal{L}\, u(\vec{r}) = f(\vec{r})$ $\qquad\qquad$ $\mathcal{L}\, G(\vec{r}, \vec{r}_0) = \delta(\vec{r} - \vec{r}_0)$

↳ linearen operator

Rešitev $\qquad u(\vec{r}) = \int G(\vec{r}, \vec{r}_0)\, f(\vec{r}_0)\, d^3\vec{r}_0$

Laplace $\quad \mathcal{L} = \nabla^2 \quad$ 2D $\quad G = \frac{1}{2\pi} \ln |\vec{r} - \vec{r}_0|$

3D $\quad G = -\frac{1}{4\pi} \frac{1}{|\vec{r} - \vec{r}_0|}$

Hanklove funkcije $\qquad H_\alpha^{(1,2)} = J_\alpha \pm i\, Y_\alpha$

Helmholtz $\quad \mathcal{L} = \nabla^2 - k^2 \quad$ 2D

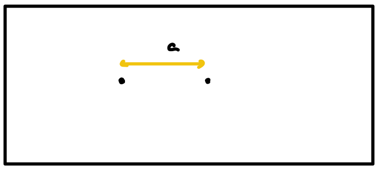
$$G = \begin{cases} -\frac{i}{4} H_0^{(1)}(k|\vec{r} - \vec{r}_0|) & \text{vpadni valovi} \\ \frac{i}{4} H_0^{(2)}(k|\vec{r} - \vec{r}_0|) & \text{valovi navzven} \\ \frac{1}{4} Y_0(k|\vec{r} - \vec{r}_0|) & \text{stoječi valovi} \end{cases}$$

3D

$$G = \begin{cases} \frac{1}{4\pi|\vec{r} - \vec{r}_0|} e^{\pm ik|\vec{r} - \vec{r}_0|} & \text{potujoči valovi} \\ \frac{\cos k|\vec{r} - \vec{r}_0|}{2\pi|\vec{r} - \vec{r}_0|} \end{cases}$$

Difuzijska $\quad \mathcal{L} = \frac{\partial}{\partial t} - D\nabla^2 \qquad G = H(t) \left(\frac{1}{4\pi D t}\right)^{N/2} e^{-|\vec{r} - \vec{r}_0|^2/4Dt} \qquad$ N-dim

26 Membrane



Točki vzbujamo z $e^{i\omega t}$

$u(r\to\infty, t) = ?$

$$u_{tt} = c^2 \nabla^2 u + \underbrace{f(\vec r, t)}_{\text{vzbujanje}}$$

$$f(\vec r,t) = \underbrace{A_0 \left(\delta^2(\vec r-(0,0)) \overset{\text{antifaza}}{-} \delta^2(\vec r-(a,0))\right)}_{F(\vec r)} e^{i\omega t}$$

Po dolgem času $u = R(\vec r)\, e^{i\omega t}$

$$(i\omega)^2 R\, e^{i\omega t} = c^2 e^{i\omega t} \nabla^2 R + F e^{i\omega t}$$

$$\Rightarrow \nabla^2 R + k^2 R = -\frac{F}{c^2} \qquad \text{Helmholtz}$$

$$-\frac{A_0}{c^2}\mathcal{L}\left(G(\vec r,(0,0)) - G(\vec r,(a,0))\right) = -\frac{A}{c^2}\left(\delta^2(\vec r-(0,0)) - \delta^2(\vec r-(a,0))\right)$$

$$\Rightarrow R = -\frac{A_0}{c^2}\left(G(\vec r,(0,0)) - G(\vec r,(a,0))\right)$$

$(1+x)^\alpha = 1+\alpha x$

$$= -\frac{A_0 i}{4c^2}\left(H_0^{(2)}\left(k\underbrace{|\vec r-(0,0)|}_{\sqrt{x^2+y^2}}\right) - H_0^{(2)}\left(k\underbrace{|\vec r-(a,0)|}_{\sqrt{(x-a)^2+y^2}}\right)\right)$$

$$= \sqrt{(x^2+y^2)\left(1-\frac{2xa}{x^2+y^2}+\cancel{\frac{a^2}{x^2+y^2}}^{\,0}\right)}$$

$$= r\left(1-\frac{xa}{r^2}\right)$$

$$= -\frac{A_0 i}{4c^2}\left(\underbrace{H_0^{(2)}(kr) - H_0^{(2)}\left(kr\left(1-\frac{xa}{r^2}\right)\right)}_{H_0^{(2)\prime}(kr)\left(-kr\frac{xa}{r^2}\right)}\right)$$

