Ocenjevanje natančnosti

$$u = f(x_1, x_2, \dots, x_n)$$

$$u + \Delta u = f(x_1+\Delta x_1, \dots, x_n+\Delta x_n) = \dots \quad ; \quad \Delta x_i \ll x_i$$

absolutna natančnost

$$\dots = \underbrace{f(x_1, \dots, x_n)}_{u} + \frac{\partial f}{\partial x_1}\Delta x_1 + \dots + \frac{\partial f}{\partial x_n}\Delta x_n + O(\Delta x_i^2)$$

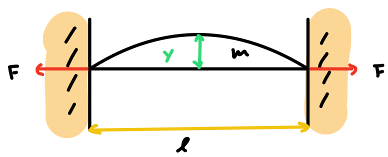
$$\boxed{\Delta u = \sum_{i=1}^{N} \frac{\partial f}{\partial x_i}\Big|_{x_i} \Delta x_i}$$

Naj bo $\vec{x} = (x_1, \dots, x_n)^T$ $\quad \Delta\vec{x} = (\Delta x_1, \dots, \Delta x_n)^T$ $\quad \frac{\partial f}{\partial \vec{x}} = \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right)$

$$\Delta u = \Delta\vec{x} \cdot \frac{\partial f}{\partial \vec{x}}\Big|_{\vec{x}}$$

$$\boxed{\frac{\Delta u}{u} = \frac{1}{f}\,\Delta\vec{x}^T \frac{\partial f}{\partial \vec{x}}\Big|_{\vec{x}}}$$ relativna natančnost

1. Oceni natančnost zvenenja strune $\quad \frac{\Delta \nu}{\nu} = ?$



$$c = \nu\lambda = \sqrt{\frac{F}{\mu}} \qquad \mu = \frac{m}{l}$$

$$\nu = \frac{1}{\lambda}\sqrt{\frac{F}{\mu}} = f(F, \mu)$$

Pri nihanju se spreminja F in l (dolžina strune)

$$\vec{x} = \begin{pmatrix} F \\ \mu \end{pmatrix} \qquad \Delta\vec{x} = \begin{pmatrix} \Delta F \\ \Delta\mu \end{pmatrix}$$

$$\frac{\partial f}{\partial \vec{x}} = \begin{bmatrix} \frac{1}{2\lambda\sqrt{\mu F}} \\ -\frac{1}{2\lambda}\sqrt{\frac{F}{\mu^3}} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\frac{\nu}{F} \\ -\frac{1}{2}\frac{\nu}{\mu} \end{bmatrix}$$

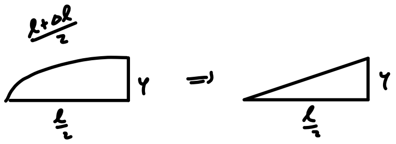
$$\frac{\Delta\nu}{\nu} = \frac{1}{\nu}\left(\Delta F \frac{1}{2}\frac{\nu}{F} - \Delta\mu \frac{1}{2}\frac{\nu}{\mu}\right) = \frac{1}{2}\left(\frac{\Delta F}{F} - \frac{\Delta\mu}{\mu}\right)$$

- $\frac{\Delta F}{S} = E\frac{\Delta l}{l}$ Hookov zakon $\Rightarrow \Delta F = ES\frac{\Delta l}{l}$
- $\mu = \frac{m}{l} \longrightarrow \Delta\mu = -\frac{m}{l^2}\Delta l \Rightarrow \frac{\Delta\mu}{\mu} = -\frac{\Delta l}{l}$

$$\Rightarrow \frac{\Delta\nu}{\nu} = \frac{1}{2}\left(\frac{ES}{F} + 1\right)\frac{\Delta l}{l}$$

Zanima nas koliko amplituda y vpliva na ν.

Ocenimo



$$\left(\frac{l+\Delta l}{2}\right)^2 = y^2 + \left(\frac{l}{2}\right)^2$$

$$l^2 + \underset{\gg}{2l\Delta l} + \underset{=0}{\Delta l^2} = 4y^2 + l^2$$

zanemarimo

$$\frac{\Delta l}{l} = 2\frac{y^2}{l^2}$$

$$\Rightarrow \frac{\Delta\nu}{\nu} = \left(\frac{ES}{F} + 1\right)\frac{y^2}{l^2}$$

Če želimo, da ν strune ne odstopa zelo mora biti $y \ll l$ in $\frac{ES}{F} \ll 1$

② Koliko časa moramo poslušati ton, da ga lahko določimo na pol tona natančno.
$\Delta t = ?$ $\quad \nu_0 = 440\,\mathrm{Hz}$ $\quad \Delta\nu$

Oktava
12 poltonov

$\nu_1 \quad 2\nu_0 \quad 880\,\mathrm{Hz}$

$\hat{\nu}_i$ ... i-ti polton

$\nu_0 \quad \Delta\nu \quad 440\,\mathrm{Hz}$

ni linearno

$\hat{\nu}_0 = \nu_0$

$\hat{\nu}_{12} = 2\nu_0 = \nu_1$

Poltoni so določeni kot

def: $\dfrac{\hat{\nu}_{i+1}}{\hat{\nu}_i} = x = \text{konst.}$

$$\Delta\nu = \hat{\nu}_1 - \hat{\nu}_0$$
$$= x\hat{\nu}_0 - \hat{\nu}_0 = \nu_0(x-1)$$

Princip nedoločenosti (za signale)

$$\underset{\text{pasovna širina}}{\Delta\nu}\ \underset{\text{časovno okno}}{\Delta t} \approx 1$$

čas poslušanja

$$\Delta t \approx \frac{1}{\Delta\nu}$$

$\hat{\nu}_1 = x\nu_0$

$\hat{\nu}_2 = x\nu_1 = x^2\hat{\nu}_0$

$\vdots$

$\hat{\nu}_{12} = x^{12}\hat{\nu}_0 \quad \Rightarrow \quad \dfrac{\nu_1}{\nu_0} = x^{12} = \dfrac{2\nu_0}{\nu_0}$

$x = \sqrt[12]{2}$

$$\Rightarrow \Delta\nu = \nu_0(\sqrt[12]{2} - 1) \approx 26\,\mathrm{Hz}$$

$$\Delta t = \frac{1}{\nu_0(\sqrt[12]{2} - 1)} \approx 40\,\mathrm{ms}$$

Optimalno združevanje (odvisnih) meritev

- Ocenjujemo/merimo neznano konstantno količino x.
- Laboratorij A izmeri meritve $z_1^{(a)}, \dots, z_m^{(a)} \Rightarrow (\bar{z}_a, \sigma_a^2)$

  B: $z_1^{(b)}, \dots, z_n^{(b)} \Rightarrow (\bar{z}_b, \sigma_b^2)$

Želimo optimalno združiti meritve v $(\hat{x}, \hat{\sigma}^2)$

$\hookrightarrow \hat{\sigma}^2$ je minimalna

porazdeljene po

$$\bar{z}_a \sim N(x, \sigma_a^2)$$
$$\bar{z}_b \sim N(x, \sigma_b^2)$$

normalna porazdelitev

$$\bar{z}_a = x + \underset{\text{šum}}{r_a} \quad ; \quad r_a \sim N(0, \sigma_a^2)$$
$$\bar{z}_b = x + r_b \quad ; \quad r_b \sim N(0, \sigma_b^2)$$

Pričakovana vrednost

$$E[\bar{z}_a] = \langle \bar{z}_a \rangle = x$$
$$\mathrm{Var}[\bar{z}_a] = \langle (\bar{z}_a - \langle \bar{z}_a \rangle)^2 \rangle = \langle (\bar{z}_a - x)^2 \rangle = \langle r_a^2 \rangle = \sigma_a^2$$

varianca

$$\langle \hat{x} \rangle = x \qquad \hat{x} = x + \hat{r} \qquad \hat{r} \sim N(0, \hat{\sigma}^2)$$

Nastavek

$$\hat{x} = a\bar{z}_a + b\bar{z}_b$$
$$\langle \hat{x} \rangle = x = a\langle \bar{z}_a \rangle + b\langle \bar{z}_b \rangle = (a+b)x \quad \Rightarrow \quad a+b = 1$$
$$\hat{x} = \bar{z}_a + b(\bar{z}_b - \bar{z}_a)$$

Iščemo optimalen b

$$\hat{x} = x + \hat{r} = a(x + r_a) + b(x + r_b) = x(a+b) + ar_a + br_b$$

$$\Rightarrow \hat{r} = ar_a + br_b = (1-b)r_a + br_b$$

$$\hat{\sigma}^2 = \langle \hat{r}^2 \rangle = \langle ((1-b)r_a + br_a)^2 \rangle = (1-b)^2 \langle r_a^2 \rangle + 2(1-b)b\langle r_a r_b \rangle + b^2 \langle r_b^2 \rangle$$

$$= (1-b)^2 \sigma_a^2 + b^2 \sigma_b^2 + 2(1-b)b \langle r_a r_b \rangle$$

$\hookrightarrow$ kovarianca $\boxed{\sigma_{ab} = \langle r_a r_b \rangle}$

$$0 = \frac{\partial \hat{\sigma}^2}{\partial b} \Rightarrow \text{optimalen } b$$

Odvisnost meritev/ocen
(zanima nas predvsem linearna odvisnost, koreliranost)

$$r_b = \alpha r_a + w$$

$\hookrightarrow$ $w \sim N(0, \sigma_w^2)$
neodvisen od r_a

$\langle w r_a \rangle = 0$
ker sta neodvisna/nekorelirana

$$\sigma_b^2 = \langle r_b^2 \rangle = \langle (\alpha r_a + w)^2 \rangle = \alpha^2 \langle r_a^2 \rangle + \langle w^2 \rangle + 2\alpha \underbrace{\langle r_a w \rangle}_{0}$$

$$\sigma_b^2 = \alpha^2 \sigma_a^2 + \sigma_w^2 \quad / \sigma_b^2$$

$$1 = \Big(\underbrace{\alpha \frac{\sigma_a}{\sigma_b}}_{\le 1}\Big)^2 + \Big(\underbrace{\frac{\sigma_w}{\sigma_b}}_{\le 1}\Big)^2$$

Korelacijski koeficient $\rho_{ab} = \alpha \frac{\sigma_a}{\sigma_b}$

$$0 \le \rho_{ab}^2 \le 1 \Rightarrow -1 \le \rho_{ab} \le 1$$

nekorelirane (neodvisne meritve) ... popolno korelirane ... antikorelacija ... korelacija

Kovarianca

$$\sigma_{12} = \langle r_1 r_2 \rangle = \langle r_1 (\alpha r_1 + w) \rangle = \alpha \langle r_1 r_1 \rangle + \langle r_1 w \rangle = \alpha \sigma_1^2 + 0$$

$$\rho_{12} = \alpha \frac{\sigma_1}{\sigma_2} \Rightarrow \alpha \sigma_1 = \rho_{12} \sigma_2 \Rightarrow \sigma_{12} = \rho_{12} \sigma_1 \sigma_2$$

Optimalno združevanje

$$\left.\begin{aligned} \bar{z}_1 &= x + r_1 \\ \bar{z}_2 &= x + r_2 \end{aligned}\right\} \quad \hat{x} = x + \hat{r}$$

izostrena ocena za x

$$\langle \hat{x} \rangle = x$$

$$\hat{\sigma}^2 = \langle (\hat{x} - x)^2 \rangle = \langle \hat{r}^2 \rangle$$

varianca izostrene ocene

$$\hat{x} = a\bar{z}_1 + b\bar{z}_2 \quad \text{(nastavek)}$$

$$\hat{x} = a(x + r_1) + b(x + r_2) = x(a+b) + ar_1 + br_2$$

$$\langle \hat{x} \rangle = (a+b)\langle x \rangle = (a+b)x = x \Rightarrow a + b = 1 \Rightarrow a = 1 - b$$

$$\hat{x} = \bar{z}_1 + b(\bar{z}_2 - \bar{z}_1)$$

Želimo, da je $\hat{\sigma}^2$ minimalen $\Rightarrow b_{opt}$

$$\hat{\sigma}^2 = \langle \hat{r}^2 \rangle = \langle ((1-b)r_1 + br_2)^2 \rangle = (1-b)^2 \sigma_1^2 + b^2 \sigma_2^2 + 2b(1-b)\langle r_1 r_2 \rangle$$

kovarianca

$$\frac{d\hat{\sigma}^2}{ds} = 0 = -2(1-s)\,\sigma_1^2 + 2s\,\sigma_2^2 + 2(1-2s)\,\sigma_{12}$$

$$\Rightarrow \quad s\,(\sigma_1^2 + \sigma_2^2 - 2\sigma_{12}) = \sigma_1^2 - \sigma_{12}$$

$$s_{opt} = \frac{\sigma_1^2 - \sigma_{12}}{\sigma_1^2 + \sigma_2^2 - 2\sigma_{12}}$$

$$\Rightarrow \quad \boxed{\hat{x} = \bar{z}_1 + \underbrace{\frac{\sigma_1^2 - \sigma_{12}}{\sigma_1^2 + \sigma_2^2 - 2\sigma_{12}}}_{\substack{\text{ojačevalni faktor} \\ K}} \underbrace{(\bar{z}_2 - \bar{z}_1)}_{\text{inovacija}}} \quad \text{optimalno}$$

$$\hat{\sigma}^2 = (1-s)^2\sigma_1^2 + s^2\sigma_2^2 + 2s(1-s)\,\sigma_{12} \,\Big|_{s=s_{opt}}$$

$$1 - s_{opt} = \frac{\sigma_1^2 + \sigma_2^2 - 2\sigma_{12} - \sigma_1^2 + \sigma_{12}}{\sigma_1^2 + \sigma_2^2 - 2\sigma_{12}} = \frac{\sigma_2^2 - \sigma_{12}}{\underbrace{\sigma_1^2 + \sigma_2^2 - 2\sigma_{12}}_{I}}$$

$$\hat{\sigma}^2 = \frac{1}{I^2}\Big((\sigma_2^2 - \sigma_{12})^2\sigma_1^2 + (\sigma_1^2 - \sigma_{12})^2\sigma_2^2 + 2(\sigma_1^2 - \sigma_{12})(\sigma_2^2 - \sigma_{12})\,\sigma_{12} \Big) =$$

$$= \frac{1}{I^2}\Big((\sigma_1^2 - \sigma_{12})\big[\underbrace{(\sigma_1^2 - \cancel{\sigma_{12}})\,\sigma_2^2 + (\cancel{\sigma_2^2} - \sigma_{12})\,\sigma_{12}}_{\sigma_1^2\sigma_2^2 - \sigma_{12}^2}\big] + (\sigma_2^2 - \sigma_{12})\big[\underbrace{(\sigma_2^2 - \cancel{\sigma_{12}})\,\sigma_1^2 + (\cancel{\sigma_1^2} - \sigma_{12})\,\sigma_{12}}_{\sigma_2^2\sigma_1^2 - \sigma_{12}^2}\big]\Big)$$

$$= \frac{1}{I^2}(\sigma_1^2\sigma_2^2 - \sigma_{12}^2)\,(\underbrace{\sigma_1^2 + \sigma_2^2 - 2\sigma_{12}}_{I}) =$$

$$= \frac{\sigma_1^2\sigma_2^2 - \sigma_{12}^2}{\sigma_1^2 + \sigma_2^2 - 2\sigma_{12}} = \qquad\qquad \sigma_{12} = \rho_{12}\,\sigma_1\sigma_2$$

$$= (1 - \rho_{12}^2)\,\frac{\sigma_1^2\sigma_2^2}{\sigma_1^2 + \sigma_2^2 - 2\sigma_{12}}$$

$$\boxed{\hat{\sigma}^2 = (1 - \rho_{12}^2)\left(\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2} - \frac{2\rho_{12}}{\sigma_1\sigma_2}\right)^{-1}} \quad \text{optimalna varianca}$$

