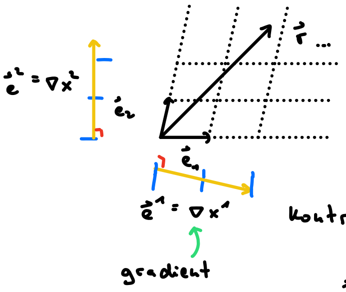
Kontravariantnost in kovariantnost



$\vec r$ ... radij vektor

dogovor

$$\vec r = x^1 \vec e_1 + x^2 \vec e_2 = \sum_i x^i \vec e_i = x^i \vec e_i$$

kontravariantne komponente

$$\vec e_i = \frac{\partial \vec r}{\partial x^i} \quad \text{... tangentni vektor}$$

$$\vec r = x_1 \vec e^1 + x_2 \vec e^2$$

$\vec e^1, \vec e^2$ ... dualna baza / recipročna baza

kovariantne komponente

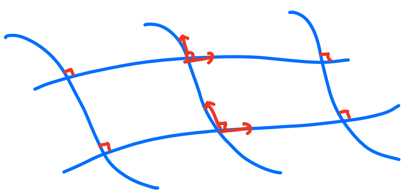
V ortogonalni bazi sta baza in dualna baza enaki, zato so tudi kontravariantne in kovariantne komponente enake.

Ortogonalni koordinatni sistemi

- Kartezični koord. sis.

$$\vec r = \sum x^i \vec e_i = x \vec e_x + y \vec e_y + z \vec e_z = x \hat i + y \hat j + z \hat k$$

- Krivočrtni koordinatni sistemi

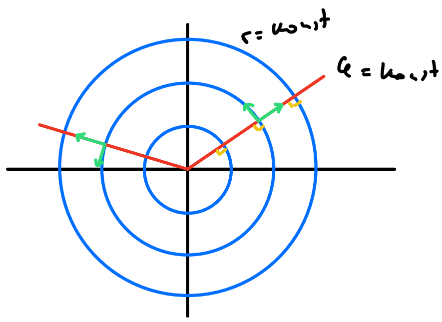


Primeri:
- polarni koord. sis.
- cilindrični koord. sis.
- sferični koord. sis.

} ortogonalni

Baza je lokalna, ni globalna

Polarni koord. sistem



$$\vec r = x \hat i + y \hat k = r \cos\varphi \, \hat i + r \sin \hat j$$

$x = r\cos\varphi$ $\qquad$ $\vec e_r = \frac{\partial \vec r}{\partial r} = \cos\varphi \, \hat i + \sin\varphi \, \hat j$

$y = r\sin\varphi$ $\qquad$ $\vec e_\varphi = \frac{\partial \vec r}{\partial \varphi} = -r\sin\varphi \, \hat i + r\cos\varphi \, \hat j$

Komentar $\vec r = \cancel{r \vec e_r + \varphi \vec e_\varphi} = r \vec e_r$

$$|\vec e_r|^2 = \cos^2\varphi \, |\hat i|^2 + \sin^2\varphi \, |\hat j|^2 = 1$$

$$|\vec e_\varphi|^2 = r^2 \sin^2\varphi \, |\hat i|^2 + r^2 \cos^2\varphi \, |\hat j|^2 = r^2$$

① Newtonov zakon v polarnih koordinatah

$$\vec F = m\vec a = m\dot{\vec v} = m\ddot{\vec r}$$

Hitrost $\vec v = \dot{\vec r} = \frac{d}{dt}\vec r = \frac{\partial \vec r}{\partial r}\frac{\partial r}{\partial t} + \frac{\partial \vec r}{\partial \varphi}\frac{\partial r}{\partial t} = \dot r\,\vec e_r + \dot\varphi\,\vec e_\varphi$

Pospešek $\vec a = \dot{\vec v} = \ddot r\,\vec e_r + \dot r\,\dot{\vec e}_r + \ddot\varphi\,\vec e_\varphi + \dot\varphi\,\dot{\vec e}_\varphi$

$$\dot{\vec e}_r = \frac{d}{dt}(\cos\varphi\,\hat i + \sin\varphi\,\hat j) = \dot\varphi(-\sin\varphi\,\hat i + \cos\varphi\,\hat j) = \dot\varphi\,\frac{\vec e_\varphi}{r}$$

$$\dot{\vec e}_\varphi = \frac{d}{dt}(-r\sin\varphi\,\hat i + r\cos\varphi\,\hat j) = \dot r(-\sin\varphi\,\hat i + \cos\varphi\,\hat j) - r\dot\varphi(\sin\varphi\,\hat i + \cos\varphi\,\hat j) =$$
$$= \dot r\,\frac{\vec e_\varphi}{r} - \vec e_r\,\dot\varphi\, r$$

$$\vec a = \vec e_r(\ddot r - \dot\varphi^2 r) + \vec e_\varphi\left(\ddot\varphi + 2\dot r\,\frac{\dot\varphi}{r}\right)$$

$$\underline{\vec F = m\left(\vec e_r(\ddot r - \dot\varphi^2 r) + \vec e_\varphi\left(\ddot\varphi + 2\dot r\,\frac{\dot\varphi}{r}\right)\right)}$$

Kinetična energija

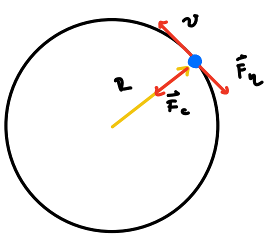
$$W_k = T = \frac{1}{2}m|v|^2 = \frac{1}{2}m(\dot r\,\vec e_r + \dot\varphi\,\vec e_\varphi)^2 = \underline{\frac{1}{2}m(\dot r^2 + r^2\dot\varphi^2)}$$

② Sferični koordinatni sistem - Izračunaj T

DN

$$x = r\cos\varphi\sin\theta$$
$$y = r\sin\varphi\sin\theta$$
$$z = r\cos\theta$$

③ Kroženje po obroču z viskoznostjo



$\vec r(t) = ?$

$$m\ddot{\vec r} = \vec F_c + \vec F_\eta \qquad \vec F_\eta = -\eta\vec v \qquad r = R \quad \dot r = \ddot r = 0$$

$$-mR\dot\varphi^2\,\vec e_r + m\ddot\varphi\,\vec e_\varphi = -F_c\,\vec e_r - \eta(\dot r\,\vec e_r + \dot\varphi\,\vec e_\varphi)$$

$\vec e_r$: $-mR\dot\varphi^2 = -F_c \Rightarrow F_c = mR\dot\varphi^2$

$\vec e_\varphi$: $m\ddot\varphi = -\eta\dot\varphi$

$x = \dot\varphi \qquad m\dot x = -\eta x \qquad \frac{dx}{x} = -\frac{\eta}{m}dt$

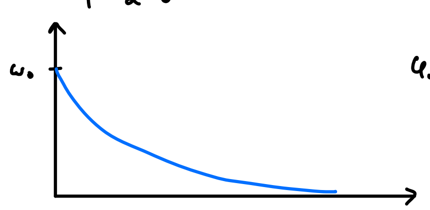
$\ln x = -\frac{\eta}{m}t$

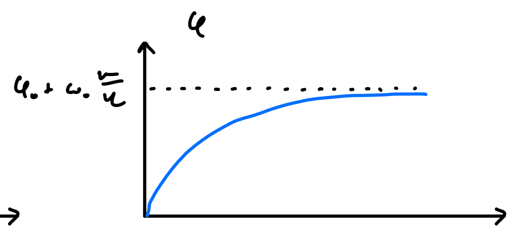
$$x = \dot\varphi = \omega_0 e^{-\frac{\eta}{m}t} \qquad \text{Pogoj } \dot\varphi(t=0) = \omega_0$$

$$\int_{\varphi_0}^{\varphi} d\varphi(t) = -\frac{m\omega_0}{\eta}e^{-\frac{\eta}{m}t}\Big|_0^t$$

$$\varphi(t) = \omega_0\frac{m}{\eta}\left(1 - e^{-\frac{\eta}{m}t}\right) + \varphi_0$$

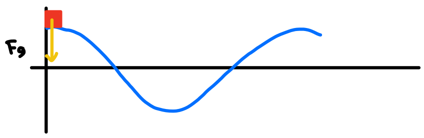
$T \propto v^2$



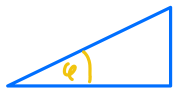


(4) Enaka naloga kot 3 le da je zraven še gravitacija. DN

(5) Klada na kosinusni podlagi - ali se klada kdaj odlepi:



$$y = y_0 \cos\left(\frac{x}{x_0}\right)$$
$$F_{tr} = 0$$



$$\tan\varphi = \frac{dy}{dx} = y'$$

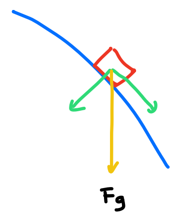
$\hat{e}_1 = \cos\varphi\,\hat{i} + \sin\varphi\,\hat{j}$ ... tangentni vektor

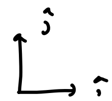
$\hat{e}_2 = -\sin\varphi\,\hat{i} + \cos\varphi\,\hat{j}$ ... normalni vektor

$$\vec{v} = v\,\vec{e}_1$$
$$\vec{a} = \dot{\vec{v}} = \dot{v}\,\vec{e}_1 + v\,\dot{\vec{e}}_1 = \dot{v}\,\vec{e}_1 + v\dot{\varphi}\,\vec{e}_2$$

$$\dot{\vec{e}}_1 = \frac{d}{dt}(\cos\varphi\,\hat{i} + \sin\varphi\,\hat{j}) = \dot{\varphi}\,\vec{e}_2$$

Odlepi se ko $F_p = 0 \Rightarrow$ $m\vec{a} = \vec{F}_g$





$$m(\dot{v}\,\vec{e}_1 + v\dot{\varphi}\,\vec{e}_2) = -mg\,\hat{j} = -mg(\sin\varphi\,\vec{e}_1 + \cos\varphi\,\vec{e}_2)$$

$$\hat{j} = \sin\varphi\,\vec{e}_1 + \cos\varphi\,\vec{e}_2$$

$\vec{e}_2$: $\quad m v\dot{\varphi} = -mg\cos\varphi$

$\tan\varphi = y'$

$$\frac{d}{dt}\tan\varphi = \frac{1}{\cos^2\varphi}\dot{\varphi} = \frac{d}{dt}y' = \frac{dy'}{dx}\frac{dx}{dt} = \dot{x}\,y''$$

$$\dot{x}^2 + \dot{y}^2 = v^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2\left(1 + \left(\frac{dt}{dx}\frac{dy}{dt}\right)^2\right) = \dot{x}^2(1+y'^2)$$

$$\Rightarrow \dot{x} = \frac{v}{\sqrt{1+y'^2}}$$

$$\Rightarrow \frac{1}{\cos^2\varphi}\dot{\varphi} = \dot{x}\,y'' = \frac{v\,y''}{1+y'^2} \qquad \cos^2\varphi = \frac{1}{1+\tan^2\varphi} = \frac{1}{1+y'^2}$$

$$\dot{\varphi} = \frac{v\,y''}{(1+y'^2)^{3/2}}$$

$$\frac{v^2 y''}{(1+y'^2)^{3/2}} = -\frac{g}{\sqrt{1+y'^2}}$$

$\underline{v^2 y'' + g(1+y'^2) = 0}$ Kdaj se telo odlepi od podlage

$$y = y_0\cos\frac{x}{x_0}$$
$$y' = -\frac{y_0}{x_0}\sin\frac{x}{x_0} = -\frac{y_0}{x_0}\sqrt{1-\cos^2\frac{x}{x_0}} = -\frac{1}{x_0}\sqrt{y_0^2 - y^2}$$
$$y'' = -\frac{y_0}{x_0^2}\cos\frac{x}{x_0} = -\frac{y}{x_0^2}$$

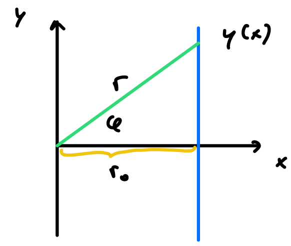
v^2: $\quad \frac{1}{2}mv^2 = mg\,\Delta h = mg(y_0 - y) \qquad v^2 = 2g(y_0 - y)$

$$-2\cancel{g}(y_0 - y)\frac{y}{x_0^2} + \cancel{g}\left(1 + \frac{1}{x_0^2}(y_0^2 - y^2)\right) = 0$$
$$2y(y - y_0) + x_0^2 + y_0^2 - y^2 = 0$$
$$(y - y_0)^2 + x_0^2 = 0$$

Enačba nima rešitve za $x_0 > 0$

zato se klada ne odlepi od podlage

⑥ Prost delec v polarnih koordinatah



$F = ma$

$m\ddot{x} = 0 \qquad v_x = \text{konst.} \qquad x = x_0 + v_x t$

$m\ddot{y} = 0 \qquad v_y = \text{konst.} \qquad y = y_0 + v_y t$

$y(x) = y_0 + \frac{v_y}{v_x}(x - x_0)$

$r(\varphi) = ?$

$r_0 = r\cos\varphi$

$r = r_0/\cos\varphi$

Newtonov zakon v pol. koord.

$\ddot{\vec{r}} = \hat{e}_r(\ddot{r} - r\dot{\varphi}^2) + \hat{e}_\varphi(r\ddot{\varphi} + 2\dot{r}\dot{\varphi}) \qquad \vec{e}_\varphi = r\,\hat{e}_\varphi$

$e_1: \quad \ddot{r} - r\dot{\varphi}^2 = 0$

$e_2: \quad r\ddot{\varphi} + 2\dot{r}\dot{\varphi} = 0$

i

$r\dot{\omega} + 2\dot{r}\omega = 0$

$\frac{\dot{\omega}}{\omega} = -2\frac{\dot{r}}{r}$

$\frac{d}{dt}\ln\omega = \frac{d}{dt}\ln r^{-2}$

$\frac{\omega}{\omega_0} = \frac{r_0^2}{r^2}$

$r^2\omega = \text{konst.}$

$r^2\dot{\varphi} = \text{konst.}$

ii

$r^2\ddot{\varphi} + 2\dot{r}r\dot{\varphi} = 0 \qquad |\cdot r$

$\frac{d}{dt}(r^2\dot{\varphi}) = 0$

$r^2\dot{\varphi} = \text{konst}$

$r^2\dot{\varphi} = p_\varphi$ ... vrtilna količina se ohranja

$mvr = mr\dot{\varphi}r$

$e_1: \quad \ddot{r} - \frac{r\,p_\varphi^2}{r^4} = 0 \qquad |\cdot\dot{r} \qquad r^3\ddot{r} = p_\varphi^2$

$\dot{r}\ddot{r} - \frac{\dot{r}\,p_\varphi^2}{r^3} = 0$

$\frac{d}{dt}\left(\frac{1}{2}\dot{r}^2 + p_\varphi^2/2r^2\right) = 0$

$\frac{\dot{r}^2}{2} + \frac{r^2\dot{\varphi}^2}{2} = \text{konst.}$ Kinetična energija v pol. koord.

integrali gib. enačb $\Leftrightarrow$ ohranjene količine

simetrije v prostoru

$\frac{1}{2}\dot{r}^2 + \frac{1}{2}\frac{p_\varphi^2}{r^2} = e \leftarrow$ neka konstanta $\qquad$ Iščemo $r(\varphi) \qquad r(\varphi(t))$

$\dot{r} = \frac{dr}{dt} = \frac{dr}{d\varphi}\dot{\varphi} = r'\frac{p_\varphi}{r^2}$

$\frac{1}{2}r'^2\frac{p_\varphi^2}{r^4} + \frac{1}{2}\frac{p_\varphi^2}{r^2} = e$

$\frac{1}{2}\frac{r'^2}{r^4} + \frac{1}{2}\frac{1}{r^2} = \frac{e}{p_\varphi^2}$

$u = \frac{1}{r} \qquad u' = -\frac{1}{r^2}r' \qquad$ oz. $\quad r' = -\frac{1}{u^2}u'$

$\frac{1}{2}u'^2 + \frac{1}{2}u^2 = \frac{e}{p_\varphi^2}$

$u'^2 + u^2 = \text{konst} \qquad u = u_0\cos(\varphi - \varphi_0)$

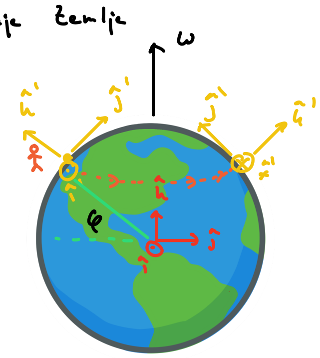
$r = r_0/\cos(\varphi - \varphi_0)$

Vrteči sistemi

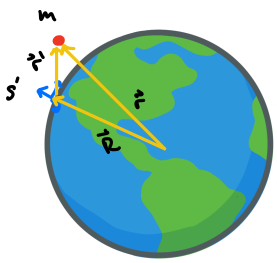
- Inercialni sistemi ... mirujoči / enakomerno gibajoči
- Neinercialni sistemi ... pospešeni sistemi (npr. vrteči sistem) S'

↳ $m\vec a_{rel} = \sum \vec F + \vec F_{sist}$

⑦ Vrtenje Zemlje



$$\hat i' = \hat i\cos(\omega t) + \hat j\sin(\omega t)$$
$$\hat j' = \hat k\cos\varphi + \sin\varphi\,(\hat j\cos(\omega t) - \hat i\sin(\omega t))$$
$$\hat k' = \hat k\sin\varphi + \cos\varphi\,(-\hat j\cos(\omega t) + \hat i\sin(\omega t))$$



2NZ: $m\ddot{\vec r} = \sum \vec F$ (S)

$$\vec r = \vec R' + \vec r'$$
$$\vec r' = x'\hat i' + y'\hat j' + z'\hat k'$$
$$\dot{\vec r}' = \vec v'_{rel} + \vec\omega\times\vec r'$$
↳ $\dot x'\hat i' + \dot y'\hat j' + \dot z'\hat k'$

$\hat i\times\hat j = \hat k$
$\hat j\times\hat k = \hat i$
$\hat k\times\hat i = \hat j$

$$\frac{d}{dt}\hat i' = \frac{d}{dt}(\cos\omega t\,\hat i + \sin\omega t\,\hat j) = \omega(-\sin\omega t\,\underbrace{\hat i}_{-\hat k\times\hat j} + \cos\omega t\,\underbrace{\hat j}_{\hat k\times\hat i})$$
$$= \underbrace{\omega\hat k}_{\vec\omega}\times\underbrace{(\sin\omega t\,\hat j + \cos\omega t\,\hat i)}_{\hat i'} = \vec\omega\times\hat i'$$

$$\dot{\hat i}' = \vec\omega\times\hat i'$$
$$\dot{\hat j}' = \vec\omega\times\hat j'$$
$$\dot{\hat k}' = \vec\omega\times\hat k'$$

$\vec\omega$ = konst.

$$\ddot{\vec r}' = \frac{d}{dt}(\vec v'_{rel} + \vec\omega\times\vec r')$$
$$= \frac{d}{dt}(\dot x'\hat i' + \dot y'\hat j' + \dot z'\hat k') + \vec\omega\times\frac{d}{dt}\vec r')$$
$$\ddot{\vec r}' = \vec a'_{rel} + \vec\omega\times\vec v'_{rel} + \vec\omega\times(\vec\omega\times\vec r')$$
↳ $\ddot x'\hat i' + \ddot y'\hat j' + \ddot z'\hat k'$

$$\vec R = R\hat k'$$
$$\dot{\vec R} = R\vec\omega\times\hat k' = \vec\omega\times\vec R \qquad \vec\omega = \text{konst.}$$
$$\ddot{\vec R} = \vec\omega\times(\vec\omega\times\vec R)$$

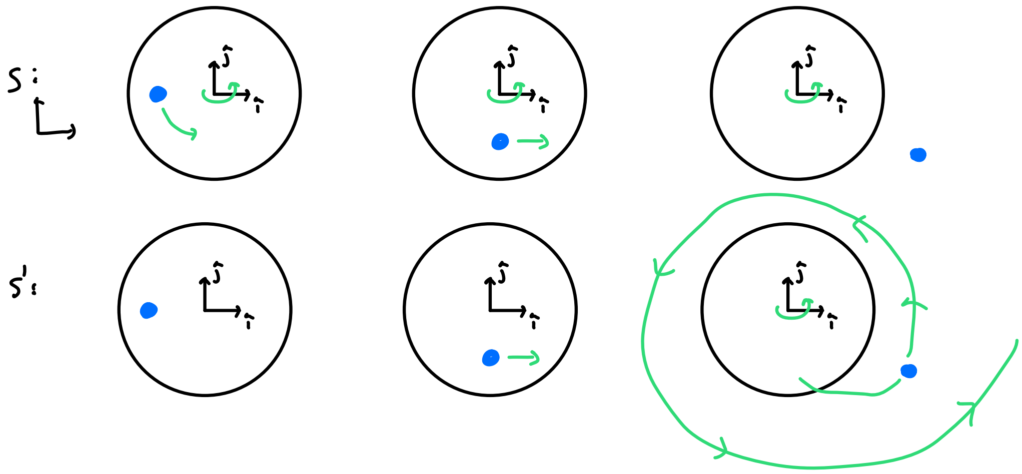
2.N.Z.

