① Naj bosta $x, y \in \mathbb{R}^n$ in $M = xy^T$. Izračunaj $\|M\|_\infty$, $\|M\|_1$, $\|M\|_F$, $\|M\|_2$

$$x = (x_1, \dots, x_n)^T \qquad y = (y_1 \dots, y_n)^T$$

$$M = xy^T = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \begin{bmatrix} y_1 & \dots & y_n \end{bmatrix} = \begin{bmatrix} x_1 y_1 & x_1 y_2 & \cdots & x_1 y_n \\ \vdots & \vdots & \ddots & \vdots \\ x_n y_1 & x_n y_2 & \cdots & x_n y_n \end{bmatrix}$$

$$\|M\|_\infty = \max_{1 \le i \le n} \sum_{j=1}^{n} |m_{ij}| = \max_i \sum_{j=1}^{n} |x_i y_j| = \max_i \left(|x_i| \sum_{j=1}^{n} |y_j| \right) = \max_i \left(|x_i| \|y\|_1 \right) =$$

$$= \|y\|_1 \max_i |x_i| = \|y\|_1 \|x\|_\infty$$

$$\|M\|_1 = \max_{1 \le j \le n} \sum_{i=1}^{n} |m_{ij}| = \dots = \|x\|_1 \|y\|_\infty$$

$$\|M\|_F = \sqrt{\sum_{i,j=1}^{n} m_{ij}^2} = \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} x_i^2 y_j^2} = \sqrt{\sum_{i=1}^{n} x_i^2 \sum_{j=1}^{n} y_j^2} = \sqrt{\sum_{i=1}^{n} x_i^2 \|y\|_2^2} = \|y\|_2 \|x\|_2$$

$$\|M\|_2^2 = \lambda_{max}(M^T M) = \lambda_{max}\left((xy^T)^T (xy^T) \right) = \lambda_{max}\left(y x^T x y^T \right) = \lambda_{max}\left(\|x\|_2^2 y y^T \right) = \dots$$

$$(M^T M) y = \left(\|x\|_2^2 y y^T \right) y = \|x\|_2^2 y (y^T y) = \underbrace{\|x\|_2^2 \|y\|_2^2}_{\lambda} y = \lambda y$$

$$\geq \|x\|_2^2 \|y\|_2^2$$

Katere pa so ostale lastne vrednosti $M^T M$?

Vse ostale lastne vrednosti so enake 0, ker je matrika ranga 1.

② LU razcep z delnim pivotiranjem. Iščemo permutacijsko matriko P, spodnjo trikotno L z enkami na diagonali in zgornjo trikotno U, da je

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 2 & 0 & 1 & -1 \\ 4 & 1 & 0 & 1 \\ 1 & 1 & 2 & -3 \end{bmatrix} \qquad PA = LU$$

Matriko P lahko predstavimo z vektorjem $p = [i_1, i_2, \dots, i_n]$, kjer je $i_1, i_2, \dots, i_n$ permutacija števil od 1 do n. Pri tem velja, da ima matrika P v k-ti vrstici element 1 namesto i_k.

$p = [1, 2, 3, 4]$

$$\begin{matrix} 1 & 0 & 0 & 1 \\ 2 & 0 & 1 & -1 \\ 4 & 1 & 0 & 1 \\ 1 & 1 & 2 & -3 \end{matrix} \xrightarrow{\text{pivot.}}$$

$[3, 2, 1, 4]$

$$\begin{matrix} 4 & 1 & 0 & 1 \\ 2 & 0 & 1 & -1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 2 & -3 \end{matrix} \xrightarrow[\text{kvocient} \; \text{odštevanje}]{\text{eliminacija}} \begin{matrix} 4 & 1 & 0 & 1 \\ 1/2 & -1/2 & 1 & -3/2 \\ 1/4 & -1/4 & 0 & 3/4 \\ 1/4 & 3/4 & 2 & -13/4 \end{matrix} \xrightarrow{\text{pivot}}$$