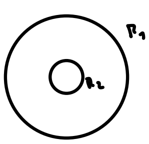
$$\frac{d}{dz}H_n^{(2)}(z) = \frac{n H_n^{(2)}(z)}{z} - H_{n+1}^{(2)}(z)$$

$$H_n^{(2)}(z) \overset{z\to\infty}{\sim} (-i)^n \frac{e^{-iz}}{z}$$

$$R = \frac{A_0 i x a k}{c^2 r} H_0^{(2)\prime}(kr) \doteq \frac{A_0 x a}{4c^2 r^2} e^{-ikr}$$

$$u(r,\varphi,t) = \frac{A_0 a}{4c^2}\frac{\cos\varphi}{r} e^{i(\omega t-kr)}$$

4



$$\frac{\partial^2 u}{\partial t^2} = c^2\nabla^2 u \qquad u(r,\varphi,t) = f(r,\varphi)\, e^{i\omega t}$$

$$-\omega^2 f = c^2\nabla^2 f$$

$$\nabla^2 f + k^2 f = 0 \qquad k = \frac{\omega}{c}$$

R.P. $u(R_1,\varphi,t) = 0$

$$0 = m\, u_{tt}\big|_{r=R_2} = \gamma 2\pi R_2 \frac{\partial u}{\partial r}\Big|_{r=R_2} \Rightarrow \frac{\partial u}{\partial r}\Big|_{r=R_2} = 0$$

$$u = \sum_{m=0} \left(A_m J_m(k_m r) + B_m Y_m(k_m r)\right)\left(C_m \cos m\varphi + D_m \sin m\varphi\right)$$

R.P. $A_m J_m(k_m R_1) + B_m Y_m(k_m R_1) = 0$ $\qquad B_m = -A_m \frac{J_m(k_m R_1)}{Y_m(k_m R_1)}$

$A_m k_m J_m'(k_m R_2) + B_m k_m Y_m'(k_m R_2) = 0$

$$\underbrace{\begin{bmatrix} J_n & Y_n \\ J_n' & Y_n' \end{bmatrix}}_{M} \begin{bmatrix} A_n \\ D_n \end{bmatrix} = 0$$

$$\det M = 0 \quad \Rightarrow \quad J_n(k_n R_1)\, Y_n'(k_n R_2) = J_n'(k_n R_2)\, Y_n(k_n R_1)$$

$$k_{m,n}$$

$$u = \sum_{m=0}^{\infty} \sum_n A_{mn} \left(J_n(k_{mn} r) - \frac{J_n(k_{mn} R_1)}{Y_n(k_{mn} R_1)} Y_n(k_{mn} r) \right) \left(C_n \cos n\varphi + D_n \sin n\varphi \right)$$

(b) $m = 0$

$$k_0 R_1 = \xi_{0n}$$

$$\omega_0 = \frac{\xi_{0n}}{R_1} c = 562\ \mathrm{Hz}$$

(c) $R_2 \ll R_1 \qquad m = 0$

$$\det M = 0 \quad \Rightarrow \quad J_0(k_0 R_1)\, Y_1(k_0 R_2) = Y_0(k_0 R_1)\, J_1(k_0 R_2)$$

$$k_0 R_1 = \xi_{01} + \Delta k R_1$$

$$J_0(\xi_{01} + \Delta k R_1) = Y_0(\xi_{01} + k R_1) \frac{J_1(\xi_{01} \frac{R_2}{R_1} + \cancel{\Delta k R_1})}{Y_1(\xi_{01} \frac{R_2}{R_1} + \cancel{\Delta k R_1})}$$

$$\underbrace{J_0(\xi_{01})}_{=0} + J_0'(\xi_{01}) \Delta k R_1 = \Big(Y_0(\xi_{01}) + \underbrace{\dots}_{\text{majhno}} \Big) \frac{\frac{1}{2} \xi_{01} \frac{R_2}{R_1}}{-\frac{2}{\pi} \frac{1}{\xi_{01} \frac{R_2}{R_1}}}$$