S: $m\ddot{\vec r} = \sum \vec F$

S': $m(\ddot{\vec R} + \ddot{\vec r}') = \sum\vec F$

$$m\vec a'_{rel} = \sum\vec F - m\Big(\underbrace{\underbrace{2\vec\omega\times\vec v'_{rel}}_{\text{coriolisova sila}} + \underbrace{\vec\omega\times(\vec\omega\times\vec r') + \vec\omega\times(\vec\omega\times\vec R)}_{\text{centrifugalna sila}}}_{\text{sistemske sile}}\Big)$$

8 Prost delec v vrtečem sistemu



1. način: v S'

$$m\vec a'_{rel} = 0 - 2m\,\vec\omega\times\vec v'_{rel} - m\,\vec\omega\times(\vec\omega\times\vec r') \qquad /:m$$

$$\vec\omega\times\vec v'_{rel} = \begin{vmatrix} \hat i' & \hat j' & \hat k' \\ 0 & 0 & \omega \\ \dot x' & \dot y' & 0 \end{vmatrix} = -\hat i'\,\dot y'\omega + \hat j'\,\dot x'\omega$$

$$\vec\omega\times\vec\omega\times\vec r' = \vec\omega(\underbrace{\vec\omega\cdot\vec r'}_{0}) - \omega^2\vec r' = -\omega^2\vec r'$$

$$\vec a'_{rel} = \ddot x'\hat i' + \ddot y'\hat j' = -2(-\dot y'\omega\,\hat i' + \dot x'\omega\,\hat j') + \omega^2(x'\hat i' + y'\hat j')$$

$$\left.\begin{aligned} \ddot x' &= 2\dot y'\omega + \omega^2 x' \\ \ddot y' &= -2\dot x'\omega + \omega^2 y' \qquad \cdot i \end{aligned}\right] +$$

$$\alpha = x' + iy' \qquad \text{Trik}$$
$$\dot\alpha = \dot x' + i\dot y'$$
$$i\dot\alpha = i\dot x' - \dot y'$$

$$\ddot\alpha + 2\omega i\dot\alpha - \omega^2\alpha = 0$$
$$\lambda^2 + 2\omega i\lambda - \omega^2 = 0$$
$$\lambda_{1,2} = \frac{1}{2}\left(-2\omega i \pm \sqrt{-4\omega^2 + 4\omega^2}\right) = -\omega i$$

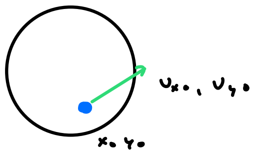
$$\alpha = Ae^{\lambda t} + Bte^{\lambda t}$$

$$\alpha(0) = x_0 + iy_0 = A$$
$$\dot x'(0) = \dot y'(0) = 0 \Rightarrow \dot\alpha = 0$$
$$\dot\alpha = A\lambda e^{\lambda t} + Be^{\lambda t} + Bt\lambda e^{\lambda t}$$
$$\dot\alpha(0) = A\lambda + B = 0 \qquad B = -A\lambda = \omega i A$$

$$\alpha(t) = Ae^{-i\omega t}(1 + ti\omega) = e^{-i\omega t}(1 + i\omega t)(x_0 + iy_0)$$
$$= (\cos\omega t - i\sin\omega t)(1 + i\omega t)(x_0 + iy_0) = (\cos\omega t - i\sin\omega t)(x_0 + iy_0 + i\omega t x_0 - \omega t y_0)$$

$$x'(t) = \mathrm{Re}(\alpha) = x_0(\cos\omega t + \omega t\sin\omega t) + y_0(\sin\omega t - \omega t\cos\omega t)$$
$$y'(t) = \mathrm{Im}(\alpha) = \cos\omega t\,(x_0\omega t + y_0) + \sin\omega t\,(-x_0 + y_0\omega t)$$

2. način $\quad S \xrightarrow{\text{transf.}} S'$



$$\vec v = \vec\omega \times \vec r$$

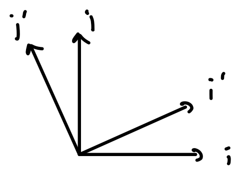
$$(v_{x0},\ v_{y0},\ 0) = \begin{vmatrix} \hat i & \hat j & \hat k \\ 0 & 0 & \omega \\ x_0 & y_0 & 0 \end{vmatrix} = (-y_0\omega,\ \omega x_0,\ 0)$$

$$\left.\begin{aligned} x(t) &= x_0 - \omega y_0 t \\ y(t) &= y_0 + \omega x_0 t \end{aligned}\right\} \vec r$$

$$\left.\begin{aligned} \hat i' &= \cos\omega t\ \hat i + \sin\omega t\ \hat j \\ \hat j' &= -\sin\omega t\ \hat i + \cos\omega t\ \hat j \end{aligned}\right\} P$$

$$\vec r\,' = P\,\vec r$$

Upodobitev rotacije v kompleksnem



$$x' = x\cos\varphi + y\sin\varphi \qquad \vec x\,' = R\,\vec x$$
$$y' = -x\sin\varphi + y\cos\varphi$$

$$\zeta' = x' + iy' \qquad \zeta = x + iy$$

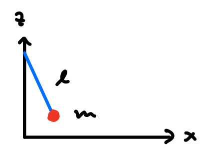
Trditev $\quad \zeta' = e^{-i\varphi}\zeta$

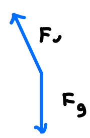
Dokaz $\quad \mathrm{Re}\,\zeta' = x' = \cos\varphi\, x + \sin\varphi\, y$

$\qquad\quad \mathrm{Im}\,\zeta' = y' = -\sin\varphi\, x + \cos\varphi\, y$

9 Nitno nihalo na severnem polu

1 $\omega = 0$ Zemlja se ne vrti





majhni odmiki

$$\sin\varphi \approx \frac{x}{l} = \varphi$$
$$z = 1 - \cos\varphi \approx \frac{\varphi^2}{2} \approx 0$$
$$\ddot z = 0$$
$$|\vec F_v| \approx mg$$

$$\left.\begin{aligned} m\ddot x = -mg\frac{x}{l} \ \Rightarrow\ \ddot x &= -\omega_0^2 x \qquad \omega_0^2 = \frac{g}{l} \\ \ddot y &= -\frac{g}{l}\, y \end{aligned}\right\} \qquad \begin{aligned} \vec r &= x\hat i + y\hat j \\ \ddot{\vec r} &= -\frac{g}{l}\vec r \end{aligned}$$

2 $\omega > 0$

$$\vec r = x'\hat i' + y'\hat j'$$
$$m\ddot{\vec r} = -\frac{mg}{l}\vec r - 2m\vec\omega\times\vec v_{rel}' - \underbrace{m\vec\omega\times\vec\omega\times\vec r}_{+m\omega^2\vec r}$$

$$\vec\omega\times\vec v_{rel} = \begin{vmatrix} \hat i & \hat j & \hat k \\ 0 & 0 & \omega \\ \dot x' & \dot y' & 0 \end{vmatrix} = (-\omega\dot y',\ \omega\dot x',\ 0)$$

$$\hat i':\quad \ddot x' = -\omega_0^2 x' + 2\omega\dot y' + \omega^2 x'$$
$$\hat j':\quad \ddot y' = -\omega_0^2 y' - 2\omega\dot x' + \omega^2 y'$$

$\xi' = x' + i y'$ $\qquad \hat{j}$ pomnožimo z i in seštejemo enačbi

$$\ddot{\xi}' = -\omega_0^2 \xi' - i\, 2\omega \dot{\xi}' + \omega^2 \xi'$$

Nastavek $\quad \xi' = \xi_0 e^{-i\alpha t}$

$$-\alpha^2 = -\omega_0^2 - 2\omega\alpha + \omega^2$$

$$\omega_0^2 = (\alpha - \omega)^2$$

$$\alpha = \omega \pm \omega_0$$

$$\begin{aligned}\xi' &= A e^{-i(\omega+\omega_0)t} + B e^{-i(\omega-\omega_0)t} \\ &= e^{-i\omega t}\underbrace{\left(A e^{-i\omega_0 t} + B e^{i\omega_0 t}\right)} \\ &= e^{-i\omega t}\, \xi\end{aligned}$$

Uzemimo neke začetne pogoje:

Ob $t=0$ prejme sunek sile v smeri $\hat{i}'$

$\dot{x}' = v_{x0} \qquad \dot{y}' = 0$

$x' = y' = 0$

$$\xi' = 0 = A + B \quad \Rightarrow \quad A = -B$$

$$\xi' = A e^{-i\omega t}\left(e^{-i\omega_0 t} - e^{i\omega_0 t}\right)$$

$$\dot{\xi}' = A e^{-i\omega t}\left(-i(\omega+\omega_0) e^{i\omega_0 t} - (-i\omega + i\omega_0) e^{-i\omega_0 t}\right)$$

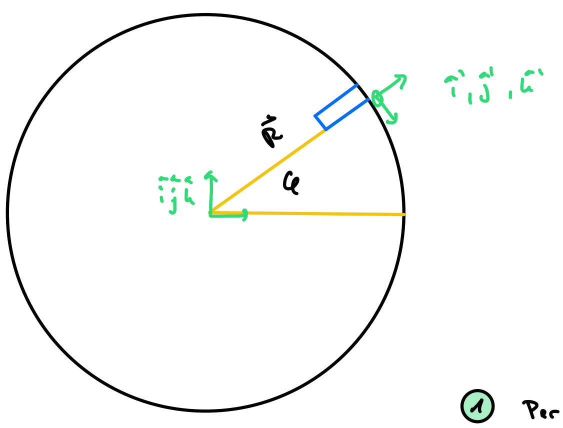
$$\dot{\xi}' = A(-i\omega_0 2) = v_{x0}$$

$$A = \frac{v_{x0}\, i}{2\omega_0}$$

$$\xi' = \frac{v_{x0}}{2\omega_0} e^{-i\omega t}\, i \left(e^{-i\omega_0 t} - e^{i\omega_0 t}\right)$$

$$= \frac{v_{x0}}{\omega_0} e^{-i\omega t} \sin(\omega_0 t)$$

10 Padanje kamna v jašek



$S: \quad m\ddot{\vec{r}} = -mg\frac{\vec{r}}{r}$

$S': \quad m\underset{\dot{\vec{v}}_{rel}}{\underset{\|}{\vec{a}_{rel}}} = -mg\hat{k}' - 2m\vec{\omega}\times\vec{v}_{rel} - \underset{0}{\underset{\approx}{m\vec{\omega}\times\vec{\omega}\times(\vec{R}+\vec{r}')}}$

odvajamo le koordinate in ne tudi baznih vektorjev. $\qquad$ Ker je $\omega \approx 10^{-5}\,s^{-1}$

1 Perturbativno / iterativno

1. približek $\quad m\dot{\vec{v}}_{rel}^{(1)} = -mg\hat{k}' \Rightarrow \vec{v}_{rel}^{(1)} = -gt\hat{k} \Rightarrow \vec{r}'(t) = -\frac{gt^2}{2}\hat{k}$

2. približek $\quad \dot{\vec{v}}_{rel} = -g\hat{k}' - 2\vec{\omega}\times\vec{v}_{rel}^{(1)}$

$$\vec\omega = \omega\hat k = \omega(\cos\varphi\,\hat j' + \sin\varphi\,\hat k')$$

$$\vec\omega\times\vec v_{rel} = \begin{vmatrix} \hat i' & \hat j' & \hat k' \\ 0 & \omega\cos\varphi & \omega\sin\varphi \\ 0 & 0 & -gt\end{vmatrix} = -gt\cos\varphi\,\hat i'\,\omega$$

Začetni pogoji:

$\vec r(0)=0$

$\vec v_{rel}(0)=0$

$$\dot{\vec v}_{rel}^{(2)} = -g\hat k' + 2gt\omega\cos\varphi\,\hat i'$$

$$\vec v_{rel}^{(2)} = -gt\,\hat k' + gt^2\omega\cos\varphi\,\hat i'$$

$$\vec r'^{(2)} = \underbrace{-\frac{gt^2}{2}}_{z'(t)}\hat k' + \underbrace{\frac{1}{3}gt^3\omega\cos\varphi}_{x'(t)}\,\hat i'$$

n-ti približek

$$\left.\begin{aligned}\dot{\vec v}_{rel}^{(n-1)} &= -g\hat k' t - 2\vec\omega\times\vec v_{rel}^{(n-2)} \\ \vec v_{rel}^{(n)} &= -g\hat k' t - 2\vec\omega\times\vec v_{rel}^{(n-1)}\end{aligned}\right] -$$

$$\underbrace{\dot{\vec v}_{rel}^{(n)} - \dot{\vec v}_{rel}^{(n-1)}}_{\delta\dot{\vec v}_{rel}^{(n)}} = -2\vec\omega\times(\underbrace{\vec v_{rel}^{(n-1)} - \vec v_{rel}^{(n-2)}}_{\substack{\text{popravek } n-1 \text{ koraka} \\ \delta\vec v_{rel}^{(n-1)}}})$$

$$\hat\omega\times\hat k' = (\cos\varphi\,\hat j' + \sin\varphi\,\hat k')\times\hat k' = \cos\varphi\,\hat i'$$

$$\hat\omega\times\hat\omega\times\hat k' = \hat\omega(\hat\omega\cdot\hat k') - \omega^2\hat k' = \sin\varphi\,\hat\omega - \hat k'$$

$$\hat\omega\times\hat\omega\times\hat\omega\times\hat k' = \hat\omega\times(\sin\varphi\,\hat\omega - \hat k') = -\hat\omega\times\hat k'$$

$$\delta\vec v_{rel}^{(n)} = -2\vec\omega\times\int_0^t\delta\dot{\vec v}_{rel}^{(n-1)}dt = -2\vec\omega\times\int_0^t\left(-2\vec\omega\times\int_0^t\delta v_{rel}^{(n-2)}dt\right)dt = \dots$$

$$\dots = \underbrace{-2\vec\omega\times(-2\vec\omega\times(-2\vec\omega\times\dots)}_{n-1}\int\dots\int\underbrace{\partial v_{rel}^{(1)}}_{-g\hat k' t}dt_1\,dt_2\dots dt_{n-1}$$

$$= (-2\omega)^{n-1}(\hat\omega\times\hat\omega\times\dots\times\hat k)\left(-g\frac{t^n}{n!}\right)$$

Kompaktni zapis

Sodi popravki $\quad \delta\vec v_{rel}^{(2n)} = (-2\omega)^{2n-1}\,\hat\omega\times(\hat\omega\times\dots(\vec\omega\times\hat k'))\left(-g\frac{t^{2n}}{(2n)!}\right)$

$$= (-2\omega)^{2n-1}(-1)^n\,\hat\omega\times\hat k'\left(-g\frac{t^{2n}}{(2n)!}\right)$$

Lihi popravki $\quad \delta\vec v_{rel}^{(2n+1)} = (-2\omega)^{2n}\,\hat\omega\times(\hat\omega\times\hat k')\left(-g\frac{t^{2n+1}}{(2n+1)!}\right)(-1)^n$

$$\vec v_{rel} = \sum_{i=1}^\infty\delta v_{rel}^{(i)} = \sum_{n=0}^\infty\delta v_{rel}^{(2n+1)} + \sum_{n=1}^\infty\delta v_{rel}^{(2n)} =$$

$$= -g\hat k' t + \sum_{n=1}^\infty\frac{(-2\omega)^{2n+1}(-1)^n}{(-2\omega)}\hat\omega\times\hat\omega\times\hat k'\left(-g\frac{t^{2n+1}}{(2n+1)!}\right) + \sum_{n=1}^\infty\frac{(-2\omega)^{2n}}{(-2\omega)}(-1)^n\,\hat\omega\times\hat k'\left(-g\frac{t^{2n}}{(2n)!}\right)$$

$$= -g\hat k' t + \frac{g}{2\omega}\hat\omega\times(\hat\omega\times\hat k')\underbrace{\sum_{n=1}^\infty\frac{(-2\omega t)^{2n+1}}{(2n+1)!}(-1)^n}_{\sin(-2\omega t)-(-2\omega t)} + \frac{g}{2\omega}\hat\omega\times\hat k\underbrace{\sum_{n=1}^\infty\frac{(-2\omega t)^{2n}}{(2n)!}(-1)^n}_{\cos(2\omega t)-1} =$$

$$= -g\hat k' t + \frac{g}{2\omega}\underbrace{\hat\omega\times(\hat\omega\times\hat k')}_{\sin\varphi\,\hat\omega - \hat k'}(-\sin(2\omega t)-(-2\omega t)) + \frac{g}{2\omega}\hat\omega\times\hat k\,(\cos(2\omega t)-1)$$

$$\vec r'(t) = -g\frac{t^2}{2}\hat k' + \frac{g}{2\omega}(\sin\varphi\,\hat\omega - \hat k')\left(\frac{\cos(2\omega t)}{2\omega} + \frac{1}{2\omega} + \omega t^2\right) + \frac{g}{2\omega}(\hat\omega\times\hat k)\left(\frac{\sin(2\omega t)}{2\omega} - 1\right)$$

$$= -g\frac{t^2}{2}\hat k' + \frac{g}{2\omega}\left(\sin\varphi(\cos\varphi\,\hat j' + \sin\varphi\,\hat k') - \hat k'\right)\left(\frac{\cos(2\omega t)-1}{2\omega} + \omega t^2\right) + \frac{g\cos\varphi}{2\omega}\hat i\left(\frac{\sin 2\omega t}{2\omega} - t\right)$$

$$= \hat i\ \frac{g\cos\varphi}{2\omega}\left(\frac{\sin 2\omega t}{2\omega} - t\right) +$$
$$\hat j\ \frac{g}{2\omega}\sin\varphi\cos\varphi\left(\frac{\cos 2\omega - 1}{2\omega} + \omega t^2\right) +$$
$$\hat k\ \left(-g\frac{t^2}{2} + \frac{g}{2\omega}(\sin^2\varphi - 1)\left(\frac{\cos 2\omega - 1}{2\omega} + \omega t^2\right)\right)$$

2. način

$$\vec a_{rel} + 2\vec\omega\times\vec v_{rel} = -g\hat k$$

$$\vec\omega\times\vec v_{rel} = \begin{vmatrix} \hat i' & \hat j' & \hat k' \\ 0 & \omega\cos\varphi & \omega\sin\varphi \\ \dot x' & \dot y' & \dot z' \end{vmatrix} = \left(\omega(\cos\varphi\,\dot z' - \sin\varphi\,\dot y'),\ \omega\sin\varphi\,\dot x',\ -\dot x'\omega\cos\varphi\right)$$

$\hat i'$: $\ddot x' + 2\omega(\cos\varphi\,\dot z' - \sin\varphi\,\dot y') = 0$

$\hat j'$: $\ddot y' + 2\omega\sin\varphi\,\dot x' = 0$ Sistem dif. en. $\left.\begin{matrix}\cdot\sin\varphi \\ \cdot\cos\varphi\end{matrix}\right]-$