3 $\sigma_1 = \sigma_2 = \sigma, \quad \rho_{12}$ poljuben

$$K = \frac{\sigma_1^2 - \sigma_{12}}{\sigma_1^2 + \sigma_2^2 - 2\sigma_{12}} = \frac{\sigma^2 - \rho\,\sigma^2}{2\sigma^2 - 2\rho\,\sigma^2} = \frac{1}{2}$$

$$\Rightarrow \quad \hat{x} = \frac{1}{2}(\bar{z}_1 + \bar{z}_2) \qquad \text{povprečenje}$$

Varianca povprečja odvisnih meritev

- imamo N odvisnih izmerkov (prej smo imeli le dva)

  $z_i \sim N(x, \sigma_i^2) \qquad \sigma_{ij} \neq 0 \qquad i = 1, \dots, N$

  $\sigma_i^2 = \langle (z_i - x)^2 \rangle \qquad \sigma_{ij} = \langle (z_i - x)(z_j - x) \rangle$

- zanima nas varianca σ_N^2 povprečja $\bar{z}$ ← ni optimalno združevanje

$$\bar{z} = \frac{1}{N}\sum_i z_i \quad : \quad \langle \bar{z} \rangle = \left\langle \frac{1}{N}\sum_i z_i \right\rangle = \frac{1}{N}\sum_i \langle z_i \rangle = x$$

varianca povprečja $\sigma_N^2 = \langle (\bar{z} - x)^2 \rangle = \left\langle \left(\frac{1}{N}\sum z_i - x\right)^2 \right\rangle = \frac{1}{N^2}\left\langle \left(\sum_i z_i - Nx\right)^2 \right\rangle =$

$$= \frac{1}{N^2}\left\langle \left(\sum_i (z_i - x)\right)^2 \right\rangle = \frac{1}{N^2}\left\langle \left(\sum_i (z_i - x)^2 + \sum_i \sum_{j,\, i \neq j} (z_i - x)(z_j - x)\right) \right\rangle =$$

$$= \frac{1}{N^2}\left\langle \left(\sum_i (z_i - x)^2 + 2\sum_i \sum_{i<j} (z_i - x)(z_j - x) \right) \right\rangle =$$

$$= \frac{1}{N^2}\left(\sum_i \sigma_i^2 + 2\sum_{i<j} \sigma_{ij} \right)$$

Primer: $\sigma_{ij} = 0$ neodvisne meritve

$$\Rightarrow \sigma_\mu^2 = \frac{1}{N^2}\sum_i \sigma_i^2$$

če $\sigma_i = \sigma \Rightarrow \sigma_\mu^2 = \frac{\sigma^2}{N} \Rightarrow \sigma_\mu = \frac{\sigma}{\sqrt{N}}$ napaka se zmanjšuje

4. Na vrhu Storžiča z GPS dvakrat izmerimo nadmorsko višino in dobimo $h_1 = (2139 \pm \underbrace{12}_{\sigma_1})\,m$ in $h_2 = (2130 \pm \underbrace{6}_{\sigma_2})\,m$, $\rho_{12} = 0$.

a) Optimalno združi meritvi

$$\bar{h} = h_1 + \frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2}(h_2 - h_1) = 2131{,}8\ m$$

$$\sigma^2 = \left(\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2} \right)^{-1} = 28{,}8\ m^2 \Rightarrow \sigma = 5{,}4\ m\ (< \sigma_2)$$

b) Izračunaj povprečno višino in napako

$$\bar{h} = \frac{h_1 + h_2}{2} = 2134{,}5\ m \qquad \sigma^2 = \frac{1}{4}(\sigma_1^2 + \sigma_2^2) = 45\ m^2 \Rightarrow \sigma = 6{,}7\ m\ (> \sigma_2)$$

nismo izkoristili meritve

Kalmanov filter

algoritem za optimalno združevanje neodvisnih izmerkov neznane konstantne količine.

$$\underbrace{z_i}_{\text{izmerki}} = x + \underbrace{r_i}_{\text{merilni šum}}; \quad \langle r_i \rangle = 0 \quad \underbrace{\langle r_i r_j \rangle = \delta_{ij}\sigma_i^2}_{\text{nekoreliran šum}}$$

- Predpostavimo: $(\hat{x}_n, \hat{\sigma}_n^2) \to$ optimalna ocena po n izmerkih
- v $n+1$ imamo novo meritev $(z_{n+1}, \sigma_{n+1}^2)$
- optimalno združimo $(\hat{x}_n, \hat{\sigma}_n^2)$ in $(z_{n+1}, \sigma_{n+1}^2)$

$$\boxed{\begin{aligned} \hat{x}_{n+1} &= \hat{x}_n + \frac{\hat{\sigma}_n^2}{\hat{\sigma}_n^2 + \sigma_{n+1}^2}(z_{n+1} - \hat{x}_n) \\ \hat{\sigma}_{n+1}^2 &= \left(\frac{1}{\hat{\sigma}_n^2} + \frac{1}{\sigma_{n+1}^2} \right)^{-1} \end{aligned}}$$

Kalmanov filter za sledenje konstantni količini

Primer $n = 0$

$$\hat{x}_1 = \hat{x}_0 + \frac{\hat{\sigma}_0^2}{\hat{\sigma}_0^2 + \sigma_1^2}(z_1 - \hat{x}_0)$$

izostreni pred 1. meritvijo

$\hat{x}_0$ ... ne poznamo, lahko je karkoli

$\hat{\sigma}_0^2 \to \infty$

$$\hat{x}_1 = \hat{x}_0 + \lim \frac{\hat{\sigma}_0^2}{\hat{\sigma}_0^2 + \sigma_1^2}(z_1 - \hat{x}_0) = z_1$$

$\Rightarrow \hat{\sigma}_1^2 = \sigma_1^2 \qquad \Rightarrow (\hat{x}_1, \hat{\sigma}_1^2) = (z_1, \sigma_1^2)$ začetna vrednost

Alternativna oblika

- Kalmanov filter je poseben primer za sprotno združevanje izmerkov (npr. senzor)
- Kaj pa v primeru, da smo zbrali vseh M meritev, npr. (z_i, σ_i^2), $i=1,\dots,M$

⇒ Uteženo povprečevanje

$$\boxed{\hat{x} = \frac{\sum_i \frac{z_i}{\sigma_i^2}}{\sum_i \sigma_i^{-2}}} \quad ; \quad w_i = \frac{1}{\sigma_i^2} \; \dots \; \text{uteži}$$

optimalno

$$\Rightarrow \quad \hat{x} = \frac{\sum w_i z_i}{\sum w_i}$$

$$\boxed{\hat{\sigma}^{-2} = \sum_i \sigma_i^{-2}}$$

DN: preveri da je ta oblika ekvivalentna Kalmanovemu filtru

Širjenje napak: povprečna absolutna napaka po Gaussu.

za 1 spremenljivko: $u = f(x,y)$

poznamo $(\bar{x}, \sigma_{\bar{x}}^2)$, $(\bar{y}, \sigma_{\bar{y}}^2)$, zanima nas $(\bar{u}, \sigma_{\bar{u}}^2)$

$$\sigma_{\bar{u}}^2 = \langle (\bar{u} - u)^2 \rangle = \langle (\overline{f(x,y)} - f(x,y))^2 \rangle$$

Razvoj po Taylorju $f(x,y) = f(\bar{x},\bar{y}) + \underbrace{\frac{\partial f}{\partial x}\Big|_{(\bar{x},\bar{y})}}_{\text{konst}} \underbrace{(x-\bar{x})}_{\langle x-\bar{x}\rangle = 0} + \frac{\partial f}{\partial y}\Big|_{(\bar{x},\bar{y})} (y-\bar{y}) + \dots$

$$\overline{f(x,y)} = f(\bar{x},\bar{y}) + 0$$

$$\Rightarrow \quad \sigma_u^2 = \langle (f(\bar{x},\bar{y}) - f(x,y))^2 \rangle$$

$$\boxed{\sigma_u^2 = \left(\frac{\partial f}{\partial x}\right)^2 \sigma_x^2 + \left(\frac{\partial f}{\partial y}\right)^2 \sigma_y^2 + 2 \frac{\partial f}{\partial x} \frac{\partial f}{\partial y} \underbrace{\sigma_{xy}}_{\rho_{xy}\sigma_x\sigma_y} \Bigg|_{(\bar{x},\bar{y})}}$$

Primer
1. $u = f(x,y) = ax + by$
2. $u = f(x,y) = A x^{\delta} y^{\varphi}$

naj bo $\rho_{\bar{x}\bar{y}} = 0$

1. $\bar{u} = f(\bar{x},\bar{y}) = a\bar{x} + b\bar{y}$
   $\sigma_u^2 = a^2 \sigma_{\bar{x}}^2 + b^2 \sigma_{\bar{y}}^2$

2. $\bar{u} = f(\bar{x},\bar{y}) = A \bar{x}^{\delta} \bar{y}^{\varphi}$
   $\sigma_{\bar{u}}^2 = (A\delta \bar{x}^{\delta-1} \bar{y}^{\varphi})^2 \sigma_{\bar{x}}^2 + (A\varphi \bar{x}^{\delta} \bar{y}^{\varphi-1})^2 \sigma_{\bar{y}}^2$

$$\left(\frac{\sigma_{\bar{u}}}{\bar{u}}\right)^2 = \delta^2 \frac{\sigma_{\bar{x}}^2}{\bar{x}^2} + \varphi^2 \frac{\sigma_{\bar{y}}^2}{\bar{y}^2} \qquad \text{relativna napaka}$$

Normalna (Gaussova) porazdelitvena funkcija

Naj bo z porazdeljena po $z \sim N(\mu, \sigma^2)$

$$z \sim p(z) = \frac{dP}{dz} = \frac{1}{\sqrt{2\pi}\,\sigma}\, e^{-\frac{(z-\mu)^2}{2\sigma^2}}$$

Verjetnost P da $z \le x$:

$$P(z \le x) = \int_{-\infty}^{x} \frac{dP}{dz}\, dz \overset{\text{def.}}{=} F(x; \mu, \sigma^2)$$

kumulativna porazdelitvena funkcija

uvedemo novo spremenljivko $u = \frac{z-\mu}{\sigma}$

$$u \sim \frac{dP}{du} = \frac{dP}{dz}\frac{dz}{du} = \sigma \frac{dP}{dz} = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{u^2}{2}} = N(0,1)$$

standardizirana normalna funkcija

def: $F(x) = \int_{-\infty}^{x} \frac{dP}{du}\, du = \mathrm{erf}(x)$

$$F(-\infty) = 0, \quad F(\infty) = 1, \quad F(0) = \tfrac{1}{2}, \quad F(-x) = 1 - F(x)$$

Zanima nas

$$P(\alpha < z < \beta) = P(z < \beta) - P(z < \alpha) = F(u_\beta) - F(u_\alpha) = F\left(\frac{\beta-\mu}{\sigma}\right) - F\left(\frac{\alpha-\mu}{\sigma}\right)$$

$u = \frac{z-\mu}{\sigma}$, $u_\alpha = \frac{\alpha-\mu}{\sigma}$, $u_\beta = \frac{\beta-\mu}{\sigma}$

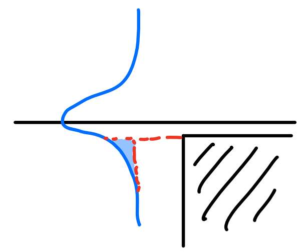
Primer $P(\mu-\sigma < z < \mu+\sigma) = F(1) - F(-1) = 2\underbrace{F(1)}_{0{,}8413} - 1 \approx 0{,}68 = 68\% \approx \frac{2}{3}$

$P(\mu-2\sigma < z < \mu+2\sigma) = 95\%$

$P(\mu-3\sigma < z < \mu+3\sigma) = 99{,}7\%$

$$P(\mu - n\sigma < z < \mu + n\sigma) = 2F(n) - 1$$

Primer: $\sigma_{Eu} = 0{,}04\, Eu$

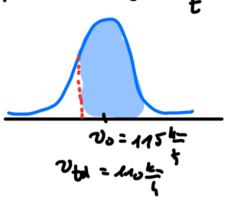


Podsoj $= P(Eu < E_{bar})$

$$= F\left(\frac{0{,}99\, Eu - Eu}{0{,}04\, Eu}\right)$$

$$= F\left(-\frac{0{,}01}{0{,}04}\right) = F(-0{,}25) = 1 - F(0{,}25) \approx 0{,}4$$

DN $\quad v = \frac{s}{t}$



$$\sigma_v^2 = \frac{1}{t^2}\sigma_s^2 + \left(\frac{l_0}{t^2}\right)^2 \sigma_t^2$$

$$\frac{\sigma_v^2}{v_0^2} = \frac{\sigma_s^2}{l_0^2} + \frac{\sigma_t^2}{t^2} = 0{,}0058^2 + 0{,}0005^2 \Rightarrow \sigma_v$$

$$\sigma_x^2 = (1-\rho_{12}^2)\left(\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2} - \frac{2\rho_{12}}{\sigma_1\sigma_2}\right)^{-1}$$

$$= (1-0{,}6^2)\left(1/10^2 + 1/20^2 + 2\cdot 0{,}6/10\cdot 20\right)^{-1}$$

$$= 34{,}6\ m^2 \qquad \sigma_x = 5{,}9\, m$$

$$P(v > v_{tol}) = 1 - P(v < v_{tol}) = 1 - F\left(\frac{v_{tol} - v_0}{\sigma_v}\right)$$

