① Integrali s parametrom

- Enakomerna konvergenca za dovolj velik t mora veljati $|f(t,x) - \lim_{t\to\infty} f(t,x)| < \varepsilon$ za vsak x

Enak. konv. za integral: f zvezna $f:[a,\infty)\times[c,d]$ $\left|\int_m^\infty f(x,t)\,dx\right| < \varepsilon$ za $\forall t$ in za dovolj velik m.

Omejeni integrali $[a,b]\times[c,d]$, f zvezna
- $\frac{d}{dt}\int f\,dx = \int f'_y\,dx$ če f' zvezna
- f zvezna $\Rightarrow \int_a\int_c = \int\int$
- f zvezna $\Rightarrow F = \int_a^d f$ je F zvezna
- Če $\exists \int_c\int_a |f|$ ali $\int_c\int_a |f| \Rightarrow \int_a\int_c f = \int_c\int_a f$

Če je F zvezna je tudi integrabilna
Če je F zvezna je $\lim \int = \int \lim$
Če je f zvezna je $\lim_{x\to a} f(x) = f(a)$
Če je f omejena na $[a,b]$ je tudi zvezna

Izlimitirani integrali $[a,\infty)\times[c,d]$, f zvezna
Če $\int_a^\infty f(x,y)\,dx$ konv. enakomerno na $[c,d]$
$\to F$ zvezna na $[c,d]$
$\to \int_c^d\int_a^\infty = \int_a^\infty\int_c^d$

$\frac{\partial f}{\partial y}$ je zvezn

Če $\int_a^\infty \frac{d}{dy} f(x,y)\,dx$ konv. enakomerno na $[c,d]$
$\to \frac{d}{dy}\int_a^\infty f(x,y)\,dx = \int_a^\infty \frac{d}{dy} f(x,y)\,dx$

- Weierstrassov kriterij za enak. konv.
$\int f(x,y)\,dx$, če $|f| < W$ in $\int_a^\infty W$ konvergira

Kriterij za konvergenco
$\int_0 \frac{g(x)}{x^a}\,dx$ konv za $a<1$ (pol je lahko tudi zamaknjen)
$\int^\infty \frac{g(x)}{x^a}\,dx$ konv. za $a>1$

Γ funkcija
$\Gamma(s) = \int_0^\infty x^{s-1} e^{-x}\,dx$
$\Gamma(n) = (n-1)!$
$\Gamma(n+1) = n\Gamma(n)$
$\Gamma(\frac{1}{2}) = \sqrt{\pi}$
$\Gamma(x+1) = \left(\frac{x}{e}\right)^x \sqrt{2\pi x}$
$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)} = B(s,1-s) \quad 0<s<1$

B funkcija
$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$
$B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}\,dx$
$B(a,b) = \int_0^\infty x^{a-1}/(1+x)^{a+b}\,dx$
$B(a,b) = 2\int_0^{\pi/2} \cos^{2a-1}x \sin^{2b-1}x\,dx$

② Dvojni in trojni integrali

$f:[a,b]\times[c,d]$ zvezna
$\iint_{[a,b]\times[c,d]} = \int_a^b\int_c^d = \int_c^d\int_a^b$

Nove spremenljivke $x,y \to u,v$ $\quad D \xrightarrow{\text{bijek.}} \Delta$
$\Phi(u,v) = (x(u,v), y(u,v))$
$\iint_D f(x,y)\,dx\,dy = \iint_\Delta f(x(u,v),y(u,v))\,|J\Phi|\,du\,dv$

$J\Phi = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$

Polarne/Cilind. koordinate
$x = r\cos\varphi$
$y = r\sin\varphi$
$z = z$
$J\Phi = r$

Sferične koord.
$x = r\sin\theta\cos\varphi$
$y = r\sin\theta\sin\varphi$
$z = r\cos\theta$
$J\Phi = r^2\sin\theta$

$V = \iiint dx\,dy\,dz$
$m = \iiint \rho(x,y,z)\,dx\,dy\,dz$
$x^T = \frac{1}{m}\iiint x\,\rho(x,y,z)\,dx\,dy\,dz$
$J_z = \iiint (x^2+y^2)\,\rho(x,y,z)\,dx\,dy\,dz$