$\hat k'$: $\ddot z - 2\omega\cos\varphi\,\dot x' = -g$

$$\gamma = \cos\varphi\,\dot z' - \sin\varphi\,\dot y'$$

$$\ddot x' + 2\omega\gamma = 0$$
$$\dot\gamma - 2\omega\dot x' = -g\cos\varphi$$

$$\xi = \dot x' + i\gamma$$

$$\ddot\xi - 2i\omega\dot\xi = -ig\cos\varphi \qquad \xi(t) = Ae^{\lambda t} + Bt + C \qquad \text{Nastavek}$$

$$\lambda_1 = 0 \qquad \lambda_2 = 2i\omega$$

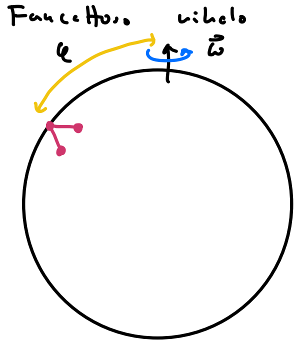
$$\xi(0) = \dot x'(0) + i\gamma(0)$$
$$\dot\xi(0) = \ddot x'(0) + i\dot\gamma(0)$$

$$\Rightarrow A, B, C$$

$$\xi(t) = \frac{g\cos\varphi}{2\omega}\left(\frac{i}{2\omega}\left(e^{2i\omega t} - 1\right) + t\right)$$

$$\dot x' = \mathrm{Re}\,\xi(t) \qquad \gamma = \mathrm{Im}\,\xi(t)$$

11 Foucaultovo nihalo



$$\vec\omega = \omega\cos\varphi\,\hat k' + \omega\sin\varphi\,\hat j'$$
$$m\ddot{\vec r}' = \vec F - m\left(2\vec\omega\times\vec v_{rel} - \underbrace{\vec\omega\times\vec\omega\times(\vec r' + \vec R)}_{=0}\right)$$

$$\vec F = -\frac{mg}{l}\vec r'$$

$$\vec r' = x'\hat i' + y'\hat j'$$

$$\vec\omega\times\vec v_{rel} = \begin{vmatrix} \hat i' & \hat j' & \hat k' \\ 0 & \omega\sin\varphi & \omega\cos\varphi \\ \dot x' & \dot y' & 0 \end{vmatrix} = \left(-\omega\cos\varphi\,\dot y',\ \omega\cos\varphi\,\dot x',\ -\omega\sin\varphi\,\dot x'\right)$$

D. Z. po komponentah

$$\ddot{x}' = -\frac{g}{l}x' + 2\omega\cos\varphi\,\dot{y}'$$

$$\ddot{y}' = -\frac{g}{l}y' - 2\omega\cos\varphi\,\dot{x}'$$

$$\xi = x' + i y'$$

$$\ddot{\xi}' = \omega_0^2\,\xi' - 2\omega\cos\varphi\, i\,\dot{\xi}'$$

$$\xi'(t) = \xi'_0\, e^{-i\alpha t}$$

$$-\alpha^2 = -\omega_0^2 - 2\omega\cos\varphi\,\alpha$$

$$\alpha_{12} = \frac{\cancel{2}\omega\cos\varphi \pm \sqrt{\cancel{4}\omega^2\cos^2\varphi + \cancel{4}\omega_0^2}}{\cancel{2}} = \omega\cos\varphi \pm \omega_0$$

($\ll$)

$$\xi'(t) = e^{-i\omega\cos\varphi\, t}\left(A e^{i\omega_0 t} + B e^{-i\omega_0 t}\right)$$

12 Virialni teorem

Za skupino delcev, ki med seboj delujejo s centralnim potencialom $V(r) = k r^\alpha$ velja

$$2\langle T\rangle = \alpha\langle V\rangle$$

$\langle f\rangle$ ... časovno povprečje $\qquad \langle f(t)\rangle = \lim_{\tau\to\infty}\frac{1}{\tau}\int_0^\tau f(t)\,dt$

T ... celotna kin. en. $\qquad V$ ... celotna pot. en.

- Za en delec

$$m\ddot{\vec{r}} = \vec{F} \qquad /\cdot\vec{r}$$

$$m\ddot{\vec{r}}\cdot\vec{r} = \vec{F}\cdot\vec{r}$$

$$m\left(\frac{d}{dt}(\dot{\vec{r}}\cdot\vec{r}) - \dot{\vec{r}}\cdot\dot{\vec{r}}\right) = \vec{F}\cdot\vec{r} \qquad \Big/\ \frac{1}{\tau}\int_0^\tau dt$$

$$\underbrace{\frac{1}{\tau}}_{\to 0}\ \underbrace{m\,\dot{\vec{r}}\cdot\vec{r}\Big|_0^\tau}_{\to\text{ konst.}} - \langle m\,\dot{\vec{r}}\cdot\dot{\vec{r}}\rangle = \langle\vec{F}\cdot\vec{r}\rangle$$

Če imamo omejeno gibanje $\vec{r}$ ostane končen

$\lim_{\tau\to\infty}$

$$0 - 2\langle T\rangle = \langle\vec{F}\cdot\vec{r}\rangle$$

$$\vec{F} = -\operatorname{grad} V(r) \qquad V(r) = k r^\alpha$$

$$= -k\alpha r^{\alpha-1}\operatorname{grad} r \qquad\qquad \operatorname{grad} r = \frac{\vec{r}}{r}$$

$$= -k\alpha r^{\alpha-1}\frac{\vec{r}}{r}$$

$$\vec{F}\cdot\vec{r} = -k\alpha r^\alpha = -\alpha V$$

$$\Rightarrow\quad 2\langle T\rangle = \alpha\langle V\rangle$$

• za n delcev

za i-ti delec

$$m\ddot{\vec r}_i = \sum_{j(\neq i)} \vec F_{ij} \qquad /\cdot \vec r_i$$

$$\sum_i \left[\frac{1}{\tau}\int dt\, \underbrace{m\left(\frac{d}{dt}(\dot{\vec r}_i\cdot\vec r_i)\right)}_{\to 0} - \dot{\vec r}_i\, \dot{\vec r}_i\right] = \sum_i\sum_{j(\neq i)}\langle \vec F_{ij}\cdot \vec r_i\rangle$$

$$-\sum_i\langle m_i\, \dot{\vec r}_i\cdot\dot{\vec r}_i\rangle = \sum_i\sum_{j(\neq i)}\langle \vec F_{ij}\cdot\vec r_i\rangle$$

$$-\sum_j\langle m_j\,\dot{\vec r}_j\cdot\dot{\vec r}_j\rangle = \sum_j\sum_{i(\neq j)}\langle\vec F_{ji}\cdot\vec r_j\rangle \qquad \Big] +$$

ker $\vec F_{ji} = -\vec F_{ij}$

$$-4\langle T\rangle = \sum_i\sum_j\langle \vec F_{ij}\cdot(\vec r_i-\vec r_j)\rangle$$

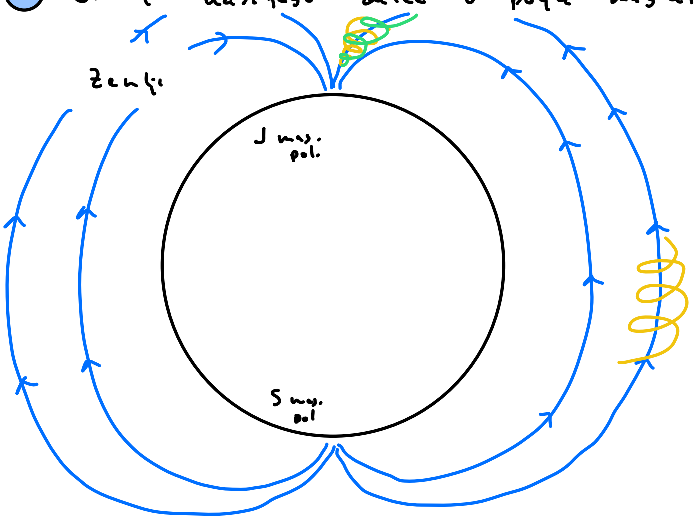
$F = -\mathrm{grad}\, V = -\alpha r_{ij}^{\alpha-1}\frac{\vec r_{ij}}{r_{ij}}$ $\quad V = k\, r_{ij}^\alpha$ $\quad \vec r_{ij} = \vec r_i - \vec r_j$

$$-4\langle T\rangle = -\sum\sum\langle \alpha k\, r_{ij}^\alpha\rangle = -\sum_{i<j}\sum 2\langle k\alpha\, r_{ij}^\alpha\rangle$$

$$2\langle T\rangle = \alpha\langle V\rangle$$

• Primer: harmonski potencial $\alpha=2$ $\Rightarrow$ $\langle V\rangle = \langle T\rangle$

(13) Gibanje nabitega delca v polju magnetnega monopola (osredinimo na severnem ali južnem polu)



$$\vec B = \frac{\mu_0 e_m}{4\pi}\frac{\vec r}{r^3}$$

$$\vec F = e(\vec E + \vec v\times\vec B) \qquad \vec E = 0$$

$$\vec F = (\vec v\times\vec B)e$$

$$m\vec a = e\vec v\times\vec B$$

$$\ddot{\vec r} = \frac{e}{m}\dot{\vec r}\times B$$

$$\ddot{\vec r} = \frac{e}{m}\frac{\mu_0 e_m}{4\pi}\dot{\vec r}\times\frac{\vec r}{r^3}$$

$$\ddot{\vec r} = \frac{e}{m}g\,\dot{\vec r}\times\frac{\vec r}{r^3}$$

Iščemo konstante gibanja

(1) $/\cdot\dot{\vec r}$ $\quad \ddot{\vec r}\cdot\dot{\vec r} = 0 = \frac{1}{2}\frac{d}{dt}(\dot{\vec r}\cdot\dot{\vec r}) \Rightarrow \dot{\vec r}\cdot\dot{\vec r} = \text{konst} = v_0^2 \Rightarrow |\dot{\vec r}| = v_0$

hitrost je vedno enaka

(2) $\vec L = \vec r\times m\dot{\vec r}$

$$\frac{d}{dt}\vec L = \overset{\to 0}{\dot{\vec r}\times m\dot{\vec r}} + \vec r\times m\ddot{\vec r} = \vec r\times m\left(\frac{e}{m}g\,\dot{\vec r}\times\frac{\vec r}{r^3}\right) = \frac{ge}{r^3}\left(\dot{\vec r}\,r^2 - \vec r\,(\dot{\vec r}\cdot\vec r)\right)$$

$$\frac{d}{dt}\frac{\vec r}{r} = \frac{\dot{\vec r}}{r} - \frac{\vec r\,\dot r}{r^2} \qquad r=\sqrt{\vec r\cdot\vec r} \qquad \dot r = \frac{1}{2}\frac{2\dot{\vec r}\cdot\vec r}{\sqrt{\vec r\cdot\vec r}} = \frac{\dot{\vec r}\cdot\vec r}{r}$$

$$= \frac{1}{r^3}\left(\dot{\vec r}\,r^2 - \vec r\,(\dot{\vec r}\cdot\vec r)\right) \qquad /\cdot\int dt$$

$$\frac{\vec L}{ge} + \frac{\vec L_{konst}}{ge} = \frac{\vec r}{r}$$

③ $\dot{\vec L} = \vec r \times \underbrace{\left(ge\, \dot{\vec r} \times \frac{\vec r}{r^3}\right)}_{-\frac{ge\vec L}{mr^3}} = -\vec r \times \vec L \,\frac{ge}{mr^3}$ $\qquad \dot{\vec L} \perp \vec L, \vec r$

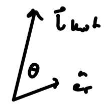
$\dot{\vec L}\cdot\vec L = 0 = \frac{d}{dt}\left(\frac12 \vec L\cdot\vec L\right) \Rightarrow \vec L\cdot\vec L = konst. \Rightarrow L_0 = |\vec L| = konst$

iz točke 2 $\qquad \frac{\vec r}{r} - \frac{\vec L_{konst}}{eg} = \frac{\vec L}{eg} \qquad |^2$

$$\underbrace{\left(\frac{\vec r}{r}\right)^2}_{konst} + \underbrace{\left(\frac{\vec L_{konst}}{eg}\right)^2}_{konst} - 2\,\frac{\vec r}{r}\cdot\frac{\vec L_{konst}}{eg} = \underbrace{\frac{L_0^2}{(eg)^2}}_{konst}$$

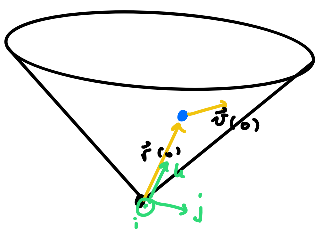
$\Downarrow$

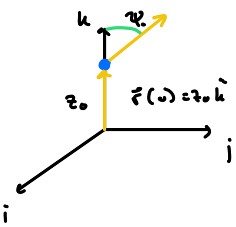
$$\frac{\vec r}{r}\cdot \vec L_{konst} = konst.$$



$$\hat e_r \cdot \vec L_{konst} = |\vec L_{konst}|\cos\theta$$

$\Rightarrow \theta = konst.$





$\vec v(0) = v_0(\sin\varphi_0\, \hat j + \cos\varphi_0\, \hat k)$

$\vec L(0) = m\vec r(0)\times\vec v(0) =$
$= m z_0 \hat k \times v_0(\sin\varphi_0\, \hat j + \cos\varphi_0\, \hat k) =$
$= -m z_0 v_0 \sin\varphi_0\, \hat i$

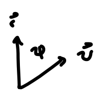
$\vec L_{konst} = -\vec L(0) + ge\,\frac{\vec r}{r}(0) =$

$\vec L_{konst} = +m z_0 v_0 \sin\varphi_0\, \hat i + ge\,\hat k$

$\theta = konst$

$\hat e_r\cdot\vec L_{konst} = |\vec L_{konst}|\cos\theta$

$$\cos\theta = \frac{\hat e_r\cdot \vec L_{konst}}{|\vec L_{konst}|} = \frac{ge}{\sqrt{(ge)^2+(mz_0v_0\sin\varphi_0)^2}}$$



$\vec r(t) = ?\qquad \varphi(t) = ?$

$\dot{\vec r}\cdot\vec r = |\dot{\vec r}||\vec r|\cos\varphi = v_0\, r\cos\varphi$

$\frac{d}{dt}(\dot{\vec r}\cdot\vec r) = \underbrace{\ddot{\vec r}\cdot\vec r}_{0} + \dot{\vec r}\cdot\dot{\vec r} = v_0^2 \qquad |\int dt$

$\dot{\vec r}\cdot\vec r = v_0^2\, t + C = v_0\, r\cos\varphi$

pri $t=0$ $\qquad C = v_0 z_0\cos\varphi_0$

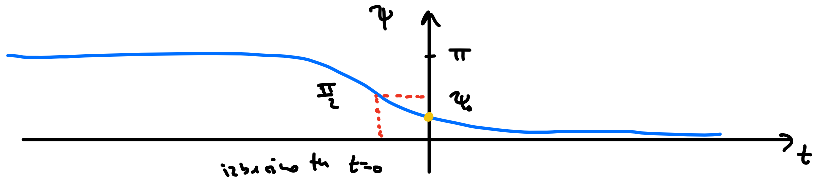
$\underline{v_0\, r\cos\varphi = v_0^2 t + v_0 z_0\cos\varphi_0}$

$\frac1m|\vec L| = \frac{L_0}{m} = |\vec r\times m\dot{\vec r}|\frac1m = |\vec r\times\dot{\vec r}| = \underline{r v_0\sin\varphi \overset{pri\ t=0}{=} z_0 v_0 \sin\varphi_0}$ $\qquad |^2$ in seštej

$$r^2 v_0 = (z_0v_0\sin\varphi_0)^2 + (v_0^2 t + v_0 z_0\cos\varphi_0)^2$$

$$r(t)=\sqrt{(z_0\sin\psi_0)^2+(v_0t+z_0\cos\psi_0)^2}$$

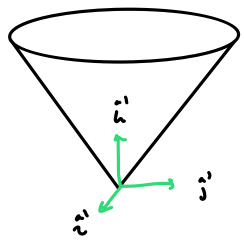
$$\tan\psi=\frac{z_0 v_0\sin\psi_0}{v_0^2t+v_0z_0\cos\psi_0}=\frac{z_0\sin\psi_0}{v_0t+z_0\cos\psi_0}$$



začetni čas $t=0$ da $\psi_0=\frac{\pi}{2}$

$$r(t)=\sqrt{z_0^2+(v_0t)^2}\qquad \tan\psi=\frac{z_0}{v_0t}\qquad \theta=\text{konst.}$$

Rešitev v sferičnih koordinatah

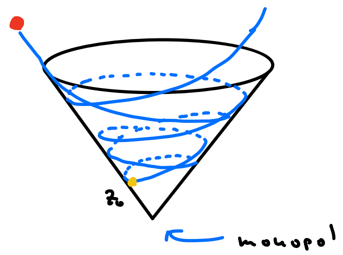


$$\vec r(t)=r(t)\,(\cos\varphi\sin\theta,\ \sin\varphi\sin\theta,\ \cos\theta)$$

$$\varphi(t)=\int\frac{z_0v_0}{r^2\sin\theta}\,dt=\frac{z_0v_0}{\sin\theta}\int\frac{dt}{z_0^2+(v_0t)^2}$$

$$\varphi(t)=\frac{1}{\sin\theta}\arctan\left(\frac{v_0t}{z_0}\right)$$

Gibanje delca



$$N=\frac{\Delta\varphi}{2\pi}\qquad \text{število ovojev}$$

$$\Delta\varphi=\varphi(\infty)-\varphi(-\infty)=\frac{\pi}{2\sin\theta}-\left(-\frac{\pi}{2\sin\theta}\right)=\frac{\pi}{\sin\theta}$$

$$N=\frac{1}{2\sin\theta}$$

Lagrangeov formalizem

T Newtonov formalizem

N-delcev

$$\left.\begin{aligned} m_1\ddot{\vec r}_1&=\vec F_1(\vec r_1,\dots,\vec r_N)\\ &\ \vdots\\ m_N\ddot{\vec r}_N&=\vec F_N(\vec r_1,\dots,\vec r_N)\end{aligned}\right\}\ 3N\ \text{enačb}$$

Vezi:

$$\begin{aligned} f_1(\vec r_1,\dots,\vec r_N)&=0\\ &\ \vdots\\ f_K(\vec r_1,\dots,\vec r_N)&=0\end{aligned}\qquad 3N-K\ \text{prostostnih stopenj}$$

$$\begin{aligned} q_1&=q_1(\vec r_1,\dots,\vec r_N)\qquad \text{posplošene koordinate}\\ &\ \vdots\\ q_{3N-K}&=q_{3N-K}(\vec r_1,\dots,\vec r_N)\end{aligned}$$

Lagrangeov formalizem

$$L = T - V = L(q_\alpha, \dot q_\alpha) = L(q_1, \dots, q_{3N-K}, \dot q_1, \dots, \dot q_{3N-K})$$

Euler-Lagrangeovi enačba

$$\boxed{\frac{d}{dt}\frac{\partial L}{\partial \dot q_\alpha} = \frac{\partial L}{\partial q_\alpha}} \left(+ Q_\alpha\right)$$

tiste sile, ki niso v V.

$$Q_\alpha = \sum_i \vec F_i \cdot \frac{\partial \vec r_i}{\partial q_\alpha}$$

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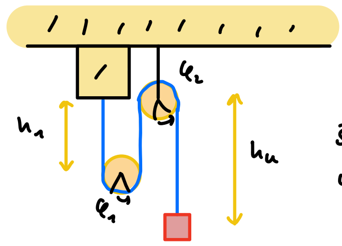


škripec : m_0, R, J

utež : m_u

$h_u(t) = ?$

Recept:

1. zapiši T in V
2. upoštevaj vezi
3. E-L enačbe

(1)

$$T = \frac{1}{2} m_u \dot h_u^2 + \frac{1}{2} J \dot\varphi_2^2 + \frac{1}{2} J \dot\varphi_1^2 + \frac{1}{2} m_0 \dot h_1^2$$

$$V = -m_u g h_u - m_0 g h_1$$

(2)

$$h_1 = -R\varphi_1 \qquad \dot h_1 = -R\dot\varphi_1$$

$$h_u = -R\varphi_2 \qquad \dot h_u = -R\dot\varphi_2$$

$l = 2h_1 + h_u + 2\pi R$ dolžina vrvice $\qquad \dot h_1 = \frac{1}{2}\dot h_u$

$$V = -m_u g h_u - \frac{m_0 g}{2}(l - 2\pi R - h_u) = \frac{m_0 g}{2} h_u - m_u g h_u + \text{konst.}$$

$$T = \frac{1}{2} m_u \dot h_u^2 + \frac{1}{2} J \frac{\dot h_u^2}{R^2} + \frac{1}{2} J \frac{\dot h_u^2}{4R^2} + \frac{1}{2} m_0 \frac{1}{4} \dot h_u^2$$

$$= \frac{1}{2}\dot h_u^2 \left(m_u + \frac{5}{4}\frac{J}{R^2} + \frac{m_0}{4}\right)$$

(3)

$$L = T - V = \frac{1}{2}\dot h_u^2 \underbrace{\left(m_u + \frac{5}{4}\frac{J}{R^2} + \frac{m_0}{4}\right)}_{\tilde m} - h_u \underbrace{\left(\frac{m_0 g}{2} - m_u g\right)}_{\tilde n} + \text{konst.}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot q} = \frac{\partial L}{\partial q}$$

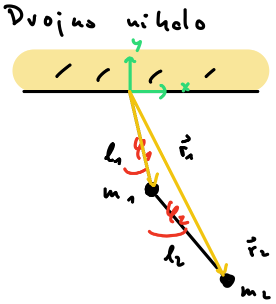
$$\frac{d}{dt}\frac{1}{2}\tilde m\, 2\dot h_u = -\frac{1}{2}\tilde n$$

$$\tilde m \ddot h_u = -\frac{1}{2}\tilde n$$

$$\ddot h_u = -\frac{1}{2}\frac{\tilde n}{\tilde m} = \tilde\alpha$$

$$h_u = \frac{\tilde\alpha t^2}{2} + v_0 t + h_u^0$$

15 Dvojno nihalo



- ravninsko gibanje
- lastne frekvence?