Optimalno filtriranje : merjenje skalarne spremenljivke

- diskretne slike
- naj bo $x = x(t)$, potem lahko dinamiko zapišem kot $\dot{x} = A(t)\, x(t) + c(t)$
- prepišimo v diskretno sliko (diferenčne enačbe)

$$\frac{x_{n+1} - x_n}{T} = A(nT)\, x_n + c(nT)$$

$$x_{n+1} = \underbrace{(1 + TA)}_{\phi_n} x_n + \underbrace{TC}_{c_n} = \phi_n x_n + c_n + \underbrace{\Gamma_n w_n}_{\text{dinamični šum}} \qquad w_n \sim N(0, Q_n)$$

- n-ti trenutek : $(\hat{x}_n, \hat{\sigma}_n^2)$
- $(n+1)$-ti trenutek : $(z_{n+1}, \sigma_{n+1}^2)$

$$\underbrace{(\hat{x}_n, \hat{\sigma}_n)}_{\text{izostrena ocena}} \longrightarrow \underbrace{(\bar{x}_{n+1}, \bar{\sigma}_{n+1}^2)}_{\text{napoved}}$$

} izostrimo s Kalmanovim filtrom

- nove oznake $\hat{\sigma}_n^2 = P_n \qquad \bar{\sigma}_{n+1}^2 = M_{n+1} \qquad \sigma_{n+1}^2 = R_{n+1}$

---

1. $(\hat{x}_n, P_n) \xrightarrow{\text{napoved}}$

$$\bar{x}_{n+1} = \Phi_n \hat{x}_n + c_n$$

$$M_{n+1} = \phi_n^2 P_n + \Gamma_n^2 Q_n$$

$\hookrightarrow$ varianca din. šuma

2. Ostrenje (z_{n+1}, R_{n+1})

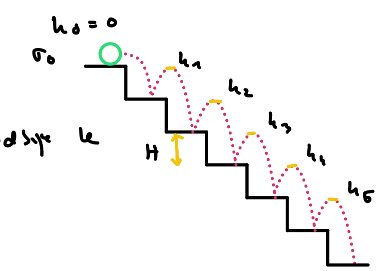
$$\hat{x}_{n+1} = \bar{x}_{n+1} + \frac{M_{n+1}}{M_{n+1} + R_{n+1}} (z_{n+1} - \bar{x}_{n+1})$$

$$P_{n+1} = \frac{M_{n+1} R_{n+1}}{M_{n+1} + R_{n+1}} = M_{n+1} - \frac{M_{n+1}^2}{M_{n+1} + R_{n+1}}$$

- v primeru, da ne merimo : z_{n+1} ... karkoli

$\Rightarrow R_{n+1} \to \infty$

$$\Rightarrow \hat{x}_{n+1} = \bar{x}_{n+1} \qquad P_{n+1} = M_{n+1}$$



Primer 5 enako visokih stopnic $H = 30\,cm$, $\sigma_H = 0$
žogica se odkotali dol, na vsaki stopnici se odbije le enkrat, pri tem izgubi $\delta = 1/2$ kin. en.
Določi h_5 in σ_5. N: meritev.

1. Poiščemo dinamiko $\quad x_{n+1} = \phi_n x_n + c_n \; (+\Gamma_n w_n)$

$$h_{n+1} = \phi_n h_n + c_n$$

1\. odboj $\quad W_{p1} = \delta W_{p0} \qquad mgh_1 = \delta mg (H + h_0)$

$$h_1 = \delta h_0 + \delta H$$

2\. odboj $\qquad h_2 = \delta h_1 + \delta H$

n. odsoj $\quad h_{n+1} = \delta \cdot h_n + \delta \cdot H$

$$\Rightarrow \phi_n = \delta \qquad c_n = \delta \cdot H$$

Opomba: Dinamični šum

če $\sigma_H \neq 0$ $\quad H = \bar{H} + \eta \qquad \langle \eta \rangle = 0 \qquad \langle \eta_n \eta_{n'} \rangle = \delta_{nn'} \sigma_H^2$

šum na višinah stopnic

$$h_{n+1} = \delta h_n + \delta \bar{H} + \delta \cdot \eta$$
$$h_{n+1} = \Phi_n h_n + c_n + \Gamma_n w_n$$

$$\langle w_n w_{n'} \rangle = \delta_{nn'} Q_n$$
$$\langle w_n^2 \rangle = Q_n$$
$$\langle \Gamma_n^2 w_n^2 \rangle = \Gamma_n^2 Q_n = \langle \delta^2 \eta^2 \rangle = \delta^2 \sigma_H^2$$

V našem primeru $\sigma_H = 0 \quad \Rightarrow \Gamma_n^2 Q_n = 0$

Našj na koloso : $h_5 = ?$

$$h_1 = \delta h_0 + \delta \bar{H}$$
$$h_2 = \delta h_1 + \delta \bar{H} = \delta^2 h_0 + \delta^2 \bar{H} + \delta \bar{H} = \delta^2 h_0 + (\delta^2 + \delta) \bar{H}$$
$$\vdots$$
$$h_n = \delta^n h_0 + (\underbrace{\delta^n + \ldots + \delta}_{S_n}) \bar{H}$$

$$S_n = \delta^n + \ldots + \delta = \delta \frac{1 - \delta^n}{1 - \delta} \quad \Rightarrow \quad h_n = \delta^n h_0 + \bar{H} \delta \frac{1 - \delta^n}{1 - \delta}$$

$$h_5 = \delta^5 \cdot 0 + 30\,\text{cm} \cdot \frac{1}{2} \frac{1 - (1/2)^5}{1 - 1/2} = \frac{31}{32} 30\,\text{cm}$$

$\sigma_5^2 = ? \qquad P_5 = ?$

$P_{n+1} = M_{n+1}$ ker ni meritev

$$= \Phi_n^2 P_n (+ \underbrace{\Gamma_n^2 Q_n}_{=0}) = \delta^2 P_n$$

$$P_0 = \sigma_0^2 \qquad \text{(podatek)}$$
$$P_1 = \delta^2 \sigma_0^2$$
$$P_2 = \delta^4 \sigma_0^2$$
$$\vdots$$
$$P_n = \delta^{2n} \sigma_0^2 = \hat{\sigma}_n^2$$

$$\hat{\sigma}_n = \sqrt{P_n} = \delta^n \sigma_0$$

Optimalno filtriranje za vektorske spremenljivke (diskretni sliki)

V n-tem trenutku imamo $\vec{x}_n$ za katerega velja dinamika $\vec{x}_{n+1} = \Phi_n \vec{x}_n + \vec{c}_n + \Gamma_n \vec{w}_n$

Pričnemo z najboljšo oceno $(\hat{x}_n, P_n)$

① napoved za $\bar{x}_{n+1}$:

$$\bar{x}_{n+1} = \Phi_n \hat{x}_n + \vec{c}_n$$

$$M_{n+1} = \Phi_n P_n \Phi_n^T + \Gamma_n Q_n \Gamma_n^T$$

② Ostrenje z meritvijo $(\vec{z}_{n+1}, R_{n+1})$

$$\hat{x}_{n+1} = \bar{x}_{n+1} + K_{n+1} (\vec{z}_{n+1} - H \bar{x}_{n+1})$$

$$P_{n+1} = M_{n+1} - M_{n+1} H^T (H M_{n+1} H^T + R_{n+1})^{-1} H M_{n+1}$$

$$K_{n+1} = P_{n+1} H^T R_{n+1}^{-1}$$

kjer je H okenska matrika (matrika senzorja)

$$\vec{z} = H \vec{x} + \vec{r}$$

Primer Opazujemo prožen trk dveh teles z masama m_1 in m_2 ter v_1 in v_2 z disperzijo σ_1 in σ_2. Izračunaj kovariančno matriko po trku za vektor $\vec{v}' = (v_1', v_2')$. Hitrosti pred trkom sta neodvisni.

$$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \qquad \vec{v}' = \begin{pmatrix} v_1' \\ v_2' \end{pmatrix}$$

$$P_0 = \begin{pmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{pmatrix} \qquad P' = ?$$

$$\underbrace{P' = M'}_{\text{kov mat napovedi}} = \Phi P_0 \Phi^T \underbrace{(+\Gamma Q \Gamma^T)}_{\text{šum =0}}$$

Iščemo Φ

iščemo v_1', v_2'

Rešujemo v težiščnem sistemu

$$v_T = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2}$$

$$u_1 = v_1 - v_T$$

$$u_2 = v_2 - v_T$$

Ohranitev gibalne količine in W_k

$$\sum p_i = 0 \qquad \sum W_{ki} = \sum W_{ki}'$$

$$m_1 u_1 + m_2 u_2 = 0 \quad \Rightarrow \quad u_2 = -\frac{m_1}{m_2} u_1$$

$$m_1 u_1' + m_2 u_2' = 0 \quad \Rightarrow \quad u_2' = -\frac{m_1}{m_2} u_1'$$

$$\frac{1}{2} m_1 u_1^2 + \frac{1}{2} m_2 u_2^2 = \frac{1}{2} m_1 u_1'^2 + \frac{1}{2} m_2 u_2'^2$$

$$m_1 u_1^2 + \frac{m_1^2}{m_2} u_1^2 = m_1 u_1'^2 + \frac{m_1^2}{m_2} u_1'^2$$

$$u_1^2 \left(1 + \frac{m_1}{m_2}\right) = u_1'^2 \left(1 + \frac{m_1}{m_2}\right)$$

$$\Rightarrow \quad u_1 = \pm u_1' \quad \Rightarrow \quad u_1 = -u_1'$$
$$u_2 = -u_2'$$

Nazaj v Lab sist.

$$u_i = v_i - v_T$$

Za 1. telo

$$u_1' = -u_1$$
$$v_1' - v_T = -(v_1 - v_T)$$
$$v_1' = -v_1 + 2v_T$$

$$\mu = \frac{m_2}{m_1}$$

$$v_1' = \frac{-m_1 v_1 - m_2 v_1 + 2 m_1 v_1 + 2 m_2 v_2}{m_1 + m_2} = \frac{v_1(m_1 - m_2) + 2 m_2 v_2}{m_1 + m_2} = \frac{v_1(\mu - 1) + 2 v_2 \mu}{1 + \mu}$$

$$v_2' = \dots \quad 1 \leftrightarrow 2 \qquad\qquad v_2' = \frac{v_2(\mu - 1) + 2 v_1}{1 + \mu}$$

$$\begin{pmatrix} v_1' \\ v_2' \end{pmatrix} = \underbrace{\frac{1}{1+\mu} \begin{pmatrix} 1-\mu & 2\mu \\ 2 & -(1-\mu) \end{pmatrix}}_{\phi} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \vec{0}$$

nove oznake

$$P' = \Phi P \Phi^T \qquad\qquad \Phi = A \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$$

$$P' = A^2 \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \begin{pmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{pmatrix} \begin{pmatrix} a & c \\ b & -a \end{pmatrix}$$

$$P' = A^2 \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \begin{pmatrix} a\sigma_1^2 & c\sigma_1^2 \\ b\sigma_2^2 & -a\sigma_2^2 \end{pmatrix}$$

$$P' = A^2 \begin{pmatrix} a^2\sigma_1^2 + b^2\sigma_2^2 & ac\sigma_1^2 - ab\sigma_2^2 \\ ac\sigma_1^2 - ab\sigma_2^2 & c^2\sigma_1^2 + a^2\sigma_2^2 \end{pmatrix} \qquad \text{simetrična} \checkmark$$

$$P' = \frac{1}{(1+\mu)^2} \begin{pmatrix} (1-\mu)^2\sigma_1^2 + 4\mu^2\sigma_2^2 & 2(1-\mu)(\sigma_1^2 - \mu\sigma_2^2) \\ 2(1-\mu)(\sigma_1^2 - \mu\sigma_2^2) & 4\sigma_1^2 + (1-\mu)^2\sigma_2^2 \end{pmatrix} \qquad \mu = \frac{m_2}{m_1}$$

Mejni primeri

a) $\sigma_1 = \sigma_2 = \sigma$
$$P' = \frac{\sigma^2}{(1+\mu)^2} \begin{pmatrix} (1-\mu)^2 + 4\mu^2 & 2(1-\mu)^2 \\ 2(1-\mu)^2 & 4 + (1-\mu)^2 \end{pmatrix}$$

$$m_1 = m_2 \Rightarrow \mu = 1 \qquad P' = \frac{\sigma^2}{4} \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} = \sigma^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = P$$

b) $\mu = 1 \quad \sigma_1 \neq \sigma_2$
$$P' = \frac{1}{4} \begin{pmatrix} 4\sigma_2^2 & 0 \\ 0 & 4\sigma_1^2 \end{pmatrix} = \begin{pmatrix} \sigma_2^2 & 0 \\ 0 & \sigma_1^2 \end{pmatrix}$$

DN: Popolnoma neprožen trk
- laboratorijski sistem
- ohranitev gibalne količine

Primer: Žarek svetlobe z začetno kov. matriko $P = \begin{pmatrix}\sigma_{y_0}^2 & 0 \\ 0 & \sigma_\varphi^2\end{pmatrix}$ upade na konveksno mejo. Kakšna je kov. matrika žarka po prehodu?