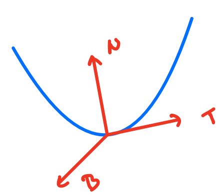
③ Krivulje

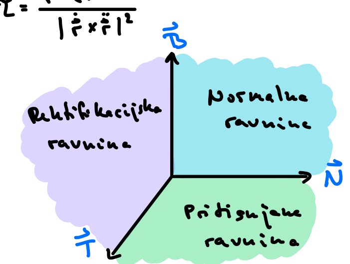
$L = \int_a^b |\dot{\vec r}(t)|\,dt$

Naravna parametrizacija $t \to s$
$s = \int_a^t |\dot{\vec r}(t)|\,dt \Rightarrow s(t) \Rightarrow t(s)$
$\vec\rho(s) = \vec r(t(s)) \qquad |\vec\rho\,'(s)| = 1$
poljuben

Frenetove baze
Naravna param. $\vec\rho$
$\vec T = \vec r\,'$
$\vec N = \frac{\vec r\,''}{|\vec r\,''|} = \frac{\vec T'}{|\vec T'|}$
$\vec B = \vec T\times\vec N = \frac{\vec r\,'\times\vec r\,''}{|\vec r\,'\times\vec r\,''|}$
$\varkappa = |\vec r\,''|$
$\tau = \frac{\vec r\,'\cdot(\vec r\,''\times\vec r\,''')}{|\vec r\,''|^2}$
$\dot{\vec T} = \varkappa\vec N$
$\vec N' = \tau\vec B - \varkappa\vec T$
$\vec B\,' = -\tau\vec N$

Polj. param.
$\vec T = \frac{\dot{\vec r}}{|\dot{\vec r}|}$ $\quad \vec B = \frac{\dot{\vec r}\times\ddot{\vec r}}{|\dot{\vec r}\times\ddot{\vec r}|}$ $\quad \vec N = \frac{(\dot{\vec r}\times\ddot{\vec r})\times\dot{\vec r}}{|(\dot{\vec r}\times\ddot{\vec r})\times\dot{\vec r}|}$
$\varkappa = \frac{|\dot{\vec r}\times\ddot{\vec r}|}{|\dot{\vec r}|^3}$ $\quad \tau = \frac{\dot{\vec r}\cdot(\ddot{\vec r}\times\dddot{\vec r})}{|\dot{\vec r}\times\ddot{\vec r}|^2}$





Radij $R = \frac{1}{\varkappa}$
Ravnina $\langle (x,y,z) - \vec r(t), \vec n\rangle = 0$

Taylorjeve vrste

$(1 \pm x)^m = 1 \pm m x + \frac{m(m-1)}{2!} x^2$

$(1 \pm x)^{-m} = 1 \mp m x + \frac{m(m+1)}{2!} x^2$

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!}$

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$

$\tan x = x + \frac{1}{3} x^3 + \frac{2}{15} x^5$

$\mathrm{sh}\, x = x + \frac{x^3}{3!} + \frac{x^5}{5!}$

$\mathrm{ch}\, x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!}$

$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!}$

$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3}$

Odvod:

${x^n}' = n x^{n-1}$

${e^x}' = e^x$

$\ln x' = \frac{1}{x}$

${a^x}' = a^x \ln a$

$\log_a x' = \frac{1}{x \ln a}$

$\tan x' = \frac{1}{\cos^2 x}$

$\cot x' = \frac{-1}{\sin^2 x}$

$\arcsin x' = \frac{1}{\sqrt{1-x^2}}$

$\arccos x' = \frac{-1}{\sqrt{1-x^2}}$

$\arctan x' = \frac{1}{1+x^2}$

$\mathrm{arccot}\, x' = \frac{-1}{1+x^2}$

Integrali

$\int x^n = \frac{x^{n+1}}{n+1}$

$\int e^x = e^x$

$\int a^x = \frac{a^x}{\ln a}$

$\int \frac{1}{x} = \ln|x|$

$\int \frac{1}{\cos^2 x} dx = \tan x$

$\int \frac{1}{\sin^2 x} dx = -\cot x$

$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \arcsin \frac{x}{a}$

$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \arctan \frac{x}{a}$

$\int \frac{1}{a^2 \mp x^2} dx = \frac{1}{2a} \ln \left| \frac{x \pm a}{x \mp a} \right|$