$(x_1, y_1), (x_2, y_2)$

Generalizirane koordinate $4 - 2$

2 vezi:

$\vec r_2' = \vec r_2 - \vec r_1$

$|\vec r_1| = l_1, \quad |\vec r_2| = l_2$

$$T = \frac{1}{2} m_1 \dot{\vec r}_1^{\,2} + \frac{1}{2} m_2 \dot{\vec r}_2^{\,2}$$

$$\vec r_1 = (l_1 \sin\varphi_1, -l_1 \cos\varphi_1)$$
$$\vec r_2' = (l_2 \sin\varphi_2, -l_2 \cos\varphi_2)$$
$$\vec r_2 = (l_1 \sin\varphi_1 + l_2 \sin\varphi_2, -l_1 \cos\varphi_1 - l_2 \cos\varphi_2)$$
$$\dot{\vec r}_1 = (l_1 \cos\varphi_1, l_1 \sin\varphi_1)\,\dot\varphi_1$$
$$\dot{\vec r}'_2 = (l_2 \cos\varphi_2, l_1 \sin\varphi_2)\,\dot\varphi_2$$

$$T = \frac{1}{2} m_1 l_1^2 (\cos^2\varphi_1 + \sin^2\varphi_1)\dot\varphi_1^2 + \frac{1}{2} m_2 (\dot{\vec r}_1^{\,2} + \dot{\vec r}_2'^{\,2} + 2\dot{\vec r}_1 \cdot \dot{\vec r}_2')$$

$$T = \frac{1}{2} m_1 l_1^2 \dot\varphi_1 + \frac{1}{2} m_2 (l_1^2 \dot\varphi_1^2 + l_2^2 \dot\varphi_2^2 + 2 l_1 l_2 \dot\varphi_1 \dot\varphi_2 \underbrace{\cos(\varphi_2 - \varphi_1)}_{\simeq 1})$$

$$V = m_1 g (-l_1 \cos\varphi_1) + m_2 g (-l_1 \cos\varphi_1 - l_2 \cos\varphi_2) \qquad \cos x = 1 - \frac{x^2}{2}$$

$$V \simeq V_0 + m_1 g l_1 \frac{\varphi_1^2}{2} + m_2 g \left(l_1 \frac{\varphi_1^2}{2} + l_2 \frac{\varphi_2^2}{2}\right)$$

$$L = T - V$$

E-L: $\varphi_1: \quad \frac{d}{dt}\frac{\partial L}{\partial \dot\varphi_1} = \frac{\partial L}{\partial \varphi_1} \qquad m_1 l_1^2 \ddot\varphi_1 + m_2 l_1^2 \ddot\varphi_1 + m_2 l_1 l_2 \ddot\varphi_2 = -m_1 g l_1 \varphi_1 - m_2 g l_1 \varphi_1$

$\varphi_2: \quad \frac{d}{dt}\frac{\partial L}{\partial \dot\varphi_2} = \frac{\partial L}{\partial \varphi_2} \qquad m_2 l_2^2 \ddot\varphi_2 + m_2 l_1 l_2 \ddot\varphi_1 + m_2 g l_2 \varphi_2 = 0$

Nastavek $\quad \varphi_1(t) = \varphi_{10} e^{-i\omega t}$
$\quad \varphi_2(t) = \varphi_{20} e^{-i\omega t}$

$$-m_1 l_1^2 \omega^2 \varphi_{10} - m_2 l_1^2 \omega^2 \varphi_{10} - m_2 l_1 l_2 \omega^2 \varphi_{20} + m_1 g l_1 \varphi_{10} + m_2 g l_1 \varphi_{10} = 0$$
$$-m_2 l_2^2 \omega^2 \varphi_{20} - m_2 l_1 l_2 \omega^2 \varphi_{10} + m_2 g l_2 \varphi_{20} = 0$$

$$A \vec\varphi_0 = A \begin{bmatrix} \varphi_{10} \\ \varphi_{20} \end{bmatrix} = 0 \qquad \det A = 0$$

$$\begin{bmatrix} -m_1 l_1^2 \omega^2 - m_2 l_1^2 \omega^2 + m_1 g l_1 + m_2 g l_1 & -m_2 l_1 l_2 \omega^2 \\ -m_2 l_1 l_2 \omega^2 & -m_2 l_2^2 \omega^2 + m_2 g l_2 \end{bmatrix} \vec\varphi_0 = 0$$

$$0 = \det A = m_1 m_2 \omega^4 + (\omega_{02}^2 + \omega_{01}^2) m_2 (-m_1 - m_2)\omega^2 + m_2 \omega_{01}^2 \omega_{02}^2 (m_1 + m_2) = 0$$

$$\omega_{01}^2 = \frac{g}{l_1} \qquad \omega_{02}^2 = \frac{g}{l_2}$$

$$\omega_\pm^2 = \frac{m_1 + m_2}{2 m_1} (\omega_{01}^2 + \omega_{02}^2) \left(1 \pm \sqrt{1 - \frac{4 m_1}{m_1 + m_2} \frac{\omega_{01}^2 \omega_{02}^2}{(\omega_{01}^2 + \omega_{02}^2)^2}}\right)$$

če $\quad m_1 = m_2, \quad l_1 = l_2 \Rightarrow \omega_{01} = \omega_{02}$

$$\omega_\pm^2 = \omega_0^2 (2 \pm \sqrt{2})$$

2 lastna načina nihanja

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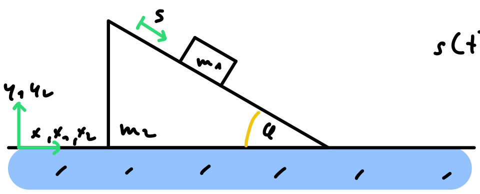


$s(t) = ?$

$x_1 = x + s\cos\varphi$
$y_1 = -s\sin\varphi$
$x_2 = x$
$y_2 = 0$ neka konst.

$$T = \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2)$$
$$T = \frac{1}{2} m_1 \left((\dot{x} + \dot{s}\cos\varphi)^2 + \dot{s}^2 \sin^2\varphi \right) + \frac{1}{2} m_2 \dot{x}^2$$
$$T = \frac{1}{2} m_1 (\dot{x}^2 + 2\dot{x}\dot{s}\cos\varphi + \dot{s}^2) + \frac{1}{2} m_2 \dot{x}^2$$
$$V = m_1 g s(-\sin\varphi)$$

generaliziran moment
x je ciklična koordinata

$$x: \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x} \Rightarrow \frac{d}{dt}\left(m_1 \dot{x} + m_1 \dot{s}\cos\varphi + m_2 \dot{x} \right) = 0 \qquad m_1 \ddot{x} + m_1 \ddot{s}\cos\varphi + m_2 \ddot{x} = 0$$

$$s: \frac{d}{dt}\frac{\partial L}{\partial \dot{s}} = \frac{\partial L}{\partial s} \Rightarrow \frac{d}{dt}\left(m_1 \dot{x}\cos\varphi + m_1 \dot{s} \right) = -\sin\varphi\, m_1 g \qquad \ddot{x}\cos\varphi + \ddot{s} = -\sin\varphi\, g$$

$$\ddot{s}\left(1 - \frac{m_1}{m_1+m_2}\cos^2\varphi\right) = -\sin\varphi\, g \qquad \ddot{x} = -\frac{m_1\cos\varphi}{m_1+m_2}\ddot{s}$$

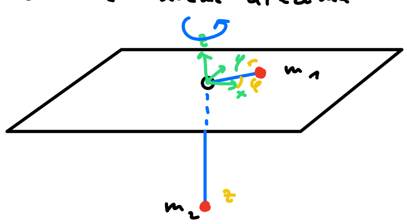
$$s = \iint \underbrace{\frac{\sin\varphi\, g}{\frac{m_1}{m_1+m_2}\cos^2\varphi - 1}}_{a} dt^2$$

$$s(t) = a\frac{t^2}{2} + At + B$$

Limite

$m_2 \to \infty \qquad a \to g\sin\varphi$
$m_2 \to 0 \qquad a \to \frac{g}{\sin\varphi}$

17 Plošča z dvema utežema



$l = r - z \qquad \Rightarrow \qquad z = r - l \qquad \dot{z} = \dot{r}$

2 gen. koordinati r, φ

$$L = \frac{1}{2} m_2 \dot{z}^2 + \frac{1}{2} m_1 (\dot{r}^2 + r^2\dot{\varphi}^2) + m_2 g z$$
$$= \frac{1}{2} m_2 \dot{r}^2 + \frac{1}{2} m_1 (\dot{r}^2 + r^2\dot{\varphi}^2) - m_2 g (r - l)$$

$$E\text{-}L: \quad r: \frac{d}{dt}(m_2\dot{r} + m_1\dot{r}) = m_1 r\dot{\varphi}^2 - m_2 g$$
$$(m_1 + m_2)\ddot{r} = m_1\dot{\varphi}^2 r - m_2 g$$

$$\varphi: \frac{d}{dt} m_1 r^2 \dot{\varphi} = 0 \qquad p_\varphi = m_1 r^2 \dot{\varphi}$$
$$m_1 2 r\dot{r}\dot{\varphi} + m_1 r^2 \ddot{\varphi} = 0$$

Stacionarno gibanje in nihanje okrog tega?

$\dot{r} = 0$, $\ddot{r} = 0$, $r = r_0$ $\Rightarrow m_1 \dot{\varphi}^2 r = m_2 g \qquad \dot{\varphi}^2 = \frac{m_2 g}{m_1 r_0} = \omega_0^2 \qquad \varphi(t) = \sqrt{\frac{m_2 g}{m_1 r_0}}\, t$

Nihanje za male popravke

$r = r_0 + x \qquad x \ll r_0$

$$(m_1 + m_2)\ddot{x} - \frac{p_\varphi^2}{m_1 r^3} + m_2 g = 0$$
$$(m_1 + m_2)\ddot{x} - \frac{p_\varphi^2}{m_1 (r_0 + x)^3} + m_2 g = 0$$

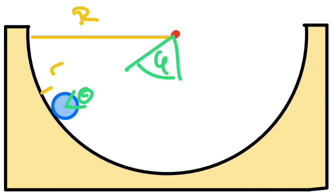
$$(m_1+m_2)\ddot{x} - \frac{p_\varphi^2}{m_1 r_0^3\left(1-\frac{x}{r_0}\right)^3} + m_2 g = 0$$

$$(m_1+m_2)\ddot{x} - \frac{p_\varphi^2}{m_1 r_0^3}\left(1-3\frac{x}{r_0}\right) + m_2 g = 0 \qquad p_\varphi^2 = m_1^2 r_0^4 \dot{\varphi}^2 = m_1^2 r_0^4 \frac{m_2 g}{m_1 r_0} = m_1 m_2 g r_0^3$$

$$(m_1+m_2)\ddot{x} + 3\frac{p_\varphi^2}{m_1 r_0^4}\, x = 0$$

$$\omega_0^2 = \frac{3 p_\varphi^2}{m_1 r_0^4 (m_1+m_2)}$$

18 Kroglica v polkrožni posodi:



Vezi:
- $x_T = (R-r)\sin\varphi$
  $y_T = -(R-r)\cos\varphi$
- ne zdrsne
  $r\dot{\theta} = R\dot{\varphi}$

$\Rightarrow x, y, \theta \qquad 3-2=1$ gen. koord φ

$$T = \frac{1}{2} m\left(\dot{x}_T^2 + \dot{y}_T^2\right) + \frac{1}{2} J\dot{\theta}^2 \qquad J = \frac{2}{5} m r^2$$

$$V = m g y_T = -mg(R-r)\cos\varphi$$

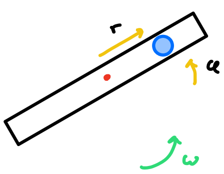
$$L = T - V = \frac{1}{2} m (R-r)^2 \dot{\varphi}^2 + \frac{1}{2} J\left(\frac{R}{r}\dot{\varphi}\right)^2 + mg(R-r)\cos\varphi$$

E-L: $$0 = \frac{d}{dt}\frac{\partial L}{\partial \dot{\varphi}} - \frac{\partial L}{\partial \varphi} = m(R-r)^2\ddot{\varphi} + 2\frac{1}{2}\frac{2}{5} m r^2 \frac{R^2}{r^2}\ddot{\varphi} + mg(R-r)\sin\varphi$$

$$0 = \left(m(R-r)^2 + \frac{2}{5} m R^2\right)\ddot{\varphi} + mg(R-r)\sin\varphi$$

$$\Rightarrow \quad \ddot{\varphi} + \omega_0^2 \sin\varphi = 0 \qquad \omega_0^2 = \frac{g(R-r)}{(R-r)^2 + \frac{2}{5}R^2}$$

19 Kroglica v vrteči se cevi



gladko drsi

$$T = \frac{1}{2} m\left(\dot{r}^2 + r^2\dot{\varphi}^2\right) = L$$

vez $\dot{\varphi} = \omega \quad V=0 \quad L = \frac{1}{2} m\left(\dot{r}^2 + r^2\omega^2\right)$

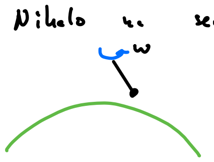
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = \frac{d}{dt} m\dot{r} = m\ddot{r} = \frac{\partial L}{\partial r} = m\omega^2 r$$

$$\ddot{r} = \omega^2 r$$

$$r(t) = A e^{\omega t} + B e^{-\omega t}$$ Nastavek

20 Nihalo na severnem polu

DN



Reši z Lagr. formalizmom in za opis gibanja uporabi kart. koord. sistem x', y'.

1 holonomne vezi

$$f(q_\alpha) = 0$$

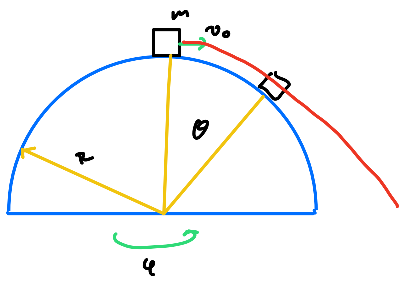
neholonomne vezi:

$$f(q_\alpha) > 0$$

21 Ukež na ledeni polkrogli

Kdaj se odlepi: ?

- sferične koordinate r, θ, φ
- φ = konst
- $r = R$ to je dokler nas $\Rightarrow r \geq R$



1 $r = R$

$$T = \frac{1}{2} m v^2 = \frac{1}{2} m R^2 \dot\theta^2$$

$$V = mg\cos\theta\, R$$

$$L = T - V = \frac{1}{2} m R^2 \dot\theta^2 - mgR\cos\theta$$

E-L θ: $m R^2 \ddot\theta + mgR(-\sin\theta) = 0$

$$\ddot\theta - \frac{g}{R}\sin\theta = 0$$ Hiperbolično padanje

2 $r > R$

$$V = mg\cos\theta\, r$$

$$T = \frac{1}{2} m (\dot r^2 + r^2\dot\theta^2)$$

$$L = T - V = \frac{1}{2} m (\dot r^2 + r^2\dot\theta^2) - mgr\cos\theta$$

holonomne vez, se spreminja s časom, ne velja vez $r = R$

$$\vec F_\perp = F_\perp \hat e_r$$

E-L $\frac{d}{dt}\frac{\partial L}{\partial \dot\theta} - \frac{\partial L}{\partial \theta} = Q_\theta$ generalizirana sila

$$\frac{d}{dt}\frac{\partial L}{\partial \dot r} - \frac{\partial L}{\partial r} = Q_r$$

$$Q_\theta = \sum_i \vec F_i \cdot \frac{\partial \vec r}{\partial \theta} = F_\perp \hat e_r \cdot \hat e_\theta = 0$$

$$Q_r = \sum_i \vec F_i \cdot \frac{\partial \vec r}{\partial r} = F_\perp \hat e_r \cdot \hat e_r = F_\perp$$

θ: $\frac{d}{dt} m r^2 \dot\theta - mgr\sin\theta = 0$

$$r^2\ddot\theta + \dot\theta\, 2 r \dot r - g r \sin\theta = 0$$

r: $\frac{d}{dt} m\dot r - mr\dot\theta^2 + mg\cos\theta = F_\perp$

nas zanima kdaj se odlepi:

$$F_\perp = 0$$

$$\dot r = \ddot r = 0 \qquad r = R$$

θ: $R^2\ddot\theta - gR\sin\theta = 0$

r: $-mR\dot\theta^2 + mg\cos\theta = F_\perp = 0$

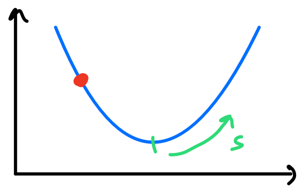
Ohranitev energije $E(t) = \text{konst} = mgR = mgR\cos\theta + \frac{1}{2} m R^2 \dot\theta^2$

$$\Rightarrow \dot{\theta}^2 = \frac{2}{mR}\, mgR\,(1-\cos\theta) = \frac{2g}{R}(1-\cos\theta)$$

$$-\cancel{m}\cancel{R}\,\frac{2\cancel{g}}{\cancel{R}}(1-\cos\theta) + \cancel{m}\cancel{g}\cos\theta = 0$$

$$\cos\theta = \frac{2}{3}$$

22 Iščemo vodilo $y(x)$, tako, da bo nihanje harmonsko (neodvisno od odmika)



$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2}m\dot{s}^2$$

$$V = \frac{ks^2}{2} = mgy \qquad y = \frac{\alpha s^2}{2} \qquad \begin{matrix} y(s) & \checkmark \\ x(s) & ? \end{matrix}$$

$$ds^2 = dx^2 + dy^2 \qquad / : ds^2$$

$$1 = \left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2$$

$$1 = \left(\frac{dx}{ds}\right)^2 + (\alpha s)^2$$

$$x = \int\sqrt{1-\alpha^2 s^2}\,ds \qquad \begin{matrix} \alpha s = \cos t \\ \alpha\, ds = -\frac{\sin t}{\alpha}\,dt \end{matrix}$$

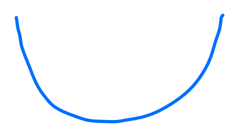
$$x = \int \sin t\left(-\frac{\sin t}{\alpha}\right)dt$$

$$x = -\frac{1}{\alpha}\int_0^t \frac{1-\cos 2t}{2}\,dt = -\frac{1}{2\alpha}\left(t - \frac{1}{2}\sin 2t\right)$$

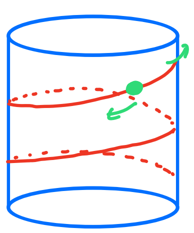
$$y = \frac{\alpha s^2}{2} = \frac{1}{2\alpha}\cos^2 t = y(t) = \frac{1}{4\alpha}(1+\cos 2t)$$

$$\tilde{t} = 2t \qquad x(\tilde{t}) = -\frac{1}{4\alpha}(\tilde{t} - \sin\tilde{t}) \qquad y(\tilde{t}) = \frac{1}{4\alpha}(1+\cos\tilde{t})$$

cikloida



23 Valj s spiralnim vodilom



Vezi:

$r = R$

$z = z_0 + p(\varphi - \phi)$

Koordinate

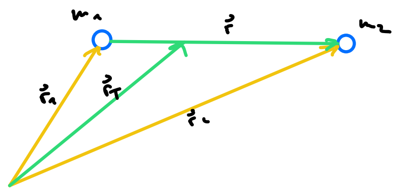
- kroglica z, φ, r
- valj ϕ ... zasuk valja