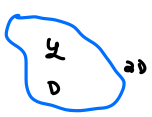
$$\Delta k R_1 = \frac{\xi_{01}^2 \pi}{4} \left(\frac{R_2}{R_1} \right)^2 \frac{Y_0(\xi_{01})}{J_1(\xi_{01})}$$

$$\Delta \omega = \Delta k c = \frac{\xi_{01}^2 \pi}{4} c \frac{R_2^2}{R_1^3} \frac{Y_0(\xi_{01})}{J_1(\xi_{01})} = 17{,}9\ \mathrm{Hz}$$

(T) Greenove funkcije na končni domeni

$$\mathcal{L}\, G(\vec r, \vec r_0) = \delta(\vec r - \vec r_0)$$

Greenove identitete



$$\int_D v \nabla^2 u - u \nabla^2 v \, dV = \oint_{\partial D} v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n} \, d\vec S$$

smerni odvod v smeri normale

$$\mathcal{L} f: \nabla^2 f = 0$$

$$G_\infty \xrightarrow{\text{zahtj}} G\big|_{\partial D} = 0 \quad , \quad \text{če} \quad u \to f \ , \ v \to G$$

$$\int_D G \underbrace{\nabla^2 f}_{=0} - f \underbrace{\nabla^2 G}_{\delta(\vec r - \vec r_0)} \, dV = \oint \underbrace{G}_{0 \text{ na robu}} \frac{\partial f}{\partial n} - f \frac{\partial G}{\partial n} \, d\vec S$$

$$f(\vec r) = \oint_{\partial D} f(\vec r_0) \frac{\partial G(\vec r, \vec r_0)}{\partial n_0} \, d\vec S$$

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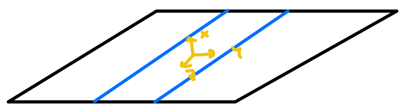
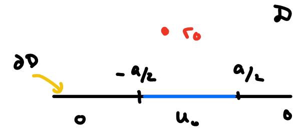




neskončen trak

$U(x>0, y, z) = ?$

RP $\quad U(0, y, z) = \begin{cases} u_0 & |y| < \frac{a}{2} \\ 0 & |y| > \frac{a}{2} \end{cases}$

$$\nabla^2 U = 0 \qquad U(x,y,z) \to U(x,y)$$

Greenova za Laplaceovo v 2D $\quad G_0 = \frac{1}{2\pi} \ln|\vec r - \vec r_0|$

$$G(x,y,x_0,y_0) = \frac{1}{2\pi} \ln \left| \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \right| - \frac{1}{2\pi} \ln \left| \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} -x_0 \\ y_0 \end{pmatrix} \right|$$

$$= \frac{1}{4\pi} \ln \frac{(x-x_0)^2 + (y-y_0)^2}{(x+x_0)^2 + (y-y_0)^2}$$

$$f(\vec r) = \oint_{\partial D} f(\vec r_0) \frac{\partial G(\vec r, \vec r_0)}{\partial n} \, d\vec S$$

$\vec n = (-1, 0)$

$$u(\vec r) = \int_{-\infty}^{\infty} U(0, y_0)\, (-1) \left. \frac{\partial G(x,y,x_0,y_0)}{\partial x_0} \right|_{x_0=0} dy_0 = \dots$$

$$\left. \frac{\partial G}{\partial x_0} \right|_{x_0=0} = \dots = \frac{1}{4\pi} \frac{-4x}{x^2 + (y-y_0)^2}$$

$z = \frac{y_0 - y}{x}$

$$\dots = + \int_{-a/2}^{a/2} u_0 \frac{1}{\pi} \frac{x}{x^2 + (y-y_0)^2} \, dy_0 = \frac{u_0}{\pi} \int_{-\frac{\frac{a}{2}+y}{x}}^{\frac{\frac{a}{2}-y}{x}} \frac{1}{1+z^2} \, dz = \frac{u_0}{\pi} \arctan z =$$