$\int \sqrt{a^2 \pm x^2}\, dx = \frac{x}{2} \sqrt{a^2 \pm x^2} \pm \frac{a^2}{2} \ln \left| \sqrt{a^2 \pm x^2} + x \right|$

$\int \ln x = x \ln x - x$

Hiperbolične funkcije

$\sinh x = \frac{e^x - e^{-x}}{2}$

$\cosh x = \frac{e^x + e^{-x}}{2}$

$\tanh x = \frac{\sinh x}{\cosh x}$

$\sinh' x = \cosh x$

$\cosh' x = \sinh x$

$\tanh' x = 1 - \tanh^2 x$

$\mathrm{ch}^2 x - \mathrm{sh}^2 = 1$

$\sin x = \frac{e^{ix} - e^{-ix}}{2i}$

$\cos x = \frac{e^{ix} + e^{-ix}}{2i}$

Trigonometrija

$\sin(\pi - x) = \sin x$ $\quad \sin(\frac{\pi}{2} \pm x) = \cos x$

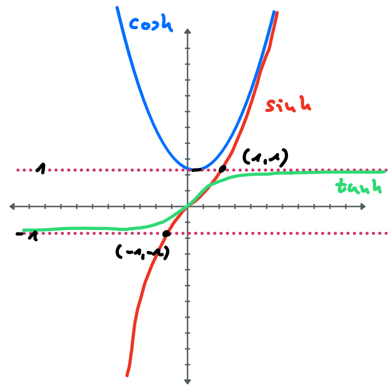
$\cos(\pi - x) = -\cos x$ $\quad \cos(\frac{\pi}{2} \pm x) = \mp \sin x$

$\tan(\pi - x) = -\tan x$ $\quad \tan(\frac{\pi}{2} \pm x) = \mp \cot x$

$\cot(\pi - x) = -\cot x$

$\sin(x \pm y) = \sin x \cos y \pm \sin y \cos x$

$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$



Trigonometrične formule:

$\sin x + \sin y = 2 \sin \frac{x+y}{2} \cos \frac{x-y}{2}$ $\qquad \sin x \cos y = \frac{1}{2}(\sin(x+y) + \sin(x-y))$

$\cos x + \cos y = 2 \cos \frac{x+y}{2} \cos \frac{x-y}{2}$ $\qquad \cos x \cos y = \frac{1}{2}(\cos(x+y) + \cos(x-y))$

$\cos x - \cos y = -2 \sin \frac{x+y}{2} \sin \frac{x-y}{2}$ $\qquad \sin x \sin y = \frac{1}{2}(\cos(x-y) - \cos(x+y))$

$\sin^2 x = \frac{1-\cos 2x}{2}$ $\qquad \cos^2 x = \frac{1+\cos 2x}{2}$

Integral:

Nastavek za integracijo racionalnih funkcij:

$$\int \frac{R(x)}{Q(x)}\, dx = \frac{A_s x^s + A_{s-1} x^{s-1} + \cdots + A_1 x + A_0}{(x-x_1)^{\alpha_1 - 1} \cdots (x-x_n)^{\alpha_n - 1} (x^2 + p_1 x + q_1)^{\beta_1 - 1} \cdots (x^2 + p_m x + q_m)^{\beta_m - 1}} +$$
$$+ B_1 \ln|x - x_1| + \cdots + B_n \ln|x - x_n| +$$
$$+ U_1 \ln(x^2 + p_1 x + q_1) + \cdots + U_m \ln(x^2 + p_m x + q_m) +$$
$$+ V_1 \operatorname{arc\,tg} \left(\frac{2x + p_1}{\sqrt{-p_1^2 + 4q_1}} \right) + \cdots + V_m \operatorname{arc\,tg} \left(\frac{2x + p_m}{\sqrt{-p_m^2 + 4q_m}} \right) + C.$$