Geometrijske optike: obosna aproksimacija

- Vsak žarek lahko opišemo z 2 parametroma

$$\vec{y} = \begin{pmatrix} y_0 \\ \tan\varphi \end{pmatrix} \approx \begin{pmatrix} y_0 \\ \varphi \end{pmatrix}$$

oddaljenost od optične osi
kot glede na optično os

- prehod žarka preko meje opišemo s prehodno matriko A

$$\vec{y}\,' = A\,\vec{y}$$

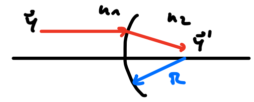
- Zaradi linearnosti velja za zaporedni prehod preko več mej (npr $D \to C \to D \to A$)

$$\vec{y}\,' = \underbrace{ADCD}_{\text{prehodne matrike}}\,\vec{y}$$

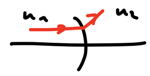
Kalman $\vec{y}\,' = \Phi\,\vec{y}$ $\qquad \hat{c} = 0$

- za konveksno mejo:

$$\Phi = A = \begin{pmatrix} 1 & 0 \\ \frac{1}{R}\left(\frac{n_1}{n_2} - 1\right) & \frac{n_1}{n_2} \end{pmatrix}$$



Konkavna meja
$R \to -R$



Prehodna matrika za ravno mejo $\begin{pmatrix} 1 & 0 \\ 0 & n_1/n_2 \end{pmatrix}$

Iščemo P'

$$P' = \Phi P \Phi^T \qquad \Phi = A = \begin{pmatrix} 1 & 0 \\ \frac{1}{R}\left(\frac{n_1}{n_2} - 1\right) & \frac{n_1}{n_2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ a & b \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ a & b \end{pmatrix} \begin{pmatrix} \sigma_y^2 & 0 \\ 0 & \sigma_\varphi^2 \end{pmatrix} \begin{pmatrix} 1 & a \\ 0 & b \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ a & b \end{pmatrix} \begin{pmatrix} \sigma_y^2 & a\sigma_y^2 \\ 0 & b\sigma_\varphi^2 \end{pmatrix} = \begin{pmatrix} \sigma_y^2 & a\sigma_y^2 \\ a\sigma_y^2 & a^2\sigma_y^2 + b\sigma_\varphi^2 \end{pmatrix}$$

$$= \begin{pmatrix} \sigma_y^2 & \frac{1}{R}\left(\frac{n_1}{n_2} - 1\right)\sigma_y^2 \\ \frac{1}{R}\left(\frac{n_1}{n_2} - 1\right)\sigma_y^2 & \frac{1}{R}\left(\frac{n_1}{n_2} - 1\right)^2\sigma_y^2 + \left(\frac{n_1}{n_2}\right)^2\sigma_\varphi^2 \end{pmatrix}$$

$\sigma_{y_0}'^2 = \sigma_y^2$ napaka lege se ne spremeni

$$\sigma_\varphi'^2 = \frac{1}{R^2}\left(\frac{n_1}{n_2} - 1\right)^2 \sigma_{y_0}^2 + \left(\frac{n_1}{n_2}\right)^2 \sigma_\varphi^2$$

Vektorske spremenljivke – zvezne slike

V zvezni sliki je Kalmanov filter:

(a) dinamika

$$\dot{\vec{x}} = A\vec{x} + \vec{c} + \underbrace{\Gamma\vec{w}}_{\text{dinamični šum}}$$

(b) filter

$$\dot{\hat{\vec{x}}} = A\hat{\vec{x}} + \vec{c} + K(\vec{z} - H\hat{\vec{x}})$$

$$K = PH^T R^{-1}$$

$$\dot{P} = \underbrace{AP + PA^T}_{\text{vpliv dinamike}} + \underbrace{\Gamma Q \Gamma^T}_{\text{din. šum}} - \underbrace{PH^T R^{-1} HP}_{\text{ostrenje z meritvami}}$$

Primer: Žogica na hrapavi mizi

Žogico spustimo iz 2 m na hrapavo mizo. Kadar jo spustimo točno na sredino ($\sigma_{x_1}(0) = 0$) žogica v povprečju po 5 sekundah pade z mize. Koliko časa potrebuje, če je $\sigma_{x_1}(0) = 20\,cm$, $\tau = ?$.
Dinamiko prečnega gibanja popiši s premim gibanjem in belim šumom, ki popisujejo naključne sile $F(t)$ pri odbojih na ploščo.

$$\dot{\vec{x}} = A\vec{x} + \vec{c} + \Gamma\vec{w}$$

Gibalna enačba $\quad m\ddot{x} = \underbrace{F(t)}_{\text{naključne sile pri odbojih}}$

$$\Rightarrow \ddot{x} = \frac{F(t)}{m} \quad \Rightarrow \vec{x} = \begin{bmatrix} x \\ v \end{bmatrix} \qquad \dot{\vec{x}} = \begin{bmatrix} v \\ F(t)/m \end{bmatrix}$$

$$\Rightarrow \dot{\vec{x}} = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{A} \vec{x} + \underbrace{\begin{bmatrix} 0 \\ w \end{bmatrix}}_{\Gamma\vec{w}} + \underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{\vec{c}} \qquad w = \frac{F(t)}{m}$$

$$\dot{P} = AP + PA^T + \Gamma Q \Gamma^T - \underbrace{PH^T R^{-1} HP}_{=0}$$

$$AP = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} p_{11}(t) & p_{12}(t) \\ p_{21}(t) & p_{22}(t) \end{bmatrix} = \begin{bmatrix} p_{21} & p_{22} \\ 0 & 0 \end{bmatrix}$$

$$PA^T = P^T A^T = (AP)^T = \begin{bmatrix} p_{21} & 0 \\ p_{22} & 0 \end{bmatrix}$$

$$\Gamma Q \Gamma^T = \langle \Gamma\vec{w} \cdot (\Gamma\vec{w})^T \rangle = \langle \Gamma \vec{w} \vec{w}^T \Gamma^T \rangle = \Gamma Q \Gamma^T$$

$$\Gamma\vec{w} = \begin{pmatrix} 0 \\ w \end{pmatrix}$$

$$\Gamma Q \Gamma^T = \langle \begin{pmatrix} 0 \\ w \end{pmatrix} (0 \;\; w) \rangle = \begin{pmatrix} 0 & 0 \\ 0 & Q \end{pmatrix} \qquad Q = \langle w^2 \rangle \text{ skalar neznana}$$

$$\dot{P} = \begin{bmatrix} p_{21} & p_{22} \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} p_{21} & 0 \\ p_{22} & 0 \end{bmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & Q \end{pmatrix}$$

$$\dot{P} = \begin{bmatrix} 2p_{21} & p_{22} \\ p_{22} & Q \end{bmatrix} = \begin{bmatrix} \dot{p}_{11} & \dot{p}_{12} \\ \dot{p}_{21} & \dot{p}_{22} \end{bmatrix}$$

$$\dot p_{11} = 2p_{21} = 2p_{12}$$
$$\dot p_{12} = p_{22}$$
$$\dot p_{22} = Q$$

Iščemo $p_{11}(t) = \sigma_x^2(t)$

$$p_{22} = Qt + p_{22}(0)$$
$$p_{12} = Q\frac{t^2}{2} + p_{22}(0)\,t + p_{12}(0)$$
$$p_{11} = Q\frac{t^3}{3} + p_{22}(0)t^2 + 2p_{12}(0) + p_{11}(0)$$

Začetni pogoji $P(t=0) = \begin{pmatrix} p_{11}(0) & p_{12}(0) \\ p_{21}(0) & p_{22}(0) \end{pmatrix}$

ⓐ $P(t=0) = 0 \Rightarrow p_{11} = Q\frac{t^3}{3}$

σ_x oceni razdaljo od središča $\quad p_{11}(t=\tau_0=5s) = R^2 = Q\frac{\tau_0^3}{3}$

$$\Rightarrow Q = \frac{3R^2}{\tau_0^3}$$

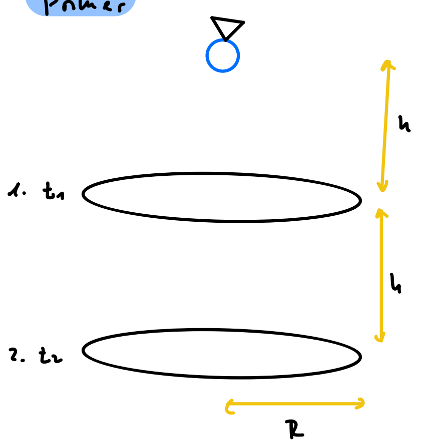
ⓑ $P(t=0) = \begin{pmatrix} \sigma_x^2(0) & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow p_{11}(t) = Q\frac{t^3}{3} + \sigma_x^2(0)$

Iščemo t $\quad R^2 = R^2\frac{t^3}{\tau_0^3} + \sigma_x^2(0)$

$$t = \sqrt[3]{\frac{\tau_0^3}{R^2}\left(R^2 - \sigma_x^2(0)\right)} = \tau_0\sqrt[3]{1 - \frac{\sigma_x^2}{R^2}}$$

$R = 2m$, $\sigma_x = 20cm$, $\tau_0 = 5s \Rightarrow t = 4{,}97s$

Primer



Kroglico spuščamo skozi svetlobna vrata. V prečni smeri je povprečje sil 0N, z dim. šumom. Ugotovimo, da skozi 1. odprtino pade 2/3 vseh kroglic. Koliko kroglic pade skozi druga vrata. Lega in hitrost na začetku znani, velikost kroglice in debelina vrat sta majhni.

$$P(-R \le x(t_1) \le R) = \frac{2}{3}$$
$$P(-R \le x(t_2) \le R) = ?$$
$$\langle F(t)\rangle = 0$$

Začetni pogoj $P(t=0) = 0$

$$x \sim N(0, \sigma_x^2(t))$$

kumulativna funkcija

$$P(-R \le x(t_2) \le R) = F\left(\frac{R-0}{\sigma_x(t_2)}\right) - F\left(\frac{-R-0}{\sigma_x(t_2)}\right) =$$
$$= F\left(\frac{R}{\sigma_x(t_2)}\right) - \left(1 - F\left(\frac{R}{\sigma_x(t_2)}\right)\right) =$$
$$= 2F\left(\frac{R}{\sigma_x(t_2)}\right) - 1$$

Uporabimo Kalmanov filter

$$\dot P = AP + PA^T + \Gamma Q \Gamma^T$$

Iščemo dinamiko v x-smeri

$m\ddot{x} = F(t)$ $\qquad\qquad$ $\ddot{x} = \frac{F(t)}{m} = w$

naključna sila $\qquad\qquad$ din. šum

$$\dot{\vec{x}} = \begin{bmatrix}\dot{x}\\ \dot{v}\end{bmatrix} = \underbrace{\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}}_{A}\underbrace{\begin{bmatrix}x\\ v\end{bmatrix}}_{\vec{x}} + \underbrace{\begin{bmatrix}0\\ 0\end{bmatrix}}_{\vec{c}} + \underbrace{\begin{bmatrix}0\\ w\end{bmatrix}}_{\Gamma\vec{u}}$$

enako kot pri prejšnji nalogi

$$\dot{P} = \begin{bmatrix}2p_{21} & p_{22}\\ p_{22} & Q\end{bmatrix} = \begin{bmatrix}\dot{p}_{11} & \dot{p}_{12}\\ \dot{p}_{21} & \dot{p}_{22}\end{bmatrix}$$

$$p_{22} = Qt + p_{22}(0)$$
$$p_{12} = Q\frac{t^2}{2} + p_{22}(0)\,t + p_{12}(0)$$
$$p_{11} = Q\frac{t^3}{3} + p_{22}(0)t^2 + 2p_{12}(0) + p_{11}(0) \qquad \text{Z.P. } P(t=0) = 0$$

$$p_{22} = Qt$$
$$p_{12} = Q\frac{t^2}{2}$$
$$p_{11} = Q\frac{t^3}{3} = \sigma_x^2$$

Iščemo Q $\qquad$ $P(-R \le x(t_1) \le R) = \frac{2}{3}$

$P(-\sigma+\mu \le x \le \mu+\sigma) \doteq \frac{2}{3}$ $\qquad$ $\mu = 0$

$\Rightarrow \sigma_x(t_1) \simeq R$

$$Q\frac{t_1^3}{3} = R^2 \qquad Q = \frac{3R^2}{t_1^3}$$

Iščemo $\sigma_x^2(t_2) = \frac{1}{3}Qt_2^3 = R^2\left(\frac{t_2}{t_1}\right)^3$

Določimo t_1, t_2 : prosti pad

$$t_1 = \sqrt{\frac{2h}{g}} \qquad t_2 = \sqrt{\frac{4h}{g}} = \sqrt{2}\,t_1$$

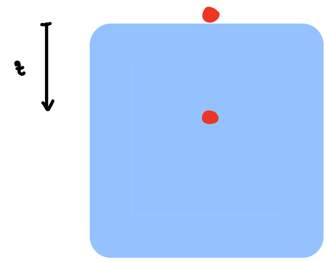
$$\sigma_x^2(t_2) = R^2\, 2^{3/2}$$
$$\sigma_x(t_2) = R\, 2^{3/4}$$

$$P(-R \le x(t_2) \le R) = 2F\left(\frac{R}{\sigma_x(t_2)}\right) - 1 = 2F(2^{-3/4}) - 1$$

iz tabel

$$= 2\cdot 0{,}724 - 1 = 45\%$$

Primer Padanje kroglice v viskozni tekočini



$$P(0) = \begin{pmatrix}0 & 0\\ 0 & \sigma_{v_0}^2\end{pmatrix}$$

$F_u = -m\beta v$

$F(t)$ ... naključna sila $\qquad\qquad$ velja $\sigma_v(t\to\infty) = \frac{1}{2}\sigma_{v_0}$

Iščemo $\vec{x} = \begin{pmatrix}z\\ v\end{pmatrix}$, σ_v ko $\beta t = 1$

Dinamika v z smeri

$$m\ddot{z} = mg - m\beta v + F(t)$$

$$\ddot{z} = g - \beta v + \underbrace{\frac{F(t)}{m}}_{w}$$

$$\dot{\vec{x}} = \begin{bmatrix} \dot{z} \\ \dot{v} \end{bmatrix} = \underset{A}{\begin{bmatrix} 0 & 1 \\ 0 & -\beta \end{bmatrix}} \underset{\vec{x}}{\begin{bmatrix} z \\ v \end{bmatrix}} + \underset{\vec{c}}{\begin{bmatrix} 0 \\ g \end{bmatrix}} + \underset{\Gamma \vec{u}}{\begin{bmatrix} 0 \\ w \end{bmatrix}}$$

$$\dot{P} = AP + PA^T + \Gamma Q \Gamma^T$$

$$AP = \begin{pmatrix} 0 & 1 \\ 0 & -\beta \end{pmatrix} \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} = \begin{pmatrix} p_{21} & p_{22} \\ -\beta p_{21} & -\beta p_{22} \end{pmatrix}$$

$$PA^T = (AP)^T \qquad \Gamma Q \Gamma^T = \begin{bmatrix} 0 & 0 \\ 0 & Q \end{bmatrix}$$

$$\dot{P} = \begin{pmatrix} 2p_{21} & p_{22} - \beta p_{21} \\ p_{22} - \beta p_{21} & Q - 2\beta p_{22} \end{pmatrix}$$

$p_{22}(t) = ?$ $\qquad \dot{p}_{22} = Q - 2\beta p_{22}$

$$u = Q - 2\beta p_{22} \qquad \dot{u} = -2\beta \dot{p}_{22}$$

$$-2\beta \dot{p}_{22} = \dot{u} = -2\beta u \qquad \frac{\dot{u}}{u} = -2\beta$$

$$u = A e^{-2\beta t}$$

$$p_{22} = -\frac{1}{2\beta}\left(A e^{-2\beta t} - Q\right)$$

$$p_{22} = \frac{Q}{2\beta} - B e^{-2\beta t}$$

Z.P. $\qquad p_{22}(t=0) = \sigma_{v_0}^2 = \frac{Q}{2\beta} - B$

$$B = \frac{Q}{2\beta} - \sigma_{v_0}^2$$

Iščemo Q $\quad t \to \infty$ $\qquad p_{22} = \frac{Q}{2\beta} - \left(\frac{Q}{2\beta} - \sigma_{v_0}^2\right) e^{-2\beta t}$

$$p_{22\infty} = \frac{Q}{2\beta}$$

$$Q = 2\beta\, p_{22\infty} = 2\beta \sigma_{v_\infty}^2 = 2\beta \left(\frac{1}{2}\sigma_{v_0}\right)^2$$

$$= \frac{1}{2}\beta \sigma_{v_0}^2$$

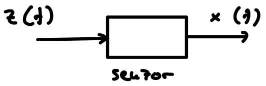
$$p_{22}(t) = \frac{\sigma_{v_0}^2}{4} + \frac{3\sigma_{v_0}^2}{4} e^{-2\beta t}$$

$$= \frac{\sigma_{v_0}^2}{4}\left(1 + 3e^{-2\beta t}\right)$$

Ob času $\beta T = 1$ $\qquad p_{22}(T) = \frac{\sigma_{v_0}^2}{4}\left(1 + 3e^{-2}\right) = \sigma_v^2(T)$