$x\, dz = dy_0$

$$= \frac{u_0}{\pi} \left(\arctan \frac{\frac{a}{2} - y}{x} + \arctan \frac{\frac{a}{2} + y}{x} \right)$$

T Laplace v 2D

$\nabla^2 f = 0 \qquad$ Harmonične funkcije

$$g(x,y) = u(x,y) + i\, v(x,y)$$

g holomorfna $\Leftrightarrow$ Cauchy-Riemannove enačbe

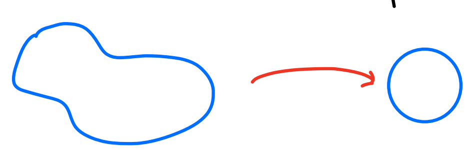
$$u_x = v_y$$
$$u_y = -v_x$$

$$\nabla^2 u = u_{xx} + u_{yy} = v_{xy} - v_{yx} = 0$$
$$\nabla^2 v = v_{xx} + v_{yy} = -u_{xy} + u_{yx} = 0$$

Konformna invarianca

Riemannov izrek za preslikave

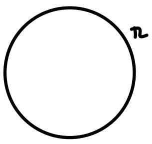
Vsako preprosto povezano odprto množico v $\mathbb{C}$ ravnini je možno konformno preslikati v odprt krog.



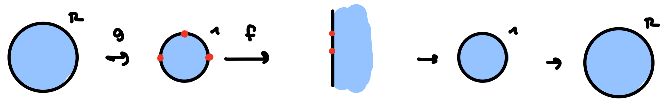
Möbiusove preslikave $f: \mathbb{C}\cup\{\infty\} \to \mathbb{C}\cup\{\infty\}$

$$f(z) = \frac{az+b}{cz+d} \qquad ad-bc\neq 0$$

28 Greenove funkcije Laplaceove enačbe na krogu



$$\nabla^2 G(\vec r,\vec r_0) = \delta(\vec r-\vec r_0)$$
$$G\big|_{|\vec r|=R} = \text{konst}$$
$$G(\vec r,\vec r_0) = ?$$



$1 \to \infty$
$i \to i$
$-1 \to 0$

$c+d=0 \qquad c=-d$

$-a+b=0 \qquad a=b$

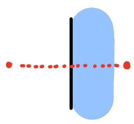
$$i = \frac{ai+b}{ci+d} = \frac{a(i+1)}{c(i-1)}$$

$$a = ic\,\frac{(i-1)}{(i+1)} = ci\,\frac{(i-1)(-i+1)}{2} =$$
$$= c\,\frac{i}{2}(1+i+i-1) = -c$$

$$g(z) = \frac{z}{R}$$

$$\Rightarrow f = \frac{az+b}{cz+d} = \frac{a(z+1)}{c(z-1)} = -\frac{z+1}{z-1} = \frac{1+z}{1-z}$$

Iščemo Greenovo funkcijo $\tilde G$ na $|/\!/$



$$\tilde G(\vec r,\vec r_0) = \frac{1}{2\pi}\ln\left|\frac{\vec r-\vec r_0}{\vec r-\vec r_*}\right| = \frac{1}{4\pi}\ln\frac{(x-x_0)^2+(y-y_0)^2}{(x+x_0)^2+(y-y_0)^2} \qquad \vec r_0 = \begin{pmatrix}x_0\\y_0\end{pmatrix} \qquad \vec r_* = \begin{pmatrix}-x_0\\y_0\end{pmatrix}$$

$f\circ g$ ( ^{R}) =  $\qquad g^{-1}\circ f^{-1}$ () =  ^{R}

$$f^{-1}: \qquad z = \frac{1+y}{1-y} \qquad (1-y)z = 1+y$$
$$y = \frac{z-1}{z+1}$$

$$\begin{tikzcd}
(a,0) \arrow[d, "f\circ g"] & (a_*,0) \\
(x_0,y_0) \arrow[r, "\text{zrcaljenje}"'] & (-x_0,y_0) \arrow[u, "(f\circ g)^{-1}"']
\end{tikzcd}$$