T ... kroglica in valj $\Rightarrow L$

V ... kroglica

$\varphi(t) = ?$ $\qquad \phi(t) = ?$

24 Gibanje dveh teles pri centralni sili



$$\vec r_1, \vec r_2 \to \vec r_T, \vec r$$

$$\vec r_T = \frac{1}{m_1+m_2}(m_1\vec r_1 + m_2\vec r_2)$$

$$\vec r = \vec r_2 - \vec r_1$$

centralni potencial

$$L = \frac{1}{2} m_1 \dot{\vec r}_1^{\,2} + \frac{1}{2} m_2 \dot{\vec r}_2^{\,2} - U(\vec r, \dot{\vec r})$$

$$L = \frac{1}{2}\mu \dot{\vec r}^{\,2} + \frac{1}{2}(m_1+m_2)\dot{\vec r}_T^{\,2} - U(\vec r, \dot{\vec r})$$

reducirana masa $\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}$

$\vec r_T = (x_T, y_T, z_T)$ ... ciklične koordinate $\Rightarrow$ posplošen impulz

$$\vec p_T = \frac{\partial L}{\partial \dot{\vec r}_T} = \text{konst}$$

$$\tilde L = \frac{1}{2}\mu\dot{\vec r}^{\,2} - U(\vec r, \dot{\vec r})$$

Zakaj je gibanje ravninsko?

$$\vec F = F\frac{\vec r}{r} = -\nabla U(\vec r) = -\frac{\partial U}{\partial r}\frac{\vec r}{r}$$

$$\vec M = \vec r \times \vec F = 0 \qquad \vec r \parallel \vec F \Rightarrow \vec\Gamma = \vec\Gamma_0 = \text{konst} \Rightarrow \text{ravninsko gibanje}$$

$$\vec\Gamma = \vec r \times \vec p$$

$$L = \frac{1}{2}\mu\dot{\vec r}^{\,2} - U(\vec r, \dot{\vec r})$$
$$= \frac{1}{2}\mu(\dot r^2 + r^2\dot\theta^2 + r^2\sin^2\theta\,\dot\varphi^2) - U(r)$$

E-L

φ: $p_\varphi = \frac{\partial L}{\partial\dot\varphi} = \mu r^2\sin^2\theta\,\dot\varphi \to \dot\varphi = \frac{p_\varphi}{\mu r^2\sin^2\theta}$

θ: $\frac{d}{dt}\frac{\partial L}{\partial\dot\theta} = \frac{\partial L}{\partial\theta} \Rightarrow \frac{d}{dt}(\mu r^2\dot\theta) = \mu r^2\sin\theta\cos\theta\,\dot\varphi^2$

$$\frac{d}{dt} = \frac{d\varphi}{dt}\frac{d}{d\varphi} = \dot\varphi\frac{d}{d\varphi}$$

$$\dot\varphi\frac{d}{d\varphi}\left(\mu r^2\dot\varphi\frac{d\theta}{d\varphi}\right) = \mu r^2\sin\theta\cos\theta\,\dot\varphi^2$$

$$\frac{d}{d\varphi}\left(\mu r^2\frac{p_\varphi}{\mu r^2\sin^2\theta}\frac{d\theta}{d\varphi}\right) = \mu r^2\sin\theta\cos\theta\frac{p_\varphi}{\mu r^2\sin^2\theta}$$

$$\frac{d}{d\varphi}\left(\frac{1}{\sin^2\theta}\frac{d\theta}{d\varphi}\right) = \cot\theta$$

$$-\frac{d^2}{d\varphi^2}(\cot\theta) = \cos\theta$$

$$\cot\theta = A\cos(\varphi - \varphi_0)$$

$$\vec r = r\sin\theta\cos\varphi\,\hat\imath + r\sin\theta\sin\varphi\,\hat\jmath + r\cos\theta\,\hat k$$
$$\hat n = \sin\theta_0\cos\varphi_0\,\hat\imath + \sin\theta_0\sin\varphi_0\,\hat\jmath + \cos\theta_0\,\hat k$$

$$\vec r\cdot\hat n = 0$$

$$r\sin\theta\cos\varphi\sin\theta_0\cos\varphi_0 + r\sin\theta\sin\varphi\sin\theta_0\sin\varphi_0 + r\cos\theta\cos\theta_0 = 0 \quad \Big| \frac{1}{r\sin\theta\sin\theta_0}$$

$$0 = \cos\varphi \cos\varphi_0 + \sin\varphi \sin\varphi_0 + \cot\theta \cot\theta_0$$

$$0 = \cos(\varphi - \varphi_0) + \cot\theta \cot\theta_0$$

$$\cos(\varphi - \varphi_0) = -\cot\theta \cot\theta_0 \qquad A = -\frac{1}{\cot\theta_0}$$

Ohranitev energije

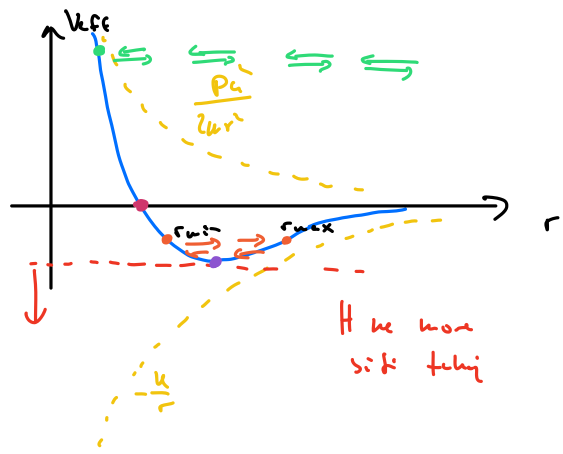
$$H = T + V = \underbrace{\frac{1}{2}\mu\dot{r}^2 + \frac{p_\varphi^2}{2\mu r^2}}_{T} + \underbrace{U(r)}_{V}$$

$$H = \frac{1}{2}\mu\dot{r}^2 + V_{eff}(r) \qquad V_{eff}(r) = U(r) + \frac{p_\varphi^2}{2\mu r^2}$$

Keplerjev problem – klasifikacija orbit

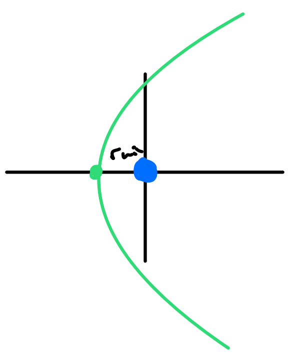
$$U(r) = -\frac{k}{r} \qquad k > 0$$

$$H = \frac{1}{2}\mu\dot{r}^2 + V_{eff}(r) \Rightarrow \frac{1}{2}\mu\dot{r}^2 = H - V_{eff}(r) \geq 0$$



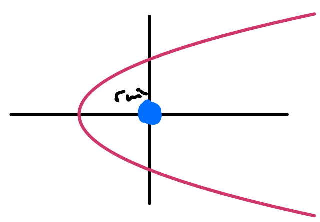
$$H \geq V_{eff}$$

$$V_{eff} = -\frac{k}{r} + \frac{p_\varphi^2}{2\mu r^2}$$



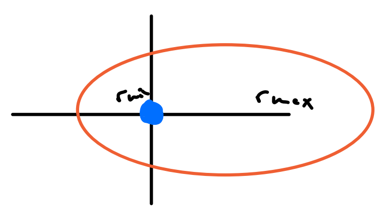
$H > 0 \qquad \varepsilon > 1$

hiperbola (nevezana orbita)



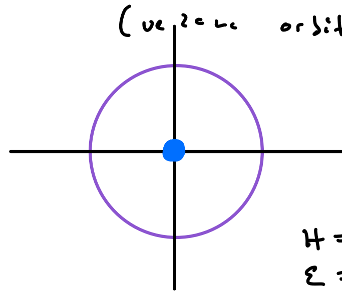
$H = 0 \qquad \varepsilon = 1$

parabola (nevezane orbite)



$H < 0 \qquad \varepsilon < 1$

elipsa (vezane orbite)



$H = V_{eff\,min}$

$\varepsilon = 0$

Orbita za Keplerjev problem

$$r(\varphi) = ?$$

$$H = \frac{1}{2}\mu \dot{r}^2 + \frac{p_\varphi^2}{2\mu r^2} - \frac{k}{r} \qquad u = \frac{1}{r} \qquad \frac{d}{dt} = \frac{d\varphi}{dt}\frac{d}{d\varphi} = \dot{\varphi}\frac{d}{d\varphi} = \frac{p_\varphi}{\mu r^2}\frac{d}{d\varphi}$$

$$\dot{r} = \frac{dr}{dt} = \frac{p_\varphi}{\mu r^2}\frac{dr}{d\varphi} = \frac{p_\varphi}{\mu} u^2 \frac{d}{d\varphi}\frac{1}{u} = \frac{p_\varphi}{\mu} u^2 \left(-\frac{1}{u^2}\right)\frac{du}{d\varphi} = -\frac{p_\varphi}{\mu} u'$$

$$H = \frac{1}{2}\mu\left(\frac{p_\varphi}{\mu}u'\right)^2 + \frac{p_\varphi^2}{2\mu}u^2 - ku = \frac{p_\varphi^2}{2\mu}u'^2 + \frac{p_\varphi}{2\mu}u^2 - ku \qquad /\cdot \frac{2\mu}{p_\varphi^2}$$

$$u'^2 + u^2 - \frac{2\mu k}{p_\varphi^2}u = \frac{2\mu H}{p_\varphi^2} \qquad / \frac{d}{d\varphi}$$

$$2u'u'' + 2uu' - \frac{2\mu k}{p_\varphi^2}u' = 0$$

$$2u'\left(u'' + u - \frac{\mu k}{p_\varphi^2}\right) = 0$$

1. $u' = 0$

$\Rightarrow u = \text{konst}$

$\Rightarrow r = \frac{1}{u} = \text{konst}$

$\Rightarrow$ krožnica

2. $u'' + u = \frac{\mu k}{p_\varphi^2}$

$$u = B\cos(\varphi - \varphi_0) + \frac{\mu k}{p_\varphi^2}$$

$$\varphi = \varphi_0 \qquad u(\varphi_0) = \frac{\mu k}{p_\varphi^2}$$

$$u'(\varphi_0) = -B\sin(\varphi - \varphi_0)\big|_{\varphi=\varphi_0} = 0$$

$\dot{r} \propto u' = 0$

$$H = \frac{1}{2}\mu \underset{0}{\dot{r}^2} + \frac{p_\varphi^2}{2\mu}u^2 - ku =$$

$$H = \frac{p_\varphi^2}{2\mu}\left(B\cos(\varphi-\varphi_0) + \frac{\mu k}{p_\varphi^2}\right)^2 - k\left(B\cos(\varphi-\varphi_0) + \frac{\mu k}{p_\varphi^2}\right)$$

$$\frac{2\mu H}{p_\varphi^2} = B^2 + \frac{2\mu k}{p_\varphi^2}B + \left(\frac{\mu k}{p_\varphi^2}\right)^2 - \frac{2\mu k}{p_\varphi^2}B - 2\left(\frac{\mu k}{p_\varphi^2}\right)^2$$

$$B^2 = \frac{2\mu H}{p_\varphi^2} + \left(\frac{\mu k}{p_\varphi^2}\right)^2$$

$$r(\varphi) = \frac{1}{u} = \frac{1}{\frac{\mu k}{p_\varphi^2} \mp \sqrt{\frac{2\mu H}{p_\varphi^2} + \left(\frac{\mu k}{p_\varphi^2}\right)^2}\cos(\varphi-\varphi_0)} =$$

ni važno ali + ali − ker je to le faza

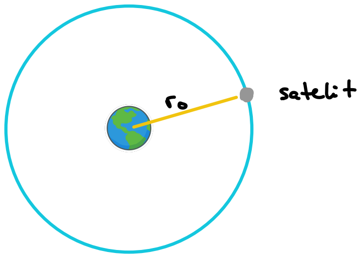
$$p = \frac{p_\varphi^2}{\mu k}$$

$$= \frac{p}{1 - \varepsilon\cos(\varphi - \varphi_0)}$$

$$\varepsilon = \sqrt{1 + \frac{2p_\varphi^2 H}{k^2\mu}}$$

ekscentričnost

25 Satelit



V satelit se zaleti asteroid tako, da je $\Delta T = -0{,}02\,T = -\eta T$

$r_{min} = ?$ (po trku)

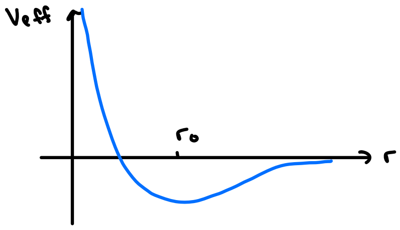
1 Pred trkom $(m = \mu)$

$$V_{eff} = \frac{p_\varphi^2}{2mr^2} - \frac{\alpha}{r}$$

$$\frac{\partial V_{eff}}{\partial r} = 0$$

$$-\frac{p_\varphi^2}{mr^3} + \frac{\alpha}{r^2} = 0 \Big|_{r=r_0}$$

$$\Rightarrow r_0 = \frac{p_\varphi^2}{\alpha m}$$



$$V_{eff}(r) = \frac{\alpha}{2}\frac{r_0}{r^2} - \frac{\alpha}{r}$$

$$E = \frac{p_\varphi^2}{2mr^2} + \frac{1}{2}m\dot{r}^2 - \frac{\alpha}{r} = \underbrace{\frac{\alpha}{2r_0}}_{T} - \underbrace{\frac{\alpha}{r_0}}_{V} = \frac{\alpha}{2r_0}$$

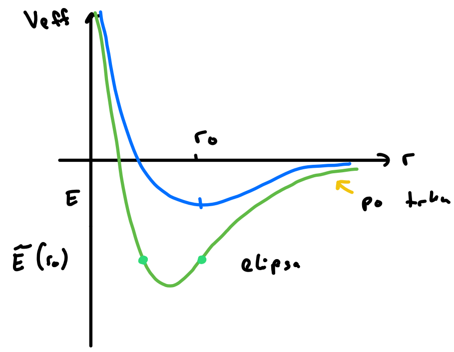
$\langle -2T = V \rangle$

2 Po trku

$$p_\varphi^2 \longrightarrow p_\varphi^2(1-\eta)$$

$$E \rightarrow \overset{\text{po trku}}{\tilde{E}} = \frac{\alpha}{2r_0}(1-\eta) - \frac{\alpha}{r_0}$$

$$V_{ef} \rightarrow \tilde{V}_{ef}(r) = \frac{\alpha}{2}\frac{r_0}{r^2}(1-\eta) - \frac{\alpha}{r}$$



V ekstremnih točkah

$$\tilde{E} = V_{ef}(r)$$

$$\frac{\alpha}{2r_0}(1-\eta) - \frac{\alpha}{r_0} = \frac{\alpha}{2}\frac{r_0(1-\eta)}{r^2} - \frac{\alpha}{r} \qquad \cdot \frac{r^2}{r_0\alpha}$$

$$\frac{r^2}{r_0^2}\left(\frac{-1-\eta}{2}\right) = \frac{1-\eta}{2} - \frac{r}{r_0}$$

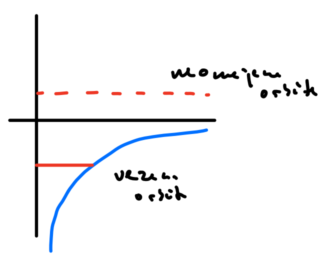
$$\frac{r}{r_0} = \frac{1 \pm \eta}{1+\eta} \Rightarrow r_1 = r_0 \qquad r_2 = r_{min} = r_0\frac{1-\eta}{1+\eta}$$

26

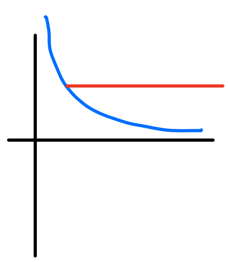
$$V = -\frac{\alpha}{r^2}$$

$$E = \frac{1}{2}m\dot{r}^2 + \underbrace{\left(\frac{p_\varphi^2}{2m} - \alpha\right)\frac{1}{r^2}}_{V_{ef}}$$

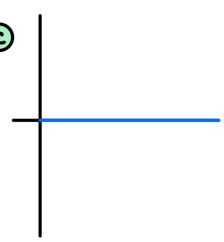
a



b



c



$$\frac{dr}{dt} = \frac{dr}{d\varphi}\dot\varphi = r'\frac{p_\varphi}{mr^2} \qquad p_\varphi = mr^2\dot\varphi$$

$$E = \frac12\frac{p_\varphi^2}{m}\frac{r'^2}{r^4} + \left(\frac{p_\varphi^2}{2m} - \alpha\right)\frac{1}{r^2} \qquad u = \frac1r \qquad r' = -\frac{1}{u^2}u' \qquad r'^2 = \frac{1}{u^4}u'^2$$

$$E = \frac{p_\varphi^2}{2m}u'^2 + \left(\frac{p_\varphi^2}{2m} - \alpha\right)u^2 \quad \Big| \frac{2m}{p_\varphi^2}$$

$$\overbrace{\frac{2mE}{p_\varphi^2}}^{e} = u'^2 + \Big(1 - \overbrace{\frac{\alpha 2m}{p_\varphi^2}}^{\tilde\alpha}\Big)u^2$$

$$e = u'^2 + (1-\tilde\alpha)u^2 \quad \Big| \frac{d}{d\varphi}$$

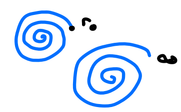
$$0 = 2u'u'' + (1-\tilde\alpha)2uu' \quad | u' \neq 0$$

$$u'' + (1-\tilde\alpha)u = 0$$

a) $\beta^2 = (1-\tilde\alpha) < 0$ $\qquad u'' - \beta^2 u = 0 \qquad u = Ae^{-\beta\varphi} + Be^{\beta\varphi}$

(i) $u = \frac{1}{r_0}\operatorname{ch}\beta\varphi \qquad r = \frac{r_0}{\operatorname{ch}\beta\varphi}$

(ii) $u = \frac{1}{r_0}\operatorname{sh}\beta\varphi \qquad r = \frac{r_0}{\operatorname{sh}\beta\varphi}$



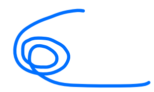
b) $1 - \tilde\alpha = 0$

$e = u'^2$ (i) $u' = \sqrt{e}$

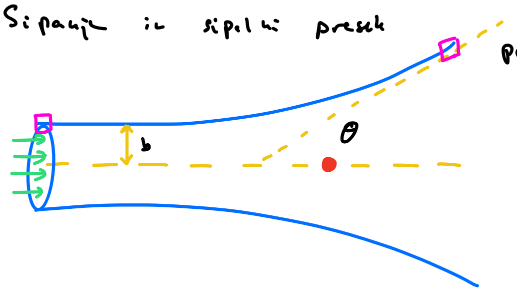
$u = \sqrt{e}(\varphi - \varphi_0) \qquad r = \frac{1}{\sqrt{e}(\varphi-\varphi_0)} \qquad e \neq 0$

(ii) $u' = $ konst. $\Rightarrow$ kroženje

c) $\beta^2 = 1 - \tilde\alpha > 0 \qquad u'' + \beta^2 u = 0 \qquad u = \frac{1}{r_0}\cos(\varphi-\varphi_0)\beta \qquad r = \frac{r_0}{\cos((\varphi-\varphi_0)\beta)}$



27 Sipanje in sipalni presek



pri neki $d\varphi$ in θ nam sipalni presek pove koliko delcev gre skozi to odprtino v nekem času.

$$\frac{dN}{dt}(\nu d\Omega) = b\,db\,d\varphi\, j$$

j — gostota toka vpadnih delcev

$$\frac{\frac{dN}{dt}(\nu d\Omega)}{j\,d\Omega} = \sigma(\Omega) = \frac{b\,db\,d\varphi}{d\Omega} = \dots$$