$[3, 4, 1, 2]$

$$\begin{matrix} 4 & 1 & 0 & 1 \\ 1/4 & 3/4 & 2 & -13/4 \\ 1/4 & -1/4 & 0 & 3/4 \\ 1/2 & -1/2 & 1 & -3/2 \end{matrix}$$

$$\longrightarrow \begin{matrix} 4 & 1 & 0 & 1 \\ 1/4 & 3/4 & 2 & -13/4 \\ 1/4 & -1/3 & 2/3 & -1/3 \\ 1/2 & -2/3 & 7/3 & -11/3 \end{matrix} \xrightarrow{\text{pivot.}}$$

$[3, 4, 2, 1]$

$$\begin{matrix} 4 & 1 & 0 & 1 \\ 1/4 & 3/4 & 2 & -13/4 \\ 1/2 & -2/3 & 7/3 & -11/3 \\ 1/4 & -1/3 & 2/3 & -1/3 \end{matrix} \xrightarrow{\text{elim.}} \begin{matrix} 4 & 1 & 0 & 1 \\ 1/4 & 3/4 & 2 & -13/4 \\ 1/2 & -2/3 & 7/3 & -11/3 \\ 1/4 & -1/3 & 2/7 & 5/7 \end{matrix}$$

$$\Rightarrow \quad P = \begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix} \qquad L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/4 & 1 & 0 & 0 \\ 1/2 & -2/3 & 1 & 0 \\ 1/4 & -1/3 & 2/7 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 4 & 1 & 0 & 1 \\ 0 & 7/4 & 2 & -13/4 \\ 0 & 0 & 7/3 & -11/3 \\ 0 & 0 & 0 & 5/7 \end{bmatrix}$$

Algoritem za reševanje tridiag. sistema

$Ax = b$, $A \in \mathbb{R}^{n \times n}$, $b \in \mathbb{R}^n$

1. $$A = \begin{bmatrix} a_{11} & a_{12} & & 0 \\ a_{21} & a_{22} & \ddots & \\ & \ddots & \ddots & \ddots \\ 0 & & & \end{bmatrix}$$

$$M = \begin{bmatrix} 0 & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} \\ a_{32} & \vdots & \vdots \\ \vdots & & a_{n-1,n} \\ a_{n,n-1} & a_{nn} & 0 \end{bmatrix}, \qquad b = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

L (first column), U (second and third columns) ← se ne spremeni

2. „Pametno" izvajamo eliminacijo na matriki M

Algoritem

$$l_{21} = \frac{a_{21}}{a_{11}} \Rightarrow \tilde{a}_{22} = a_{22} - l_{21} a_{12}$$

$$\vdots$$

3. Računamo zaporedje približkov $x_{r+1} = g(x_r)$, $r = 0, 1, 2, \dots$ kjer je $g(x) = \frac{x^2 + a}{2x}$, $a > 0$

a) Kam lahko konvergira $(x_r)_{r=0}^{\infty}$?

b) Kakšen je red konvergence?

a) Konvergira lahko le k α: $\alpha = g(\alpha)$, α je negibna točka

Recimo, da je $\alpha = \lim_{r\to\infty} x_r$. Potem je $\lim_{r\to\infty} x_{r+1} = \lim_{r\to\infty} g(x_r)$

$$\alpha = g(\lim_{r\to\infty} x_r) = g(\alpha) = \frac{\alpha^2 + a}{2\alpha}$$

$$\Rightarrow 2\alpha^2 = \alpha^2 + a$$

$$\alpha = \pm\sqrt{a} \qquad \text{(uporabno za računanje korenov)}$$

b) Red konvergence:

- $g(\sqrt{a}) = \sqrt{a}$ preverimo negibnost točk

  $g(\sqrt{a}) = \frac{(\sqrt{a})^2 + a}{2\sqrt{a}} = \sqrt{a}$

- $g'(x) = \frac{2x \cdot 2x - (x^2 + a)2}{4x^2} = 1 - \frac{x^2 + a}{2x^2}$

  $g'(\sqrt{a}) = 1 - \frac{a + a}{2a} = 0 \Rightarrow$ red konvergence je vsaj 2.