$\vdots$

$$G^{(kr)}(\vec r, a, 0) = G_\infty(\vec r,(a,0)) - G_\infty\left(\vec r, \left(\frac{R^2}{a}, 0\right)\right)$$

29 Difuzija na konačnom intervalu

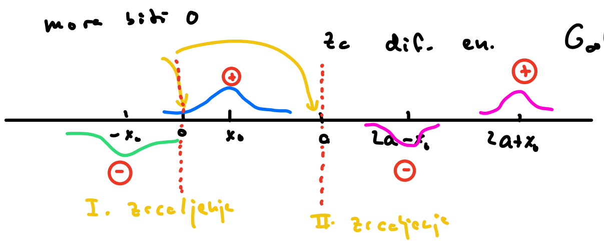
$$\frac{\partial T}{\partial t} = D\frac{\partial^2 T}{\partial x^2}$$

$T(0,t) = T(a,t) = 0$

$T(x,0) = \delta(x - x_0)$

$T(x,t) = ?$

$$G_\infty(x,x_0,t,t_0) = \frac{1}{\sqrt{4\pi D(t-t_0)}}\, e^{-\frac{(x-x_0)^2}{4D(t-t_0)}}\, H(t-t_0)$$



... zrcaljenje ponavljamo beskonačnost

Pozicije

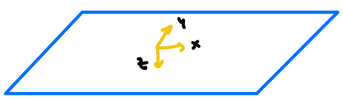
⊕ $x_0 + n\,2a$

⊖ $-x_0 + m\,2a$

$$G = +\sum_{n=-\infty}^{\infty} G_\infty(x, x_0 + n2a, t, t_0) - \sum_{m=-\infty}^{\infty} G_\infty(x, -x_0 + m2a, t, t_0)$$

$$= \frac{1}{\sqrt{4\pi D(t-t_0)}}\, H(t-t_0) \sum_{n=-\infty}^{\infty} e^{-[(x-x_0-2na)^2 - (x+x_0-2na)^2]/4D(t-t_0)}$$

30 Lloydovo zrcalo



gladina vode

izvor pri $z=a$
s frek. ω

povprečen energijski tok

$\bar{\vec{j}}(\vec r) = ?$ za $\vec r \to \infty$

T Akustika

stisljiv, neviskozen

$$\rho\frac{D\vec v}{Dt} = -\nabla p$$

$$\frac{\partial\rho}{\partial t} + \nabla(\vec v \rho) = 0$$

majhne perturbacije

$p(\vec r,t) = p_0 + \delta p(\vec r,t)$

$\rho(\vec r,t) = \rho_0 + \delta\rho(\vec r,t)$

Adiabatne relacije

$$\frac{\delta\rho}{\rho_0} = \chi_s\, \delta p$$

$$\Rightarrow \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\delta p = \nabla^2 \delta p \qquad c^2 = \frac{1}{\rho_0\chi_s}$$

zvočni pomak

$$\vec j = \frac{1}{2}\operatorname{Re}(\delta p\, \vec v^{*})$$

kompleksna konjugacija

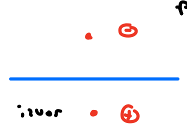
$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\delta p = \nabla^2\delta p + A\delta(\vec r - a\hat e_z)\, e^{i\omega t}$$

$$\delta p = P(\vec r)\, e^{i\omega t}$$

$$-\underbrace{\frac{\omega^2}{c^2}}_{k^2}\frac{\partial^2}{\partial t^2} P = \nabla^2 P + A\,\delta(\vec r - a\hat e_z)$$