Senzorji:

Zanima nas odziv senzorjev na različne vhodne signale $z(t)$



dif. en. v časovni domeni

$\downarrow \mathcal{L}$

algeb. en.

$$\begin{array}{ccccc} z(t) & \rightarrow & DE & \rightarrow & x(t) \\ \mathcal{L}\downarrow & & & & \uparrow \mathcal{L}^{-1} \\ z(s) & \cdot & H(s) & = & x(s) \end{array}$$

prenosna funk.

$$f(t) \xrightarrow{\mathcal{L}} F(s) \qquad \mathcal{L}(f(t)) = \int_0^\infty f(t)\, e^{-st}\, dt = F(s)$$

$$\mathcal{L}(\dot{f}) = -f(0) + sF(s)$$

privzamemo $f(0)=0$ ker ni bilo signala na začetku

$$\Rightarrow \quad \mathcal{L}(f^{(n)}) = s^n F(s)$$

Senzorji 1. reda

DE: $\tau \dot{x}(t) + x(t) = z(t)$ $\quad \downarrow \mathcal{L}$ $\quad X(s), Z(s)$

$$\tau s X(s) + X(s) = Z(s)$$

$$X(s)(1+\tau s) = Z(s)$$

$$X(s) = \underbrace{\frac{1}{1+\tau s}}_{H(s)} Z(s)$$

Odzivi senzorjev 1. reda na nekaj značilnih vhodnih signalov

a) $z(t) = \delta(t)$ $\qquad x(t) = ?$ $\qquad H(s) = \frac{1}{1+\tau s}$

$$X(s) = H(s)\, Z(s)$$

$$Z(s) = 1 \qquad X(s) = \frac{1}{1+\tau s}$$

$$x(t) = \mathcal{L}^{-1}\left(\frac{1}{1+\tau s}\right) = \mathcal{L}^{-1}\left(\frac{1}{\tau}\, \frac{1}{s+1/\tau}\right) = \frac{1}{\tau} e^{-t/\tau}$$

s pomočjo tabele

b) $z(t) = kt$ $\qquad Z(s) = k\,\mathcal{L}(t) = k\frac{1}{s^2}$

$$X(s) = \frac{1}{1+\tau s}\,\frac{k}{s^2} \overset{\text{parcialni ulomki}}{=} \frac{A}{1+\tau s} + \frac{Bs+C}{s^2} = \frac{+k\tau^2}{1+\tau s} + \frac{-k\tau s + k}{s^2} = \ldots$$

$$\frac{As^2 + Bs + C + B\tau s^2 + C\tau s}{(1+\tau s)\, s^2}$$

$C = k$

$A + B\tau = 0$

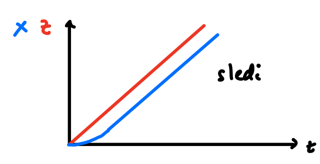
$B + C\tau = 0$

$A = +k\tau^2$

$B = -k\tau$

$$\ldots = +k\tau\, \frac{1}{1/\tau + s} - k\tau\, \frac{1}{s} + k\,\frac{1}{s^2}$$

$$x(t) = k\tau e^{-t/\tau} - k\tau + kt = k\left(t - \tau\left(1 - e^{-t/\tau}\right)\right)$$



(c) $z(t) = at^2$ $\qquad Z(s) = a\frac{2}{s^3}$

$$X(s) = \frac{1}{1+\tau s}\frac{2a}{s^3} = \frac{A}{1+\tau s} + \frac{Bs^2+Cs+D}{s^3} = \dots$$

$$2a = s^3(A + B\tau) + s^2(B + \tau C) + s(C + \tau D) + D$$

$$D = 2a \qquad A = -B\tau = -\tau^3 2a$$
$$B = -\tau C = \tau^2 2a$$
$$C = -\tau D = -\tau 2a$$

$$\dots = 2a\left(\frac{-\tau^3}{1+\tau s} + \frac{\tau^2 s - \tau s + 1}{s^3}\right) = 2a\left(\frac{1}{s^3} - \frac{\tau}{s^2} + \frac{\tau^2}{s} - \frac{\tau^3}{1+\tau s}\right)$$

$$x(t) = at^2 - 2a\tau t + 2a\tau^2 - 2a\tau^2 e^{-t/\tau}$$

Ali sledi? $\quad t \to \infty \qquad x(t\to\infty) = at^2 - 2a\tau t + 2a\tau^2$

$$z(t\to\infty) - x(t\to\infty) = 2a\tau t \qquad \text{ne sledi}$$

Opomba: $\quad x(0^-) \equiv 0 \quad$ če ni premaknemo v času in vrsti

Primera

(a) $z(t) = \underline{z_0} + kt \qquad x(t=0) = \underline{x_0} \neq 0 \quad$ začetk

$$\tau\dot{x} + x = z \qquad /\mathcal{L}$$
$$\tau(sX(s) - x(0)) + X(s) = Z(s)$$

$$X(s) = \frac{Z(s) + \tau X(0)}{1+\tau s} \qquad Z(s) = \frac{z_0}{s} + \frac{k}{s^2} = \frac{z_0 s + k}{s^2}$$

$$Z(s) + \tau X(0) = \frac{\tau X(0) s^2 + z_0 s + k}{s^2}$$

$$X(s) = \frac{1}{1+\tau s}\frac{\tau X(0) s^2 + z_0 s + k}{s^2}$$
$$= \frac{A}{1+\tau s} + \frac{Bs + C}{s^2}$$

$$\tau X(0) s^2 + z_0 s + k = s^2(A + \tau B) + s(B + \tau C) + C$$

$$\Rightarrow C = k \qquad B = z_0 - \tau k \qquad A = \tau X(0) - \tau z_0 + k\tau^2$$

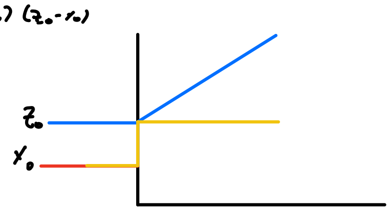
$$X(s) = \frac{x_0 - z_0 + k\tau}{s + 1/\tau} + \frac{k}{s^2} + \frac{z_0 - k\tau}{s}$$

$$x(t) = kt + z_0 - k\tau + (x_0 - z_0 + k\tau)e^{-t/\tau}$$

enak odziv kot prej

$$= kt + x_0 - \underbrace{(x_0 - z_0)(1 - e^{-t/\tau})}_{\text{odziv na step } H(t)(z_0 - x_0)} - k\tau(1 - e^{-t/\tau})$$

$$X(0) = x_0 \checkmark$$



Senzorji: 2. reda

dif. en. $\ddot{x} + 2\zeta\omega_0\dot{x} + \omega_0^2 x = \omega_0^2 z + \cancel{2\zeta\omega_0\dot{z}}$

ω_0 ... lastna frekvenca

ζ ... koeficient dušenja

Optimalno dušenje $\zeta_{opt} = \frac{1}{\sqrt{2}}$

$x(0^-) = 0 \qquad z(0^-) = 0$

$$s^2 X(s) + 2\zeta\omega_0 s X(s) + \omega_0^2 X(s) = \omega_0^2 Z(s)$$

$$X(s) = \frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2} Z(s) = H(s)Z(s)$$

(a) $z(t) = \delta(t) \qquad Z(s) = 1 \Rightarrow X(s) = H(s) \qquad x(t) = ?$

$$\mathcal{L}(\sin(\omega t)e^{at}) = \frac{\omega}{(s-a)^2+\omega^2} \quad \Leftrightarrow \quad a = -\zeta\omega_0 \quad \omega^2 = \omega_0^2(1-\zeta^2)$$

$$s^2 + 2\zeta\omega_0 s + \omega_0^2 = (s+\zeta\omega_0)^2 - \zeta^2\omega_0^2 + \omega_0^2 = (s+\zeta\omega_0)^2 + \omega_0^2(1-\zeta^2)$$

$$X(s) = \frac{\omega_0}{\sqrt{1-\zeta^2}} \frac{\omega_0\sqrt{1-\zeta^2}}{(s+\zeta\omega_0)^2 + \omega_0^2(1-\zeta^2)}$$

$$x(t) = \frac{\omega_0}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_0 t} \sin(\omega_0^2(1-\zeta^2)t)$$

(b) $z(t) = kt \qquad Z(s) = \frac{k}{s^2}$

$$X(s) = \frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}\frac{k}{s^2} = k\left(\frac{As + \mathcal{D}}{s^2 + 2\zeta\omega_0 s + \omega_0^2} + \frac{Cs+D}{s^2}\right)$$

$$\omega_0^2 = (A+C)s^3 + s^2(\mathcal{D} + D + C2\zeta\omega_0) + s(C\omega_0^2 + 2\zeta\omega_0 D) + D\omega_0^2$$

$$D = 1 \qquad C = -\frac{2\zeta}{\omega_0} \qquad \mathcal{D} = 4\zeta^2 - 1 \qquad A = \frac{2\zeta}{\omega_0}$$

$$X(s) = k\left(\frac{1}{s^2} - \frac{2\zeta}{\omega_0}\frac{1}{s} + \frac{2\zeta}{\omega_0}\frac{s}{s^2+2\zeta\omega_0 s+\omega_0^2} + \frac{4\zeta^2-1}{s^2+2\zeta\omega_0 s+\omega_0^2}\right)$$

- s pomočjo tabel $\mathcal{L}(e^{at}\sin\omega t) = \frac{\omega}{(s-a)^2+\omega^2}$

  $\mathcal{L}(e^{at}\cos\omega t) = \frac{s-a}{(s-a)^2+\omega^2}$

$$\frac{2\zeta}{\omega_0}\frac{s}{(s+\zeta\omega_0)^2 + \omega_0^2(1-\zeta^2)} + \frac{4\zeta^2-1}{s^2+2\zeta\omega_0 s+\omega_0^2} =$$

$$= \frac{2\zeta}{\omega_0}\frac{s+\zeta\omega_0}{-\|-} - \frac{2\zeta}{\cancel{\omega_0}}\frac{\zeta\cancel{\omega_0}}{-\|-} + \frac{4\zeta^2-1}{-\|-} =$$

$$= \frac{2\zeta}{\omega_0}\frac{s+\zeta\omega_0}{-\|-} + \frac{2\zeta^2-1}{\omega_0\sqrt{1-\zeta^2}}\frac{\omega_0\sqrt{1-\zeta^2}}{-\|-}$$

$$X(s) = k\left(\frac{1}{s^2} - \frac{2\zeta}{\omega_0}\frac{1}{s} + \frac{2\zeta}{\omega_0}\frac{s+\zeta\omega_0}{-\|-} + \frac{2\zeta^2-1}{\omega_0\sqrt{1-\zeta^2}}\frac{\omega_0\sqrt{1-\zeta^2}}{-\|-}\right)$$

$$x(t) = k\left(t - \frac{2\xi}{\omega_0} + \frac{2\xi}{\omega_0} e^{-\xi\omega_0 t}\cos(\omega t) + \frac{2\xi^2-1}{\omega} e^{-\xi\omega_0 t}\sin\omega t\right) \qquad \omega = \omega_0\sqrt{1-\xi^2}$$

Za optimalen filter $\xi_{opt} = \frac{1}{\sqrt{2}}$

$$x_{opt}(t) = k\left(t - \frac{\sqrt{2}}{\omega_0} + \frac{\sqrt{2}}{\omega_0} e^{-\omega_0 t/\sqrt{2}}\cos\frac{\omega_0 t}{\sqrt{2}}\right)$$

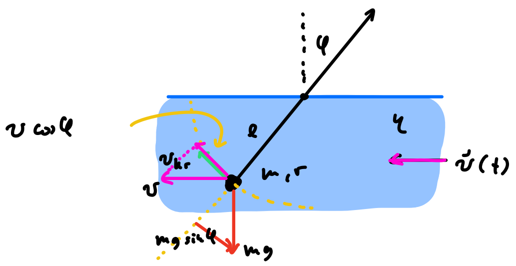
Po dolgem času $x(t) = k\left(t - \frac{2\xi}{\omega_0}\right)$ $\qquad z(t) = kt$

časovni zamik

Primer Merilnik rečne hitrosti

(gib. enačba)

Določi dinamiko in red senzorja



$$\sum_i M_i = J\ddot\varphi$$

$$F_u = 6\pi\eta r v$$

$v_{kr} = \dot\varphi l \qquad F_{u\perp} = 6\pi\eta r(v\cos\varphi - \dot\varphi l)$

Gibalna en. $\quad J\ddot\varphi = -mg\sin\varphi\, l + 6\pi\eta r(v\cos\varphi - \dot\varphi l)l$

$$J\ddot\varphi + 6\pi\eta r l^2\dot\varphi + mgl\sin\varphi = 6\pi r\eta v\cos\varphi$$

za $\varphi \ll 1$

$$\ddot\varphi + \frac{6\pi\eta r l^2}{J}\dot\varphi + \frac{mgl}{J}\varphi = \frac{6\pi r\eta l^2}{J}\frac{v}{l}$$ senzor 2. reda

$$\ddot x + 2\xi\omega_0\dot x + \omega_0^2 x = \omega_0^2 z + 2\xi\omega_0\dot z$$

$$\Rightarrow \dot z = \frac{v}{l} \qquad \omega_0^2 = \frac{mgl}{J} \qquad 2\xi\omega_0 = \frac{6\pi\eta r l^2}{J}$$

$$\xi = \frac{6\pi\eta r l^2}{2J}\sqrt{\frac{J}{mgl}}$$

Naj bo senzor optimalen

$$\xi = \frac{1}{\sqrt{2}} = \frac{6\pi\eta r l^2}{2\omega_0 J} \qquad /^2$$

$$\frac{1}{2} = \frac{9\pi^2\eta^2 r^2 l^4}{\omega_0^2 J^2}$$

$$\frac{mgl}{J} J^2 = 18\pi^2\eta^2 r^2 l^4$$

$$mg = \frac{18\pi^2 r^2\eta^2 l^3}{J}$$

$$\frac{mJ}{r^2 l^3} = 18\pi^2\frac{\eta^2}{g} = \text{konst.}$$

Odziv senzorja na periodični vhodni signal

Pomen $H(s)$ pri periodičnih signalih

$z(t) = z_0 e^{i\omega t}$ $\quad$ ω... fr. vhodnega sig.