- $g''(x) = -\frac{2x \cdot 2x^2 - (x^2 + a)4x}{4x^4} = -\frac{x^3 - x^3 - ax}{x^4} = \frac{a}{x^3} = a^{1 - 3/2} = \frac{1}{\sqrt{a}} \neq 0$

$$\Rightarrow \text{red konvergence je 2.}$$

Posplošitev metode za računanje k-tih korenov

$$g(x) = \frac{(k-1)\, x_r^{k} + a}{k\, x_r^{k-1}} \quad ; \quad a>0,\ k \in \mathbb{N} \quad k \geq 2$$

$\Rightarrow \quad (x_r)_{r=0}^{\infty}, \quad x_{r+1} = g(x_r)$ konvergira k k-temu korenu iz a $(\sqrt[k]{a})$

k Konvergenca Newtonove iteracije

$$f(x) = 0 \qquad x_{r+1} = x_r - \frac{f(x_r)}{f'(x_r)} \qquad r = 0, \dots$$

1 Naj bo α enostavna ničla f

$$f(x) = (x-\alpha)\, h(x) \qquad h(\alpha) \neq 0$$

Iteracijska funkcija za Newtonovo iteracijo je $g(x) = x - \frac{f(x)}{f'(x)}$

$$g(\alpha) = \alpha - \frac{\overbrace{f(\alpha)}^{=0}}{\underbrace{f'(\alpha)}_{\neq 0}} \quad \Rightarrow \quad g(\alpha) = \alpha \qquad \alpha \text{ negibna točka}$$

$$g'(x) = 1 - \frac{f'(x)^2 - f(x)\, f''(x)}{f'(x)^2} = 1 - 1 + \frac{f(x)\, f''(x)}{f'(x)^2}$$

$$g'(\alpha) = \frac{\overbrace{f(\alpha)}^{=0}\ \overbrace{f''(\alpha)}^{\neq 0}}{\underbrace{f'(\alpha)^2}_{\neq 0}} = 0 \quad \Rightarrow \quad \text{red konvergence vsaj } 2$$

2 Naj bo α m-kratna ničla ; $m \geq 2$

$$f(x) = (x-\alpha)^m\, h(x) \quad ; \quad h(\alpha) \neq 0$$

$$f'(x) = m\,(x-\alpha)^{m-1}\, h(x) + (x-\alpha)^m\, h'(x)$$
$$f''(x) = m\,(m-1)(x-\alpha)^{m-2}\, h(x) + m\,(x-\alpha)^{m-1}\, h'(x) + m\,(x-\alpha)^{m-1}\, h'(x) + (x-\alpha)^m\, h''(x)$$

$$g(x) = x - \frac{(x-\alpha)^m\, h(x)}{m\,(x-\alpha)^{m-1}\, h(x) + (x-\alpha)^m\, h'(x)}$$
$$= x - \frac{(x-\alpha)\, h(x)}{m\, h(x) + (x-\alpha)\, h'(x)}$$

$$g(\alpha) = \alpha - \frac{(\alpha-\alpha)\, h(\alpha)}{m\, h(\alpha)} = \alpha \qquad \text{negibna točka}$$

$$g'(x) = \frac{f(x)\, f''(x)}{f'(x)^2} = \frac{(x-\cancel{\alpha})^m\, h(x)\, (x-\cancel{\alpha})^{m-2} \left[m(m-1)\, h(x) + h'(x)\, 2m\,(x-\alpha) + (x-\alpha)^2\, h''(x) \right]}{(x-\cancel{\alpha})^{2m-2} \left(m\, h(x) + (x-\alpha)\, h'(x) \right)^2}$$

$$g'(\alpha) = \frac{h^2(\alpha)\ m\,(m-1)}{m^2\ h^2(\alpha)} = \frac{m-1}{m} = 1 - \frac{1}{m} \neq 0 \qquad m \geq 2$$