Helmholtzova enačba

RP $\quad \delta p|_{z=0} = 0 \quad$ na gladini konstanten tlak



Za Helm. $\quad G_\infty(\vec r, \vec r_0) = -\frac{1}{4\pi|\vec r - \vec r_0|}\, e^{ik|\vec r - \vec r_0|}$

$$\Rightarrow P(\vec r) = -A\, G_\infty(\vec r, a\hat e_z) + A\, G_\infty(\vec r, -a\hat e_z)$$

$$P(\vec r) = A\frac{e^{ik|\vec r - a\hat e_z|}}{4\pi|\vec r - a\hat e_z|} - A\frac{e^{ik|\vec r + a\hat e_z|}}{4\pi|\vec r + a\hat e_z|}$$

$$r^{\pm} = |\vec r \pm a\hat e_z| = \sqrt{x^2+y^2+(z\pm a)^2} = \sqrt{\underbrace{x^2+y^2+z^2}_{r^2} \pm 2a\underbrace{z}_{r\cos\theta} + \underbrace{a^2}_{\approx 0}} = r\sqrt{1 \pm \frac{2a\cos\theta}{r}} = r\left(1 \pm \frac{1}{2}\frac{2a\cos\theta}{r}\right)$$

$$= r \pm a\cos\theta$$

$$\delta p = e^{i\omega t} A\left(\frac{e^{ikr^-}}{4\pi r^-} - \frac{e^{ikr^+}}{4\pi r^+}\right)$$

Iščemo $\vec{v}(\vec{r})$

$$\frac{D\vec{v}}{Dt} = \frac{\partial \vec{v}}{\partial t} + \underbrace{(\vec{v}\cdot\nabla)}\vec{v}$$

$= 0$ celotna hitrost v mali spremembi

$$\rho_0 \frac{\partial \vec{v}}{\partial t} = -\nabla \delta p \qquad \vec{v} = e^{i\omega t}\,\vec{V}(\vec{r})$$

$$i\omega\rho_0 \vec{V}(\vec{r}) = -\nabla P(\vec{r}) \Rightarrow \vec{V}(\vec{r}) = \frac{i\nabla P(\vec{r})}{\omega\rho_0}$$

$$\nabla f = \frac{\partial f}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial f}{\partial \varphi}\hat{e}_\varphi + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \theta}\hat{e}_\theta$$

$$\frac{\partial r^\pm}{\partial r} = 1 \qquad \frac{\partial r^\pm}{\partial \varphi} = 0 \qquad \frac{\partial r^\pm}{\partial \theta} = \mp a\sin\theta$$

daleč stran $\frac{1}{r^2} \sim 0$

$$\vec{v}(\vec{r},t) = \frac{Ai}{4\pi\omega\rho_0} e^{i\omega t}\Big(\big(e^{ikr^-}\frac{ik}{r^-} - \underbrace{e^{ikr^-}\frac{1}{r^{-2}}}_{0} - e^{ikr^+}\frac{ik}{r^+} + \underbrace{\frac{e^{ikr^+}}{r^{+2}}}_{0}\big)\hat{e}_r + \hat{e}_\theta\,\underbrace{O(\tfrac{1}{r})}_{0}\Big)$$

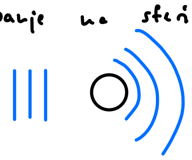
$$\vec{v}(\vec{r},t) = -\frac{A}{4\pi\rho_0 c} e^{i\omega t}\left(\frac{e^{ikr^-}}{r^-} - \frac{e^{ikr^+}}{r^+}\right)\hat{e}_r$$

$$\bar{j}_E = -\frac{1}{2}\frac{A^2}{(4\pi)^2\rho_0 c}\hat{e}_r \,\mathrm{Re}\left(\frac{e^{ikr^-}}{r^-} - \frac{e^{ikr^+}}{r^+}\right)\left(\frac{e^{-ikr^-}}{r^-} - \frac{e^{-ikr^+}}{r^+}\right)$$

$$= -\frac{A^2\hat{e}_r}{32\pi^2\rho_0 c}\mathrm{Re}\left(\frac{1}{r^{-2}} + \frac{1}{r^{+2}} - \frac{2\cos k(r^- - r^+)}{r^- r^+}\right)$$

$$\doteq -\frac{A^2\hat{e}_r}{32\pi^2\rho_0 c}\frac{2}{r^2}\big(1 - \cos(k 2a\cos\theta)\big)$$