$z_0 \in \mathbb{R}$ ... amplituda vhodnega sig.

$H(s)$ $\quad$ $X(s) = H(s)\, Z(s)$

$$Z(s) = z_0 \frac{1}{s - i\omega} \qquad H(s) = A \frac{P(s)}{D(s)} = A \frac{\prod_a (s - s_a)}{\prod_b (s - s_b)} \qquad s_a, s_b \in \mathbb{C}$$

s_a, s_b ničle pol $P(s)$, $D(s)$

$$X(s) = A z_0 \frac{\prod_a (s - s_a)}{\prod_b (s - s_b)} \frac{1}{s - i\omega} =$$

razcep na parcialne ulomke

$$= A z_0 \left(\frac{d_0}{s - i\omega} + \sum_b \frac{d_b}{s - s_b} \right)$$

$$x(t) = A z_0 \left(d_0 e^{i\omega t} + \sum_b \underbrace{d_b e^{s_b t}} \right)$$

s_b ... poli $H(s)$

$s_b = \sigma_b + i\omega_b$

$d_b e^{\sigma_b t} e^{i\omega_b t}$

prehodni pojav $\Leftarrow$ $d_b e^{-|\sigma_b| t} e^{i\omega_b t}$ $\quad$ $\sigma_b < 0$! $\quad$ ne želimo exp naraščanja

Po dovolj dolgem času $\to 0$

$$x(t) = A z_0 d_0 e^{i\omega t} \qquad t \gg 1$$

$$= x_0 e^{i\omega t} \qquad x_0 \in \mathbb{C}$$

$x_0 = |x_0| e^{i\delta}$ $\quad$ δ... fazni zamik

$$= |x_0| e^{i\delta} e^{i\omega t}$$

d_0 še iščemo

$$X(s) = A z_0 \frac{\prod_a (s - s_a)}{\prod_b (s - s_b)} \frac{1}{s - i\omega} = A z_0 \left(\frac{d_0}{s - i\omega} + \sum_b \frac{d_b}{s - s_b} \right)$$

Gledamo števce $\quad$ $\prod_a (s - s_a) = d_0 \prod_b (s - s_b) + (s - i\omega) \prod_b (s - s_b) \sum_b \frac{d_b}{s - s_b}$

Naj bo $s = i\omega$

$$d_0 = \frac{\prod_a (i\omega - s_a)}{\prod_b (i\omega - s_b)}$$

$$x(t) = A z_0 \frac{\prod_a (i\omega - s_a)}{\prod_b (i\omega - s_b)} e^{i\omega t}$$

$$= \underbrace{A \frac{\prod_a (i\omega - s_a)}{\prod_b (i\omega - s_b)}}_{H(i\omega)} z(t)$$

$$x(t) = |x_0| e^{i\delta} e^{i\omega t} = H(i\omega)\, z_0\, e^{i\omega t}$$

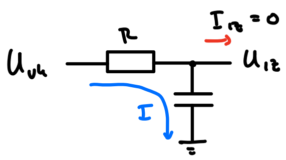
$$H(i\omega) = \frac{|x_0|}{z_0} e^{i\delta} \in \mathbb{C}$$

Koj nas zanima pri filtrih / senzorjih?

1. razmerje amplitud $\frac{|x_0|}{z_0} = |H(i\omega)|$ (ojačanje)

2. fazni zamik $\tan\delta = \frac{\operatorname{Im} H(i\omega)}{\operatorname{Re} H(i\omega)}$

Primer Nizko frekvenčni prepustni filter (low pass filter)



$$H(i\omega) = \frac{U_{iz}}{U_{vh}}$$

$$U_{iz} = Z_C\, I$$

$$I = \frac{U_{vh}}{Z_R + Z_C}$$

$$U_{iz} = \frac{Z_C}{R + Z_C} U_{vh} \quad \Rightarrow \quad H(i\omega) = \frac{Z_C}{R + Z_C} \qquad Z_C = \frac{1}{i\omega C}$$

$$H(i\omega) = \frac{1}{i\omega RC + 1} = \frac{1}{1 + i\omega\tau}$$

senzor 1. reda

Ojačanje, razmerje amplitud

$$|H(i\omega)|^2 = \frac{1}{1 + i\omega\tau} \cdot \frac{1}{1 - i\omega\tau} = \frac{1}{1 + (\omega\tau)^2}$$

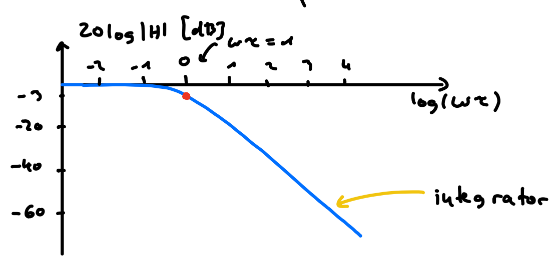
$$|H| = \frac{1}{\sqrt{1 + (\omega\tau)^2}}$$

Če $\omega\tau \ll 1$: $|H| \to 1$ $\qquad$ $\omega\tau \gg 1$: $|H| \to \frac{1}{\omega\tau}$

$\omega\tau = 1$ $\qquad |H| = \frac{1}{\sqrt{2}}$

$20 \log \frac{1}{\sqrt{2}} \approx -3\,dB$

Bodejev diagram



Idealni integrator

$$U_{iz}(t) = k \int_0^t U_v(t')\, dt' \qquad \Big/ \frac{d}{dt}$$

$$\dot{U}_{iz} = k\, U_v(t) \qquad / \mathcal{L} \qquad \text{privzamem } U_v(0) = 0$$

$$s U_{iz}(s) = k\, U_v(s)$$

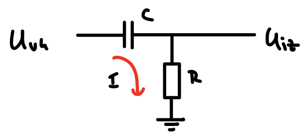
$$H_{id.\,int.} = \frac{U_{iz}}{U_{vh}} = \frac{k}{s}$$

$$H(i\omega) = \frac{k}{i\omega}$$

$\Updownarrow$ LPF pri $\omega\tau \gg 1$ se obnaša kot integrator

$$H_{LPF}(i\omega) = \frac{1}{1 + i\omega\tau} \xrightarrow{\omega\tau \gg 1} \frac{1}{i\omega\tau}$$

Primer Visoko frekvenčni prepustni filter



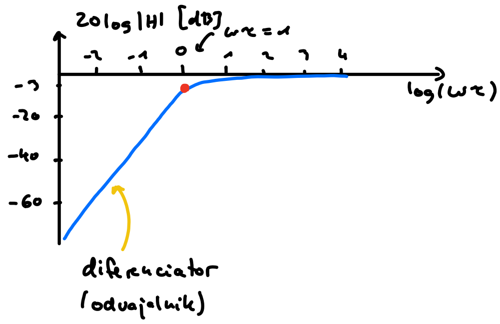
$U_{iz} = RI \;\; = U_{vh} \dfrac{R}{Z_C + R}$

$I = \dfrac{U_{vh}}{Z_C + R}$

$$H_{HPF}(i\omega) = \frac{U_{iz}}{U_{vh}} = \frac{R}{\frac{1}{i\omega C} + R} = \frac{i\omega RC}{1 + i\omega RC} = \frac{i\omega\tau}{1+i\omega\tau} \qquad \tau = RC$$

$$|H_{HPF}| = \frac{\omega\tau}{\sqrt{1+(\omega\tau)^2}}$$

		$20\log\lvert H\rvert$
$\lvert H_{HPF}\,(\omega\tau \gg 1)\rvert$	$\to 1$	0
$(\omega\tau \ll 1)$	$\to \omega\tau$	$20\log \omega\tau$
$(\omega\tau = 1)$	$= \frac{1}{\sqrt{2}}$	$\sim -3\,dB$



Idealni diferenciator

$U_{iz}(t) = k\,\dot{U}_{vh}(t) \qquad /\mathcal{L}$

$U_{iz}(s) = ks\,U_{vh}(s)$

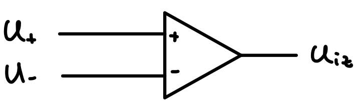
$H(i\omega) = ki\omega$

$$\Updownarrow$$

$$H_{HPF}(i\omega) = \frac{i\omega\tau}{1+i\omega\tau} \overset{\omega\tau \ll 1}{=} i\omega\tau$$

Aktivna vezja (1. red)

Operacijski ojačevalec



prenosna funkcija

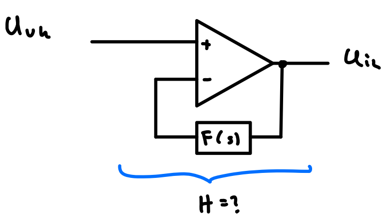
$U_{iz} = A(s)\,(U_+ - U_-)$

Idealni	Realni
$A(s) = A_0 \to \infty$	$A(s) = A_0 H_{LPF}$ $\quad A_0 \sim 10^4 - 10^6$
$Z_+, Z_- \to \infty$	$Z_+, Z_- \sim 10^{10} - 10^{12}\,\Omega$
$I_+, I_- \to 0$	$I_+, I_- < 10^{-12}\,A$
$Z_{iz} \to 0$	$Z_{iz} \sim 10^{-3}\,\Omega$

Težava $A(s)$ ni dobro definirana

$A_0 = f(T, Z, \dots)$

Uporaba v negativni povratni zanki

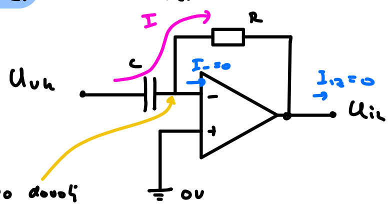


$U_{iz} = A_0\,(U_{vh} - F U_{iz})$

$U_{iz}\,(1 + A_0 F) = A_0 U_{vh}$

$$H = \frac{A_0}{1 + A_0 F} = \frac{1}{\frac{1}{A_0} + F} \overset{A_0 \to \infty}{=} \frac{1}{F}$$

Primer diferenciator

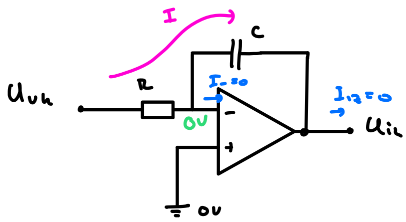


po dovolj dolgem času bo 0V zaradi negativne povratne zanke

$$\frac{U_{vh} - 0V}{Z_C} = I = \frac{0V - U_{iz}}{R}$$

$$\frac{U_{iz}}{U_{vh}} = -\frac{R}{Z_C} = -i\omega RC = -i\omega\tau = H_{dif.}$$

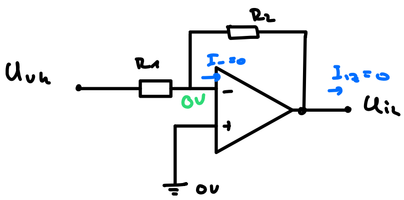
Primer integrator



$$\frac{U_v}{R} = -\frac{U_{iz}}{Z_C}$$

$$H = -\frac{Z_C}{R} = -\frac{1}{i\omega RC} = -\frac{1}{i\omega\tau}$$

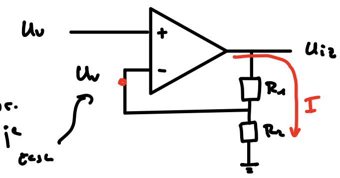
Primer invertiran ojačevalnik



$$\frac{U_v}{R_1} = -\frac{U_{iz}}{R_2}$$

$$H = -\frac{R_2}{R_1}$$

Primer neinvertirajoči ojačevalnik



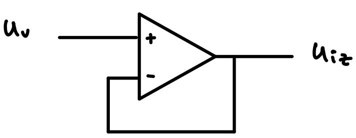
ker je neg. povr. zanka je po dolgem času

$$I = \frac{U_{iz} - U_v}{R_1} = \frac{U_v}{R_2}$$

$$U_i = U_v\left(1 + \frac{R_1}{R_2}\right)$$

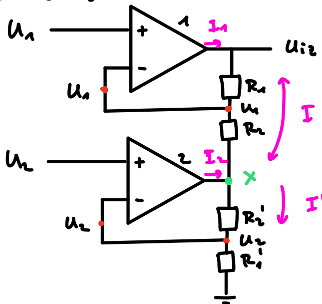
$$H = \frac{U_i}{U_v} = 1 + \frac{R_1}{R_2}$$

Primer Napetostni sledilnik



$\Rightarrow$ $U_{iz} = U_v$ $H = 1$

Primer odštevalnik



Izberi R_1, R_2, R_1', R_2' da bo $U_{iz} \propto U_1 - U_2$

$$I' = \frac{x - U_2}{R_2'} = \frac{U_2}{R_1'} \qquad x = U_2\left(1 + \frac{R_2'}{R_1'}\right)$$

$$I = \frac{U_{iz} - U_1}{R_1} = \frac{U_1 - x}{R_2}$$

$$U_{iz} = U_1\left(1 + \frac{R_1}{R_2}\right) - \frac{R_1}{R_2}x$$

$$= \left(1 + \frac{R_1}{R_2}\right)U_1 - \frac{R_1}{R_2}\left(1 + \frac{R_2'}{R_1'}\right)U_2$$

Želim $U_{iz} = K(u_1 - u_2)$

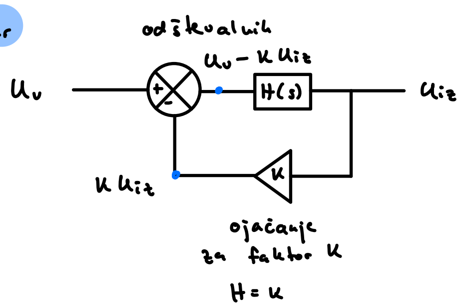
$$\Rightarrow \quad 1 + \frac{R_1}{R_2} = \frac{R_1}{R_2}\left(1 + \frac{R_2'}{R_1'}\right)$$

$$1 = \frac{R_1}{R_2}\frac{R_2'}{R_1'} \quad \Rightarrow \quad \frac{R_1}{R_2} = \frac{R_1'}{R_2'}$$

$$\Rightarrow \quad U_{iz} = \underbrace{\left(1 + \frac{R_1}{R_2}\right)}_{\text{ojačanje}}(u_1 - u_2)$$

$$H = \frac{U_{iz}}{U_{vh}} = \frac{U_{iz}}{U_1 - U_2} = 1 + \frac{R_1}{R_2}$$

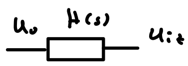
Primer



$H(s) = \frac{\tau s}{1+\tau s}$ diferenciator

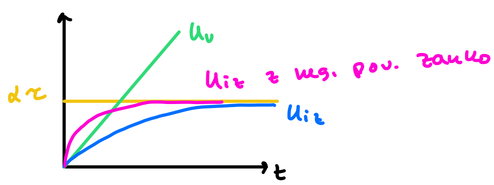
Želimo 3x hitreje slediti odvodu signala $U_i = \alpha t$