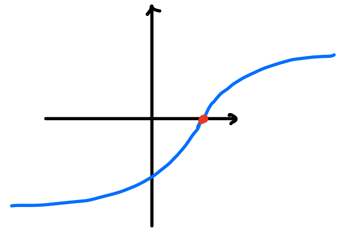
$\Rightarrow$ Konvergenca je linearna

Kdaj je red konvergence pri enostavnih ničlah več kot 2?

$$g'(x) = \frac{f(x)\, f''(x)}{f'(x)^2}$$

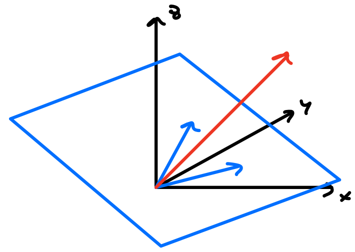
$$g''(x) = \frac{(f' f'' + f f''') f'^2 - f f'' 2 f' f''}{(f')^4}$$

$$g''(\alpha) = \frac{f'^3 f''}{f'^4} = \frac{f''(\alpha)}{f'(\alpha)}$$



Red konvergence bo vsaj 3, če je $f''(\alpha)=0$ (prevoj)

5 Kateri vektor je v ravnini, ki jo razpenjata $\vec{a} = (1,1,0)$ in $\vec{b} = (0,1,1)$, najbližji vektorju $\vec{c} = (1,2,3)$.



a) analitična rešitev

Iščem α, β $\|\alpha\vec{a} + \beta\vec{b} - \vec{c}\|_2$ minimum

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \vec{x} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \quad \vec{c} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\|A\vec{x} - \vec{c}\|_2 = \left\| \begin{bmatrix} \alpha - 1 \\ \alpha + \beta - 2 \\ \beta - 3 \end{bmatrix} \right\| = \sqrt{\underbrace{(\alpha-1)^2 + (\alpha+\beta-2)^2 + (\beta-3)^2}_{f(\alpha,\beta)}}$$

iščemo $\min\limits_{\alpha,\beta} \sqrt{f(\alpha,\beta)}$, min nastopi v isti točki kot $\min\limits_{\alpha,\beta} f(\alpha,\beta)$

$$\frac{\partial f}{\partial \alpha}(\alpha,\beta) = \cancel{2}(\alpha-1) + \cancel{2}(\alpha+\beta-2) = 2\alpha + \beta - 3 = 0$$

$$\frac{\partial f}{\partial \beta}(\alpha,\beta) = \cancel{2}(\alpha+\beta-2) + \cancel{2}(\beta-3) = \alpha + 2\beta - 5 = 0$$

$$\begin{aligned} 2\alpha + \beta &= 3 \\ \alpha + 2\beta &= 5 \end{aligned} \quad \Rightarrow \quad \alpha = \frac{1}{3} \quad \beta = \frac{7}{3}$$

Normalni sistem enačb $A^T A \vec{x} = A^T \vec{c}$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \qquad A^T c = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

b) QR razcep

$A = QR$ $\quad Q \in \mathbb{R}^{m \times n}$ z ortonormiranimi stolpci, $R \in \mathbb{R}^{n \times n}$ zgornje trikotna

6 Reši sistem enačb s QR razcepom s Householderjevimi zrcaljenji

$$\begin{bmatrix} 2 & 2 & 6 \\ 2 & 1 & -2 \\ 1 & 6 & -2 \end{bmatrix} X = \begin{bmatrix} 6 \\ -1 \\ -7 \end{bmatrix}$$

$Ax = b \Rightarrow QRx = b \Rightarrow Rx = Q^T b \Rightarrow$ rešimo zgornje trikotni sistem.