($\mathrm{st} R < \mathrm{st} Q$, $Q(x) = (x-x_1)^{\alpha_1} \cdots (x-x_n)^{\alpha_n} (x^2 + p_1 x + q_1)^{\beta_1} \cdots (x^2 + p_m x + q_m)^{\beta_m}$ je razcep na paroma različne realne nerazcepne člene)

Integrali iracionalnih funkcij:

$$\int \frac{dx}{\sqrt{x^2 + px + q}} = \ln \left| x + \frac{p}{2} + \sqrt{x^2 + px + q} \right| + C$$

$$\int \frac{dx}{\sqrt{-x^2 + px + q}} = \arcsin \left(\frac{2x - p}{\sqrt{p^2 + 4q}} \right) + C$$

$$\int \frac{P(x)\, dx}{\sqrt{ax^2 + bx + c}} = T(x) \sqrt{ax^2 + bx + c} + K \int \frac{dx}{\sqrt{ax^2 + bx + c}}, \ (a \neq 0,\ P \text{ in } T \text{ polinoma})$$

Univerzalna trigonometrična substitucija: $\operatorname{tg} \frac{x}{2} = t$, $dx = \frac{2\,dt}{1+t^2}$, $\cos x = \frac{1-t^2}{1+t^2}$, $\sin x = \frac{2t}{1+t^2}$

	Ime	Pravokoten na	Formula	Naravna param
$\vec{T}$	tang. vektor	normalno ravnino	$\frac{\dot{\vec r}}{\lvert\dot{\vec r}\rvert}$	$\vec{\rho}\,'$
$\vec{N}$	glavna normala	rektifikacijsko ravnino	$\frac{(\dot{\vec r} \times \ddot{\vec r}) \times \dot{\vec r}}{\lvert(\dot{\vec r} \times \ddot{\vec r}) \times \dot{\vec r}\rvert}$	$\frac{\vec{\rho}\,''}{\lvert\vec{\rho}\,''\rvert}$
$\vec{B}$	binormala	pritisnjeno ravnino	$\frac{\dot{\vec r} \times \ddot{\vec r}}{\lvert\dot{\vec r} \times \ddot{\vec r}\rvert}$	$\frac{\vec{\rho}\,' \times \vec{\rho}\,''}{\lvert\vec{\rho}\,' \times \vec{\rho}\,''\rvert}$

Ukrivljenosti

	Ime	formula	Naravna param.
κ	fleksijska ukrivljenost	$\frac{\lvert\dot{\vec r} \times \ddot{\vec r}\rvert}{\lvert\dot{\vec r}\rvert^3}$	$\lvert\vec{\rho}\,''\rvert$
τ	torzijska ukrivljenost	$\frac{(\dot{\vec r}, \ddot{\vec r}, \dddot{\vec r})}{\lvert\dot{\vec r} \times \ddot{\vec r}\rvert^2}$	$\frac{(\vec{\rho}\,', \vec{\rho}\,'', \vec{\rho}\,''')}{\lvert\vec{\rho}\,''\rvert^2}$

Ploskve $u,v \to x,y,z$

tangentna vektorja $\frac{\partial \vec r}{\partial u}, \frac{\partial \vec r}{\partial v}$

tang. ravnina $\langle (x,y,z) - \vec r(u,v), (\vec r_u \times \vec r_v)(u,v)\rangle = 0$

normalni vektor $\vec r_u \times \vec r_v / |\vec r_u \times \vec r_v|$

Površina ploskve $P = \iint_D |\vec r_u \times \vec r_v| \, du\,dv = \iint_D \sqrt{EG - F^2}$

$E = \langle \vec r_u, \vec r_u\rangle \quad F = \langle \vec r_u, \vec r_v\rangle$

$G = \langle \vec r_v, \vec r_v\rangle$

Ploskovni integral skalarnega polja

$$\int_P f\,dS = \iint_D f(\vec r(u,v))\,|\vec r_u \times \vec r_v|\,du\,dv$$

$$= \iint_D f(\vec r(u,v))\sqrt{EG-F^2}\,du\,dv$$

ρ ... površinska gostota $\quad \iint_P \rho\,dS = $ masa

$\bar x = \frac{1}{m}\int_P x\rho\,dS \qquad J_z = \iint_P (x^2+y^2)\rho\,dS$

Ploskovni integral vektorskega polja

$$\int_P \vec F\cdot d\vec S = \int_P \langle \vec F, \vec n\rangle\,dS = \iint \langle \vec F(\vec r(u,v)), \vec r_u \times \vec r_v\rangle\,du\,dv$$