ⓐ brez povratne zanke



$U_v = \alpha \cdot t$ $\quad U_v(s) = \frac{\alpha}{s^2}$ $\quad U_{iz} = \frac{\tau s}{1+\tau s}\frac{\alpha}{s^2} = \alpha\tau\left(\frac{A}{s} + \frac{B}{1+\tau s}\right)$

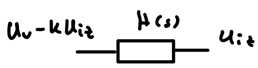
$A = 1 \quad B = -\tau$



$$U_{iz}(s) = \alpha\tau\left(\frac{1}{s} - \frac{1}{s + 1/\tau}\right)$$

$$U_{iz}(t) = \alpha\tau\left(1 - e^{-t/\tau}\right)$$

ⓑ z neg. povratno zanko



$$U_{iz} = H(s)\left(U_v - kU_{iz}\right)$$

$$\tilde{H}(s) = \frac{U_{iz}}{U_v} = \frac{H(s)}{1 + kH(s)} = \frac{\tau s}{1 + \tau s + k\tau s} = \frac{\tau s}{(1+k)\tau s + 1}\frac{(1+k)}{(1+k)} = \frac{1}{1+k}\frac{\tau' s}{1 + \tau' s}$$

$\tau' = \tau(1+k)$

Želim $\tau' = \frac{1}{3}\tau$

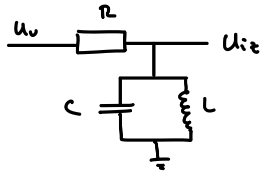
$$\frac{1}{3}\tau = \tau(1+k) \quad \Rightarrow \quad k = -\frac{2}{3}$$

$$U_{iz} = \frac{1}{1+k}\alpha\tau'\left(1 - e^{-t/\tau'}\right)$$

$$= \alpha\tau\left(1 - e^{-t/\tau'}\right)$$

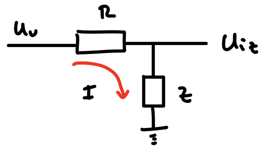
Resonančne vezje (filtri 2. reda)

Pasovno prepustni filter (BPF)



$H(s) = ?$

$\Rightarrow$ ekvivalentno vezje



$U_{iz} = I Z$

$I = \frac{U_v}{R+Z}$ $\Rightarrow$ $H = \frac{U_{iz}}{U_v} = \frac{Z}{R+Z}$

$$\frac{1}{Z} = \frac{1}{Z_C} + \frac{1}{Z_L} = i\omega C + \frac{1}{i\omega L} = \frac{1-\omega^2 LC}{i\omega L}$$

$$\Rightarrow H = \frac{1}{1 + \frac{R}{i\omega L}(1-\omega^2 LC)}$$

$$= \frac{i\omega L}{R - \omega^2 LRC + i\omega L} \qquad s = i\omega$$

$$H(s) = \frac{sL}{R + s^2 LRC + sL}$$

$$= \frac{s\,\frac{1}{RC}}{s^2 + s\frac{1}{RC} + \frac{1}{LC}} \qquad \omega_0^2 = \frac{1}{LC}$$

$$= \frac{s\omega_c}{s^2 + s\omega_c + \omega_0^2} \qquad \omega_c = \frac{1}{RC}$$

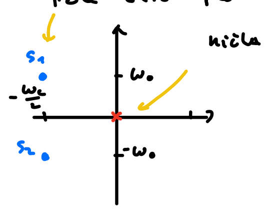
Iščemo pole $H(s)$

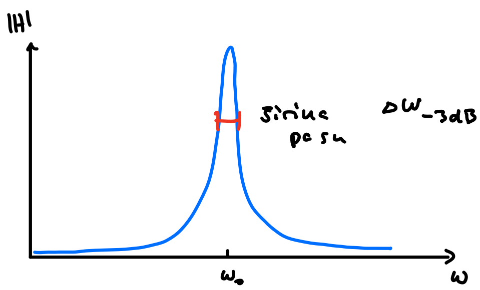
$$s_{1,2} = \frac{-\omega_c \pm \sqrt{\omega_c^2 - 4\omega_0^2}}{2} = -\frac{\omega_c}{2} \pm \frac{i}{2}\sqrt{4\omega_0^2 - \omega_c^2}$$

Ponavadi želimo $\omega_0 \gg \omega_c$: $\quad s_{1,2} = -\frac{\omega_c}{2} \pm i\omega_0$

$$\Delta\omega_{-3dB} = 2\,|\mathrm{Re}\, s_1|$$

Pole-zero plot





$$H(s) = \frac{s\omega_c}{(s-s_1)(s-s_2)}$$

$$|H(i\omega)| = \frac{\omega\omega_c}{|i\omega - s_1|\,|i\omega - s_2|}$$

Približek $\quad \tilde{H} = \frac{1}{s-s_1} \quad |\tilde{H}| = \frac{1}{|i\omega - s_1|} \quad$ za $\omega \approx \omega_0$

$$\tilde{H} = \frac{1}{s-s_1} = \frac{1}{i\omega - i\omega_0 + \frac{\omega_c}{2}} \Rightarrow |\tilde{H}| = \frac{1}{\sqrt{(\omega-\omega_0)^2 + (\frac{\omega_c}{2})^2}}$$

$\Delta\omega_{-3dB}$: $\quad \Delta\omega$ pri $\quad 20\log\left(\frac{|\tilde{H}|}{|\tilde{H}|_{max}}\right)dB = -3dB$

$$10\log\left(\frac{|\tilde{H}|^2}{|\tilde{H}|^2_{max}}\right) = -3dB$$

$$|\tilde{H}|^2_{max}\Big|_{\omega=\omega_0} = \left(\frac{2}{\omega_c}\right)^2$$

$$|\tilde{H}|^2 = |\tilde{H}|^2_{max}\,\underbrace{10^{-0,3}}_{\sim 1/2} = \frac{2}{\omega_c^2}$$

$$(\omega-\omega_0)^2 + \left(\frac{\omega_c}{2}\right)^2 = \frac{\omega_c^2}{2}$$

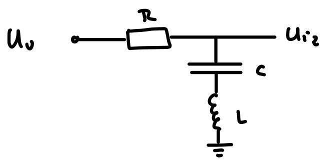
$$(\omega-\omega_0)^2 = \frac{\omega_c^2}{4}$$

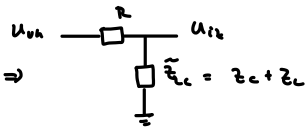
$$\omega_{1,2} = \omega_0 \pm \frac{\omega_c}{2} \quad \Rightarrow \quad \Delta\omega = \omega_c$$

Q ... quality $\qquad Q = \frac{\omega_0}{\Delta\omega_{-3dB}} = \frac{\omega_0}{\omega_c} >> 1$

$$Q = \frac{RC}{\sqrt{LC}}$$

Pasovni neprepustni filter



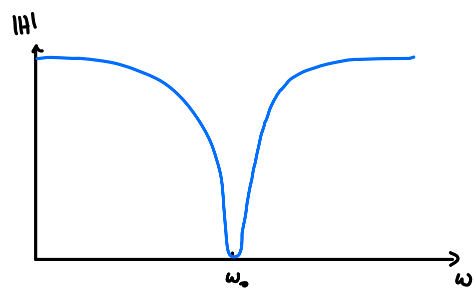


$$\Rightarrow \quad H = \frac{\tilde{Z}_{LC}}{R + \tilde{Z}_{LC}} \qquad Z_{LC} = \frac{1}{i\omega C} + i\omega L = \frac{1}{sC} + sL = \frac{1+s^2LC}{sC}$$

$$= \frac{1+s^2LC}{RCs + 1 + s^2LC}$$

$$= \frac{s^2 + \frac{1}{LC}}{s^2 + s\frac{R}{L} + \frac{1}{LC}} \qquad \omega_0^2 = \frac{1}{LC} \quad \omega_c = \frac{R}{L}$$

$$= \frac{s^2 + \omega_0^2}{s^2 + s\omega_c + \omega_0^2}$$



$$\Delta\omega_{-3dB} = \omega_c = \frac{R}{L}$$

$$Q = \frac{\omega_0}{\Delta\omega} = \frac{\omega_0}{\omega_c} = \frac{L}{R}\frac{1}{\sqrt{LC}}$$

Statistika

Ocenjevanje parametrov normalne porazdelitve

$$z_i \sim N(a, \sigma^2)$$

Vzorec $\{z_i\}$, N izmerkov

Vzorčni statistiki

ⓐ cenilka (estimator) za pričakovano vrednost a

$u \Rightarrow \langle u \rangle = a$ $\qquad$ (nepristranskost)

Če je $u = k\sum z_i$, želim $a = \langle u \rangle = k\sum\langle z_i\rangle = kNa \Rightarrow k = \frac{1}{N}$

$\Rightarrow$ cenilka u je povprečje

⑥ standardne deviacije

$$s^2 = \frac{1}{N-1}\sum (z_i-\bar z)^2 \quad , \quad \langle s^2\rangle = \sigma^2 \quad \text{nepristranskost}$$

smo za porabili eno dimenzijo za povprečje

Kaj če poznamo a?

$$w^2 = k\sum (z_i-a)^2 \qquad \langle w^2\rangle = \sigma^2$$

$$\langle w^2\rangle = \sigma^2 = k\sum \underbrace{\langle (z_i-a)^2\rangle}_{\sigma^2} = kN\sigma^2 \quad \Rightarrow \quad k=\frac{1}{N}$$

χ^2 porazdelitvene funkcije

Ocenjevanje variance σ^2:

št. prost. stopenj

$$X_i \sim N(0,1) \quad , \quad \chi^2 = \sum X_i^2 \sim \frac{dP}{d\chi^2}(k) = \chi^2(k)$$

sprem. χ^2 funkcija χ^2

$$z_i \sim N(a,\sigma^2)$$

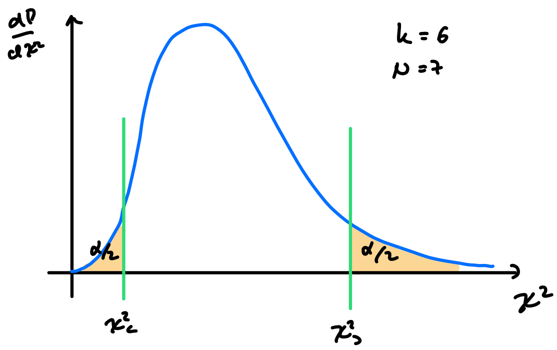
$$X_i = \frac{z_i-a}{\sigma} \sim N(0,1) \qquad \chi^2 = \sum\frac{(z_i-a)^2}{\sigma^2} \sim \chi^2(k)$$

$k=N$

$$\chi^2 = \sum \frac{(z_i-\bar z)^2}{\sigma^2} \sim \chi^2(N-1)$$

$$= (N-1)\frac{s^2}{\sigma^2} \sim \chi^2(N-1)$$

Iščemo interval zaupanja za σ^2



α ... stopnja tveganja (0'01, 0'05, 0'1)

Iščemo χ_s^2

χ_s^2 : $P(\chi^2 > \chi_s^2) = \frac{\alpha}{2}$

χ_l^2 : $P(\chi^2 < \chi_l^2) = \frac{\alpha}{2}$

$\Leftrightarrow P(\chi^2 > \chi_l^2) = 1-\frac{\alpha}{2}$

$$\sigma^2 = (N-1)\frac{s^2}{\chi^2}$$

$\Rightarrow \boxed{\sigma^2_{l,s} = (N-1)\frac{s^2}{\chi^2_{s,l}}}$ Interval zaupanja pri stopnji tveganja α

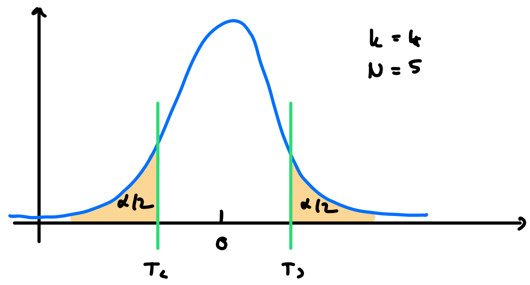
Studentova porazdelitvena funkcija

Ocenjevanje intervala zaupanja za a

$$T = \frac{\bar z - a}{s}\sqrt{N} \qquad \bar z, s \text{ - vzorčni statistiki}, \; N \text{ - velikost vzorca}$$

$$T \sim \frac{dP}{dT}(k) = S(k) \qquad k \text{ ... št. prost. stopenj}$$

$k = N-1$



$$P(|T| > T_>) = \alpha$$

$$a = \bar z \pm T_> \frac{s}{\sqrt{N}}$$

$$T_< = -T_>$$

Alternativni del. T-statistike:

$$T = \frac{X}{\sqrt{\frac{Y}{N-1}}} \sim S(N-1) \qquad X \sim N(0,1), \; Y \sim \chi^2(N-1)$$

Testiranje hipotez

Privzemimo neko hipotezo $H_0 : a = a_0$ (nulta hipoteza)

Imamo vzorec $\{z_i\}_1^N \Rightarrow \bar z, s^2$

$$T_* = \frac{\bar z - a_0}{s}\sqrt{N} \text{ ... dobimo vrednost } T \text{ statistike ob privzetku da je } a = a_0$$

Poiščemo $T_>$, $P(|T| > T_>) = \alpha \Rightarrow T_>(\alpha, N-1)$

Če je $|T_*| > T_>$ (v repu porazdelitve) hipotezo zavržemo

Če testiramo varianco (σ^2); $H_0 : \sigma^2 = \sigma_0^2$

$$\chi^2 = (N-1)\frac{s^2}{\sigma^2} \sim \chi^2(N-1)$$

$$\chi_*^2 = (N-1)\frac{s^2}{\sigma_0^2} \text{ ... vrednost } \chi^2 \text{ ob privzetku } \sigma^2 = \sigma_0^2$$

Poiščemo $\chi_>^2$ in $\chi_<^2$, $P(\chi^2 > \chi_>^2) = \frac{\alpha}{2} \Rightarrow \chi_>^2$

Če je χ_*^2 v repih, zavržemo hipotezo pri stopnji tveganja α.

Primerjava dveh vzorcev (dvakrat merimo isto stvar in primerjamo)

$\{x_i\}$: $X_i \sim N(a_x, \sigma^2)$, hipoteza $a_x = a_y$
$\{y_i\}$: $Y_i \sim N(a_y, \sigma^2)$, med meritvami se ni zgodilo nič

$$T_* = \frac{(\bar{x}-\bar{y}) - \overbrace{(a_x - a_y)}^{\text{po hipotezi}}}{\sqrt{\frac{1}{N_x}+\frac{1}{N_y}}} \frac{\sqrt{N_x+N_y-2}}{\sqrt{s_x^2(N_x-1)+s_y^2(N_y-1)}} \sim S(N_x+N_y-2)$$

Preverimo da T_* ni v repih.