1 Iščemo $P^{(1)} A x = P^{(1)} b$ $\quad \begin{bmatrix} x & x & x \\ 0 & x & x \\ 0 & x & x \end{bmatrix} x = \begin{bmatrix} x \\ x \\ x \end{bmatrix}$

$$\left[\begin{array}{ccc|c} 2 & 2 & 6 & 6 \\ 2 & 1 & -2 & -1 \\ 1 & 6 & -2 & -7 \\ a_1 & a_2 & a_3 & \end{array}\right]$$

P_1 mora preslikati a_1 v $\begin{bmatrix} \lambda \\ 0 \\ 0 \end{bmatrix}$

$$P^{(1)} = I - \frac{2}{w^{(1)T} w^{(1)}} w^{(1)} w^{(1)T}$$

$$a_1 = (2, 2, 1)^T$$

$$w^{(1)} = a_1^{(1)} \pm \begin{bmatrix} \|a_1^{(1)}\|_2 \\ 0 \\ 0 \end{bmatrix}$$

izberemo tako, da je $\|w\|_2$ čim večja

$$w^{(1)} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} + \begin{pmatrix} \sqrt{2^2+2^2+1} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \\ 1 \end{pmatrix}$$

$$w^{(1)T} w^{(1)} = 30$$

$$P^{(1)} a_1^{(1)} = \left(I - \frac{2}{30} w^{(1)} w^{(1)T} \right) a^{(1)} =$$

$$= a^{(1)} - \frac{2}{30} w^{(1)} \left(w^{(1)T} a^{(1)} \right)$$

$$= \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} - \frac{2}{30} \begin{pmatrix} 5 \\ 2 \\ 1 \end{pmatrix} (2 \cdot 5 + 2 \cdot 2 + 1)$$

$$= \begin{pmatrix} -3 \\ 0 \\ 0 \end{pmatrix}$$

$$P^{(1)} a_2^{(1)} = a_2^{(1)} - \frac{2}{30} \left(w^{(1)T} a_2^{(1)} \right) w^{(1)}$$

$$= \begin{pmatrix} 2 \\ 1 \\ 6 \end{pmatrix} - \frac{2}{30} \cdot 18 \begin{pmatrix} 5 \\ 2 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ 1 \\ 6 \end{pmatrix} - \frac{6}{5} \begin{pmatrix} 5 \\ 2 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} -4 \\ -7/5 \\ 24/5 \end{pmatrix}$$

$$P^{(1)} a_3^{(1)} = a_3^{(1)} - \frac{2}{30} \left(w^{(1)T} a_3^{(1)} \right) w^{(1)}$$

$$= \begin{pmatrix} 6 \\ -2 \\ -2 \end{pmatrix} - \underbrace{\frac{2}{30} 24}_{\frac{8}{5}} \begin{pmatrix} 5 \\ 2 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} -2 \\ -26/5 \\ -18/5 \end{pmatrix}$$

$$P^{(1)} b = \begin{pmatrix} 6 \\ -1 \\ -7 \end{pmatrix} - \underbrace{\frac{2}{30} \cdot 21}_{\frac{7}{5}} \begin{pmatrix} 5 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -19/5 \\ -42/5 \end{pmatrix}$$

$$\begin{bmatrix} -3 & -4 & -2 \\ 0 & -\frac{7}{5} & -\frac{26}{5} \\ 0 & \frac{24}{5} & -\frac{18}{5} \end{bmatrix} x = \begin{bmatrix} -1 \\ -\frac{19}{5} \\ -\frac{42}{5} \end{bmatrix}$$