Diferencialni operatorji:

$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$

$\operatorname{grad} u = \nabla u = (u_x, u_y, u_z)$ sk → vek

$\operatorname{div}\vec F = \nabla\cdot\vec F = U_x + V_y + W_z$ vek → sk

$\operatorname{rot}\vec F = \nabla\times\vec F = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ U & V & W\end{vmatrix}$ vek → vek

$\begin{vmatrix} 1 & 4 \\ 3 & 2 \end{vmatrix}$

- $\operatorname{rot}(\operatorname{grad} u) = 0$
- $\operatorname{div}(\operatorname{rot}\vec F) = 0$
- $\operatorname{div}(\operatorname{grad} u) = (\partial^2_{xx}u + \partial^2_{yy}u + \partial^2_{zz}u)$

Gaussov izrek $\oiint_{\partial P} \vec F\cdot d\vec S = \iiint_V \operatorname{div}\vec F\,dV$

Stokesova formula $\oint_{\partial P} \vec F\cdot d\vec s = \iint_P \operatorname{rot}\vec F\cdot d\vec S$

Greenova formula $\oint_{\partial D} \vec v\cdot d\vec s = \iint_D (Y_x - X_y)\,dx\,dy$

Orientacija v smeri normale $\vec n$ $\vec r$

Krivuljni integral skalarne funkcije

$$\int_K f\,ds = \int_a^b f(\vec r(t))\,|\dot{\vec r}(t)|\,dt$$

parametriziramo

$\int_K \rho\,ds = $ masa krivulje $\qquad \rho(x,y,z)$ ... dolžinska gostota

$\int_K ds = $ dolžina krivulje $\qquad \bar x = \dfrac{\int_K x\rho\,ds}{\int_K \rho\,ds} = $ težišče krivulje

$\int_K (x^2+y^2)\rho\,ds = J_z$

Krivuljni integral vektorskega polja $\vec v = (U,V,W)$

$$\int_K \vec v\cdot d\vec s = \int_a^b \langle \vec v(\vec r(t)), \dot{\vec r}(t)\rangle\,dt$$

$$\int_K \vec v\cdot d\vec s = \int_K U\,dx + V\,dy + W\,dz$$

Potencialno vektorsko polje

$\operatorname{rot}\vec v = 0 \ \overset{\Rightarrow}{\cancel{\Leftarrow}} \ \vec v = \operatorname{grad} f = \nabla f = (f_x, f_y, f_z)$

$\int_K \vec v\cdot d\vec s = \int_K \operatorname{grad} f\,d\vec s = f(B) - f(A) \qquad \oint \operatorname{grad} f\,d\vec s = 0$

Diferencialne enačbe

- Ločljive spremenljivke
- Homogene dif. en.

$y' = f\left(\frac{y}{x}\right) \Rightarrow z = \frac{y}{x},\ y = xz \Rightarrow z + xz' = f(z) \Rightarrow$

$\frac{z'}{f(z)-z} = \frac{1}{x} \Rightarrow z(x,c) \Rightarrow y = x\,z(x,c)$

- Linearna dif. en. 1. reda $\quad p(x)y' + q(x)y = r(x)$
  - Homogena rešitev $py' + qy = 0 \Rightarrow \frac{y'}{y} = -\frac{q}{p} \Rightarrow$ $y_H = D e^{-\int q/p\,dx} = D\varphi(x)$
  - Partikularna rešitev

$y_p = D(x)\varphi(x) \Rightarrow py' + qy = r \Rightarrow p(D'\varphi + \overbrace{D\varphi') + qD\varphi}^{0} = r$

$\Rightarrow D' = \frac{r}{p\varphi} \Rightarrow D \cancel{+C}$

Splošna rešitev $y = D\varphi(x) + D(x)\varphi(x)$

- Bernoullijeve dif. en. $\quad py' + qy = ry^\alpha$

$p y^{-\alpha} y' + q y^{1-\alpha} = r \Rightarrow z = y^{1-\alpha} \Rightarrow$

$\frac{p}{1-\alpha} z' + qz = r \xRightarrow{LDE}$ poiščemo splošno rešitev $\Rightarrow y = z^{\frac{1}{1-\alpha}}$