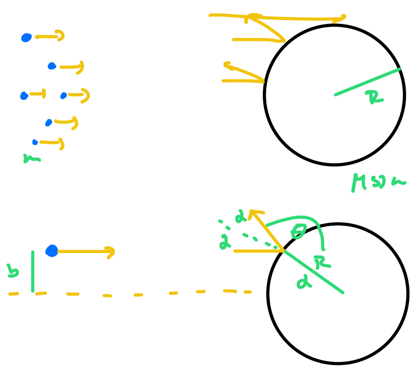
diferencialni sipalni presek $\qquad \sigma(\Omega) = \frac{d\sigma_{TOT}}{d\Omega}$ — celotni sipalni presek

s tem delimo da dobimo rezultat neodvisen od j in $d\Omega$

$$\dots = \frac{b\,db\,d\varphi}{d(\cos\theta)\,d\varphi} =$$

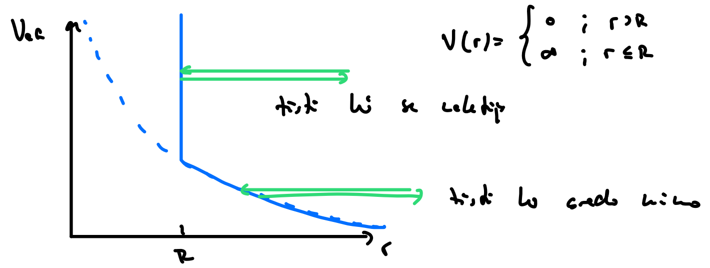
$$\sigma(\Omega) = \left|\frac{b\,db}{\sin\theta\,d\theta}\right|$$

28 Sipanje na togi mirujoči krogli



$$H = \frac{1}{2} m \dot{r}^2 + V_{ef}(r) \qquad V_{ef} = \frac{p_\varphi^2}{2mr^2} + V(r)$$

$$V(r) = \begin{cases} 0 & ; r > R \\ \infty & ; r \leq R \end{cases}$$



Odbojni zakon

$$2\alpha + \theta = \pi$$

$$\frac{b}{R} = \sin\alpha$$

$$\Rightarrow \quad b(\theta) = R \sin\alpha = R \sin\left(\frac{\pi}{2} - \frac{\theta}{2}\right) = R \cos\frac{\theta}{2}$$

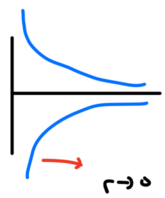
$$\sigma(\theta) = \frac{b}{\sin\theta} \left| \frac{db}{d\theta} \right| = \frac{R \cos\frac{\theta}{2}}{\sin\theta} \left| \frac{R}{2} \sin\frac{\theta}{2} \right| = \frac{R^2}{4} = \text{konst.}$$

19. 4. 2. ura ? manjka σ_{TOT} izotropno sipanje

29 Presek za zrušitev delcev

$$V(r) = -\frac{k}{r^n} \qquad n \geq 2, \; k > 0$$

1. $n = 2$ $\qquad H = \frac{1}{2} m \dot{r}^2 + V_{eff}$



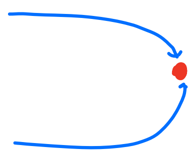
$$V_{eff} = \frac{1}{r^2}\left(\frac{p_\varphi^2}{2m} - k\right) \leq 0$$

$r \to 0$ to pomeni zrušitev delcev

$$k \geq \frac{p_\varphi^2}{2m}$$

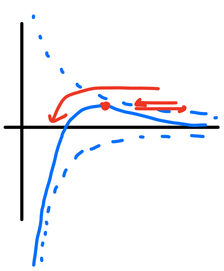
mejni primer $\quad k = \frac{p_\varphi^2}{2m} = \frac{(m b v_\infty)^2}{2m} \Rightarrow b^2 = \frac{2k}{m v_\infty^2}$

$$p_\varphi = m b v_\infty$$



$$\sigma_{TOT} = \pi b^2$$

2. $n \geq 3$ $\qquad$ vzemimo $n = 3$



$$V_{ef} = \frac{p_\varphi^2}{2mr^2} - \frac{k}{r^3}$$

$$0 = \left.\frac{dV_{ef}}{dr}\right|_{r=r_0} = -\frac{p_\varphi^2}{m r_0^3} + \frac{3k}{r_0^4} \Rightarrow r_0 = \frac{3km}{p_\varphi^2}$$

$$V_{eff}(r_0) = \frac{1}{r_0^2}\left(\frac{p_\varphi^2}{2m} - \frac{k p_\varphi^2}{3km}\right) = \frac{p_\varphi^2}{m r_0^2} \frac{1}{6}$$

$$H = \frac{1}{2} m v_\infty^2 \geq V_{eff}(r_0)$$

mejni primer $\quad \frac{1}{2} m v_\infty^2 = \frac{p_\varphi^2}{6 m r_0^2} = \frac{(m^2 b_{min}^2 v_\infty^2)^3}{6m \, (3km)^2} =$

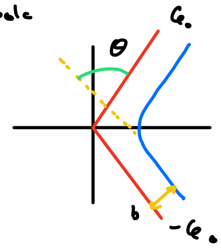
$$= \frac{m^3 b_{min}^6 v_\infty^6}{2 \cdot 27 \, k^2} \Rightarrow b_{min}^2 = 3\left(\frac{k}{m v_\infty^2}\right)^{2/3} \Rightarrow \sigma_{TOT} = \pi b_{min}^2 = 3\pi \left(\frac{k}{m v_\infty^2}\right)^{2/3}$$

30 Rutherfordovo sipanje

$$V(r) = \frac{\alpha}{r} \qquad \alpha = \frac{e_1 e_2}{4\pi\varepsilon_0}$$

$$\alpha < 0 \qquad r(\varphi) = \frac{p}{1+\varepsilon\cos\varphi} \qquad p = \frac{p_\varphi^2}{m\alpha} \qquad \varepsilon = \sqrt{1+\frac{2p_\varphi^2 E}{m\alpha^2}}$$

$$\alpha > 0 \qquad r(\varphi) = \frac{p}{-1+\varepsilon\cos\varphi}$$ hiperbole



$$\theta = \pi - 2\varphi_0$$

$$\pm\varphi_0 \to r\to\infty$$

$$\frac{d\sigma}{d\Omega} = \left|\frac{b\,db}{\sin\theta\, d\theta}\right| =$$

$$\sin\frac{\theta}{2} = \cos\varphi_0 = \frac{1}{\varepsilon}$$

$$\varepsilon\cos\varphi_0 = 1$$

$$\varphi_0 = \arccos\frac{1}{\varepsilon}$$

$$\cot^2\frac{\theta}{2} = \frac{\cos^2\theta/2}{\sin^2\theta/2} = \frac{1}{\sin^2\theta/2} - 1 = \varepsilon^2 - 1 =$$

$$p_\varphi = m v_\infty b \qquad = \frac{2p_\varphi^2 E}{m\alpha^2} = \frac{2m v_\infty^2 E b^2}{\alpha^2} = \frac{4E^2 b^2}{\alpha^2}$$

$$\cot\frac{\theta}{2} = \frac{2Eb}{\alpha} \qquad b = \frac{\alpha}{2E}\cot\frac{\theta}{2}$$

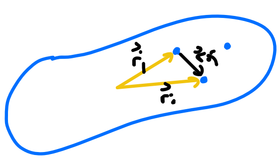
$$\frac{db}{d\theta} = \frac{\alpha}{2E}\left(-\frac{1}{\sin^2\theta/2}\right)\cdot\frac{1}{2}$$

$$\sigma(\Omega) = \left(\frac{\alpha}{2E}\right)^2 \frac{\cos\theta/2}{\sin\theta/2}\,\frac{1}{2}\,\frac{1}{\sin^2\theta/2}\,\frac{1}{2\sin\theta/2\cos\theta/2} = \left(\frac{\alpha}{4E}\right)^2\frac{1}{\sin^4\theta/2}$$

$$\sigma_{TOT} = \int\sigma(\Omega)\,d\varphi\,\sin\theta\,d\theta = 2\pi\left(\frac{\alpha}{4E}\right)^2\int_0^\pi\frac{1}{\sin^4\theta/2}\sin\theta\,d\theta = 2C\int_0^\pi\frac{\cos\theta/2}{\sin^3\theta/2}\,d\theta/2 =$$

$$= 4C\left(-\frac{1}{\sin^2\theta/2}\right)\Big|_0^\pi = \infty$$

31 Dinamika togega telesa



$|\vec{r}_{ij}| = c_{ij}$ ... vezi

$\exists$ prostorske stopnje - rotacije

φ, θ, ψ

$$\dot{\vec{L}} = \vec{M}$$

$$\vec{L} = J\vec{\omega}$$

tenzor vztrajnostnega momenta

$$(J)_{ij} = \int dm\,(r^2\delta_{ij} - r_i r_j)$$

$$\vec{L} = \sum_i m_i \vec{r}_i \times \dot{\vec{r}}_i \qquad \dot{\vec{r}}_i = \underbrace{\vec{v}_{rel}}_{0\ \text{toga telo}} + \vec{\omega}\times\vec{r}_i$$

$$= \sum_i m_i \vec{r}_i \times \vec{\omega}\times\vec{r}_i =$$

$$= \sum_i m_i\left(r_i^2\vec{\omega} - (\vec{r}_i\cdot\vec{\omega})\vec{r}_i\right) = J\vec{\omega}$$

J simetričen $\Rightarrow$ $\exists$ lastne vrednosti in $\exists$ lastne osi $(\hat{i}', \hat{j}', \hat{k}')$

$$\vec{\omega} = \omega_x'\hat{i}' + \omega_y'\hat{j}' + \omega_z\hat{k}' = \sum_i \omega_i'\hat{e}_i'$$

$$\vec{L} = \sum_i J_i \omega_i' \hat{e}_i'$$

$$\dot{\vec{L}} = M$$

$$\dot{\vec{L}} = \sum_i J_i \dot{\omega}_i' \hat{e}_i + \sum_i J_i \omega_i' \dot{\hat{e}}_i'$$

$$\dot{\hat{e}}_i' = \vec{\omega} \times \hat{e}_i'$$

Levi-Civita simbol

$$(\vec{a} \times \vec{b})_k = \varepsilon_{kij} a_i b_j$$

$$\dot{\vec{L}} = \sum_i J_i \dot{\omega}_i' \hat{e}_i' + \sum_i J_i \omega_i \hat{e}_k' \varepsilon_{kji} \omega_j =$$

$$\varepsilon_{kij} = \begin{cases} 1 & (1,2,3), (2,3,1), (3,1,2) \\ -1 & (3,2,1), (2,1,3), (1,3,2) \\ 0 & \text{sicer} \end{cases}$$

$$\dot{\vec{L}} = \sum_i \left(J_i \dot{\omega}_i \hat{e}_i' + \sum_{j,k} J_i \omega_i' \omega_j' \varepsilon_{kji} \hat{e}_k' \right)_{i \leftrightarrow k}$$

$$\boxed{\dot{\vec{L}} = \sum_i \left(J_i \dot{\omega}_i' + \sum_{jk} J_k \omega_k' \omega_j' \varepsilon_{ijk} \right) \hat{e}_i'}$$

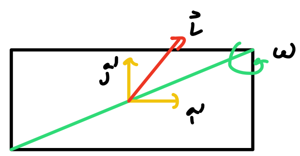
$$x: \quad M_x = \dot{L}_x = J_x \dot{\omega}_x' + \omega_y' \omega_z' (J_z - J_y)$$

$$y: \quad M_y = \dot{L}_y = J_y \dot{\omega}_y' + \omega_z' \omega_x' (J_x - J_z)$$

$$z: \quad M_z = \dot{L}_z = J_z \dot{\omega}_z' + \omega_x' \omega_y' (J_y - J_x)$$

Eulerjeve enačbe

32 Navor na togo ploščo



$$|\vec{M}| = ? \qquad |\vec{F}| = ?$$

$$\vec{L} = J_x \omega_x' \hat{i}' + J_y \omega_y' \hat{j}'$$

$$J_{ij} = \int dm \left(r^2 \delta_{ij} - r_i r_j \right) = \int dr\, \rho(\vec{r}') \left(r^2 \delta_{ij} - r_i r_j \right) = \int dS\, \rho_s \left(r^2 \delta_{ij} - r_i r_j \right)$$

$$= \frac{m}{ab} \int_{-a/2}^{a/2} dx \int_{-b/2}^{b/2} dy \left((x^2 + y^2) \delta_{ij} - r_i r_j \right)$$

$$J_{12} = \frac{m}{ab} \iint (0 - xy)\, dx\, dy = 0$$

$$J_{11} = \frac{m}{ab} \iint (x^2 + y^2) - x^2 = \frac{m}{ab} a \frac{1}{3} \left(\left(\frac{b}{2}\right)^3 + \left(\frac{b}{2}\right)^3 \right) = \frac{m b^2}{12}$$

$$J_{22} = \frac{m a^2}{12}$$

$$J_{33} = \frac{m}{12} (a^2 + b^2)$$

$$J = \frac{m}{12} \begin{pmatrix} b^2 & & \\ & a^2 & \\ & & a^2 + b^2 \end{pmatrix}$$

$$M_x = J_x \dot{\omega}_x' + (J_z - J_y) \omega_y' \omega_z' = 0$$

$$M_y = J_y \dot{\omega}_y' + (J_x - J_z) \omega_z' \omega_x' = 0$$

$$M_z = J_z \dot{\omega}_z' + (J_y - J_x) \omega_x' \omega_y'$$

$$\dot{\omega} = 0$$

$$\omega = \left(\frac{\omega a}{\sqrt{a^2 + b^2}}, \frac{\omega b}{\sqrt{a^2 + b^2}}, 0 \right)$$

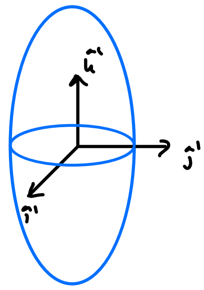
$$M_z = (J_y - J_x) \omega_x' \omega_y'$$

$$M_z = \frac{m}{12} (a^2 - b^2) \frac{\omega^2 ab}{a^2 + b^2}$$

$$\vec{M} = \vec{r} \times \vec{F} = r F = \sqrt{a^2 + b^2}\, F$$

$$\Rightarrow \quad F = \frac{m}{12} (a^2 - b^2) \frac{\omega^2 ab}{(a^2 + b^2)^{3/2}}$$

33 Prosta simetrična vrtavka



$$J = \begin{pmatrix} J & & \\ & J & \\ & & J' \end{pmatrix} \qquad \vec{M} = 0 \qquad J_x = J_y$$

Eulerjeve enačbe

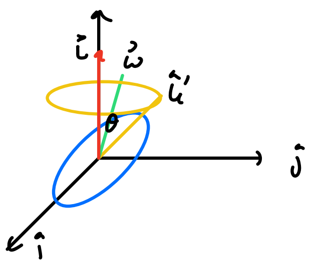
$$M_x = J_x \dot{\omega}_x' + \omega_y' \omega_z' (J_z - J_y) = 0$$
$$M_y = J_y \dot{\omega}_y' + \omega_z' \omega_x' (J_x - J_z) = 0$$
$$M_z = J_z \dot{\omega}_z + \omega_x' \omega_y' (J_y - J_x) = 0 \Rightarrow J_z \dot{\omega}_z = 0 \qquad \omega_z' = \text{konst}$$

$$\dot{\omega}_x' = -\frac{J'-J}{J} \omega_z' \omega_y'$$
$$\dot{\omega}_y' = \frac{J'-J}{J} \omega_z' \omega_x' \qquad \Omega_{pr} = \frac{J'-J}{J} \omega_z'$$

$$\dot{\omega}_x' = -\Omega_{pr} \omega_y'$$
$$\dot{\omega}_y' = +\Omega_{pr} \omega_x' \qquad /\cdot i \qquad \xi = \omega_x' + i\omega_y'$$
$$\dot{\xi} = i \Omega_{pr} \xi \Rightarrow \xi = \xi_0 e^{i\Omega_{pr} t}$$
$$\xi_0 = \omega_{x0} + i\omega_{y0}$$

Kako izgleda gibanje v zunanjem sistemu?

$$\vec{M} = 0 \Rightarrow \vec{L} = \text{konst} \qquad \vec{L} = L\hat{k}$$



① $\theta = ?$

$$\vec{L} \cdot \hat{k}' = L\cos\theta = L_z' = J_z \overset{\text{konst}}{\omega_z'} = \text{konst}$$
$$\Rightarrow \theta = \text{konst.}$$

② $\vec{L}, \vec{\omega}, \hat{k}'$ ležijo v isti ravnini?

$$(\vec{L} \times \vec{\omega}) \cdot \hat{k} \overset{?}{=} 0$$

$$(\vec{L} \times \vec{\omega})_{z'} = 0 = \sum_{ij} \varepsilon_{zij} L_i \omega_j = L_x \omega_y - L_y \omega_x =$$
$$= L_x \omega_x' \omega_y' - J_y \omega_y' \omega_x' = 0 \qquad \text{ker } J_x = J_y$$

③ $$\dot{\hat{k}}' = \vec{\omega} \times \hat{k}' = (\omega_x' \hat{i}' + \omega_y' \hat{j}' + \omega_z' \hat{k}') \times \hat{k}' = -\omega_x' \hat{j}' + \omega_y' \hat{i}'$$

$$|\dot{\hat{k}}'| = \sqrt{\omega_x'^2 + \omega_y'^2} = |\xi| = \text{konst.}$$
$\theta = \text{konst}$ } gibanje enakomerno po stožcu

$$\vec{\Omega}_e = \Omega_e \hat{k} \qquad \dot{\hat{k}}' = \vec{\Omega}_e \times \hat{k} = \Omega_e \hat{k} \times \hat{k}'$$
$$|\dot{\hat{k}}| = \Omega_e |\hat{k} \times \hat{k}'| = \Omega_e \sin\theta = |\xi|$$

izberemo začetno tako $\omega_y' = 0$

$$|\xi| = \omega_x' = \frac{L_x}{J_x} = \frac{L\sin\theta}{J}$$

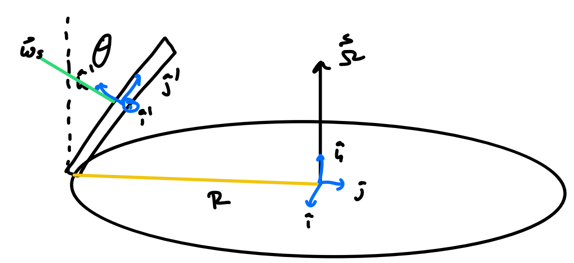
$$\Omega_e \sin\theta = \frac{L\sin\theta}{J} \qquad \Omega_e = \frac{L}{J}$$

$$\omega_z' = \frac{L_z}{J_z} = \frac{L\cos\theta}{J'} \Rightarrow L = \frac{\omega_z' J'}{\cos\theta}$$

$$\Omega_e = \frac{J'}{J\cos\theta} \omega_z'$$

① krožnik / frizbi $\quad J' = \frac{1}{2} m R^2, \quad J = \frac{1}{4} m R^2 \Rightarrow \Omega_e = 2\omega_z'$ ② Zemlja – Chandlerjeva precesija

34 Krožeč kovanec



$\theta = ?$

$$\vec{L} = \underline{\underline{J}}\,\vec{\omega}$$

$$\dot{\vec{L}} = \Sigma \vec{M}$$

$$\underline{\underline{J}} = \begin{pmatrix} J & & \\ & J & \\ & & J' \end{pmatrix} \qquad J = \frac{1}{4} m r^2 \qquad J' = \frac{1}{2} m r^2$$

$$\vec{\omega} = \vec{\Omega} + \vec{\omega}_s$$

$$\vec{\Omega} = \Omega \hat{k} = \Omega(\cos\theta\, \hat{j}' + \sin\theta\, \hat{k}')$$

$$\vec{\omega}_s = \omega_s \hat{k}'$$

$$\vec{\omega} = \omega_s \hat{k}' + \Omega(\cos\theta\, \hat{j}' + \sin\theta\, \hat{k}')$$

Kotaljenje (brez spodrsavanja)

↳ hitrost ob podlagi enaka 0 ($\vec{v}_P = 0$)

$$0 = \vec{v}_P = \vec{\Omega} \times \vec{R} + \vec{\omega}_s \times \vec{r} = \Omega R\, \hat{i}' + \omega_s r\, \hat{i}' = (\Omega R + \omega_s r)\hat{i}' \Rightarrow \omega_s = -\frac{R}{r}\Omega$$

$$\vec{\omega} = \Omega(\cos\theta\, \hat{j}' + \sin\theta\, \hat{k}') - \Omega \frac{R}{r} \hat{k}' = \overset{0}{\overset{\parallel}{\omega_x}}\, \hat{i}' + \omega_y \hat{j}' + \omega_z \hat{k}'$$