A_2 $\qquad$ b_2

$$A_2 = \left[a_1^{(2)} \quad a_2^{(2)} \right]$$

② $P^{(2)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & & \\ 0 & P_2 & \end{bmatrix}$: $P_2 A_2 = \begin{bmatrix} x & x \\ 0 & x \end{bmatrix}$ max $\|w_2\|$

$$P_2 = I_2 - \frac{2}{w_2^T w_2} w_2 w_2^T \ : \quad w_2 = \begin{bmatrix} -7/5 - \overset{5}{\left\| \begin{matrix} -7/5 \\ 24/5 \end{matrix} \right\|} \\ 24/5 \end{bmatrix} = \begin{bmatrix} -32/5 \\ 24/5 \end{bmatrix}$$

$$w_2 w_2^T = 64$$

$$P_2 a_1^{(2)} = a_1^{(2)} - \frac{2}{64}\frac{1}{25}\left(32 \cdot 7 + 24^2\right) w_2 = \dots = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$$

$$P_2 a_2^{(2)} = \dots = \begin{bmatrix} -2 \\ -6 \end{bmatrix}$$

$$P_2 b_2 = \dots = \begin{bmatrix} -7 \\ -6 \end{bmatrix}$$

$$P^{(1)} P^{(2)} A x = \begin{bmatrix} -1 \\ -7 \\ -6 \end{bmatrix}$$

$$\begin{bmatrix} -3 & -4 & -2 \\ 0 & 5 & -2 \\ 0 & 0 & -6 \end{bmatrix} x = \begin{bmatrix} -1 \\ -7 \\ -6 \end{bmatrix}$$

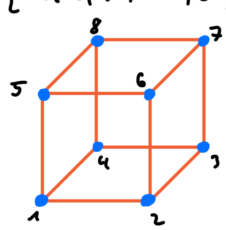
obratno vstavljanje

$\Rightarrow x_3 = 1 \Rightarrow x_2 = -1 \Rightarrow x_1 = 1$ $\qquad x = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

⑦ „Lepo" risanje grafov (diskretnih objektov) v 3D

Def Neusmerjeni graf G je par (V, E), kjer je V množica vozlišč in $E \subseteq V \times V$ pa množica povezav

Primer $V = \{1, 2, 3, \dots, 8\}$ $\quad E$ množica povezav



$E = \{(1,2), (2,3), \dots\}$

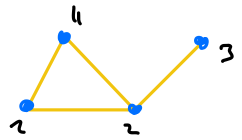
Reprezentacija grafa z matriko sosednjosti

$V = \{1, 2, \dots, n\}$ $\quad E = \{(i,j) ;\ i \in V_1 \subseteq V,\ j \in V_2 \subseteq V\}$

Definiramo mat. $A = (a_{ij})_{i,j=1}^{n}$

$$a_{ij} = \begin{cases} 1 ; & \text{če } i \text{ in } j \text{ povezani} \\ 0 ; & \text{sicer} \end{cases}$$

Primer



$$A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

Kako lahko A uporabimo za risanje grafa v $\mathbb{R}^3$?

Privzemimo, da je graf G kubičen (vsaka točka ima natanko 3 sosede)

V vsaki vrstici mat A so natanko 3 enke, zato je

$$A\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ \vdots \\ 3 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \Rightarrow 3 \text{ je lastna vrednost}$$

Ker je A realna simetrična ima n realnih lastnih vred. $\lambda_1 = 3 \ge |\lambda_2| \ge |\lambda_3| \dots |\lambda_n|$
Ima tudi n ortogonalnih lastnih vektorjev

Naj bodo x_2, x_3, x_4 normirani last. vekt., ki pripadajo lastnim vrednostim $\lambda_2, \lambda_3, \lambda_4$

$$x_j = \begin{bmatrix} x_{1j} \\ \vdots \\ x_{nj} \end{bmatrix} \qquad j = 2, 3, 4$$

Če za koordinate točk grafa izberemo $T_i = (x_{i2}, x_{i3}, x_{i4}) \in \mathbb{R}^3$ $i = 1, 2, \dots, n$ ter narišemo ustrezni graf, dobimo „lepo" sliko grafa.