Primerjava disperzij σ^2 (F-statistika)

$$\left.\begin{array}{l} X_1 \sim \chi^2(k_1) \\ X_2 \sim \chi^2(k_2) \end{array}\right\} F = \frac{\frac{X_1}{k_1}}{\frac{X_2}{k_2}} \sim \frac{dP}{dF}(k_1,k_2) = F(k_1,k_2)$$

Fisher - Snedecor porazdelitev

Vemo $X_1 = k_1 \frac{s_1^2}{\sigma_1^2} \sim \chi^2(k_1)$ $\quad k_1 = N_1 - 1$

$$F = \frac{s_1^2/\sigma_1^2}{s_2^2/\sigma_2^2} \sim F(k_1,k_2) = F(N_1-1, N_2-1)$$

$H_0: \sigma_1^2 = \sigma_2^2 \Rightarrow F_* = \frac{s_1^2}{s_2^2}$

Kritični vrednosti F : $F_{c>} = F_{\alpha/2}(k_1,k_2)$
$F_{c<} = F_{1-\frac{\alpha}{2}}(k_1,k_2) = (F_{\alpha/2}(k_2,k_1))^{-1}$

Preverimo če je F_* v repih

Primer $x_i = \{0'44, 0'46, 0'50, 0'47, 0'48, 0'47\}$

(a) $N = 6$, $\alpha = 10\%$, $H_0: a = a_0 = 0'50$
$k = 5$

$\bar{x} = \frac{1}{N}\sum x_i = 0'47$ $\qquad s^2 = \frac{1}{N-1}\sum(x_i-\bar{x})^2 = (0{,}02)^2$

$T_* = \frac{\bar{x}-a_0}{s}\sqrt{N} = -3{,}67$

tabela

$T \sim S(k=5)$ $\qquad P(|T| > T_c) = \alpha = 10\% \Rightarrow T_c(\alpha=10\%, k=5) = 2{,}015$

$|T_*| > T_c$ ✓ $\quad T_*$ je v repu , $a \neq a_0$, hipotezo zavrnemo , pri $\alpha = 10\%$

(b) Ponovimo meritev

$y_i = \{0'52, 0'50, 0'55, 0'53, 0'52, 0'53\}$ $\qquad N_y = 6$

$H_0: a_x = a_y$ $\qquad \bar{y} = \dots = 0{,}525$ $\qquad s_y^2 = \dots = (0{,}016)^2$

$$T_* = \frac{(\bar{x}-\bar{y})}{\sqrt{\frac{1}{N_x}+\frac{1}{N_y}}} \frac{\sqrt{N_x+N_y-2}}{\sqrt{s_x^2(N_x-1)+s_y^2(N_y-1)}} = 5{,}7 \quad k = N_x+N_y-2 = 10 \Rightarrow T_c(\alpha=10\%, k=10) = 1{,}81$$

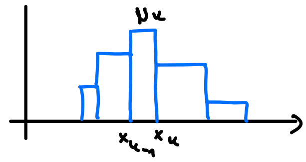
hipotezo ovržemo

Oblikovni testi

Testiranje porazdelitve $\Rightarrow$ $H_0 : \frac{dP}{dx} = f(x)$ privzeta porazdelitev

Pearsonov χ^2 (χ^2_P) test

$\rightarrow$ izračunamo histogram iz $\{z_i\}_1^N$



N_k ... št. izmerkov iz vzorca, ki padejo v k-ti razred

Pričakovano število izmerkov v k-tem razredu:

$N p_k$ $\qquad N = \sum N_k$

verjetnost, da izmerek pade v k-ti razred

$$p_k = \int_{x_{k-1}}^{x_k} \frac{dP}{dx}\, dx$$

Def:

$$\chi^2_P = \sum_{k=1}^{g} \frac{(N_k - Np_k)^2}{N p_k} \sim \chi^2(g-1) \qquad g \text{ ... št. razredov}$$

Velja za $N \cdot p_k \geq 5$!

$\Rightarrow$ Enostranski test

$$P(\chi^2 > \chi_c^2) = \alpha$$

- $\chi^2_P \geq \chi^2_c$ $\Rightarrow$ zavržem privzeto porazdelitev pri α

Primer Met kocke

Ali je kocka pravična, pri $\alpha = 10\%$?

#	št. dosedkov N_k
1	5
2	8
3	9
4	8
5	10
6	20

$p_k = \frac{1}{6}$, $N \cdot p_k = 60 \cdot \frac{1}{6} = 10 \geq 5$ ✓ , $g = 6$

$$\chi^2 = \frac{(5-10)^2}{10} + \frac{(8-10)^2}{10} + \dots + \frac{(20-10)^2}{10} = 13{,}4$$

$P(\chi^2 > \chi_c^2) = \alpha = 10\%$ $\Rightarrow$ $\chi_5^2 = 15{,}086$

$\Rightarrow$ $\chi^2 < \chi_5^2$ ✓ kocka je pravična

Linearna metoda najmanjših kvadratov

Na podlagi izmerkov $\vec{z}$ iščemo $\vec{x}$ da velja $\vec{z} = H\vec{x} + \vec{r}$.

$2J = \vec{r}^T R^{-1} \vec{r}$ $\qquad$! $R = \sigma^2 I$

$2J = (\vec{z} - H\vec{x})^T R^{-1} (\vec{z} - H\vec{x})$ $\qquad$ $\frac{\partial 2J}{\partial \vec{x}} = 0 \Rightarrow \hat{\vec{x}}$

$\Rightarrow$ $\boxed{\hat{\vec{x}} = (H^T H)^{-1} H^T \vec{z} \qquad P = (H^T H)^{-1} \sigma^2}$

Na primer

$$\vec{z} = \begin{pmatrix} z_1 \\ \vdots \\ z_N \end{pmatrix} \; \dots \text{ meritve os različnih trenutkih}$$

$$\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_M \end{pmatrix} \; \dots \text{ parametri modela}$$

model: $z_i = x_0 f_0(t_i) + x_1 f_1(t_i) + \dots + x_{M-1} f_{M-1}(t_i) + r_i$

$$\Rightarrow \quad \vec{z} = H\vec{x} + \vec{r} \qquad H \dots \; N \times M \text{ matrika}$$

Kaj pa v primeru neenakih varianc? Uteženo linearno metodo najmanjših kvadratov

$$(R)_{ij} = \sigma_i^2 \delta_{ij}$$

$$\hat{x} = (H^T R^{-1} H)^{-1} H^T R^{-1} \vec{z}$$
$$P = (H^T R^{-1} H)^{-1}$$

$$2J = (\vec{z} - H\vec{x}) R^{-1} (\vec{z} - H\vec{x})^T \sim \chi^2(N-M)$$

Primer

$$z_i = x_0 \underbrace{t_i}_{f(t_i)} + r_i \qquad \text{en parameter}$$

$$\Rightarrow H = \begin{pmatrix} t_1 \\ \vdots \\ t_N \end{pmatrix} \qquad \Rightarrow \quad H^T H = t_1^2 + \dots + t_N^2 = \sum t_i^2$$

$$P = \frac{\sigma^2}{\sum t_i^2} \qquad \hat{x}_0 = \frac{\sum t_i z_i}{\sum t_i^2}$$

Primer

$$z_i = x_0 t_i + x_1 + r_i$$

$$\vec{x} = \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} \qquad H = \begin{pmatrix} t_1 & 1 \\ \vdots & \vdots \\ t_N & 1 \end{pmatrix} \qquad H^T H = \begin{pmatrix} \sum t_i^2 & \sum t_i \\ \sum t_i & N \end{pmatrix}$$

Inverz 2x2 matrike

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\det} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$P = \sigma^2 (H^T H)^{-1} = \frac{1}{N\sum t_i^2 - (\sum t_i)^2} \begin{pmatrix} N & -\sum t_i \\ -\sum t_i & \sum t_i^2 \end{pmatrix} \sigma^2$$

$$H^T z = \begin{pmatrix} t_1 & \dots & t_N \\ 1 & \dots & 1 \end{pmatrix} \begin{pmatrix} z_1 \\ \vdots \\ z_N \end{pmatrix} = \begin{pmatrix} \sum t_i z_i \\ \sum z_i \end{pmatrix}$$

$$\hat{x} = (H^T H)^{-1} H^T z = \frac{1}{N\sum t_i^2 - (\sum t_i)^2} \begin{pmatrix} N \sum t_i z_i - \sum t_i \sum z_i \\ -\sum t_i \sum t_i z_i + \sum t_i^2 \sum z_i \end{pmatrix}$$

Poenostavimo

$$\hat{x}_0 = \frac{1}{I} \left(N \sum t_i z_i - \sum t_i \sum z_i \right) \qquad / \cdot \sum t_i$$
$$\hat{x}_1 = \frac{1}{I} \left(\sum t_i^2 \sum z_i - \sum t_i \sum t_i z_i \right) \qquad / \cdot N$$

$$\left.\begin{aligned} I x_0 \sum t_i &= N \sum t_i \sum t_i z_i - (\sum t_i)^2 \sum z_i \\ I x_1 N &= N \sum t_i^2 \sum z_i - N \sum t_i \sum t_i z_i \end{aligned}\right] +$$

$$I (x_0 \sum t_i + x_1 N) = \sum z_i \underbrace{\left(N \sum t_i^2 - (\sum t_i)^2 \right)}_{I}$$

$$\hat{x}_0 \sum t_i + \hat{x}_1 N = \sum z_i$$

$$\hat{x}_1 = \underbrace{\frac{1}{N} \sum z_i}_{\bar{z}} - \underbrace{\frac{1}{N} \sum t_i}_{\bar{t}} \hat{x}_0 \qquad \Rightarrow \qquad \boxed{\hat{x}_1 = \bar{z} - \bar{t}\, \hat{x}_0}$$

Poenostavimo še x_0

$$\hat{x}_0 = \frac{1}{N\sum t_i^2 - \underbrace{(\sum t_i)^2}_{N^2\bar{t}^2}}\left(N\sum t_i z_i - \underbrace{\sum t_i \sum z_i}_{N^2\bar{t}\,\bar{z}}\right)$$

$$= \frac{1}{N\sum t_i^2 - N^2\bar{t}^2}\left(N\sum t_i z_i - N^2\bar{t}\,\bar{z}\right)$$

$$= \frac{\overline{tz} - \bar{t}\,\bar{z}}{\overline{t^2} - \bar{t}^2}$$

Primer

poraba goriva P (l/100km)	hitrost avta v (km/h)
4,8	60
5,0	72
7,1	90
8,2	120
11,0	150

Napaka ne presega 0,5 L/100km. Ali lahko pri stopnji tveganja $\alpha = 10\%$ zavrnemo tezo, da se poraba goriva spreminja s kvadratom hitrosti;
$P = P_0 + \beta v^2$

(a) Iščemo $\hat{P}_0$, $\hat{\beta}$ (najmanjši kvadrati)

$$P_i = \underbrace{v_i^2}_{t_i}\underbrace{\beta}_{x_0} + 1\cdot\underbrace{P_0}_{x_1} + r_i$$ (lahko uporabimo prejšnje formule)

$$\hat{x} = \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} = \begin{pmatrix} \beta \\ P_0 \end{pmatrix} \qquad t = \begin{pmatrix} v_1^2 \\ \vdots \\ v_N^2 \end{pmatrix}$$

$$\hat{\beta} = \hat{x}_0 = \frac{\overline{tz} - \bar{t}\,\bar{z}}{\overline{t^2} - \bar{t}^2} \qquad \hat{P}_0 = \hat{x}_1 = \bar{z} - \bar{t}\,\hat{x}_0$$

$z = P$	v	$t = v^2$	$t^2\,[10^4]$	tz
4,8	60	3600	1296	17280
5,0	72	5184	2687	25920
7,1	90	8100	6561	57510
8,2	120	14400	20736	118080
11,0	150	22500	50625	247500

Povprečja
$\bar{z} = 7{,}22$
$\bar{t} = 10756{,}8$
$\overline{t^2} = 16381\cdot 10^4$
$\overline{tz} = 93258$

$$\Rightarrow \quad \hat{\beta} = \hat{x}_0 = 3{,}24\cdot 10^{-4}\ \frac{l}{100km}\,(km/h)^{-2}$$
$$\hat{P}_0 = \hat{x}_1 = 3{,}73\ \frac{l}{100km}$$

(b) Preverimo hipotezo

$$2J = \chi^2 = (\vec{z} - H\hat{x})^T R^{-1} (\vec{z} - H\hat{x}) \qquad R = \sigma^2 I \qquad \sigma = 0{,}5\ \frac{l}{100km}$$

$z = P_0$	$t = v^2$	$P(v_i, \hat{x}) = \beta v^2 + P_0$
4,8	3600	4,90
5,0	5184	5,41
7,1	8100	6,35
8,2	14400	8,40
11,0	22500	11,02

$$\chi^2 = \sum \frac{(P_i - P(v_i,\hat{x}))^2}{\sigma^2} = \ldots = 3{,}1$$

Naredim χ^2 test (enostranski test)

$P(\chi^2 > \chi_c^2) = \alpha$, $k = N - M = 5 - 2 = 3$, $\alpha = 10\%$

$\Rightarrow \chi_c^2 = 6{,}25$ ... modela ne zavrnemo na st. tveganja 10%

ⓒ Če zaupamo modelu in ne poznamo napak σ

$$2J = \chi^2 = \frac{\sum (p_i - p(v_i))^2}{\sigma^2}$$

lahko poiščemo interval zaupanja za σ^2

$$\sigma^2_{\substack{<\\>}} = \frac{1}{\chi^2_{\substack{<\\>}}} \sum (p_i - p(v_i))^2$$

ⓓ Reduciran χ^2_r : če je $\chi^2 \sim \chi^2(N-M)$ je $\langle \chi^2 \rangle = N-M$

$$\Rightarrow \quad \chi^2_r = \frac{\chi^2}{N-M} \quad \Rightarrow \quad \langle \chi_r^2 \rangle = 1$$

Primer meritve t_i, I_i, model $I_i = I_0 e^{-t_i/\tau}$

Ali velja model pri $\alpha = 10\%$.

Iščemo τ, problem je ker je nelinearen model

Lineariziramo model $\underbrace{\ln\left(\frac{I_i}{I_0}\right)}_{z_i} = \underbrace{-\frac{1}{\tau}}_{x_0} t_i \quad \Rightarrow \quad z_i = x_0 t_i$

Poznamo že model $\hat{x}_0 = \frac{\sum z_i t_i}{\sum t_i^2} \quad \Rightarrow \quad \hat{x}_0 = -3{,}45 \cdot 10^{-3}\,\text{min} \quad \Rightarrow \quad \hat{\tau} = 290\,\text{min}$