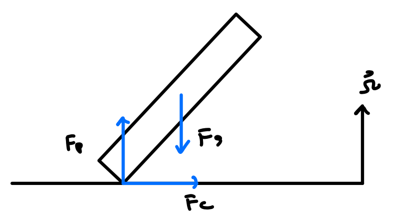
$$\dot{\vec{L}} = \frac{d}{dt}\left(J\omega_y \hat{j}' + J'\omega_z \hat{k}'\right) = J\omega_y \dot{\hat{j}}' + J'\omega_z \dot{\hat{k}}'$$

$\theta = \text{konst.}$

$$\dot{\hat{j}}' = \vec{\Omega} \times \hat{j}' = \Omega(\cos\theta\, \hat{j}' + \sin\theta\, \hat{k}') \times \hat{j}' = -\Omega \sin\theta\, \hat{i}'$$

$$\dot{\hat{k}}' = \vec{\Omega} \times \hat{k}' = \Omega(\cos\theta\, \hat{j}' + \sin\theta\, \hat{k}') \times \hat{k}' = \Omega \cos\theta\, \hat{i}'$$

$$\Rightarrow \dot{\vec{L}} = \frac{1}{4} m r^2 \Omega \cos\theta (-\Omega \sin\theta)\, \hat{i}' + \frac{1}{2} m r^2 \left(\Omega \sin\theta - \Omega \frac{R}{r}\right) \Omega \cos\theta\, \hat{i}'$$



$$\vec{F}_g = mg(-\hat{k})$$

$$\vec{F}_\perp = mg\, \hat{k}$$

$$\vec{F}_c = m\vec{a}_c = m\Omega^2 (R - r\sin\theta)\, \hat{j} = m\Omega^2 (R - r\sin\theta)(\sin\theta\, \hat{j}' - \cos\theta\, \hat{k}')$$

$$\vec{M} = \vec{r} \times (\vec{F}_\perp + \vec{F}_c) = r(-\hat{j}') \times \left(mg(\cos\theta\, \hat{j}' + \sin\theta\, \hat{k}') + m\Omega^2 (R - r\sin\theta)(\sin\theta\, \hat{j}' - \cos\theta\, \hat{k}')\right)$$

$$= \hat{i}'\, r\left(-mg\sin\theta + m\Omega^2 (R - \sin\theta\, r)\cos\theta\right)$$

$$\dot{\vec{L}} = \vec{M}$$

$\hat{i}'$:

$$-\frac{1}{4} m r^2 \Omega^2 \cos\theta \sin\theta + \frac{1}{2} m r^2 \Omega^2 \sin\theta\cos\theta - \frac{1}{2} m r^2 \Omega^2 \frac{R}{r}\cos\theta = -rmg\sin\theta + m\Omega^2 r(R - \sin\theta\, r)\cos\theta$$

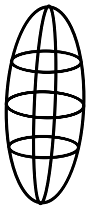
$$\frac{5}{4} m r^2 \Omega^2 \sin\theta\cos\theta - \frac{3}{2} m r R \Omega^2 \cos\theta = -mgr\sin\theta$$

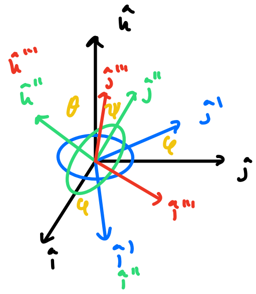
$$\frac{5}{4} r\Omega^2 \cos\theta\sin\theta - \frac{3}{2} R\Omega^2 \cos\theta = -g\sin\theta$$

$r \ll R$ $\qquad \frac{3}{2} R\Omega^2 \cos\theta = g\sin\theta$

$$\tan\theta = \frac{3}{2}\frac{R\Omega^2}{g}$$

35 Eulerjevi koti





$\hat{i}, \hat{j}, \hat{k}$ ... mirujoč sistem
$\hat{i}''', \hat{j}''', \hat{k}'''$ ... sistem pripet na vrtavko

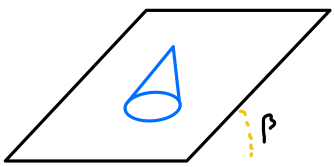
$$\vec{\omega} = \dot{\varphi}\,\hat{k} + \dot{\theta}\,\hat{i}'' + \dot{\psi}\,\hat{k}'''$$

$$\vec{\omega}' = \hat{i}'''(\dot{\varphi}\sin\theta\sin\psi + \dot{\theta}\cos\psi) + \hat{j}'''(\dot{\varphi}\sin\theta\cos\psi - \dot{\theta}\sin\psi) + \hat{k}'''(\dot{\psi} + \dot{\varphi}\cos\theta)$$

$$T = \frac{1}{2}\vec{\omega}\,\underline{\underline{J}}\,\vec{\omega}$$

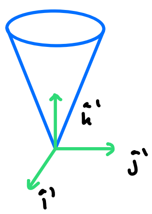
$$T = \frac{1}{2}J(\dot{\varphi}^2\sin^2\theta + \dot{\theta}^2) + \frac{1}{2}J'(\dot{\varphi}\cos\theta + \dot{\psi})^2$$

36 Stožec na nagnjeni podlagi

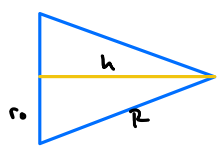


$$V = \frac{\pi r_0^2 h_0}{3} \qquad \tan\alpha = \frac{r_0}{h}$$

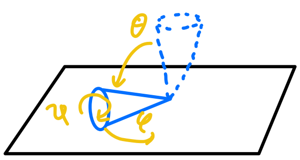
$$h^* = h_t = \frac{3}{4}h_0$$



$$J' = J_z = \frac{3}{10}m r_0^2$$
$$J = J_x = J_y = \frac{3}{20}m r_0^2 + \frac{3}{5}m h_0^2$$



$$R = \sqrt{r_0^2 + h^2}$$



Nastavek za θ $\qquad \theta = \frac{\pi}{2} - \alpha = \text{konst.}$

ne zdrsava $\qquad \dot{\varphi} R = -\dot{\psi} r_0$

$$T = \frac{1}{2}J(\dot{\varphi}^2\sin^2\theta + \dot{\theta}^2) + \frac{1}{2}J'(\dot{\varphi}\cos\theta + \dot{\psi})^2$$
$$= \frac{1}{2}J(\dot{\varphi}^2\cot^2\alpha) + \frac{1}{2}J'(\dot{\varphi}\sin\alpha - \frac{R}{r_0}\dot{\varphi})^2$$

$$T = \frac{1}{2}\tilde{J}\dot{\varphi}^2 \qquad \tilde{J} = \frac{h^2}{h^2 + r_0^2}\left(\frac{3}{20}m r_0^2 + \frac{9}{10}m h^2\right)$$

$$\vec{g} = (\cos\beta\,(-\hat{k}) + \sin\beta\,\hat{i})\,g$$

$$V = -m\vec{g}\cdot\vec{r}_T$$

$$\vec{r}_T = \frac{3}{4}h\left(\hat{k}\sin\alpha + \cos\alpha(\hat{i}\cos\varphi + \hat{j}\sin\varphi)\right)$$

$$V = \frac{3}{4}mg\,h\sin\alpha - \frac{3}{4}mgh\cos\alpha\cos\varphi\sin\beta$$

$$L = \frac{1}{2} \tilde{J} \dot{\varphi}^2 + \frac{3}{4} mgh \cos\alpha \sin\beta \overset{1-\frac{\varphi^2}{2}}{\cos\varphi} + C$$

$$L = \frac{1}{2} \tilde{J} \dot{\varphi}^2 - \frac{3}{4} mgh \cos\alpha \sin\beta \frac{\varphi^2}{2} + C'$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\varphi}} = \tilde{J}\ddot{\varphi} = -\frac{3}{4} mgh \cos\alpha \sin\beta \, \varphi$$

$$\omega^2 = \frac{3\, mgh \cos\alpha \sin\beta}{4 \tilde{J}}$$

Mala nihanja

Posplošene koordinate $q_1, \dots, q_n$ $\qquad \underline{q} = \begin{pmatrix} q_1 \\ \vdots \\ q_n \end{pmatrix}$ odmik

Potencial $V(q_1, \dots, q_n)$ $\qquad \underline{q} = \underline{q}_0 + \underline{\eta}$

ravnovesna lega

$$V(\underline{\eta}) = V(\underline{q}_0) + \frac{1}{2}\sum_{i,j} \frac{\partial^2 V}{\partial q_i \partial q_j}\Big|_{\underline{q}=\underline{q}_0} \eta_i \eta_j$$

Matrična oblika

$$V = V_0 + \frac{1}{2}\underline{\eta}^T \underline{\underline{V}}\, \underline{\eta} \qquad (\underline{\underline{V}})_{ij} = \frac{\partial^2 V}{\partial q_i \partial q_j}\Big|_{q=q_0}$$

Kinetična energija $\quad T = \frac{1}{2}\underline{\dot{\eta}}^T \underline{\underline{T}}\, \underline{\dot{\eta}}$

$\underline{\underline{V}}, \underline{\underline{T}}$ simetrični, realni, pozitivno definitni

$$L = \frac{1}{2}\underline{\dot{\eta}}^T \underline{\underline{T}}\, \underline{\dot{\eta}} - \frac{1}{2}\underline{\eta}^T \underline{\underline{V}}\, \underline{\eta}$$

E-L enačba $\quad \underline{\underline{T}}\, \underline{\ddot{\eta}} + \underline{\underline{V}}\, \underline{\eta} = 0$

$$\underline{\eta} = \underline{a}\, \alpha\, e^{-i\omega t} \qquad \text{Nastavek}$$

relativni odmiki $\quad$ amplituda

$$(-\omega^2 \underline{\underline{T}} + \underline{\underline{V}})\underline{a} = 0 \qquad \text{posplošen lastni problem}$$

$$\det(-\omega^2 \underline{\underline{T}} + \underline{\underline{V}}) = 0 \quad \Rightarrow \quad \omega_n^2 \dots \text{lastne frekvence}$$

↳ $\underline{a}_n \dots$ lastni vektorji / normalni načini nihanja

Splošna rešitev

$$\underline{\eta}(t) = \sum_n \underline{a}_n \begin{cases} \alpha_n e^{-i\omega_n t} & \omega_n^2 > 0 \\ b_n + c_n t & \omega_n^2 = 0 \quad \text{translacija} \end{cases}$$

Ortogonalnost zveze

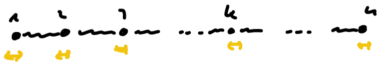
$$\underline{a}_n \qquad \underline{a}_k^T \underline{\underline{T}}\, \underline{a}_l = \underline{\underline{T}}' \delta_{kl} \overset{\text{če so normirani}}{=} \delta_{kl}$$

(matrična oblika) $\quad \underline{\underline{A}} = (\underline{a}_1, \underline{a}_2, \dots, \underline{a}_n)$

$$\underline{\underline{A}}^T \underline{\underline{T}}\, \underline{\underline{A}} = \underset{\text{diagonalna}}{\underline{\underline{T}}'} \overset{\text{normirano}}{=} I$$

$$\underline{A}^T \underline{V} \underline{A} = \underline{V}' \overset{\text{če so } a_n \text{ normir.}}{=} \begin{pmatrix} \omega_1^2 & & \\ & \ddots & \\ & & \omega_n^2 \end{pmatrix}$$

diagonalna



$$\underline{u}(t) = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} = \sum_k u_k \hat{e}_k \qquad \hat{e}_k = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}$$

$$\underline{u}(t) = \sum_k \mathrm{Re}(\underline{a}_k \underbrace{\alpha_k e^{-i\omega_k t}}_{\alpha_k(t)}) = \sum_k \mathrm{Re}(\alpha_k(t))\, \underline{a}_k$$

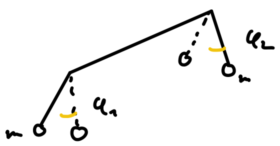
normalne koordinate — normalni načini

$$L = \sum_k \left(\frac{1}{2} \dot{\alpha}_k^2 \, \underline{T}'_k - \frac{1}{2} \underline{V}'_k \alpha_k^2\right) \overset{\text{če so } a_k \text{ normirani}}{=} \sum_k \left(\frac{1}{2}\dot{\alpha}_k^2 - \frac{1}{2}\omega_k^2 \alpha_k^2\right)$$

vsota k modalnih harmonskih oscilatorjev

$$H = \sum_k \left(\frac{1}{2}\dot{\alpha}_k^2 + \frac{1}{2}\omega_k^2 \alpha_k^2\right)$$

37 2 torzijsko sklopljeni matematični nihali



Iz strani

$$V = mgl\big((1-\cos\varphi_1) + (1-\cos\varphi_2)\big) + \frac{d}{2}(\varphi_1 - \varphi_2)^2$$

Iščemo ravnovesne lege

$$\frac{dV}{d\varphi_1} = mgl\sin\varphi_1 + d(\varphi_1 - \varphi_2) = 0$$

$$\frac{dV}{d\varphi_2} = mgl\sin\varphi_2 - d(\varphi_1 - \varphi_2) = 0$$

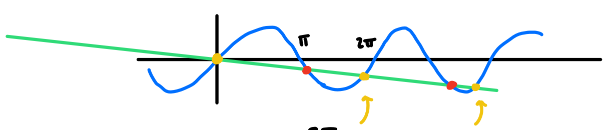
v ravnovesni legi; $\pm$

$$mgl(\sin\varphi_1 + \sin\varphi_2) = 0 \quad \Rightarrow \quad \varphi_1 = -\varphi_2 + 2k\pi \qquad \varphi_1 = \varphi_2 + \pi + 2k\pi$$

$$mgl(\sin\varphi_1 - \sin\varphi_2) = -2d(\varphi_1 - \varphi_2)$$

1) Ravnovesna lega, kjer $\varphi_1^0 = -\varphi_2^0$

$$2\sin\varphi_1 = -\frac{4d}{mgl}\varphi_1$$



φ_1^r ko r sod ≥ 2 $\quad \varphi_1 \in [\frac{3\pi}{2}, 2\pi] + 2k\pi \quad$ stabilna

φ_1^r ko r lih $\quad \varphi_1 \in [\pi, \frac{3\pi}{2}] + 2k\pi \quad$ labilna

$$\frac{\partial^2 V}{\partial\varphi_i \partial\varphi_j} = \begin{pmatrix} mgl\cos\varphi_1 + d & -d \\ -d & mgl\cos\varphi_2 + d \end{pmatrix} \overset{\varphi_2 = -\varphi_1}{=} \begin{pmatrix} mgl\cos\varphi_1 + d & -d \\ -d & mgl\cos\varphi_1 + d \end{pmatrix} = \underline{V}$$

Diagonaliziramo $\quad \underline{a}_1 = (1,1) \quad \underline{a}_2 = (1,-1)$

Last. vred. $\quad mgl\cos\varphi_1 \quad mgl\cos\varphi_1 + 2d$

Izračunajmo lastni frekvenci okrog $\varphi_1^0 = \varphi_2^0 = 0$

$$T = \frac{1}{2}m(l\dot{\varphi}_1)^2 + \frac{1}{2}m(l\dot{\varphi}_2)^2 = \frac{1}{2}ml^2(\dot{\varphi}_1, \dot{\varphi}_2)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{pmatrix} \dot{\varphi}_1 \\ \dot{\varphi}_2 \end{pmatrix}$$

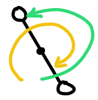
$$\underline{\underline{T}} = m\ell^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\omega^2 \underline{\underline{T}}\ \underline{a} = \underline{\underline{V}}\ \underline{a}$$

① $\quad \underline{a}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \omega^2 m \ell^2 = mg\ell \cos \overset{\overset{0}{\shortparallel}}{\varphi_1} = mg\ell \qquad \omega_1^2 = \frac{g}{\ell}$

② $\quad \underline{a}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad \omega^2 m \ell^2 = mg\ell + 2d \qquad \omega_2^2 = \frac{g}{\ell} + \frac{2d}{m\ell^2}$

② $\quad \varphi_1^\circ = \varphi_2^\circ + \pi \quad (+\, 2k\pi)$



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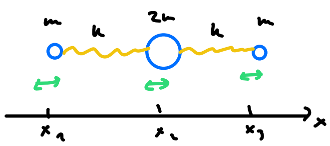


Obravnavamo v 1D

V ravnovesju je razdalja ℓ $\qquad \omega_k,\ \underline{a}_k = ? \qquad H_k = ?$

$$x_1 = 0 + \eta_1 \qquad \dot{x}_1 = \dot{\eta}_1$$
$$x_2 = \ell + \eta_2 \qquad \dot{x}_2 = \dot{\eta}_2$$
$$x_3 = 2\ell + \eta_3 \qquad \dot{x}_3 = \dot{\eta}_3$$

Kin. $\quad T = \frac{1}{2} m \dot{x}_1^2 + \frac{1}{2} 2m \dot{x}_2^2 + \frac{1}{2} m \dot{x}_3^2 = \frac{m}{2}\left(\dot{\eta}_1^2 + 2\dot{\eta}_2^2 + \dot{\eta}_3^2\right) = \frac{1}{2}\dot{\underline{\eta}}^T\, \underline{\underline{T}}\, \dot{\underline{\eta}} \qquad \underline{\underline{T}} = m\begin{array}{c|c} \begin{matrix} \eta_1 & \eta_2 & \eta_3 \end{matrix} & \\ \begin{bmatrix} 1 & & \\ & 2 & \\ & & 1 \end{bmatrix} & \begin{matrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{matrix} \end{array}$

Pot. $\quad V = \frac{1}{2}k\left((x_2 - x_1) - \ell\right)^2 + \frac{1}{2}k\left((x_3 - x_2) - \ell\right)^2 = \frac{k}{2}\left((\eta_2 - \eta_1)^2 + (\eta_3 - \eta_2)^2 \right) =$

$$= \frac{k}{2}\left(\eta_2^2 + \eta_1^2 - 2\eta_1\eta_2 + \eta_3^2 + \eta_2^2 - 2\eta_2\eta_3 \right)$$

$$= \frac{1}{2}\underline{\eta}^T\, \underline{\underline{V}}\, \underline{\eta} \qquad \underline{\underline{V}} = k\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

namesto -2 simetriziramo

Posplošeni lastni problem $\qquad \underline{\eta}(t) = \alpha\, \underline{a}\, e^{-i\omega t}$

$$\left(-\omega^2 \underline{\underline{T}} + \underline{\underline{V}}\right)\underline{a} = 0$$

$$\left(\frac{\omega}{\omega_0}\right)^2 = \lambda$$

$$0 = \det\left(\underline{\underline{V}} - \omega^2 \underline{\underline{T}}\right) = \det\left(k\left(\tilde{V} - \omega^2 \underbrace{\frac{m}{k}}_{\frac{1}{\omega_0^2}} \tilde{T}\right)\right)$$

$$0 = \det\left(\tilde{V} - \lambda \tilde{T}\right)$$

$$\det\left(\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & & \\ & 2 & \\ & & 1 \end{bmatrix} \right) = \det \begin{bmatrix} 1-\lambda & -1 & 0 \\ -1 & 2-2\lambda & -1 \\ 0 & -1 & 1-\lambda \end{bmatrix} =$$

$$= (1-\lambda)\begin{vmatrix} 2-2\lambda & -1 \\ -1 & 1-\lambda \end{vmatrix} - (-1)\begin{vmatrix} -1 & -1 \\ 0 & 1-\lambda \end{vmatrix} =$$

$$= (1-\lambda)\left((2-2\lambda)(1-\lambda) - 1\right) - (1-\lambda) = (1-\lambda)(2-\lambda)2\lambda$$

$$\lambda_1 = 1 \qquad \lambda_2 = 2 \qquad \lambda_3 = 0$$

$$\omega_1^2 = \omega_0^2 \qquad \omega_2^2 = 2\omega_0^2 \qquad \omega_3^2 = 0$$

$\lambda_1 = 1 \qquad (\hat{V} - \lambda \hat{I})\, \underline{a} = 0$

$$\begin{pmatrix} 1-1 & -1 & 0 \\ -1 & 2-2 & -1 \\ 0 & -1 & 1-1 \end{pmatrix} \begin{pmatrix} a_{11} \\ a_{12} \\ a_{13} \end{pmatrix} = 0$$

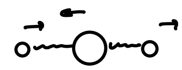
$$\begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} a_{11} \\ a_{12} \\ a_{13} \end{pmatrix} = 0 \qquad a_{12} = 0 \qquad a_{13} = -a_{11}$$

$$\underline{a}_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \qquad \omega_1^2 = \omega_0^2$$