- Eksaktne DE $\qquad \vec v = (P,Q)$

$y' = f(x,y) \Rightarrow -f(x,y) + y' = 0 \Rightarrow P\,dx + Q\,dy = 0$

$(P,Q) = \operatorname{grad} u \Rightarrow \operatorname{rot}(P,Q) = 0 \Rightarrow$ rešitev je nivojnica $u(x,y) = C$

$\operatorname{rot}(P,Q) = 0 \Leftrightarrow Q_x = P_y \Rightarrow (P,Q) = \operatorname{grad} u \Rightarrow u \Rightarrow u(x,y) = C$

- Integrirajoči množitelj η (ko $Q_x \neq P_y$)

$\frac{\partial}{\partial x}(\eta Q) = \frac{\partial}{\partial y}(\eta P) \Rightarrow \exists u,\ \operatorname{grad} u = (\eta P, \eta Q) \Rightarrow u(x,y) = C$

Iskanje η: poskusimo $\eta = \eta(x)$

$\eta = \exp\int \underbrace{\frac{P_y - Q_x}{Q}}\,dx$ če je odvisno le od x ot y.

Splošno $z = z(x,y) \quad \mu = \mu(x,y)$

$$\frac{1}{\mu}\frac{d\mu}{dz} = \frac{\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}}{Q\frac{\partial z}{\partial x} - P\frac{\partial z}{\partial y}}$$

- Sistem lin. dif. en. $\dot{x} = Ax + f$
  - $f = 0$ $\quad x_H = C_1 e^{tA} \quad C \in \mathbb{R}^n$

    $x_H = \Theta(t)\, C_1$
  - $f \neq 0$ $\quad x_p = \Theta(t)\, C_1(t)$

    $\dot{C}_1 = \Theta^{-1}(t)\, f(t) \to C(t)$

  $x = C_1 e^{tA} + C_1(t) e^{tA}$

$$\dot{x} = Ax \to \det(A - \lambda I) = 0 \to \lambda \to v$$

$$x = \begin{bmatrix} | & & | \\ e^{\lambda_1 t} v_1 & \cdots & e^{\lambda_n t} v_n \\ | & & | \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \Theta(t)\, C$$

$$x = C_1 e^{\lambda_1 t} v_1 + \dots$$

Če $e^{\lambda t} v \in \mathbb{C}$ $\quad x = C_1 e^{\lambda t} v + C_2 e^{\bar{\lambda} t} \bar{v}$

$$x = C_1 \operatorname{Re}(e^{\lambda t} v) + C_2 \operatorname{Im}(e^{\lambda t} v)$$

Korenski vektorji:

$\det(A - \lambda I) = 0 \to \lambda_1, \lambda_2 \dots$

Vzemimo neko λ

$0 < \dim(\ker \underset{k-1}{(A - \lambda I)}) < \dim(\ker (A - \lambda I)^2) < \dots$

$k \longrightarrow$ # last. vekt.

$\lambda_1 \quad v_1^{(1)} \in \ker(A - \lambda I)^k - \ker(A - \lambda I)^{k-1}$ $\quad$ izsamemo

$v_1^{(2)} = (A - \lambda I) v_1^{(1)}$

$\vdots$

$v_1^{(k)} = (A - \lambda I) v_1^{(k-1)}$ $\quad$ last. vektor

$v_2^{(1)} \in \ker(A - \lambda I)^{k-1} - \ker(A - \lambda I)^{k-2} -$ $\quad$ lin. neodv. od vseh zgoraj

Enako za ostale λ

$$x = C_1 e^{\lambda t} v_1^{(k)} + C_2 e^{\lambda t}(t v_1^{(k)} + v_1^{(k-1)}) + \dots$$
$$+ C_k e^{\lambda t}\left(\frac{t^{k-1}}{(k-1)!} v_1^{(k)} + \dots + t v_1^{(2)} + v_1^{(1)}\right)$$

$+$ za ostale λ.