(31) Sipanje na sferi



$$\frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} = \nabla^2\varphi \quad \text{hitrostni potencial}$$

$$\text{RP} \quad \frac{\partial\varphi}{\partial r}\Big|_{r=a} = 0$$

$$\varphi(\vec{r},t) = \varphi_i + \varphi_s = e^{i(kx-\omega t)} + \varphi_s \qquad \varphi_s = ?$$

Razvoj ravnih po sfer. $e^{i\vec{k}\vec{r}} = \sum_{l=0}^{\infty}(2l+1)\,i^l\, j_l(kr)\,P_l(\hat{k}\,\hat{r})$ — sferične Besselove f.

hanklova funk. – navzven potujoči valovi

Nastavek za rešitev Helmholtzove en. za odboj val

$$\varphi_s = e^{-i\omega t}\sum_l c_l P_l(\cos\theta)\, h_l^{(1)}(kr)$$

$$\varphi_i = e^{i(kx-\omega t)} = \sum_{l=0}^{\infty}(2l+1)\,i^l\, j_l(kr)\,P_l(\cos\theta)\; e^{-i\omega t}$$

$$\text{R.P.} \quad \frac{\partial\varphi}{\partial r}\Big|_{r=a} = 0 = \sum_{l=0}^{\infty}(2l+1)\,i^l\, j_l'(ka)\,k\,P_l(\cos\theta) + \sum_{l=0}^{\infty} c_l\, h_l^{(1)\prime}(ka)\,k\,P_l(\cos\theta)$$

$$c_l = -(2l+1)\,i^l\,\frac{j_l'(ka)}{h_l^{(1)\prime}(ka)}$$

V lim $ka \to 0$, $r \to \infty$

Velja $\frac{j_0'(ka)}{h_0'(ka)} = i\frac{(ka)^3}{3}$ $\quad \frac{j_1'(ka)}{h_1'(ka)} = -i\frac{(ka)^3}{6}$ $\quad h_0^{(1)}(z) \approx -\frac{i}{z}e^{iz}$ $\quad h_1^{(1)}(z) \sim -\frac{1}{z}e^{iz}$

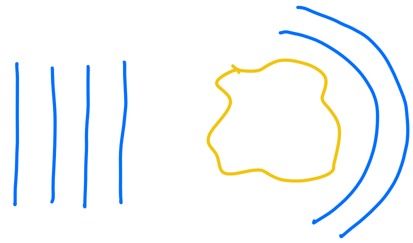
$$u_s = e^{-i\omega t}\left(-\frac{j_0'(ka)}{h_0^{(1)\prime}(ka)}h_0(kr) - 3\frac{j_1'(ka)}{h_1'(ka)}\, i\, h_1(kr)\cos\theta\right)$$

$$= e^{-i\omega t}\left(-i\frac{(ka)^3}{3}\left(-\frac{i}{kr}\right)e^{ikr} - 3\left(-\frac{i}{6}\right)(ka)^3 i\left(-\frac{1}{kr}\right)e^{ikr}\cos\theta\right)$$

$$= -e^{-i(\omega t - kr)}\frac{k^2a^3}{3r}\left(1-\frac{3}{2}\cos\theta\right)$$

T Sipalna teorija

$$-\frac{\hbar^2}{2m}\nabla^2\Psi + V(\vec{r})\Psi = E\Psi$$



Lippmann-Schwinger

$$\Psi(\vec{x}) = e^{i\vec{k}\vec{x}} + \int d^3y\, G(\vec{x}-\vec{y}, k)\, V(\vec{y})\, \Psi_k(\vec{y})$$ Integralske enačbe

$$G(\vec{x}) = -\frac{m}{2\pi\hbar^2}\frac{e^{i|\vec{k}||\vec{x}|}}{|\vec{x}|}$$

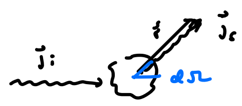
Bornova serija

$$\Psi_k^{(0)}(\vec{x}) = 0 \qquad \Psi_k^{(n+1)}(\vec{x}) = e^{i\vec{k}\cdot\vec{x}} + K\,\Psi_k^{(n)}(\vec{x})$$

$$K f(\vec{x}) = \int d^3y\, G(\vec{x}-\vec{y})\, V(\vec{y})\, f(\vec{y})$$

$$\frac{d\sigma}{d\Omega} = \frac{|j_s|}{|j_i|} r^2$$

sipalni presek



$$\frac{d\sigma}{d\Omega}(\vec{k},\vec{k}') = |f(\vec{k}-\vec{k}')|^2$$

$$f(\vec{k}-\vec{k}') = -\frac{m}{2\pi\hbar^2}\int d^3y\, V(\vec{y})\, e^{i(\vec{k}-\vec{k}')\vec{y}}$$