8 Nevillov algoritem - interpolacija polinoma

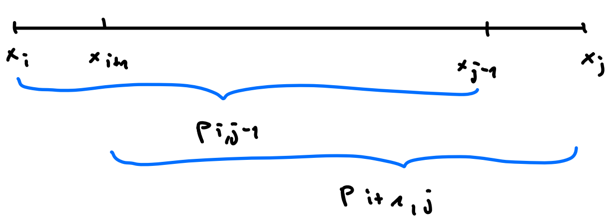
Podatki: $(x_0, f_0), (x_1, f_1), \dots (x_n, f_n)$ $\quad x_i \neq x_j$ za $\forall\, i \neq j$ $\quad x \in \mathbb{R}$

Rezultat: $p_n(x)$ vrednost interpolacijskega polinoma v danih točkah pri x

Oznaka: p_{ij} polinom st. $\le j-i$, ki interpolira vrednosti $(x_i, f_i), \dots, (x_j, f_j)$ $\quad j \ge i$

Lema: Polinom p_{ij} zadošča zvezam:

1. $p_{ii}(x) = f_i \qquad 0 \le i \le n$
2. $p_{ij}(x) = \dfrac{(x - x_j)\, p_{i,j-1}(x) - (x - x_i)\, p_{i+1,j}(x)}{(x_i - x_j)} \qquad 0 \le i < j \le n$



Dokaz: 1 je očitno

2 Velja: $i+1 \le k \le j-1 \Rightarrow p_{i,j-1}(x_k) = p_{i+1,j}(x_k) = f_k$

Opazimo: če je $\gamma \in \mathbb{R} \Rightarrow \gamma f_k + (1-\gamma) f_k = f_k$

Poskusimo p_{ij} zapisati kot

$$p_{ij}(x) = (\alpha x + \beta)\, p_{i,j-1} + (1 - (\alpha x + \beta))\, p_{i+1,j}(x)$$

$$p_{ij}(x_k) = f_k, \qquad i+1 \le k \le j-1$$

Za x_i in x_j pa more veljati:

$$p_{ij}(x_i) = (\alpha x_i + \beta)\, \overbrace{p_{i,j-1}(x_i)}^{f_i} + (1 - (\alpha x_i + \beta))\, p_{i+1,j}(x_i) = f_i$$

$$p_{ij}(x_j) = (\alpha x_j + \beta)\, p_{i,j-1}(x_j) + (1 - (\alpha x_j + \beta))\, \underbrace{p_{i+1,j}(x_j)}_{f_j} = f_j$$

$$\Rightarrow \quad 1 - \alpha x_i - \beta = 0 \qquad \alpha x_j + \beta = 0$$

$$\Rightarrow \quad \alpha = \frac{1}{x_i - x_j} \qquad \beta = -\frac{x_j}{x_i - x_j}$$

Če α in β vstavimo v enačbo dobimo Nevilleovo formulo

Primer: Izračunaj vrednost inter. polinoma st. ≤ 2, ki interp.

x_i	-1	0	2
y_i	-2	-1	7

pri $x = 1$

x_i	f_i		
-1	-2		
		0	
0	-1		$\boxed{2}$
		3	
2	7		

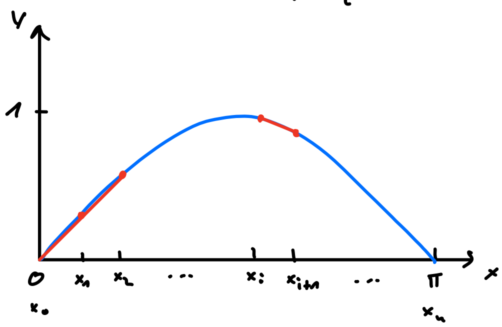
$$P_{01}(1) = \frac{(1-0)(-2) - (1+1)(-1)}{-1-0} = 0$$