$\lambda_2 = 2$

$$\begin{pmatrix} -1 & -1 & 0 \\ -1 & -2 & -1 \\ 0 & -1 & -1 \end{pmatrix} \begin{pmatrix} a_{21} \\ a_{22} \\ a_{23} \end{pmatrix} = 0 \qquad a_{21} = -a_{22} \qquad a_{23} = -a_{22}$$

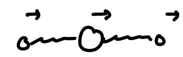
$$\underline{a}_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$



$$\omega_2^2 = 2\omega_0^2$$

$\lambda_3 = 0$

$$\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} a_{31} \\ a_{32} \\ a_{33} \end{pmatrix} = 0 \qquad \underline{a}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \text{Translacija}$$

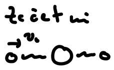


Lastni načini nihanja

$$\eta_1 = \underline{a}_1 \left(\tilde{A}_1 e^{i\omega_0 t} + \tilde{B}_1 e^{-i\omega_0 t} \right) = \underline{a}_1 \left(A_1 \cos(\omega_0 t) + B_1 \sin(\omega_0 t) \right)$$
$$\eta_2 = \underline{a}_2 \left(\tilde{A}_2 e^{i\sqrt{2}\omega_0 t} + \tilde{B}_2 e^{-i\sqrt{2}\omega_0 t} \right) = \underline{a}_2 \left(A_2 \cos(\sqrt{2}\omega_0 t) + B_2 \sin(\sqrt{2}\omega_0 t) \right)$$
$$\eta_3 = \underline{a}_3 \left(\tilde{A}_3 + \tilde{B}_3 t \right)$$

Splošna rešitev $\qquad \eta(t) = \sum_{k=1}^{3} \eta_k(t)$

Začetni pogoj



$$\eta(0) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \dot{\eta}(0) = \begin{pmatrix} v_0 \\ 0 \\ 0 \end{pmatrix}$$

$$\eta(0) = \begin{pmatrix} \underline{a}_1 & \underline{a}_2 & \underline{a}_3 \end{pmatrix} \begin{pmatrix} A_1 \cos \omega_0 t + \dots \\ A_2 \cos \sqrt{2}\omega_0 t + \dots \\ A_3 + B_3 t \end{pmatrix} \overset{t=0}{=} \begin{pmatrix} \underline{a}_1 & \underline{a}_2 & \underline{a}_3 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} =$$

$$= \begin{pmatrix} -1 & 1 & 1 \\ 0 & -1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow A_1 = A_2 = A_3 = 0$$

$$\dot{\eta}(0) = \begin{pmatrix} v_0 \\ 0 \\ 0 \end{pmatrix} = (\underline{a}_1, \underline{a}_2, \underline{a}_3) \begin{pmatrix} B_1 \omega_0 \dots \\ \vdots \end{pmatrix} = \begin{pmatrix} -B_1 \omega_0 + B_2 \sqrt{2}\omega_0 + B_3 \\ 0 - B_2 \sqrt{2}\omega_0 + B_3 \\ B_1 \omega_0 + B_2 \sqrt{2}\omega_0 + B_3 \end{pmatrix}$$

$$\Rightarrow \quad B_1 = -\frac{1}{2} \frac{v_0}{\omega_0} \qquad B_2 = \frac{1}{4\sqrt{2}} \frac{v_0}{\omega_0} \qquad B_3 = \frac{1}{4} v_0$$

Splošna rešitev

$$\eta(t) = \underbrace{\frac{1}{2} \frac{v_0}{\omega_0} \sin \omega_0 t}_{\alpha_1(t)} \underbrace{\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}}_{\underline{a}_1} + \underbrace{\frac{1}{4\sqrt{2}} \frac{v_0}{\omega_0} \sin \sqrt{2}\omega_0 t}_{\alpha_2(t)} \underbrace{\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}}_{\underline{a}_2} + \underbrace{\frac{v_0}{4} t}_{\alpha_3(t)} \underbrace{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}_{\underline{a}_3}$$

hitrost težišča je $\frac{v_0}{4}$

Energija, ki jih prispevajo posamezni lastni načini

$$H_k = \frac{1}{2} \dot{\alpha}_k^2 T_k' + \frac{1}{2} \alpha_k^2 V_k' \qquad \underline{T}' = \underline{A}^T \underline{T} \underline{A} \qquad \underline{V}' = \underline{A}^T \underline{V} \underline{A} \qquad \underline{A} = (\underline{a}_1 \ \underline{a}_2 \ \underline{a}_3)$$

$$T' = \begin{pmatrix} 2 & & \\ & 4 & \\ & & 4 \end{pmatrix} m \qquad V' = \begin{pmatrix} 2 & & \\ & 8 & \\ & & 0 \end{pmatrix} k$$

$$H_1 = \tfrac{1}{2}\dot{a}_1^2\, 2m + \tfrac{1}{2} a_1^2\, 2k = \left(\tfrac{1}{2} v_0 \cos\omega_0 t\right)^2 m + \left(\tfrac{1}{2} v_0 \sin \omega_0 t\right)^2 k = \tfrac{1}{4} m v_0^2$$
$$H_2 = \tfrac{1}{2}\dot{a}_2^2\, 4m + \tfrac{1}{2} a_2^2\, 8k \quad \dots = \tfrac{1}{8} m v_0^2$$
$$H_3 = \tfrac{1}{2}\dot{a}_3^2\, 4m + 0 \quad \dots = \tfrac{1}{8} m v_0^2$$

$$H = H_1 + H_2 + H_3 = \tfrac{1}{2} m v_0^2$$

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$$\underline{\underline{V}}\, \underline{a} = \omega^2 \underline{\underline{T}}\, \underline{a}$$

$$k\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} \underline{a} = \omega^2 m \begin{pmatrix} 1 & & \\ & M/m & \\ & & 1 \end{pmatrix} \underline{a}$$

Lastne načine si ugotovili z uporabo simetrije

$$\underline{a}_i^T\, \underline{\underline{T}}\, \underline{a}_j = \delta_{ij}\, \underline{a}_i^T\, \underline{\underline{T}}\, \underline{a}_i$$

1. translacija $\quad \underline{a}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad \underline{\underline{V}}\,\underline{a}_1 = 0 \Rightarrow \omega^2 = 0$

2. zrcalna simetrija $\quad \underline{a}_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \qquad \underline{a}_1^T \underline{\underline{T}}\, \underline{a}_2 = 0 \qquad \underline{\underline{V}}\,\underline{a}_2 = k\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \omega^2 \begin{pmatrix} m \\ 0 \\ -m \end{pmatrix} \Rightarrow \omega^2 = \frac{k}{m}$

3. ortogonalen na prva dva $\quad \underline{a}_3 = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \qquad \underline{a}_3^T \underline{\underline{T}}\, \underline{a}_2 = 0 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}^T \begin{pmatrix} m \\ 0 \\ -m \end{pmatrix} = am - mc$

$$\Rightarrow a = c$$

$$\underline{a}_3 = \begin{pmatrix} 1 \\ b \\ 1 \end{pmatrix} \qquad \underline{a}_3^T \underline{\underline{T}}\, \underline{a}_1 = \begin{pmatrix} 1 \\ b \\ 1 \end{pmatrix}^T \begin{pmatrix} m \\ M \\ m \end{pmatrix} = 2m + bM = 0$$
$$b = -2\frac{m}{M}$$

$$\Rightarrow \underline{a}_3 = \begin{pmatrix} 1 \\ -2\frac{m}{M} \\ 1 \end{pmatrix}$$

$$\underline{\underline{V}}\,\underline{a}_3 = k\begin{pmatrix} 1 + 2m/M \\ -2 - 4m/M \\ 1 + 2m/M \end{pmatrix} = \omega^2\, \underline{\underline{T}} \begin{pmatrix} 1 \\ -2m/M \\ 1 \end{pmatrix} = \begin{pmatrix} m \\ -2m \\ m \end{pmatrix}\omega^2 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}\omega^2 m$$

$$k\left(1 + \frac{2m}{M}\right)\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}\omega^2 m$$

$$\omega^2 = \frac{k}{m}\left(1 + \frac{2m}{M}\right)$$

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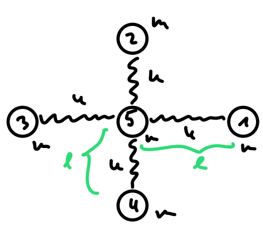
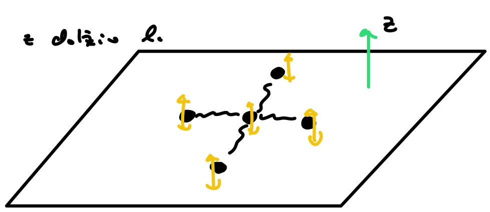


Vzmet z dolžino l.



$$\underline{z} = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_5 \end{pmatrix}$$

$$T = \frac{1}{2} m \sum_{i=1}^{5} \dot{z}_i^2 = \frac{1}{2}\dot{\underline{z}}^T\, \underline{\underline{T}}\, \dot{\underline{z}} = \frac{1}{2}\dot{\underline{z}}^T\, m\underline{\underline{I}}\, \dot{\underline{z}}$$

$\vec{r}_1 = (l, 0, z_1)$
$\vec{r}_2 = (0, l, z_2)$
$\vec{r}_3 = (-l, 0, z_3)$
$\vec{r}_4 = (0, -l, z_4)$
$\vec{r}_5 = (0, 0, z_5)$

$l_0 < l$: to pomeni da so vzmeti prednapete

$$V = \frac{1}{2} k \sum_{i=1}^{4} \left(|\vec r_i - \vec r_5| - \ell_0 \right)^2$$

$$|\vec r_i - \vec r_5| = \sqrt{\ell^2 + (z_i - z_5)^2} = \ell \sqrt{1 + \left(\frac{z_i - z_5}{\ell}\right)^2} \approx \ell \left(1 + \frac{1}{2} \left(\frac{z_i - z_5}{\ell} \right)^2 \right)$$ majhni nihaji

$$V = \frac{1}{2} k \sum_{i=1}^{4} \left((\ell - \ell_0) + \frac{1}{2\ell} (z_i - z_5)^2 \right)^2 = \frac{1}{2} k \left((\ell-\ell_0)^2 + 2(\ell-\ell_0) \frac{1}{2\ell} (z_i - z_5)^2 + \xrightarrow{0} \cancel{\frac{1}{4\ell^2} (z_i - z_5)^4} \right)$$

$$\approx \frac{1}{2} k \sum_{i=1}^{4} \underbrace{\frac{\ell - \ell_0}{\ell}}_{>0} (z_i - z_5)^2 + \text{konst.} = \frac{1}{2} \tilde k \sum_{i=1}^{4} (z_i - z_5)^2 + \text{konst.}$$

$$= \frac{1}{2} \tilde k \left(z_1^2 + z_2^2 + z_3^2 + z_4^2 + 4 z_5^2 - 2 z_1 z_5 - 2 z_2 z_5 - 2 z_3 z_5 - 2 z_4 z_5 \right) =$$

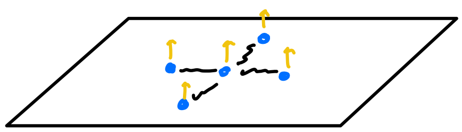
$$= \frac{1}{2} \underline{z}^T \underline{V}\, \underline{z}$$

$$\underline{V} = \tilde k \begin{bmatrix} 1 & & 0 & & -1 \\ & 1 & & & -1 \\ & & 1 & & -1 \\ 0 & & & 1 & -1 \\ -1 & -1 & -1 & -1 & 4 \end{bmatrix}$$

E-L enačba $\quad (\underline{V} - \omega^2 \underline{T}) \underline{s} = 0 \qquad \underline{T} = m \underline{I}$

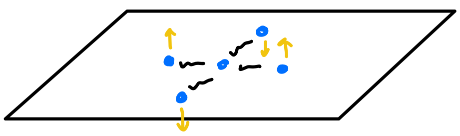
$$(\underline{V} - \omega^2 m \underline{I}) \underline{s} = 0$$

Rešitve ugibamo



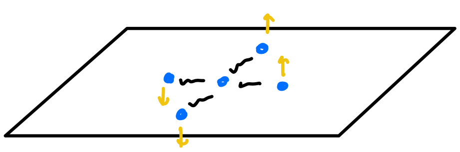
$$a_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \qquad \underline{V}\, \underline{a}_1 = \tilde k \begin{pmatrix} 1 & & & & -1 \\ & 1 & & & -1 \\ & & 1 & & -1 \\ & & & 1 & -1 \\ -1 & -1 & -1 & -1 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = m\omega^2 \underline{a}_1$$

$$\Rightarrow \omega_1^2 = 0$$

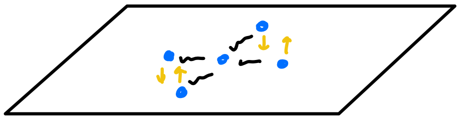


$$a_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 0 \end{pmatrix} \qquad \underline{V}\, \underline{a}_2 = \tilde k \begin{bmatrix} 1 & & 0 & & -1 \\ & 1 & & & -1 \\ & & 1 & & -1 \\ 0 & & & 1 & -1 \\ -1 & -1 & -1 & -1 & 4 \end{bmatrix} \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 0 \end{pmatrix} = \tilde k \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 0 \end{pmatrix} = m\omega^2 \underline{a}_2$$

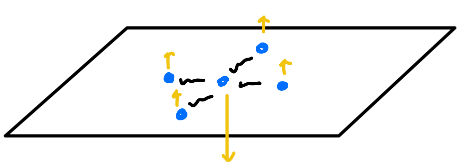
$$\Rightarrow \omega_2^2 = \frac{\tilde k}{m} = \omega_0^2$$



$$a_3 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \\ 0 \end{pmatrix} \qquad \underline{V}\, \underline{a}_3 = \dots = \omega_3^2 m \underline{a}_3 \Rightarrow \omega_3^2 = \omega_0^2$$



$$a_4 = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} \qquad \dots \Rightarrow \omega_4^2 = \omega_0^2$$



$$\underline{a}_5 = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \end{pmatrix}$$

$$\left. \begin{array}{l} a_1 \cdot a_5 = 0 \\ \vdots \\ a_4 \cdot a_5 = 0 \end{array} \right\} \Rightarrow d_1 = d_2 = d_3 = d_4 = 1 \qquad \underline{a}_5 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ d_5 \end{pmatrix}$$

$$a_1 \cdot a_5 = 0 = 1 + 1 + 1 + 1 + d_5 = 0 \qquad d_5 = -4 \qquad \underline{a}_5 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ -4 \end{pmatrix}$$

$$\underline{\underline{V}}\, \underline{q}_s = \dots = \omega_s^2\, m\, \underline{q}_s \qquad \Rightarrow \qquad \omega_s^2 = 5\omega_0^2$$

(T) Hamiltonov formalizem

$$L = L(q_\alpha, \dot q_\alpha) \qquad p_\alpha = \frac{\partial L}{\partial \dot q_\alpha}$$

$$H = \sum_\alpha p_\alpha \dot q_\alpha - L(q_\alpha, \dot q_\alpha) = H(q_\alpha, p_\alpha)$$

Primer: cilindrične koordinate

$$L = \frac{1}{2} m \dot r^2 + \frac{1}{2} m r^2 \dot\varphi^2 - V(r)$$

$$p_r = m \dot r \qquad p_\varphi = m r^2 \dot\varphi$$

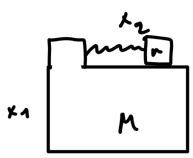
$$H = m\dot r^2 + m r^2 \dot\varphi^2 - \frac{1}{2} m \dot r^2 - \frac{1}{2} m r^2 \dot\varphi^2 + V(r) = \frac{1}{2} m \dot r^2 + \frac{1}{2} m r^2 \dot\varphi^2 + V(r) = T + V$$

če $V \neq V(\dot q) \Rightarrow H = T + V$

$$H = \frac{p_\varphi^2}{2 m r^2} + \frac{p_r^2}{2m} + V(r)$$

$$\boxed{\dot q_\alpha = \frac{\partial H}{\partial p_\alpha} \qquad \dot p_\alpha = -\frac{\partial H}{\partial q_\alpha}} \qquad \text{Hamilt. en.}$$

(41)



$$L = \frac{1}{2} M \dot x_1^2 + \frac{1}{2} m (\dot x_1 + \dot x_2)^2 - \frac{k x_2^2}{2}$$

$$p_1 = \frac{\partial L}{\partial \dot x_1} = M \dot x_1 + m(\dot x_1 + \dot x_2)$$

$$p_2 = \frac{\partial L}{\partial \dot x_2} = m(\dot x_1 + \dot x_2)$$

$$H = T + U = \frac{1}{2} M \dot x_1^2 + \frac{1}{2} m (\dot x_1 + \dot x_2)^2 + \frac{k x_2^2}{2} = \frac{1}{2}\frac{(p_1 - p_2)^2}{M} + \frac{p_2^2}{2m} + \frac{k x_2^2}{2}$$

$$\dot x_1 = \frac{\partial H}{\partial p_1} = \frac{p_1 - p_2}{M} \qquad \dot x_2 = \frac{\partial H}{\partial p_2} = \frac{-(p_1 - p_2)}{M} + \frac{p_2}{m}$$

$$\dot p_1 = -\frac{\partial H}{\partial x_1} = 0 \qquad \dot p_2 = -\frac{\partial H}{\partial x_2} = -k x_2$$

$$\ddot x_2 = -\frac{\overset{=0}{\dot p_1}}{M} + \dot p_2 \left(\frac{1}{M} + \frac{1}{m}\right) = -k x_2 \left(\frac{1}{M} + \frac{1}{m}\right) \qquad \text{nihanje}$$

(42) Delec v homogenem magnetnem polju

$$L = \frac{1}{2} m v^2 - e\varphi + e\vec v \cdot \vec A$$

$$p_x = \frac{\partial L}{\partial \dot x} = m \dot x + e A_x$$

$$\frac{dp_x}{dt} = m \ddot x + e \dot A_x = \frac{\partial L}{\partial x} = -e \frac{\partial \varphi}{\partial x}$$

$$m\ddot{x} = e\left(-\frac{\partial \varphi}{\partial x} - \frac{\partial A_x}{\partial t}\right)$$

$$m\dot{\vec{v}} = e\left(-\nabla \varphi - \frac{\partial \vec{A}}{\partial t}\right) = e\vec{E}$$

$$H = \sum_i p_i \dot{x}_i - L = \vec{p}\cdot\vec{v} - L = mv^2 + e\vec{A}\cdot\vec{v} - \frac{1}{2}mv^2 + e\varphi - e\vec{A}\,\vec{v} = \frac{1}{2}mv^2 + e\varphi \neq T + V$$

$$\varphi = 0 \qquad \vec{B} = B\hat{k} \qquad H = \frac{(\vec{p} - e\vec{A})^2}{2m} \qquad \vec{A} = B(-y\hat{i})$$

$$\vec{A} = \frac{1}{2}B\left(-y\hat{i} + x\hat{j}\right) \qquad \vec{B} = \nabla \times \vec{A} = (0, 0, B)$$

$$H = \frac{1}{2m}\left(p_x + \frac{eBy}{2}\right)^2 + \frac{1}{2m}\left(p_y - \frac{eBx}{2}\right)^2$$

$$\dot{x} = \frac{\partial H}{\partial p_x} = \frac{1}{m}\left(p_x + \frac{eBy}{2}\right) \qquad \dot{p}_x = -\frac{\partial H}{\partial x} = \frac{1}{m}\left(p_y - \frac{eBx}{2}\right)\left(-\frac{eB}{2}\right) = \frac{eB}{2}\dot{y}$$

$$\dot{y} = \frac{\partial H}{\partial p_y} = \frac{1}{m}\left(p_y - \frac{eBx}{2}\right) \qquad \dot{p}_y = -\frac{\partial H}{\partial y} = \frac{1}{m}\left(p_x + \frac{eBy}{2}\right)\frac{eB}{2} = \frac{eB}{2}\dot{x}$$

$$\ddot{x} = \frac{\dot{p}_x + \frac{eB}{2}\dot{y}}{m} = \frac{eB}{m}\dot{y} = \omega_c \dot{y} \qquad \ddot{y} = -\frac{eB}{m}\dot{x} = -\omega_c \dot{x}$$

$$\Rightarrow \quad v_y = A\cos(\omega_c t)$$
$$v_x = A\sin(\omega_c t)$$