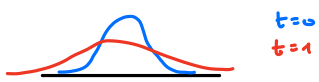
T Nelinearne PDE

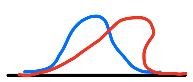
disperzija → nelinearnost

integrabilne PDE

NLS, sine-Gordon, LL, KdV enačbe

$$\frac{\partial u}{\partial t} = \nabla^2 u \qquad \frac{\partial u}{\partial t} = u\frac{\partial u}{\partial x} \quad \text{Burgerjeva en.}$$





burgers difuzija

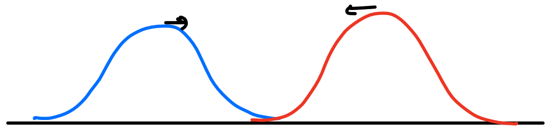
$$u_t - 6uu_x + u_{xxx} = 0$$

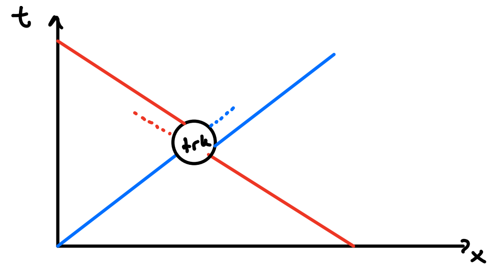
Enačba za valovanje pri nizki globini

Rešitve so solitoni

$$u(x,t) = \frac{c}{\mathrm{ch}^2\left(\sqrt{2c}\,(x-ct-x_0)\right)}$$

Kaj se zgodi, če trčita dva solitona





Klein-Gordon $\longleftrightarrow$ sine-Gordon

$$H = \frac{1}{2}\int_{\mathbb{R}} dx\, (\pi^2 + \varphi_x^2 + m^2\varphi^2) \; ; \quad \xleftarrow{\beta \to 0} \quad H = \int_{\mathbb{R}} dx\, (\pi^2 + \varphi_x^2) + \frac{m}{\beta^2}(1 - \cos\beta\varphi)$$

$\pi(x) = \varphi_t(x)$

$\{\varphi(x), \pi(y)\} = \delta(x-y)$

$\frac{df}{dt} = \{f, H\}$

KG: $\varphi_{tt} - \varphi_{xx} + m^2\varphi = 0$

sG: $\varphi_{tt} - \varphi_{xx} + \frac{m}{\beta}\sin\varphi = 0$

KG $\quad \varphi(x,0) \xrightarrow{\text{Fourier}} \hat{\varphi}(k,0) \qquad \varphi(x,t) = \frac{1}{2\pi}\int \hat{\varphi}(k,t)\, e^{ikx}\, dk$

$$\frac{1}{2\pi}\int \partial_t^2 \hat{\varphi}\, e^{ikx}\, dk + k^2 \frac{1}{2\pi}\int \hat{\varphi}\, e^{ikx}\, dk + m^2 \frac{1}{2\pi}\int \hat{\varphi}\, e^{ikx}\, dk = 0$$

$$(\partial_t^2 + (k^2 + m^2))\,\hat{\varphi}(k,t) = 0 \qquad \text{Navadna DE}$$

$$\frac{\partial^2}{\partial t^2}\hat{\varphi} + \omega_k^2\hat{\varphi} = 0$$

$$\hat{\varphi}(k,t) = A e^{i\omega_k t} + B e^{-i\omega_k t}$$

A in B iz z.p.

$\longrightarrow$ Inverzna Fourierova $\rightarrow \varphi(x,t)$

Podobno naredimo za sG