$$P_{12}(1) = \frac{(1-2)(-1) - (1-0)7}{0-2} = 3$$

$$P_{02}(1) = \frac{(1-2)0 - (1+1)3}{-1-2} = \boxed{2}$$

rešitev

9. Funk. $f = \sin x$ aproks. z odsekoma linearno funkcijo ℓ na $n+1$ ekvidistančnih točkah $x_0 = 0 < x_1 < \ldots < x_n = \pi$ tako, da ℓ interpolira dano točk, na vsakem podintervalu $[x_i, x_{i+1}]$, $i = 0, \ldots, n-1$ pa je ℓ daljica med interp. točkami. Kolikšen mora biti n, da so napake aproks. pod 10^{-7}.



Zanima nas napaka

$$\ell : [x_0, x_n] \to \mathbb{R} \qquad \ell\big|_{[x_i, x_{i+1}]} = \ell_i \quad \text{daljica}$$

Zanima nas

$$\max_{x_0 \leq x \leq x_n} |f(x) - \ell(x)| = ?$$

Vemo, da je $x_{i+1} - x_i = h$

$$\max_{x_0 \leq x \leq x_n} |f(x) - \ell(x)| = \max_{0 \leq i \leq n-1} \max_{x_i \leq x \leq x_{i+1}} |f(x) - \ell_i(x)|$$

Naj bo $x \in [x_i, x_{i+1}]$:

$$|f(x) - \ell_i(x)| = \left| \frac{f^{(2)}(\xi_i)}{2!} \omega_i \right| \leq \ldots$$

$$\omega_i(x) = (x - x_i)(x - x_{i+1}) \quad ; \quad \xi_i \in [x_i, x_{i+1}]$$

$$\left| \frac{f^{(2)}(\xi_i)}{2!} \right| \leq \frac{1}{2} \max_{x_i \leq \xi \leq x_{i+1}} |-\sin(\xi)| = \frac{1}{2} S_i$$

$$\omega_i(x) = (x - x_i)(x - x_{i+1}) = (x - x_i)(x - x_i - h)$$

$$\omega_i'(x) = (x - x_i - h) + (x - x_i) = 2x - 2x_i - h = 0 \qquad x = x_i + \frac{h}{2}$$

$$\omega_i\left(x_i + \frac{h}{2}\right) = \left(x_i + \frac{h}{2} - x_i\right)\left(x_i + \frac{h}{2} - x_i - h\right) = -\frac{h^2}{4}$$

$$\dots \leq \frac{1}{2} s_i \frac{h^2}{4}$$

$$\max_{0 \leq i \leq n-1} \max_{x_i \leq x \leq x_{i+1}} |f(x) - l_i(x)| \leq \max_i \frac{h^2}{8} s_i = \frac{h^2}{8}$$

$$h = \sqrt{8 \cdot 10^{-3}} \Rightarrow h < 0{,}0894 \qquad n \geq \frac{\pi}{h} = 35{,}12$$

10 Modeliranje procesa pada padalca

Začetni problem višjega reda

$$y^{(n)}(x) = f(x, y(x), y'(x), \dots, y^{(n-1)}(x))$$
$$y(a) = y_a \quad y'(a) = y_a' \quad \dots \quad y^{(n-1)}(a) = y_a^{(n-1)} \qquad \text{Začetni pogoji}$$

$$y_1 = y \quad y_2 = y' \quad \dots \quad y_n = y^{(n-1)}$$

$$\begin{aligned} y_1' &= y' = y_2 \\ y_2' &= y'' = y_3 \\ &\vdots \\ y_{n-1}' &= y^{(n-1)} = y_n \\ y_n' &= y^{(n)} = f(x, y_1, \dots, y_n) \end{aligned} \qquad \text{Začetni problem 1. reda (sistem)}$$

Model padanja

$$\ddot{y} = -g + \frac{1}{2} \frac{\rho S c}{m} \dot{y}^2$$